Assignment 1

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2.1

Supppose $E(X) = 2$, $Var(X) = 9$, $E(Y) = 0$, $Var(Y) = 4$, and $Corr(X, Y) = .25$. Find:

```r
mean_x <- 2
mean_y <- 0
var_x <- 9
var_y <- 4
corr_xy <- .25
```

(a) $Var(X + Y)$

$Var(X, Y) = Var(X) + Var(Y) + 2Cov(X, Y)$

$Corr(X, Y) = \frac{Cov(X,Y)}{\sigma_x \sigma_y}$

Solve for $Cov(X, Y)$ and use $Var(X, Y)$ equation for solution:

```r
cov_xy <- sqrt(var_x * var_y) * corr_xy
glue("Answer: {(var_x_plus_y <- var_x + var_y + 2 * cov_xy)}")
```

```
## Answer: 16
```

(b) $Cov(X, X + Y)$

$Cov(X, X + Y) = Cov(X, X) + Cov(X, Y)$ $Cov(X, X + Y) = Var(X) + Cov(X, Y)$

```r
glue("Answer: {var_x + cov_xy}")
```

```
## Answer: 10.5
```

(c) $Corr(X + Y, X - Y)$

$$Corr(X + Y, X - Y) = \frac{Cov(X + Y, X - Y)}{\sqrt{Var(X + Y)Var(X - Y)}}$$

$$\begin{aligned} Cov(X + Y, X - Y) &= Cov(X, X - Y) + Cov(Y, X - Y) \\ &= Cov(X, X) - Cov(X, Y) + Cov(Y, X) - Cov(Y, Y) \\ &= Cov(X, X) - Cov(Y, Y) \\ &= Var(X) - Var(Y) \\ &= 9 - 4 \\ &= 5 \end{aligned}$$

```r
(cov_x_plus_y_x_minus_y <- var_x - var_y)
```

```
## [1] 5
```

$$Var(X - Y) = Var(X) + Var(Y) - 2Cov(X, Y)$$

```
(var_x_minus_y <- var_x + var_y - 2 * cov_xy)
```

```
## [1] 10
```

```
glue("Answer: {cov_x_plus_y_x_minus_y / (sqrt(var_x_plus_y * var_x_minus_y))}")
```

```
## Answer: 0.395284707521047
```

2.2

If X and Y are dependent but $Var(X) = Var(Y)$, find $Cov(X + Y, X - Y)$.

$$\begin{aligned}
Cov(X+Y, X-Y) &= Cov(X, X-Y) + Cov(Y, X-Y) \\
&= Cov(X, X) - Cov(X, Y) + Cov(Y, X) - Cov(Y, Y) \\
&= Cov(X, X) - Cov(Y, Y) \\
&= Var(X) - Var(Y)
\end{aligned}$$

However, the variance of X Y are equal, so:

```
glue("Answer: {0}")
```

```
## Answer: 0
```

2.5

Suppose $Y_t = 5 + 2t + X_t$, where $\{X_t\}$ is a zero-mean stationary series with autocovariance function γ_k.

(a) Find the mean function for $\{Y_t\}$

$$\begin{aligned}
E[Y_t] &= E[5 + 2t + X_t] \\
&= E[5] + E[2t] + E[X_t] \\
&= 5 + 2t + 0 \\
&= 5 + 2t
\end{aligned}$$

(b) Find the autocovariance function for $\{Y_t\}$.

$$\begin{aligned}
Cov(Y_t, Y_{t-k}) &= Cov(5 + 2t + X_t, 5 + 2(t-k) + X_{t-k}) \\
&= Cov(X_t, X_{t-k}) \\
&= \gamma_k
\end{aligned}$$

(c) Is $\{Y_t\}$ stationary? Why or why not? $\{Y_t\}$ is not stationary because the mean value depends on time.

2.6

Let $\{X_t\}$ be a stationary time series, and define $Y_t = \begin{cases} X_t \\ X_t + 3 \end{cases}$ for t odd and t even, respectively.

(a) Show that $Cov(Y_t, Y_{t-k})$ is free of t for all lags k.

Even:

$$\begin{aligned} &= Cov(X_t + 3, X_{t-k} + 3) \\ &= Cov(X_t, X_{t-k}) \\ &= Cov(X_t) \end{aligned}$$

Odd:

$$\begin{aligned} &= Cov(X_t, X_{t-k}) \\ &= Cov(X_t) \end{aligned}$$

(b) Is $\{Y_t\}$ stationary?

The time series is not stationary due to the expected value depending on t:

$$E[Y_t] = \begin{cases} E[X_t] \\ E[X_t + 3] = E[X_t] + E[3] \end{cases}$$

2.7

Suppose that $\{Y_t\}$ is stationary with autocovariance function γ_k.

(a) Show that $W_t = \nabla Y_t = Y_t - Y_{t-1}$ is stationary by finding the mean and autocovariance function for $\{W_t\}$.

Mean:

$$\begin{aligned} E[W_t] &= E[Y_t - Y_{t-1}] \\ &= E[Y_t] - E[Y_{t-1}] \\ &= 0 \end{aligned}$$

Auto covariance only depends on k, concluding the stationarity of $\{W_t\}$:

$$\begin{aligned} Cov[W_t] &= Cov(Y_t - Y_{t-1}, Y_{t-k} - Y_{t-k-1}) \\ &= Cov(Y_t, Y_{t-k}) + Cov(Y_t, -Y_{t-k-1}) + Cov(-Y_{t-1}, Y_{t-k}) + Cov(-Y_{t-1}, -Y_{t-k-1}) \\ &= \gamma_k - \gamma_{k-1} - \gamma_{k+1} + \gamma_k \\ &= 2\gamma_k - \gamma_{k-1} - \gamma_{k+1} \end{aligned}$$

(b) Show that $U_t = \nabla^2 Y_t = \nabla[Y_t - Yt - 1] = Y_t - 2Y_{t-1} + Y_{t-2}$ is stationary (You need not find the mean and autocovariance function for $\{U_t\}$.)

In (a), the derived that the difference between two stationary time series concludes ∇Y_t as stationary also. This holds true for $\nabla^2 Y_t$

2.8

Suppose that $\{Y_t\}$ is stationary with autocovariance function γ_k. Show that for any fixed positive integer n and any constants $c_1, c_2, ..., c_n$, the process $\{W_t\}$ defined by $W_t = c_1 Y_t + c_2 Y_{t-1} + \cdots + c_n Y_{t-n+1}$ is stationary. (Note that Exercise 2.7 is a special case of this result.)

The expected value is constant:

$$
\begin{aligned}
E[W_t] &= c_1E[Y_t] + c_2E[Y_t] + \cdots + c_nE[Y_t] \\
&= E[Y_t](c_1 + c_2 + \cdots + c_n)
\end{aligned}
$$

$Cov(W_t)$ is free from t:

$$
\begin{aligned}
Cov[W_t] &= Cov[c_1Y_t + c_2Y_{t-1} + \cdots + c_nY_{t-k}, c_1Y_{t-k} + c_2Y_{t-k-1} + \cdots + c_nY_{t-k-n}] \\
&= \sum_{i=0}^{n} c_i \sum_{j=0}^{n} c_j Cov[Y_{t-j}Y_{t-i-k}] \\
&= \sum_{i=0}^{n} c_i \sum_{j=0}^{n} c_j \gamma_{j-k-i},
\end{aligned}
$$

Therefore, W_t is stationary.

2.11

Suppose $Cov(X_t, X_{t-k}) = \gamma_k$ is free of t but that $E[X_t] = 3t$.

(a) Is $\{X_t\}$ stationary?

$\{X_t\}$ varies with t. Therefore, it is not stationary.

(b) Let $Y_t = 7 - 3t + X_t$. Is $\{Y_t\}$ stationary?

$$
\begin{aligned}
E[Y_t] &= 3 - 3t + E[X_t] \\
&= 7 - 3t - 3t \\
&= 7
\end{aligned}
$$

$$
\begin{aligned}
Cov[Y_t, Y_{t-k}] &= [7 - 3t + X_t, 7 - 3(t-k) + X_{t-k}] \\
&= Cov[X_t, X_{t-k}] \\
&= \gamma_k
\end{aligned}
$$

Since the mean is constant and the autocovariance is free of t, $\{Y_t\}$ is stationary.

2.12

Suppose that $Y_t = e_t - e_{t-12}$. Show that $\{Y_t\}$ is stationary and that, for $k > 0$, its autocorrelation function is nonzero only for lag $k = 12$.

Mean:

$$
\begin{aligned}
E[Y_t] &= E[e_t - e_{t-12}] \\
&= E[e_t] - E[e_{t-12}] \\
&= 0
\end{aligned}
$$

Autocovariance:

$$
Cov(Y_t, Y_{t-k}) = Cov(e_t - e_{t-12}, e_{t-k} - e_{t-12-k})
$$

$k = 12$:

$$
\begin{aligned}
&= Cov(e_t, e_{t-12}) + Cov(e_t, -e_{t-12-12}) + Cov(-e_{t-12}, e_{t-12}) + Cov(-e_{t-12-12}) \\
&= 0 + 0 - Cov(e_{t-12}, e_{t-12}) + 0 \\
&= -\sigma_\epsilon^2
\end{aligned}
$$

All terms are correlated except when $k = 12$ (autocorrelation is 1).

2.14

Evaluate the mean and covariance function for each of the follow processes. In each case, determine whether or not the process is stationary.

(a) $Y_t = \theta_0 + te_t$.

Mean:

$$
\begin{aligned}
E[Y_t] &= \theta_0 + tE[e_t] \\
&= \theta_0
\end{aligned}
$$

Covariance:

$$
\begin{aligned}
Cov(Y_t, Y_{t-k}) &= Cov(\theta_0 + te_t, \theta_0 + (t-k)e_{t-k}) \\
&= Cov(te, (t-k)e_{t-k}) \\
&= Cov(te_t, te_{t-k} - ke_{t-k}) \\
&= Cov(te_t, te_{t-k}) + Cov(te_t, -ke_{t-k}) \\
&= t^2 Cov(e_t, e_{t-k}, -tk(e_t, e_{t-l})
\end{aligned}
$$

This time series is not stationary due to autocovariance depending on t^2.

(b) $W_t = \triangledown Y_t$, where Y_t is as given in part (a).

Mean:

$$
\begin{aligned}
E[W_t] &= E[\triangledown Y_t] \\
&= E[\theta_0 + te_t - \theta_0 - (t-1)e_{t-1}] \\
&= tE[e_t] - tE[e_{t-1} + E[e_{t-1}] \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
Var[\triangledown Y_t] &= Var[te_t] \\
&= -Var[(t-1)e_{t-1}] \\
&= t^2\sigma_e^2 - (t-1)^2\sigma_e^2 = \sigma_e^2(t^2 - t^2 + 2t - 1) = (2t-1)\sigma_e^2
\end{aligned}
$$

Varies with t. Therefore, it is not stationary.

(c) $Y_t = e_t e_{t-1}$ (You may assume that e_t is normal white noise.)

$$
\begin{aligned}
E[Y_t] &= E[e_t e_{t-1}] \\
&= E[e_t]E[e_{t-1}] \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
Cov[Y_t, Y_{t-1}] &= Cov[e_t e_{t-1}, e_{t-1} e_{t-2}] \\
&= E[(e_t e_{t-1} - \mu_t^2)(e_{t-1} e_{t-2} - \mu_t^2)] \\
&= E[e_t] E[e_{t-1}] E[e_{t-1}] E[e_{t-2}] \\
&= 0
\end{aligned}
$$

Since mean and covariance are zero, the time series is stationary.

2.15

Suppose that X is a random variable with zero mean. Define a time series by $Y_t = (-1)^t X$.

(a) Find the mean function for $\{Y_t\}$.

$$
\begin{aligned}
E[Y_t] &= E[(-1)^t X] \\
&= (-1)^2 E[X] \\
&= 0
\end{aligned}
$$

(b) Find the covariance function for $\{Y_t\}$.

$$
\begin{aligned}
Cov(Y_t) &= Cov((-1)^t X, (-1)^{t-k} X) \\
&= (-1)^t (-1)^{t-k} Cov(X, X) \\
&= (-1)^{2t-k} \sigma_x^2
\end{aligned}
$$

For even or odd k, covariance does not depend on t.

(c) Is {Y_t} stationary? The time series is stationary since the mean is 0 and the covariance is free of t.