Convergence in real function. $f: \mathbb{R} \to [0,1]$

① pointwise

$$f_n(x) \to f(x) \quad \forall\, x \in [0,1]$$

② Convergence in norm. $\|\cdot\| \quad V \to [0,\infty)$

Space of all fun. on $[0,1]$

$f_n \to f$ in $\|\cdot\|$

if $\|f_n - f\| \to 0$

eg. $\|\ \| = \|\cdot\|_p$

$$\left(\int |f_n(x) - f(x)|^p \, dx \right)^{\frac{1}{p}} \to 0$$

$\|\ \|_\infty$.

$$\sup_{x \in [0,1]} |f_n(x) - f(x)| \to 0$$

$\|\ \|_1$

$$\int |f_n(x) - f(x)| \, dx \to 0$$

③. Convergence in Lebesgue measure.

$$\forall \varepsilon > 0 \quad \mu(\{x \in [0,1] : |f_n(x) - f(x)| > \varepsilon\}) \to 0$$

$X_n \quad X : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}), \cdot)$

①. a.s. convergence. $X_n \xrightarrow{a.s.} X$

Relaxation of Pointwise convergence.

$$\mathbb{P}(\{\omega \in \Omega : X_n(\omega) \not\to X(\omega)\}) = 0.$$

$$\Updownarrow$$

$$\mathbb{P}(\{\omega \in \Omega : X_n(\omega) \to X(\omega)\}) = 1$$

If sth. has prob $= 0$

we can omit it in

prob. Theory

② Convergence in r-th. mean. $\Leftrightarrow$ L^r norm

$X_n \xrightarrow{r} X$

$$\int |X_n(\omega) - X(\omega)|^r \, d\mathbb{P}(\omega) \to 0 \qquad r \geq 1$$

$$\Updownarrow$$

$$\mathbb{E}|X_n - X|^r$$

③ Convergence in prob. $X_n \xrightarrow{P} X$

$\mathbb{P}(\{\omega : |X_n(\omega) - X(\omega)| > \varepsilon\}) \to 0 \quad \forall \varepsilon > 0$

④. Convergen in distribution. $X_n \xrightarrow{D} X$ $\quad (X_n \xrightarrow{d} X)$

(weak convergence) $\quad F_{X_n} \quad F_X \quad X_n \Rightarrow X$

real c.d.f. pointwise for those continuity point of $F_X(x)$

$F_{X_n}(x) \to F_X(x)$ for continuity point x of $F_X(x)$

$$\Updownarrow$$

$X_n \xrightarrow{D} F_X$

eg. $X_n = \frac{1}{n}$ $\qquad F_{X_n}(x) = \begin{cases} 0 & \text{if } x < \frac{1}{n} \\ 1 & \text{if } x \geq \frac{1}{n} \end{cases}$

$Y_n = -\frac{1}{n}$ $\qquad F_{Y_n}(x) = \begin{cases} 0 & \text{if } x < -\frac{1}{n} \\ 1 & \text{if } x \geq -\frac{1}{n} \end{cases}$

$F_{X_n}(x) \to \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$ $\qquad F_{Y_n}(x) \to \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases} \to Y \equiv 0$

Not a d.f. $\qquad\qquad$ a d.f.

△ The first three convergence mode have to be defined on the same prob space

eg. X_1 is die rolling X_2 is coin flipping X_2 ✗

while convergence in distribution is ok for the above.

Thm.

① $(X_n \xrightarrow{a.s.} X) \Rightarrow$
$(X_n \xrightarrow{r} X) \Rightarrow$ $(X_n \xrightarrow{P} X) \Rightarrow (X_n \xrightarrow{D} X)$

② $(X_n \xrightarrow{r} X) \Rightarrow (X_n \xrightarrow{s} X)$ if $r > s \geq 1$

③ Converse statement are not true in general.

Proof of ③

1° $(X_n \xrightarrow{D} X) \not\Rightarrow (X_n \xrightarrow{\mathbb{P}} X)$ $P(|Y_n - Y| > \varepsilon) \to 0$

$\Updownarrow$

$F_{X_n}(x) \to F_X(x)$. $\mathbb{P}(\{w \in \Omega : |X_n(w) - X(w)| > \varepsilon\}) \to 0$ $\forall \varepsilon > 0$

$\Omega = \{H, T\}$

$$X(w) \equiv X_{2n}(w) = \begin{cases} 1 & \text{if } w = H \\ 0 & \text{if } w = T \end{cases}$$

$$X_{2n+1}(w) = \begin{cases} 1 & \text{if } w = T \\ 0 & \text{if } w = H \end{cases}$$

coin is fair

Bernoulli r.v. with $p = \frac{1}{2}$.

$\forall w$ $\mathbb{P}(|X_{2n+1}(w) - X_n(w)| = 1) = 1$

$2°$ $(X_n \xrightarrow{\mathbb{P}} X) \not\Rightarrow (X_n \xrightarrow{L^r} X)$

wrt. $\mathbb{P}$ wrt P

not wrt size of $|X_n - X|$ & the size of $|X_n - X|$

$$X_n = \begin{cases} n & \text{with. prob. } \frac{1}{n} \\ 0 & \cdots \quad 1-\frac{1}{n} \end{cases} \qquad X \equiv 0$$

$$\mathbb{P}(|X_n - X| > \varepsilon) = \frac{1}{n} \to 0$$

$$\mathbb{E}|X_n - X| = n \cdot \frac{1}{n} + 0 \cdot (1-\frac{1}{n}) = 1 \not\to 0$$

If. $s<r$ $(X_n \xrightarrow{L^s} X) \not\Rightarrow (X_n \xrightarrow{L^r} X)$

$$\mathbb{E}|X_n - X|^s = \int |Y_n|^s \, d\mu_{Y_n} = \int |y|^s f_{Y_n}(y)\, dy \quad ①$$

$$\mathbb{E}|X_n - X|^r \quad \cdots \quad = \int |y|^r f_{Y_n}(y)\, dy \quad ②$$

$f_{Y_n}(y)$ is small enough for ① but not for ②

Choose

$$X_n = \begin{cases} n & \text{with. prob } n^{-\frac{r+s}{2}} \\ 0 & \text{with. prob. } 1-n^{-\frac{r+s}{2}} \end{cases} \qquad X \equiv 0$$

$$\mathbb{E}|X_n - X|^s = n^s \cdot n^{-\frac{r+s}{2}} + 0 \cdot (1-n^{-\frac{r+s}{2}})$$

$$= n^{\frac{s-r}{2}} \to 0 \qquad n \to \infty$$

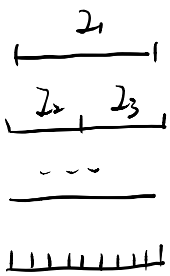
$$\mathbb{E}|X_n - X|^r = n^r \cdot n^{-\frac{r+s}{2}} + 0 \cdot (1-n^{-\frac{r+s}{2}})$$

$$= n^{\frac{r-s}{2}} \to \infty \qquad n \to \infty.$$

$(X_n \xrightarrow{\mathbb{P}} X) \not\Rightarrow (X_n \xrightarrow{a.s} X)$

$\Omega = [0,1]$ $\quad \omega \sim \text{Uniform}[0,1]$

$$X_n(\omega) = \begin{cases} 1 & \omega \in I_n. \\ 0 & \text{otherwise} \end{cases}$$



$X \equiv 0 \quad \forall \omega \in [0,1].$

$$\frac{m(m-1)}{2} = n(m)$$

$I_{n(m)+1} \cdots I_{n(m)+m}$. m-th. step.

$X_n \xrightarrow{\mathbb{P}} X$

$\forall \varepsilon > 0 \quad \mathbb{P}(|X_n - X| > \varepsilon) \to 0$

$X_n \xrightarrow{a.s} X$

$\mathbb{P}(X_n \to X) = 1$

Claim. $X_n \xrightarrow{\mathbb{P}} X \qquad X_n \not\xrightarrow{a.s.} X$

$$\mathbb{P}(\{\omega : |X_n(\omega) - X(\omega)| > \varepsilon\}) = \mathbb{P}(I_n) \to 0 \quad n \to \infty.$$

Every step there is a I_{xx} st. $X_{xx}(\omega) = 0$

$\Rightarrow$ infinite many 1

So. For any ω. $X_n \to X$ does not hold.

Δ The def of $\xrightarrow{P}$ only care about the size of bad event but it doesn't care about the relation of bad event

Δ If ω fall into bad event occurs for only finite many times. then $X_n(\omega) \to X(\omega)$.

Δ In this eg. the bad event of X_n together covers. the $[0,1]$ again & again.

Pf of $(X_n \xrightarrow{\mathbb{P}} X) \Rightarrow (X_n \xrightarrow{D} X)$

Aim. $\mathbb{P}(X_n \le x) \to \mathbb{P}(X \le x)$ for all continuity point of $\mathbb{P}(X \le x)$

condition

$$\mathbb{P}(|X_n - X| > \varepsilon) \to 0 \quad \forall \varepsilon > 0$$

$$\begin{aligned}
\mathbb{P}(X_n \le x) &= \mathbb{P}(X_n \le x, |X_n - X| > \varepsilon) \\
&\quad + \mathbb{P}(X_n \le x, |X_n - X| \le \varepsilon) \\
&\le \mathbb{P}(|X_n - X| > \varepsilon) + \mathbb{P}(X_n \le x, X - \varepsilon \le X_n \le X + \varepsilon) \\
&\le \mathbb{P}(|X_n - X| > \varepsilon) + \mathbb{P}(X_n \le x, X \le X_n + \varepsilon) \\
&= \mathbb{P}(|X_n - X| > \varepsilon) + \mathbb{P}(X \le x + \varepsilon) \quad ①
\end{aligned}$$

Switch X_n and X

$$\mathbb{P}(X \le \underbrace{y}_{x-\varepsilon}) \le \mathbb{P}(|X_n - X| > \varepsilon) + \mathbb{P}(X_n \le \underbrace{y+\varepsilon}_{y+\varepsilon = x})$$

$$\mathbb{P}(X \le x - \varepsilon) - \mathbb{P}(|X_n - X| > \varepsilon) \le \mathbb{P}(X_n \le x) \quad ②$$

Discuss liminf limsup

$$\mathbb{P}(X \le x - \varepsilon) - 0 \le \liminf_{n \to \infty} \mathbb{P}(X_n \le x) \le \limsup_{n \to \infty} \mathbb{P}(X_n \le x) \le \mathbb{P}(X \le x + \varepsilon) + 0$$

send $\varepsilon \downarrow 0$

Lemma. $(X_n \xrightarrow{1} X) \Rightarrow (X_n \xrightarrow{\mathbb{P}} X)$

$(X_n \xrightarrow{r} X) \Rightarrow (X_n \xrightarrow{s} X) \quad r \ge s \ge 1$

Markov ineq

$\mathbb{P}(|X_n - X| > \varepsilon) \le \frac{\mathbb{E}|X_n - X|}{\varepsilon} \to 0$

For any r.v. Z any $a > 0$

$\mathbb{P}(|Z| > a) < \frac{\mathbb{E}|Z|}{a}$

Lyapunov ineq

$(\mathbb{E}|Z|^s)^{\frac{1}{s}} \le (\mathbb{E}|Z|^r)^{\frac{1}{r}} \quad r \ge s > 0$

$\Uparrow$

Jensen.

φ is convex Then. $\varphi(\mathbb{E}(X)) \le \mathbb{E}(\varphi(X))$

$X = |Z|^s \quad \varphi = x^{\frac{r}{s}}$

then, $\varphi(X) = |Z|^r$

$\therefore (\mathbb{E}|Z|^s)^{\frac{r}{s}} \le \mathbb{E}|Z|^r$

$\therefore (\mathbb{E}|Z|^s)^{\frac{1}{s}} \le (\mathbb{E}|Z|^r)^{\frac{1}{r}}$

$\Uparrow$ Hölder

$\mathbb{E}|XY| \le (\mathbb{E}|X|^p)^{\frac{1}{p}} (\mathbb{E}|Y|^q)^{\frac{1}{q}}$

choose. $X = Z^s. \quad Y = 1 \quad p = \frac{r}{s} \quad q = \frac{1}{1 - \frac{1}{p}} = \frac{1}{\frac{r-s}{r}} = \frac{r}{r-s}$

$\therefore (\mathbb{E}|Z|^s)^{\frac{1}{s}} \le (\mathbb{E}|X|^{\frac{r}{s}})^{r}$ 上课讲的不对.

$\mathbb{P}(\{\underbrace{\omega : |X_n(\omega) - X(\omega)| > \varepsilon}\}) \to 0 \quad \forall \varepsilon > 0$

$\triangleq A_n(\varepsilon)$ function of both. n and ε.

event of ω.

Δ

Lemma. :

Let $B_m(\varepsilon) = \bigcup_{n=m}^{\infty} A_n(\varepsilon)$ Take the union of the tail part of the event.

① $X_n \xrightarrow{a.s.} X$ iff. $\boxed{\lim_{m\to\infty} \mathbb{P}(B_m(\varepsilon)) = 0 \quad \forall \varepsilon > 0}$ ①

② $X_n \xrightarrow{a.s.} X$ if $\boxed{\sum_{n=1}^{\infty} \mathbb{P}(A_n(\varepsilon)) < \infty \quad \forall \varepsilon > 0}$ ②

③ $(X_n \xrightarrow{a.s.} X) \Rightarrow (X_n \xrightarrow{\mathbb{P}} X)$

$$B_m(\varepsilon) = \bigcup_{n=m} A_n(\varepsilon)$$

subadditivity

$$\mathbb{P}(B_m(\varepsilon)) = \mathbb{P}\left(\bigcup_{n=m} A_n(\varepsilon)\right) \le \sum_{n=m}^{\infty} \mathbb{P}(A_n(\varepsilon))$$

if ② $\Rightarrow$ ① $\Leftrightarrow$ $X_n \xrightarrow{a.s.} X$

So. we only need to prove ①

⚠ If $A_n(\varepsilon)$ overlaps a lot ② may fail though $X_n \xrightarrow{a.s.} X$

③ Because. $A_m(\varepsilon) \subset B_m(\varepsilon)$.

If. $\lim_{m\to\infty} \mathbb{P}(B_m(\varepsilon)) = 0$. $\forall \varepsilon > 0$

of course. $\lim_{m\to\infty} A_m(\varepsilon) = 0 \quad \forall \varepsilon > 0 \Leftrightarrow X_n \xrightarrow{P} X$

Pf of ①

$X_n \xrightarrow{a.s.} X$

$$\Leftrightarrow \mathbb{P}(\underbrace{\{w : X_n(w) \longrightarrow X(w)\}}_{C}) = 1$$

$$C^c = \{w : X_n(w) \not\to X(w)\}$$

$\Leftrightarrow \mathbb{P}(C^c)=0$

if. $\omega \in C$

$X_n(\omega) \to X(\omega) \quad n\to\infty.$

$\forall\, \varepsilon>0. \quad |X_n(\omega)-X(\omega)| > \varepsilon$ for only finite many n.

if $\omega \notin C$.

$\boxed{\exists\, \varepsilon>0}$ st. $|X_n(\omega)-X(\omega)|>\varepsilon$. for infinite many n.

$\therefore\ C^c = \bigcup_{\varepsilon>0} \{\omega: |X_n(\omega)-X(\omega)|>\varepsilon$ for infinite many $n\}$

$B_m(\varepsilon) \downarrow$

$\therefore\ ① = \lim_{m\to\infty} \mathbb{P}(B_m(\varepsilon)) = \mathbb{P}\Big(\underbrace{\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty} A_n(\varepsilon)}_{}\Big)$

$A_n(\varepsilon)$ i.o. $= \limsup A_n(\varepsilon)$.

$\|$

$\{\omega\in\Omega: |X_n(\omega)-X(\omega)|>\varepsilon\}$ for infinite many n.

$LHS = P(C^c) = 0$

$RHS = P(\{\omega: |X_n(\omega)-X(\omega)|>\varepsilon$ for i.o. $n.\}) = 0 \quad \forall \varepsilon.$

LHS is stronger than RHS

uncountable.

decreasing in ε.

$LHS = \mathbb{P}\big(\bigcup_{\varepsilon>0}\{\underline{A_n(\varepsilon)\text{ i.o.}}\}\big) = 0.$

Just choose a sequence to do the union.

$RHS = \mathbb{P}(\underline{A_n(\varepsilon)\text{ i.o.}}) = 0 \quad \forall \varepsilon>0.$

$\limsup_{n\to\infty} A_n(\varepsilon)$,

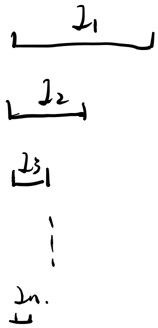
$A_n(\varepsilon)\downarrow$ if $\varepsilon\uparrow$

$\bigcup_{\varepsilon>0}\{A_n(\varepsilon)\text{ i.o.}\}$

$= \bigcup_{k=1}^{\infty}\{A_n(\tfrac{1}{k})\text{ i.o.}\}$

$$\mathbb{P}\left(\bigcup_{\varepsilon>0}\{A_n(\varepsilon)\ \text{i.o.}\}\right)$$

$$=\mathbb{P}\left(\bigcup_{k=1}^{\infty}\{A_n(\tfrac{1}{k})\ \text{i.o.}\}\right)\le\sum_{k=1}^{\infty}\mathbb{P}\left(A_n(\tfrac{1}{k})\ \text{i.o.}\right)$$



$I_n=[0,\frac{1}{n}]$ $\quad X_n(\omega)=\begin{cases}1 & \omega\in I_n\\ 0 & \text{otherwise}\end{cases}$ $\quad X\equiv 0$.

$I_n\subset I_{n-1}\subset\cdots\subset I_1$

$$B_m(\varepsilon)=\bigcup_{n=m}^{\infty}A_n(\varepsilon)=\bigcup_{n=m}^{\infty}I_n=I_m.$$

$$\lim_{m\to\infty}\mathbb{P}(B_m(\varepsilon))=\lim_{m\to\infty}\mathbb{P}(I_m)=\lim_{m\to\infty}\frac{1}{m}=0.\ \checkmark$$

We cannot use sufficient condition (2)

Because $A_n(\varepsilon)$ overlaps a lot.

From weak to strong

① If $X_n\xrightarrow{D}c$ then $X_n\xrightarrow{\mathbb{P}}c$

In LLN the limit is a number / no stochastic

② If $X_n\xrightarrow{\mathbb{P}}X$, $\mathbb{P}(|X_n|\le k)=1$ for some k uniformly in n.

then $X_n\xrightarrow{r}X$ $\forall r\ge 1$

(Bounded Convergence Thm).

③ If $\sum_{n=1}^{\infty}\mathbb{P}(|X_n-X|>\varepsilon)<\infty$ $\forall\varepsilon>0$ then $X_n\xrightarrow{a.s.}X$

Pf: $\mathbb{P}(|X_n-c|>\varepsilon)=\mathbb{P}(X_n<c-\varepsilon)+\mathbb{P}(X_n>c+\varepsilon)$

$=\mathbb{P}(X_n<c-\varepsilon)+1-\mathbb{P}(X_n\le c+\varepsilon)$

$\xrightarrow{D}\underline{\mathbb{P}(X<c-\varepsilon)}+1-\underline{\mathbb{P}(X\le c+\varepsilon)}$

$=0+1-1=0$

$\mathbb{P}(c\le c+\varepsilon)$

$P(X_n \le c - 2\varepsilon) \le \qquad\qquad \le \mathbb{P}(X_n \le c - \frac{\varepsilon}{2})$

$\downarrow$

$\mathbb{P}(c \le c - 2\varepsilon)$