Truncation (at level M)

$$\bar{X} := X\,\mathbb{1}(|X| \le M) = \begin{cases} X & \text{if } |X| \le M \\ 0 & \text{otherwise.} \end{cases}$$

Thm. Let $\underbrace{X_{n,k} \quad 1 \le k \le n}_{\text{independent}}$ be a triangular array Δ choose a nice b_n.

Let $b_n > 0$ and let $\overline{X_{n,k}} = X_{n,k}\,\mathbb{1}(|X_{n,k}| \le b_n)$

Suppose as $n \to \infty$

(i) $\sum_{k=1}^{n} \mathbb{P}(|X_{n,k}| > b_n) \to 0$

~~(ii)~~ $b_n^{-2} \sum_{k=1}^{n} \mathbb{E}\,\overline{X_{n,k}}^2 \to 0$ replaced by $b_n^{-2} \sum_{k=1}^{n} \mathrm{Var}(\overline{X_{n,k}}) \to 0$

If $S_n = \sum_{i=1}^{n} X_{n,i}$ and

If. $S_n = \sum_{i=1}^{n} X_{n,k}$ and put $a_n = \sum_{k=1}^{n} \mathbb{E}\,\overline{X_{n,k}}$ then.

$$\frac{S_n - a_n}{b_n} \xrightarrow{\mathbb{P}} 0$$

only about $\mathbb{E}\,\overline{X_{n,k}}$

we remove the assumption of $\mathbb{E}\,X_{n,k}$ $\mathrm{Var}\,X_{n,k}$ existence.

pf: $\bar{S}_n = \sum_{k=1}^{n} \overline{X_{n,k}}$

$$\mathbb{P}\left(\left|\frac{S_n - a_n}{b_n}\right| > \varepsilon\right) = \mathbb{P}\left(\underbrace{\left|\frac{S_n - a_n}{b_n}\right| > \varepsilon,\ S_n = \bar{S}_n}_{\subset \left|\frac{\bar{S}_n - a_n}{b_n}\right| > \varepsilon}\right)$$

$$+\, \mathbb{P}\left(\left|\frac{S_n - a_n}{b_n}\right| > \varepsilon,\ S_n \ne \bar{S}_n\right)$$

$$\subset \{S_n \ne \bar{S}_n\} \subset \bigcup_{k=1}^{n} \{X_{nk} \ne \overline{X_{nk}}\}$$

$$\le \mathbb{P}\left(\left|\frac{\bar{S}_n - a_n}{b_n}\right| > \varepsilon\right) + \mathbb{P}\left(\bigcup_{k=1}^{n} \{X_{n,k} \ne \overline{X_{nk}}\}\right)$$

$$\le \frac{\operatorname{Var} \bar{S_n}}{b_n^2 \varepsilon^2} \qquad\qquad \le \sum_{k=1}^{n} \mathbb{P}(|X_{n,k}| > b_n)$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$0 \text{ by (ii)} \qquad\qquad 0 \text{ by (i)}$$

Thm. Let $X_1\ X_2 \cdots$ be $\boxed{\text{i.i.d.}}$ with.

$$x\,\mathbb{P}(|X_1| > x) \to 0 \quad \text{if } x \to \infty$$

Denote $\quad a_n = \mathbb{E}X_1 \mathbb{I}(|X_1| \le n)$ $\quad \boxed{\text{选择 } b_n = n}$

$$S_n = \sum_{i=1}^{n} X_i$$

Then.

$$\frac{S_n}{n} - a_n \xrightarrow{\mathbb{P}} 0$$

Rough discuss on the condition.

Relation? $\begin{cases} x\,\mathbb{P}(|X_1| > x) \to 0 & (1) \\ \mathbb{E}|X_1| < \infty & (2) \end{cases}$ $\quad$ $\mathbb{I}$ is Bern so $\mathbb{E}\mathbb{I} = \mathbb{P}$

$$\int_0^\infty \mathbb{P}(|X_1| > x)\,dx < \infty.$$

$$\mathbb{P}(|X_1| > x) \le \frac{1}{x} \quad x \to \infty.$$

$\because \frac{1}{x}$ is not integrable.

DCT $\quad (2) \to (1)$

$$x\,\mathbb{P}(|X_1| > x) = x\,\mathbb{E}\mathbb{I}(|X_1| > x)$$

$$\le \mathbb{E}|X_1| \mathbb{I}(|X_1| > x)$$

Dominated Convergence Thm

$$|X_1| \mathbb{I}(|X_1| > x) \le |X_1| \qquad \mathbb{E}|X_1| < \infty$$

need a func to bound ....

$$|X_1| \mathbb{I}(|X_1| > x) \xrightarrow{a.s.} 0 \quad \text{if } x \to 0$$

If $Y \ge 0$

$$\mathbb{E}Y = \int_0^\infty \mathbb{P}(Y > y)\,dy$$

$$\int_0^\infty \int_y^\infty f_Y(x)\,dx\,dy$$

$x \to y$

$y \to x$

$x \ge y$

$\therefore \quad y \le x$

$$\int_0^\infty \int_0^x f_Y(x)\,dy\,dx$$

$$= \int_0^\infty f_Y(x) \left(\int_0^x dy \right) \cdot dx$$

$$= \int_0^\infty x f_Y(x)\,dx$$

$$= \mathbb{E}Y$$

if $\mathbb{P}(|X_1| > x)$

$$\sim \frac{1}{x \log\log x}$$

$x \to \infty$.

4 By DCT $x\mathbb{P}(|X_1|>x) \to 0$ $x\to\infty$

4 Thm → Even the $\mathbb{E}|X|$ does not exist,

we still have some ~~kind~~ of WLLN.

what is the a_n.

Relation to a_n and $\mathbb{E}X$?

① $X_1 > 0$

$\mathbb{E}X_1 = \infty$

$a_n \to \infty$ as $n\to\infty$.

↓

grow along n. Monotonic Convergence T

Thm. Above tell us though. $a_n \to \infty$ (first order term)

The second order term. is still negligible.

② X_1 is symmetric.

$a_n = 0$ (Feller Vol II (1971) pp 234-236)

$\mathbb{E}|X_1| = \infty$

proof. $b_n = n$ $X_{n,k} = X_k$.

check:

①. $\sum_{k=1}^{n} \mathbb{P}(|X_{n,k}| > b_n) = n\mathbb{P}(|X| > n) \to 0$ $n\to\infty$.

② $b_n^{-2} \cdot \sum_{k=1}^{n} \mathbb{E}(\bar{X}_{nk})^2$.

$= n^{-2}\, n \cdot \mathbb{E}(\bar{X}_k)^2$

Tail sum formula.

$Y > 0. \quad Y^2 > 0$

$\mathbb{E}Y = \int_0^\infty \mathbb{P}(Y > y)\,dy$

$\mathbb{E}Y^2 = \int_0^\infty \mathbb{P}(Y^2 > y)\,dy = \int_0^\infty \mathbb{P}(Y > \sqrt{y})\,dy.$

$t = \sqrt{y} \qquad y = t^2 \qquad dy = 2t\,dt.$

$= \int_0^\infty \mathbb{P}(Y > t) \cdot 2t\,dt \cdots =$

$Y \geq 0$

$\mathbb{E}Y^p = p \cdot \int y^{p-1}\,\mathbb{P}(Y > y)\,dy$

$n^{-2} \cdot \sum_{k=1}^{n} \mathbb{E}(\overline{X}_{nk})^2.$ 其中 $\mathbb{E}(\overline{X_k}^2) = 2 \cdot \int_0^\infty y \cdot \mathbb{P}(\overline{X_k} > y)\,dy$

$= n^{-2}\; n \cdot \mathbb{E}(\overline{X_k})^2$

$= \frac{2}{n} \cdot \int_0^\infty y \cdot \mathbb{P}(\overline{X_k} > y)\,dy$ → Since $\overline{X_k}$ is truncated

$= \frac{2}{n} \int_0^n y\,\mathbb{P}(X_k > y)\,dy$

$= \frac{2}{n} \int_0^n y\,\mathbb{P}(\overline{X_1} > y)\,dy$

$\leq \frac{2}{n} \int_0^n y\,\mathbb{P}(X_1 > y)\,dy$

If $a_1\ a_2 \cdots$

$a_n \to 0$ as $n \to \infty$.

$\frac{a_1 + \cdots + a_n}{n} \to 0.$

Since $y\,\mathbb{P}(X_1 > y) \to 0.$ By condition.

$\therefore \left(\frac{1}{n} \int \cdots\cdots\right) \to 0$ ✓ $\int_0^1 + \int_1^2 + \cdots + \int_{n-1}^n$

Thm. Let $X_1\ X_2 \cdots$ be i.i.d. $\mathbb{E}|X_1| < \infty$

$\mathbb{E}X_1 = a$ Then. $\frac{S_n}{n} - a \xrightarrow{\mathbb{P}} 0$

$a_n \to a$ is done by DCT

St. petersberg paradox

Let X_1 X_2 ⋯ i.i.d. r.v.s. with.

$$\mathbb{P}(X_1 = 2^j) = 2^{-j} \quad j=1,\cdots$$

prob. mass func.

A Game: Dealer flips the coin. until the first time the H shows up.

X_j : winning of the j-th game.

j-th. H ✓ then gains 2^j

Dealer always wins. So Casino need ticket.

Q: the reasonable price for each game.

$$\mathbb{E}X_1 = \sum_{j=1}^{\infty} 2^j \cdot 2^{-j} = \infty$$

The prob of the tail of the event is too small.

①

Truncate at 50.

$$\mathbb{E}X_1 \mathbb{I}(|X_1| \le 2^{50})$$
$$= \sum_{j=1}^{50} 2^j \cdot 2^{-j} = 50.$$

② Assume casino has ∞ money

X_1 X_2 ⋯ X_n n cannot go to ∞.

People really care is $\frac{S_n}{n}$ empirical average - $\frac{S_n}{n}$

①. Choose the Truncation. b_n.

so. 1°. $\sum_{k=1}^{n} \mathbb{P}(|X_{n,k}| > b_n) \to 0$

2° $b_n^{-2} \cdot \sum_{k=1}^{n} \mathbb{E}(\overline{X_{n,k}}^2) \to 0$

3° $(b_n \le a_n)$ in order sense

find a b_n such that.

①. $\sum_{k=1}^{n} \mathbb{P}(|X_k| > b_n) = n\mathbb{P}(|X_1| > b_n) \to 0$

②. $b_n^{-2} \sum_{k=1}^{n} \mathbb{E}(\overline{X_k}^2) = b_n^{-2} n \cdot \mathbb{E}(\overline{X_1}^2) = b_n^{-2} n \mathbb{E} X_1^2 \mathbb{I}(|X_1| \le b_n)$
$\to 0$

③. $b_n \le a_n = \sum_{k=1}^{n} \mathbb{E}(\overline{X_k}) = n \cdot \mathbb{E} X_1 \mathbb{I}(|X_1| \le b_n)$

X only take value of power of 2

$\therefore b_n = 2^{m(n)}$

① $n\mathbb{P}(|X_1| > 2^{m_n}) = n \cdot \sum_{j=m_n+1}^{\infty} 2^{-j} \sim n \cdot 2^{-m_n} \to 0$

②. $2^{-2m_n} \cdot n \sum_{j=1}^{m_n} 2^{2j} 2^{-j} \sim \boxed{n \cdot 2^{-m_n} \to 0}$ $\quad \frac{2^{m_n}}{n} \to \infty$

$m_n - \log_2 n \to \infty.$

③ $2^{m_n} \le n \cdot \sum_{j=1}^{m_n} 2^j 2^{-j} = n m_n.$ $\quad m_n < \log_2 n + \log_2 m_n.$

Δ choose. $m_n = \log_2 n + \log_2 \log_2 n$

$b_n = 2^{m_n} = n \log_2 n$

$\frac{S_n - a_n}{b_n} \xrightarrow{\mathbb{P}} 0 \qquad a_n = n \mathbb{E} \overline{X} = n \cdot m_n = n\log_2 n + n \cdot \log_2 m_n.$

$b_n = n \log_2 n$

$$\frac{S_n - n(\log_2 n + \log_2 m_n)}{n \log n} \xrightarrow{\mathbb{P}} 0$$

$$\therefore \frac{S_n}{n \log n} \xrightarrow{\mathbb{P}} 1$$

$$\therefore \frac{S_n}{n} = \log_2 n (1 + o(1))$$