0-1 Law

Borel 0-1 Law   Kolmogorov.

Kolmogorov 0-1 Law (tail event)

Tail Function

Three applications.

0-1 Law

① By BC Lemma If $A_1\ A_2 \cdots \in \mathcal{F}$ are independent

$\mathbb{P}(A_n \text{ i.o.}) = 0$ or 1

② If $X_1\ X_2 \cdots$ i.i.d.

$\mathbb{P}(\lim_{n\to\infty} \frac{S_n}{n} \text{ exists}) = 0$ or 1

depends. on. $\mathbb{E} < \infty$ or $\mathbb{E} = \infty$.

Borel's 0-1 Law. About a sequence of events.

$(\Omega, \mathcal{F}, \mathbb{P})$

Let $A_1\ A_2 \cdots \in \mathcal{F}$. keep doing $\cup\ \cap\ ^c$

def. $\mathcal{A} := \sigma(A_1\ A_2 \cdots) \subset \mathcal{F}$

Assume ① $A \in \mathcal{A}$

② $A \perp\!\!\!\perp$ any $\boxed{\text{finite}}$ collection of A_i's.

Then,

$\mathbb{P}(A) = 0$ or 1

Δ Do not assume $A_1 \cdots \perp\!\!\!\perp$

Δ. See A_n i.o. as $\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_n$

① By BC Lemma If $A_1\ A_2 \cdots \in \mathcal{F}$ are independent

$\mathbb{P}(A_n \text{ i.o.}) = 0$ or 1

choose. $\mathcal{A} = \sigma(A_1\ A_2 \cdots)$

Let. $A = A_n$ i.o.

① $A \in \mathcal{A}$ Because $A = \bigcap_{m=1}^{\infty} \underbrace{\bigcup_{n=m}^{\infty} A_n}_{B_m}$.

$$= \bigcap_{m=k}^{\infty} \bigcup_{n=m}^{\infty} A_n \perp\!\!\!\perp \{A_1 \cdots A_{k-1}\}$$

$\because k$ is arbitary

$\therefore A \perp\!\!\!\perp$ any finite $A_1 \cdots A_k \cdots$

① We have considered the limsup.

We can also consider the liminf.

Extension.

We can consider more event A

$$A = \liminf = \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} A_n.$$

$$= \bigcup_{m=k}^{\infty} \bigcap_{n=m}^{\infty} A_n.$$

Any other things?

eg. $(\limsup A_{2n}) \cup (\liminf A_{2n+1})$

② We no longer require $A_1 \cdots$ are $\perp\!\!\!\perp$

Extension. $A_1 \cdots$ do no need to be $\perp\!\!\!\perp$

$A_1 = A_2 = A \quad A_3 \cdots \perp\!\!\!\perp$ is also OK.

① By BC Lemma If $\underline{A_1\ A_2 \cdots \in \mathcal{F}}$ are independent

$\mathbb{P}(A_n \text{ i.o.}) = 0$ or 1

Borel's 0-1 Law About a sequence of events.

$(\Omega, \mathcal{F}, \mathbb{P})$

Let $A_1\ A_2 \cdots \in \mathcal{F}$. keep doing $\cup \cap {}^c$

Let. $\mathcal{A} := \sigma(A_1\ A_2 \cdots) \subset \mathcal{F}$

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② $A \perp\!\!\!\perp$ any $\boxed{\text{finite}}$ collection of A_i's.

Then.

$\mathbb{P}(A) = 0$ or 1

① A is generated on sequence $A_1 \cdots A_n \cdots$

② A is $\perp\!\!\!\perp$ $\Rightarrow$ A is only about the tail part.

A is kind of like something $\xrightarrow{\lim}$

Comparison. $a_1 \ a_2 \cdots \geq 0$

$$\sum_{n=1}^{\infty} a_n = \infty.$$

$$\xrightarrow{\text{or}} \text{const} > 0$$

$$\lim_{m\to\infty} \sum_{n=m}^{\infty} a_n = 0 \quad \text{if } \sum_{n=1}^{\infty} a_n \to \text{const}.$$

$$= \infty \quad \text{if } \sum_{n=1}^{\infty} a_n = \infty.$$

$$\lim_{m\to\infty} \sum_{n=m}^{\infty} a_n = \infty \text{ or } 0$$

Heuristic proof of BC-01

① $A \in \sigma(A_1, A_2, \cdots)$

② $A \perp\!\!\!\perp A_1 \cdots A_K$

$\Rightarrow A \perp\!\!\!\perp B \in \sigma\{A_1 \cdots A_K\}$ $K \to \infty$

$A \perp\!\!\!\perp B \in \sigma\{A_1 \cdots\}$

$\Rightarrow A \perp\!\!\!\perp A \quad \therefore P(A \cap A) = P(A) \cdot P(A)$

$$\underset{P(A).}{\parallel}$$

$P(A) = 0$ or 1

Kolmogorov 0-1 law

Let $X_1\ X_2\ \cdots\ \Omega \to \mathbb{R}$ independent.

$\mathcal{H}_n = \sigma\{X_{n+1}\ X_{n+2}\ \cdots\}$

$\ne \bigcup_{i=n+1}^{\infty} \sigma(X_i)$

$= \sigma\{\bigcup_{i=n+1}^{\infty} \sigma(X_i)\}$

if $E_n \in \mathcal{H}_n$

E_n is given in terms of $X_i^{-1}(B_i)$ $i \ge n+1$

$X: \Omega \to \mathbb{R}$

$\sigma(X) = \{\underbrace{X^{-1}(C)}\ \ C \in \mathcal{B}(\mathbb{R})\}$

$\{\omega: X(\omega) \in C\}$

$\subset \mathcal{F}$

$\sigma(X)$: info you can tell through the value of $X(\omega)$.

eg. $X(\omega) = \begin{cases} 1 & \omega = 2, 4, 6. \\ 0 & \omega = 1, 3, 5. \end{cases}$

$H_n \supset H_{n+1} \supset \cdots$

so we can take the intersection.

$H_\infty = \bigcap_{i=0}^{\infty} \mathcal{H}_i$: tail σ-field.

$\underline{E \in \mathcal{H}_\infty}$ tail event.

① E is given in terms of $X_i^{-1}(B_i)$ $i \ge 1$

② E is $\perp$ of any finite collection of $X_i^{-1}(B_i)$

eg. ①

$\{X_n > 0\ \text{i.o.}\} = \{\omega \in \Omega : X_n(\omega) > 0\ \text{i.o.}\}$

$= \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \underbrace{\{X_n > 0\}}_{X_n^{-1}((0,\infty))}$

satisfies ① ②

②

$\{\limsup_{n\to\infty} X_n = \infty\} = \{\omega \in \Omega : \limsup_{n\to\infty} X_n(\omega) = \infty\}$

① in terms of X_i

② You can change/delete the first finite X_i the thing is the same.

③ $\{\sum_{n=1}^{\infty} X_n$ converges$\}$ ① ✓ ② ✓

④ $\{\sum_{n=1}^{\infty} X_n$ converges to $1\}$ ① ② ✗ It dependends on the fist finite X_i

$X_1 \; X_2 \cdots$ i.i.d. What's conclusion of General tail event?

$\mathcal{H}_n, \mathcal{H}_{n+1} \cdots \mathcal{H}_\infty$ tail σ-field

eg. $\limsup X_i = \infty$ tail event.

Komoglor 0-1 Law.

Thm. If E is a tail event then.

$\mathbb{P}(E) = 0$ or 1

Tail Function.

$Y: \Omega \to \mathbb{R} \cup \{\pm\infty\}$ is $\mathcal{H}_\infty$-measurable.

$\mathcal{H}_n \subset \mathcal{F}$

$\mathcal{H}_\infty \subset \mathcal{F}$

$\mathcal{H}_\infty$ much smaller than $\mathcal{F}$.

$\mathcal{H}_\infty$-measurable stronger than. $\mathcal{F}$-measurable.

is a func of $X_1 \; X_2 \cdots$

but its value. does not depend on

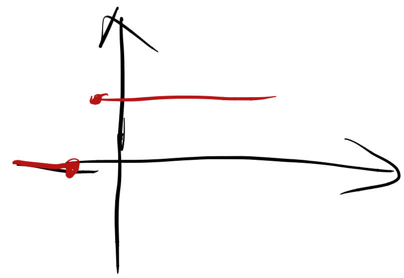
any finite collection of X_i

eg. $\limsup_{n\to\infty} X_n$. is $\Omega \to \mathbb{R} \cup \{\pm\infty\}$ ✓

$X_1 + X_2$ ✗

$\sum_i X_i$ ✗

$$\boxed{\begin{gathered} F_Y(x) = \mathbb{P}(\underbrace{Y \le x}_{\text{tail event}}) = 0 \text{ or } 1 \\[2em] \therefore\ \mathbb{P}(Y \equiv c) = 1 \end{gathered}}$$



App. Let $X_1, X_2, \dots$ be independent.

$\lim\limits_{n\to\infty} \frac{S_n}{n}(\omega)$ exist ?

only require $X_1 \cdots \perp$

$$\mathbb{P}\left(\omega : \lim_{n\to\infty} \frac{S_n}{n}(\omega) \text{ exist}\right) \overset{?}{=} 0 \text{ or } 1$$

not necessarily identical.

$$\mathbb{P}\left(\lim_{n\to\infty} \frac{S_n}{n} \text{ exists}\right) = 0 \text{ or } 1$$

pf. $= \mathbb{P}\left(\underline{\limsup\limits_{n\to\infty} \frac{S_n}{n}} = \underline{\liminf\limits_{n\to\infty} \frac{S_n}{n}}\right) \sim$ tail event.

$= 0$ or 1

tail function

$$= \limsup_{n\to\infty} \frac{S_n}{n} = \limsup_{n\to\infty} \underbrace{\frac{X_1+\dots+X_k}{n}}_{\to\, 0 \text{ a.s.}} + \frac{X_{k+1}+\dots+X_n}{n}$$

Application. 2. (Random Power Series)

$X_1, X_2, \dots$ i.i.d. $\sim Exp(1)$

$f(z) = \sum\limits_{n=0}^{\infty} X_n(\omega) z^n$ $\qquad \mathbb{P}(X_i > y) = e^{-y}$ $\quad y \ge 0$

$$R(\omega) = \frac{1}{\limsup\limits_{n\to\infty} |X_n(\omega)|^{\frac{1}{n}}} \rightarrow \text{tail function.}$$

① n terms of X_n

②. has nothing to do with first finite X_n.

By Kolmogorov 0-1 law $\mathbb{P}(R=C)=1$

Claim $\mathbb{P}(R=1)=1 \Leftrightarrow \mathbb{P}(\limsup_{n\to\infty} |X_n|^{\frac{1}{n}}=1)=1$

$\Leftrightarrow$ ① $\mathbb{P}(|X_n|^{\frac{1}{n}} > 1+\varepsilon \text{ i.o.})=0 \quad \forall \varepsilon>0$

② $\mathbb{P}(|X_n|^{\frac{1}{n}} > 1-\varepsilon \text{ i.o.})=1$

$\Leftarrow$ ①. $\sum_{n=1}^{\infty} \mathbb{P}(|X_n| > (1+\varepsilon)^n) < \infty \Leftarrow \sum_{n=1}^{\infty} e^{-(1+\varepsilon)^n} < \infty$

②. $\sum_{n=1}^{\infty} \mathbb{P}(|X_n| > (1-\varepsilon)^n) = \infty \Leftarrow \sum_{n=1}^{\infty} e^{-(1-\varepsilon)^n} = \infty.$

$X \sim \exp(1)$