Notes on Numerical Integration

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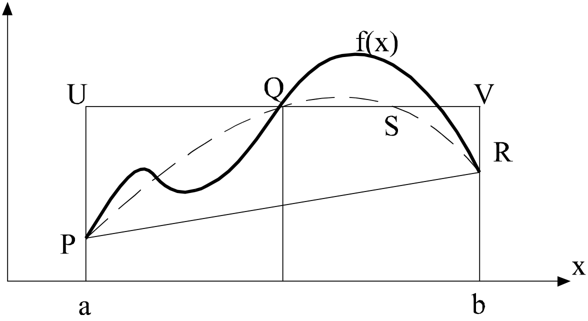
Numerical Integration

We are interested in lots of problems that require computing difficult integrals (e.g.: evaluating expectations)

1. Midpoint/Trapezoid Rules
2. Simpson's Rule
3. Gaussian Rules
4. Higher-Dimensional Rules

Quadrature Rules

Basic idea of quadrature is to approximate complicated functions with something easier to integrate, and then integrate that exactly.



- Constant $f(x)$ at midpoint of $[a, b]$ $aUQVb$ for box.
- Linear: Trapezoid $aPRb$
- Parabola through $f(x)$ at a, b and $\frac{a+b}{2}$ for $aPQRb$

Simpsons Rule (Newton-Cotes)

Piecewise Quadratic Approximation at some $\xi \in [a,b]$

$$\int_a^b f(x)dx \approx \left(\frac{b-a}{6}\right)\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{(b-a)^5}{2880} f^{(4)}(\xi)$$

With approximation error

$$\frac{1}{90}\left(\frac{b-a}{2}\right)^5 |f^{(4)}(\xi)|$$

Works well when quadratic approximation is good $f^{(4)}$ is small or interval is small.

Gaussian Quadrature

Formulas of the form

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} w_i f(x_i)$$

for some quadrature nodes $x_i \in [a, b]$ and weights w_i.

- Let $\mathcal{F}_k$ be the space of degree k polynomials
- Quadrature formulas are exact of degree k if it correctly integrates each function in $\mathcal{F}_k$
- Gaussian quadrature formulas use n points and are exact of degree $2n - 1$.

Approximation Error

$$(f, g) = \int_a^b w(x)f(x)dx - \sum_{i=1}^{n} w_i f(x_i) = \frac{f^{(2n)}(\xi)}{(2n)!}(p_n, p_n)$$

Gaussian Quadrature

Legendre Domain: $[-1, 1]$, $w(x) = 1$

Chebyshev Domain: $[-1, 1]$, $w(x) = \frac{1}{\sqrt{1-x^2}}$

Laguerre Domain: $[0, \infty]$, $w(x) = \exp[-x]$ (useful for present value)

Hermite Domain: $[-\infty, \infty]$, $w(x) = \exp[-x^2]$ (useful for normal)

Helpful if function is C^∞ or analytic.

Gauss Herrmite

Let $Y \sim N(\mu, \sigma^2)$ and apply COV $x = (y-\mu)/\sqrt{2}\sigma$

$$E[f(Y)] = (2\pi\sigma^2)^{-\frac{1}{2}} \int_{-\infty}^{\infty} f(y) \exp\left[-\frac{(y-\mu)^2}{2\sigma^2}\right] dy$$

$$\int_{-\infty}^{\infty} f(y) \exp\left[-\frac{(y-\mu)^2}{2\sigma^2}\right] dy = \int_{-\infty}^{\infty} f(\sqrt{2}\sigma x + \mu) e^{-x^2} \sqrt{2}\sigma dx$$

Gives the quadrature formula using Gauss Hermite (x_i, w_i).

$$E[f(Y)] = \frac{1}{\sqrt{\pi}} \sum_{i=1}^{n} w_i f(\sqrt{2}\sigma x_i + \mu)$$

Higher Dimensional Integration

- In higher dimension we can use product rules of 1-D integrals.
- This grows exponentially in dimension D (Curse of Dimensionality)
- Monte Carlo is not cused but slow to converge $\frac{1}{\sqrt{n}}$ vs $\frac{1}{2n!} f^{(2n)}$
- Some monomial rules (Judd), (Skrainka and Judd) aren't cursed
- Sparse Grids show how to combine 1-D rules more efficiently (www.sparse-grids.de)