probability space $(\Omega, \mathcal{F}, \mathbb{P})$

measure space $(\Omega, \mathcal{F}, \mu)$

mearable space $(\Omega, \mathcal{F})$

$\mathbb{P}: \mathcal{F} \to [0,1]$

set function.

$f: \mathbb{R} \to \mathbb{R}$

$x_1 \; x_2 \cdots x_n \cdots$

$\lim_n x_n = x$ exists.

Continuity of $\mathbb{P}$ (a set function)

given. $A_1, A_2 \cdots A_n \cdots \in \mathcal{F}$

Suppose. $\lim_n A_n$. exists.

If. $\lim_n f(x_n) = f(\lim_n x_n)$

then we say f is continuous

let $x = \lim_n x_n$,

$$\lim_n \mathbb{P}(A_n) \overset{?}{=} \mathbb{P}(\lim_n A_n)$$

Def of lim of set: lim sup? / lim inf?

$$\limsup_{n\to\infty} A_n = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_n = \{\omega \in \Omega : \omega \in A_n \text{ for infinitely many } n\}$$

$= A_n$ i.o.

(infinitely often)

$$\liminf_{n\to\infty} A_n = \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} A_n = \{\omega \in \Omega : \omega \in A_n \text{ for all but finitely many } n\}$$

$= A_n$. a.b.f.o.

$\liminf_{n\to\infty} A \subseteq \limsup_{n\to\infty} A_n$

Def. $\lim_{n\to\infty} A_n$ exists if $\limsup_{n\to\infty} A_n = \liminf_{n\to\infty} A_n$.

let $\lim_{n\to\infty} A_n = \limsup_{n\to\infty} A_m = \liminf_{n\to\infty} A_n$.

eg. $A_1 \subset A_2 \subset \cdots$

$$\limsup A_n = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_n = \bigcup_{n=1}^{\infty} A_n.$$

$\Rightarrow \lim_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} A_n.$

$$\liminf A_n = \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} A_n = \bigcup_{m=1}^{\infty} A_m.$$

$A_1 \supset A_2 \supset \cdots$

$$\limsup A_n = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_n = \bigcap_{m=1}^{\infty} A_m \quad \Rightarrow \lim_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} A_n$$

$$\liminf A_n = \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} A_n = \bigcap_{n=1}^{\infty} A_n.$$

Thm. If $\lim A_n$ exist then. $\leftarrow$ Continuity of $\mathbb{P}$

$$\lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}(\lim_{n\to\infty} A_n)$$

Pf for special case $A_1 \subset A_2 \subset \cdots$

It suffices to check

$$\mathbb{P}(\underbrace{\bigcup_{n=1}^{\infty} A_n}_{\lim_{n\to\infty} A_n}) = \lim_{n\to\infty} \mathbb{P}(A_n).$$

let $B_1 = A_1$

$B_i = A_i \setminus A_{i-1} = A_i \cap (A_i)^c$.

mon incr, bounded $\Rightarrow$ lim exists.

PUA $\bigcup A_n = \bigcup B_n$.

countable additivity

$$1 \geq \boxed{\mathbb{P}(\bigcup A_n)} = \mathbb{P}(\bigcup B_n) = \sum_{n=1}^{\infty} \mathbb{P}(B_n) = \lim_{n\to\infty} \sum_{i=1}^{n} \mathbb{P}(B_i)$$

$= \lim_{n\to\infty} \mathbb{P}\left(\bigcup_{i=1}^{n} B_i\right) = \lim_{n\to\infty} \mathbb{P}(A_n)$

finite additivity

Continuity $\Leftrightarrow$ countable additivity

$(\Omega, \mathcal{F}, \mathbb{P})$

probability space on $\mathbb{R}^d$: take $\Omega = \mathbb{R}^d$.

- case $d=1$

$(\mathbb{R}, ?\ \mathcal{B}(\mathbb{R}))$

$\mathcal{F} = \{\phi, \mathbb{R}\}$

$\mathcal{F} = 2^{\mathbb{R}} = \{0,1\}^{\mathbb{R}}$

$\mathcal{B}(\mathbb{R})$

probability of intervals ??? interested events.

Borel σ-field (on $\mathbb{R}$) = smallest σ-field containing all intervals. (collection of)

σ-field generated by $\mathcal{A}$: $\mathcal{A}$ is any collection of events.

$\sigma(\mathcal{A})$

take $\mathcal{A}$ as a basis, keep taking countable union, and complement. until we cannot enlarge the collection.

Borel σ-field on $\mathbb{R}$ can be generated from.

① collection of all open interval (a, b)

② closed $[a, b]$

③ half closed $(a, b]$

④. $(-\infty, b]$

⑤ collection of all open sets.

Borel sets any event in Borel σ-field.

eg. intervals.

$\{b\} = [a,b] \cap [b,c]$

$\mathbb{Q}$

$\mathbb{R} \setminus \mathbb{Q}$

$(\mathbb{R}, \mathcal{B}(\mathbb{R}), ?)$

How to determine a $\mathbb{P}$ on $\mathbb{R}$ $\mathcal{B}(\mathbb{R})$?

Stieltjes measure function. (way to fix a measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$

(eg. distribution func).

Def. (SMF) $F: \mathbb{R} \to \mathbb{R}$

① F is nondecreasing

② F is right continuous, that is $\lim_{y \downarrow x} F(y) = F(x)$.

Thm. given a SMF $F: \mathbb{R} \to \mathbb{R}$ there exists a unique.

measure μ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ s.t.

$$\mu((a,b]) = F(b) - F(a).$$

eg. Counting measu on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$

$\mu(A) = $ cardinality of A

$\Rightarrow \mu((a,b]) = \infty \notin \mathbb{R}$ $\forall a < b$.

$\therefore F$ does not exist

eg Lebesgue measure

$F(x) = x + c \quad \mu((a,b]) = b-a.$

You can change c.

$\therefore$ F is not unique.

$\{(a,b] \ a<b \in \mathbb{R}\} \leftarrow$ semi algebra.

extension of μ. from. semi algebra. to Borel σ-field (generated by semialgebra) is unique

Caratheodory extension thm.

Why Right Continuity should be necessary

If F is used to identify a measure in the way like (SMF Th)

T it has to be right continuous.

$x>a$

$$\lim_{y \downarrow x} (F(y) - F(a)) = \lim_{n\to\infty} (F(x+\tfrac{1}{n}) - F(a)) = \lim_{n\to\infty} \mu((a, x+\tfrac{1}{n}])$$

$\downarrow$ $\therefore$ take lim $\Leftrightarrow$ take intersection.

$$= \mu(\lim_{n\to\infty} (a, x+\tfrac{1}{n}]) = \mu(\bigcap_{n=1}^{\infty} (a, x+\tfrac{1}{n}])$$

$$= \mu((a,x]) = F(x) - F(a)$$

$$\lim_{y \uparrow x} (F(y) - F(a)) = \lim_{n\to\infty} (F(x-\tfrac{1}{n}) - F(a)) = \lim_{n\to\infty} \mu((a, x-\tfrac{1}{n}])$$

$$= \mu(\lim_{n\to\infty} (a, x-\tfrac{1}{n}]) \uparrow$$

$\therefore$ take lim $\Leftrightarrow$ take union.

$$= \mu(\bigcup_{n=1}^{\infty} (a, x-\tfrac{1}{n}])$$

$$= \mu((a,x)) = \mu((a,x]) - \mu\{x\}$$

Probability space on $\mathbb{R}^d$.

$\Omega = \mathbb{R}^d$

$\mathcal{F} = \mathcal{B}(\mathbb{R}^d)$: σ-field. generated by all open sets in $\mathbb{R}^d$.

equivalently all rectangles in $\mathbb{R}^d$

$[a_1, b_1] \times \cdots \times [a_d, b_d]$

$(a_1, b_1] \times \cdots \times (a_d, b_d]$.

$(-\infty, b_1] \times \cdots \times (-\infty, b_d]$

How to fix a measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$: μ.

Def. Stieltjes measure func on $\mathbb{R}^d$.

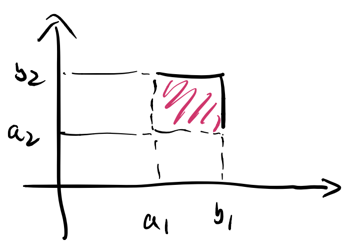
$F : \mathbb{R}^d \to \mathbb{R}$ eg. j.c.d.f.

(i) F is non decreasing i.e. $F(x) \le F(y)$ $x \le y$ $\begin{matrix} x = (x_1 \cdots x_d) \\ y = (y_1, \cdots y_d) \end{matrix}$

(i.e. $x_i \le y_i$)

(ii) continuity from above $\lim_{y \downarrow x} F(y) = F(x)$ $y \downarrow x$ mean $y_i \downarrow x_i$ $\forall x_i = 1 \cdots d$.

eg. j.c.d.f.



$F(b_1, b_2) = 1$

$F(a_1, b_2) = 1$

$F(b_1, a_2) = 1$

$F(a_1, a_2) = 0$

$\mathbb{P}(\square) =$ inclusion exclution formula.

$F(b_1 b_2) - F(a_1, b_2) - F(b_1, a_2) + F(b_1, b_2) = -1$ X

(iii) $A = (a_1 b_1] \times \cdots \times (a_d, b_d]$

$V = \{a_1, b_1\} \times \cdots \times \{a_d, b_d\}$ $\to$ collection of vertices.

$\forall v \in V$ let.

$\operatorname{sgn}(v) = (-1)^{\#\text{ of } a\text{'s in } v}$

(iii) $\Delta_A F = \sum_{v \in V} \text{sign}(v) F(v) \geq 0$

Thm. Suppose $F : \mathbb{R}^d \to \mathbb{R}$ satisfies. (i) ⋯ (iii)

Then there exists a unique measure μ. on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$

s.t.

$\mu(A) = \Delta_A F$ μ is uniquely identified by F

$A = (a_1, b_1] \times \cdots \times (a_d, b_d]$