Problem 5.6
If $x \sim Mult_L(n, \pi)$, use the Poisson trick (5.44) to appropriate the mean and variance of x_1/x_2. (Here we are assumming that $n\pi_2$ is large enough to ignore the possibility $x_2 = 0$.) Hint: In notation (5.41),

$$\frac{S1}{S2} \doteq \frac{\mu_1}{\mu_2}(1 + \frac{S_1 - \mu_1}{\mu_1} - \frac{S_2 - \mu_2}{\mu_2}).$$

$$S_l \overset{ind}{\sim} Poi(\mu_l), \quad l = 1, 2, \ldots, L \qquad (5.41)$$
$$Mult_L(N, \pi) \sim Poi(n\pi) \qquad (5.44)$$

Solution Let $X = (x_1, x_2, \ldots, x_L) \sim Mult_L(n, \pi)$ where $\pi = (\pi_1, \pi_2, \ldots, \pi_L)$ and $N \sim Poi(n)$. Then, by using Poisson trick, we can approximate $X = (x_1, x_2, \ldots, x_L) \sim Poi(n\pi)$. In other words, $x_i \overset{indep}{\sim} Poi(n\pi_i)\ \forall i = 1, 2, \ldots, L$. Define $\mu_i = n\pi_i\ \forall i$. We know that $E_\pi(x_i) = n\pi_i = \mu_i$ and $Var_\pi = n\pi_i = \mu_i$. Next, we use the hint given to the problem. Then, we can calculate the mean and variance of x_1/x_2.

First, the mean of x_1/x_2 is

$$\begin{aligned} E_\pi(\frac{x_1}{x_2}) &= E_\pi\left(\frac{\mu_1}{\mu_2}(1 + \frac{x_1 - \mu_1}{\mu_1} - \frac{x_2 - \mu_2}{\mu_2}\right) \\ &= \frac{\mu_1}{\mu_2}(1 + E_\pi\left(\frac{x_1 - \mu_1}{\mu_1}\right) - E_\pi\left(\frac{x_2 - \mu_2}{\mu_2}\right)) \\ &= \frac{\mu_1}{\mu_2} \end{aligned}$$

Second, the variance of x_1/x_2 is

$$\begin{aligned} Var_\pi(\frac{x_1}{x_2}) &= Var_\pi\left(\frac{\mu_1}{\mu_2}(1 + \frac{x_1 - \mu_1}{\mu_1} - \frac{x_2 - \mu_2}{\mu_2}\right) \\ &= \frac{\mu_1^2}{\mu_2^2}(\frac{1}{\mu_1^2}Var_\pi(x_1) + \frac{1}{\mu_2^2}Var_\pi(x_2)) \\ &= \frac{\mu_1^2}{\mu_2^2}(\frac{1}{\mu_1} + \frac{1}{\mu_2}) \\ &= \frac{\mu_1(\mu_1 + \mu_2)}{\mu_2^3} \end{aligned}$$

Problem 5.7
Show explicitly how the binomial density bi(12, 0.3) is an exponential tilt of bi(12, 0.6).

Solution Let $p, p_0 \in \mathbb{P} = \{p : 0 < p < 1\}$. The pmfs of bi(n, p) and bi(n, p_0) are $f_p(x) = \binom{n}{x}p^x(1-p)^{n-x}$ and $f_{p_0}(x) = \binom{n}{x}p_0^x(1-p_0)^{n-x}$. Suppose that p_0 is given. Then,

$$\begin{aligned} \frac{f_p(x)}{f_{p_0}(x)} &= (\frac{p}{p_0})^x(\frac{1-p}{1-p_0})^{n-x} \Leftrightarrow \\ f_p(x) &= (\frac{p}{p_0})^x(\frac{1-p}{1-p_0})^{n-x} f_{p_0}(x) \Leftrightarrow \\ f_p(x) &= exp\left(xlog(\frac{p}{p_0}) + (n-x)log(\frac{1-p}{1-p_0})\right) f_{p_0}(x) \Leftrightarrow \\ f_p(x) &= exp\left(xlog(\frac{p/(1-p)}{p_0/(1-p_0)}) + nlog(\frac{1-p}{1-p_0})\right) f_{p_0}(x) \end{aligned}$$

Define the $\alpha = log\frac{p/1-p}{p_0/1-p_0}$. Then, $f_p(x) = exp\left(xlog(\frac{p/(1-p)}{p_0/(1-p_0)}) + nlog(\frac{1-p}{1-p_0})\right) f_{p_0}(x) = exp\left(\alpha x - \psi(\alpha)\right) f_{p_0}(x)$. where $\psi(\alpha) = nlog(\frac{1-p_0}{1-p})$.

Let $\tilde{f}_p(x) = e^{\alpha x} f_{p_0}(x)$. Then, we can express the pmf of bi(n, p) as follows. $f_p(x) = \frac{\tilde{f}_{p_0}(x)}{e^{\psi(\alpha)}} = e^{\alpha x - \psi(\alpha)} f_{p_0}(x)$. Here, $e^{\psi(\alpha)}$ satisfies the following equation. $e^{\psi(\alpha)} = \sum_{x=0}^{n} \tilde{f}_p(x) = \sum_{x=0}^{n} e^{\alpha x} f_{p_0}(x)$. This means $e^{\psi(\alpha)}$ is moment generating function of $f_{p_0}(x)$, i.e , $\psi(\alpha)$ is cumulant generating function of $f_{p_0}(x)$. Thus, we can find another form of $e^{\psi(\alpha)}$ by binomial theorem.

$$\begin{aligned} e^{\psi(\alpha)} &= \sum_{x=0}^{n} e^{\alpha x} f_{p_0}(x) = \sum_{x=0}^{n} e^{\alpha x} \binom{n}{x} p_0^x (1-p_0)^{n-x} \\ &= \sum_{x=0}^{n} \binom{n}{x} (p_0 e^{\alpha})^x (1-p_0)^{n-x} = (p_0 e^{\alpha} + 1 - p_0)^n \\ \therefore \psi(\alpha) &= nlog(p_0 e^{\alpha} + 1 - p_0) \quad \text{(The same result as the one above)} \end{aligned}$$

By using the above equation, we can show the problem. Substituting $n = 12, p = 0.3, p_0 = 0.6$ into the equation above. Then, $\alpha = log\frac{0.3/0.7}{0.6/0.4} \simeq -1.2527$ and $\psi(\alpha) = 12log(0.6e^{\alpha} + 0.4) \simeq -6.7153$.

Therefore, $f_p(x) = exp\left(\alpha x - \psi(\alpha)\right) f_{p_0}(x) = e^{-1.2527x + 6.7153} f_{p_0}(x)$.