2020 Qualifying exam: Statistics

1. Let T_i be a UMVUE of θ_i, $j = 1, ..., k$. Then
   (a) $T = \sum_{i=1}^{k} c_i T_i$ is a UMVUE of $\theta = \sum_{i=1}^{k} c_i \theta_i$, for constants c_i's.
   (b) $T = \prod_{i=1}^{k} T_i$ is a UMVUE of $\theta = ET = E\left(\prod_{i=1}^{k} T_i\right)$.

2. Let $X_1, \cdots, X_n \sim_{iid} F$. Find the limiting d.f. of the sample median $\hat{m} = F^{-1}(1/2)$.

3. Suppose $X \sim F$, and $m = F^{-1}(1/2)$, $\mu = EX$, $\sigma^2 = Var(x)$. Show that $|m - \mu| \leq \sigma$.

4. Let $(X, Y), (X_1, Y_1), ..., (X_n, Y_n)$ be i.i.d. with mean $\mu = (\mu_x, \mu_y)^\tau$ and covariance matrix Σ. Let $r = \mu_x/\mu_y$ with $\mu_y \neq 0$, and an estimate of r is $\hat{r} = \bar{X}/\bar{Y}$. Investigate the limiting behavior of $\hat{r}$ (i.e. both consistency and asymptotic d.f.).

5. Let $U_n \sim_{iid}$ Uniform$[0, 1]$, and $X_n = \left(\prod_{i=1}^{n} U_i\right)^{-1}$. Show that $\sqrt{n}(X_n - e) \to_d N(0, e^2)$.

6. Suppose $X_1, \ldots, X_n \sim_{iid}$ with pdf $f(x) = \theta(\theta+1)x^{\theta-1}(1-x)$ $\quad (0 < x < 1, \theta > 0)$. Write the sample mean $\bar{X} = \sum_{i=1}^{n} X_i/n$.
   (a) By equating $\bar{X}$ with its population mean, show that the method of moments estimator of θ is $T_n = 2\bar{X}/(1 - \bar{X})$.
   (b) Show that $\sqrt{n}(T_n - \theta)/\sigma(\theta) \longrightarrow_d N(0, 1)$, where $\sigma^2(\theta) = \dfrac{\theta(\theta+2)^2}{2(\theta+3)}$.

   (Hint: If $X \sim Beta(\alpha, \beta)$ distribution, then $EX = \alpha/(\alpha + \beta)$.)

7. If independent lifetimes T_1 and T_2 have proportional hazards, say $\lambda_i(t) = \lambda_0(t)\eta_i$ for $i = 1, 2$, respectively, then show that
$$P(T_1 < T_2) = \eta_1/(\eta_1 + \eta_2),$$
   regardless of the shape of the baseline hazard function $\lambda_0(t)$.

8. Consider a random effects model
$$y_{ij} = \mu + b_i + \epsilon_{ij}, \qquad 1 \leq i \leq q, \qquad 1 \leq j \leq n,$$
   where μ is a fixed overall mean effect, $b_i \sim_{iid} N(0, \sigma_b^2)$ is the individual random effect, and $e_{ij} \sim_{iid} N(0, \sigma^2)$ are the error terms, and they are independent. Find Bayes estimators for b_i's, assuming $(\mu, \sigma_2, \sigma_b^2)$ are known.