MATH 5431: Advanced Mathematical Statistics I

Final examination

1. Let $X \sim \text{Poisson}(\lambda)$ with pdf

$$P_\lambda(X = x) = \lambda^x e^{-\lambda}/x! \qquad x = 0, 1, 2, \cdots.$$

Find a variance stabilizing transformation for X.

2. Suppose $X_1, \cdots, X_n$ are an i.i.d. sample from $N(\mu, \sigma^2)$ with both parameters unknown.

   (a) Write down the likelihood function of (θ, σ^2).

   (b) Find the profile likelihood of μ. (Definition given below.)

   (c) Derive the profile MLE for μ.

Definition: Given the joint likelihood $L(\theta, \eta)$, the profile likelihood of θ is $L(\theta) = \max_\eta L(\theta, \eta)$.

3. Suppose we have multinomial data $(N_1, N_2, N_3) \sim \text{multinomial}(p_1, p_2, p_3)$, but it is known that $p_3 = 1/3$. Find the MLE of the unknown probabilities p_1, p_2, and interpret the result.

4. Given iid sample $X_1, \cdots, X_n$, let $S_n^2 = n^{-1} \sum_{i=1}^n (X_i - \bar{X})^2$ be an estimator of the population variance σ^2. Use S_n^2 to derive a Jackknife estimator of σ^2.

Definition: Suppose that $T_n = T(X_1, ..., X_n)$ estimates $\theta = T(F)$. Let $T_{n-1}^{(-i)}$ be the estimator of θ based on the data after deleting X_i, i.e., $T_{n-1}^{(-i)} = T(X_1, ..., X_{i-1}, X_{i+1}, ..., X_n)$. The jackknife estimate of $\theta = T(F)$ is

$$T_n^{Jack} = nT_n - (n-1)T_{n-1}^{(\cdot)}, \qquad \text{where} \quad T_{n-1}^{(\cdot)} = \frac{1}{n}\sum_{i=1}^n T_{n-1}^{(-i)}.$$

5. Let $X \sim N(\mu, \sigma^2)$, and let $T(X) = X - g(X)$ be an estimator of μ. Denote $g'(x)$ to be the derivative of g. Show that

$$E(T(x) - \mu)^2 = \sigma^2 + Eg^2(x) - 2\sigma^2 Eg'(X).$$

Hence, Stein's unbiased risk estimate is $SURE(T(X)) = \sigma^2 + g^2(x) - 2\sigma^2 g'(X)$.

6. Let Y_i be independent $\text{Poisson}(\lambda_i)$ with $\lambda_i = \exp\{\beta x_i\}$. Derive the score function and find an implicit solution for the MLE of β, and its asymptotic distribution.