SLLN

Let $X_1 \cdots$ i.i.d. $\mathbb{E} X_i = \mu$. $\mathbb{E} X_i^4 < \infty$.

If. $S_n = X_1 + \cdots + X_n$ then.

Finally we will remove it with the help of.

- Δ Truncation.
- Δ Chebyshev.
- Δ B-C Lemma.
- Δ Sparsification.

$$\frac{S_n}{n} \xrightarrow{a.s.} \mu.$$

Pf: $\frac{S_n}{n} \xrightarrow{a.s.} \mu.$

$\Leftrightarrow \mathbb{P}(A_n(\varepsilon) \text{ i.o.}) = 0$

$A_n(\varepsilon) = \{ |\frac{S_n}{n} - \mu| > \varepsilon \}$

$\mathbb{P}\left(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_n(\varepsilon) \right) = 0$

$= \lim_{m \to \infty} \mathbb{P}(B_m(\varepsilon)) = 0.$

By. BC(1) it sufficients to show.

$$\sum_{n=1}^{\infty} P\left(\left|\frac{S_n}{n} - \mu\right| > \varepsilon \right) < \infty \quad \boxed{\forall \varepsilon > 0}$$

$\mathbb{P}\left(\left|\frac{S_n}{n} - \mu\right| > \varepsilon \right) \leq \frac{\mathrm{Var}(S_n)}{\varepsilon^2 n^2} \leq \frac{nC}{\varepsilon^2 n^2} \sim \frac{1}{n}$ not summable.

$\therefore$ use Chebyshev is not enough.

Use the 4-th moment.

Why should we do that.

$$\mathbb{P}\left(\left|\frac{S_n - n\mu}{n}\right|^4 > \varepsilon^4 \right) \leq \frac{\mathbb{E}\left|\frac{S_n - n\mu}{n}\right|^4}{\varepsilon^4} = \frac{\mathbb{E}|S_n - n\mu|^4}{n^4 \varepsilon^4}$$

$\sim \frac{n^2}{n^4 \varepsilon^4}$

$\sim \frac{1}{n^2 \varepsilon^4}$

$$\mathbb{E}|S_n - n\mu|^k = \mathbb{E}\Big|\sum_i \underbrace{(X_i - \mu)}_{Y_i}\Big|^4 = \sum_{i,j,k,l} \mathbb{E}[Y_i Y_j Y_k Y_l]$$

iid + centralized

Y_i have to equal to others.

$i=j \neq k=l$

$i=k \neq j=l$

$i=l \neq j=k$

$i=j=k=l$.

$$= 3 \sum_{i \neq j} (\mathbb{E} Y_i^2)^2 + \sum_i \mathbb{E} Y_i^4$$

$$= n(n-1)\, \mathbb{E} \cdots \quad + n \cdot \mathbb{E} \cdots$$

$$\sim n^2 \qquad\qquad \sim n$$

Recall (A WLLN): Let $X_1 \cdots$ i.i.d. and

$$x \mathbb{P}(|X_1| > x) \to 0 \qquad x \to \infty$$

Let $a_n = \mathbb{E} X_1 \mathbb{1}(|X_1| \le n)$

Then.

$$\frac{S_n}{n} - a_n \xrightarrow{\mathbb{P}} 0$$

eg. $X_1 \geq 0$

$\lim_{n\to\infty} \frac{S_n}{n}$ does not exists in $(-\infty, \infty) = \mathbb{R}$

eg. X_1 symmetric

$\frac{S_n}{n} \xrightarrow{\mathbb{P}} 0$

Consider the case. $\mathbb{E}|X_1| = \infty$

Thm. If. $X_1\ X_2 \cdots$ i.i.d. $\boxed{\mathbb{E}|X_1| = \infty}$

Let. $S_n = \sum_{i=1}^n X_i$

Q: $\frac{S_n}{n} \xrightarrow{a.s.} 0$

① $\mathbb{P}\left(\left\{\omega : \lim \frac{S_n(\omega)}{n} \text{ exists in } (-\infty, \infty)\right\}\right) = 0$ $\quad \mathbb{P}\left(\left\{\omega : \frac{S_n(\omega)}{n} \text{ converges}\right\}\right)$

then. ②

$$\mathbb{P}(\{|X_n| \geq n\} \text{ i.o.}) = 1$$

(A_n above $\{|X_n| \geq n\}$)

Pf:

$$\infty = \mathbb{E}|X_1| = \int_0^\infty \mathbb{P}(|X_1| > y)\, dy$$

$$= \left(\int_0^1 + \int_1^2 + \cdots\right)$$

$$\leq \sum_{n=0}^{\infty} \mathbb{P}(|X_1| \geq n)$$

take. Left point value.

Replace $|X_1|$ to $|X_n|$ i.i.d.

$$\Rightarrow \sum_{n=1}^{\infty} \mathbb{P}(|X_n| > n) = \infty$$

$\Rightarrow \mathbb{P}(\{|X_n| > n \text{ i.o.}\}) = 1$ by BC (II)

Relation Between ① ② ?

$$\frac{S_n}{n} - \frac{S_{n+1}}{n+1} = \frac{S_n}{n(n+1)} - \frac{X_{n+1}}{n+1}$$

Heuristic For any ω if $\frac{S_n(\omega)}{n}$ converges.

Cauchy $\rightarrow \frac{S_n(\omega)}{n} - \frac{S_{n+1}(\omega)}{n} \to 0$

Further $\frac{S_n(\omega)}{(n+1)n} \to 0$

$\Downarrow$

$\frac{X_{n+1}(\omega)}{n+1} \to 0$. contradictory

$\therefore \frac{S_n(\omega)}{n}$ does not converge

$C = \{\omega \in \Omega : |X_n(\omega)| > n \text{ for infinite many } n\}$

$\mathbb{P}(C) = 1$

If. $\omega \in C$ $\frac{X_n(\omega)}{n} \not\to 0$

$\Rightarrow \frac{S_n(\omega)}{n}$ does not converge.

$\mathbb{P}(\{\omega : \lim_{n\to\infty} \frac{S_n(\omega)}{n}$ does not exist in $(-\infty, -\infty)\})$

$\geq \mathbb{P}(C) = 1$

$\mathbb{P}(A_n \text{ i.o.}) = 1$ if $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$. and $A_1, A_2 \cdots$ are $\perp\!\!\!\perp$

equ

• $\perp\!\!\!\perp \Rightarrow$ pairwise $\perp\!\!\!\perp$

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• more quantitative way

$$\mathbb{P}\left(\underline{\sum_{m=1}^{\infty} \mathbb{1}_{A_m}} = \infty \right) = 1$$

How often A_m shows up.

Thm. If $A_1 \cdots$ are pairwise indept and $\sum_{m=1}^{\infty} \mathbb{P}(A_m) = \infty$ then as $n \to \infty$.

$$\frac{\sum_{m=1}^{n} \mathbb{1}_{A_m}}{\sum_{m=1}^{n} \mathbb{P}(A_m)} \xrightarrow{a.s.} 1.$$

① try $\xrightarrow{\mathbb{P}}$

② upgrade to $\xrightarrow{a.s.}$

Pf. of weak version: $\dfrac{\sum_{m=1}^{n} \mathbb{1}_{A_m}}{\sum_{m=1}^{n} \mathbb{P}(A_m)} \xrightarrow{\mathbb{P}} 1$

Let. $X_m = \mathbb{1}_{A_m}$ $\quad S_n = \sum_{m=1}^{n} X_m$ $\quad$ Triangular Array

① Compute $\mathbb{E}$ Var.

② Choose $b_n \leq \mathbb{E}$

$$\mathbb{E} S_n = \sum_{m=1}^{n} \mathbb{E} X_m = \sum_{m=1}^{n} \mathbb{P}(A_m)$$

$Y \sim Be(p)$

$\mathbb{E} Y = p \quad Var Y = p(1-p) < \mathbb{E} Y$

$$Var(S_n) = \sum_{m=1}^{n} Var X_m \leq \mathbb{E} S_n = \sum_{m=1}^{n} \mathbb{P}(A_m).$$

$$\mathbb{P}\left(\left| \frac{S_n - \mathbb{E} S_n}{\mathbb{E} S_n} \right| > \varepsilon \right) \leq \frac{Var(S_n)}{(\mathbb{E} S_n)^2 \cdot \varepsilon^2} \leq \frac{1}{(\mathbb{E} S_n) \cdot \varepsilon^2} \to 0$$

as. $n \to \infty$.

$$\therefore \frac{S_n - \mathbb{E} S_n}{\mathbb{E} S_n} \xrightarrow{\mathbb{P}} 0. \qquad \frac{S_n}{\mathbb{E} S_n} \xrightarrow{\mathbb{P}} 1$$

SLLN used $\mathbb{E}X^4 < \infty$

Be(p) have all moments. $\Rightarrow$ Raise the order

pf. $\frac{S_n}{\mathbb{E}S_n} \xrightarrow{a.s.} 1.$

$$\lim_{m\to\infty} \mathbb{P}\Big(\underbrace{\bigcup_{n=m}^{\infty} \{\underbrace{|\tfrac{S_n}{n}-\mu|>\varepsilon}_{A_n(\varepsilon)}\}}_{B_m(\varepsilon)}\Big)=0 \quad \forall \varepsilon>0.$$

$$\frac{S_n}{n}-\mu \xrightarrow{a.s.} 0 \iff \mathbb{P}\left(|\tfrac{S_n}{n}-\mu|>\varepsilon \ \text{ i.o.}\right)=0 \ \forall \varepsilon>0$$

$\Uparrow$ suff

$$\sum_{n=1}^{\infty} \mathbb{P}\left(|\tfrac{S_n}{n}-\mu|>\varepsilon\right)<\infty \quad \forall \varepsilon>0.$$

$A_n(\varepsilon)=\{\omega: |\frac{S_n(\omega)}{n}-\mu|>\varepsilon\}$ $\qquad \frac{S_{n+1}}{n+1}=\frac{S_n}{n+1}+\frac{X_{n+1}}{n+1}$ negligible.

$A_{n+1}(\varepsilon)=\{\omega \mid |\frac{S_{n+1}(\omega)}{n+1}-\mu|>\varepsilon\}$

$$\frac{S_1}{\mathbb{E}S_1} \quad \frac{S_2}{\mathbb{E}S_2} \quad \frac{S_{n_1}}{\mathbb{E}S_{n_1}} \cdots\cdots \frac{S_{n_2}}{\mathbb{E}S_{n_2}} \cdots \frac{S_{n_k}}{\mathbb{E}S_{n_k}} \cdots$$

$$\frac{S_{n_k}}{\mathbb{E}S_{n_k}} \xrightarrow{a.s.} 1$$

① choose sparse enough. Σ of bad events is summable.

② --- Dense enough. st. the sub seq can represent the original series.

Let. $n_k=\inf\{n: \mathbb{E}S_n \geq k^2\}$

$\mathbb{E}S_n=\sum_{m=1}^{n}\mathbb{P}(A_m) \uparrow n$

$\therefore \mathbb{E}S_{n_k-1}<k^2 \leq \mathbb{E}S_{n_k} \leq k^2+1$

claim. $\frac{S_{n_k}}{\mathbb{E}S_{n_k}} \xrightarrow{a.s.} 1$ as $k\to\infty$

$$\sum \mathbb{P}\left(\left| \frac{S_{n_k} - \mathbb{E} S_{n_k}}{\mathbb{E} S_{n_k}} \right| > \varepsilon \right) \le \sum \frac{\operatorname{Var}(S_n)}{\varepsilon^2 (\mathbb{E} S_{n_k})^2} \le \sum \frac{1}{\varepsilon^2 \, \mathbb{E} S_{n_k}}$$

$$\le \sum_{k=1}^{\infty} \frac{1}{\varepsilon^2 k^2} < \infty.$$

Thm. If $A_1 \cdots$ are pairwise indept

and $\sum_{m=1}^{\infty} \mathbb{P}(A_m) = \infty$ then as $n \to \infty$.

$$\frac{\sum_{m=1}^{n} \mathbb{1}_{A_m}}{\sum_{m=1}^{n} \mathbb{P}(A_m)} \xrightarrow{a.s.} 1.$$

① try $\xrightarrow{\mathbb{P}}$

② upgrade to $\xrightarrow{a.s.}$

① The conclusion can be seen as extention of BC(II).

②. only requires pairwise independent

③. written in a more quantitative way

$$\mathbb{P}\left(\left\{ \omega : \frac{S_{n_k}(\omega)}{\mathbb{E} S_{n_k}} \to 1 \quad k \to \infty \right\} \right) = 1$$

If $\omega \in \Omega_0$ $\quad n_k \le n \le n_{k+1}$

$$\frac{S_{n_k}(\omega)}{\mathbb{E} S_{n_{k+1}}} \le \frac{S_n(\omega)}{\mathbb{E} S_n} \le \frac{S_{n_{k+1}}(\omega)}{\mathbb{E} S_{n_k}}.$$

$$\frac{\mathbb{E} S_{n_k}}{\mathbb{E} S_{n_{k+1}}} \cdot \frac{S_{n_k}(\omega)}{\mathbb{E} S_{n_k}} \to 1 \qquad \text{squeezed} \qquad \frac{\mathbb{E} S_{n_{k+1}}}{\mathbb{E} S_{n_{k+1}}} \cdot \frac{\mathbb{E} S_{n_{k+1}}}{\mathbb{E} S_{n_k}} \to 1$$

$$\left\{ \omega : \frac{S_n(\omega)}{\mathbb{E} S_n} \to 1 \right\} \supset \Omega_0.$$

$\therefore \mathbb{P}(\{\omega: \frac{S_n(\omega)}{\mathbb{E}S_n} \to 1\}) \geq \mathbb{P}(\Omega_0) = 1$

1"

Thm. $X_1, X_2, \cdots$ i.i.d. $\mathbb{E}|X_1| = \infty$.

$$\mathbb{P}(\{\omega: \lim \frac{S_n(\omega)}{n} \text{ exists}\}) = 0$$

Thm. (SLLN) Let. $X_1, X_2, \cdots$ be identically distributed pairwise indept $\mathbb{E}|X_1| < \infty$ $\mathbb{E}X_1 = \mu$.

Let. $S_n = \sum_{i=1}^{n} X_i$ Then. $\frac{S_n}{n} \xrightarrow{a.s.} \mu$.

$\Downarrow$

$\mathbb{P}(\{\omega: \lim \frac{S_n}{n} = \mu\}) = 1$

Roughly BC-Lemma at certain stage.

Finally we need to prove for $\sum_{k=1}^{\infty} \mathbb{P}(|\frac{S_n}{n} - \mu| > \varepsilon) < \infty$.

$\Downarrow$ $\overline{X_k} = X_k \mathbb{I}(|X_k| \leq ?)$

only have first moment. $\sum_{k=1}^{\infty} \mathbb{P}(|\frac{\overline{S_n}}{n} - \mu| > \varepsilon) < \infty$.

$\overline{S_n} = \sum_{k=1}^{n} \overline{X_k}$

Δ pairwise indept can be reserved after doing the truncation.

but not the same with uncorrelated.

$X_{1,1}, \cdots X_{1,n}$.

$\cdots$

$\overline{X_{nk}} = X_{nk} \mathbb{I}(|\cdot| \leq b_n)$

low — hard to show $\frac{S_n}{n} - \frac{\overline{S_n}}{n} \xrightarrow{a.s.} 0$.

truncation. level

high. $\mathbb{P}(|\frac{\overline{S_{n_k}}}{n} - \mu| > \varepsilon) \uparrow \leq \frac{\operatorname{Var} \overline{S_{n_k}}}{n_k^2 \varepsilon^2}$ $\rightsquigarrow$ hard to control. the $\operatorname{Var}(\overline{S_{n_k}})$

$\Downarrow$

need to choose $\{n_k\}$ sufficiently large.

approx $\frac{S_n}{n}$ by $\frac{S_{n_k}}{n_k}$ $\to$ need to choose. $\{n_k\}$ $\cdots$ dense.

pf. Let. $\overline{X_k} = X_k \mathbb{I}(|X_k| \le k)$

$\overline{S_n} = \sum_{k=1}^{n} \overline{X_k}$

① $\frac{\overline{S_n}}{n} \xrightarrow{a.s.} \mu.$

$\Downarrow$

② $\frac{S_n}{n} \xrightarrow{a.s.} \mu.$

Convention. $X_k \ge 0$ $\forall k = 1, 2 \cdots$

$X_k = X_k^+ - X_k^-$

$S_n^+ = \sum_{k=1}^{n} X_k^+$ $S_n^- = \sum_{k=1}^{n} X_k^-$

more sparse than the previous. k^2

Let $n_k = \lfloor \alpha^k \rfloor$ $\alpha > 1$

① sparse for using B-C Lemma.
② dense for plug in. original sequence.

First. show. $\frac{\overline{S_{n_k}} - \mathbb{E}\overline{S_{n_k}}}{n_k} \xrightarrow{a.s.} 0.$ $k \to \infty.$

$\Uparrow$ BC(I).

$$\sum_{k=1}^{\infty} \mathbb{P}\left(\left| \frac{\overline{S_{n_k}} - \mathbb{E}\overline{S_{n_k}}}{n_k} \right| > \varepsilon \right) < \infty \quad \forall \varepsilon > 0.$$

$\Uparrow$ Chebyshev.

$$\sum_{k=1}^{\infty} \mathbb{P}(| \quad | > \varepsilon) \le \sum_{k=1}^{\infty} \frac{Var(\overline{S_{n_k}})}{\varepsilon^2 n_k^2} = \varepsilon^{-2} \sum_{k=1}^{\infty} n_k^{-2} \sum_{i=1}^{n_k} Var(\overline{X_i})$$

$i \le n_k$

$Y \ge 0.$

$\mathbb{E} Y^p = p \int_0^\infty y^{p-1} \mathbb{P}(Y \ge y) dy$

$$= \varepsilon^{-2} \sum_{i=1}^{\infty} var(\overline{X_i}) \underbrace{\left(\sum_{k: n_k \ge i}^{\infty} n_k^{-2} \right)}$$

$\alpha^k \ge i$

$k \ge \log_\alpha i$

$\Downarrow$

$$\sum_{k=\log_\alpha i}^{\infty} [\alpha^{-2}]^k \le C \cdot \frac{[\alpha^{-2}]^{\log_\alpha i}}{1-\alpha}$$

$\Uparrow$

$$\sum_{i=1}^{\infty} i^{-2} var(\overline{X_i}) \le \sum_{i=1}^{\infty} i^{-2} \mathbb{E} \overline{X_i}^2$$

$\le C \cdot \frac{1}{1-\alpha^{-1}} \cdot i^{-2}$

truncation

$$= \sum_{i=1}^{\infty} i^{-2} 2 \cdot \int_0^{\infty} y \, \mathbb{P}(|\overline{X_i}| > y) dy$$

i for ∞

$\le \int_0^i y \, \mathbb{P}(|X_1| > y) dy.$

$$\le 2 \sum_{i=1}^{\infty} i^{-2} \int_0^i y \, \mathbb{P}(|X_1| > y) dy$$

$y \le i.$

$$\le \int_0^\infty \Big(\underbrace{\sum_{i=y}^{\infty} i^{-2}}_{\sim \frac{1}{y}} \Big) y \, \mathbb{P}(|X_1|>y)\, dy \le C \int_0^\infty \mathbb{P}(|X_1|>y)\, dy$$

$$= C\, \mathbb{E}|X_1| < \infty.$$

If $n_k = k^c$

$n_k = k^c \ge i \Rightarrow k \ge i^{\frac{1}{c}}$ $\quad \sum_{k=i^{\frac{1}{c}}}^{\infty} k^{-2c} = i^{\frac{1}{c} \times (-2c+1)} = i^{-2+\frac{1}{c}}$ $\quad y^{-1+\frac{1}{c}}.$

Then. show. $\frac{\overline{S_{n_k}}}{n_k} \xrightarrow{a.s.} \mu.$

$\Uparrow$

$\mathbb{E} \frac{S_{n_k}}{n} \to \mu.$

$$\mathbb{E}\overline{S_n} = \sum_{k=1}^{n} \mathbb{E}\overline{X_k}$$

$$= \sum_{k=0}^{n} \mathbb{E} X_k \mathbb{I}(|X_k| \le k)$$

$$= \sum_{k=1}^{n} \mathbb{E} X_1 \mathbb{I}(|X_1| \le k)$$

$X_1 \mathbb{I}(|X_1| \le k) \xrightarrow{a.s.} X_1 \quad k \to \infty$

$|X_1 \mathbb{I}(|X_1| \le k)| \le |X_1| \quad \mathbb{E}|X_1| < \infty.$

By DCT

$\mathbb{E} X_1 \mathbb{I}(|X_1| \le k) \to \mathbb{E} X_1 = \mu. \quad k \to \infty.$

DCT: ① The r.v converges to a r.v. almost surely

② The r.v. is bounded by an integrable. r.v.

③. Then we have the expectation of that guy goes to the number

$\frac{\mathbb{E}\overline{S_n}}{n} = \frac{a_1 + \cdots + a_n}{n} \qquad a_k \triangleq \mathbb{E} X_1 \mathbb{I}(|X_1| \le k)$

$a_n \to \mu.$

$\therefore \frac{\mathbb{E}\overline{S_n}}{n} \to \mu.$

Consequently

$\frac{S_{n_k}}{n_k} \xrightarrow{a.s.} \mu. \qquad \mathbb{P}(\underbrace{\{\omega : \frac{S_{n_k}(\omega)}{n_k} \to \mu\}}_{\Omega_0}) = 1$

Next. we show $\frac{\overline{S_n}}{n} \xrightarrow{a.s.} \mu$.

$\forall \omega \in \Omega_0 \quad \forall n_k \le n \le n_{k+1}$

$$\frac{\overline{S_{n_k}}}{n_{k+1}} \le \frac{\overline{S_n}}{n} \le \frac{\overline{S_{n_{k+1}}}}{n}$$

$n_k = \alpha^k$

$$\frac{n_k}{n_{k+1}} \frac{\overline{S_{n_k}}(\omega)}{n_k} \qquad \frac{\overline{S_{n_{k+1}}}(\omega)}{n_k} \frac{n_{k+1}}{n_k}.$$

$\frac{n_k}{n_{k+1}} \downarrow \frac{1}{\alpha}$, $\frac{\overline{S_{n_k}}(\omega)}{n_k} \downarrow \mu$; $\frac{\overline{S_{n_{k+1}}}(\omega)}{n_k} \downarrow \mu$, $\frac{n_{k+1}}{n_k} \downarrow \alpha$.

Take the liminf or limsup first.

$$\frac{n_k}{n_{k+1}} \frac{\overline{S_{n_k}}(\omega)}{n_k} \le \liminf_{n\to\infty} \frac{\overline{S_n}(\omega)}{n} \le \limsup_{n\to\infty} \frac{\overline{S_n}(\omega)}{n} \le \frac{\overline{S_{n_{k+1}}}(\omega)}{n_{k+1}} \frac{n_{k+1}}{n_k}$$

Δ liminf limsup has nothing to do with ω.

Further send $\alpha \downarrow 1$

$$\lim_n \frac{\overline{S_n}(\omega)}{n} = \mu. \quad \omega \in \Omega_0$$

$$\lim_{n\to\infty} \frac{\overline{S_n}(\omega)}{n} = \mu. \quad \omega \in \Omega_0.$$

$$\frac{\overline{S_n}}{n} \xrightarrow{a.s.} \mu.$$

② $\frac{S_n}{n} \xrightarrow{a.s.} \mu$.

claim. $\boxed{\mathbb{P}(X_k \ne \overline{X_k} \ \text{i.o.}) = 0}$

$\Omega' = \{\omega : X_k(\omega) \ne \overline{X_k}(\omega) \text{ for infinitely many } k\}$

$$\frac{S_n - \overline{S_n}}{n} \xrightarrow{a.s.} 0 \qquad S_n = X_1 + X_2 + \cdots + X_n$$

$$\overline{S_2} = \overline{X_1} + \overline{X_2} + \cdots + \cdots \overline{X_n}$$

$\Omega_0 = \Omega \setminus \Omega' = \{\omega : X_k(\omega) \ne \overline{X_k}(\omega) \text{ for only finitely many } k\}$

$\omega \in \Omega_0$

$$\frac{S_n(\omega) - \overline{S_n}(\omega)}{n} \to 0 \qquad n \to \infty,$$

Show the claim. $\mathbb{P}(\underbrace{X_k \neq \overline{X_k} \text{ . i.o. }}_{\text{typical in BC(I2)}}) = 0$

Need to check

$$\sum_k \mathbb{P}(X_k \neq \overline{X_k}) = \sum_{k=1}^{\infty} \mathbb{P}(|X_k| > k) = \sum_{k=1}^{\infty} \mathbb{P}(|X_1| > k)$$

$$\leq \int_0^{\infty} \mathbb{P}(|X_1| > y)\, dy$$

$$= \mathbb{E}|X_1| < \infty$$

$$\int_0^{1} + \int_1^{2}.$$