§16.1

1、第一类曲线积分计算

EXAMPLE 1 Integrate $f(x, y, z) = x - 3y^2 + z$ over the line segment C joining the origin to the point $(1, 1, 1)$ (Figure 16.2). P15

Example 2 Integrate $f(x, y, z) = x - 3y^2 + z$ over $C_1 \cup C_2$.

P17

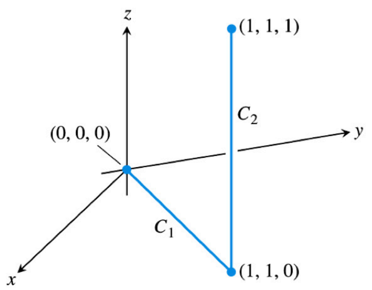


FIGURE 16.3 The path of integration in Example 2.

8. $\mathbf{r}(t) = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{k}, \quad 0 \le t \le \pi$

Evaluating Line Integrals over Space Curves

§16.1

9. Evaluate $\int_C (x + y)\,ds$ where C is the straight-line segment $x = t, y = (1 - t), z = 0$, from $(0, 1, 0)$ to $(1, 0, 0)$.

10. Evaluate $\int_C (x - y + z - 2)\,ds$ where C is the straight-line segment $x = t, y = (1 - t), z = 1$, from $(0, 1, 1)$ to $(1, 0, 1)$.

11. Evaluate $\int_C (xy + y + z)\,ds$ along the curve $\mathbf{r}(t) = 2t\mathbf{i} + t\mathbf{j} + (2 - 2t)\mathbf{k}, 0 \le t \le 1$.

Line Integrals over Plane Curves

§16.1

19. Evaluate $\int_C x\,ds$, where C is

a. the straight-line segment $x = t, y = t/2$, from $(0, 0)$ to $(4, 2)$.

b. the parabolic curve $x = t, y = t^2$, from $(0, 0)$ to $(2, 4)$.

20. Evaluate $\int_C \sqrt{x + 2y}\,ds$, where C is

a. the straight-line segment $x = t, y = 4t$, from $(0, 0)$ to $(1, 4)$.

b. $C_1 \cup C_2$; C_1 is the line segment from $(0, 0)$ to $(1, 0)$ and C_2 is the line segment from $(1, 0)$ to $(1, 2)$.

21. Find the line integral of $f(x, y) = ye^{x^2}$ along the curve $\mathbf{r}(t) = 4t\mathbf{i} - 3t\mathbf{j}, -1 \le t \le 2$.

22. Find the line integral of $f(x, y) = x - y + 3$ along the curve $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}, 0 \le t \le 2\pi$.

8 16、2

2、第二类曲线积分计算

Example 2 Evaluate

$$\int_C \boldsymbol{F} \bullet d\boldsymbol{r}$$ P151

where $\boldsymbol{F}(x, y, z) = (z, xy, -y^2)$ along the directional curve C given by $\boldsymbol{r}(t) = (t^2, t, \sqrt{t}), t: 0 \to 1$.

Example 3 Evaluate $\int_C -y dx + z dy + 2x dz$ P152

where the directional curve C is the helix

$$\boldsymbol{r}(t) = (\cos t, \sin t, t), t: 0 \to 2\pi.$$

EXAMPLE 4 Find the work done by the force field $\mathbf{F} = (y - x^2)\mathbf{i} + (z - y^2)\mathbf{j} + (x - z^2)\mathbf{k}$ along the curve $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$, $0 \le t \le 1$, from ~~(0, 0, 0) to (1, 1, 1)~~ (Figure 16.18). P153

(1,1,1) to (0,0,0)?

Solution First we evaluate $\mathbf{F}$ on the curve $\mathbf{r}(t)$:

Example 5 Find the work done by the force $\boldsymbol{F}(x, y, z) = (x, y, z)$ in moving an object along the curve C parametrized by

$$\boldsymbol{r}(t) = (\cos \pi t, t^2, \sin \pi t), t: 0 \to 1.$$

Exercises 16.2

Vector Fields

Find the gradient fields of the functions in Exercises 1–4.

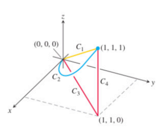
1. $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$
2. $f(x, y, z) = \ln \sqrt{x^2 + y^2 + z^2}$
3. $g(x, y, z) = e^z - \ln(x^2 + y^2)$
4. $g(x, y, z) = xy + yz + xz$
5. Give a formula $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ for the vector field in the plane that has the property that $\mathbf{F}$ points toward the origin with magnitude inversely proportional to the square of the distance from (x, y) to the origin. (The field is not defined at $(0, 0)$.)
6. Give a formula $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ for the vector field in the plane that has the properties that $\mathbf{F} = \mathbf{0}$ at $(0, 0)$ and that at any other point (a, b), $\mathbf{F}$ is tangent to the circle $x^2 + y^2 = a^2 + b^2$ and points in the clockwise direction with magnitude $|\mathbf{F}| = \sqrt{a^2 + b^2}$.

Line Integrals of Vector Fields

In Exercises 7–12, find the line integrals of $\mathbf{F}$ from $(0, 0, 0)$ to $(1, 1, 1)$ over each of the following paths in the accompanying figure.

a. The straight-line path C_1: $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}$, $0 \le t \le 1$
b. The curved path C_2: $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^4\mathbf{k}$, $0 \le t \le 1$
c. The path $C_3 \cup C_4$ consisting of the line segment from $(0, 0, 0)$ to $(1, 1, 0)$ followed by the segment from $(1, 1, 0)$ to $(1, 1, 1)$

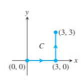
7. $\mathbf{F} = 3y\mathbf{i} + 2x\mathbf{j} + 4z\mathbf{k}$
8. $\mathbf{F} = [1/(x^2 + 1)]\mathbf{j}$
9. $\mathbf{F} = \sqrt{z}\mathbf{i} - 2x\mathbf{j} + \sqrt{y}\mathbf{k}$
10. $\mathbf{F} = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$
11. $\mathbf{F} = (3x^2 - 3x)\mathbf{i} + 3z\mathbf{j} + \mathbf{k}$
12. $\mathbf{F} = (y + z)\mathbf{i} + (z + x)\mathbf{j} + (x + y)\mathbf{k}$



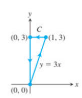
Line Integrals with Respect to x, y, and z

In Exercises 13–16, find the line integrals along the given path C.

13. $\int_C (x - y)\, dx$, where C: $x = t$, $y = 2t + 1$, for $0 \le t \le 3$
14. $\int_C \frac{x}{y}\, dy$, where C: $x = t$, $y = t^2$, for $1 \le t \le 2$
15. $\int_C (x^2 + y^2)\, dy$, where C is given in the accompanying figure



16. $\int_C \sqrt{x + y}\, dx$, where C is given in the accompanying figure



17. Along the curve $\mathbf{r}(t) = t\mathbf{i} - \mathbf{j} + t^2\mathbf{k}$, $0 \le t \le 1$, evaluate each of the following integrals.
   a. $\int_C (x + y - z)\, dx$ b. $\int_C (x + y - z)\, dy$
   c. $\int_C (x + y - z)\, dz$
18. Along the curve $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} - (\cos t)\mathbf{k}$, $0 \le t \le \pi$, evaluate each of the following integrals.
   a. $\int_C xz\, dx$ b. $\int_C xz\, dy$ c. $\int_C xyz\, dz$

Work

In Exercises 19–22, find the work done by $\mathbf{F}$ over the curve in the direction of increasing t.

19. $\mathbf{F} = xy\mathbf{i} + y\mathbf{j} - yz\mathbf{k}$
   $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}$, $0 \le t \le 1$
20. $\mathbf{F} = 2y\mathbf{i} + 3x\mathbf{j} + (x + y)\mathbf{k}$
   $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (t/6)\mathbf{k}$, $0 \le t \le 2\pi$
21. $\mathbf{F} = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$
   $\mathbf{r}(t) = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} + t\mathbf{k}$, $0 \le t \le 2\pi$
22. $\mathbf{F} = 6z\mathbf{i} + y^2\mathbf{j} + 12x\mathbf{k}$
   $\mathbf{r}(t) = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} + (t/6)\mathbf{k}$, $0 \le t \le 2\pi$

Line Integrals in the Plane

23. Evaluate $\int_C xy\, dx + (x + y)\, dy$ along the curve $y = x^2$ from $(-1, 1)$ to $(2, 4)$.
24. Evaluate $\int_C (x - y)\, dx + (x + y)\, dy$ counterclockwise around the triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$.
25. Evaluate $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$ for the vector field $\mathbf{F} = x^2\mathbf{i} - y\mathbf{j}$ along the curve $x = y^2$ from $(4, 2)$ to $(1, -1)$.
26. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ for the vector field $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$ counterclockwise along the unit circle $x^2 + y^2 = 1$ from $(1, 0)$ to $(0, 1)$.

§16.4

1、格林公式计算线积分、二重积分

EXAMPLE 3 Verify both forms of Green's Theorem for the vector field P83

$$\mathbf{F}(x, y) = (x - y)\mathbf{i} + x\mathbf{j}$$

and the region R bounded by the unit circle

$$C: \quad \mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}, \qquad 0 \le t \le 2\pi.$$

Supple. Example 1 $\oint_C x^2 y\,\mathrm{d}x - xy^2\,\mathrm{d}y$ P86

$x^2 + y^2 = a^2$

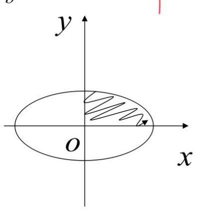
Supple. Example 3 Area of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ P102

Solution 1

$$A = 4\int_0^a f(x)\,\mathrm{d}x = \pi ab.$$

Solution 2

$$A = \iint_R \mathrm{d}x\,\mathrm{d}y = \pi ab.$$



EXAMPLE 4 Evaluate the line integral P101

Class Work?

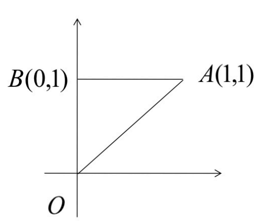
$$\oint_C xy\,dy - y^2\,dx,$$

where C is the square cut from the first quadrant by the lines $x = 1$ and $y = 1$.

Supple. Example 2 $\iint_R e^{-y^2}\,\mathrm{d}x\,\mathrm{d}y$ P98

x-type?

$$\iint_R e^{-y^2}\,\mathrm{d}x\,\mathrm{d}y = \int_0^1\left[\int_x^1 e^{-y^2}\,\mathrm{d}y\right]\mathrm{d}x$$



y-type(ok)?

Applying Green Formula:

Class Work Evaluate the line integral P97

$\oint_C (2xy - x^2)dx + (y^2 + x)\,dy$, where C is boundary of the region bounded by $y = x^2$ and $x = y^2$ with positive direction(i.e. counterclockwise forward direction).

Answer: $\frac{1}{30}$

Supple. Example 4 Show that

$$\oint_C \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2 + y^2} = \begin{cases} 0 & (0,0) \notin R \\ 2\pi & (0,0) \in R \end{cases}$$

P104

Solution $M = \frac{-y}{x^2 + y^2}, N = \frac{x}{x^2 + y^2}.$

EXAMPLE 5 Calculate the outward flux of the vector field $\mathbf{F}(x, y) = x\mathbf{i} + y^2\mathbf{j}$ across the square bounded by the lines $x = \pm 1$ and $y = \pm 1$. counterclockwise P107

20. $\mathbf{F} = (4x - 2y)\mathbf{i} + (2x - 4y)\mathbf{j}$

C: The circle $(x - 2)^2 + (y - 2)^2 = 4$

§16.4

Using Green's Theorem

Apply Green's Theorem to evaluate the integrals in Exercises 21–24.

21. $\oint_C (y^2\,dx + x^2\,dy)$

C: The triangle bounded by $x = 0$, $x + y = 1$, $y = 0$

22. $\oint_C (3y\,dx + 2x\,dy)$

C: The boundary of $0 \le x \le \pi$, $0 \le y \le \sin x$

23. $\oint_C (6y + x)\,dx + (y + 2x)\,dy$

C: The circle $(x - 2)^2 + (y - 3)^2 = 4$

24. $\oint_C (2x + y^2)\,dx + (2xy + 3y)\,dy$

30. **Integral dependent only on area** Show that the value of

$$\oint_C xy^2\,dx + (x^2y + 2x)\,dy$$

around any square depends only on the area of the square and not on its location in the plane.

31. Evaluate the integral

$$\oint_C 4x^3y\,dx + x^4\,dy$$

for any closed path C.

32. Evaluate the integral

$$\oint_C -y^3\,dy + x^3\,dx$$

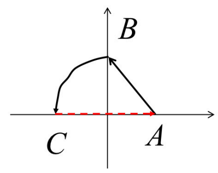
for any closed path C.

33. **Area as a line integral** Show that if R is a region in the plane bounded by a piecewise smooth, simple closed curve C, then

Class Work

- Evaluate the line integral $\int_C \left(x + e^{\sin y}\right) dy - \left(y - \frac{1}{2}\right) dx$, where C is consisted of line segment from A(1, 0) to B(0, 1) and arc from B(0, 1) to C(-1, 0) along $y = \sqrt{1 - x^2}$.

- Answer: $\frac{\pi}{2}$



§16.5

1、求曲面面积

Example 7 Find the area of the surface cut from the bottom of the paraboloid $z = x^2 + y^2$ by the plane $z = 4$. P10

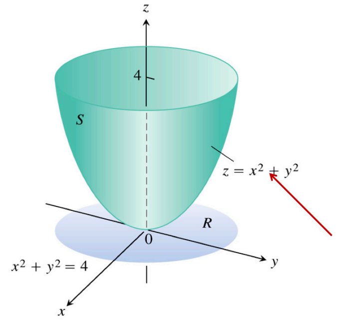
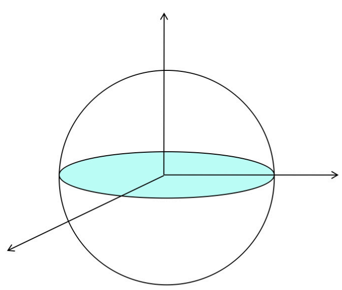


FIGURE 16.46 The area of this parabolic

Example 5 Find the surface area of a sphere of radius a.

P13



$$x^2 + y^2 + z^2 = a^2$$