eg. CLT Let $X_1 X_2 \cdots$ i.i.d. $\mathbb{E}|X|^2 < \infty$ only works for i.i.d. r.v.s.

$\mathbb{E}X = \mu$. $\operatorname{Var} X = \sigma^2$ Let $S_n = \sum_{i=1}^{n} X_i$ $\Downarrow$

Then. $\frac{S_n - n\mu}{\sqrt{n}} \xrightarrow{D} N(0, \sigma^2)$ all r.v. have small effects.

CLT

Lindeberg - Feller CLT

Allow r.v. to have diff size.

$$\begin{matrix} X_{1,1} & & \\ \vdots & \ddots & \\ X_{n,1} & \cdots & X_{n,n} \end{matrix}$$

$X_{n,m}$. $1 \le m \le n$ be independant

$\mathbb{E}X_{n,m} = 0$ Let. $S_n = \sum_{m=1}^{n} X_{n,m}$

(i) $\sum_{m=1}^{n} \mathbb{E}X_{n,m}^2 \to \sigma^2$ variance

(ii) $\forall \varepsilon > 0$ $\left(\sum_{m=1}^{n} \mathbb{E}X_{n,m}^2 \mathbb{1}(|X_{n,m}| > \varepsilon)\right) \to 0$

Non $X_{n,m}$ should play the significant role.

$X_{n,m}$ is not scaled by $\frac{1}{\sqrt{n}}$

no $\frac{1}{\sqrt{n}}$ helps us to do the Taylor expansion.

Then we have.

$$S_n \xrightarrow{D} N(0, \sigma^2)$$

$\Downarrow$ use the smallest of r.v.s.

An alternative Taylor expansion for c.f. itself.

对. $\mathbb{E}|X|^k < \infty$

$$\left|\mathbb{E}e^{itX} - \sum_{j=0}^{k} \frac{\mathbb{E}X^j}{j!} (it)^j\right| \le \mathbb{E}\min\left(|tX|^{k+1}, 2|tX|^k\right)$$

t could be large.

proof. $\forall$ given $t \in \mathbb{R}$

$$\mathbb{E}e^{itS_n} = \prod_{m=1}^{n} \mathbb{E}e^{itX_{n,m}} = \prod_{m=1}^{n} \varphi_{n,m}(t).$$

Aim. $\prod_{m=1}^{n} \varphi_{n,m}(t) - e^{-\frac{\sigma^2 t^2}{2}} \to 0$

$$\left| \varphi_{n,m}(t) - \sum_{j=0}^{2} \frac{\mathbb{E} X_{n,m}^j}{j!} (it)^j \right| \le \mathbb{E} \min\left(\underline{|t X_{n,m}|^3}, 2 |t X_{n,m}|^2 \right) \quad k=2.$$

Truncation. $\Leftarrow$ can not do this not enough moments.

$$\left| \varphi_{n,m}(t) - \left(1 - \frac{t^2}{2} \underline{\mathbb{E} x_{n,m}^2}_{\sigma_{n,m}^2}\right) \right| \le \mathbb{E} \min(\quad) \mathbb{I}(|x_{n,m}| \le \varepsilon) \Rightarrow |x_{\cdot\cdot}| \text{ Bounded}$$

$$+ \mathbb{E} \min(\quad) \mathbb{I}(|x_{n,m}| > \varepsilon)$$

Do not need to worry about the moments.

We do truncation if we do not have enough moments

Due to (ii) Truncate at level ε.

take one $x_{n,m}$ out Bounded By

$$\le \mathbb{E} |t x_{n,m}|^3 \mathbb{I}(|x_{n,m}| \le \varepsilon)$$

$$+ \mathbb{E}(2 |t x_{n,m}|^2) \mathbb{I}(|x_{n,m}| > \varepsilon)$$

$$\le \varepsilon \cdot |t|^3 \mathbb{E} |x_{n,m}|^2 \mathbb{I}(|x_{n,m}| \le \varepsilon)$$

$$+ t^2 \cdot \mathbb{E} |x_{n,m}|^2 \mathbb{I}(|x_{n,m}| > \varepsilon)$$

(i) (ii) about the sum of terms not individual ones.

$\Rightarrow$ ① take the sum.

② translate the sum into product

$$\sum_{m=1}^{n} \left| \varphi_{n,m}(t) - \left(1 - \frac{t^2}{2} \sigma_{n,m}^2\right) \right| \le \varepsilon |t|^3 \left(\sum_{m=1}^{n} \mathbb{E} |x_{n,m}|^2 \mathbb{I}(|\;| \le \varepsilon) \right)^{①}$$

$$+ t^2 \left(\sum_{m=1}^{n} \mathbb{E} |x_{n,m}|^2 \mathbb{I}(|\;| > \varepsilon) \right)$$

(ii) $\to$ ② $\to 0$

take limsup. (i) $\to$ ①+② $\to \sigma^2$.

$$\limsup_{n\to\infty} \sum_{m=1}^{n} \left| \varphi_{n,m}(t) - \left(1 - \frac{t^2 \sigma_{n,m}^2}{2}\right) \right| \le \varepsilon |t|^3 \sigma^2.$$

Send $\varepsilon \to 0$ $\quad \lim_{n\to\infty} \sum_{m=1}^{n} \left| \varphi_{n,m}(t) - \left(1 - \frac{t^2 \sigma_{n,m}^2}{2}\right) \right| = 0$

Translate the property of sum to product.

$$\left|\prod_{m=1}^{n} z_m - \prod_{m=1}^{n} w_m\right| \le \theta \sum_{m=1}^{n} |z_m - w_m|$$

$|z_m|, |w_m| \le \theta \quad \forall\, m = 1 \cdots n.$

$$x_1 x_2 - y_1 y_2 = x_1 x_2 - x_1 y_2 + x_1 y_2 - y_1 y_2.$$

$$= x_1 (x_2 - y_2) + y_2 (x_1 - y_1) \le \theta \cdot \sum_{m=1}^{2} |x_m - y_m|$$

$$\lim_{n\to\infty} \sum_{m=1}^{n} \Big|\underbrace{\varphi_{n,m}(t)}_{z_m} - \underbrace{(1 - \frac{t^2 \sigma_{n,m}^2}{2})}_{w_m}\Big| = 0$$

$$\left|\prod_{m=1}^{n} \underbrace{\varphi_{n,m}(t)}_{|\varphi_{n,m}(t)| \le 1} - \prod_{m=1}^{n} (1 - \underbrace{\frac{t^2 \sigma_{n,m}^2}{2}}_{t \uparrow \text{ may be very negative.}})\right| \le \theta \sum_{m=1}^{n} \left|\varphi_{n,m}(t) - (1 - \frac{t^2 \sigma_{n,m}^2}{2})\right|$$

But $\sigma_{n,m}^2$ small. $\therefore \frac{t^2 \sigma_{n,m}^2}{2} \to 0.$

$$\sigma_{n,m}^2 = E X_{n,m}^2 = E X_{n,m}^2 \mathbb{1}(|X_{n,m}| \le \varepsilon) \rightsquigarrow \le \varepsilon^2$$

$$+ E X_{n,m}^2 \mathbb{1}(|\ \ | > \varepsilon) \rightsquigarrow 0$$

$\to 0 \quad \because \text{sum} \to 0.$

$\therefore \sigma_{n,m}^2 \to 0 \quad \varepsilon \to 0 \quad n \to \infty.$

$$\lim_{n\to\infty} \sup_{1 \le m \le n} \sigma_{n,m}^2 = 0$$

we n is sufficiently large

$$\left|\prod_{m=1}^{n} \varphi_{n,m}(t) - \prod_{m=1}^{n} (1 - \frac{t^2 \sigma_{n,m}^2}{2})\right| \rightsquigarrow |\varphi_{n,m}(t)| \le 1 \quad \left|1 - \frac{t^2 \sigma_{n,m}^2}{2}\right| \le 1$$

$\therefore$ take $\theta = 1$

$$\lim_{n\to\infty} \left|\prod_{m=1}^{n} \varphi_{n,m}(t) - \underbrace{\prod_{m=1}^{n} (1 - \frac{t^2 \sigma_{n,m}^2}{2})}_{e^{\sum_{m=1}^{n} \log(1 - \frac{t^2 \sigma_{n,m}^2}{2})}}\right| \le \lim_{n\to\infty} \sum_{m=1}^{n} \left|\varphi_{n,m}(t) - (1 - \frac{t^2 \sigma_{n,m}^2}{2})\right|$$

$$= 0$$

$$= e^{-\sum_m \frac{t^2 \sigma_{n,m}^2}{2}} + \text{error}$$

$$\longrightarrow e^{-\frac{\sigma^2 t^2}{2}} \quad \text{when. } n \to \infty.$$

From

对. $\mathbb{E}|X|^k < \infty$

$$\left|\mathbb{E}e^{itx} - \sum_{j=0}^{k} \frac{\mathbb{E}X^j}{j!}(it)^j\right| \le \mathbb{E}\min\left(|tX|^{k+1}, 2|tX|^k\right)$$

to (Taylor Expansion to c.f.)

对. $\mathbb{E}|X|^k < \infty$

$$\left|\mathbb{E}e^{itx} - \sum_{j=0}^{k} \frac{\mathbb{E}X^j}{j!}(it)^j\right| \le \mathbb{E}\min\left(\frac{|tX|^{k+1}}{(k+1)!}, \frac{2|tX|^k}{k!}\right)$$

f: $(k+1)$ - times differentiable.

$$f(x) = \sum_{j=0}^{k} \frac{f^{(j)}(0)}{j!} x^j + \frac{1}{k!}\int_0^x (x-a)^k f^{(k+1)}(a)\, da.$$

eg. $f(x) = e^{ix}$

$$\left|e^{ix} - \sum_{j=0}^{k} \frac{(ix)^j}{j!}\right| = \left|\frac{1}{k!}\int_0^x (x-a)^k e^{ia}\, da\right| \to \frac{1}{k!}\int_0^x (x-a)^k da = \frac{1}{(k+1)!} x^{k+1}$$

What if the Lindeberg condition fail?

Let $X_1\ X_2 \cdots$ i.i.d. $\mathbb{E}X_1 = 0$ $\mathbb{E}X_1^2 = \sigma^2$

$$\sum_m \boxed{\frac{X_m}{\sqrt{n}}} \Rightarrow N(0, \sigma^2)$$

$\tilde{X}_{n,m}$ $\mathbb{E}\tilde{X}_{n,m} = 0$

(i)

$$n \cdot \mathbb{E}\tilde{X}_{n,m}^2 = \frac{n}{n}\mathbb{E}(X_m^2) = \sigma^2$$

when n is large $\mathbb{D}$ (rare event) is small

(ii)

$$n \cdot \mathbb{E}\tilde{X}_{n,m}^2 \mathbb{1}(|\tilde{X}_{n,m}| > \varepsilon) = \mathbb{E}\boxed{X_m^2 \mathbb{1}(|X_m| > \varepsilon\sqrt{n})}$$

$\square \le X_m^2$ $\mathbb{E}X_m^2 = \sigma^2$ integrable.

$\square \to 0$ a.s. as $n \to \infty$.

By DCT $\mathbb{E}\square \to \mathbb{E}0 = 0$

What if the $X_{n,m}^2$ small $\sum X_{n,m}^2 \to \sigma^2$ But Var mainly comes from the rare event

Poisson Convergence for Rare Events

Let. $Y_{n,m} \sim Be(p_n)$ $p = p_n = \frac{c}{n}$ $Y_{n,m}$ iid $1 \le m \le n$

Let $X_{n,m} = Y_{n,m} - p$ $E X_{n,m} = 0$ $E X_{n,m} = Var(Y_{n,m}) = p(1-p)$.

$\tilde{S}_n = \sum_{m=1}^{n} X_{n,m}$ (i) $\sum_{m=1}^{n} E X_{n,m}^2 = n \times \underset{= \frac{c}{n}}{p}(1-p) \to c$

(ii) $\sum_{m=1}^{n} E X_{n,m}^2 \mathbb{1}(|X_{n,m}| > \varepsilon)$

$$X_{n,m} = \begin{cases} 1-p & \text{with prop } p \\ \underbrace{-p}_{\to 0} & \text{with prop } \underbrace{1-p}_{\to 1} \end{cases}$$

$\therefore$ $X_{n,m}$ is small

But with. prop p $1-p$ is big $\to 1$ >> than typical value.

$$E X_{n,m}^2 = \underbrace{\underbrace{(1-p)^2}_{1^2} \cdot \underbrace{p}_{\text{small } 0}}_{\text{main.}} + \underbrace{\underbrace{(-p)^2}_{0^2} \cdot \underbrace{(1-p)}_{=1}}_{\to 0}.$$

(ii) $\sum_{m=1}^{n} E X_{n,m}^2 \mathbb{1}(|X_{n,m}| > \varepsilon) \to c$ Lindeberg con fails.

$\tilde{S}_n = \sum X_{n,m} = \sum_{m=1}^{n} Y_{n,m} - \overset{c}{\overset{\shortparallel}{np}}$

$S_n = \sum_{m=1}^{n} Y_{n,m}$ $Y_{n,m} \sim Be(p)$ $p = \frac{c}{n}$

Since we know the Distribution of $Y_{n,m}$.

We do not need to use the c.f. we can use d.f. directly on p.m.f.

$\mathbb{P}(S_n \le x)$

$$\mathbb{P}(S_n = k) = \binom{n}{k} \beta^k (1-\beta)^{n-k} = \underbrace{\frac{n!}{k!(n-k)!} \frac{c^k}{n^k}}_{\to 1} \cdot \frac{\overbrace{(1-\frac{c}{n})^n}^{e^{-c}}}{\underbrace{(1-\frac{c}{n})^k}_{\to 1}}$$

$$\to \frac{c^k e^{-c}}{k!} \quad \text{pmf. poisson. (c).}$$

$pn = c$

n is large

p is small

np is moderate

$\downarrow$

Poisson Law.

n is large

$np(1-p) \to \infty$.

$\downarrow$

CLT

Let. $X_{n,m}$ iid $\mathbb{E} X_{n,m} = 0$ $\mathrm{Var}\, X_{n,m} = \frac{\sigma^2}{n}$

$S_n = \sum_{m=1}^{n} X_{n,m}$ may not go to Gaussian

If $\widetilde{X_{n,m}} = \frac{X_m}{\sqrt{n}}$ CLT $\mathbb{E}|X_m|^3 < \infty$

If $X_{n,m} \sim Be(\frac{c}{n}) \cdot \frac{c}{n}$ Poisson.

$$\mathbb{E} e^{it\widetilde{X_{n,m}}} = \mathbb{E}\, e^{i\frac{t}{\sqrt{n}} X_m} = 1 - \frac{t^2}{2n}\sigma^2 + \boxed{\frac{t^3}{6n^{3/2}} \mathbb{E} X_m^3}$$

$n^{-\frac{3}{2}}$

$$\mathbb{E}\, e^{itX_{n,m}} = 1 - \frac{t^2}{2n} c + \frac{(it)^3}{6} \mathbb{E} X_{n,m}^3 + \cdots$$

$\sim \frac{1}{n}$. Doesn't decay

$$\mathbb{E} X_{n,m}^2 = (1-p)^3 \times p + (-p)^3 (1-p)$$

Thm For each n let $X_{n,m}$ $1 \le m \le n$ be ⫫ with

$\mathbb{P}(X_{n,m}=1) = p_{n,m}$ $\mathbb{P}(X_{n,m}=0) = 1-p_{n,m}$

① sum of r.v.s^2 $\to$ const. $\mathbb{E}X_{n,m} = p_{n,m}$

② sum of truncated r.v.s^2 $\to 0$ $Var(X_{n,m}) = p_{n,m}\underbrace{(1-p_{n,m})}_{\text{small}}$

$\sim p_{n,m}$

(i) $\sum_{m=1}^{n} p_{n,m} \to \lambda \in (0,\infty)$

(ii) $\max_{1\le m\le n} p_{n,m} \to 0$ Nothing dominate

If $S_n = \sum_{m=1}^{n} X_{n,m}$ then

$S_n \Rightarrow Pois(\lambda)$

$$\mathbb{E}e^{itS_n} = \mathbb{E}\exp\{it\sum_{m=1}^{n} X_{n,m}\} = \mathbb{E}\prod_{m=1}^{n}\exp\{itX_{n,m}\} = \prod_{m=1}^{n}\mathbb{E}e^{itX_{n,m}}$$

Why Pois distribution is every where.

$$\mathbb{E}e^{itX_{n,m}} = \mathbb{E}\sum_{j=0}^{2}\frac{(itX_{nm})^j}{j!} = \mathbb{E}\sum_{j=0}^{2}\frac{i^j t^j X_{nm}^j}{j!} = 1+\mathbb{E}\frac{itX_{nm}}{1!} + \mathbb{E}\frac{i^2t^2X_{nm}^2}{2!} = 1+it\,\mathbb{E}X_{n,m} + \frac{i^2t^2}{2!}\mathbb{E}X_{nm}^2$$

Proof: $\varphi_{n,m}(t) = \mathbb{E}e^{itX_{n,m}} = p_{nm}e^{it} + (1-p_{nm})$

$$\mathbb{E}e^{itS_n} = \prod_{m=1}^{n}(1+p_{nm}(e^{it}-1)) \to e^{\sum p_{nm}(e^{it}-1)}$$

c.f. of $Pois(\lambda)$.

$$\left|\prod_{m=1}^{n}(1+p_{nm}(e^{it}-1)) - \prod_{m=1}^{n}e^{p_{n,m}(e^{it}-1)}\right|$$
$$\le \theta\sum_{m=1}^{n}\left|1+p_{nm}(e^{it}-1) - e^{p_{nm}(e^{it}-1)}\right|$$

$|1+p_{n,m}(e^{it}-1)| = |1-p_{n,m}+p_{nm}e^{it}| \le (1-p_{nm}) + |p_{nm}| = 1$

$|e^{p_{nm}(e^{it}-1)}| = e^{p_{nm}\,\underbrace{Re(e^{it}-1)}_{\le 0}} \le 1$

$$\left|\prod_{m=1}^{n}(1+p_{nm}(e^{it}-1)) - \prod_{m=1}^{n}e^{p_{n,m}(e^{it}-1)}\right|$$
$$\le \sum_{m=1}^{n}\left|1+p_{nm}(e^{it}-1) - e^{p_{nm}(e^{it}-1)}\right|$$

$p_{nm}\to 0$.

Taylor around 0

$$1+p_{nm}(e^{it}-1) + O(p_{nm}^2)$$

$$\le C\cdot\sum_{m=1}^{n}p_{n,m}^2 \le C\cdot\underbrace{\max p_{n,m}}_{0}\underbrace{\sum p_{nm}}_{\lambda} \to 0 \quad (n\to\infty)$$

The fail of the Lindeberg condition.

tough X_{nm} small with large prob.

the rare events take big value for X_{nm}^2

so the $\sum X_{nm}^2 \mathbb{I}(\cdots) \nrightarrow 0$.