I did the homework on two days. The first part is written in LaTeX but the second part is written by hand tonight.

Six problems are solved in this assignment.

Assignment 1 MATH5431

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1 Ex 2.6

(1)
From $x^{(1)} = -0.76$ and $x^{(n)} = 0.5$ where $n = 7$, we know that only $\theta \in [-0.5, 0.24]$ makes sense in term of the likelihood.

$$L(X;\theta) = \begin{cases} \left(\frac{1}{2}\right)^7 & \theta \in (-0.5, 0.24) \\ 0 & otherwise \end{cases} \tag{1}$$

(2)
From $\max\{|x^{(1)}|, |x^{(n)}|\} = 0.76$ where $n = 7$, we know that only $\theta > 0.76$ makes sense in term of the likelihood.

$$L(X;\theta) = \begin{cases} \left(\frac{1}{2\theta}\right)^7 & \theta > 0.76 \\ 0 & otherwise \end{cases} \tag{2}$$

(3)

$$p(x) = \frac{1}{\sqrt{2\pi\theta}} exp(-\frac{x^2}{2\theta}) \tag{3}$$

$$L(x;\theta) = \prod_{i=1}^{n} L(x_i;\theta) \tag{4}$$

$$= (2\pi\theta)^{\frac{7}{2}} exp\{-\frac{1.6614}{2\theta}\} \tag{5}$$

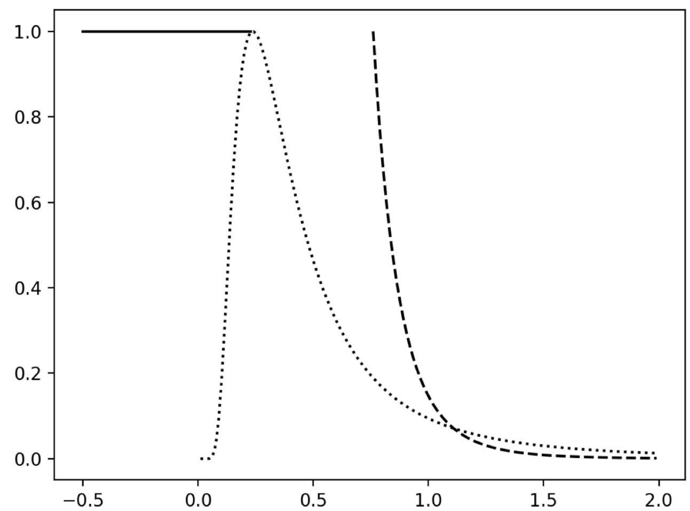


Figure 1: The likelihood of θ after normalization.

2 Ex 2.10

$$\log L(X;\lambda) = \sum_{i=1}^{n} \log L(x_i;\lambda) = \sum_{i=1}^{n} \log(\frac{e^{-\frac{x_i}{\lambda}}}{\lambda}) \tag{6}$$

$$\begin{aligned} S(\lambda) &= \frac{\partial \log L(\lambda)}{\partial \lambda} &(7)\\ &= \sum_{i=1}^{n} \Big[\lambda e^{\frac{x_i}{\lambda}} \times \frac{e^{-x_i/\lambda} \times \frac{x_i}{\lambda} - e^{-x_i/\lambda}}{\lambda^2}\Big] &(8)\\ &= \sum_{i=1}^{n} [\frac{x_i - \lambda}{\lambda^2}] &(9) \end{aligned}$$

Let $S(\lambda)$=0, We have

$$\hat{\lambda} = \frac{\sum_{i=1}^{n} x_i}{n} \tag{10}$$

$$I(\lambda) = -\frac{\partial^2 \log L(X;\lambda)}{(\partial \lambda)^2} = \frac{n\lambda - 2\sum_{i=1}^{n} x_i}{\lambda^3} \tag{11}$$

$$Se(\hat{\lambda}) = I(\hat{\lambda})^{-\frac{1}{2}} = \frac{\sum_{i=1}^{n} x_i}{n^{\frac{3}{2}}} \tag{12}$$

3 Ex 2.19

Set $\eta = g(\theta)$

$$L(\theta) = L(g^{-1}g(\theta)) = L(g^{-1}(\eta)) \tag{13}$$

When g is a one-to-one map

$$Find\ a\ \theta\ to\ minimize(13) \iff Find\ a\ g^{-1}(\eta)\ to\ minimize(13) \iff Find\ a\ \eta\ to\ minimize(13) \tag{14}$$

When g is a many-to-one map

The $\hat{\theta}$ which maximize the $L(\theta)$ still corresponds a a $\hat{\eta}$ that minimize $L(g^{-1}\eta)$.
Though, for this $\hat{\theta}$ we may have more than one $theta$'s, say, $theta_1$ and θ_2 so that $\hat{\eta} = g(\theta_1) = g(\theta_2)$
But we could just pick one from θ_1 and θ_2
Therefore, $g(\theta_1) = g(\theta_2) = g(g^{-1}\hat{\eta})$ will still be the solution of MLE of $g(\theta)$

4 Ex 2.20

$$L_g(g(\theta)) = L_\theta(\theta) \tag{15}$$

$$\begin{aligned} \frac{\partial L_g(g(\theta))}{\partial g(\theta)} &= \frac{\partial L_\theta(\theta)}{\partial g(\theta)} && (16) \\ &= \frac{\partial L_\theta(\theta)}{\partial \theta} \times \frac{\partial \theta}{\partial g(\theta)} && (17) \end{aligned}$$

$$\frac{\partial^2 L(g(\theta))}{(g(\theta))^2} = \frac{\partial^2 L(\theta)}{(\partial\theta)^2} \times \frac{\partial \theta}{\partial g(\theta)} \times \frac{\partial \theta}{\partial g(\theta)} \tag{18}$$

Take the negative sign for both sides

$$-I^*(g(\theta)) = -I(\theta) \cdot |\frac{\partial g(\theta)}{\partial \theta}|^{-2} \tag{19}$$

From the conclusion of exercise 2.19, we have

$$\widehat{g(\theta)} = g(\hat{\theta}) \tag{20}$$

Therefore,

$$-I^*(\widehat{g(\theta)}) = -I^*(g(\hat{\theta})) = -I(\hat{\theta}) \cdot |\frac{\partial g(\theta)}{\partial \theta}|^{-2} \tag{21}$$

Problem 5 Ex. 2.7

$x^{(1)} = 2.31 < 2.32 < 2.51 < 2.97 < 3.01 < 3.07 < 3.08 < 3.41 < 3.86 < 3.98 = x^{(n)}$ where $n=10$

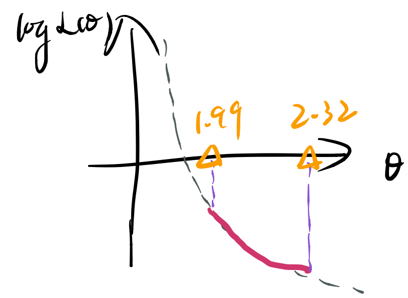
For θ, $\begin{cases} \theta \le 2.32 \\ 2\theta \ge 3.98 \end{cases} \Rightarrow 1.99 \le \theta \le 2.32$

$x_i \overset{iid}{\sim} U(\theta, 2\theta)$ $\quad X = (x_1, x_2, \cdots x_{10})$

$$L(X) = \prod_{i=1}^{10} L(x_i) = \begin{cases} \frac{1}{\theta^{10}} & \text{when } \theta \in [1.99, 2.32] \\ 0 & \text{otherwise.} \end{cases}$$

under the assumption that $\theta \in [1.99, 2.32]$

we have $\hat{\theta} = 1.99$



$\log L(\theta) = -10 \log \theta$

$$S(\theta) = \frac{\partial \log L(\theta)}{\partial \theta} = -10\,\theta^{-11}$$

$$I(\theta) = -\frac{\partial^2 \log L(\theta)}{(\partial \theta)^2} = -110\,\theta^{-12}$$

$I(\hat{\theta})$ should be negative and the loglikelihood function is not regular. Thus, the curvature of the log-likelihood at the MLE or the standard error is not meaningful.

Actually, likelihood-based intervals are better supplement of MLE. More specifically, while the likelihood is not regular, it's still possible to provide an exact theoretical justification for a confidence interval interpretation.

$$P\left(\frac{L(\theta)}{L(\hat{\theta})} > c\right) = P\left(\frac{\theta^{-10}}{\left(\frac{X^{(10)}}{2}\right)^{-10}} > c\right) = P\left(\frac{X^{(10)}}{2\theta} > c^{\frac{1}{10}}\right)$$

$$= 1 - P\left(\frac{X^{(10)}}{2\theta} < c^{\frac{1}{10}}\right)$$

where $Y_i \triangleq \frac{X_i}{2\theta} \overset{iid}{\sim} U(\frac{1}{2}, 1)$

$$P(Y^{(10)} < C^{\frac{1}{10}}) = P(\bigcap_{i=1}^{10}\{Y^{(i)} < C^{\frac{1}{10}}\}) \overset{Y^{(i)}}{\underset{iid.}{=\!=}} \prod_{i=1}^{10} P(Y^{(i)} < C^{\frac{1}{10}})$$

$$\overset{iid}{=} P(Y_1 < C^{\frac{1}{10}})^{10} = \left(\frac{C^{\frac{1}{10}} - \frac{1}{2}}{\frac{1}{2}}\right)^{10} = (2C^{\frac{1}{10}} - 1)^{10}$$

Thus, $P(\frac{L(\theta)}{L(\hat{\theta})} > c)$

$$= 1 - (2C^{\frac{1}{10}} - 1)^{10}$$

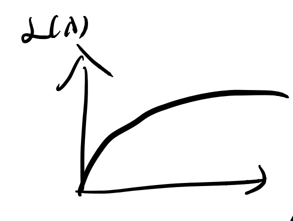
Thus. The pure likelihood interval still works.

Note that, since the log-likelihood function is far from quadratic, Wald interval is deficient since it includes values with lower likelihood compare to values outside the interval.

Problem 6 Ex. 2.1

For c.d.f. of life time exponential distribution.

$F(x) = \int_{-\infty}^{x} p(t)\,dt = 1 - e^{-\frac{x}{\lambda}}$



(1). $P(X > 30) = 1 - F(30) = e^{-\frac{30}{\lambda}}$

(2). $X_{(1)} = 6 \quad X_{(2)} = 28$

$L(\lambda) = P(X=6)\cdot P(X=28)\,[P(X \geq 28)]^{8} \quad \frac{10!}{8!} \quad 90\,\lambda^{-2}\cdot \exp\{-\frac{258}{\lambda}\}$

(3). let $L(\lambda) = 0$ we have $\hat{\lambda} = 129$

(4) $F(t) = 0.9$

$1-\exp\{-\frac{t}{129}\}=0.9$

$t \approx 297.03$

$\Rightarrow P(X \leq 30)^2$

$= 1-2e^{-\frac{30}{\lambda}}+e^{-\frac{60}{\lambda}}$