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Contents

Chapter 1

Introduction

In this thesis . . .

Chapter 2

Electromagnetism and Laser Profiles

2.1 Classical Electrodynamics

The main principles and laws that govern the phenomena behind lasers, plasma and their interaction are those of classical electrodynamics. As such, like many others tackling this area of research, I find that adding an overview of electrodynamics is simply mandatory. My aim when it comes to differentiating this introductory review from the millions of others out there, if at all possible, is to offer thorough calculations and explanations on some aspects where I personally felt like I wanted to see things from a clearer perspective.

2.1.1 Maxwell's Equations

The Maxwell equations are (Jackson 1999):

$$\nabla \cdot \mathbf{D} = \rho \tag{2.1a}$$
$$\nabla \cdot \mathbf{B} = 0 \tag{2.1b}$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.1c}$$
$$\nabla \times \mathbf{H} = \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t}\,. \tag{2.1d}$$

In the absence of magnetic and polarizable media, $\mathbf{D} = \varepsilon_0 \mathbf{E}$ and $\mathbf{B} = \mu_0 \mathbf{H}$ and the equations become:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \tag{2.2a}$$
$$\nabla \cdot \mathbf{B} = 0 \tag{2.2b}$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.2c}$$
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}\,, \tag{2.2d}$$

While most readers probably have already had at least a basic introduction to the phenomena from which these equations arise and are well acquainted to how to make use of these equations, I would direct those who haven't towards the book by Fleisch 2008

By extracting the current density from equation (2.2d), computing its divergence and then replacing the electric field term using equation (2.2a) one obtains the continuity equation, which relates only the field sources to one another:

$$\boldsymbol{\nabla} \cdot \mathbf{j}(\mathbf{r}, t) + \frac{\partial \rho(\mathbf{r}, t)}{\partial t} = 0\,. \tag{2.3}$$

These equations are also complemented by the Lorentz force, which describes how the fields act on the sources. The expression of the Lorentz force in the contiuous case is:

$$\mathbf{F} = \int_{\mathcal{V}} \mathrm{d}\mathbf{r}' \left[\rho(\mathbf{r}', t)\mathbf{E}(\mathbf{r}', t) + \frac{1}{c}\mathbf{j}(\mathbf{r}', t) \times \mathbf{B}(\mathbf{r}', t)\right].$$

2.1.2 The Scalar and Vector Potentials

Since the electric ($\mathbf{E}$) and magnetic ($\mathbf{B}$) fields are vectors, they can be described together by a total of six quantities. The sources on the other hand can be described using only four quantities: the electric charge density ρ and the three components of the electric current density $\mathbf{j}$. This points to the fact that there is a more convenient way to describe the fields. In finding this alternative, we shall employ the following basic results from algebra:

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{v}) = 0 \tag{2.4a}$$
$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \cdot \mathbf{v}) = 0 \tag{2.4b}$$
$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} f) = 0\,, \tag{2.4c}$$

which are valid for any vector function $\mathbf{v}$ and for any scalar function f.

From equations (2.2b) and (2.4a) one can define the vector potential $\mathbf{A}$ such that

$$\mathbf{B}(\mathbf{r}, t) = \boldsymbol{\nabla} \times \mathbf{A}(\mathbf{r}, t)\,. \tag{2.5}$$

By substituting (2.5) in (2.2c) one obtains

$$\boldsymbol{\nabla} \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0 \tag{2.6}$$

which together with equation (2.4c) defines the scalar potential ϕ

$$\boldsymbol{\nabla}\phi(\mathbf{r}, t) = -\mathbf{E}(\mathbf{r}, t) - \frac{\partial \mathbf{A}}{\partial t}\,. \tag{2.7}$$

Using this in equation (2.2a)

$$\nabla^2\phi + \frac{\partial}{\partial t}\boldsymbol{\nabla} \cdot \mathbf{A} = -\frac{\rho}{\varepsilon_0}\,. \tag{2.8}$$

Similarly, using equation (2.7) in equation (2.2d) and making use of the following vector identity

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{v}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{v}) - \nabla^2\mathbf{v}\,, \tag{2.9}$$

another equation of the potentials is obtained

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0\mathbf{j} + \boldsymbol{\nabla}\left(\boldsymbol{\nabla} \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t}\right). \tag{2.10}$$

Considering that at every step in the derivation of equations (2.8) and (2.10) we only imposed the Maxwell equations and basic algebraic identities, it follows that equations (2.8) and (2.10) and equation (2.2) are completely equivalent. We now have reduced the six quantities describing the fields to only four: the scalar potential ϕ and the three components of the vector potential $\mathbf{A}$. This description of the fields through the potentials is quite useful since it is easily integrated in the formalism of special relativity. One can define the electomagnetic potential 4-vector such that the scalar field is the time-like component and the vector field is the space-like component.

In general, when studying the dynamics of particles in an electomagnetic field, once the potentials are computed using equations (2.8) and (2.10) the fields are obtained from equations (2.5) and (2.7) and can be used further in the expression of the Lorentz force.

2.1.3 Gauge Transformation

By a direct application of equation (2.4) one can show that a simultaneous transformation by an arbitrary well-behaved (continuous with continuous derivatives) scalar function $f = f(\mathbf{r}, t)$ of the potentials:

$$\phi \to \phi + \frac{\partial f}{\partial t} \tag{2.11a}$$

$$\mathbf{A} \to \mathbf{A} - \boldsymbol{\nabla} f\,, \tag{2.11b}$$

leaves the electric and magnetic field unchanged. This is actually a quite natural equivalent of the intuitive fact that any potential is defined up to a constant. In the particular case of the electromagnetic potential, equation (2.11) define a gauge transformation. There are two widely used gauges.

Lorenz gauge

$$\boldsymbol{\nabla} \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0 \tag{2.12}$$

This gauge cancels the gradient in equation (2.10). If one works in the usual Mikovski metric (Weinberg 1972)

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.13}$$

the d'Alembert operator is then defined as

$$\Box = \partial^\mu \partial_\mu = \eta^{\mu\nu}\partial_\nu\partial_\mu = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\,,$$

where μ, $\nu = \overline{0,3}$ with 0 being the temporal index and 1, 2, 3 being the spatial indices (note: in this thesis I use Einstein's summation convention whenever there is and index repeated once up and down, *i.e.* is appears as both variant and covariant in a product). By replacing this definition in equations (2.8) and (2.10), it is easy to see that both $\mathbf{A}$ and ϕ obey a free wave equation:

$$\Box\mathbf{A} = -\mu_0 \mathbf{j} \tag{2.14a}$$

$$\Box\phi = -\frac{\rho}{\varepsilon_0}\,. \tag{2.14b}$$

Coulomb Gauge (sometimes found as transversal/velocity gauge)

$$\boldsymbol{\nabla} \cdot \mathbf{A} = 0 \tag{2.15}$$

Under this gauge, the potential equations equations (2.8) and (2.10) take the form:

$$\Box \mathbf{A} = -\mu_0 \mathbf{j} + \frac{1}{c^2} \boldsymbol{\nabla} \frac{\partial \phi}{\partial t} \tag{2.16a}$$

$$\nabla^2 \phi = -\frac{\rho}{\varepsilon_0}\,. \tag{2.16b}$$

Getting Back the Fields

In principle, one can choose the most convenient gauge for his system, solve the corresponding potential equations, then recover the electric and magnetic fields with

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \boldsymbol{\nabla} \phi \tag{2.17a}$$

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}\,. \tag{2.17b}$$

2.1.4 The Poynting Theorem

The Poynting theorem is the form of the conservation of energy in the case of electromagnetic fields interacting with charges and currents. Since it is such an important and general result, this presentation of it will start from the more general form of the Maxwell equations equation (2.1).

In the derivation of this theorem, one usually starts from the local form of the Lorentz force (Griffiths 1999):

$$\mathbf{F} = \delta q \mathbf{E} + \delta q \mathbf{v} \times \mathbf{B}$$

The work done by the electric field part of the force on the volume element with charge δq and velocity $\mathbf{v} = \frac{\mathrm{d}\mathbf{l}}{\mathrm{d}t}$ is

$$\mathrm{d}W_e = q\,\mathrm{d}\mathbf{l} \cdot \mathbf{E}$$

and the corresponding rate of work done is

$$\frac{\mathrm{d}W_e}{\mathrm{d}t} = q\mathbf{v} \cdot \mathbf{E}$$

while for the magnetic part we have (as expected)

$$\mathrm{d}W_m = \mathrm{d}\mathbf{l} \cdot \mathbf{F_b} = q\,\mathrm{d}\mathbf{l} \cdot (\mathbf{v} \times \mathbf{B})$$

$$\frac{\mathrm{d}W_b}{\mathrm{d}t} = q\mathbf{v} \cdot (\mathbf{v} \times \mathbf{B}) = 0\,.$$

Adding these contributions and generalizing for the case of a distribution of charges and currents one obtains

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \int_{\mathcal{V}} \mathrm{d}\mathbf{r}\, \mathbf{E} \cdot \mathbf{j} \tag{2.18}$$

By extracting **j** from equation (2.1d) and replacing in the above equation we have

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \int_{\mathcal{V}} \mathrm{d}\mathbf{r} \left[\mathbf{E} \cdot (\boldsymbol{\nabla} \times \mathbf{H}) - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} \right]$$

Employing here the vector identity here

$$\boldsymbol{\nabla}(\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\boldsymbol{\nabla} \times \mathbf{u}) - \mathbf{u} \cdot (\boldsymbol{\nabla} \times \mathbf{v}) \tag{2.19}$$

gives

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \int_{\mathcal{V}} \mathrm{d}\mathbf{r} \left[\mathbf{H} \cdot (\boldsymbol{\nabla} \times \mathbf{E}) - \boldsymbol{\nabla}(\mathbf{E} \times \mathbf{H}) - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} \right] .$$

Replacing the curl of **E** using Faraday's law (2.1c) we finally obtain

$$\frac{\mathrm{d}W}{\mathrm{d}t} = -\int_{\mathcal{V}} \mathrm{d}\mathbf{r} \left[\boldsymbol{\nabla}(\mathbf{E} \times \mathbf{H}) + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} \right] .$$

If we restric the discussion now only to linar media (*i.e.* $\mathbf{D} = \varepsilon \mathbf{E}$ and $\mathbf{B} = \mu \mathbf{H}$) a new important quantity can be defined

$$w_{em} = \frac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B}) \tag{2.20}$$

which leads to a new way to write the expression of the rate of work done by the electromagnetic field

$$\frac{\mathrm{d}W}{\mathrm{d}t} = -\int_{\mathcal{V}} \mathrm{d}\mathbf{r} \left[\boldsymbol{\nabla}(\mathbf{E} \times \mathbf{H}) + \frac{\partial w_{em}}{\partial t} \right] , \tag{2.21}$$

where the Poynting vector is

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} . \tag{2.22}$$

In order to complete the derivation of Poynting's theorem, an we must see how it is to be interpreted. As such, a short paranthesis concerning w_{em} is in order.

Electrostatic field energy density

For a system of N stationary point-like charged particles of charges q_i placed at $\mathbf{r_i}$, $i = \overline{1, N}$ in a medium with permittivity ε, the total potential energy of the system, when neglecting the infinite self-interaction terms, is (Jackson 1999)

$$W_e = \frac{1}{2} \sum_{i,j=1, i \neq j}^{N} \frac{q_i q_j}{4\pi\varepsilon |\mathbf{r_i} - \mathbf{r_j}|}$$

or, factoring out the scalar potential $\phi(\mathbf{r_i})$ generated by all the other particles at the position of particle i,

$$W_e = \frac{1}{2} \sum_{i=1}^{N} q_i \phi(\mathbf{r_i})$$

This is easily generalized in integral form

$$W_e = \frac{1}{2}\int_{\mathcal{V}} \mathrm{d}\mathbf{r}\, \rho(\mathbf{r})\phi(\mathbf{r})\,,$$

where we use the delta-Dirac function for pointlike particles if needed.

Using the fact that the electrostatic potential is defined by $\mathbf{E} = -\boldsymbol{\nabla}\phi$ and replacing this in equation (2.1a) one obtains the poison equation

$$\nabla^2\phi = -\frac{\rho}{\varepsilon}\,. \tag{2.23}$$

With this, the integral above becomes

$$W_e = \frac{\varepsilon}{2}\int_{\mathcal{V}} \mathrm{d}\mathbf{r}\, \phi\nabla^2\phi = -\frac{\varepsilon}{2}\int_{\mathcal{V}} \mathrm{d}\mathbf{r}\, \phi\boldsymbol{\nabla}\phi + \frac{\varepsilon}{2}\int_{\mathcal{V}} \mathrm{d}\mathbf{r}\, |\boldsymbol{\nabla}\phi|^2\,,$$

where integration by parts has been used.

In order to reach the desired result, we still have to perform one more integration by parts

$$\int_{\mathcal{V}} \mathrm{d}\mathbf{r}\, \phi\boldsymbol{\nabla}\phi = \frac{1}{2}\int_{\mathcal{V}} \mathrm{d}\mathbf{r}\, \boldsymbol{\nabla}\phi^2 = \int_{\mathcal{S}_{\mathcal{V}}} \mathrm{d}\mathbf{a}\, \phi^2\,,$$

where in the last step we used Gauss' theorem. Now, if we itegrate over the entire space and keep in mind that the electrostatic potential should be zero at infinity, the above integral becames null. Using again the realtion between the gradient of the potential and the electric field we get

$$W_e = \frac{\varepsilon}{2}\int_{\mathcal{V}} \mathrm{d}\mathbf{r}\, \mathbf{E}^2 \tag{2.24}$$

or, equivalently,

$$W_e = \frac{1}{2}\int_{\mathcal{V}} \mathrm{d}\mathbf{r}\, \mathbf{E}\cdot\mathbf{D}\,. \tag{2.25}$$

This leads to the definition of the energy density of the electrostatic field

$$w_e = \frac{1}{2}\mathbf{E}\cdot\mathbf{D}\,. \tag{2.26}$$

Magnetostatic field energy density

This time around we start with a current loop in the case of magnetostatics ($\boldsymbol{\nabla}\cdot\mathbf{j} = 0$). No matter the current distribution in space, since the current density is rotational, we can always divide it in individual infinitesimal current loops. A change in the magnetic flux through such a loop is given by the integral form of Faraday's law (2.1c)

$$e = \oint_{\gamma} \mathrm{d}\mathbf{l}\cdot\mathbf{E} = -\frac{\mathrm{d}\phi_B}{\mathrm{d}t}\,, \tag{2.27}$$

where γ is the closed curve describing the loop and ϕ_B is the magnetic flux through the loop.

Since the autoinduced magnetic flux is $\phi_B = LI$, where L is the inductance of the loop and I the intensity of the electric current flowing in it, the electromotive force caused by autoinduction is

$$e = -L\frac{\mathrm{d}I}{\mathrm{d}t}\,.$$

Thus the rate of work against the increase of the current is

$$\frac{\mathrm{d}W_B}{\mathrm{d}t} = -Ie = LI\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{LI^2}{2}\right).$$

With this result we obtain the energy necessary to get a current of intensity I starting through a loop:

$$W_B = \frac{LI^2}{2}.$$

We will now eliminate L the same way we introduced it

$$\phi_B = LI = \int_{\mathcal{S}_\gamma} \mathrm{d}\mathbf{a} \cdot \mathbf{B} = \int_{\mathcal{S}_\gamma} \mathrm{d}\mathbf{a} \cdot (\boldsymbol{\nabla} \times \mathbf{A}) = \oint_\gamma \mathrm{d}\mathbf{l} \cdot \mathbf{A},$$

where the vector potential was introduced and Stokes' theorem was applyed.

$$W_B = \frac{1}{2} I \oint_\gamma \mathrm{d}\mathbf{l} \cdot \mathbf{A} = \frac{1}{2}\int_\mathcal{V} \mathrm{d}\mathbf{r}\, \mathbf{j} \cdot \mathbf{A}.$$

Here we naturally introduced the electric current density in our calculations. It can be replaced though using equation (2.1d) (we work in the confinements of magnetostatics, so there is no time dependent electric field)

$$W_B = \frac{1}{2}\int_\mathcal{V} \mathrm{d}\mathbf{r}\, \mathbf{A} \cdot (\boldsymbol{\nabla} \times \mathbf{H}).$$

We employ here the identity (2.19) to reach

$$W_B = \frac{1}{2}\int_\mathcal{V} \mathrm{d}\mathbf{r}\, \mathbf{H} \cdot (\boldsymbol{\nabla} \times \mathbf{A}) - \frac{1}{2}\int_\mathcal{V} \mathrm{d}\mathbf{r}\, \boldsymbol{\nabla} \cdot (\mathbf{A} \times \mathbf{H}) = \frac{1}{2}\int_\mathcal{V} \mathrm{d}\mathbf{r}\, \mathbf{H} \cdot \mathbf{B} - \frac{1}{2}\int_{\mathcal{S}_\mathcal{V}} \mathrm{d}\mathbf{a} \cdot (\mathbf{A} \times \mathbf{H}).$$

The same trick as in the previous subsection is appliable here. By extending the integration volume over the entire space and using the fact that the vector potential should be zero at infinity, the second integral vanishes.

$$W_B = \frac{1}{2}\int_\mathcal{V} \mathrm{d}\mathbf{r}\, \mathbf{H} \cdot \mathbf{B} \tag{2.28}$$

The energy density of the magnetostatic field is defined to be

$$w_B = \frac{1}{2}\mathbf{H} \cdot \mathbf{B}. \tag{2.29}$$

Interpretation of the Poynting theorem

We can see now that (2.20) is simply the sum of equation (2.26) and equation (2.29). Summing up all the previous considerations, w_{em} holds the meaning of the energy density of the electromagnetic field itself, that is, the energy density present in space due to the presence of the electric and magnetic fields.

The Poynting theorem (2.21) can be rewritten using Gauss' theorem in its integral form

$$\frac{\mathrm{d}W}{\mathrm{d}t} = -\int_{\mathcal{S}_\mathcal{V}} \mathrm{d}\mathbf{a} \cdot \mathbf{S} - \frac{\mathrm{d}}{\mathrm{d}t}\int_\mathcal{V} \mathrm{d}\mathbf{r}\, w_{em} \tag{2.30}$$

or in its differential form by eliminating the integrals

$$\mathbf{E}\cdot\mathbf{j} = -\boldsymbol{\nabla}\cdot\mathbf{S} - \frac{\partial w_{em}}{\partial t}\,. \tag{2.31}$$

The Poynting vector has units of $\frac{J}{m^2 s}$ and decribes the flux of energy through a surface. From this, we can conclude that the physical meaning behind equations (2.30) and (2.31) is that the rate of change in time of the energy inside a volume added with the flow of energy in and out of that volume is equal to minus the work done by the fields on the sources inside the volume.

2.1.5 Momentum of a System of Fields and Field Sources

By taking the vector product of $\mathbf{D}$ with equation (2.1c) and of $\mathbf{B}$ with equation (2.1d) and then adding them up the following equality can be obtained:

$$\mathbf{D}\times(\boldsymbol{\nabla}\times\mathbf{E}) + \mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{H}) = -\mathbf{j}\times\mathbf{B} - \frac{\partial}{\partial t}(\mathbf{D}\times\mathbf{B})\,. \tag{2.32}$$

We will restric this discussion to the case where there is no polarizable or magnetic media:

$$\varepsilon_0\mathbf{E}\times(\boldsymbol{\nabla}\times\mathbf{E}) + \frac{1}{\mu_0}\mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{B}) = -\mathbf{j}\times\mathbf{B} - \varepsilon_0\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B})\,. \tag{2.33}$$

Considering that the speed of light in vacuum is $c = \frac{1}{\sqrt{\varepsilon_0\mu_0}}$, equation (2.34) becomes

$$\varepsilon_0(\mathbf{E}\times(\boldsymbol{\nabla}\times\mathbf{E}) + c^2\mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{B})) = -\mathbf{j}\times\mathbf{B} - \varepsilon_0\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B})\,. \tag{2.34}$$

In order to proceed, some vector algebra must be discussed. In particular, we would like to evaluate the following expression:

$$\mathbf{v}(\boldsymbol{\nabla}\cdot\mathbf{v}) - \mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{v})$$

We have

$$\mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{v}) = \mathbf{e}^i\varepsilon_{ijk}v^j(\boldsymbol{\nabla}\times\mathbf{v})_k = \mathbf{e}^i\varepsilon_{ijk}v^j\varepsilon^{lmk}\partial_l v_m =$$

$$= \mathbf{e}^i(\delta_i^l\delta_j^m - \delta_i^m\delta_j^l)v^j\partial_l v_m = \mathbf{e}^i\left[v^j\partial_i v_j - v^j\partial_j v_i\right]$$

and

$$\mathbf{v}(\boldsymbol{\nabla}\cdot\mathbf{v}) = \mathbf{e}^i v_i\partial_j v^j\,,$$

where ε_{ijk} is the Levi-Civita tensor, δ_i^j is the Kroneker-delta symbol and $\mathbf{e}^i$, $i=\overline{1,3}$ are the Carthesian versors. Subtracting these two expressions leads to

$$\mathbf{v}(\boldsymbol{\nabla}\cdot\mathbf{v}) - \mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{v}) = \mathbf{e}^i\left[v_i\partial_j v^j - v^j\partial_i v_j + v^j\partial_j v_i\right] = \mathbf{e}^i\left[\partial_j(v_i v^j) - v^j\partial_i v_j\right]\,.$$

Since

$$\partial_i(v^j v_j) = 2v^j\partial v_j\,,$$

we have

$$\mathbf{v}(\boldsymbol{\nabla}\cdot\mathbf{v}) - \mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{v}) = \mathbf{e}^i\left[\partial_j(v_i v^j) - \frac{1}{2}\partial_i(v_j v^j)\right] ,$$

The second term can be stylized by introducing a Kroneker-delta and writing $v_j v^j$ as $\mathbf{v}^2$, which leads to the desired final result

$$\mathbf{v}(\boldsymbol{\nabla}\cdot\mathbf{v}) - \mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{v}) = \mathbf{e}^i\partial_j\left[v_i v^j - \frac{1}{2}\mathbf{v}^2\delta_i^j\right] = \mathbf{e}_i\partial_j\left[v^i v^j - \frac{1}{2}\mathbf{v}^2\delta^{ij}\right] . \tag{2.35}$$

The last step is possible due to the fact that we only work with space-like components and we chose the convenient metric (2.13).

By defining the Maxwell stress tensor as

$$T^{ij} = \varepsilon_0\left[E^i E^j + c^2 B^i B^j - \frac{1}{2}(\mathbf{E}^2 + c^2\mathbf{B}^2)\delta^{ij}\right] \tag{2.36}$$

and using it along with equation (2.35) in equation (2.34) one gets

$$\mathbf{e}_i\partial_j T^{ij} = \varepsilon_0\mathbf{E}(\boldsymbol{\nabla}\cdot\mathbf{E}) + \frac{1}{\mu_0}\mathbf{B}(\boldsymbol{\nabla}\cdot\mathbf{B}) + \mathbf{j}\times\mathbf{B} + \varepsilon_0\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B})$$

Using Maxwell's equations (2.2a) and (2.2b) together with the definition of the Poynting vector (2.22) the above expression is simplified to the law of momentum conservation

$$\mathbf{e}_i\partial_j T^{ij} = \rho\mathbf{E} + \mathbf{j}\times\mathbf{B} + \frac{1}{c^2}\frac{\partial\mathbf{S}}{\partial t} \tag{2.37}$$

or

$$\mathbf{e}_i\partial_j T^{ij} = \rho\mathbf{E} + \mathbf{j}\times\mathbf{B} + \frac{\partial\mathbf{g}}{\partial t} , \tag{2.38}$$

where the volumic density of the fields' electromagnetic momentum $\mathbf{g}$ is defined to be

$$\mathbf{g} = \frac{1}{c^2}\mathbf{S} = \mathbf{D}\times\mathbf{B} . \tag{2.39}$$

By observing that when integrating over a volume, the $\rho\mathbf{E} + \mathbf{j}\times\mathbf{B}$ is simply the Lorentz force and that $\mathbf{e}_i\partial_j T^{ij} = \boldsymbol{\nabla}\cdot\hat{T}$ we reach an integral form of the momentum conservation

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{P_{em}} + \mathbf{P_{mech}}) = \int_{\mathcal{V}}\mathrm{d}\mathbf{r}\,\boldsymbol{\nabla}\cdot\hat{T} = \int_{\mathcal{S}_\mathcal{V}}\mathrm{d}\mathbf{a}\cdot\hat{T} , \tag{2.40}$$

where $\mathbf{P_{em}}$ and $\mathbf{P_{mech}}$ are the electromagnetic and mechanical momenta, respectively. If we integrate over the entire space and use the fact that the stress tensor vanishes at infinity, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{P_{em}} + \mathbf{P_{mech}}) = 0 . \tag{2.41}$$

2.2 Electromagnetic Waves

The short review of classical electrodynamics had as an ultimate goal to introduce the definitions, equations and formalism required in order to study electromagnetic waves. In this section I will start from the definition and properties of an electromagnetic wave and I will follow up with how one can describe laser pulses. The second part will contain a short introduction to the laser profiles used in research.

2.2.1 Maxwell's Equations in Vacuum

The concept of electromagnetic waves arises naturally from the Maxwell equations equation (2.1) if we consider them in the absence of any sources

$$\boldsymbol{\nabla} \cdot \mathbf{D} = 0 \tag{2.42a}$$
$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \tag{2.42b}$$
$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.42c}$$
$$\boldsymbol{\nabla} \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} . \tag{2.42d}$$

The last equation (2.42d) can be rewritten using $\mathbf{H} = \frac{1}{\mu}\mathbf{B}$ and $\mathbf{D} = \varepsilon\mathbf{E}$ as

$$\boldsymbol{\nabla} \times \mathbf{B} = \varepsilon\mu\frac{\partial \mathbf{E}}{\partial t} \tag{2.43}$$

By taking the curl of equation (2.42c) one gets

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{E}) + \frac{\partial}{\partial t}(\boldsymbol{\nabla} \times \mathbf{B}) = 0 ,$$

which, using the vector identity

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{v}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{v}) - \nabla^2\mathbf{v} \tag{2.44}$$

and equation (2.42a), becomes

$$\left[\nabla^2 - \varepsilon\mu\frac{\partial^2}{\partial t^2}\right]\mathbf{E} = 0 . \tag{2.45}$$

Through an analogous procedure, one obtains that the magnetic field $\mathbf{B}$ satisfies the same equation. Using the D'Alembertian defined in section 2.1.3 we can conclude that both the electric and magnetic fields satisfy the Helmholtz equation

$$\Box\mathbf{E} = 0 \tag{2.46a}$$
$$\Box\mathbf{B} = 0 , \tag{2.46b}$$

with $v^2 = \frac{1}{\varepsilon\mu}$ giving the speed of the wave (also called phase velocity) and $c^2 = \frac{1}{\varepsilon_0\mu_0}$ the speed of electromagnetic waves in vacuum.

The reader most probably has encountered waves in various contexts before, but I will add a reminder of the relevant parameters describing solutions of the Helmholtz equation just for the sake of completeness:

- if $\mathbf{n}$ is the unit vector along the direction of propagation, the wave vector is defined as $\mathbf{k} = \mathbf{n}k$, where $k = \frac{2\pi}{\lambda}$ is the wave number and λ is the wavelength;
- if T is the period in time of the wave, the frequency is defined as $\nu = \frac{1}{T}$ and, equivalently, the angular frequency is defined as $\omega = 2\pi\nu$;
- $v = \lambda\nu = \frac{\omega}{k} = \frac{1}{\sqrt{\varepsilon\mu}}$ is the phase velocity and $v_g = \frac{\partial\omega}{\partial k}$ is the group velocity.

There is one more property we can derive before discussing the particular solutions of equation (2.46), namely the transverse character of electromagnetic waves in vacuum.

A very general form for a solution of equation (2.46) can be written as

$$\mathbf{E} = \mathbf{E_0} f(\mathbf{k} \cdot \mathbf{r} - \omega t)\,,$$

where $\mathbf{E_0}$ is a constant vector. Using it in equation (2.42c) leads to the following developement

$$0 = \boldsymbol{\nabla} \cdot \mathbf{E} = \boldsymbol{\nabla} \cdot [\mathbf{E_0} f(\mathbf{k} \cdot \mathbf{r} - \omega t)] = \mathbf{E_0} \boldsymbol{\nabla} f(\mathbf{k} \cdot \mathbf{r} - \omega t) = \mathbf{k} \cdot \mathbf{E_0} f'(\mathbf{k} \cdot \mathbf{r} - \omega t)$$

which concludes that

$$\mathbf{k} \cdot \mathbf{E} = 0\,. \tag{2.47}$$

Similarly, equation (2.42c) leads to

$$-\frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla} \times \mathbf{E} = \boldsymbol{\nabla} \times [\mathbf{E_0} f(\mathbf{k} \cdot \mathbf{r} - \omega t)] = \mathbf{k} \times \mathbf{E_0} f'(\mathbf{k} \cdot \mathbf{r} - \omega t)\,,$$

where in the last step this identity was used

$$\boldsymbol{\nabla} \times (a\mathbf{v}) = a(\boldsymbol{\nabla} \times \mathbf{v}) + (\boldsymbol{\nabla} a) \times \mathbf{v}\,. \tag{2.48}$$

This suggests that

$$\mathbf{B} \propto \mathbf{k} \times \mathbf{E}\,. \tag{2.49}$$

Looking at equations (2.47) and (2.49) it is easy to conclude that the electromagnetic waves are transverse and that at any moment, the magnetic and electric fields are perpendicular to one another.

Note: Electromagnetic waves can only be transversal in “free space” or homogeneous media (Heaviside 1971). Longitudinal modes can also be achieved in special conditions, like inside confined spaces and in plasmas (Jackson 1999; Griffiths 1999). However, there has been work done on the production of longitudinal waves in vacuum (Wang *et al.* 2008) as a consequence of theoretical work showing the possibility of having a small longitudinal component in electromagnetic waves in vacuum (Cicchitelli, Hora, and Postle 1990) using an improved paraxial approximation.

2.2.2 Plane Waves

The simplest solution to the Helmholtz equation (2.46) is the plane wave

$$\sin(\mathbf{k} \cdot \mathbf{r} - \omega t + \delta) \tag{2.50}$$

In the research literature it is common to employ a complex formulation (Vrejoiu 1987). Thus, the complex fields are defined as

$$\tilde{\mathbf{E}}(\mathbf{r}, t) = E_0 \mathbf{s}\, \mathrm{e}^{\mathrm{i}(\mathbf{k} \cdot \mathbf{r} - \omega t)} \tag{2.51a}$$

$$\tilde{\mathbf{B}}(\mathbf{r}, t) = B_0 \mathbf{n} \times \mathbf{s}\, \mathrm{e}^{\mathrm{i}(\mathbf{k} \cdot \mathbf{r} - \omega t)}\,, \tag{2.51b}$$

where E_0 and B_0 are the real amplitudes and $\mathbf{s}$ is a complex vector of norm one

$$\mathbf{s} = \mathbf{s_r} + \mathrm{i}\mathbf{s_i},\ |\mathbf{s}|^2 = \mathbf{s}^* \cdot \mathbf{s} = \mathbf{s_r}^2 + \mathbf{s_i}^2 = 1$$

With this setup, the real fields are to be obtained as

$$\mathbf{E}(\mathbf{r},t) = \mathrm{Re}\left\{\tilde{\mathbf{E}}(\mathbf{r},t)\right\} = E_0\left[\mathbf{s_r}\cos(\mathbf{k}\cdot\mathbf{r}-\omega t) - \mathbf{s_i}\sin(\mathbf{k}\cdot\mathbf{r}-\omega t)\right] \tag{2.52a}$$

$$\mathbf{B}(\mathbf{r},t) = \mathrm{Re}\left\{\tilde{\mathbf{B}}(\mathbf{r},t)\right\} = B_0\mathbf{n}\times\left[\mathbf{s_r}\cos(\mathbf{k}\cdot\mathbf{r}-\omega t) - \mathbf{s_i}\sin(\mathbf{k}\cdot\mathbf{r}-\omega t)\right]\ . \tag{2.52b}$$

In what follows we are interested in analyzing the plane wave solution from the perspective of energy in the formalism developed in sections 2.1.4 and 2.1.5.

From the discussion in the previous subsection it is easy to deduce the relation between the magnetic and electric fields of a wave (it is the same for both the real and complex fields)

$$\tilde{\mathbf{B}} = \frac{1}{c}\mathbf{n}\times\tilde{\mathbf{E}}\,. \tag{2.53}$$

With this, the energy density (2.20) of the fields is

$$w_{em} = \frac{1}{2}\varepsilon_0\mathbf{E}^2 + \frac{1}{2\mu_0}\mathbf{B}^2 = \varepsilon_0\mathbf{E}^2\,, \tag{2.54}$$

which can be computed using equation (2.52a) to be

$$w_{em} = \varepsilon_0 E_0^2\left[\mathbf{s_r}^2\cos^2(\mathbf{k}\cdot\mathbf{r}-\omega t) + \mathbf{s_i}^2\sin^2(\mathbf{k}\cdot\mathbf{r}-\omega t) - \mathbf{s_r}\cdot\mathbf{s_i}\sin(2\mathbf{k}\cdot\mathbf{r}-2\omega t)\right]\ . \tag{2.55}$$

This quantity could vary quite wildly in time depending on the wave's frequency, so we would rather compute a quantity that can be easured experimentally, which is of course the time average of the energy density

$$\langle w_{em}\rangle = \frac{1}{T}\int_0^T \mathrm{d}t\, w_{em} = \frac{\varepsilon_0 E_0^2}{T}\int_0^T \mathrm{d}t\left[\mathbf{s_r}^2\cos^2(\mathbf{k}\cdot\mathbf{r}-\omega t) + \mathbf{s_i}^2\sin^2(\mathbf{k}\cdot\mathbf{r}-\omega t) - \mathbf{s_r}\cdot\mathbf{s_i}\sin(2\mathbf{k}\cdot\mathbf{r}-2\omega t)\right]\ . \tag{2.56}$$

Since we know that the average of sine over one period is zero and the averages of both sine and cosine squared over one period are one half, we get

$$\langle w_{em}\rangle = \frac{\varepsilon_0 E_0^2}{2}\ . \tag{2.57}$$

But looking at definition (2.51a) we see that

$$\langle w_{em}\rangle = \frac{\varepsilon_0}{2}\tilde{\mathbf{E}}^*\cdot\tilde{\mathbf{E}}\,. \tag{2.58}$$

In a very similar way we have for the Poynting vector the following developement

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\cdot\mathbf{B} = \frac{1}{\mu_0 c}\mathbf{n}\mathbf{E}^2 \tag{2.59}$$

$$\langle\mathbf{S}\rangle = \sqrt{\frac{\varepsilon_0}{\mu_0}}\left\langle\mathbf{E}^2\right\rangle\mathbf{n} = \frac{1}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}}E_0^2\mathbf{n} = c\left\langle w_{em}\right\rangle\mathbf{n} \tag{2.60}$$

$$\langle\mathbf{S}\rangle = \frac{1}{2\mu_0}\tilde{\mathbf{E}}\times\tilde{\mathbf{B}}^*\,. \tag{2.61}$$

And, obviously, the electromagnetic momentum (2.39) is

$$\langle g\rangle = \left\langle\frac{1}{c^2}\mathbf{S}\right\rangle = \frac{\langle w_{em}\rangle}{c^2}\mathbf{n}\,. \tag{2.62}$$

Polarization of Plane Waves

For any arbitrary complex field we can find a decomposition of the real field in orthogonal components. In oreder to do that, we make the following notations concerning the complex vector of (2.51a)

$$\mathbf{s} \cdot \mathbf{s} = \alpha^2 \, \mathrm{e}^{2\mathrm{i}\theta} \ \ \textit{with}\ \alpha,\, \theta \in \mathbb{R}\,. \tag{2.63}$$

We can define $\mathbf{u} = \mathbf{s}\,\mathrm{e}^{-\mathrm{i}\theta}$ such that $\mathbf{u_r} \cdot \mathbf{u_i} = 0$. In this way the orthogonal coordinates system can be chosen as

$$\mathbf{e_x} = \frac{\mathbf{u_r}}{|\mathbf{u_r}|},\ \mathbf{e_y} = \pm\frac{\mathbf{u_i}}{|\mathbf{u_i}|},\ \mathbf{e_z} = \mathbf{n}\,, \tag{2.64}$$

with the sign of $\mathbf{e_y}$ being conveniently chosen in order to have a right-handed system (*i.e.* $\mathbf{e_x} \times \mathbf{e_y} = \mathbf{e_z}$).

The real field (2.52a) is in this basis

$$\mathbf{E} = E_0 \left[u_r \mathbf{e_x} \cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \theta) \mp u_i \mathbf{e_y} \sin(\mathbf{k} \cdot \mathbf{r} - \omega t + \theta) \right]\,. \tag{2.65}$$

For time-independent $\mathbf{e_x}$, $\mathbf{e_y}$ and $\mathbf{e_z}$, the following cases are to be distinguished:

- u_r, u_i arbitrary and non-zero: eliptically polarized wave;
- $u_r = u_i \neq 0$: circularly polarized wave;
- either $u_r = 0$ or $u_i = 0$: linearly polarized wave.

2.2.3 Paraxial Approximation

This and the next section discuss the ways in which we can describe beams of electromagnetic waves (like, say, laser beams) and follows ideas from Goldsmith 1998.

The paraxial approximation aims to simplify the Helmholtz equation

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \right] \psi = 0\,. \tag{2.66}$$

We can treat this equation by the method of separation of variables $\psi = \zeta(\mathbf{r})T(t)$ in order for it to become

$$\frac{1}{\zeta}\nabla^2\zeta = \frac{1}{c^2}\frac{T''}{T} = -k^2\,, \tag{2.67}$$

where k is simply the wave number. It is clear that the solution of the time equation is a combination of sine and cosine functions, so the real problem consists in solving the spatial equation

$$\nabla^2\zeta + k^2\zeta = 0\,. \tag{2.68}$$

In the particular case of electromagnetic waves, this equation must hold for the complex vector $\tilde{\mathbf{E}}$, so it must hold for each of its components. The Helmholtz equation for the electric field can be reduced by considering a solution of the form

$$\zeta(\mathbf{r}) = u(\mathbf{r})\,\mathrm{e}^{-\mathrm{i}kz}\,, \tag{2.69}$$

where the z-axis was chosen as the propagation direction for the wave.

Inserting (2.69) in equation (2.68) we get

$$0 = \nabla^2\zeta + k^2\zeta = (\partial_x^2 u + \partial_y^2 u)\,\mathrm{e}^{-\mathrm{i}kz} + \partial_z^2(u\,\mathrm{e}^{-\mathrm{i}kz}) + k^2 u\,\mathrm{e}^{-\mathrm{i}kz} =$$

$$= (\partial_x^2 u + \partial_y^2 u)\,\mathrm{e}^{-\mathrm{i}kz} + (\partial_z^2 u)\,\mathrm{e}^{-\mathrm{i}kz} - 2\mathrm{i}k(\partial_z u)\,\mathrm{e}^{-\mathrm{i}kz} - k^2 u\,\mathrm{e}^{-\mathrm{i}kz} + k^2 u\,\mathrm{e}^{-\mathrm{i}kz} =$$

$$= \mathrm{e}^{-\mathrm{i}kz}\,\nabla^2 u - 2\mathrm{i}k\,\mathrm{e}^{-\mathrm{i}kz}\,\partial_z u\,.$$

Multiplying with $\mathrm{e}^{\mathrm{i}kz}$ leads to

$$\nabla^2 u - 2\mathrm{i}k\partial_z u = 0\,. \tag{2.70}$$

The first paraxial approximation argument says that, due to diffraction, the variation of the amplitude u along the direction of propagation is very small compared to distances of the order of the wave's wavelength. This can be summarized by the mathematical condition

$$\lambda\frac{\Delta(\partial_z u)}{\Delta z} \ll \partial_z u\,, \tag{2.71}$$

which indicates that the double partial derivative with respect to z (the propagation axis) is negligible compared to the $2\mathrm{i}k\partial_z u$ term. The second argument says that in the laplacian, the double partial derivative with respect to the z-axis can be neglected, such that one obtains

$$\partial_x^2 u + \partial_y^2 u - 2\mathrm{i}k\partial_z u = 0\,. \tag{2.72}$$

which is the paraxial wave equation.

2.2.4 Gaussian Beams

One can find solutions to equation (2.72) working in various coordinate systems, but the most convenient and useful for our purpose (and in practical aplications in general) is to work in cylindrical coordinated. In this case, the equation becomes

$$\partial_r^2 u + \frac{1}{r}\partial_r u + \frac{1}{r}\partial_\varphi^2 u - 2\mathrm{i}k\partial_z u = 0\,. \tag{2.73}$$

To simplify our calculations even more, we can remove the φ dependence of u, which is to imply axial symmetry for the wave. This gives

$$\partial_r^2 u + \frac{1}{r}\partial_r u - 2\mathrm{i}k\partial_z u = 0\,. \tag{2.74}$$

The radial part of the equation suggests that we should have a dependence of a complex exponential of r^2. An educated guess would be a Gaussian distribution-like function of the form

$$u(r,z) = G(z)\,\mathrm{e}^{-i\frac{kr^2}{2q(z)}}\,, \tag{2.75}$$

where the complex functions $G(z)$ and $q(z)$ are to be determined. Let us do just that by inserting (2.75) in equation (2.74):

$$\partial_r u = -\frac{\mathrm{i}kr}{q(z)}G(z)\,\mathrm{e}^{-i\frac{kr^2}{2q(z)}}$$

$$\partial_r^2 u = -\frac{\mathrm{i}k}{q(z)} G(z) \left[1 - \frac{\mathrm{i}kr^2}{q(z)}\right] \mathrm{e}^{-i\frac{kr^2}{2q(z)}}$$

$$\partial_z u = \left[G'(z) + \frac{\mathrm{i}kr^2}{2q^2(z)} G(z) q'(z)\right] \mathrm{e}^{-i\frac{kr^2}{2q(z)}} \ .$$

Replacing these results and ridding ourselves of the exponential leads to

$$-2\mathrm{i}k\left(\frac{G}{q} + G'\right) + \frac{k^2r^2G}{q^2}(q' - 1) = 0\,, \tag{2.76}$$

which gives the following differential equations for G and q:

$$\frac{\mathrm{d}q}{\mathrm{d}z} = 1 \tag{2.77a}$$

$$\frac{\mathrm{d}G}{\mathrm{d}z} = -\frac{G}{q}\,. \tag{2.77b}$$

The solution of equation (2.77a) is trivial

$$q(z) = q(z_0) + z - z_0\,,$$

which can be simplified by choosing our origin at z_0

$$q(z) = q(0) + z\,. \tag{2.78}$$

The quantity q (which is actually complex) is often called *Gaussian beam parameter*. Since it appears in (2.75) as $\frac{1}{q}$, it is convenient to express it in the form

$$\frac{1}{q} = \left(\frac{1}{q}\right)_r - \mathrm{i}\left(\frac{1}{q}\right)_i\,. \tag{2.79}$$

If we now substitute this in the guessed solution (2.75) we obtain

$$u(r,z) = G(z)\,\mathrm{e}^{-\frac{\mathrm{i}kr^2}{2}\left[\left(\frac{1}{q}\right)_r - \mathrm{i}\left(\frac{1}{q}\right)_i\right]} = G(z)\,\mathrm{e}^{-\frac{kr^2}{2}\left(\frac{1}{q}\right)_i}\,\mathrm{e}^{-\frac{\mathrm{i}kr^2}{2}\left(\frac{1}{q}\right)_r}\,. \tag{2.80}$$

The real part of $\frac{1}{q}$ has physical significance. In order to see this, imagine that at a point z on the propagation direction we draw a plane perpendicular to the z-axis. If R would be the radius of curvature of the wavefront at point z (with respect to the position of the source), we can define $\phi(r) = k\delta x$ to be the difference in phase between the wavefront and the plane as a function of r. Since we work in the paraxial approximation, we can consider that $r \ll R$, such that, using as reference figure 2.1, we have

$$\alpha \approx \frac{r}{R} \tag{2.81a}$$

$$\delta x = -R(1 - \cos(\alpha)) \approx R\frac{\alpha^2}{2} \tag{2.81b}$$

$$\phi(r) \approx -\frac{kr^2}{2}\frac{1}{R}\,. \tag{2.81c}$$

We can conclude now that

$$\left(\frac{1}{q}\right)_r = \frac{1}{R}\,. \tag{2.82}$$

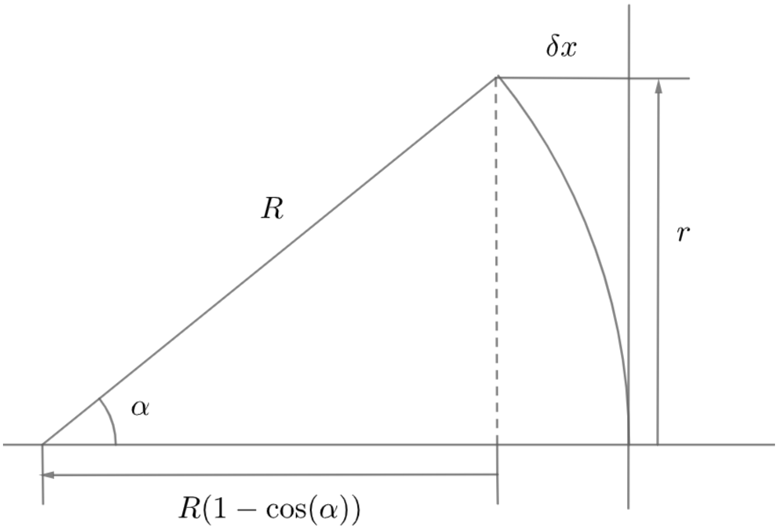


Figure 2.1: A drawing showing how to compute $\phi(r)$

The imaginary part of $\frac{1}{q}$ appears in the real exponential. This exponential should thus give the Gaussian distribution form of the wave, that is it should look like

$$\mathrm{e}^{-\left(\frac{r}{r_0}\right)^2}, \tag{2.83}$$

where r_0 is proportional to the standard deviation. In this case we can write

$$\left(\frac{1}{q}\right)_i = \frac{2}{kw^2(z)} = \frac{\lambda}{\pi w^2(z)}. \tag{2.84}$$

This defines the *beam radius* $w(z)$ as the value of r at which the field falls to $\frac{1}{\mathrm{e}}$ of its value on the z-axis. Putting these results together, we reach a final formula for $\frac{1}{q}$

$$\frac{1}{q} = \frac{1}{R(z)} - \mathrm{i}\frac{\lambda}{\pi w^2(z)}. \tag{2.85}$$

It is conventional to take $\lim_{z\to 0} R(z) \to \infty$, such that $\frac{1}{q(0)} = -\mathrm{i}\frac{\lambda}{\pi w_0^2}$, and $w_0 = w(0)$ is usually interpreted as the *beam waist radius*. If we look back at the solution (2.78), we can rewrite q in this formalism as

$$q = z + \mathrm{i}\frac{\pi w_0^2}{\lambda}. \tag{2.86}$$

Playing around with equations (2.85) and (2.86) we have the following developement

$$\frac{1}{q} = \frac{1}{R} - \mathrm{i}\frac{\lambda}{\pi w^2} = \frac{1}{z + \mathrm{i}\frac{\pi w_0}{\lambda}} = \frac{z - \mathrm{i}\frac{\pi w_0}{\lambda}}{z^2 + \left(\frac{\pi w_0^2}{\lambda}\right)^2}$$

$$\frac{1}{R} = \frac{z}{z^2 + \left(\frac{\pi w_0^2}{\lambda}\right)^2} \Rightarrow R = z + \frac{1}{z}\left(\frac{\pi w_0^2}{\lambda}\right)^2$$

$$\frac{1}{w^2} = \frac{\frac{\pi^2 w_0^2}{\lambda^2}}{z^2 + \left(\frac{\pi w_0^2}{\lambda}\right)^2} \Rightarrow w = w_0\sqrt{1 + \left(\frac{\lambda z}{\pi w_0^2}\right)^2}.$$

For the sake of clarity, I write again the expressions obtained for the radius of curvature and the beam radius

$$R = z + \frac{1}{z}\left(\frac{\pi w_0^2}{\lambda}\right)^2 \tag{2.87}$$

$$w = w_0\sqrt{1 + \left(\frac{\lambda z}{\pi w_0^2}\right)^2}\,. \tag{2.88}$$

Turning back now to equation (2.77b), using (2.86), we can rewrite it as

$$\frac{\mathrm{d}G}{G} = -\frac{\mathrm{d}\left(z + \mathrm{i}\frac{\pi w_0^2}{\lambda}\right)}{z + \mathrm{i}\frac{\pi w_0^2}{\lambda}}\,,$$

which, after integration, becomes

$$\ln\frac{G(z)}{G(0)} = \ln\frac{z + \mathrm{i}\frac{\pi w_0^2}{\lambda}}{\mathrm{i}\frac{\pi w_0^2}{\lambda}}$$

or

$$\frac{G(z)}{G(0)} = \frac{1}{1 - \mathrm{i}\frac{\lambda z}{\pi w_0^2}} = \frac{1 + \mathrm{i}\frac{\lambda z}{\pi w_0^2}}{1 + \left(\frac{\lambda z}{\pi w_0^2}\right)^2}\,. \tag{2.89}$$

For convenience, this is usually expressed interms of a phasor (commonly called Gouy phase) defined as

$$\tan(\phi_0) = \frac{\lambda z}{\pi w_0^2}\,. \tag{2.90}$$

Now the solution for G is stylized to be

$$\frac{G(z)}{G(0)} = \frac{w_0}{w}\,\mathrm{e}^{\mathrm{i}\phi_0}\,. \tag{2.91}$$

Putting together equations (2.75), (2.85) and (2.91) we finally find u

$$u(r,z) = G(0)\frac{w_0}{w}\exp\left(-\frac{r^2}{w^2} - \mathrm{i}\frac{\pi r^2}{\lambda R} + \mathrm{i}\phi_0\right) \tag{2.92}$$

and, consequently, the solution to the paraxial wave equation with axial symmetry

$$\zeta(r,z) = G(0)\frac{w_0}{w}\exp\left(-\frac{r^2}{w^2} - \mathrm{i}kz - \mathrm{i}\frac{\pi r^2}{\lambda R} + \mathrm{i}\phi_0\right). \tag{2.93}$$

2.2.5 Gaussian Beam Packets

In the reaserch literature, it is a custom to use a parameter called *confocal distance* or *Reyleigh range*

$$z_0 = \frac{\pi w_0^2}{\lambda}\,. \tag{2.94}$$

Including it, all the relevant auxiliary functions become

$$R(z) = z + \frac{z_0^2}{z} \tag{2.95a}$$

$$w(z) = w_0\sqrt{1 + \left(\frac{z}{z_0}\right)^2} \tag{2.95b}$$

$$\phi_0(z) = \arctan\left(\frac{z}{z_0}\right). \tag{2.95c}$$

We can immediately observe that $w(z)$ at $z = z_0$ is actually equal to $\sqrt{2}w_0$. From this we can deduce that the z_0 indicates how far from the origin the beam is collimated. It is very simple to understand by looking at figure figure 2.2

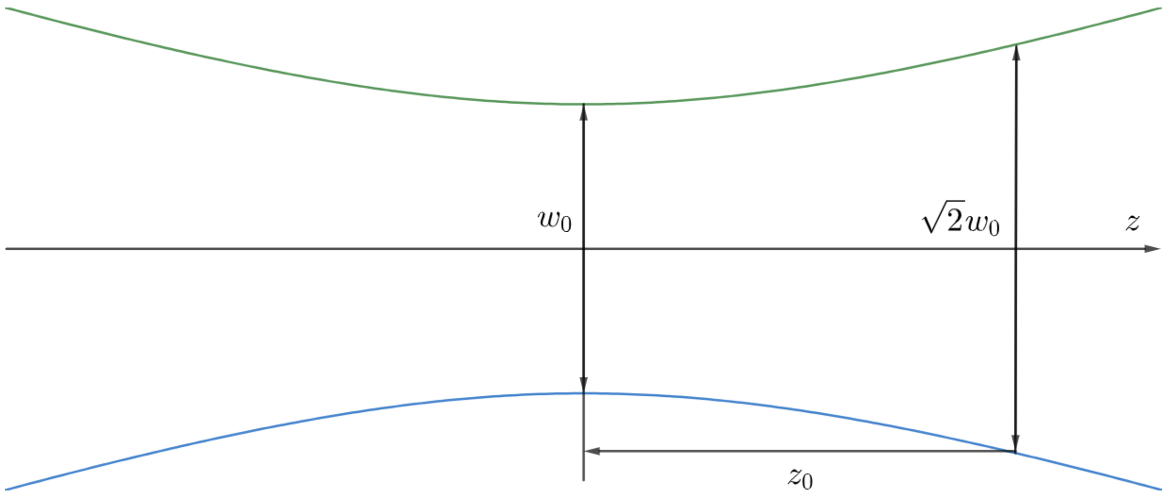


Figure 2.2: Gaussian beam radius w as a function of z

We can also differentiate three cases of interest for the curvature radius $R(z)$. At $z \to 0$, *i.e.* near the waist, $R \to \infty$, so the profile is that of a plane wave. At the Reyleigh range, the curvature ($\frac{1}{R}$) is maximum and, consequently, the radius itself is minimum ($2z_0$). Finally, at very large distances away from the waist, the radius is equal to z, so the profile is spherical.

The Gouy phase is an important parameter in theoretical considerations, especially when it comes to higher order Gaussian modes, but is hard to observe experimentally. Physically, it modifies the wavelength near the waist (Paschotta 2020). This results also in a change of the phase velocity. As a consequence, the phase velocity near the waist can exceed the velocity of light in the medium, just as it might inside a waveguide.

Now that we understood the shape and behaviour of the Gauss beam, we are almost ready to define the electric and magnetic fields. But before that we must talk about normalization. While a look at (2.93) might not suggest the need for any normalization, physical arguments request it. We would like to not have unexplained losses of power as a funtion of z (remember that we are basically just setting the dependence of the fields on position right now). As such, a normalization of $\zeta(r, z)$ over the transversal surface is necessary. That is, we want the intensity as a function of z to be just a constant. While in literature is very common to impose norm 1, I find it more useful to norm it to πw_0^2, as suggested by Dondera 2020, such that the final result is adimensional and it is easier to introduce the amplitude of the field. We have

$$\frac{1}{2}\pi w_0^2 = \iint \mathrm{d}r\,\mathrm{d}\varphi\, r|\zeta|^2 = 2\pi|G(0)|^2 \int \mathrm{d}r \left|\frac{w_0}{w}\right|^2 r\,\mathrm{e}^{-2\frac{r^2}{w^2}} =$$

$$= \frac{1}{2}\pi \left(\frac{w_0}{w}\right)^2 |G(0)|^2 w^2 \int \mathrm{d}\left(2\frac{r^2}{w^2}\right) \mathrm{e}^{-2\frac{r^2}{w^2}} =$$

$$= \frac{1}{2}\pi w_0^2 G^2(0)$$

which gives the normalization constant

$$G(0) = 1\,. \tag{2.96}$$

The Electric Field

The x and y components of the electric field are now expressed using equation (2.93) as

$$E_x(r,z) = \alpha_x E_0 \frac{w_0}{w} \exp\left(-\frac{r^2}{w^2} - \mathrm{i}kz - \mathrm{i}\frac{kr^2}{2R} + \mathrm{i}\phi_0\right) \tag{2.97a}$$

$$E_y(r,z) = \alpha_y E_0 \frac{w_0}{w} \exp\left(-\frac{r^2}{w^2} - \mathrm{i}kz - \mathrm{i}\frac{kr^2}{2R} + \mathrm{i}\phi_0\right), \tag{2.97b}$$

where we choose $\alpha_x = 1$, $\alpha_y = 0$ for linear polarization, and $\alpha_x = \frac{1}{\sqrt{2}}$, $\alpha_y = \pm\frac{\mathrm{i}}{\sqrt{2}}$ for right and left-handed circular polarization, respectively. We see that E_0 is simply the value of the field at $z = 0$ and $t = 0$. In order to obtain the z component, we have to impose the condition $\boldsymbol{\nabla}\cdot\mathbf{E} = 0$ and to use the approximation $\partial_z E_z \approx -\mathrm{i}kE_z$ (which holds if the pulse is long enough or quasi-rectangular). The immediate result is

$$E_z(r,z) = -\frac{\mathrm{i}}{k}\left(\partial_x E_x(r,z) + \partial_y E_y(r,z)\right), \tag{2.98}$$

or, explicitly

$$E_z(r,z) = \frac{2\left(\mathrm{i} - \frac{z}{z_0}\right)}{kw^2(z)}\left[xE_x(r,z) + yE_y(r,z)\right]. \tag{2.99}$$

The Magnetic Field

In order to derive the magnetic, one can impose the relation (2.53)

$$\mathbf{B} = \frac{1}{c}\mathbf{n}\times\mathbf{E} = \frac{1}{c}\mathbf{e_z}\times\mathbf{E} = \frac{1}{c}E_x\mathbf{e_z}\times\mathbf{e_x} + \frac{1}{c}E_y\mathbf{e_z}\times\mathbf{e_y} = -\frac{1}{c}E_y\mathbf{e_x} + \frac{1}{c}E_x\mathbf{e_y}\,,$$

which indicates that

$$B_x(r,z) = -\frac{1}{c}E_y(r,z) \tag{2.100a}$$

$$B_y(r,z) = \frac{1}{c}E_x(r,z)\,. \tag{2.100b}$$

The third component is found exactly in the same way as for the electric field, taking into account that $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$ and using the same approximation $\partial_z B_z \approx -\mathrm{i}kB_z$ (the conditions for its validity are the same):

$$B_z(r,z) = -\frac{\mathrm{i}}{k}\left(\partial_x B_x(r,z) + \partial_y B_y(r,z)\right) = -\frac{\mathrm{i}}{ck}\left(\partial_y E_x(r,z) - \partial_x E_y(r,z)\right)\,, \tag{2.101}$$

or rather

$$B_z(r,z) = \frac{2\left(\mathrm{i} - \frac{z}{z_0}\right)}{ckw^2(z)}\left[yE_x(r,z) - xE_y(r,z)\right]. \tag{2.102}$$

The Temporal Profile

One observation must be made now. These expressions only describe the spatial part of the field. In order to give the exact field we must add the time-dependent part of the solution $\mathrm{e}^{\mathrm{i}\omega t}$. However, this is not all there is to it. Since we are interested in describing laser beams, we must take into consideration the fact that the pulse has a finite duration. One does this by adding a Gaussian envelope over time. The time-dependent part will now be

$$g(z,t) = \exp\left(\mathrm{i}\omega t - \left(\frac{t - \frac{z-z_F}{c}}{\tau_0}\right)^2\right), \tag{2.103}$$

where τ_0 is the duration of the pulse and z_F is the original position of the intensity peak.

In what follows, I aim to provide a short proof of the fact that even with this envelope, the final fields are still solutions of Helmholtz equation under the paraxial approximation.

Let $f(r,z)$ be the solution for

$$\partial_r^2 f + \frac{1}{r}\partial_r f - 2\mathrm{i}k\partial_z f = 0\,. \tag{2.104}$$

The solution for the complete Helmholtz equation

$$\partial_r^2 u + \frac{1}{r}\partial_r u + \partial_z^2 u - \frac{1}{c}\partial_t^2 u = 0 \tag{2.105}$$

is proposed to be $u(r,z) = f(r,z)g(z,t)\,\mathrm{e}^{-\mathrm{i}kz}$ such that we have

$$\partial_r^2(fg\,\mathrm{e}^{-\mathrm{i}kz}) + \frac{1}{r}\partial_r(fg\,\mathrm{e}^{-\mathrm{i}kz}) + \partial_z^2(fg\,\mathrm{e}^{-\mathrm{i}kz}) - \frac{1}{c^2}\partial_t^2(fg\,\mathrm{e}^{-\mathrm{i}kz}) = 0$$

$$(\partial_r^2 f + \frac{1}{r}\partial_r f)g\,\mathrm{e}^{-\mathrm{i}kz} + (\partial_z^2 f)g\,\mathrm{e}^{-\mathrm{i}kz} + 2(\partial_z f)\partial_z(g\,\mathrm{e}^{-\mathrm{i}kz}) + f\partial_z^2(g\,\mathrm{e}^{-\mathrm{i}kz}) - \frac{1}{c^2}f\partial_t^2(g\,\mathrm{e}^{-\mathrm{i}kz}) = 0\,.$$

The paraxial approximation allows us to ignore the $\partial_z^2 f$ term

$$(\partial_r^2 f + \frac{1}{r}\partial_r f)g\,\mathrm{e}^{-\mathrm{i}kz} + 2(\partial_z f)\partial_z(g\,\mathrm{e}^{-\mathrm{i}kz}) + f\partial_z^2(g\,\mathrm{e}^{-\mathrm{i}kz}) - \frac{1}{c^2}f\partial_t^2(g\,\mathrm{e}^{-\mathrm{i}kz}) = 0$$

$$(\partial_r^2 f + \frac{1}{r}\partial_r f - 2\mathrm{i}k\partial_z f)g\,\mathrm{e}^{-\mathrm{i}kz} + 2(\partial_z f)\,\mathrm{e}^{-\mathrm{i}kz}\,\partial_z g + f\partial_z^2(g\,\mathrm{e}^{-\mathrm{i}kz}) - \frac{1}{c^2}f\partial_t^2(g\,\mathrm{e}^{-\mathrm{i}kz}) = 0$$

$$(\partial_r^2 f + \frac{1}{r}\partial_r f - 2\mathrm{i}k\partial_z f)g\,\mathrm{e}^{-\mathrm{i}kz} + f\left[2\frac{\partial_z f}{f}\,\mathrm{e}^{-\mathrm{i}kz}\,\partial_z g + \partial_z^2(g\,\mathrm{e}^{-\mathrm{i}kz}) - \frac{1}{c^2}\partial_t^2(g\,\mathrm{e}^{-\mathrm{i}kz})\right] = 0\,.$$

The first term is zero since f is a solution, so we must have the second term also equal to zero

$$2\frac{\partial_z f}{f}\,\mathrm{e}^{-\mathrm{i}kz}\,\partial_z g + \partial_z^2(g\,\mathrm{e}^{-\mathrm{i}kz}) - \frac{1}{c^2}\partial_t^2(g\,\mathrm{e}^{-\mathrm{i}kz}) = 0$$

$$2\frac{\partial_z f}{f}\,\mathrm{e}^{-\mathrm{i}kz}\,\partial_z g + (\partial_z^2 g)\,\mathrm{e}^{-\mathrm{i}kz} - 2\mathrm{i}k(\partial_z g)\,\mathrm{e}^{-\mathrm{i}kz} - k^2 g\,\mathrm{e}^{-\mathrm{i}kz} - \frac{1}{c^2}(\partial_t^2 g)\,\mathrm{e}^{-\mathrm{i}kz} = 0$$

and eliminating the exponential

$$2\frac{\partial_z f}{f}\partial_z g + \partial_z^2 g - 2\mathrm{i}k\partial_z g - k^2 g - \frac{1}{c^2}\partial_t^2 g = 0$$

or, using $ck = \omega$

$$2\mathrm{i}\omega c\partial_z g + \partial_t^2 g + \omega^2 g = c^2\partial_z^2 g + 2c^2\frac{\partial_z f}{f}\partial_z g\,. \tag{2.106}$$

Based on equation (2.103) we have

$$\partial_z g = \frac{2}{c}\frac{t - \frac{z-z_F}{c}}{\tau_0^2}g \Rightarrow 2\mathrm{i}\omega c\partial_z g = 4\mathrm{i}\omega\frac{t - \frac{z-z_F}{c}}{\tau_0^2}g \tag{2.107}$$

$$\partial_z^2 g = -\frac{2}{c^2}\frac{1}{\tau_0^2}g + \frac{4}{c^2}\left(\frac{t - \frac{z-z_F}{c}}{\tau_0^2}\right)^2 g \Rightarrow c^2\partial_z^2 g = -2\frac{1}{\tau_0^2}g + 4\left(\frac{t - \frac{z-z_F}{c}}{\tau_0^2}\right)^2 g \tag{2.108}$$

$$\partial_t g = \left(\mathrm{i}\omega - 2\frac{t - \frac{z-z_F}{c}}{\tau_0^2}\right)g \tag{2.109}$$

$$\partial_t^2 g = -2\frac{1}{\tau_0^2}g + \left(\mathrm{i}\omega - 2\frac{t - \frac{z-z_F}{c}}{\tau_0^2}\right)^2 g = -4\mathrm{i}\omega\frac{t - \frac{z-z_F}{c}}{\tau_0^2}g - 2\frac{1}{\tau_0^2}g - \omega^2 g + 4\left(\frac{t - \frac{z-z_F}{c}}{\tau_0^2}\right)^2 g \tag{2.110}$$

Summing everything up, we still remain with a

$$\frac{\partial_z f}{f}\partial_z g = 0\,, \tag{2.111}$$

which is true under the paraxial approximation.

It is important to mention that this profile is not always usable. According to Quesnel and Mora 1998 finite pulse effects are important for $c\tau_0 \lessapprox 2w_0$, and in this situation we must put an additional $2\partial_z g\partial_z\mathbf{E}$ in the paraxial wave equation.

The Final Fields

The final relations are straightforward

$$\mathbf{E}(r,z,t) = \mathbf{E}(r,z)g(t,z) \tag{2.112a}$$
$$\mathbf{B}(r,z,t) = \mathbf{B}(r,z)g(t,z)\,. \tag{2.112b}$$

2.2.6 Laguerre-Gauss Beams

This tipe of beam represents a correction for the Gauss one in order to remove the axial symmetry approximation. That is, we want fo find a function $f(r,z,\varphi) = \zeta(r,z)s(r,z,\varphi)$ to be a solution of

$$\partial_r^2 f + \frac{1}{r}\partial_r f + \frac{1}{r}\partial_\varphi^2 f - 2\mathrm{i}k\partial_z f = 0\,, \tag{2.113}$$

where ζ is found in (2.93). Actually, we would rather make an educated guess for a trial solution

$$f(r,z,\varphi) = \zeta(r,z)S(r)\,\mathrm{e}^{im\varphi}\,, \tag{2.114}$$

with m can be a real number. In this case, it is straightforward to find that $S(r)$ turns out to satisfy an equation similar to that of the associated Laguerre polynomials. One can reach through not so short computations the expression

$$S(r) = \left(\frac{\sqrt{2}r}{w(z)}\right)^m L_{pm}\left(\frac{2r^2}{w^2(z)}\right)\exp\left(\mathrm{i}(2p+m)\arctan\left(\frac{z}{z_0}\right)\right)\,. \tag{2.115}$$

The associated (or sometimes also called generalized) Laguerre polynomials are a solution of (Abramowitz and Stegun 2013)

$$xL''_{pm}(x) + (m+1-x)L'_{pm}(x) + pL_{pm}(x) = 0\,, \tag{2.116}$$

where $p \in \mathbb{N}$ and $m \in \mathbb{R}$. Their expression can be obtained using the Rodrigues formula

$$L_{pm}(x) = \frac{x^{-m}e^x}{p!}\frac{\mathrm{d}^p}{\mathrm{d}x^p}\left(e^{-x}x^{m+p}\right)\,. \tag{2.117}$$

We will now apply the same normalization criterion we used for the Gauss mode for $f(r,z,\varphi)$

$$\frac{1}{2}\pi w_0^2 = \iint \mathrm{d}r\,\mathrm{d}\varphi\, r|f|^2\,. \tag{2.118}$$

The integration is done as such

$$\iint \mathrm{d}r\,\mathrm{d}\varphi\, r|f|^2 = 2\pi\int \mathrm{d}r\, r|\zeta|^2|S|^2 = 2\pi|G(0)|^2\left(\frac{w_0}{w}\right)^2\int \mathrm{d}r\, r\,\mathrm{e}^{-\frac{2r^2}{w^2}}\left(\frac{2r^2}{w^2}\right)^m L_{pm}^2\left(\frac{2r^2}{w^2}\right) =$$

$$= \frac{1}{2}\pi|G(0)|^2\left(\frac{w_0}{w}\right)^2 w^2\int \mathrm{d}\left(\frac{2r^2}{w^2}\right)\mathrm{e}^{-\frac{2r^2}{w^2}}\left(\frac{2r^2}{w^2}\right)^m L_{pm}^2\left(\frac{2r^2}{w^2}\right) =$$

$$= \frac{\pi w_0^2}{2}|G(0)|^2\int_0^\infty \mathrm{d}x\, x^m\,\mathrm{e}^{-x}\,L_{pm}^2(x)\,.$$

The remaining integral is trivial if we take a look at the orthogonality relation of the associated Laguerre polynomials

$$\int_0^\infty \mathrm{d}x\, x^m\,\mathrm{e}^{-x}\,L_{pm}(x)L_{p'm}(x) = \frac{(p+m)!}{p!}\delta_{pp'}\,. \tag{2.119}$$

This gives the result

$$G(0) = \sqrt{\frac{p!}{(p+m)!}} \,. \tag{2.120}$$

The final expressions for the electric field, after dealing with normalization, are simply

$$E_x^{pm}(r,z,\varphi) = E_x(r,z)\sqrt{\frac{p!}{(|m|+p)!}}\left(\frac{\sqrt{2}r}{w(z)}\right)^{|m|} L_{p|m|}\left(\frac{2r^2}{w^2(z)}\right)\exp\left(\mathrm{i}(2p+|m|)\arctan\left(\frac{z}{z_0}\right)\right)\mathrm{e}^{-\mathrm{i}m\varphi} \tag{2.121a}$$

$$E_y^{pm}(r,z,\varphi) = E_y(r,z)\sqrt{\frac{p!}{(|m|+p)!}}\left(\frac{\sqrt{2}r}{w(z)}\right)^{|m|} L_{p|m|}\left(\frac{2r^2}{w^2(z)}\right)\exp\left(\mathrm{i}(2p+|m|)\arctan\left(\frac{z}{z_0}\right)\right)\mathrm{e}^{-\mathrm{i}m\varphi} \tag{2.121b}$$

$$E_z^{pm}(r,z,\varphi) = -\frac{i}{k}\left(\partial_x E_x^{pm}(r,z,\varphi) + \partial_y E_y^{pm}(r,z,\varphi)\right) \,, \tag{2.121c}$$

where $E_x(r,z)$ and $E_y(r,z)$ are to be taken from the Gaussian beam (2.97). For the sake of use in numerical simulations (mainly to eliminate the need for numerical differentiation) the z component of the field can be computed explicitly to be

$$E_z^{pm} = -\frac{\mathrm{i}}{k}\left(-2\frac{1+\mathrm{i}\frac{z}{z_0}}{w^2(z)} + \frac{\partial L_{p|m|}\left(\frac{2r^2}{w^2(z)}\right)}{L_{p|m|}\left(\frac{2r^2}{w^2(z)}\right)}\right)\left(xE_x^{pm} + yE_y^{pm}\right) -$$
$$-\frac{\mathrm{i}}{k}\sqrt{\frac{p!}{(|m|+p)!}}\left(\frac{\sqrt{2}}{w(z)}\right)^{|m|} L_{p|m|}\left(\frac{2r^2}{w^2(z)}\right)\exp\left(\mathrm{i}(2p+|m|)\arctan\left(\frac{z}{z_0}\right)\right)(T_xE_x + T_yE_y) \,, \tag{2.122}$$

with

$$T_x = \begin{cases} l(x-\mathrm{i}y)^{l-1}, \; l>0 \\ 0, \; l=0 \\ -l(x+\mathrm{i}y)^{-l-1}, \; l<0 \end{cases} \tag{2.123a}$$

$$T_y = \begin{cases} -\mathrm{i}l(x-\mathrm{i}y)^{l-1}, \; l>0 \\ 0, \; l=0 \\ -\mathrm{i}l(x+\mathrm{i}y)^{-l-1}, \; l<0 \,. \end{cases} \tag{2.123b}$$

The magnetic field is derived exactly as in the case of the Gaussian beam

$$B_x(r,z,\varphi) = -\frac{1}{c}E_y(r,z,\varphi) \tag{2.124a}$$

$$B_y(r,z,\varphi) = \frac{1}{c}E_x(r,z,\varphi) \tag{2.124b}$$

$$B_z(r,z,\varphi) = -\frac{\mathrm{i}}{ck}\left[\partial_y E_x(r,z,\varphi) - \partial_x E_y(r,z,\varphi)\right] \,. \tag{2.124c}$$

The explicit third component of $\mathbf{B}$ is

$$B_z^{pm} = -\frac{\mathrm{i}}{ck}\left(-2\frac{1+\mathrm{i}\frac{z}{z_0}}{w^2(z)} + \frac{\partial L_{p|m|}\left(\frac{2r^2}{w^2(z)}\right)}{L_{p|m|}\left(\frac{2r^2}{w^2(z)}\right)}\right)\left(yE_x^{pm} - xE_y^{pm}\right) -$$
$$-\frac{\mathrm{i}}{ck}\sqrt{\frac{p!}{(|m|+p)!}}\left(\frac{\sqrt{2}}{w(z)}\right)^{|m|} L_{p|m|}\left(\frac{2r^2}{w^2(z)}\right)\exp\left(\mathrm{i}(2p+|m|)\arctan\left(\frac{z}{z_0}\right)\right)(T_yE_x - T_xE_y)\,. \tag{2.125}$$

These expressions are easier to derive if we bring together the terms $\left(\frac{\sqrt{2}r}{w(z)}\right)^{|m|}$ and $\mathrm{e}^{-\mathrm{i}m\varphi}$ and replace $r\,\mathrm{e}^{\pm\mathrm{i}\varphi}$ with $x \pm \mathrm{i}y$ before computing the derivatives. This step also helps to eliminate the apparent singularity that might appear on the z-axis.

Be aware that $\partial L_{p|m|}$ is not the derivative of the Laguerre generalized polynomial with respect to anything, but rather a notation for the common factor of the partial derivatives with respect to x and y. Its expression is

$$\partial L_{p|m|}\left(\frac{2r^2}{w^2(z)}\right) = \begin{cases} -\frac{4}{w^2(z)}L_{p-1,|m|+1}\left(\frac{2r^2}{w^2(z)}\right),\ 1 \leq p \\ 0,\ otherwise\,. \end{cases} \tag{2.126}$$

Of course, in order to use these relations in simulations involving laser beams, one must attach also the temporal profile (2.103).

It is important to mention that the Laguerre-Gauss laser beams are not simply a solution resulting from mathematical sleight of hand, but they can be obtained in practice. Some examples of articles that deal with the production of Laguerre-Gauss modes (only a few selected from an expansive literature) are Bolze and Nuernberger 2018, Sueda *et al.* 2004 and Zeylikovich *et al.* 2007.

2.2.7 Other Types of Beams

Gauss Beams

One can also work directly in Carthesian coordinates in order to solve the paraxial wave equation. This will lead to obtaining the rectangular Gauss beam mode. The extension of this solution, in the same manner used to extend the cylindrical mode to the Laguerre-Gauss mode, is the Hermite-Gauss beam and, as the name suggests, uses the Hermite polynomials. However, these modes are not of interest in our endeavours. If the reader is interested to look upon them, the book by Goldsmith 1998 has a very concise and straightforward presentation of them in its first chapter.

Bessel Beams

These are a direct solution to the paraxial equation without axial symmetry. These beams have two noteworthy and peculiar properties: they are immune (at least to some degree) to diffraction and they can "heal" after being obstructed by an obstacle. A good summary on how they are described and how their properties manifest are found in the fifth chapter of S Simon 2016.

2.3 Angular Momentum of Electromagnetic Waves

In this section I will simply mention how the orbital angular momentul can be defined. A thorough derivation of its expression for each of the presented laser profiles is a simple, but lengthy and redundant work. If one is interested in computing it in numerical simulations (which is the case for the present thesis), it can be computed from the values of the fields.

By having the electromagnetic momentum defined in (2.39), we can simply extend the expression of the angular momentum as is used in classical mechanics to our case, such that we have

$$\mathbf{J} = \varepsilon_0 \int \mathrm{d}\mathbf{r}\, \mathbf{r} \times (\mathbf{E} \times \mathbf{B}) \tag{2.127}$$

Details about the actual derivation and how to apply this definition in particular cases are found in Belinfante 1940 and Humblet 1943.

Chapter 3

The Interaction Between Electromagnetic Radiation and Matter

Now that the aspects related to the formalism and theory behind the modeling of laser produced electromagnetic waves has been presented, we must naturally turn our attention towards the interaction of those wave pulses with matter. This chapter only deals with the dinamics of particles under the action of electromagnetic fields and the ponderomotive force, since these topics offer great insight and intuition for the physical behaviour of high intensity laser-plasma interaction. The specific phenomena arising from the properties of plasma as a medium are to be presented later.

3.1 Electron Dynamics in Electromagnetic Fields

This section deals with analyzing the motion of a single electron in the fields of a wave. For simplicity, I will only talk about the case of linearly polarized plane waves, since this entire discussion has the purpose of building up intuition and getting a feel for the scale of the relevant quantities. Most of what is to be presented is following the lecture notes of Karsch 2018.

The fact that we want to study dynamics and we are using a very simple type of wave means that it is actually more convenient this time around to work with real fields, rather than complex ones. As per usual, the direction of propagation is chosen to be the z-axis such that the fields are

$$\mathbf{E} = \mathbf{e_x} E_0 \cos(kz - \omega t) \tag{3.1a}$$

$$\mathbf{B} = \mathbf{e_y} B_0 \cos(kz - \omega t)\,. \tag{3.1b}$$

Just as an exercise, it can be observed that these fields are generated by the following choice for the 4-potential

$$\begin{cases} \phi = 0 \\ \mathbf{A} = \mathbf{e_x} A_0 \sin(kz - \omega t)\,, \end{cases} \tag{3.2}$$

where $A_0 = \frac{E_0}{\omega} = \frac{B_0}{k}$.

3.1.1 Classical Treatment

We start from the classical equation of motion given by Newton's second principle using the Lorentz force

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \frac{\mathrm{d}(m_e \mathbf{v_e})}{\mathrm{d}t} = -e\left(\mathbf{E} + \mathbf{v_e} \times \mathbf{B}\right) , \tag{3.3}$$

with m_e and $\mathbf{v_e}$ the mass and the velocity of the electron, respectively, and e the elementary charge. Since we have $B \propto \frac{E}{c}$ and also $v_e \ll c$ (which is implied in order to have a classical treatment), we can safely remove the second term in the right-hand side of the equation above, remaining with

$$\frac{\mathrm{d}\mathbf{v_e}}{\mathrm{d}t} = -\frac{e}{m_e}\mathbf{E} = -\frac{e}{m_e}E_0 \mathbf{e_x} \cos(kz - \omega t) . \tag{3.4}$$

Simply integrating with initial conditions x_0, y_0, $z_0 = 0$ and $\mathbf{v_e(0)} = \mathbf{0}$ leads to

$$\mathbf{v_e}(t) = \frac{e}{\omega m_e} E_0 \mathbf{e_x} \sin(kz - \omega t) \tag{3.5a}$$

$$x(t) = \frac{e}{\omega^2 m_e} E_0 \left[\cos(kz - \omega t) - 1\right] . \tag{3.5b}$$

It is important now to see when the classical treatment breaks down. Let us impose that

$$v_e^{max} = c , \tag{3.6}$$

such that we have

$$a_0 \equiv \frac{eE_0}{\omega m_e c} = \frac{eA_0}{m_e c} = 1 . \tag{3.7}$$

The parameter a_0 is called the normalized or dimensionless vector potential. From its definition it is easy to see that it can only take values between 0 and 1. We can use it to describe the amplitude of the electric field as such

$$E_0 = a_0 \frac{\omega m_e c}{e} . \tag{3.8}$$

It is very convenient in practice to use the wavelength and to extract the rest mass to charge ratio of the electron as follows

$$E_0 = \frac{a_0}{\lambda} 2\pi \frac{m_e c^2}{e} = \frac{a_0}{\lambda} 2\pi \cdot 511 kV . \tag{3.9}$$

The normalized vector field helds also an important significance. One can see that its definition actually boils down to

$$a_0 = \frac{v_{max}^{classical}}{c} , \tag{3.10}$$

so we can use it to find a boundary for the validity of the classical treatment. For simplicity, lets disect the $a_0 = 1$ case, for which the motion should be completely relativistic, keeping in mind that the classical description stops being reliable well befor that. From the result concerning the Poynting vector of a plane wave (2.60), we can find the intensity of the pulse in this limiting case to be

$$I = c\frac{\varepsilon_0}{2} E_0^2 \propto \frac{a_0^2}{\lambda^2} 10^{18} W \frac{\mu m^2}{cm^2} , \tag{3.11}$$

which says that already at intensities of $10^{18} \frac{W}{cm^2}$ the motion of the electron should be treated completely within the grounds of special relativity.

3.1.2 Relativistic Treatment

In the light of our discussion in the previous subsection, we see that in order to study how electrons interact with high-intensity laser beams (namely, terwatt and petawatt lasers), we should do all our calculations relativistically. The equation of motion remains the same, but the relativistic momentum is $\mathbf{p} = \gamma m_e \mathbf{v_e}$, where γ is the usual Lorentz factor. By taking the scalar product of equation (3.3) with $\mathbf{p}$ we get

$$\frac{1}{2}\frac{\mathrm{d}\mathbf{p}^2}{\mathrm{d}t} = -e\mathbf{p}\cdot\mathbf{E}\,, \tag{3.12}$$

where we used the fact that $\mathbf{p}\cdot(\mathbf{v_e}\times\mathbf{B})$, since $\mathbf{p}$ is proportional to $\mathbf{v_e}$. Now, it is useful to write the Lorenz factor in terms of momentum like this

$$\gamma = \frac{1}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}} \Rightarrow \frac{1}{\gamma^2} = 1 - \frac{\mathbf{v}^2}{c^2} = 1 - \frac{1}{\gamma^2}\frac{\mathbf{p}^2}{m_e^2c^2} \Rightarrow \gamma = \sqrt{1+\left(\frac{\mathbf{p}}{m_e c}\right)^2}\,.$$

Now we can expect to find a $\frac{\mathrm{d}\mathbf{p}^2}{\mathrm{d}t}$ by taking the derivative of γ with respect to time

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\sqrt{1+\left(\frac{\mathbf{p}}{m_e c}\right)^2} = \frac{1}{\sqrt{1+\left(\frac{\mathbf{p}}{m_e c}\right)^2}}\frac{1}{m_e^2c^2}\frac{1}{2}\frac{\mathrm{d}\mathbf{p}^2}{\mathrm{d}t} = \frac{1}{\gamma m_e^2c^2}\frac{1}{2}\frac{\mathrm{d}\mathbf{p}^2}{\mathrm{d}t} \Rightarrow$$

$$\Rightarrow m_e c^2\frac{\mathrm{d}\gamma}{\mathrm{d}t} = -e\frac{\mathbf{p}}{\gamma m_e}\cdot\mathbf{E} = -e\mathbf{v_e}\cdot\mathbf{E}\,.$$

Remembering that the kinetic energy in special relativity is obtained as $K = (\gamma-1)m_e c^2$, we can reach an equation for it

$$\frac{\mathrm{d}K}{\mathrm{d}t} = -e\mathbf{v_e}\cdot\mathbf{E}\,. \tag{3.13}$$

This equation can also be rewritten as

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = -\frac{eE_0}{m_e c}\frac{\mathbf{v_e}\cdot\mathbf{e_x}}{c}\cos(kz-\omega t) = -a_0\omega\frac{v_x}{c}\cos(kz-\omega t)\,. \tag{3.14}$$

In order to procees, we should also take a better look at the equation of motion as it is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{p}}{m_e c}\right) = -\frac{e}{m_e c}\left[E_0\mathbf{e_x} + B_0\mathbf{v}\times\mathbf{e_y}\right]\cos(kz-\omega t) =$$

$$= -\frac{eE_0}{m_e c}\left[\mathbf{e_x} + \frac{\mathbf{v}}{c}\times\mathbf{e_y}\right]\cos(kz-\omega t) =$$

$$= -a_0\omega\left[\left(1-\frac{v_z}{c}\right)\mathbf{e_x} + \frac{v_x}{c}\mathbf{e_z}\right]\cos(kz-\omega t)\,.$$

By defining $\tilde{\mathbf{p}} = \frac{\mathbf{p}}{m_e c}$ to be the normalized momentum, we can write the equations for its components as follows

$$\frac{\mathrm{d}\tilde{p_x}}{\mathrm{d}t} = -a_0\omega\left(1-\frac{v_z}{c}\right)\cos(kz-\omega t) \tag{3.15a}$$
$$\frac{\mathrm{d}\tilde{p_y}}{\mathrm{d}t} = 0 \tag{3.15b}$$
$$\frac{\mathrm{d}\tilde{p_z}}{\mathrm{d}t} = -a_0\omega\frac{v_x}{c}\cos(kz-\omega t)\,. \tag{3.15c}$$

The y is trivial, and since we took the initial velocity to be zero, the y-velocity will be zero at all times.

From equations (3.14) and (3.15c) we have the following developement

$$\frac{\mathrm{d}(\tilde{p_z}-\gamma)}{\mathrm{d}t} = 0 \Leftrightarrow \tilde{p_z}-\gamma = C\,. \tag{3.16}$$

Again, making use of our choice of initial conditions, which translate here as $\gamma(0) = 1$ and $\tilde{p_z}(0) = 0$, we obtain C to be -1.

To summarize so far, we already know that

$$\tilde{p_y} = 0 \tag{3.17a}$$
$$\tilde{p_z} = \gamma - 1\,. \tag{3.17b}$$

By squaring up the very last equation, yet another useful relation can be derived

$$\gamma = 1+\tilde{p_z} = \sqrt{1+\tilde{\mathbf{p}}} \Rightarrow 1+2\tilde{p_z}+\tilde{p_z}^2 = 1+\tilde{p_x}+\tilde{p_y}+\tilde{p_z}$$

$$\tilde{p_z} = \frac{1}{2}\tilde{p_x}^2\,. \tag{3.18}$$

Since p_z is the normalized momentum along the propagation direction of the wave, we can see that for $\tilde{p_z} = \frac{1}{2}\tilde{p_x}^2 \ll 1$ (classical regime) the transversal momentum is more important, while for $\tilde{p_z} = \frac{1}{2}\tilde{p_x}^2 \gg 1$ (highly relativistic regime) the longitudinal momentum is more important.

Now we would like to find p_x. There is actually a more efficient way to do so than solving equations (3.15a) and (3.15c) together. But for that we shall make use of the electromagnetic potential (3.2). We also need the relations that define the fields from the potential, as detailed in section (2.1.3). With this in mind, we return to the equation of motion

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = -e\left(\mathbf{E}+\mathbf{v_e}\times\mathbf{B}\right) = -e\left(-\partial_t\mathbf{A}+\mathbf{v_e}\times(\boldsymbol{\nabla}\times\mathbf{A})\right)\,. \tag{3.19}$$

We make use of the vector identity

$$\mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{u}) = \boldsymbol{\nabla}(\mathbf{v}\cdot\mathbf{u}) - (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{u}\,, \tag{3.20}$$

and the total derivative of A with respect to time

$$\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} = \partial_t\mathbf{A} + \partial_{x_i}A^{x_i}\partial_t x_i = \frac{\partial\mathbf{A}}{\partial t} + (\mathbf{v_e}\cdot\boldsymbol{\nabla})\mathbf{A}\,. \tag{3.21}$$

Thus,

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = -e\left[-\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} + (\mathbf{v_e}\cdot\boldsymbol{\nabla})\mathbf{A} + \boldsymbol{\nabla}(\mathbf{v_e}\cdot\mathbf{A}) - (\mathbf{v_e}\cdot\boldsymbol{\nabla})\mathbf{A}\right] = -e\left[-\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} + \boldsymbol{\nabla}(\mathbf{v_e}\cdot\mathbf{A})\right]. \quad (3.22)$$

Since

$$\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} = -\mathbf{e_x}\omega A_0\cos(kz-\omega t)$$

$$\boldsymbol{\nabla}(\mathbf{v_e}\cdot\mathbf{A}) = \mathbf{e_z}kv_xA_0\cos(kz-\omega t)\,,$$

we can extract the p_x equation

$$\frac{\mathrm{d}p_x}{\mathrm{d}t} = -e\omega A_0\cos(kz-\omega t) = e\frac{\mathrm{d}A}{\mathrm{d}t}\,, \quad (3.23)$$

which gives

$$p_x = eA + C'\,. \quad (3.24)$$

With the initial conditions this becomes

$$p_x = eA = eA_0\sin(kz-\omega t)\,. \quad (3.25)$$

Bringing together equations (3.17b), (3.18) and (3.25), using the normalized vector potential $a = \frac{eA_0}{m_ec}\sin(kz-\omega t)$

$$K = (\gamma-1)m_ec^2 = \frac{a^2}{2}m_ec^2 \Rightarrow \gamma = 1 + \frac{a^2}{2}\,. \quad (3.26)$$

With this, all the puzzle pieces are in place, so we can collect the following results concerning the motion of the electron

$$\gamma = 1 + \frac{a^2}{2} \quad (3.27a)$$

$$\tilde{p_x} = a \quad (3.27b)$$

$$\tilde{p_y} = 0 \quad (3.27c)$$

$$\tilde{p_z} = \frac{a^2}{2}\,. \quad (3.27d)$$

It is more convenient, and general, to work with the derivative with respect to $\tau = t - \frac{z(t)}{c}$, which is to choose the convenient reference frame of the wave to simplify the computations. This derivative is developed as such

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}\tau}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}\tau} = \left(1 - \frac{1}{c}\frac{\mathrm{d}z}{\mathrm{d}t}\right)\frac{\mathrm{d}}{\mathrm{d}\tau} = \left(1 - \frac{a^2}{2}\right)\frac{\mathrm{d}}{\mathrm{d}\tau} = \frac{1}{\gamma}\frac{\mathrm{d}}{\mathrm{d}\tau}\,. \quad (3.28)$$

We can also write the phase of a in terms of this time

$$kz - \omega t = kz - \omega\tau - \frac{\omega}{c}z = -\omega\tau\,. \quad (3.29)$$

A simple substitution gives the equations for the coordinates

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = ca \tag{3.30a}$$

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = 0 \tag{3.30b}$$

$$\frac{\mathrm{d}z}{\mathrm{d}\tau} = c\frac{a^2}{2}\,, \tag{3.30c}$$

which have the solutions (of course using the initial conditions we chose at the begining)

$$x(\tau) = \frac{ca_0}{\omega}\left[\cos(\omega\tau) - 1\right] \tag{3.31a}$$

$$y(\tau) = 0 \tag{3.31b}$$

$$z(\tau) = \frac{ca_0^2}{4}\left[\tau - \frac{1}{2\omega}\sin(2\omega\tau)\right]\,. \tag{3.31c}$$

These results show us that the motion in the transversal plane is the same as in the classical motion. However, the motion on the propagation direction is more complex, being a superposition of an oscillation and a drift motion. The drift velocity $v_{drift} = \langle v_z \rangle = \left\langle \frac{z}{t} \right\rangle$ can be computed fairly easy

$$z_{drift} = \frac{ca_0^2}{4}\tau = \frac{ca_0^2}{4}\left(t - \frac{z}{c}\right) = \frac{ca_0^2}{4}t - \frac{a_0^2}{4}z_{drift}$$

$$z_{drift} = \frac{ca_0^2}{4}\frac{t}{1+\frac{a_0^2}{4}} = \frac{ca_0^2}{4+a_0^2}t$$

$$v_{drift} = \frac{ca_0^2}{4+a_0^2}\,. \tag{3.32}$$

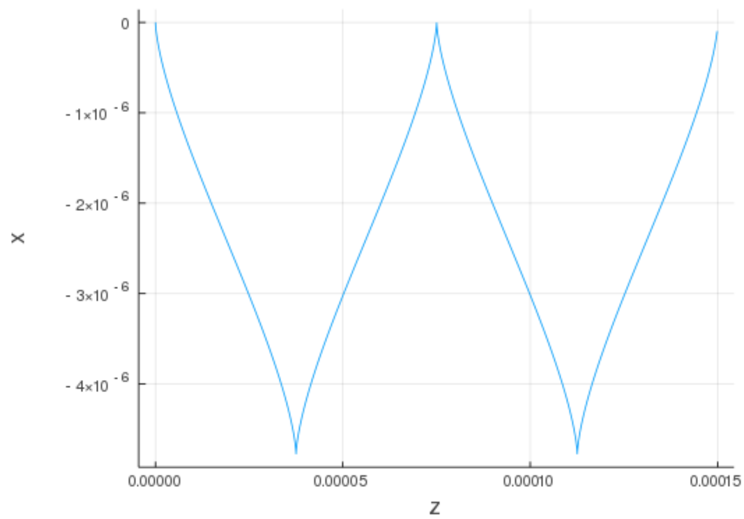


Figure 3.1: The motion in the laboratory frame over two periods of the pulse for some practical parameters: $a_0 = 20$ and $\nu = \frac{\omega}{2\pi} = 400$ THz

In finite length pulses, a certain phenomenon occurs. In order to obtain a finite plane wave pulse, we simply add a Gaussian envelope

$$a(\tau) = a_0 \exp\left(-\left(\frac{\tau}{\tau_0}\right)^2\right) \sin(\omega\tau)\,. \tag{3.33}$$

Under the fields given by this potential, the electron starts moving, but it stops when the wave passes it. Thus, although the electron is moved forward, the net energy gain in its interaction with the field is zero. The theoretical calculations that prove this are quite lengthy, so, in order to visualize this effect, I offer a numerical solution of this motion solved using a standard Euler method in figure 3.2. We can see that the trajectory of the electron converges to a point after a time longer than the pulse duration.

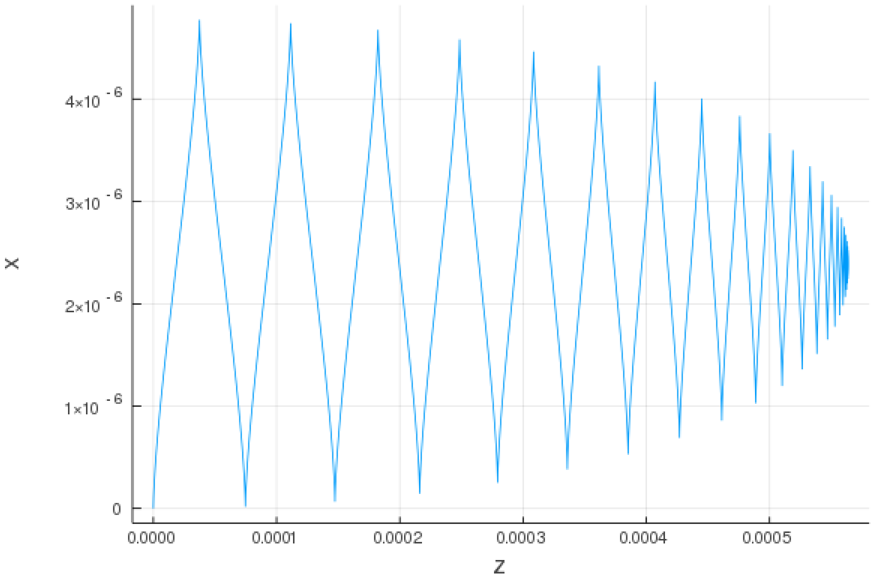


Figure 3.2: The motion in the laboratory frame over a long time compared to the pulse duration τ_0. The parameters used were: $a_0 = 20$, $\tau_0 = 30$ fs and $\nu = \frac{\omega}{2\pi} = 400$ THz

While a rigurous and complete discussion on dinamics of electrons in wave fields can take an entire book, I will offer here the Lawson-Woodward theorem, which analyzes the posibility for an electron to be accelerated by an electromagnetic wave.

Theorem(Lawson-Woodward)

A reletivistic electron does not gain energy from an electromagnetic wave if

- if the interaction takes place in vacuum, in the abence of any walls or any other particles;
- if the electron is in the highly relativistic regime during the entire acceleration process;
- if there are no static electromagnetic fields;
- if the interaction region is infinitely large;
- if the ponderomotive force can be neglected.

If at least one of these conditions are violated, acceleration can happen.

In laser-plasma interaction in particular, all the conditions of this theorem are not valid, which is why the possibility of developing a plasma based, laser powered accelerator is appealing. More on the prospects of this technology will be presented in the next sections.

3.2 The Ponderomotive Force

In this section we will delve into the most relevant physical phenomenon for our studies, namely the ponderomotive force. Intuitively, if we have a focused laser beam, its field will strongly vary in the transverse direction. For an electron that is initially found on the axis of the pulse, the deflection is stronger than the restoring force. Because of this, the electron will be completely removed from the focal region after a couple of oscillations and will retain a non-zero kinetic energy. This effect is called ponderomotive scattering. Some results from simulations of electron movement in various laser beam profiles are presented later in section 3.3. In these simulations we solved numerically the equations of motion for the electron, but in this section we will do some approximated computations to get a better understanding of the physics that generate this effect.

3.2.1 Classical Derivation

We already solved the motion of the electron in a plane wave, but for our current purpose, we will need a more general expression for the field. Namely, we would like to have a periodic dependence on time and a spatial dependence of the amplitude

$$\mathbf{E} = \mathbf{E_0}(\mathbf{r})\cos(\omega t)\,. \tag{3.34}$$

Note that the plane wave field is just a particular case of this ansaz, but now the equation of motion (3.3) is valid only locally, at the electron position $\mathbf{r_0}$. Let us approxiamates the field around $\mathbf{r_0}$ by Taylor expansion

$$\mathbf{E_0}(\mathbf{r}) = [\mathbf{E_0}(\mathbf{r})]_{\mathbf{r}=\mathbf{r_0}} + [(\delta\mathbf{r}\cdot\boldsymbol{\nabla})\mathbf{E_0}(\mathbf{r})]_{\mathbf{r}=\mathbf{r_0}} + \mathcal{O}(\delta\mathbf{r}^2)\,. \tag{3.35}$$

From the Maxwell equation $\partial_t\mathbf{B} = -\boldsymbol{\nabla}\times\mathbf{E}$, by integration over time, we can also find the magnetic field

$$\mathbf{B}(\mathbf{r}) = \frac{1}{c}\,[\mathbf{e_z}\times\mathbf{E_0}(\mathbf{r})]_{\mathbf{r}=\mathbf{r_0}}\cos(\omega t) - \frac{1}{\omega}\,[\boldsymbol{\nabla}\times\mathbf{E_0}(\mathbf{r})]_{\mathbf{r}=\mathbf{r_0}}\sin(\omega t)\,. \tag{3.36}$$

We can now write the equation of motion in first order and in second oreder, respectively

$$m_e\frac{\mathrm{d}\mathbf{v}^{(0)}}{\mathrm{d}t} = -e\left[\mathbf{E_0}(\mathbf{r_0})\cos(\omega t) + \frac{\mathbf{v}^{(0)}}{c}\times(\mathbf{e_z}\times\mathbf{E_0}(\mathbf{r_0}))\cos(\omega t)\right] \approx -e\mathbf{E_0}(\mathbf{r_0})\cos(\omega t) \tag{3.37a}$$

$$m_e\frac{\mathrm{d}\mathbf{v}^{(1)}}{\mathrm{d}t} = -e\,[(\delta\mathbf{r}\cdot\boldsymbol{\nabla})\mathbf{E_0}(\mathbf{r})]_{\mathbf{r}=\mathbf{r_0}}\cos(\omega t) + e\frac{1}{\omega}\mathbf{v}^{(0)}\times[\boldsymbol{\nabla}\times\mathbf{E_0}(\mathbf{r})]_{\mathbf{r}=\mathbf{r_0}}\sin(\omega t)\,, \tag{3.37b}$$

where $\frac{\mathrm{d}(\delta\mathbf{r})}{\mathrm{d}t} = \mathbf{v}^{(0)}$ and we have used the fact that $v^{(0)} \ll c$.Equation (3.37a) is solved by direct integration

$$\mathbf{v}^{(0)} = -\frac{e}{m_e\omega}\mathbf{E_0}(\mathbf{r_0})\sin(\omega t) \tag{3.38a}$$

$$\delta\mathbf{r} = \frac{e}{m_e\omega^2}\mathbf{E_0}(\mathbf{r_0})\cos(\omega t)\,. \tag{3.38b}$$

These are pretty much the results from the plane wave, that is the oscillatory motion induced by the wave on the electron. The second order equation describes an additional effectthat arises. Replacing the above solutions in equation (3.37b) leads to

$$m_e\frac{\mathrm{d}\mathbf{v}^{(1)}}{\mathrm{d}t} = -\frac{e^2}{m_e\omega^2}\left[(\mathbf{E_0}\cdot\boldsymbol{\nabla})\mathbf{E_0}\cos^2(\omega t) + \mathbf{E_0}\times(\boldsymbol{\nabla}\times\mathbf{E_0})\sin^2(\omega t)\right]_{\mathbf{r}=\mathbf{r_0}}\,. \tag{3.39}$$

So far this discussion followed the microscopic motion. We are intrested in average effects, so we should find the time average of the expression obtained. Keeping in mind that both the averages of sine squared and cosine squared over one period are one half, we can deduce that

$$\left\langle m_e\frac{\mathrm{d}\mathbf{v}^{(1)}}{\mathrm{d}t}\right\rangle_T = -\frac{e^2}{2m_e\omega^2}\left[(\mathbf{E_0}\cdot\boldsymbol{\nabla})\mathbf{E_0} + \mathbf{E_0}\times(\boldsymbol{\nabla}\times\mathbf{E_0})\right]_{\mathbf{r}=\mathbf{r_0}}\,, \tag{3.40}$$

where $T = \frac{2\pi}{\omega}$. We aplly now the vector identity (3.20) and the fact that $\boldsymbol{\nabla}\mathbf{v}^2 = 2\mathbf{v}(\boldsymbol{\nabla}\cdot\mathbf{v})$ to reach the final result

$$\left\langle m_e\frac{\mathrm{d}\mathbf{v}^{(1)}}{\mathrm{d}t}\right\rangle_T = -\frac{e^2}{4m_e\omega^2}\boldsymbol{\nabla}\mathbf{E_0}^2 = \mathbf{F_{pond}}\,, \tag{3.41}$$

which gives the expression of the ponderomotive force $\mathbf{F_{pond}}$.

3.2.2 Relativistic Ponderomotive Force

We already saw that for high intensity lasers the classical treatment is insuficient. In the case of the ponderomotive force a relativistic derivation leads to the same dependence on the field. More precisely, the ponderomotive force is directly proportional to the gradient of the squared electric field. Nonetheless, I will try to present in this section a rigurous relativistic treatment based on the discussion in the 5th chapter of Mulser and Bauer 2010.

We will start with the Lagrangian for an electron in an arbitrary 4-vector potential $(\phi, \mathbf{A})$

$$L(\mathbf{x}, \mathbf{v}, t) = -\frac{m_e c^2}{\gamma} - e\mathbf{v}\cdot\mathbf{A} + e\phi\,. \tag{3.42}$$

Since our system has an oscillation part given by the periodic time dependence of the 4-vector potential of the driving electromagnetic wave, we can change to action-angle variables $S = S(\mathbf{x}, t)$ and $\eta = \eta(\mathbf{x}, t)$, which are Lorentz invariant, so the motion will be governed by Hamilton's principle

$$\delta S = \delta\int_{\eta_i}^{\eta_f}\mathrm{d}\eta\, L(\mathbf{x}(\eta), \mathbf{v}(\eta), t(\eta))\frac{\mathrm{d}t}{\mathrm{d}\eta}\,. \tag{3.43}$$

This change of variables introduces a new Lagrangian (a "Lagrangian angular density" if you want)

$$\mathcal{L}(\eta) = \frac{L(\eta)}{\frac{\mathrm{d}\eta}{\mathrm{d}t}}\,. \tag{3.44}$$

With all these things set up and choosing η to be normalized to 2π, we introduce the cycle averaged Lagrangian

$$\mathcal{L}_0(\eta) = \frac{1}{2\pi}\int_{\eta}^{\eta+2\pi} \mathrm{d}\eta'\,\mathcal{L}(\eta')\,. \tag{3.45}$$

We denote by $N = \frac{n_f - n_1}{2\pi}$ to be the number of cycles over which $\mathcal{L}_0$ undergoes a significant change. We are interested now to see how N imposes a boundary on the action of $\mathcal{L}_0$. From equations (3.43) and (3.44) we see that

$$\delta\int_{\eta_i}^{\eta_f} \mathrm{d}\eta\,\mathcal{L} = 0\,,$$

which is useful in order to do the following mathematical arguments

$$\begin{aligned}
\left|\delta\int_{\eta_i}^{\eta_f} \mathrm{d}\eta\,\mathcal{L}_0\right| &= \left|\delta\int_{\eta_i}^{\eta_f} \mathrm{d}\eta\,(\mathcal{L}_0 - \mathcal{L})\right| = \left|\int_{\eta_i}^{\eta_f} \mathrm{d}\eta\,\delta(\mathcal{L}_0 - \mathcal{L})\right| = \\
&= \left|\int_{\eta_i}^{\eta_f} \mathrm{d}\eta\,[\mathcal{L}_0(\eta+\delta\eta) - \mathcal{L}(\eta+\delta\eta)] - \int_{\eta_i}^{\eta_f} \mathrm{d}\eta\,[\mathcal{L}_0(\eta) - \mathcal{L}(\eta)]\right| \leq \\
&\leq \sum_{n=1}^{N}\left|\int_{\eta_n}^{\eta_n+2\pi} \mathrm{d}\eta\,[\mathcal{L}_0(\eta+\delta\eta) - \mathcal{L}(\eta+\delta\eta)] - \int_{\eta_n}^{\eta_n+2\pi} \mathrm{d}\eta\,[\mathcal{L}_0(\eta) - \mathcal{L}(\eta)]\right|,
\end{aligned}$$

where we discretized the angle variable as $\eta_n = \eta_i + 2\pi n$, $n \in \mathbb{N}^*$ and used the triangle inequality for the modulus. We will now replace $\mathcal{L}$ from equation (3.45) and continue our endeavours

$$\begin{aligned}
\left|\delta\int_{\eta_i}^{\eta_f} \mathrm{d}\eta\,\mathcal{L}_0\right| &\leq \sum_{n=1}^{N}\left|\int_{\eta_n}^{\eta_n+2\pi} \mathrm{d}\eta\,\mathcal{L}_0(\eta+\delta\eta) - 2\pi\mathcal{L}_0(\eta_n+\delta\eta_n) - \int_{\eta_n}^{\eta_n+2\pi} \mathrm{d}\eta\,\mathcal{L}_0(\eta) - 2\pi\mathcal{L}_0(\eta_n)\right| = \\
&= \sum_{n=1}^{N}\left|\int_{\eta_n}^{\eta_n+2\pi} \mathrm{d}\eta\,[\mathcal{L}_0(\eta+\delta\eta) - \mathcal{L}_0(\eta)] - 2\pi\,[\mathcal{L}_0(\eta_n+\delta\eta_n) - \mathcal{L}_0(\eta_n)]\right| \approx \\
&\approx \sum_{n=1}^{N}\left|\int_{\eta_n}^{\eta_n+2\pi} \mathrm{d}\eta\,\frac{\partial\mathcal{L}_0(\eta)}{\partial\eta}\delta\eta - 2\pi\frac{\partial\mathcal{L}_0(\eta_n)}{\partial\eta_n}\delta\eta_n\right|,
\end{aligned}$$

where we kept only the first terms in $\delta\eta$ and $\delta\eta_n$. Now, it is time to use the mean value theorem to evaluate the integral

$$\int_{\eta_n}^{\eta_n+2\pi} \mathrm{d}\eta\,\frac{\partial\mathcal{L}_0(\eta)}{\partial\eta}\delta\eta = 2\pi\frac{\partial\mathcal{L}_0(\eta_n^c)}{\partial\eta_n^c}\delta\eta_n^c\,,$$

with $\eta_n^c \in (\eta_n, \eta_n + 2\pi)$, to obtain

$$\left|\delta\int_{\eta_i}^{\eta_f} \mathrm{d}\eta\,\mathcal{L}_0\right| \leq 2\pi\sum_{n=1}^{N}\left|\frac{\partial\mathcal{L}_0(\eta_n^c)}{\partial\eta_n^c}\delta\eta_n^c - \frac{\partial\mathcal{L}_0(\eta_n)}{\partial\eta_n}\delta\eta_n\right| = 2\pi\sum_{n=1}^{N}\left|\left[\frac{\partial\mathcal{L}_0(\eta)}{\partial\eta}\delta\eta\right]_{\eta=\eta_n^c} - \left[\frac{\partial\mathcal{L}_0(\eta)}{\partial\eta}\delta\eta\right]_{\eta=\eta_n}\right|.$$

We can also use a first order expansion to simplify the quantity in the modulus

$$\left|\left[\frac{\partial\mathcal{L}_0(\eta)}{\partial\eta}\delta\eta\right]_{\eta=\eta_n^c} - \left[\frac{\partial\mathcal{L}_0(\eta)}{\partial\eta}\delta\eta\right]_{\eta=\eta_n}\right| =$$

$$= \left|\left[\frac{\partial\mathcal{L}_0(\eta)}{\partial\eta}\delta\eta\right]_{\eta=\eta_n^c} - \left[\frac{\partial\mathcal{L}_0(\eta)}{\partial\eta}\delta\eta\right]_{\eta=\eta_n^c} - \left[\frac{\partial^2\mathcal{L}_0(\eta)}{\partial\eta^2}\delta\eta\right]_{\eta=\eta_n^c}(\eta_n - \eta_n^c)\right| =$$

$$= \left|\left[\frac{\partial^2\mathcal{L}_0(\eta)}{\partial\eta^2}\delta\eta\right]_{\eta=\eta_n^c}\right| |\eta_n - \eta_n^c)| \leq$$

$$\leq 2\pi\left|\left[\frac{\partial^2\mathcal{L}_0(\eta)}{\partial\eta^2}\delta\eta\right]_{\eta=\eta_n^c}\right| ,$$

so that in the end we remain with

$$\left|\delta\int_{\eta_i}^{\eta_f} \mathrm{d}\eta\,\mathcal{L}_0\right| \leq (2\pi)^2\sum_{n=1}^{N}\left|\left[\frac{\partial^2\mathcal{L}_0(\eta)}{\partial\eta^2}\right]_{\eta=\eta_n^c}\right||\delta\eta_n^c| \leq (2\pi)^2 N max\left\{\left|\left[\frac{\partial^2\mathcal{L}_0(\eta)}{\partial\eta^2}\right]_{\eta=\eta_n^c}\right|\right\} max\left\{|\delta\eta_n^c|\right\} . \tag{3.46}$$

We would want now to impose the condition that $\mathcal{L}_0^{max}$ changes at most by $\mathcal{L}_0$ over N cycles, that is

$$N = \frac{1}{2\pi}\frac{|\mathcal{L}_0^{max}|}{max\left\{\left|\left[\frac{\partial\mathcal{L}_0(\eta)}{\partial\eta}\right]_{\eta=\eta_n^c}\right|\right\}} . \tag{3.47}$$

By imposing a similar constraint on the first derivative

$$2\pi\frac{max\left\{\left|\left[\frac{\partial^2\mathcal{L}_0(\eta)}{\partial\eta^2}\right]_{\eta=\eta_n^c}\right|\right\}}{max\left\{\left|\left[\frac{\partial\mathcal{L}_0(\eta)}{\partial\eta}\right]_{\eta=\eta_n^c}\right|\right\}} \leq \frac{1}{N} , \tag{3.48}$$

we obtain that the action of $\mathcal{L}_0$ is bounted by a function $\mathcal{O}(N^{-1})$

$$\left|\delta\int_{\eta_i}^{\eta_f} \mathrm{d}\eta\,\mathcal{L}_0\right| \leq \frac{|\mathcal{L}_0^{max}|}{N} max\left\{|\delta\eta|\right\} . \tag{3.49}$$

Let us observe that for the regular averaged lagrangian defined by $L_0 = \mathcal{L}_0\frac{\mathrm{d}\eta}{\mathrm{d}t}$ we have

$$\int_{\eta_i}^{\eta_f} \mathrm{d}\eta\,\mathcal{L}_0 = \int_{t_i}^{t_f} \mathrm{d}t\,L_0 . \tag{3.50}$$

If this would be zero, we would get an Euler-Lagrange set of equations for L_0

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_0}{\partial\mathbf{v_0}}\right) - \frac{\partial L_0}{\partial\mathbf{x_0}} = 0 , \tag{3.51}$$

where the coordinates $\mathbf{x_0}$ and $\mathbf{v_0}$ correspond to the oscillation center. However, working with the inequality (3.49) we would obtain an extra term in the right-hand side of equation (3.51) whose absolute value does not exceed

$$\frac{|\mathcal{L}_0^{max}|}{N^2} max \{|\delta\eta|\} \ .$$

If the Lagrangian $\mathcal{L}_0$ doesn't depend on time, energy is conserved on average in time and the right-hand side of equation (3.51) is adiabatically zero. For $N \to \infty$ it is exactly zero. This extra term is the generalized force associated to the ponderomotive force.

The relativistic Hamiltonian obtained from equation (3.42) by the definition $H(\mathbf{x}, \mathbf{p}, t) = \mathbf{pv} - L(H(\mathbf{x}, \mathbf{v}, t)$ is

$$H = \sqrt{m^2c^4 + c^2(\mathbf{p} + e\mathbf{A})^2} - e\phi\,, \tag{3.52}$$

with the canonical momentum defined as

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}} = \gamma m\mathbf{v} - e\mathbf{A}\,. \tag{3.53}$$

By defining the effective mass as

$$m_{eff} = \frac{\gamma_0 \mathcal{L}_0 \frac{\mathrm{d}\eta}{\mathrm{d}t}}{c^2} = \frac{\gamma_0 L_0}{c^2}\,, \tag{3.54}$$

we obtain that the Lagrangian and Hamiltonian in an arbitrary reference system in which the oscillation center moves with velocity $\mathbf{v_0}$ are those of a free particle with an effective mass that depends on space-time

$$L_0(\mathbf{x_0}, \mathbf{v_0}, t) = -\frac{m_{eff}c^2}{\gamma_0} \tag{3.55a}$$

$$H_0(\mathbf{x_0}, \mathbf{p_0}, t) = \gamma_0 m_{eff} c^2\,, \tag{3.55b}$$

with $\gamma_0 = \sqrt{1 - \frac{\mathbf{v_0}^2}{c^2}}$ and $\mathbf{p_0} = \gamma_0 m_{eff}\mathbf{v_0}$. In the oscillation center system, the ponderomotive force is defined as

$$\mathbf{f_p} \equiv \frac{\partial L_0}{\partial \mathbf{x_0}} = -c^2 \boldsymbol{\nabla} m_{eff}\,. \tag{3.56}$$

For a strong monochromatic electromagnetic wave the effective mass is

$$m_{eff} = m_e\sqrt{1 + \frac{4}{m_e c^2}\mathbf{A}^2}\,. \tag{3.57}$$

With this, the classical dependence obtained in the previous subsection still holds

$$\mathbf{F_p} \propto \boldsymbol{\nabla}\mathbf{E}^2\,. \tag{3.58}$$

3.3 Simulations for the Visualization of the Ponderomotive Force

Chapter 4

Plasma Physics

4.1 The Definition of Plasma

It is common that people, when asked about what is plasma, their definition stops at the fact that it is a **partially ionized gas**. But this is just one of the three defining characteristics. After all, even the air is partially ionized. The other two proprties that a medium should satisfy in order to be considered plasma are quasi-neutrality and collective behaviour.

To be **quasi-neutral** means that the medium has an equal number of positive and negative charges in its entire volume, but small deviations from neutrality are possible locally. That is, if n_p, n_e are the positive charge density and the electron density, respectively, in the whole region of the plasma we have $n_p = n_e$, yet in small regions in the space inside we have $n_p \approx n_e$. A small remark should be made here. While I say that the plasma as a whole is neutral, it is so by approximation still. If one starts building plasma by pumping energy into a gaseous medium for example, some of the first ionized electrons can actually escape the medium. It is only after a certain positive charge density has been achieved that no electrons can not escape anymore due to Coulomb attraction. Once enough ionization electrons are produced, the charge imbalance becomes incredibly small (*i.e.* $\frac{n_p - n_e}{n_e} \ll 1$). This is though a very hard to observe charge imbalance in practice, so we can say that plasma as a whole is neutral. The localized imbalance in turn is not constantly small; it can vary widely due to the disordered motion of the constituent particles, but statistically speaking neutrality is maintained locally when we look at the time averages.

Collective behaviour is a consequence of the fact that the main type of interaction between the particles constituting the plasma, namely Coulomb interaction, is long range. As such, we can say that any particle in the plasma feels all the other ones. This leads to many important properties specific to plasma, like particle and momentum transport. The simplest response is plasma oscillation, which arises when plasma is placed in a constant electric field. The electrons are pushed by the electric field, but the surplus of positive charge left behind pulls them back, creating an oscillatory motion (we should take into consideration that the positive ions are at least a couple thousand times heavier than the electrons, so it is harder to influence their motion).This property also influences the the way in which plasma interacts with electromagnetic radiation, giving rise to radiation transport phenomena for example.

It is important to note from the very begining that in a plasma we have quite many different species of particles. The most simple model would only include electrons, neutral atoms and ions that have just one missing electron, but in reality we can have all the possible types of ions (so also with two or more missing electrons) and photons (which arise from the excitations and de-excitations that happen in this very energetic medium).

In the following sections we aim to go deeper into the parameters that characterize plasmas and the basic models for it. The discussion brings together ideas from Karsch 2018 and Mulser and Bauer 2010.

4.2 Temperature

We would like now to study the statistics of electrons. First of all, we should realize that the interparticle distances in plasmas are quite large, but also the temperature needed to sustain ionization is quite high. So high in fact that working with the Fermi-Dirac statistics is not necesary, since this quantum mechanically derived distribution can be approximized very well by the classical Maxwell-Boltzmann distribution in this particular situation.

The number of electron with x-axis velocity between $v_{e,x}$ and $v_{e,x} + \mathrm{d}v_{e,x}$ is then given by

$$f_e(v_{e,x})\,\mathrm{d}v_{e,x} = n_e \sqrt{\frac{m_e}{2\pi K_B T_e}}\,\mathrm{e}^{-\frac{K_x}{K_B T_e}}\,, \tag{4.1}$$

where n_e, m_e and T_e are the electrons' density, mass and temperature, respectively, K_B is the Boltzmann constant and $K_x = \frac{m_e v_{e,x}^2}{2}$ is the kinetic energy of the photons. The normalization constant was obtained from the electron density, since $n_e = \int_{-\infty}^{+\infty} \mathrm{d}v_{e,x}\, f_e(v_{e,x})$. This gives an average kinetic energy of

$$K_x^{avg} = \frac{\int_{-\infty}^{+\infty} \mathrm{d}v_{e,x}\, K_x f_e(v_{e,x})}{\int_{-\infty}^{+\infty} \mathrm{d}v_{e,x}\, f_e(v_{e,x})} = \frac{m_e}{2n_e}\int_{-\infty}^{+\infty} \mathrm{d}v_{e,x}\, v_{e,x}^2 f_e(v_{e,x}) = \frac{1}{2} K_B T_e\,. \tag{4.2}$$

This is extended in 3D easily, since the distribution of velocity in this case should not have any preferential direction

$$K^{avg} = \frac{3}{2} K_B T_e\,. \tag{4.3}$$

As it can be seen, we can basically treat the electrons inside the plasma as we would an ideal gas and we have obtained that the average kinetic energy is proportional to the temperature.

A simple numerical application shows us that in order to have $K_B T_e = 1$ eV, the temperature would be around 11600 K. Thus, since the ionized electrons are above the energy level of outer bounded states (so above 1 eV), using Kelvin or degrees Celsius is not that handy. In practice, we will rather use eV (energy units) temperature, which is to be converted to the usual temperature by dividing to K_B.

We could actually treat the ions and the neutral atoms inside the plasma in the same manner. Considering this, we must make the remark that we can have different temperature scales in plasmas. While at thermodynamic equilibrium the system of electrons, ions and neutrals should have a uniform temperature, under the action of an electric field, lets say, the motion of the electrons is influenced more than that of the ions due to the difference in mass, while the neutrals are not affected at all, so we have $T_n \neq T_i \neq T_e$. Of course, equilibration between species can be achieved through collisions or radiation emission and absorption. In complete models, one should also consider the temperatures of photons and individual ion subspecies that can appear. Considering this, in general, thermal equilibration can take a long time (How long exactly is hard to say. Even a rough estimate should account for many types of collisions that take place in a plasma. A comperhensive list of the processes occuring can be found in Braithwaite 2000). It is also important to visualize that the temperature can be directional depending on the orientation of the fields we apply.

Regarding the photons in the system, we must point out that they never reach an equilibrium state. That is because photons are not maintained in the plasma like the other forces (say, electrons and ions pulled back in the plasma by Coulomb forces, neutral particles which have to deal with surface effects). This effect is easy to see in practice. Since the photons leave the plasma medium, the plasma itself is radiating light. This property is the basis for building plasma lamps and plasma displays.

4.3 Debye Shielding

In this section we will derive a common criterion for quasi-neutrality. Let us consider an infinite medium filled with plasma at thermal equilibrium, $T = T_e = T_i$ and with one ion species with charge Ze, such that we have $n_e = Zn_i$. We are interested to see what happens if we introduce an infinite plane with constant positive surface charge density σ in this system (see figure 4.1).

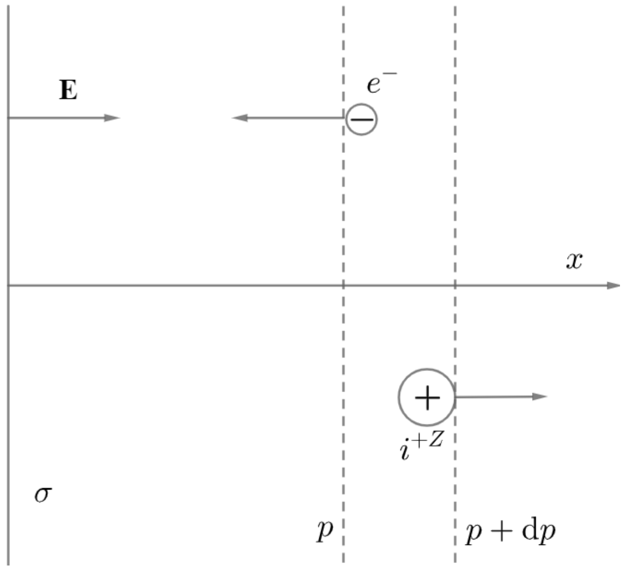


Figure 4.1: A schematic figure that shows the action of introducing the charged sheet in the plasma

The constant electric $E = \frac{\sigma}{2\varepsilon_0}$ generated by the plate will act to locally separate sheets of electrons and ions, until equilibrium between the pressures $p_e = n_e K_B T$ and $p_i = n_i K_B T$ is achieved. For the electrons in a sheet of thickness δx we have

$$\mathrm{d}p_e = K_B T \,\mathrm{d}n_e = -e n_e E \delta x \,, \tag{4.4}$$

which can be rewritten as

$$\frac{1}{n_e}\partial_x n_e = -\frac{e}{K_B T}E = \frac{e}{K_B T}\partial_x \phi \,, \tag{4.5}$$

where ϕ is the electrostatic potential. Solving this equation for the density of electrons, we obtain

$$n_e(x) = \bar{n}_e \,\mathrm{e}^{\frac{e\phi}{K_B T}} \,,\ \bar{n}_e = n_e(x \to \infty) \,. \tag{4.6}$$

In the same manner one obtains a similar expression for the ion density

$$n_i(x) = \frac{\bar{n}_e}{Z}\,\mathrm{e}^{-\frac{Ze\phi}{K_BT}}\ . \tag{4.7}$$

Writing the Poisson equation in terms of these results gives us

$$\partial_x^2\phi = \frac{e\bar{n}_e}{\varepsilon_0}\left(\mathrm{e}^{\frac{e\phi}{K_BT}} - \mathrm{e}^{-\frac{Ze\phi}{K_BT}}\right)\ . \tag{4.8}$$

If the potential energy arising from the field is small compared the the kinetic energy of the particles in the plasma, *i.e.* $e\phi \ll K_BT$, the potential equation can be simplified in approximation to

$$\partial_x^2\phi = \frac{e\bar{n}_e}{\varepsilon_0}\left(1 + \frac{e\phi}{K_BT} - 1 + \frac{Ze\phi}{K_BT}\right) = \frac{e^2\bar{n}_e(Z+1)}{\varepsilon_0 K_BT}\phi\,. \tag{4.9}$$

Obtaining the solution if this is trivial

$$\phi(x) = \phi_0\,\mathrm{e}^{-\frac{x}{\lambda_D}}\ , \tag{4.10}$$

where we introduced the Debye length

$$\lambda_D = \sqrt{\frac{\varepsilon_0 K_BT}{\bar{n}_e e^2(Z+1)}}\,. \tag{4.11}$$

This shows us that at a distance of λ_D away from the plate, the electric field generated by it, as well as the corresponding potential, is screened by about 63%. This offers us great insight in how to obtain quasi-neutrality.

From this discussion we can conclude that quasi-neutrality holds if the spatial extension of our ionized gas is at least a couple times larger than the Debye length, since in this case the local deviations from neutrality $n_p \approx n_e$ are screened. For dimensions smaller than λ_D, there is no quasi-neutrality very high-intensity localized fields can occur giving rise to interesting physical phenomena.

In practice, one uses another form for λ_D which neglects the Z and uses the temperature and particle density with more convenient units

$$\lambda_D = \sqrt{\frac{\varepsilon_0 K_BT}{\bar{n}_e e^2}} = 6.9\sqrt{\frac{T_e[\mathrm{K}]}{n_e[\mathrm{cm}^3]}}\,, \tag{4.12}$$

the last expression giving λ_D in cm.

4.4 Plasma Frequency

The simplest plasma wave is a longitudinal oscilation of the electrons (caused by an external electric field). Considering how heavy all the other particle species are compared to them, we can treat this problem by considering the ions stationary. Thus, the electrons simply move in a static positive background. The local changes in electron density cause imbalance and, as we mentioned before, restoring Coulombian forces arise. In what follows we will work purely electrostatic by considering that the electric field is always paralel to the x-axis and the external electric field. We also consider that the electron temperature is close to zero.

We make use of the equation of motion for a single electron, the continuity equation and the Poisson equation

$$m_e \frac{\mathrm{d}v_x}{\mathrm{d}t} = m_e \left(\partial_t v_x + v_x \partial_x v_x\right) = -eE \tag{4.13a}$$
$$\partial_t n_e + \partial_x (n_e v_e) = 0 \tag{4.13b}$$
$$\partial_x E = \frac{e}{\varepsilon_0}(n_p - n_e)\,, \tag{4.13c}$$

where n_p is the positive charge density divided by the unit charge. In equation (4.13a) we can neglect the second term by considering that the velocity is small. We have $v_x \partial_x v_x = \frac{1}{2}\partial_x v_x^2 \approx 0$, since $v_x^2 \ll v_x$ (remember that we work with $T_e \approx 0$), so

$$\partial_t v_x = -\frac{e}{m_e} E\,. \tag{4.14}$$

Similarly, considering that the particle density is also slowly changing around an equilibrium position $n_0 = n_p$ (*i.e.* $n_e = n_0 + n(x,t)$), we can simplify equation (4.13b)

$$\partial_t n_e + n_0 \partial_x v_x = 0\,. \tag{4.15}$$

These approximations also change equation (4.13c)

$$\partial_x E = -\frac{e}{\varepsilon_0} n\,. \tag{4.16}$$

Using a plane wave ansaz for E, v_x and n

$$E = E_0\, \mathrm{e}^{\mathrm{i}(kx-\omega t)} \tag{4.17a}$$
$$v_x = v_0\, \mathrm{e}^{\mathrm{i}(kx-\omega t)} \tag{4.17b}$$
$$n = n_{10}\, \mathrm{e}^{\mathrm{i}(kx-\omega t)}\,, \tag{4.17c}$$

The approximated equation become

$$-\mathrm{i}\omega v_0 = -\frac{e}{m_e} E_0 \tag{4.18a}$$
$$-\mathrm{i}\omega n_{10} + \mathrm{i} n_0 k v_0 = 0 \tag{4.18b}$$
$$\mathrm{i}k E_0 = -\frac{e}{\varepsilon_0} n_{10}\,. \tag{4.18c}$$

The frequency ω can be extracted from these relations as follows

$$-\,\mathrm{i}\omega v_0 = -\frac{e}{m_e} E_0 = \frac{e^2}{\mathrm{i}\varepsilon_0 k m_e} n_{10} = \frac{e^2}{\mathrm{i}\varepsilon_0 k m_e} \frac{k n_0}{\omega} v_0\,, \tag{4.19}$$

or

$$\omega_p = \sqrt{\frac{n_0 e^2}{\varepsilon_0 m_e}}\,. \tag{4.20}$$

This is called the cold plasma frequency. This result can be used to derive the plasma waves dispersion relation for the thermal motion of electrons

$$\omega^2 = \omega_p^2 + 3k^2 \frac{K_B T_e}{m_e} = \omega_p^2 + \frac{3}{2} k^2 v_{th}^2 \,, \tag{4.21}$$

where v_{th} is the thermal velocity. As usual, the dispersion relation is useful for finding the phase and group veloities

$$v_{ph} = \frac{\omega}{k} = \sqrt{\frac{\omega_p^2}{k^2} + \frac{3}{2} v_{th}^2} \tag{4.22a}$$

$$v_{gr} = \frac{\partial \omega}{\partial k} = \frac{3}{2} \frac{v_{th}^2}{v_{ph}} \,. \tag{4.22b}$$

All these velocities satisfy the inequality

$$v_{gr} < \sqrt{\frac{3}{2}} v_{th} < v_{ph} \,. \tag{4.23}$$

It is also important to remark that the cold plasma frequency (4.20) and the Debye length (4.12) can be related with each other

$$\omega_p \cdot \lambda_D = \sqrt{\frac{K_B T_e}{m_e}} \propto v_{th} \,. \tag{4.24}$$

4.5 Electromagnetic Waves in Plasma

We will now try to discuss how a plane wave interacts with a plasma medium. We shall start assumung that we have a wave that enters the plasma having initially the properties

$$\mathbf{E},\, \mathbf{B} \propto \mathrm{e}^{\mathrm{i}(\mathbf{kr} - \omega t)} \tag{4.25a}$$

$$\mathbf{B} = \frac{1}{c} \mathbf{k} \times \mathbf{E} \,. \tag{4.25b}$$

For simplicity, we will assume that no longitudinal field components arise from the interaction with the plasma. While this may not be a perfectly realistic assumtion, the longitudinal components that appear would be very small and would not affect to much the result we obtain.

We will make use of the already familiar Maxwell equations (two of them, to be precise)

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2} \partial_t \mathbf{E} \Rightarrow \boldsymbol{\nabla} \times (\partial_t \mathbf{B}) = \frac{1}{\varepsilon_0 c^2} \partial_t \mathbf{j} + \partial_t^2 \mathbf{E} \tag{4.26a}$$

$$\boldsymbol{\nabla} \times \mathbf{E} = -\partial_t \mathbf{B} \Rightarrow \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{E}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\boldsymbol{\nabla} \times (\partial_t \mathbf{B}) \,. \tag{4.26b}$$

Bringing these results together gives

$$\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{E}) - \nabla^2 \mathbf{E} + \frac{1}{\varepsilon_0 c^2} \partial_t \mathbf{j} + \partial_t^2 \mathbf{E} = 0 \,. \tag{4.27}$$

From equation (4.25) we can compute the derivatives to obtain

$$\mathrm{i} \boldsymbol{\nabla}(\mathbf{k} \cdot \mathbf{E}) + k^2 \mathbf{E} - \mathrm{i} \frac{\omega}{\varepsilon_0 c^2} \mathbf{j} - \frac{\omega^2}{c^2} \mathbf{E} = 0 \,. \tag{4.28}$$

Here the assumption of no longitudinal field component comes in handy and removes the first term, leaving us with

$$(\omega^2 - c^2k^2)\mathbf{E} = -\mathrm{i}\frac{\omega}{\varepsilon_0}\mathbf{j}\,. \tag{4.29}$$

We shall deviate a bit in order to find an expression for the current density. We can find the average electron velocity from the force equation

$$m_e\partial_t\mathbf{v_e} = -e\mathbf{E} \Rightarrow \mathbf{v_e} = \frac{e}{\mathrm{i}\omega m_e} \tag{4.30}$$

and introduce it in the definition of the current density in terms of drift velocity

$$\mathbf{j} = -en_0\mathbf{v_e}\,. \tag{4.31}$$

By putting together these last three results we obtain the dispersion relation for plane electromagnetic waves in plasma

$$\omega^2 = \omega_p^2 + k^2c^2\,. \tag{4.32}$$

From this, obtaining also the phase velocity is trivial

$$v_{ph} = \frac{c}{\sqrt{1-\frac{\omega_p^2}{w^2}}} = \frac{c}{\eta}\,. \tag{4.33}$$

Here we have also defined the plasma refractive index $\eta = \frac{kc}{\omega}$. It gives very important criteria for propagation of waves in plasma. If $\omega > \omega_p$, then $\eta < 1$ and the wave can propagate through. Otherwise, when $\omega < \omega_p$ and η is imaginary, the wave can not propagate; it will drop exponentially after it enters the plasma medium. The group velocity is also obtained to be

$$v_{gr} = c\eta < c\,. \tag{4.34}$$

There is also a physical way to think about the possibility of propagation of a wave. If the electromagnetic oscillation occur at a faster rate than that to which the plasma can react, the charged particles in the plasma will not respond fast enough to shield the radiation and it will penetrate. In the oposite case, the incoming radiation is shielded and energy conservation dictates that the wave must be relfected. Whatever small part of the wave would enter the plasma will drop very fast.

4.6 The Vlasov Equation

There is an alternative way to treat the statistics of plasma populations without using the Maxwell-Boltzmann distribution. More precisely, one can derive an equation for the distribution function in phase space. In order to do that, we must start from the Liouville theorem (Arnold 1997, p. 68).

Theorem(Liouville)

The *phase flow* is the one-parameter group of transformations of phase space $g^t : (\mathbf{p}(0), \mathbf{q}(0)) \to (\mathbf{p}(t), \mathbf{q}(t))$, where $\mathbf{p}(t)$ and $\mathbf{q}(t)$ are solutions of the Hamilton equations. The phase flow preserves the volume of any region. That is, for any region D, $V(g^tD) = V(D)$.

In more layman terms, for a Hamiltonian system, the volume in phase space is conserved in time. In this theorem we also defined the time propagation transformation group $\{g^t\}$. Before delving into the proof of Liouville's theorem let us see that g^t is indeed a group (just as a warming up exercise).

Let $g^{t_a}\,, g^{t_b} \in \{g^t\}$ three arbitrary transformations (the time t is just a real parameter) and $\mathbf{v}\,, \mathbf{p}$ solutions of Hamilton's equations. For the closure property we have

$$(g^{t_a} \cdot g^{t_b}) : (\mathbf{p}(0), \mathbf{q}(0)) \equiv g^{t_a} : (g^{t_b} : (\mathbf{p}(0), \mathbf{q}(0))) \equiv g^{t_a} : (\mathbf{p}(t_b), \mathbf{q}(t_b)) \equiv$$

$$\equiv (\mathbf{p}(t_b + t_a), \mathbf{q}(t_b + t_a)) \equiv g^{t_b + t_a} : (\mathbf{p}(0), \mathbf{q}(0))\,.$$

We can see from this that composing two transformations gives another transformation whose parameter is the sum of the parameters of the initial transformations. Now it is easy to argue that associativity holds because the parameters are real numbers and their addition is associative. The identity element is g^0 and the inverve of the element g^t is g^{-t}. Even more, this group is abelian. This is consistent to the physical reality of time evolution, since, starting from exactly the same initial conditions, progressing in time first up to t_a and then up to $t_a + t_b$ is equivalent to progressing in time first up to t_b and then up to $t_a + t_b$.

Now let us turn back to proving the theorem. I will deonote by $V(t)$ the volume of $D(t)$, where $D(t) = g^t D$ and D is the initial volume. Supposing we have the set on n ordinary differential equations

$$\mathbf{u} = \mathbf{F}(\mathbf{u}),\ \mathbf{u} = (u_1,\, u_2,\, \ldots,\, u_n)\,, \tag{4.35}$$

we can define the corresponding group of transformations $\{g^t\}$ for an infinitesimal time propagation $\delta t \to 0$ as

$$g^{\delta t} : \mathbf{u} = \mathbf{u} + \mathbf{F}(\mathbf{x})\delta t + \mathcal{O}(\delta t^2)\,. \tag{4.36}$$

In the space of the soluions $\mathbf{u}$, the initial and final volumes can be written as

$$V(0) = \int_D \mathrm{d}\mathbf{u} \tag{4.37a}$$

$$V(\delta t) = \int_{D(\delta t)} \mathrm{d}\mathbf{u}(\delta t)\ , \tag{4.37b}$$

with $\mathrm{d}\mathbf{u} = \mathrm{d}u_1\,\mathrm{d}u_2 \ldots \mathrm{d}u_n$. The key argument of the proof is that we can write the integral at time δt using as integration variables the coordinates at time 0 through a change of variables, that is, employing the Jacobian $\frac{\partial g^{\delta t}:\mathbf{u}}{\partial \mathbf{u}}$

$$V(\delta t) = \int_D \mathrm{d}\mathbf{u} \left| \frac{\partial g^{\delta t} : \mathbf{u}}{\partial \mathbf{u}} \right|. \tag{4.38}$$

This Jacobian can found from equation (4.36) to be

$$\frac{\partial g^{\delta t} : \mathbf{u}}{\partial \mathbf{u}} = I + \frac{\partial \mathbf{F}}{\partial \mathbf{u}}\delta t + \mathcal{O}(\delta t^2)\,, \tag{4.39}$$

where we denoted with I the corresponding identity matrix.

Here it is obvious that we can employ the Cayley-Hamilton theorem, or rather a direct consequence of it

Theorem

For any matrix A

$$|I + \delta t A| = 1 + \delta t\, trA + \mathcal{O}(\delta t^2)\,, \tag{4.40}$$

when $\delta t \to 0$.

This immediately gives

$$\left|\frac{\partial g^{\delta t} : \mathbf{u}}{\partial \mathbf{u}}\right| = 1 + \delta t\, tr\frac{\partial \mathbf{F}}{\partial \mathbf{u}} + \mathcal{O}(\delta t^2)\,. \tag{4.41}$$

Since $tr\frac{\partial \mathbf{F}}{\partial \mathbf{u}} = \sum_{i=1}^{n} \partial_{x_i} F_i = \boldsymbol{\nabla} \cdot \mathbf{F}$, equation (4.38) can be rewritten as

$$V(\delta t) = \int_D \mathrm{d}\mathbf{u} \left(1 + \delta t\, \boldsymbol{\nabla} \cdot \mathbf{F} + \mathcal{O}(\delta t^2)\right)\,, \tag{4.42}$$

so the variation in volume in the solution space is (by taking the derivative)

$$\partial_t V = \int_D \mathrm{d}\mathbf{u}\, \boldsymbol{\nabla} \cdot \mathbf{F}\,. \tag{4.43}$$

In the case of the Hamilton equations

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = -\frac{\partial H}{\partial \mathbf{q}} \tag{4.44a}$$

$$\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} = \frac{\partial H}{\partial \mathbf{p}}\,, \tag{4.44b}$$

we have

$$\boldsymbol{\nabla} \cdot \mathbf{F} = \frac{\partial}{\partial \mathbf{p}}\left(-\frac{\partial H}{\partial \mathbf{q}}\right) + \frac{\partial}{\partial \mathbf{q}}\left(\frac{\partial H}{\partial \mathbf{p}}\right) = 0\,, \tag{4.45}$$

which proves the Liouville theorem.

Now, finding the Vlasov equation is only a matter of using basic and principial satistical mechanics reasoning. If we have a system of particles obeying Hamiltonian dynamics, the probability to fin this system in a certain volume element $\mathrm{d}\mathbf{p}\,\mathrm{d}\mathbf{q}$ in phase space is denoted as $w(\mathrm{d}\mathbf{p}\,\mathrm{d}\mathbf{q})$. This can be written in terms of a distribution function ρ as $w(\mathrm{d}\mathbf{p}\,\mathrm{d}\mathbf{q}) = \rho(\mathbf{p}, \mathbf{q}, t)\,\mathrm{d}\mathbf{p}\,\mathrm{d}\mathbf{q}$. Since at equilibrium, w is constant, we can conclude that

$$\rho(\mathbf{p}, \mathbf{q}, t)\,\mathrm{d}\mathbf{p}\,\mathrm{d}\mathbf{q} = \rho(\mathbf{p_0}, \mathbf{q_0}, t_0)\,\mathrm{d}\mathbf{p_0}\,\mathrm{d}\mathbf{q_0}\,, \tag{4.46}$$

or, using Liouville's theorem,

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = 0 = \frac{\partial \rho}{\partial t} + \frac{\partial \mathbf{p}}{\partial t}\frac{\partial \rho}{\partial \mathbf{p}} + \frac{\partial \mathbf{q}}{\partial t}\frac{\partial \rho}{\partial \mathbf{q}}\,. \tag{4.47}$$

While this equation is sometimes called the Liouville equation, Boltzmann was the first to use it for the purpose of statistical physics. In the original Boltzmann equation, a collision term is considered. For plasma, this collision term is neglected, since the intercation mechanism is the long range Coulomb force. The colisionless Boltzmann equation is named the Vlasov equation, since Anatoly Vlasov was the first to implement this formalism in studies of plasma physics.

Since $\mathbf{p}$ and $\mathbf{q}$ are functions of time only, their partial derivatives can be written as total derivatives. Using Newton's second principle and defining the forces acting on the particles as $\mathbf{f}$, we can stylize the Vlasov equation so it may look more familiar (Feix and Bertrand 2005)

$$\frac{\partial \rho}{\partial t} + \mathbf{f} \cdot \frac{\partial \rho}{\partial \mathbf{p}} + \mathbf{v} \cdot \frac{\partial \rho}{\partial \mathbf{r}} = 0 \,. \tag{4.48}$$

An imortant remark must be made. The distribution function ρ is actually one that describe all the particles in the system together. In practice, in order ease the solving of this equation, one uses the approximation that this equation is valid for the single-particle distribution. The interaction between particles is reintroduced by including in the scalar potential the contribution of each particle (Silim and Büchner 2003). While this is a better way of describing the plasma system than using the Canonical ensamble (*i.e.* the Maxwell-Boltzmann statistics), this approximation neglects some inherent correlations that exist in real systems.

4.7 Two Stream Instability

In this chapter we present the results from a sample Particle in Cell simulation that can be found in the documentation (*Basic Examples of Using EPOCH* 2020) for the software we are using, EPOCH. The main purpose is to test that the code itself is working properly while also learning some extra stuff about plasma physics.

The figures 4.2 and 4.3 illustrate this effect in phase space. Two streams of electrons with the same drift velocity and particle densities are set to move in opposite directions. The images were obtained through a 1D Particle in Cell simulation in EPOCH with periodic boundary conditions. More on the EPOCH software and how to use it can be found in chapter 5. Red points represent the particles going to the right, while blue points represent the particles going to the left.

Before delving into the mathematical description of this phenomenon, I will present a qualitative explanation for it, as I understood it from the considerations I found in the seventh chapter of Stix and Stix 1992. Suppose we have two counter-flowing streams of charged particles (they can be both electrons, one electron and one ion, it doesn't make a difference). If a point of disturbance appears, a plasma wave will start emanating from it ab will propagate in the medium. The electric field of this wave accelerates the beam particles and some of those will be carried back to the initial disturbance center by the zero-order streaming motion. During the time it take to reach the disturbance point, the coherent acceleration they are undergoing is generating a fluctuation in their density. When the phase of this fluctuation has the right value, the initial disturbance is amplified. The production of these fluctuations in density is named bunching and it ditinguishes itself as a velocity dependent process. This means that plasma components with different velocities will be bunching with different amplitudes and phases.

For what follows, we will treat the beams as collisionless continuous cold fluids of charged particles. We will only deal with two beams, although an analogous treatment can be done for three or more. The electric field is assumed to have a null zero order approximation, since no external field is present in the system. The zero order particle densities and velocities will de denoted with capital letters (N and V, respectively), while the first order parts will be denoted with lowercase letters (n, v). The first order functions are assumed to have a dependence of the form

$$\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-\omega t)} \,.$$

For each stream we can write the cold fluid equations (continuity and dynamics) in the similar manner we did in the previous sections

$$-\omega n_{1,2} + N_{1,2}\mathbf{k}\cdot\mathbf{v_{1,2}} + n_{1,2}\mathbf{k}\cdot\mathbf{V_{1,2}} = 0 \tag{4.49a}$$

$$-\omega\mathbf{v_{1,2}} + \mathbf{v_{1,2}}(\mathbf{k}\cdot\mathbf{V_{1,2}}) = -\mathrm{i}\frac{q_{1,2}\mathbf{E}}{m_{1,2}}\,, \tag{4.49b}$$

while the electric field is described by the Gauss law

$$\mathrm{i}\mathbf{k}\cdot\mathbf{E} = 4\pi(n_1q_1 + n_2q_2)\,. \tag{4.50}$$

In the last equation we assumed that the system was initially completely charge neutral. From equation (4.49b) the first orde velocity can be immediately extracted

$$\mathbf{v_{1,2}} = \mathrm{i}\frac{q_{1,2}\mathbf{E}}{m_{1,2}(\omega - \mathbf{k}\cdot\mathbf{V_{1,2}})}\,. \tag{4.51}$$

We can introduce this in equation (4.49a) to obtain also the density

$$n_{1,2} = \mathrm{i}\frac{q_{1,2}N_{1,2}\mathbf{k}\cdot\mathbf{E}}{m_{1,2}(\omega - \mathbf{k}\cdot\mathbf{V_{1,2}})^2}\,. \tag{4.52}$$

This dependence showcases that the buncing phenomenon depends on the zero order velocity. Now, we substitute this in equation (4.50) to obtain

$$\mathbf{k}\cdot\mathbf{E} = 4\pi\mathbf{k}\cdot\mathbf{E}\left(\frac{q_1^2N_1}{m_1(\omega - \mathbf{k}\cdot\mathbf{V_1})^2} + \frac{q_2^2N_2}{m_2(\omega - \mathbf{k}\cdot\mathbf{V_2})^2}\right)\,, \tag{4.53}$$

or, rather

$$\frac{q_1^2N_1}{m_1(\omega - \mathbf{k}\cdot\mathbf{V_1})^2} + \frac{q_2^2N_2}{m_2(\omega - \mathbf{k}\cdot\mathbf{V_2})^2} = \frac{1}{4\pi}\,. \tag{4.54}$$

This equation is hareder to solve than one might guess from a first look. In order to get a feeling for how it behaves, I restric the discussion to analyzing the particular case in which we have streams of eaqual strengths, *i.e.* that have the product of the total charge and the charge-to-mass ratio equal

$$N_1\frac{q_1^2}{m_1} = N_2\frac{q_2^2}{m_2} = \alpha\,. \tag{4.55}$$

An easy way to see why the terms in this equality characterize the strength of the beam is the following: suppose we look at how a beam's particles interact with themselves and we choose one while all the others are concentrated to one point. The acceleration that the lonely particle will get from the bunch of particles is proportional to $(N_1 - 1)\frac{q_1^2}{m_1} \approx N_1\frac{q_1^2}{m_1}$ for large N_1. We can say that this quantity reflects the maximum amount of acceleration that can be gained through self-interaction. Sure, since we also assumed the total system to be charge neutral, what we really equate is just the charge-to-mass ratio, but we should bear in mind that this is not a general occurance, but rather a particular simplification in the case of just two beams. For more than two, the condition needs to be written similarly to equation (4.55).

We now get a biquadratic equation

$$\frac{1}{(\omega - \mathbf{k}\cdot\mathbf{V_1})^2} + \frac{1}{(\omega - \mathbf{k}\cdot\mathbf{V_2})^2} = \frac{1}{4\pi\alpha}\,. \tag{4.56}$$

The solution follows easily after some arithmetics

$$(\omega - \mathbf{k} \cdot \langle \mathbf{V} \rangle)^2 = \omega_{p0}^2 \left(\frac{1}{2} + x^2 \pm \frac{\sqrt{1 + 8x^2}}{2} \right) , \tag{4.57}$$

where

$$\omega_{p0} = 4\pi\alpha \tag{4.58a}$$

$$\langle \mathbf{V} \rangle = \frac{\mathbf{V_1} + \mathbf{V_2}}{2} \tag{4.58b}$$

$$x = \frac{\mathbf{k} \cdot (\mathbf{V_1} - \mathbf{V_2})}{2\omega_{p0}} . \tag{4.58c}$$

A quick look at figure 4.4 shows us that both solutions are stable for $|x| > 1$, that is for the case in which the zero order velocity of one stream is much larger than that of the other beam (on a scale of ω_{p0}, which is quite large for large N). The domain of iterest in which the intability appears is $|x| > 1$. Here, the plus-sign solution gives the dispersion relation for longitudinal plasma waves. The minus-sign solution gives an unstable mode. In the simulation we see both these effect together. The instability evolves much slower than the longitudinal mode formation, so that is why in figures 4.2 and 4.3 we first see the formation of some plasma waves. But at some point the instability overtakes the oscillation and we see the formation of the characteristic cat-eye structures. An important thing to have in mind is that the growth rate of instability (which is derived from the imaginary part of the frequency in the minus-sign solution) is proportional to the wavelength.

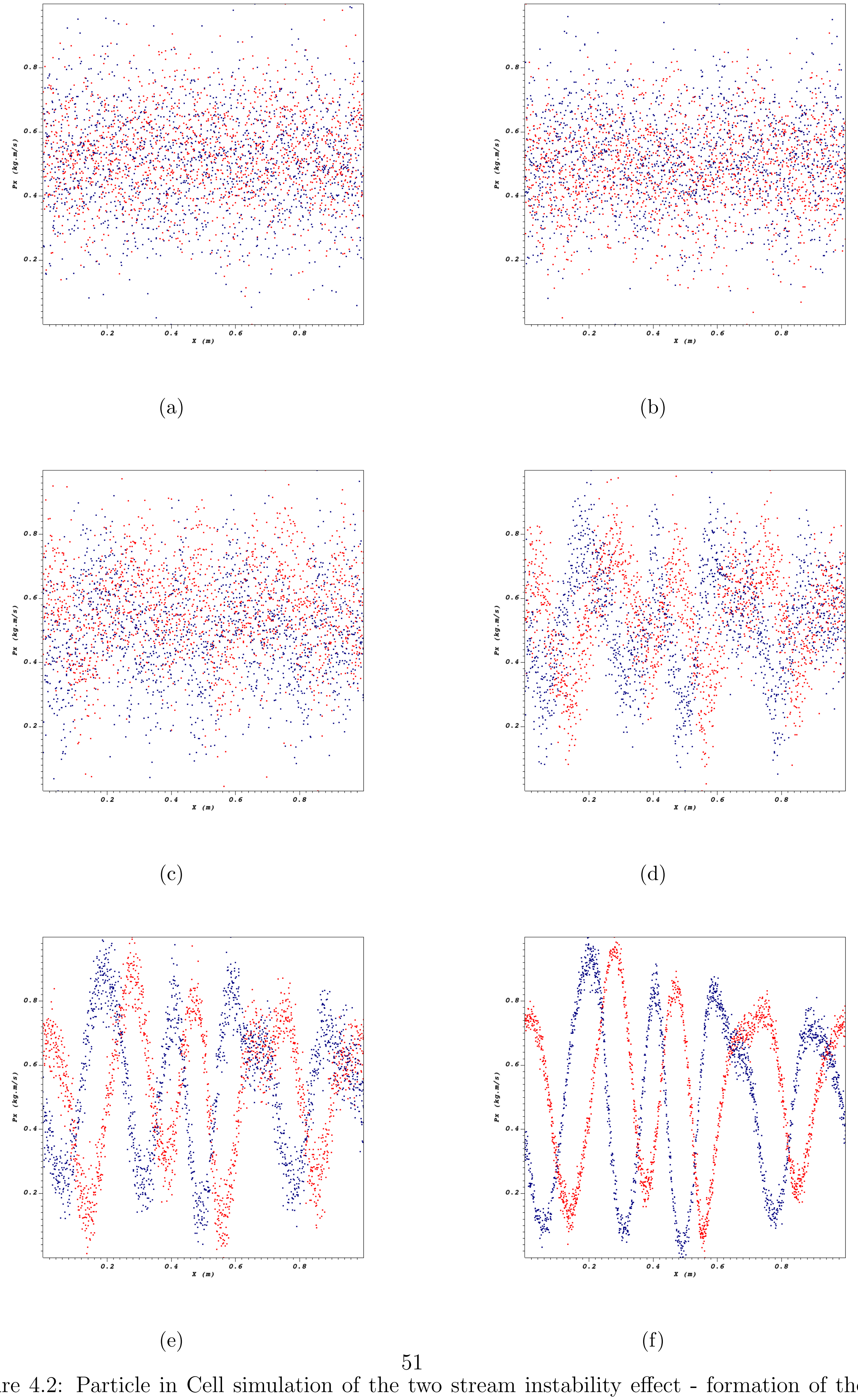
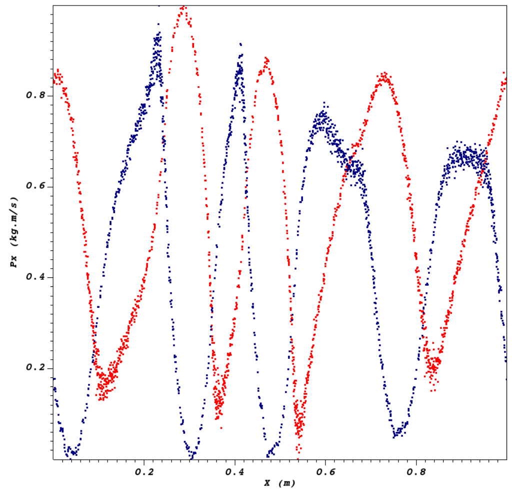
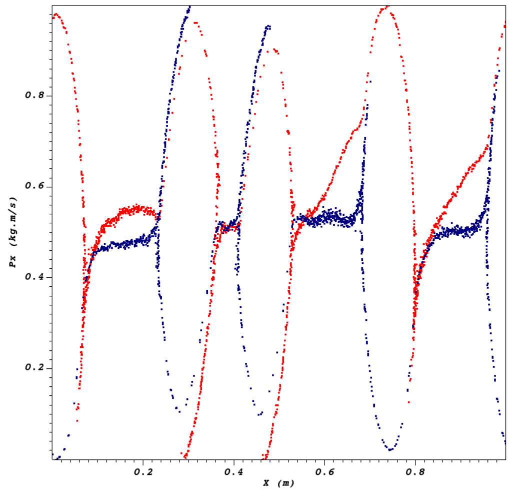


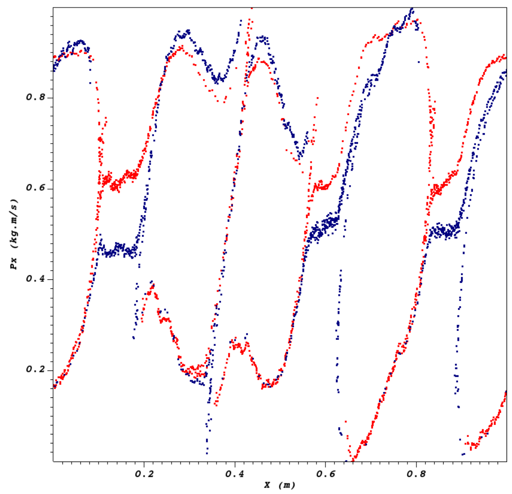
Figure 4.2: Particle in Cell simulation of the two stream instability effect - formation of the waves



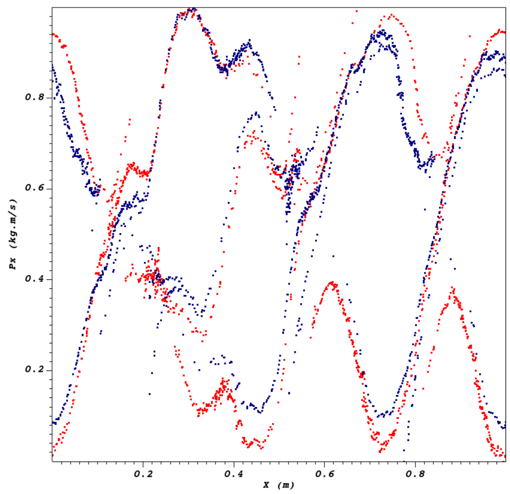
(a)



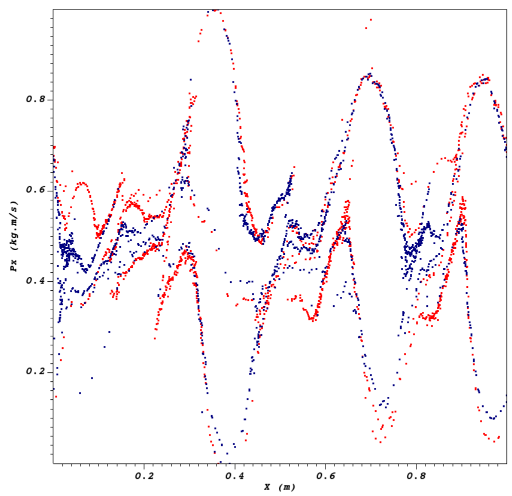
(b)



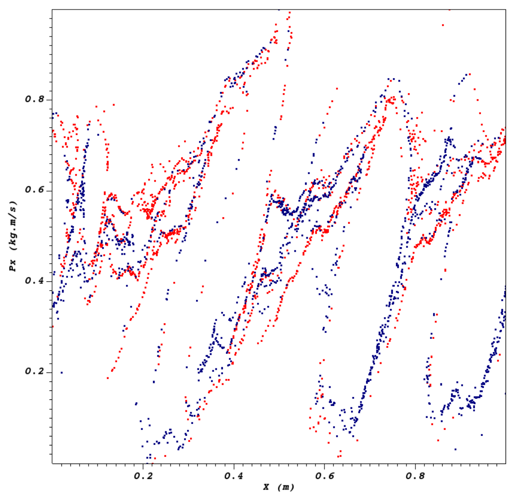
(c)



(d)



(e)



(f)

Figure 4.3: Particle in Cell simulation of the two stream instability effect - continuation

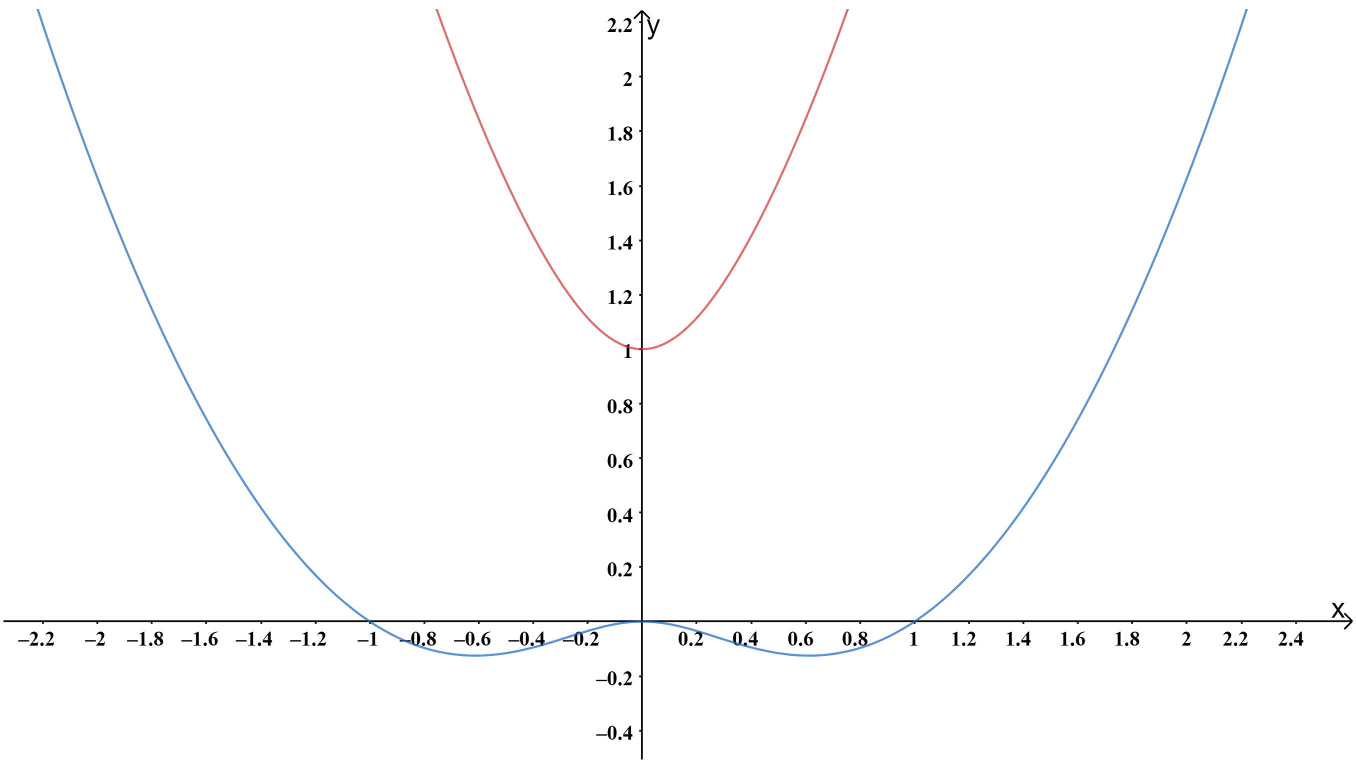


Figure 4.4: A plot of the quantity $\left(\frac{1}{2}+x^2 \pm \frac{\sqrt{1+8x^2}}{2}\right)$. With red is the expression with the plus sign, while with blue is the expression with the minus sign.

4.8 Laser Wakefield Acceleration

While there have been many proposals for electron accelerators that use electromagnetic radiation, the most promising and reliable one is that by Tajima and Dawson 1979. The mechanism is the following (Mangles 2020): One shoots an intense laser pulse towards an underdense plasma. Since the ions are much heavier than the electrons, they will react much slower to the pulse. Thus, the drive pulse will push away electrons (mainly by the ponderomotive force), leaving behind the positively charged particles. At the same time, the positive charge accumulation will pull back the electrons. While returning, the electrons will overshoot, creating a plasma wave. Some electrons will be trapped in the wakefield of the pulse and will be pulled by it. A strong enough drive beam can expell all the electrons from the region in which the pulse is found. This situation corresponds to a so called blow-out regime. A bubble of just positive charges is formed and follows the laser. The high intensity laser will produce ionization in this bubble and thus electrons will be "injected" in it. This way the amount of accelerated electrons can be increased.

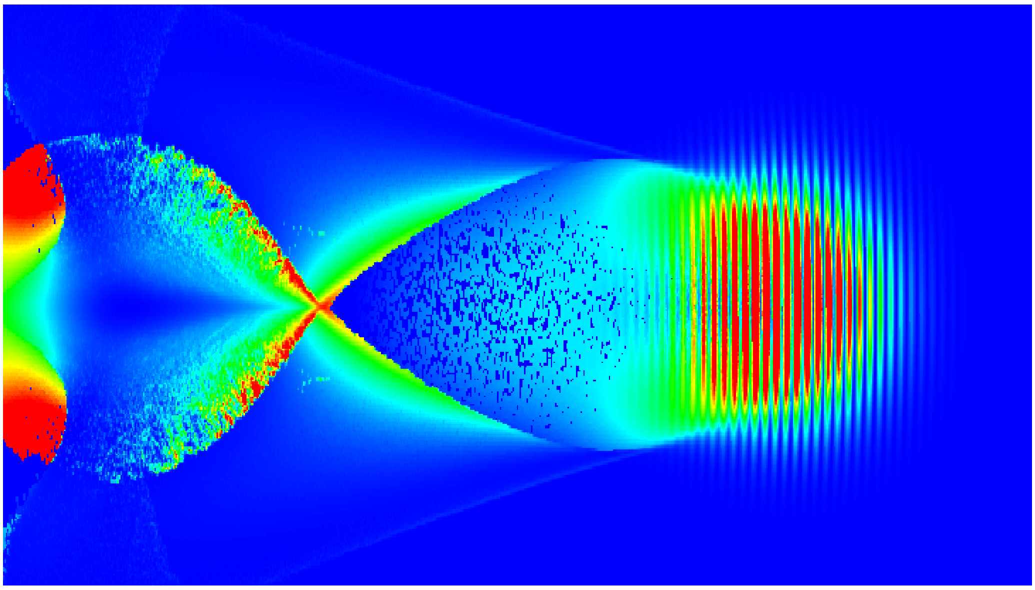


Figure 4.5: An illustrative picture of the laser wakefield acceleration effect obtained from a 3D particle in cell simulation showing the pseudocolor plot of the electron energy density inside the plasma. The laser pulse (modeled as a Gaussian pulse in this case) was traveling from left to right.

In figure 4.5 we can see to the right the profile of the laser beam since that region shows the electrons that are currently interacting with the pulse. We can also see quite clearly the formation of the plasma wave. The simulation was done with la low intensity laser, so we can see that the bubble regions are not completely empty in this case.

Classical particle accelerators can only transfer a certain amount of energy per meter. Because of this, in order to obtain high kinetic energies, their size must be very large. In a plasma based accelerator, the transfer of energy is much faster over small distances. Because of this they are of great interest especially for medical applications. Obviously, laser wake field acceleration is not perfect. It is limited by three effects: dephasing, pump deplition and diffraction. A concise and useful description of these limitations can be found in ONeil 2017.

While, the mechanism is well understood, there is much work to be done in order to optimize it, mostly related to injection, the stability of the beam, and the understanding and exploitation of phemomena that arise at higher energies inside the plasma.

Chapter 5

Numerical Methods and Particle in Cell Simulations

Chapter 6

Results

In this chapter we present the main results ...

Chapter 7

Conclusions

In conclusion . . .

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