Logical Reasoning

Russell Chap. 6, 8, 9

Luger Chap. 2, 13, 14

Propositional Calculus

- Proposition
  - A statement that is true or false
  - Ex> "It is raining", "1+1 = 2"
- Symbols

  Propositional symbols: P, Q, R, …

  Truth symbols: T, F

  Connectives: $\wedge, \vee, \sim, \Rightarrow, =$

Propositional Calculus

- Sentences
  1. Propositional symbols → sentence
  2. If s1, s2 is a sentence,
     $\sim$s1, s1 $\wedge$ s2, s1 $\vee$ s2, s1 $\Rightarrow$ s2, s1 $=$ s2 → sentence
  3. (sentence) → sentence

⇨ Well-formed formula (WFF)

- $P \vee Q$, $(P \wedge Q) \Rightarrow R$ O
- $P \vee \wedge Q$ X

Propositional Calculus

- Interpretation (semantics)

P	Q	$\sim P$	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

Ex> If $a = b$ then $a^2 = b^2$: T $\rightarrow$ $a = 1, b = -1$: ?

Propositional Calculus

- Inference
  - Generating new sentences X from a set of sentences S

$P \Rightarrow Q$

P

Q

  - Inference can be made by using valid implications

P	Q	$P \Rightarrow Q$	$((P \Rightarrow Q) \wedge P) \Rightarrow Q$
T	T	T	**T**
T	F	F	**T**
F	T	T	**T**
F	F	T	**T**

    - If we know $P \Rightarrow Q$ is T and P is T, then we can conclude Q is T

Propositional Calculus

- Example
  - If you get a cold, then you get a fever
    He get a cold ⇒ He must have a fever
  - If you get a cold, then you get a fever
    He have a fever ⇒ He must get a cold

P	Q	$P \Rightarrow Q$	$((P \Rightarrow Q) \wedge Q) \Rightarrow P$
T	T	T	**T**
T	F	F	**T**
F	T	T	**F**
F	F	T	**T**

Propositional Calculus

- Inference rules
  - (S1 $\vdash$ S2 : from S1, generate S2)

  - Modus Ponens $P \Rightarrow Q,\quad P \quad \vdash \quad Q$
  - Modus Tolens $P \Rightarrow Q,\quad \sim Q \quad \vdash \quad \sim P$
  - And Elimination $P \wedge Q \quad \vdash \quad P, Q$
  - Unit Resolution $P \vee Q,\quad \sim P \quad \vdash \quad Q$

→ Program?

```
char* modus_ponens(char* s1, char* s2) {
    if(is_implication(s1))
        if(equal(left_side(s1), s2))
            return(right_side(s1));
    else …
```

Propositional Calculus

- Example
  - $L \vee \sim(M \wedge N)$

    M
  - $A \wedge (B \vee C) \Rightarrow D \wedge E$

    $B \vee C$

    A
  - $R \Rightarrow V \wedge W$

    $\sim Q \vee R \vee S$

    $\sim S \wedge Q$

Propositional Calculus

- Example
  - $L \vee \sim(M \wedge N)$

    M

  - $A \wedge (B \vee C) \Rightarrow D \wedge E$

    $B \vee C$

    A

  - $R \Rightarrow V \wedge W$

    $\sim Q \vee R \vee S$

    $\sim S \wedge Q$

Propositional Calculus

- Modus Ponens $P \Rightarrow Q,\quad P \quad \vdash \quad Q$
- Modus Tolens $P \Rightarrow Q,\quad \sim Q \quad \vdash \quad \sim P$
- And Elimination $P \wedge Q \quad \vdash \quad P, Q$
- Unit Resolution $P \vee Q,\quad \sim P \quad \vdash \quad Q$

- Example
  - $R \Rightarrow V \wedge W$

    $\sim Q \vee R \vee S$

    $\sim S \wedge Q$

A.E: $\sim S \wedge Q \quad \vdash \quad \sim S, Q$

U.R: $\sim Q \vee R \vee S, Q \quad \vdash \quad R \vee S$

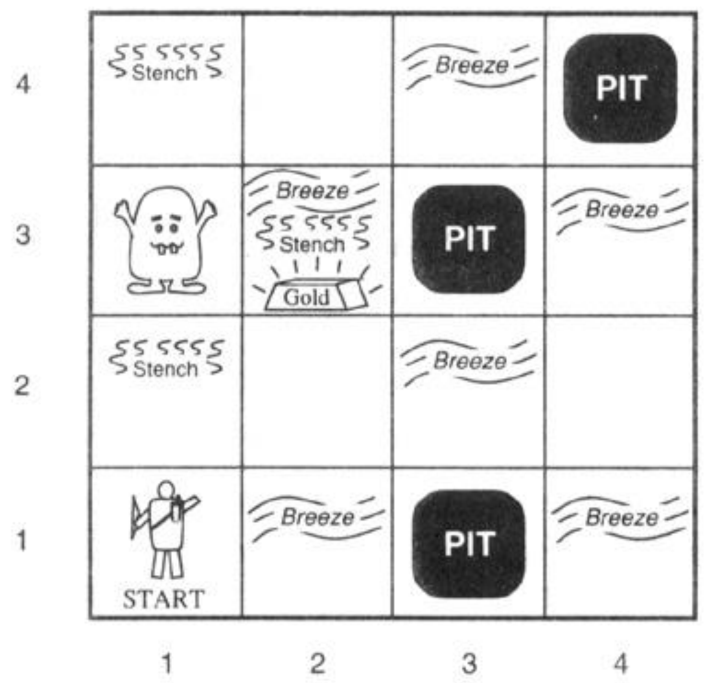
U.R: $R \vee S, \sim S \quad \vdash \quad R$

M.P: $R \Rightarrow V \wedge W, R \quad \vdash \quad V \wedge W$

$V \wedge W$

Example

- Wumpus game

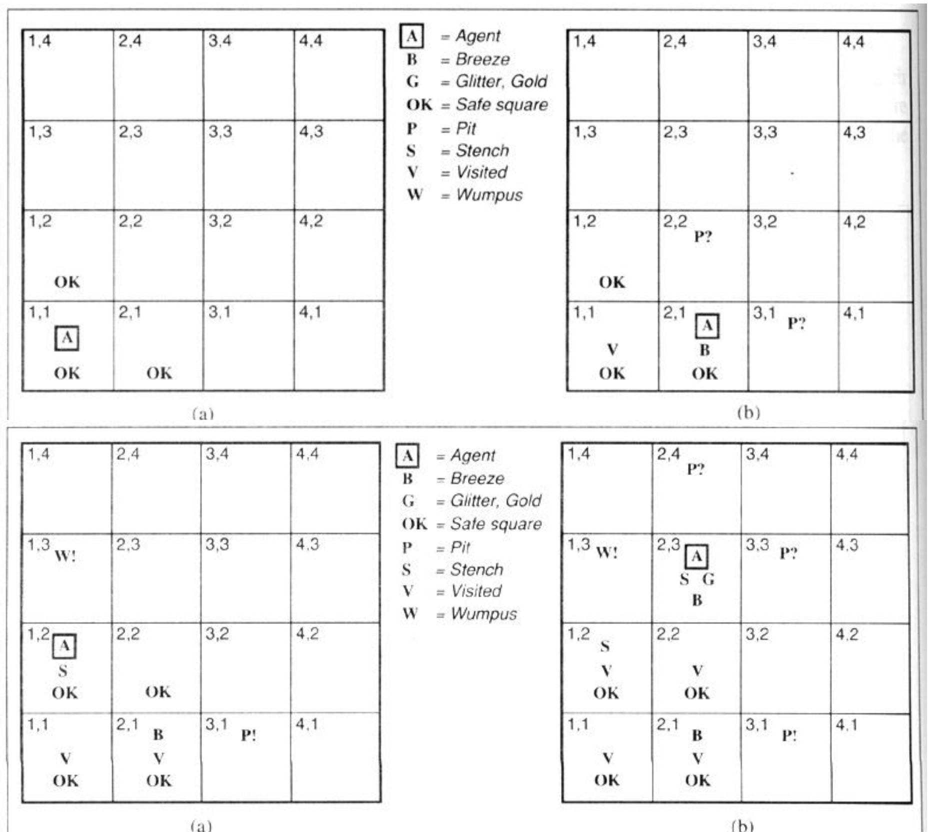


	S	
S	**W**	S
	S	

	B	
B	**P**	B
	B	

Example

S



B

Figure 6.4 Two later stages in the progress of the agent. (a) After the third move, with percept [*Stench, None, None, None, None*]. (b) After the fifth move, with percept [*Stench, Breeze, Glitter, None, None*].

Example

- Rules
  - $S12 \Rightarrow W13 \vee W12 \vee W22 \vee W11$
  - $\sim S11 \Rightarrow \sim W11 \wedge \sim W12 \wedge \sim W21$
  - $\sim S21 \Rightarrow \sim W11 \wedge \sim W21 \wedge \sim W22 \wedge \sim W31$
  - $\sim S12 \Rightarrow \sim W11 \wedge \sim W12 \wedge \sim W22 \wedge \sim W13$
  - …
- Facts
  - ~S11, ~B11
  - ~S21, B21
  - S12, ~B12

W?			
S	W?		
W?			

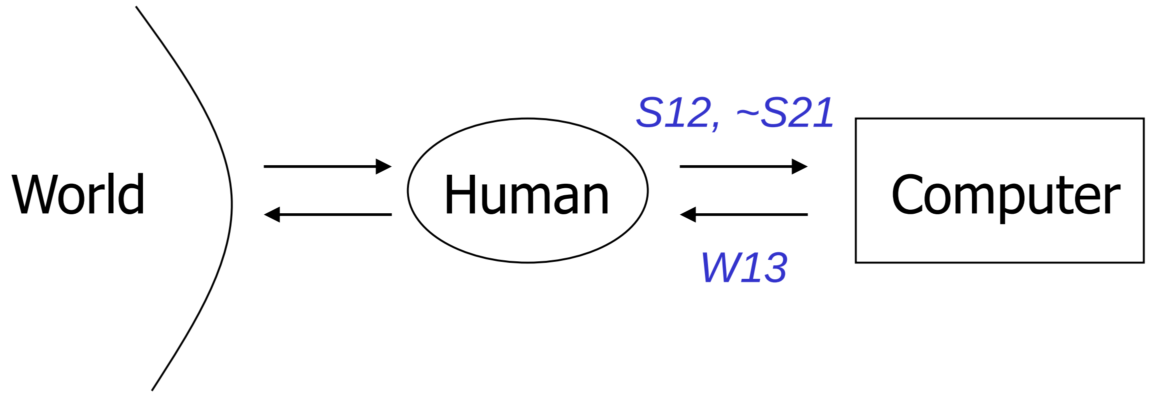
Example

- Inference

1. $(\sim S11), (\sim S11 \Rightarrow \sim W11 \wedge \sim W12 \wedge \sim W21)$ **M. P.** , then **A. E.**
   $\rightarrow$ **$\underline{\sim W11, \sim W12, \sim W21}$**
2. $(\sim S21), (\sim S21 \Rightarrow \sim W11 \wedge \sim W21 \wedge \sim W22 \wedge \sim W31)$ **M. P.** , then **A. E.**
   $\rightarrow$ **$\underline{\sim W11, \sim W21, \sim W22, \sim W31}$**
3. $(S12), (S12 \Rightarrow W13 \vee W12 \vee W22 \vee W11)$ **M. P.**
   $\rightarrow$ **$\underline{W13 \vee W12 \vee W22 \vee W11}$**
4. $(\sim W11), (W13 \vee W12 \vee W22 \vee W11)$ **U. R.**
   $\rightarrow$ **$\underline{W13 \vee W12 \vee W22}$**
5. $(\sim W22), (W13 \vee W12 \vee W22)$ **U. R.**
   $\rightarrow$ **$\underline{W13 \vee W12}$**
6. $(\sim W12), (W13 \vee W12)$ **U. R.**
   $\rightarrow$ **$\underline{W13}$** !!!

Propositional Logic

- Inference in computers
  - *Without any understanding, valid conclusion can be made by using logical inference*



Predicate Calculus

- Predicate Calculus
  - Introduce variables

    → Access components of an individual proposition
  - Example
    - Tom is a man: **P** → **man(tom)**

      John is a man: **Q** → **man(john)**

      man(X)
    - S11 $\Rightarrow$ W11

      S12 $\Rightarrow$ W12 → **S(X,Y) $\Rightarrow$ W(X,Y)**

      S21 $\Rightarrow$ W21

Predicate Calculus

- Terms
  - Constants: refers to an object (tom, 123, …)
  - Variables: refers to a set of objects (X, Person, …)
  - Functions: maps an object to another (father_of(tom), …)

    represents *objects*
- Predicates
  - Predicates: T or F (red(apple), like(tom, jane), …)

    represents *properties* or *relations*

Predicate Calculus

- Sentences
  - Combination of atomic sentences(predicates)
  - Connectives: $\wedge$, $\vee$, ~, $\Rightarrow$, =
  - Quantifiers: constrain the scope of variables
    - $\forall$ (universal): For all
    - $\exists$ (existential): For some, there exist
    - Example

      Even(1): Even(2): Even(X):

      $\forall$ X Even(X):

      $\exists$ X Even(X):

Predicate Calculus

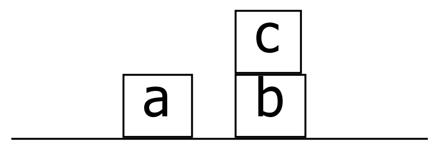
- Properties of quantifiers
  - Universal: $\forall X\, p(X) \equiv p(x1) \wedge p(x2) \wedge p(x3) \ldots$
  - Existential: $\exists X\, p(X) \equiv p(x1) \vee p(x2) \vee p(x3) \ldots$
- Negation of quantification
  - $\sim\forall x\, p(X) = \sim [p(x1) \wedge p(x2) \wedge p(x3) \ldots]$
    $= \sim p(x1) \vee \sim p(x2) \vee \sim p(x3) \ldots$
    $= \exists X \sim p(X)$
  - $\sim\exists X\, p(X) = \sim [p(x1) \vee p(x2) \vee p(x3) \ldots]$
    $= \sim p(x1) \wedge \sim p(x2) \wedge \sim p(x3) \ldots$
    $= \forall X \sim p(X)$

Examples

- Simple examples
  - Tom is a student →
  - Tom takes CS101 →
  - Everyone take CS101 →
  - Someone takes CS101 →
  - Some girls take CS101 →
  - All girls take CS101 →
  - Tom is a male student →
  - All students are male →
  - Graduate students are students →

Examples

- Blocks world



  - ontable(a), ontable(b), on(c, b), clear(a), clear(c)
  - $\forall$ X,Y ~on(X, Y) $\Rightarrow$ clear(Y)
- Family relationship
  - parent(tom, jane), parent(mike, tom)
  - $\forall$ X,Y,Z parent(X, Y) $\wedge$ parent(Y, Z) $\Rightarrow$ grand_parent(X, Z)

Inference Rules S $\vdash$ X

- Logically follows
  - X logically follows from S $\equiv$ If S is T, then X is T
    - (Q) logically follows from ($P \Rightarrow Q$, P)
- Sound rules
  - It <u>generates X that is logically follows from S</u>
    - ($P \Rightarrow Q$, P) $\vdash$ Q →
    - ($P \Rightarrow Q$, Q) $\vdash$ P →
- Complete rules
  - It can <u>generate all X that is logically follows from S</u>
    - From ($P \Rightarrow Q$, $Q \Rightarrow R$, P), (Q, R, $P \Rightarrow R$) are logically follows
    - But Modus Ponens can not generate $P \Rightarrow R$ (not complete)

Inference Rules

- Inference Rules
  - Modus Ponens $P(a) \Rightarrow q(a), \quad p(a) \quad \vdash$
  - Modus Tolens $p(a) \Rightarrow q(a), \quad \sim q(a) \quad \vdash$
  - And Elimination $p(a) \wedge q(a) \quad \vdash$
  - Unit Resolution $p(a) \vee q(a), \quad \sim p(a) \quad \vdash$
  - And Introduction $p(a), q(a) \quad \vdash$
  - *Universal Instantiation* $\forall X, p(X) \quad \vdash$
    - Ex> $\forall X$ like(X, tom) $\vdash$ like(jane, tom)

Inference Rules

- Example

1. $\forall$ X man(X) $\Rightarrow$ mortal(X)
2. man(tom)

U. I. on 1 : 3. man(tom) $\Rightarrow$ mortal(tom)

M. P. with 2, 3: 4. mortal(tom)

Unification

- Substitution
  - { B / A }: substitute A by B
  - ∀ variable → terms(constant, variable, function)
    - ∀ X man(X) → {Y/X}: ∀ Y man(Y)
      → {tom/X}: man(tom)
- Unification
  - Finding substitution that makes two expressions equal
    - unify(X, Z) = {Z/X} or {X/Z}
    - unify(X, tom) = {kim/X}
    - unify(tom, jane) = fail
    - unify(like(tom, X), like(Y, jane)) = {tom/Y, jane/X}

Inference Examples

- 1. $\forall$ X know(tom, X) $\Rightarrow$ hate(tom, X)
  2. $\forall$ Y know(Y, jane)
  Unify 1, 2 : { } →

  Apply M.P. 3, 4→

- 1. $\forall$ X know(tom, X) $\Rightarrow$ hate(tom, X)
  2. $\forall$ Y know(Y, f(Y))
  Unify 1, 2 : { } →

  Apply M.P. 3, 4 →

Inference Examples

- 1. $\forall$ X,Y,Z p(X,Y) $\wedge$ p(Y,Z) $\Rightarrow$ gp(X,Z)
  2. p(tom, jane)
  3. p(mike, tom)

Unify 1, 2 : {tom/X, jane/Y} → p(tom, jane) $\wedge$ p(,) $\Rightarrow$ gp(tom,Z)

→ No more unification

Unify 1, 3 : {mike/X, tom/Y} → p(mike, tom) $\wedge$ p(tom, Z) $\Rightarrow$ gp(mike,Z)

→ Unify with 2 : {jane/Z} → 4.

Apply A.I. (3, 2) → 5. p(mike, tom) $\wedge$ p(tom, jane)

Apply M.P. (4, 5) →

Forward/Backward Chaining

- Forward chaining

$$
\begin{array}{l}
P \\
\;P \Rightarrow Q \\
\qquad Q \Rightarrow R \\
\qquad\qquad \mathbf{R}
\end{array}
$$

- Backward chaining

$$
\begin{array}{l}
\qquad\qquad \mathbf{R}? \\
\qquad Q \Rightarrow R \\
\;P \Rightarrow Q \\
P
\end{array}
$$

Forward/Backward Chaining

- Example
  - $p(X) \Rightarrow q(X)$, $p(X) \Rightarrow r(X)$, $r(X) \Rightarrow t(X)$, $r(X) \Rightarrow u(X)$
  - $p(a)$, $q(b)$
  - $t(a)$?
  - Forward chaining
    - $p(a), p(X) \Rightarrow r(X) \quad \vdash \quad r(a)$
    - $r(a), r(X) \Rightarrow t(X) \quad \vdash \quad$ **t(a)**
  - Backward chaining
    - $t(a), r(X) \Rightarrow t(X) \quad \rightarrow \quad$ find $r(a)$
    - $r(a), p(X) \Rightarrow r(X) \quad \rightarrow \quad$ find $p(a) \quad \rightarrow$ **true !**