Chapter 3

查梦月
ZHA Mengyue
No. 2066890|

1. $T=(X_1, X_2 \cdots X_n)$ $\quad X_1, \cdots X_n \overset{iid}{\sim} f(x)$

$f_X(x) = \prod_{i=1}^{n} f_{X_i}(x=x_i)$

So $T=(X_1, X_2 \cdots X_n)$ is sufficient,

However $E[g(T)]=0 \nRightarrow g(T)=0$

For example, if $X_1 \cdots X_n \sim N(0, \sigma^2)$

Then $g(T) \triangleq \bar{X} \sim N(0, \frac{\sigma^2}{n})$

It's easy to see $E\, g(T)=0$ is always true

However it's not necessary for $\bar{X}=0$

Thus T is sufficient but not complete.

2. T_i is the UMVUE for θ_i $\forall i$

$\overset{\text{characterization}}{\underset{\text{Theorem.}}{\Longleftrightarrow}}$ ① $E T_i = \theta_i$

② $\forall u \in \mathcal{U}\{0\}$ $E[T_i u]=0$

a)

1° $E T = \sum_{i=1}^{k} c_i E T_i = \sum_{i=1}^{k} c_i \theta_i = \theta$ $\quad T$ is unbiased

2° $\forall u \in \mathcal{U}\{0\}$

$E[T \cdot u] = \sum_{i=1}^{k} c_i E[T_i u] = 0$

3° By characterization Theorem.

T is the UMVUE of θ

b)

1° $E T = E(\prod_{i=1}^{k} T_i) = \theta$.

2° $\forall u \in \mathcal{U}\{0\}$

$$\mathbb{E}[TU] = \mathbb{E}\mathbb{E}[(\prod_{i=1}^{k} T_i) U \mid T_1 \cdots T_{k-1}]$$

$$= \mathbb{E}(\prod_{i=1}^{k-1} T_i)\mathbb{E}[T_n \cdot U] = 0.$$

3° By characterization Theorem.

T is the UMVUE of θ

Chapter 6.

1. ① $g(x) = e^x \quad g'(x) = e^x \quad g'(\mu) = e^\mu$

$$e^{\bar{x}} \sim AN(e^\mu, n^{-1}(e^{2\mu})\sigma^2)$$

② $g(x) = \ln|\bar{x}|$

$$g'(x) = \frac{\partial g(x)}{\partial |x|} \cdot \frac{\partial |x|}{\partial x} = \frac{1}{|x|}(\pm 1)$$

$$g'(\mu)^2 = \frac{1}{|\mu|^2}$$

$$\ln|\bar{x}| \sim AN(\ln|\mu|, n^{-1}\sigma^2 \frac{1}{|\mu|^2})$$

③ $g(x) = \cos x \quad g'(x) = -\sin x \quad g'(\mu) = -\sin\mu$

$$\cos(\bar{x}) \sim AN(\cos\mu, n^{-1}\sigma^2 \sin^2\mu)$$

6. $\mu = np \qquad p = \frac{\mu}{n}$

$$\sigma^2 = np(1-p) = n \cdot \frac{\mu}{n}(1 - \frac{\mu}{n}) = \frac{1}{n}\mu(n-\mu)$$

$$(g'(\mu))^2 \cdot \frac{1}{n}\mu(n-\mu) = c \triangleq 1$$

$$\Rightarrow g'(\mu)^2 = \frac{n}{\mu(n-\mu)}$$

$$g'(\mu) = \frac{\sqrt{n}}{\sqrt{\mu(n-\mu)}}$$

$$g(\mu) = \int g'(\mu) \cdot d\mu = \sqrt{n}\int \frac{1}{\sqrt{\mu(n-\mu)}} d\mu$$

$$= -2\sqrt{n} \sin^{-1}\sqrt{n-\mu}$$

$$\therefore g(T) = -2\sqrt{n} \cdot \sin^{-1}\sqrt{n-T}$$

Section 7.7

1. From the L_1 optimal property of the median.

$$\mathbb{E}|X-m| \le \mathbb{E}|X-c|$$

$c \triangleq \mu$

$$\mathbb{E}|X-m| \le \mathbb{E}|X-\mu|$$

$$\mathbb{E}|X-m|^2 \le \mathbb{E}|X-\mu|^2 = \sigma^2$$

VI By Jensen's inequality

$$|\mathbb{E}X-\mathbb{E}m|^2$$

$$\|$$

$$|\mu-m|^2$$

$$\|$$

$$|m-\mu|^2$$

$\therefore\ |m-\mu|^2 \le \sigma^2$

$$|m-\mu| \le \sigma.$$

QED.