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Introduction

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Chapter 2

Electromagnetism and Laser Profiles

2.1 Classical Electrodynamics

The main principles and laws that govern the phenomena behind lasers, plasma and their interaction are those of classical electrodynamics. As such, like many others tackling this area of research, I find that adding an overview of electrodynamics is simply mandatory. My aim when it comes to differentiating this introductory review from the millions of others out there, if at all possible, is to offer thorough calculations and explanations on some aspects where I personally felt like I wanted to see things from a clearer perspective.

2.1.1 Maxwell's Equations

The Maxwell equations are (Jackson 1999):

$$\nabla \cdot \mathbf{D} = \rho \tag{2.1a}$$
$$\nabla \cdot \mathbf{B} = 0 \tag{2.1b}$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.1c}$$
$$\nabla \times \mathbf{H} = \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t} \,. \tag{2.1d}$$

In the absence of magnetic and polarizable media, $\mathbf{D} = \varepsilon_0 \mathbf{E}$ and $\mathbf{B} = \mu_0 \mathbf{H}$ and the equations become:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \tag{2.2a}$$
$$\nabla \cdot \mathbf{B} = 0 \tag{2.2b}$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.2c}$$
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \,, \tag{2.2d}$$

While most readers probably have already had at least a basic introduction to the phenomena from which these equations arise and are well acquainted to how to make use of these equations, I would direct those who haven't towards the book by Fleisch 2008

By extracting the current density from equation (2.2d), computing its divergence and then replacing the electric field term using equation (2.2a) one obtains the continuity equation, which relates only the field sources to one another:

$$\boldsymbol{\nabla} \cdot \mathbf{j}(\mathbf{r}, t) + \frac{\partial \rho(\mathbf{r}, t)}{\partial t} = 0\,. \tag{2.3}$$

These equations are also complemented by the Lorentz force, which describes how the fields act on the sources. The expression of the Lorentz force in the contiuous case is:

$$\mathbf{F} = \int_{\mathcal{V}} \mathrm{d}\mathbf{r}' \left[\rho(\mathbf{r}', t)\mathbf{E}(\mathbf{r}', t) + \frac{1}{c}\mathbf{j}(\mathbf{r}', t) \times \mathbf{B}(\mathbf{r}', t)\right].$$

2.1.2 The Scalar and Vector Potentials

Since the electric ($\mathbf{E}$) and magnetic ($\mathbf{B}$) fields are vectors, they can be described together by a total of six quantities. The sources on the other hand can be described using only four quantities: the electric charge density ρ and the three components of the electric current density $\mathbf{j}$. This points to the fact that there is a more convenient way to describe the fields. In finding this alternative, we shall employ the following basic results from algebra:

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{v}) = 0 \tag{2.4a}$$
$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \cdot \mathbf{v}) = 0 \tag{2.4b}$$
$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} f) = 0\,, \tag{2.4c}$$

which are valid for any vector function $\mathbf{v}$ and for any scalar function f.

From equations (2.2b) and (2.4a) one can define the vector potential $\mathbf{A}$ such that

$$\mathbf{B}(\mathbf{r}, t) = \boldsymbol{\nabla} \times \mathbf{A}(\mathbf{r}, t)\,. \tag{2.5}$$

By substituting (2.5) in (2.2c) one obtains

$$\boldsymbol{\nabla} \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0 \tag{2.6}$$

which together with equation (2.4c) defines the scalar potential ϕ

$$\boldsymbol{\nabla}\phi(\mathbf{r}, t) = -\mathbf{E}(\mathbf{r}, t) - \frac{\partial \mathbf{A}}{\partial t}\,. \tag{2.7}$$

Using this in equation (2.2a)

$$\nabla^2\phi + \frac{\partial}{\partial t}\boldsymbol{\nabla} \cdot \mathbf{A} = -\frac{\rho}{\varepsilon_0}\,. \tag{2.8}$$

Similarly, using equation (2.7) in equation (2.2d) and making use of the following vector identity

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{v}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{v}) - \nabla^2\mathbf{v}\,, \tag{2.9}$$

another equation of the potentials is obtained

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{j} + \boldsymbol{\nabla}\left(\boldsymbol{\nabla} \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t}\right). \tag{2.10}$$

Considering that at every step in the derivation of equations (2.8) and (2.10) we only imposed the Maxwell equations and basic algebraic identities, it follows that equations (2.8) and (2.10) and equation (2.2) are completely equivalent. We now have reduced the six quantities describing the fields to only four: the scalar potential ϕ and the three components of the vector potential $\mathbf{A}$. This description of the fields through the potentials is quite useful since it is easily integrated in the formalism of special relativity. One can define the electomagnetic potential 4-vector such that the scalar field is the time-like component and the vector field is the space-like component.

In general, when studying the dynamics of particles in an electomagnetic field, once the potentials are computed using equations (2.8) and (2.10) the fields are obtained from equations (2.5) and (2.7) and can be used further in the expression of the Lorentz force.

2.1.3 Gauge Transformation

By a direct application of equation (2.4) one can show that a simultaneous transformation by an arbitrary well-behaved (continuous with continuous derivatives) scalar function $f = f(\mathbf{r}, t)$ of the potentials:

$$\phi \to \phi + \frac{\partial f}{\partial t} \tag{2.11a}$$

$$\mathbf{A} \to \mathbf{A} - \boldsymbol{\nabla} f \,, \tag{2.11b}$$

leaves the electric and magnetic field unchanged. This is actually a quite natural equivalent of the intuitive fact that any potential is defined up to a constant. In the particular case of the electromagnetic potential, equation (2.11) define a gauge transformation. There are two widely used gauges.

Lorenz gauge

$$\boldsymbol{\nabla} \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0 \tag{2.12}$$

This gauge cancels the gradient in equation (2.10). If one works in the usual Mikovski metric (Weinberg 1972)

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.13}$$

the d'Alembert operator is then defined as

$$\Box = \partial^\mu \partial_\mu = \eta^{\mu\nu}\partial_\nu \partial_\mu = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \,,$$

where μ, $\nu = \overline{0,3}$ with 0 being the temporal index and 1, 2, 3 being the spatial indices (note: in this thesis I use Einstein's summation convention whenever there is and index repeated once up and down, *i.e.* is appears as both variant and covariant in a product). By replacing this definition in equations (2.8) and (2.10), it is easy to see that both $\mathbf{A}$ and ϕ obey a free wave equation:

$$\Box \mathbf{A} = -\mu_0 \mathbf{j} \tag{2.14a}$$

$$\Box \phi = -\frac{\rho}{\varepsilon_0} \,. \tag{2.14b}$$

Coulomb Gauge (sometimes found as transversal/velocity gauge)

$$\boldsymbol{\nabla} \cdot \mathbf{A} = 0 \tag{2.15}$$

Under this gauge, the potential equations equations (2.8) and (2.10) take the form:

$$\Box \mathbf{A} = -\mu_0 \mathbf{j} + \frac{1}{c^2} \boldsymbol{\nabla} \frac{\partial \phi}{\partial t} \tag{2.16a}$$

$$\nabla^2 \phi = -\frac{\rho}{\varepsilon_0} \,. \tag{2.16b}$$

2.1.4 The Poynting Theorem

The Poynting theorem is the form of the conservation of energy in the case of electromagnetic fields interacting with charges and currents. Since it is such an important and general result, this presentation of it will start from the more general form of the Maxwell equations equation (2.1).

In the derivation of this theorem, one usually starts from the local form of the Lorentz force (Griffiths 1999):

$$\mathbf{F} = \delta q \mathbf{E} + \delta q \mathbf{v} \times \mathbf{B}$$

The work done by the electric field part of the force on the volume element with charge δq and velocity $\mathbf{v} = \frac{\mathrm{d}\mathbf{l}}{\mathrm{d}t}$ is

$$\mathrm{d}W_e = q \, \mathrm{d}\mathbf{l} \cdot \mathbf{E}$$

and the corresponding rate of work done is

$$\frac{\mathrm{d}W_e}{\mathrm{d}t} = q \mathbf{v} \cdot \mathbf{E}$$

while for the magnetic part we have (as expected)

$$\mathrm{d}W_m = \mathrm{d}\mathbf{l} \cdot \mathbf{F_b} = q \, \mathrm{d}\mathbf{l} \cdot (\mathbf{v} \times \mathbf{B})$$

$$\frac{\mathrm{d}W_b}{\mathrm{d}t} = q \mathbf{v} \cdot (\mathbf{v} \times \mathbf{B}) = 0 \,.$$

Adding these contributions and generalizing for the case of a distribution of charges and currents one obtains

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \int_{\mathcal{V}} \mathrm{d}\mathbf{r} \, \mathbf{E} \cdot \mathbf{j} \tag{2.17}$$

By extracting $\mathbf{j}$ from equation (2.1d) and replacing in the above equation we have

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \int_{\mathcal{V}} \mathrm{d}\mathbf{r} \left[\mathbf{E} \cdot (\boldsymbol{\nabla} \times \mathbf{H}) - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} \right]$$

Employing here the vector identity here

$$\boldsymbol{\nabla}(\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\boldsymbol{\nabla} \times \mathbf{u}) - \mathbf{u} \cdot (\boldsymbol{\nabla} \times \mathbf{v}) \tag{2.18}$$

gives

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \int_{\mathcal{V}} \mathrm{d}\mathbf{r} \left[\mathbf{H} \cdot (\boldsymbol{\nabla} \times \mathbf{E}) - \boldsymbol{\nabla}(\mathbf{E} \times \mathbf{H}) - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} \right] .$$

Replacing the curl of $\mathbf{E}$ using Faraday's law (2.1c) we finally obtain

$$\frac{\mathrm{d}W}{\mathrm{d}t} = - \int_{\mathcal{V}} \mathrm{d}\mathbf{r} \left[\boldsymbol{\nabla}(\mathbf{E} \times \mathbf{H}) + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} \right] .$$

If we restric the discussion now only to linar media (*i.e.* $\mathbf{D} = \varepsilon \mathbf{E}$ and $\mathbf{B} = \mu \mathbf{H}$) a new important quantity can be defined

$$w_{em} = \frac{1}{2} (\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B}) \tag{2.19}$$

which leads to a new way to write the expression of the rate of work done by the electromagnetic field

$$\frac{\mathrm{d}W}{\mathrm{d}t} = - \int_{\mathcal{V}} \mathrm{d}\mathbf{r} \left[\boldsymbol{\nabla}(\mathbf{E} \times \mathbf{H}) + \frac{\partial w_{em}}{\partial t} \right] , \tag{2.20}$$

where the Poynting vector is

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} . \tag{2.21}$$

In order to complete the derivation of Poynting's theorem, an we must see how it is to be interpreted. As such, a short paranthesis concerning w_{em} is in order.

Electrostatic field energy density

For a system of N stationary point-like charged particles of charges q_i placed at $\mathbf{r_i}$, $i = \overline{1, N}$ in a medium with permittivity ε, the total potential energy of the system, when neglecting the infinite self-interaction terms, is (Jackson 1999)

$$W_e = \frac{1}{2} \sum_{i,j=1, i \neq j}^{N} \frac{q_i q_j}{4 \pi \varepsilon |\mathbf{r_i} - \mathbf{r_j}|}$$

or, factoring out the scalar potential $\phi(\mathbf{r_i})$ generated by all the other particles at the position of particle i,

$$W_e = \frac{1}{2} \sum_{i=1}^{N} q_i \phi(\mathbf{r_i})$$

This is easily generalized in integral form

$$W_e = \frac{1}{2} \int_{\mathcal{V}} \mathrm{d}\mathbf{r} \, \rho(\mathbf{r}) \phi(\mathbf{r}) ,$$

where we use the delta-Dirac function for pointlike particles if needed.

Using the fact that the electrostatic potential is defined by $\mathbf{E} = -\boldsymbol{\nabla} \phi$ and replacing this in equation (2.1a) one obtains the poison equation

$$\nabla^2 \phi = - \frac{\rho}{\varepsilon} . \tag{2.22}$$

With this, the integral above becomes

$$W_e = \frac{\varepsilon}{2}\int_{\mathcal{V}} \mathrm{d}\mathbf{r}\,\phi\nabla^2\phi = -\frac{\varepsilon}{2}\int_{\mathcal{V}} \mathrm{d}\mathbf{r}\,\phi\boldsymbol{\nabla}\phi + \frac{\varepsilon}{2}\int_{\mathcal{V}} \mathrm{d}\mathbf{r}\,|\boldsymbol{\nabla}\phi|^2\,,$$

where integration by parts has been used.

In order to reach the desired result, we still have to perform one more integration by parts

$$\int_{\mathcal{V}} \mathrm{d}\mathbf{r}\,\phi\boldsymbol{\nabla}\phi = \frac{1}{2}\int_{\mathcal{V}} \mathrm{d}\mathbf{r}\,\boldsymbol{\nabla}\phi^2 = \int_{\mathcal{S}_{\mathcal{V}}} \mathrm{d}\mathbf{a}\,\phi^2\,,$$

where in the last step we used Gauss' theorem. Now, if we itegrate over the entire space and keep in mind that the electrostatic potential should be zero at infinity, the above integral becames null. Using again the realtion between the gradient of the potential and the electric field we get

$$W_e = \frac{\varepsilon}{2}\int_{\mathcal{V}} \mathrm{d}\mathbf{r}\,\mathbf{E}^2 \tag{2.23}$$

or, equivalently,

$$W_e = \frac{1}{2}\int_{\mathcal{V}} \mathrm{d}\mathbf{r}\,\mathbf{E}\cdot\mathbf{D}\,. \tag{2.24}$$

This leads to the definition of the energy density of the electrostatic field

$$w_e = \frac{1}{2}\mathbf{E}\cdot\mathbf{D}\,. \tag{2.25}$$

Magnetostatic field energy density

This time around we start with a current loop in the case of magnetostatics ($\boldsymbol{\nabla}\cdot\mathbf{j} = 0$). No matter the current distribution in space, since the current density is rotational, we can always divide it in individual infinitesimal current loops. A change in the magnetic flux through such a loop is given by the integral form of Faraday's law (2.1c)

$$e = \oint_{\gamma} \mathrm{d}\mathbf{l}\cdot\mathbf{E} = -\frac{\mathrm{d}\phi_B}{\mathrm{d}t}\,, \tag{2.26}$$

where γ is the closed curve describing the loop and ϕ_B is the magnetic flux through the loop.

Since the autoinduced magnetic flux is $\phi_B = LI$, where L is the inductance of the loop and I the intensity of the electric current flowing in it, the electromotive force caused by autoinduction is

$$e = -L\frac{\mathrm{d}I}{\mathrm{d}t}\,.$$

Thus the rate of work against the increase of the current is

$$\frac{\mathrm{d}W_B}{\mathrm{d}t} = -Ie = LI\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{LI^2}{2}\right)\,.$$

With this result we obtain the energy necessary to get a current of intensity I starting through a loop:

$$W_B = \frac{LI^2}{2}\,.$$

We will now eliminate L the same way we introduced it

$$\phi_B = LI = \int_{\mathcal{S}_\gamma} \mathrm{d}\mathbf{a} \cdot \mathbf{B} = \int_{\mathcal{S}_\gamma} \mathrm{d}\mathbf{a} \cdot (\boldsymbol{\nabla} \times \mathbf{A}) = \oint_\gamma \mathrm{d}\mathbf{l} \cdot \mathbf{A}\,,$$

where the vector potential was introduced and Stokes' theorem was applyed.

$$W_B = \frac{1}{2} I \oint_\gamma \mathrm{d}\mathbf{l} \cdot \mathbf{A} = \frac{1}{2} \int_{\mathcal{V}} \mathrm{d}\mathbf{r}\, \mathbf{j} \cdot \mathbf{A}\,.$$

Here we naturally introduced the electric current density in our calculations. It can be replaced though using equation (2.1d) (we work in the confinements of magnetostatics, so there is no time dependent electric field)

$$W_B = \frac{1}{2} \int_{\mathcal{V}} \mathrm{d}\mathbf{r}\, \mathbf{A} \cdot (\boldsymbol{\nabla} \times \mathbf{H})\,.$$

We employ here the identity (2.18) to reach

$$W_B = \frac{1}{2} \int_{\mathcal{V}} \mathrm{d}\mathbf{r}\, \mathbf{H} \cdot (\boldsymbol{\nabla} \times \mathbf{A}) - \frac{1}{2} \int_{\mathcal{V}} \mathrm{d}\mathbf{r}\, \boldsymbol{\nabla} \cdot (\mathbf{A} \times \mathbf{H}) = \frac{1}{2} \int_{\mathcal{V}} \mathrm{d}\mathbf{r}\, \mathbf{H} \cdot \mathbf{B} - \frac{1}{2} \int_{\mathcal{S}_{\mathcal{V}}} \mathrm{d}\mathbf{a} \cdot (\mathbf{A} \times \mathbf{H})\,.$$

The same trick as in the previous subsection is appliable here. By extending the integration volume over the entire space and using the fact that the vector potential should be zero at infinity, the second integral vanishes.

$$W_B = \frac{1}{2} \int_{\mathcal{V}} \mathrm{d}\mathbf{r}\, \mathbf{H} \cdot \mathbf{B} \tag{2.27}$$

The energy density of the magnetostatic field is defined to be

$$w_B = \frac{1}{2} \mathbf{H} \cdot \mathbf{B}\,. \tag{2.28}$$

Interpretation of the Poynting theorem

We can see now that (2.19) is simply the sum of equation (2.25) and equation (2.28). Summing up all the previous considerations, w_{em} holds the meaning of the energy density of the electromagnetic field itself, that is, the energy density present in space due to the presence of the electric and magnetic fields.

The Poynting theorem (2.20) can be rewritten using Gauss' theorem in its integral form

$$\frac{\mathrm{d}W}{\mathrm{d}t} = -\int_{\mathcal{S}_{\mathcal{V}}} \mathrm{d}\mathbf{a} \cdot \mathbf{S} - \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{V}} \mathrm{d}\mathbf{r}\, w_{em} \tag{2.29}$$

or in its differential form by eliminating the integrals

$$\mathbf{E} \cdot \mathbf{j} = -\boldsymbol{\nabla} \cdot \mathbf{S} - \frac{\partial w_{em}}{\partial t}\,. \tag{2.30}$$

The Poynting vector has units of $\frac{J}{m^2 s}$ and decribes the flux of energy through a surface. From this, we can conclude that the physical meaning behind equations (2.29) and (2.30) is that the rate of change in time of the energy inside a volume added with the flow of energy in and out of that volume is equal to minus the work done by the fields on the sources inside the volume.

2.1.5 Momentum of a System of Fields and Field Sources

By taking the vector product of **D** with equation (2.1c) and of **B** with equation (2.1d) and then adding them up the following equality can be obtained:

$$\mathbf{D} \times (\boldsymbol{\nabla} \times \mathbf{E}) + \mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{H}) = -\mathbf{j} \times \mathbf{B} - \frac{\partial}{\partial t}(\mathbf{D} \times \mathbf{B}) . \tag{2.31}$$

We will restric this discussion to the case where there is no polarizable or magnetic media:

$$\varepsilon_0 \mathbf{E} \times (\boldsymbol{\nabla} \times \mathbf{E}) + \frac{1}{\mu_0}\mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{B}) = -\mathbf{j} \times \mathbf{B} - \varepsilon_0 \frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}) . \tag{2.32}$$

Considering that the speed of light in vacuum is $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$, equation (2.33) becomes

$$\varepsilon_0(\mathbf{E} \times (\boldsymbol{\nabla} \times \mathbf{E}) + c^2 \mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{B})) = -\mathbf{j} \times \mathbf{B} - \varepsilon_0 \frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}) . \tag{2.33}$$

In order to proceed, some vector algebra must be discussed. In particular, we would like to evaluate the following expression:

$$\mathbf{v}(\boldsymbol{\nabla} \cdot \mathbf{v}) - \mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{v})$$

We have

$$\mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{v}) = \mathbf{e}^i \varepsilon_{ijk} v^j (\boldsymbol{\nabla} \times \mathbf{v})_k = \mathbf{e}^i \varepsilon_{ijk} v^j \varepsilon^{lmk} \partial_l v_m =$$

$$= \mathbf{e}^i (\delta_i^l \delta_j^m - \delta_i^m \delta_j^l) v^j \partial_l v_m = \mathbf{e}^i \left[v^j \partial_i v_j - v^j \partial_j v_i \right]$$

and

$$\mathbf{v}(\boldsymbol{\nabla} \cdot \mathbf{v}) = \mathbf{e}^i v_i \partial_j v^j ,$$

where ε_{ijk} is the Levi-Civita tensor, δ_i^j is the Kroneker-delta symbol and $\mathbf{e}^i$, $i = \overline{1,3}$ are the Carthesian versors. Subtracting these two expressions leads to

$$\mathbf{v}(\boldsymbol{\nabla} \cdot \mathbf{v}) - \mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{v}) = \mathbf{e}^i \left[v_i \partial_j v^j - v^j \partial_i v_j + v^j \partial_j v_i \right] = \mathbf{e}^i \left[\partial_j (v_i v^j) - v^j \partial_i v_j \right] .$$

Since

$$\partial_i (v^j v_j) = 2 v^j \partial v_j ,$$

we have

$$\mathbf{v}(\boldsymbol{\nabla} \cdot \mathbf{v}) - \mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{v}) = \mathbf{e}^i \left[\partial_j (v_i v^j) - \frac{1}{2} \partial_i (v_j v^j) \right] ,$$

The second term can be stylized by introducing a Kroneker-delta and writing $v_j v^j$ as $\mathbf{v}^2$, which leads to the desired final result

$$\mathbf{v}(\boldsymbol{\nabla} \cdot \mathbf{v}) - \mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{v}) = \mathbf{e}^i \partial_j \left[v_i v^j - \frac{1}{2} \mathbf{v}^2 \delta_i^j \right] = \mathbf{e}_i \partial_j \left[v^i v^j - \frac{1}{2} \mathbf{v}^2 \delta^{ij} \right] . \tag{2.34}$$

The last step is possible due to the fact that we only work with space-like components and we chose the convenient metric (2.13).

By defining the Maxwell stress tensor as

$$T^{ij} = \varepsilon_0 \left[E^i E^j + c^2 B^i B^j - \frac{1}{2}(\mathbf{E}^2 + c^2\mathbf{B}^2)\delta^{ij} \right] \tag{2.35}$$

and using it along with equation (2.34) in equation (2.33) one gets

$$\mathbf{e}_i \partial_j T^{ij} = \varepsilon_0 \mathbf{E}(\boldsymbol{\nabla} \cdot \mathbf{E}) + \frac{1}{\mu_0}\mathbf{B}(\boldsymbol{\nabla} \cdot \mathbf{B}) + \mathbf{j} \times \mathbf{B} + \varepsilon_0 \frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B})$$

Using Maxwell's equations (2.2a) and (2.2b) together with the definition of the Poynting vector (2.21) the above expression is simplified to the law of momentum conservation

$$\mathbf{e}_i \partial_j T^{ij} = \rho \mathbf{E} + \mathbf{j} \times \mathbf{B} + \frac{1}{c^2}\frac{\partial \mathbf{S}}{\partial t} \tag{2.36}$$

or

$$\mathbf{e}_i \partial_j T^{ij} = \rho \mathbf{E} + \mathbf{j} \times \mathbf{B} + \frac{\partial \mathbf{g}}{\partial t}\,, \tag{2.37}$$

where the volumic density of the fields' electromagnetic momentum $\mathbf{g}$ is defined to be

$$\mathbf{g} = \frac{1}{c^2}\mathbf{S} = \mathbf{D} \times \mathbf{B}\,. \tag{2.38}$$

By observing that when integrating over a volume, the $\rho\mathbf{E} + \mathbf{j} \times \mathbf{B}$ is simply the Lorentz force and that $\mathbf{e}_i \partial_j T^{ij} = \boldsymbol{\nabla} \cdot \hat{T}$ we reach an integral form of the momentum conservation

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{P_{em}} + \mathbf{P_{mech}}) = \int_{\mathcal{V}} \mathrm{d}\mathbf{r}\, \boldsymbol{\nabla} \cdot \hat{T} = \int_{\mathcal{S}_{\mathcal{V}}} \mathrm{d}\mathbf{a} \cdot \hat{T}\,, \tag{2.39}$$

where $\mathbf{P_{em}}$ and $\mathbf{P_{mech}}$ are the electromagnetic and mechanical momenta, respectively. If we integrate over the entire space and use the fact that the stress tensor vanishes at infinity, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{P_{em}} + \mathbf{P_{mech}}) = 0\,. \tag{2.40}$$

2.2 Electromagnetic Waves

The short review of classical electrodynamics had as an ultimate goal to introduce the definitions, equations and formalism required in order to study electromagnetic waves. In this section I will start from the definition and properties of an electromagnetic wave and I will follow up with how one can describe laser pulses. The second part will contain a short introduction to the laser profiles used in research.

2.2.1 Maxwell's Equations in Vacuum

The concept of electromagnetic waves arises naturally from the Maxwell equations equation (2.1) if we consider them in vacuum

$$\nabla \cdot \mathbf{D} = 0 \tag{2.41a}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2.41b}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.41c}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}\,. \tag{2.41d}$$

2.3 Electron Dynamics in Electromagnetic Fields

2.4 The Ponderomotive Force

2.5 Simulations for the Visualization of the Ponderomotive Force

Chapter 3

Results

In this chapter we present the main results . . .

Chapter 4

Conclusions

In conclusion ...

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