3. A particle moves in the x–y plane so that

$$x(t) = at, \qquad y(t) = bt^2,$$

where a, b are positive constants. Find its trajectory, velocity, and acceleration (its tangential and normal components).

4. The velocities of two particles observed from a fixed frame of reference are given in the Cartesian coordinates by vectors $\mathbf{v}_1(t) = (0, 2, 0) + (3, 1, 2)t^2$ and $\mathbf{v}_2(t) = (1, 0, 1)$. At the initial instant of time $t = 0$, the positions of these particles are $\mathbf{r}_1(0) = (1, 0, 0)$ and $\mathbf{r}_2(0) = (0, 1, 1)$.

   Find: the positions of both particles and the acceleration of particle 1 (and its tangential and normal components), relative position, and relative acceleration of particle 1 with respect to particle 2 at any instant of time t.

5. (*solution provided*) A disc of radius R rotates about its axis of symmetry (perpendicular to the disk surface) with constant angular velocity $\dot{\varphi} = \omega = \text{const}$. At the instant of time $t = 0$ a beetle starts to walk with constant speed v_0 along a radius of the disk, from its center to the edge. Find

   (a) the position of the beetle and its trajectory in the Cartesian and polar coordinate systems,

   (b) its velocity both systems,

   (c) its acceleration in both systems (Cartesian components, polar components, as well as tangential and normal components).

6. A particle moves along a hyperbolic spiral (i.e. a curve $r = c/\varphi$, where c is a positive constant), so that $\varphi(t) = \varphi_0 + \omega t$, where φ_0 and ω are positive constants. Find its velocity and acceleration (all components and magnitudes of both vectors).

7. For the situation discussed in problem 5 answer the following questions.

   (a) What is the distance covered by the beetle (write down the integral only, do not evaluate it)?
   Answer: $s = \int_0^T v\,\mathrm{d}t = \int_0^T \sqrt{\dot{x}^2 + \dot{y}^2}\,\mathrm{d}t$, where $T = R/v_0$ is the time needed to travel from the center to the edge of the disc.

   (b) What is the curvature of the trajectory?
   Answer: $R_c = \frac{v^2}{a_n} = \frac{v_0(1+\omega^2 t^2)^{3/2}}{\omega(2+\omega^2 t^2)}$ or as a function of the polar angle $R_c = \frac{v_0(1+\varphi^2)^{3/2}}{\omega(2+\varphi^2)}$.

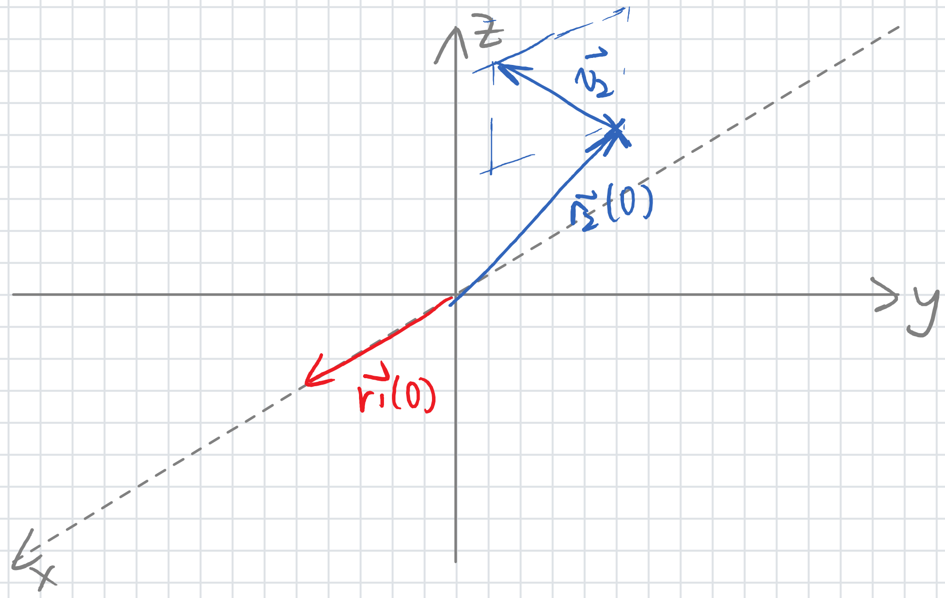
8. Four spiders are initially placed at the four corners of a square with side length a. The spiders crawl counter-clockwise at the same speed and each spider crawls directly toward the next spider at all times. They approach the center of the square along spiral paths. Find

   (a) polar coordinates of a spider at any instant of time, assuming the origin is at the center of the square.

   (b) the time after which all spiders meet,

   (c) the trajectory of a spider in polar coordinates.

Problem 4



$$\vec{r_1} = \vec{r_1}(0) + \int_0^t v_1(t)\,dt$$

$$= \begin{pmatrix}1\\0\\0\end{pmatrix} + \int_0^t \begin{pmatrix}3t^2\\2+t^2\\2t^2\end{pmatrix} dt$$

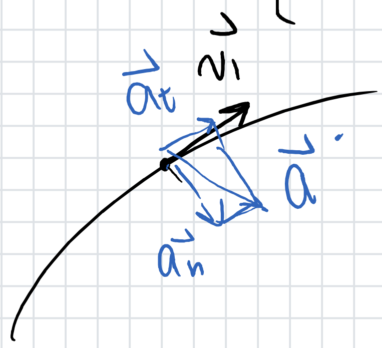
$$= \begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}3\cdot\frac{1}{3}t^3\\2t+\frac{1}{3}t^3\\2\cdot\frac{1}{3}t^3\end{pmatrix}$$

$$= \begin{pmatrix}1+t^3\\2t+\frac{1}{3}t^3\\\frac{2}{3}t^3\end{pmatrix}$$

$$\vec{v_1} = \dot{\vec{r_1}} \qquad \vec{a_1} = \dot{\vec{v_1}} = \ddot{\vec{r_1}}$$

$$\vec{a_1} = \frac{d}{dt}\vec{v_1} = \frac{d}{dt}\left[\begin{pmatrix}0\\2\\0\end{pmatrix} + \begin{pmatrix}3\\1\\2\end{pmatrix}t^2\right] = 2\begin{pmatrix}3\\1\\2\end{pmatrix}t = \begin{pmatrix}6\\2\\4\end{pmatrix}t^2$$

$$\vec{r_2} = r_2(0) + \int_0^t v_2(t)\,dt$$
$$= \begin{pmatrix}0\\1\\1\end{pmatrix} + \begin{pmatrix}1\\0\\1\end{pmatrix}t$$



$$\vec{a_t} = \frac{\vec{a_1}\cdot\vec{v_1}}{v_1}\ \frac{\vec{v_1}}{v_1}$$

$$\vec{a_n} = \vec{a} - \vec{a_t}$$

$$\vec{r_{12}} = \vec{r_1} - \vec{r_2} = \begin{pmatrix}1+t^3\\2t+\frac{1}{3}t^3\\\frac{2}{3}t^3\end{pmatrix} - \begin{pmatrix}t\\1\\t+1\end{pmatrix} = \begin{pmatrix}t^3-t+1\\\frac{1}{3}t^3+2t-1\\\frac{2}{3}t^3-t-1\end{pmatrix}$$

$$\vec{v_{12}} = \vec{v_1} - \vec{v_2} = \begin{pmatrix}0\\2\\0\end{pmatrix} + \begin{pmatrix}3\\1\\2\end{pmatrix}t^2 - \begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}-1\\2\\-1\end{pmatrix} + \begin{pmatrix}3\\1\\2\end{pmatrix}t^2$$

$$\vec{a_{12}} = \frac{d}{dt}\vec{v_{12}} = 2\begin{pmatrix}3\\1\\2\end{pmatrix}t = \begin{pmatrix}6\\2\\4\end{pmatrix}t$$

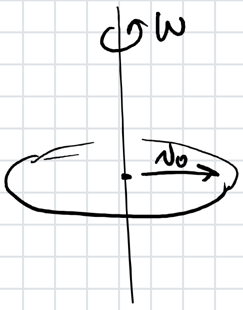
Problem 5

阿基米德螺线

A rotating plane

$\downarrow$

polar Coordinate



a) $\dot{\varphi} = \omega$. $(\varphi = \omega t$

$\dot{r} = v_0$. $|\ r = v_0 t$

$$\boxed{r = v_0 \cdot \frac{\varphi}{\omega}}$$

$$\begin{cases} x = r\cos\varphi = v_0 t \cos\omega t \\ y = r\sin\varphi = v_0 t \sin\omega t \end{cases}$$

param etric equ

b). $\vec{v} = \dot{\vec{r}} = \frac{d}{dt}(r\hat{n}_r) = \dot{r}\hat{n}_r + r\cdot\dot{\hat{n}}_r = \dot{r}\hat{n}_r + r\cdot\dot{\varphi}\hat{n}_\varphi = v_0\hat{n}_r + v_0 t\cdot\omega\hat{n}_\varphi$

$\vec{v} = \dot{\vec{r}} = \dot{x}\hat{n}_x + \dot{y}\hat{n}_y = (v_0\cos\omega t + v_0 t\,\omega(-\sin\omega t))\hat{n}_x + (v_0\sin\omega t + v_0 t\,\omega\cos\omega t)\hat{n}_y$

$= (v_0\cos\omega t - v_0 t\omega\sin\omega t)\hat{n}_x + (v_0\sin\omega t + v_0 t\omega\cos\omega t)\hat{n}_y$

c). Cartansian :

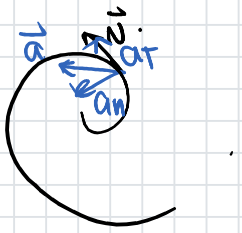
Polar : $\vec{a} = \dot{\vec{v}} = v_0\dot{\hat{n}}_r + v_0\omega\hat{n}_\varphi + v_0 t\omega\dot{\hat{n}}_\varphi$

$= v_0\cdot\dot{\varphi}\hat{n}_\varphi + v_0\omega\hat{n}_\varphi + v_0 t\omega(-\dot{\varphi}\hat{n}_r)$

$= 2v_0\omega\hat{n}_\varphi - v_0 t\omega^2\hat{n}_r$

Nature :

$v = \sqrt{v_0^2 + (v_0 t\omega)^2} = v_0\sqrt{\omega^2 t^2 + 1}$



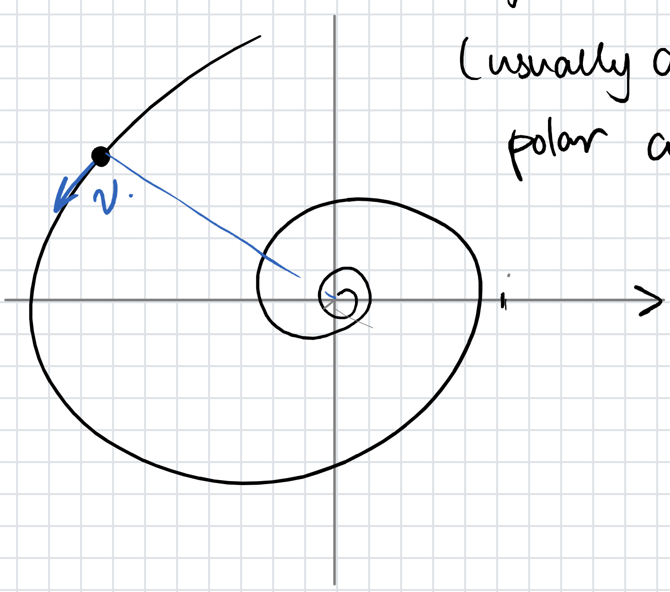
$a_\tau = \frac{dv}{dt} = \frac{d}{dt} v_0\sqrt{\omega^2 t^2 + 1} = v_0\cdot\frac{1}{2}(\omega^2 t^2 + 1)^{-\frac{1}{2}}\cdot(\omega^2\cdot 2t) = \frac{\omega^2 t v_0}{\sqrt{\omega^2 t^2 + 1}}$

$a_n = \sqrt{a^2 - a_\tau^2}$

Problem 6

hyperbolic spiral

(usually described under polar coordinates)



$r(\varphi) = \frac{c}{\varphi}$. $\varphi(t) = \varphi_0 + \omega t$ $r(t) = \frac{c}{\varphi_0 + \omega t}$

$\vec{v} = \dot{\vec{r}} = \dot{r}\hat{n}_r + r\dot{\hat{n}}_r = \dot{r}\hat{n}_r + r\cdot\dot{\varphi}\hat{n}_\varphi$

$= -c(\varphi_0 + \omega t)^{-2}\cdot\omega\hat{n}_r + r\omega\hat{n}_\varphi$

$= \frac{-c\omega}{(\varphi_0 + \omega t)^2}\hat{n}_r + \frac{c}{\varphi_0 + \omega t}\omega\hat{n}_\varphi$

$\vec{a} = \dot{\vec{v}} = \frac{d}{dt}\left(\frac{-c\omega}{(\varphi_0 + \omega t)^2}\right)\hat{n}_r + \frac{-c\omega}{(\varphi_0 + \omega t)^2}\cdot\dot{\varphi}\hat{n}_\varphi$

$+ \frac{d}{dt}\left(\frac{c\omega}{\varphi_0 + \omega t}\right)\cdot\hat{n}_\varphi + \frac{c\omega}{\varphi_0 + \omega t}\cdot(-\dot{\varphi}\hat{n}_r)$

Problem 8.