Problem 6.1

Suppose that instead of the Poisson model (6.1), we assume a binomial model

$$Pr\{x_k = x\} = \binom{n}{x}\theta_k^x(1-\theta_k)^{n-x},$$

n some fixed and known integer such as n = 10. What is the equivalent of Robbins' formula (6.5)?

Solution By Robbins' formula, $\mathbb{E}(\theta|x) = \dfrac{\int_0^1 \theta p_\theta(x)g(\theta)d\theta}{f_g(x)}$ where $f_g(x) = \int_0^1 p_\theta(x)g(\theta)d\theta$. Then,

$$\begin{aligned}
\int_0^1 \theta p_\theta(x)g(\theta)d\theta &= \int_0^1 \binom{10}{x}\theta^{x+1}(1-\theta)^{10-x}g(\theta)d\theta \\
&= \binom{10}{x}\left(\int_0^1 (\theta^{x+1}(1-\theta)^{9-x} - \theta^{x+2}(1-\theta)^{9-x})g(\theta)d\theta\right) \\
&= \binom{10}{x}\left(f_g(x+1)/\binom{10}{x+1} - \int_0^1 \theta^{x+2}(1-\theta)^{9-x}g(\theta)d\theta\right) \\
&= \frac{x+1}{10-x}f_g(x+1) - \binom{10}{x}\left(\int_0^1 (\theta^{x+2}(1-\theta)^{8-x} - \theta^{x+3}(1-\theta)^{8-x})g(\theta)d\theta\right) \\
&= \frac{x+1}{10-x}f_g(x+1) - \frac{(x+2)(x+1)}{(10-x)(9-x)}f_g(x+2) + \binom{10}{x}\int_0^1 \theta^{x+3}(1-\theta)^{8-x}g(\theta)d\theta \\
&= \cdots \text{ (Continue this process)} \\
&= \sum_{k=1}^{10-x} \frac{(x+k)(x+k-1)\cdots(x+1)}{(10-x)(9-x)\cdots(11-x-k)}(-1)^{k-1}f_g(x+k) \\
&= \sum_{k=1}^{10-x} \frac{(x+k)!(10-x-k)!}{(10-x)!x!}(-1)^{k-1}f_g(x+k)
\end{aligned}$$

Therefore, the Robbins' formula is equivalent to

$$\mathbb{E}(\theta|x) = \frac{1}{f_g(x)}\sum_{k=1}^{10-x} \frac{(x+k)!(10-x-k)!}{(10-x)!x!}(-1)^{k-1}f_g(x+k)$$

Problem 6.2

Define $\mathbb{V}\{\theta|x\}$ as the variance of θ given x. In the Poisson situation (6.1), show that

$$\mathbb{V}\{\theta|x\} = \mathbb{E}\{\theta|x\}\cdot(\mathbb{E}\{\theta|x+1\} - \mathbb{E}\{\theta|x\}),$$

where $\mathbb{E}\{\theta|x\}$ is as given in (6.5).

Solution By Robbins' formula, $\mathbb{E}(\theta|x) = (x+1)\frac{f_g(x+1)}{f_g(x)}$, where $f_g(x) = \int_0^\infty p_\theta(x)g(\theta)d\theta$ is marginal distribution of x. Then, we know that $\mathbb{V}\{\theta|x\} = \mathbb{E}(\theta^2|x) - (\mathbb{E}(\theta|x))^2$. So we'll show that $\mathbb{E}(\theta^2|x) = \mathbb{E}(\theta|x)\cdot\mathbb{E}(\theta|x+1)$.

$$
\begin{aligned}
\mathbb{E}(\theta^2|x) &= \frac{\int_0^\infty \theta^2 p_\theta(x)g(\theta)d\theta}{f_g(x)} = \frac{\int_0^\infty \theta^2 \dfrac{\theta^x e^{-\theta}}{x!} g(\theta)d\theta}{f_g(x)} \\
&= (x+2)(x+1)\frac{\int_0^\infty \theta^{x+2} \dfrac{e^{-\theta}}{(x+2)!} g(\theta)d\theta}{f_g(x)} = (x+2)(x+1)\frac{f_g(x+2)}{f_g(x)} \qquad = (x+2)(x+1)\frac{f_g(x+2)}{f_g(x)} \\
&= (x+2)(x+1)\frac{f_g(x+2)}{f_g(x+1)}\frac{f_g(x+1)}{f_g(x)} = \mathbb{E}(\theta|x+1)\cdot\mathbb{E}(\theta|x)
\end{aligned}
$$

Therefore, we have shown that the above equation holds.

Problem 6.3

Instead of (6.8), assume $g(\theta) = (1/\sigma)e^{-\theta/\sigma}$ for $\theta > 0$.

(a) Numerically find the maximum likelihood estimate $\hat{\sigma}$ for the Poisson model (6.1) fit to the count data in Table 6.1.

(b) Calculate the estimates of $\hat{E}\{\theta|x\}$, as in the third row of Table 6.1.

$$g(\theta) = \frac{\theta^{\nu-1}e^{\theta/\sigma}}{\sigma^{\nu}\Gamma(\nu)}, \text{for } \theta \geq 0, \quad (6.8)$$

Solution

(a) We'll find the maximum likelihood estimates of σ. Tha marginal probability density function of x is $f_g(x) = \int_0^\infty \frac{\theta^x e^{-\theta}}{x!} g(\theta)d\theta = \int_0^\infty \frac{\theta^x e^{-\theta}}{x!}(1/\sigma)e^{-\theta/\sigma}d\theta = \frac{\gamma^{x+1}}{\sigma}\int_0^\infty \frac{1}{\Gamma(x+1)\gamma^{x+1}}\theta^x e^{-\theta/\gamma}d\theta$, where $\gamma = \frac{\sigma}{\sigma+1}$. Since the function in the integral follows Gamma(x+1, γ) distribution, $f_g(x) = \frac{\gamma^{x+1}}{\sigma} = \frac{\sigma^x}{(\sigma+1)^{x+1}}$. So the marginal likelihood function of x is $L(\sigma) = \prod_{i=1}^N f_g(x_i)$. Define the log likelihood function $l(\sigma) = log(L(\sigma)) = \sum_{i=1}^N (x_i log\sigma - (1+x_i)log(1+\sigma))$. The score function is $S(\sigma) = \dot{l}(\sigma) = -\frac{n}{\sigma+1} + \frac{n\bar{x}}{\sigma} - \frac{n\bar{x}}{\sigma+1}$ where $\bar{x} = \sum_{i=1}^N x_i$. So we'll find the mle of σ satisfying $S(\sigma) = 0$.

$$S(\sigma) = 0 \Leftrightarrow -n\sigma + n\bar{x}(\sigma+1) - n\bar{x}\sigma = 0 \Leftrightarrow -n\sigma + n\bar{x} = 0 \Leftrightarrow \sigma = \bar{x}$$

Therefore, the mle of σ is $\bar{x}$, denoted $\hat{\sigma} = \bar{x}$. So, we plug $\hat{\sigma}$ in $f_g(x)$. Then,

$$f_{\hat{g}}(x) = \frac{\hat{\sigma}^x}{(\hat{\sigma}+1)^{x+1}}$$

Actually, $\hat{\sigma} = \bar{x} = \frac{1}{9461}\sum_{i=1}^{9461} x_i = \frac{1}{9461}(1317 \cdot 1 + 239 \cdot 2 + 42 \cdot 3 + 14 \cdot 4 + 4 \cdot 5 + 4 \cdot 6 + 1 \cdot 7) = 0.2143537$
Then, $\hat{y}_x = 9461 f_{\hat{g}}(x)$ where $x = 0, 1, \ldots, 7$ By using R, we can calculate $\forall y_x$.

In summary,

$\hat{y}_x$	**Fitting value**
$\hat{y}_0$	7790.976
$\hat{y}_1$	1375.237
$\hat{y}_2$	242.7523
$\hat{y}_3$	42.84982
$\hat{y}_4$	7.563708
$\hat{y}_5$	1.33512
$\hat{y}_6$	0.235671
$\hat{y}_7$	0.04159986

(b) Since $\hat{E}\{\theta|x\} = (x+1)\frac{f_{\hat{g}}(x+1)}{f_{\hat{g}}(x)} = (x+1)\frac{\hat{\sigma}}{1+\hat{\sigma}}$. So we can calculate the estimates of posterior mean for all $x = 0, 1, \ldots, 6$

The R code related to the above is attached to the last page.

$\hat{E}(\theta\|x)$	**Fitting value**
$\hat{E}(\theta\|x=0)$	0.1765167
$\hat{E}(\theta\|x=1)$	0.3530333
$\hat{E}(\theta\|x=2)$	0.5295500
$\hat{E}(\theta\|x=3)$	0.7060667
$\hat{E}(\theta\|x=4)$	0.8825833
$\hat{E}(\theta\|x=5)$	1.0591000
$\hat{E}(\theta\|x=6)$	1.2356167
$\hat{E}(\theta\|x=7)$	.

Problem 6.7

The nodes data of Section 6.3 consists of 844 pairs (n_i, x_i).

(a) Plot x_i versus n_i

(b) Perform a cubic regression of x_i versus n_i and add it to the plot.

(c) What would you expect the plot to look like if the values of n_i were assigned randomly before surgery?

Solution

(a)

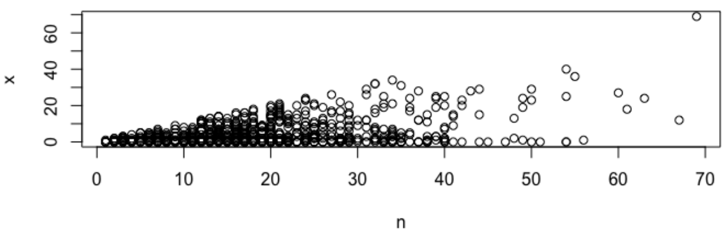

Figure 1:

(b)

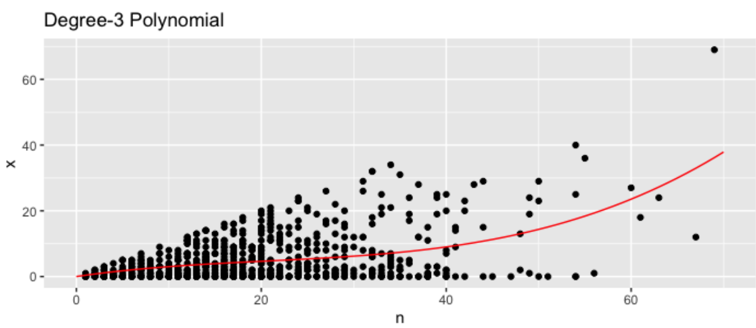

Figure 2:

Problem 2

Show that the marginal distribution of "x"(in the missing species problem) is negative binomial.

Solution In the book, the distribution of $x|\theta$ and θ are $x|\theta \sim Poisson(\theta)$ and $\theta \sim Gamma(\nu, \sigma)$. Then, the joint probability density function of (x, θ) is $f(x, \theta) = p_\theta(x)g(\theta)$ where $p_\theta(x)$ and $g(\theta)$ are pdf of $x|\theta$ and θ. Then, the marginal pdf of x is

$$\begin{aligned}
f(x) &= \int_0^\infty f(x,\theta)d\theta = \int_0^\infty \frac{\theta^x e^{-\theta}}{x!}\frac{1}{\Gamma(\nu)\sigma^\nu}\theta^{\nu-1}e^{-\theta/\sigma}d\theta \\
&= \frac{\Gamma(x+\nu)\gamma^{x+\nu}}{\Gamma(\nu)x!\sigma^\nu}\int_0^\infty \frac{1}{\Gamma(x+\nu)\gamma^{x+\nu}}\theta^{x+\nu-1}e^{-\theta/\gamma}d\theta \quad (\text{where } \gamma = (1+\frac{1}{\sigma})^{-1}) \\
&= \frac{(\nu+x-1)(\nu+x-2)\cdots(\nu+1)\nu}{x!}(\frac{1}{\sigma+1})^\nu(\frac{\sigma}{1+\sigma})^x \\
&= \binom{x+\nu-1}{x}(\frac{1}{\sigma+1})^\nu(\frac{\sigma}{1+\sigma})^x \quad \text{if } \nu \text{ is natural number}
\end{aligned}$$

Therefore, if ν is natural number, then the marginal pdf of x follows negative binomial distribution, denoted $Nebin(\nu, \frac{1}{1+\sigma})$.

Problem 3
Verify $E(t) = e_1\frac{1-(1+\gamma t)^{-\nu}}{\gamma\nu}$ where $\gamma = (1/\sigma+1)^{-1}$. (See the lecture note Chapter 6-2, page 5, parametric approach)

Solution In lecture note Chapter 6-2 page 5, we assume that the prior distribution of θ is $Gamma(\nu, \sigma)$. Let the pdf of θ be $g(\theta) = \frac{1}{\Gamma(\nu)\sigma^\nu}\theta^{\nu-1}e^{-\frac{\theta}{\sigma}}$. The moment generating function of θ is $M_\theta(t) = E(e^{t\theta}) = (1-\sigma t)^{-\nu} \quad \forall t < \frac{1}{\sigma}$. To calculate $E(t)$, we use the mgf of θ.

$$\begin{aligned}
E(t) &= S\int_0^\infty e^{-\theta}(1-e^{-\theta t}g(\theta)d\theta = SE(e^{-\theta}(1-e^{-\theta t})) \\
&= S(E(e^{-\theta}) - E(e^{-\theta(1+t)})) = S((1+\sigma)^{-\nu} - (1+\sigma(1+t))^{-\nu}) \\
&= S(1+\sigma)^{-\nu}(1-(1+\frac{\sigma}{1+\sigma}t)^{-\nu}) \\
&= S(1+\sigma)^{-\nu}(1-(1+\gamma t)^{-\nu})
\end{aligned}$$

Next, we'll show that $e_1 = E(y_1) = S\gamma\nu(1+\sigma)^{-\nu}$.

$$\begin{aligned}
e_1 = E(y_1) &= S\int_0^\infty \theta e^{-\theta}g(\theta)d\theta \\
&= S\frac{\Gamma(\nu+1)\gamma^{\nu+1}}{\Gamma(\nu)\sigma^\nu}\int_0^\infty \frac{1}{\Gamma(\nu+1)\gamma^{\nu+1}}\theta^\nu e^{-\frac{\theta}{\gamma}}d\theta \\
&= S\frac{\nu\gamma^{\nu+1}}{\sigma^\nu} = S\nu\frac{(\sigma/(1+\sigma))^{\nu+1}}{\sigma^\nu} \\
&= S\nu\frac{\sigma}{(1+\sigma)^{\nu+1}} = S\nu\frac{\sigma}{1+\sigma}\frac{1}{(1+\sigma)^\nu} \\
&= S\nu\gamma(1+\sigma)^{-\nu}
\end{aligned}$$

Therefore,

$$E(t) = S(1+\sigma)^{-\nu}(1-(1+\gamma t)^{-\nu}) = e_1\frac{(1-(1+\gamma t)^{-\nu})}{\gamma\nu}$$