SLLN

$\overline{X_k} = X_k \mathbb{1}(|X_k| \le k) \quad k=1\cdots$

$\frac{S_n}{n} \xrightarrow{a.s.} \mu. \Leftarrow \frac{\overline{S_n}}{n} \xrightarrow{a.s.} \mu.$

Additional task.

$\frac{S_n - \overline{S_n}}{n} \xrightarrow{a.s.} 0$

$\Updownarrow$

$$\mathbb{P}\left(\bigcup_{n=m}^{\infty} \left\{ \left| \frac{S_n - \overline{S_n}}{n} \right| > \varepsilon \right\}\right) \to 0 \quad \forall \varepsilon > 0 \quad m \to \infty.$$

$$\subset \bigcup_{k=1}^{n} \{ (X_k \neq \overline{X_{n,k}}) \}$$

$\Uparrow$

$$\mathbb{P}\left(\bigcup_{n=m}^{\infty} \bigcup_{k=1}^{n} \{ X_k \neq \overline{X_{n,k}} \}\right) \to 0 \quad m \to \infty$$

$\Updownarrow$

$\{ X_k > n \}$

$$\mathbb{P}\left(\bigcup_{n=m}^{\infty} \bigcup_{k=1}^{n} \{ |X_k| > n \}\right) \to 0 \quad m \to \infty$$

$n = m. \quad \bigcirc = \{|X_1| > m\} \cup \cdots \cup \{|X_m| > m\}$

$\bigcup_{n=m+1}$ $\quad = \{|X_1| > m+1\} \cup \cdots \cup \{|X_m| > m+1\} \cup \{|X_{m+1}| > m+1\}$

$$\therefore \bigcup_{n=m}^{\infty} \{|X_n| > n\}$$

$$\lim_{m \to \infty} \mathbb{P}\left(\bigcup_{n=m}^{\infty} \{|X_n| > n\}\right) \to 0$$

$\to \mathbb{P}(\{|X_n| > n\} \text{ i.o.}) = 0 \quad \Leftarrow \sum_{n=1}^{\infty} \mathbb{P}(|X_n| > n)$

$\Updownarrow$

$$\boxed{\sum_{n=1}^{\infty} \mathbb{P}(|X_1| > n) \sim \mathbb{E}|X_1| < \infty.}$$

WLLN

$\overline{X_{n,k}} = X_{n,k} \mathbb{1}(|X_{n,k}| \le b_n)$

$\overline{S_n} = \sum_{k=1}^{n} \overline{X_{n,k}}$

choose of b_n.

Why do we care about SLLN

The conclusion cannot be derived if we only have SLLN.

Renewal Theory eq.

Let $X_1, X_2, \cdots$ i.i.d. $\mathbb{E}|X_1| < \infty$, $\mathbb{E}X_1 = \mu$. ≥ 0.

$X_i : \Omega \to \mathbb{R}$

given $\omega \in \Omega$

$X_1(\omega)\ X_2(\omega) \cdots$

$T_1(\omega)\ T_2(\omega) \cdots$

$T_n = \sum_{i=1}^{n} X_i$: lifetime of the first n bulb.

X_i : life time of i-th bulb.

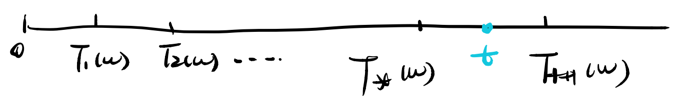
By SLLN $\frac{T_n}{n} \xrightarrow{a.s.} \mu$. as $n \to \infty$

$N_t := \sup\{n : T_n \leq t\}$ → number of bulbs fails just before time t.

No bulbs fail between $T_{N_t}(\omega)$ and t.

$\downarrow$

N_t.



$T_{N_t} : \Omega \to \mathbb{R}$

$\omega \to T_{N_t(\omega)}(\omega)$,

$\downarrow$

$= \sum_{n=0}^{\infty} T_n(\omega) \cdot \mathbb{1}(n = N_t(\omega))$

① How many bulbs fails. → $N_t(\omega)$.

② What's the exactly time of N_t-th bulb fails

$\frac{t}{N_t} \sim \mu$.

Thm.: $\frac{N_t}{t} \to \frac{1}{\mu}$ $t \to \infty$.

not func of ω

$T_{N_t} \leq t < \underbrace{T_{N_t+1}}$ func of ω.

So. f.w. $t \to \infty$. $T_{N_t+1} \to \infty$.

$\forall \omega \in \Omega \quad \text{if } t \to \infty$

$$\frac{T_{N_t}}{N_t} \le \frac{t}{N_t} < \frac{T_{N_t+1}}{N_t+1} \cdot \frac{N_t+1}{N_t}.$$

If we only have WLLN we cannot conclude the behavior of T_{N_t}

$T_{N_t(\omega)+1}(\omega) \to \infty$

$\| $

$X_1(\omega) + \cdots + X_{N_t(\omega)+1}(\omega)$

$\to N_t(\omega) \to \infty$. for $\forall\, \omega$

$\therefore N_t \xrightarrow{a.s.} \infty$

① $\frac{T_{N_t}}{N_t} \xrightarrow{a.s.} \mu. \quad t \to \infty.$

② $\frac{T_{N_t+1}}{N_t+1} \xrightarrow{a.s.} \mu. \quad t \to \infty.$

③ $\frac{N_t+1}{N_t} \xrightarrow{a.s.} 1 \quad t \to \infty.$

④ $N_t \xrightarrow{a.s.} \infty. \quad t \to \infty.$

SLLN $\approx$ pointwise convergence $\forall \varepsilon \cdots\cdots$

By SLLN $\frac{T_n}{n} \xrightarrow{a.s.} \mu.$

$$\mathbb{P}(\underbrace{\{\omega : \tfrac{T_n}{n} \to \mu\}}_{\Omega_0}) = 1$$

$\forall\, \omega \in \Omega_0 \quad \frac{T_{N_t(\omega)}(\omega)}{N_t(\omega)} \to \mu.$

for fixed ω

the sequence is the same.

$$\mathbb{P}(\{\omega : \frac{T_{N_t(\omega)}}{N_t(\omega)} \to \mu\}) \ge \mathbb{P}(\Omega_0) = 1$$

$f(x) \to a \quad x \to \infty$

$f(X(t)) \to a \quad t \to \infty$

$X(t) \to \infty \quad t \to \infty.$

$X_n \xrightarrow{a.s.} X$ (as $n \to \infty$) $N_t \xrightarrow{a.s.} \infty$. as. $t \to \infty$.

then. $X_{N_t} \xrightarrow{a.s.} X$ as $t \to \infty$.

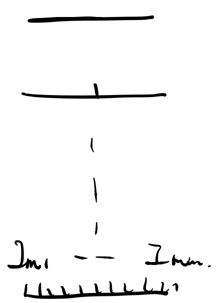
What if we only have WLLN.

$X_n \xrightarrow{\mathbb{P}} X$ (as $n \to \infty$) $\quad N_t \xrightarrow{a.s.} \infty$. as $t \to \infty$.

It's not necessarily true $X_{N_t} \xrightarrow{a.s} X$ as $t \to \infty$.

But true only for $X_{N_t} \xrightarrow{\mathbb{P}} X$

$\omega \in [0, 1)$ uniform.



Let $I_{m,k} = [\frac{k-1}{m}, \frac{k}{m})$

$$Y_{m,k} = \mathbb{1}_{I_{m,k}} = \begin{cases} 1 & \omega \in I_{m,k} \\ 0 & \text{otherwise} \end{cases}.$$

X_i : enumerate of $Y_{m,k}$

$X_1 = Y_{1,1}$ $X_2 = Y_{2,1}$ $X_3 = Y_{2,2}$.

$X_{\sum_{i=1}^{m-1} i + k} = Y_{m,k}$

$X_n \xrightarrow{\mathbb{P}} X \equiv 0$

$X_n \not\xrightarrow{a.s.} X \equiv 0$

$\forall \omega \in [0, 1)$

$\forall m$ $\exists k_m \equiv k_m(\omega)$ s.t. $\forall m$ $\omega \in$ 某区间 那个编号就是 k_m.

$\omega \in I_{m, k_m} = [\frac{k_m - 1}{m}, \frac{k_m}{m})$

Let $N_m(\omega) = \sum_{i=1}^{m-1} i + k_m(\omega)$,

$X_{N_m}(\omega) = Y_{m, k_m(\omega)}(\omega) \equiv 1$

$\therefore$ Tough $X_{N_m} \xrightarrow{\mathbb{P}} X \equiv 0$ $m \to \infty$.

But $X_{N_m} \not\xrightarrow{a.s.} X \equiv 0$ $m \to \infty$.

$X \sim F$ (cdf) $\qquad X_i : \Omega \to \mathbb{R}$

Let $X_1 \cdots$ i.i.d. samples of $X \sim F$

empirical $\quad F_n(x) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}(X_i \le x) \quad \forall$ fixed x

$$\Downarrow$$

$$F_n(X, \omega) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}(X_i(\omega) \le x) : \Omega \to \mathbb{R}$$

change ω. It's no deterministic

By SLLN for a fixed x $\Rightarrow$ for fixed x ✓

$$F_n(x) \xrightarrow{a.s.} F(x)$$

But does not tell sth. in a uniformly sense.

$\because \mathbb{E}\,\mathbb{1}(X_i \le x) = \mathbb{P}(X_i \le x) = F(x)$.

$\Omega_x = \{\omega : F(x, \omega) \to F(x)\} \qquad \mathbb{P}(\Omega_x) = 1$

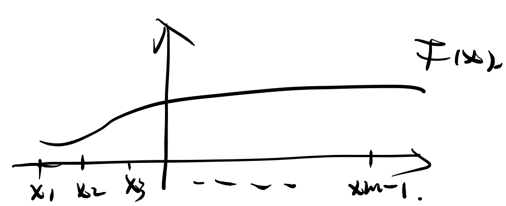
Glivenko - Cantelli thm.

$$\sup_{x \in \mathbb{R}} |F_n(x) - F(x)| \xrightarrow{a.s.} 0$$

$\underbrace{\bigcap^{\infty} \Omega_x}$ uncountable intersection

Approx by countable. discrete ---

Pf for the continuous case.



$\forall m \in \mathbb{N}$

$\forall x \in [x_{j-1}, x_j)$

$F(x_j) - F(x_{j-1}) = \frac{1}{m}$.

$$F_n(x) - F(x) \le F_n(x_j) - F(x_{j-1})$$
$$= F_n(x_j) - F(x_j) + \frac{1}{m}$$

$$\Downarrow$$

$$\geq F_n(x_{j-1}) - F(x_{j-1}) - \frac{1}{m}.$$

$$\therefore -\sup_j |F_n(x_{j-1}) - F(x_{j-1})| - \frac{1}{m} \leq \sup_{x \in \mathbb{R}} |F_n(x) - F(x)| \leq \sup_j |F_n(x_j) - F(x_j)| + \frac{1}{m}$$

Let. $\Omega_{x_j} := \{\omega : F_n(x_j) \to F(x_j)\}$ $\quad P(\Omega_{x_j}) = 1$

$$P\left(\bigcap_{j=0}^{m} \Omega_{x_j}\right) = 1 - P\left(\bigcup_{j=0}^{m} \Omega_{x_j}^c\right) \geq 1 - \sum_{j=0}^{m} P(\Omega_{x_j}^c) = 1$$

$\forall \omega \in \bigcap_{j=0}^{m} \Omega_{x_j}$ $\quad -\frac{1}{m} \leq \liminf |F_n(x) - F(x)| \leq \limsup |F_n(x) - F(x)| \leq \frac{1}{m}$

send m to ∞

$\forall \omega \in \bigcap_{m=1}^{\infty} \bigcap_{j=1}^{m} \Omega_{x_j}(m)$

$$\lim_{n \to \infty} |F_n(x) - F(x)| = 0$$

$$P\left(\bigcap_{m=1}^{\infty} \bigcap_{j=0}^{m} \Omega_{x_j, m}\right) = 1 - P\left(\bigcup_{m=1}^{\infty} \left(\bigcup_{j=0}^{m} \Omega_{x_j, m}^c\right)\right)$$

$$\geq 1 - \sum_{m=1}^{\infty} \underbrace{P\left(\bigcup_{j=0}^{m} \Omega_{x_j, m}^c\right)}_{0} = 1$$