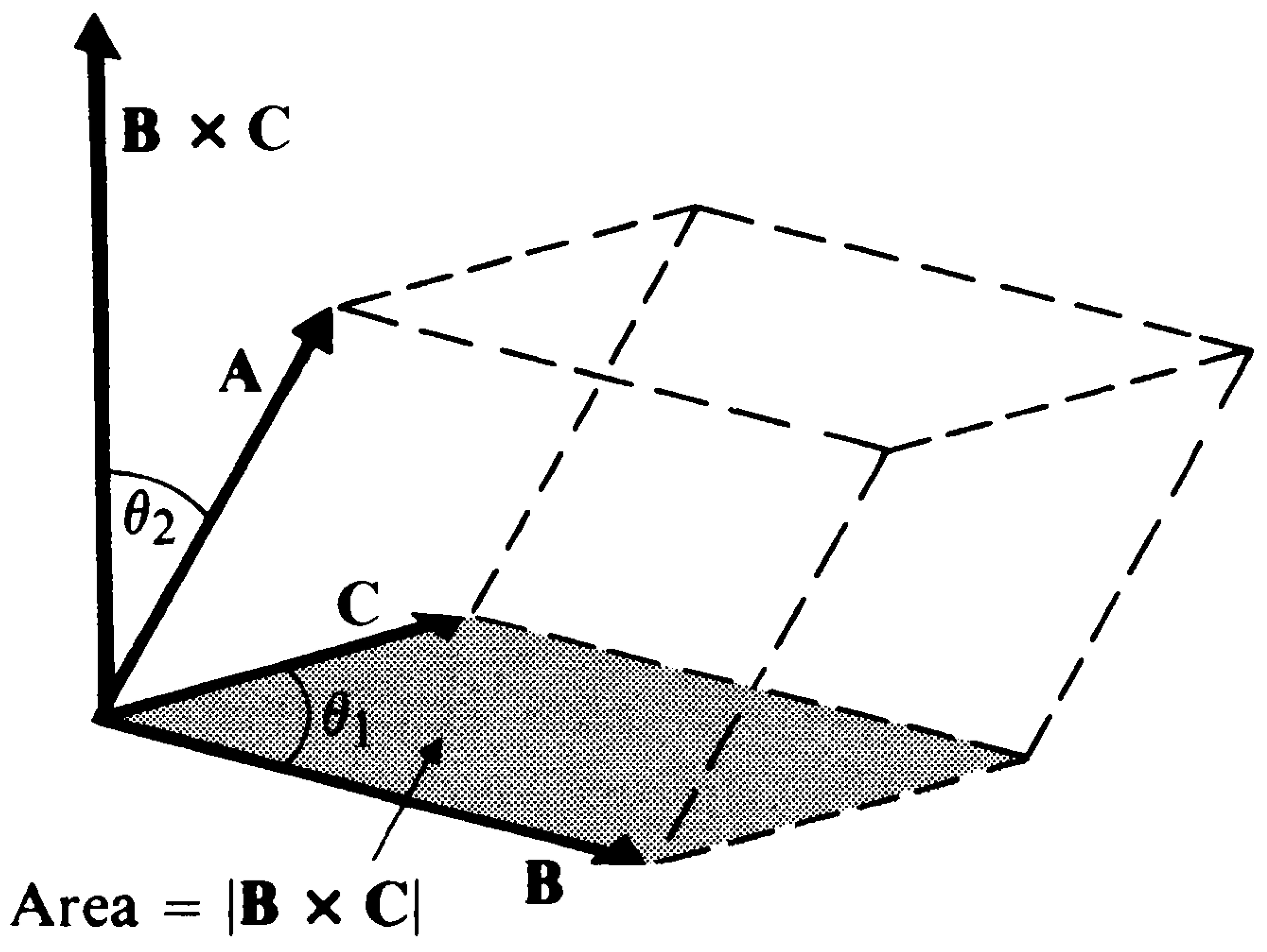

FIGURE 2-8
Illustrating scalar triple product $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$.

2-3.3 PRODUCT OF THREE VECTORS

There are two kinds of products of three vectors; namely, the ***scalar triple product*** and the ***vector triple product***. The scalar triple product is much the simpler of the two and has the following property:

$$\boxed{\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}).} \tag{2-18}$$

Note the cyclic permutation of the order of the three vectors **A**, **B**, and **C**. Of course,

$$\begin{aligned} \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) &= -\mathbf{A} \cdot (\mathbf{C} \times \mathbf{B}) \\ &= -\mathbf{B} \cdot (\mathbf{A} \times \mathbf{C}) \\ &= -\mathbf{C} \cdot (\mathbf{B} \times \mathbf{A}). \end{aligned} \tag{2-19}$$

As can be seen from Fig. 2-8, each of the three expressions in Eq. (2-18) has a magnitude equal to the volume of the parallelepiped formed by the three vectors **A**, **B**, and **C**. The parallelepiped has a base with an area equal to $|\mathbf{B} \times \mathbf{C}| = |BC \sin \theta_1|$ and a height equal to $|A \cos \theta_2|$; hence the volume is $|ABC \sin \theta_1 \cos \theta_2|$.

The vector triple product $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ can be expanded as the difference of two simple vectors as follows:

$$\boxed{\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}).} \tag{2-20}$$

Equation (2-20) is known as the "***back-cab*** ***rule***" and is a useful vector identity. (Note "BAC-CAB" on the right side of the equation!)

EXAMPLE 2-3† Prove the back-cab rule of vector triple product.

† The back-cab rule can be verified in a straightforward manner by expanding the vectors in the Cartesian coordinate system (Problem P.2-12). Only those interested in a general proof need to study this example.

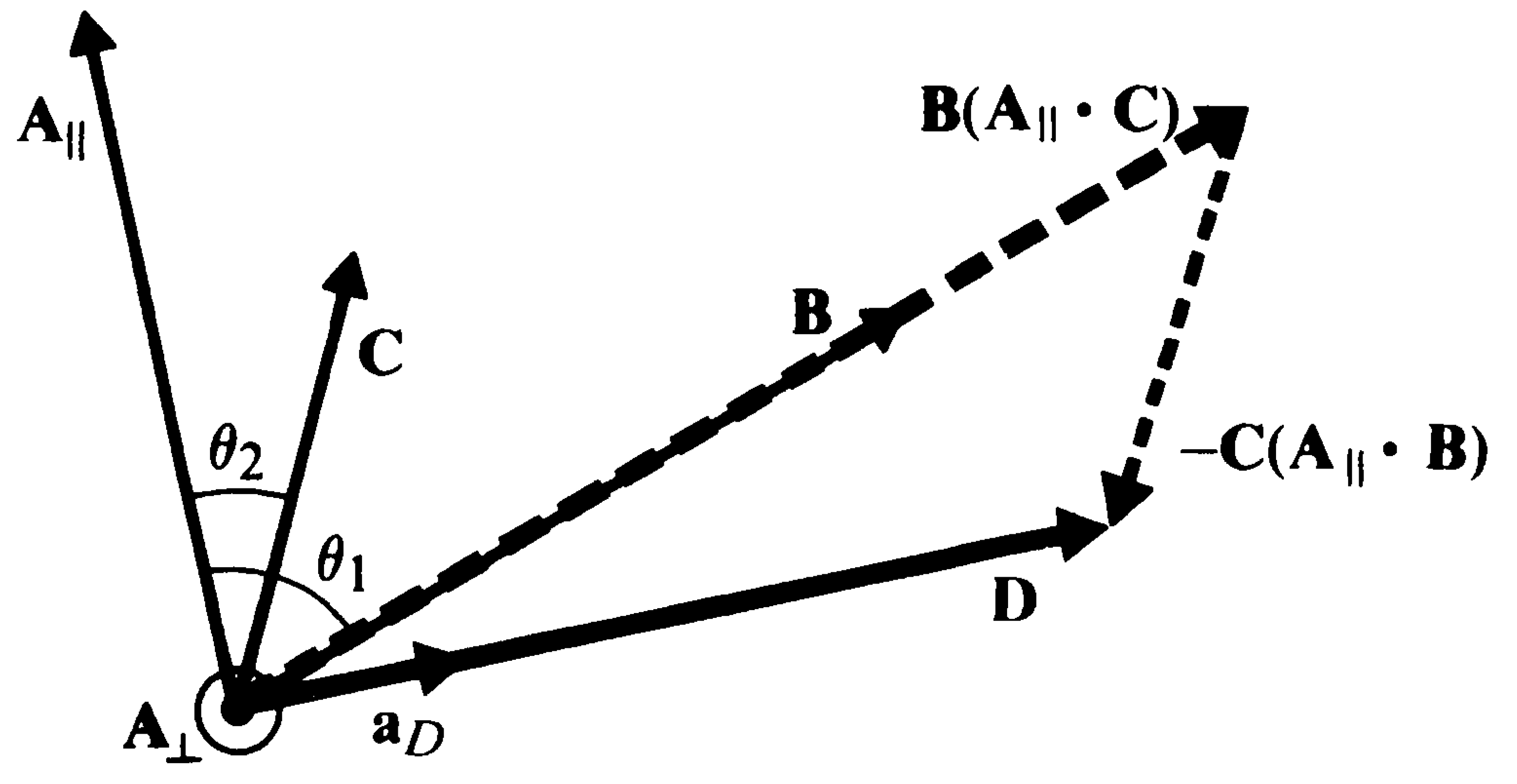

FIGURE 2–9
Illustrating the back-cab rule of vector triple product.

Solution In order to prove Eq. (2–20) it is convenient to expand **A** into two components:

$$\mathbf{A} = \mathbf{A}_{||} + \mathbf{A}_{\perp},$$

where $\mathbf{A}_{||}$ and $\mathbf{A}_{\perp}$ are parallel and perpendicular, respectively, to the plane containing **B** and **C**. Because the vector representing $(\mathbf{B} \times \mathbf{C})$ is also perpendicular to the plane, the cross product of $\mathbf{A}_{\perp}$ and $(\mathbf{B} \times \mathbf{C})$ vanishes. Let $\mathbf{D} = \mathbf{A} \times (\mathbf{B} \times \mathbf{C})$. Since only $\mathbf{A}_{||}$ is effective here, we have

$$\mathbf{D} = \mathbf{A}_{||} \times (\mathbf{B} \times \mathbf{C}).$$

Referring to Fig. 2–9, which shows the plane containing **B**, **C**, and $\mathbf{A}_{||}$, we note that **D** lies in the same plane and is normal to $\mathbf{A}_{||}$. The magnitude of $(\mathbf{B} \times \mathbf{C})$ is $BC \sin(\theta_1 - \theta_2)$, and that of $\mathbf{A}_{||} \times (\mathbf{B} \times \mathbf{C})$ is $A_{||}BC \sin(\theta_1 - \theta_2)$. Hence,

$$\begin{aligned} D = \mathbf{D} \cdot \mathbf{a}_D &= A_{||}BC \sin(\theta_1 - \theta_2) \\ &= (B \sin \theta_1)(A_{||}C \cos \theta_2) - (C \sin \theta_2)(A_{||}B \cos \theta_1) \\ &= [\mathbf{B}(\mathbf{A}_{||} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A}_{||} \cdot \mathbf{B})] \cdot \mathbf{a}_D. \end{aligned}$$

The expression above does not alone guarantee the quantity inside the brackets to be **D**, since the former may contain a vector that is normal to **D** (parallel to $\mathbf{A}_{||}$); that is, $\mathbf{D} \cdot \mathbf{a}_D = \mathbf{E} \cdot \mathbf{a}_D$ does not guarantee $\mathbf{E} = \mathbf{D}$. In general, we can write

$$\mathbf{B}(\mathbf{A}_{||} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A}_{||} \cdot \mathbf{B}) = \mathbf{D} + k\mathbf{A}_{||},$$

where k is a scalar quantity. To determine k, we scalar-multiply both sides of the above equation by $\mathbf{A}_{||}$ and obtain

$$(\mathbf{A}_{||} \cdot \mathbf{B})(\mathbf{A}_{||} \cdot \mathbf{C}) - (\mathbf{A}_{||} \cdot \mathbf{C})(\mathbf{A}_{||} \cdot \mathbf{B}) = 0 = \mathbf{A}_{||} \cdot \mathbf{D} + kA_{||}^2.$$

Since $\mathbf{A}_{||} \cdot \mathbf{D} = 0$, then $k = 0$ and

$$\mathbf{D} = \mathbf{B}(\mathbf{A}_{||} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A}_{||} \cdot \mathbf{B}),$$

which proves the back-cab rule, inasmuch as $\mathbf{A}_{||} \cdot \mathbf{C} = \mathbf{A} \cdot \mathbf{C}$ and $\mathbf{A}_{||} \cdot \mathbf{B} = \mathbf{A} \cdot \mathbf{B}$. ■

Division by a vector is not defined, and expressions such as $k/\mathbf{A}$ and $\mathbf{B}/\mathbf{A}$ are meaningless.