# Pass-through

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Internal Notes

# Villas Boas (ReStud 2007)

Retailer and Wholesaler FOC given by:

$$\mathbf{p^r} = \underbrace{\mathbf{p^w} + \mathbf{c^r}}_{\mathbf{mc^r}} - (\mathcal{H}_r \odot \Delta_r(\mathbf{p^r}))^{-1}\mathbf{s}(\mathbf{p^r})$$

$$\mathbf{p^w} = \mathbf{mc^w} + \left(\mathcal{H}_w \odot \left(\frac{\partial \mathbf{p^r}}{\partial \mathbf{p^w}} \cdot \Delta_r(\mathbf{p^r})\right)\right)^{-1} \mathbf{s}(\mathbf{p^r})$$

- $\Delta_r$ is matrix of (retail) demand derivatives $\frac{\partial\, \mathbf{s}}{\partial\, \mathbf{p}}$.
- $\mathcal{H}_r, \mathcal{H}_w$ ownership matrix $(j,k) = 1$ if both products sold by same retailer/wholesaler.
- $\frac{\partial\, \mathbf{p^r}}{\partial\, \mathbf{p^w}}$ is the pass-through matrix (NEW!)

Challenge: We want $\mathbf{p^r}(\mathbf{p^w})$ and $\mathbf{mc^w}$ but we only have implicit solution for retailer FOC.

# How do we get pass-through?

The pass-through matrix $\frac{\partial \mathbf{p^r}}{\partial \mathbf{p^w}}$ can be obtained in one of two ways:

1. Numerically: perturbing the retailer's marginal costs for each possible choice of $k$ and solving

$$\mathbf{p^r} = \mathbf{mc^r} + e_k - (\mathcal{H}_r \odot \Delta_r(\mathbf{p^r}))^{-1}\mathbf{s}(\mathbf{p^r})$$

   (Use Morrow Skerlos (2011) formulation and solve for every $(j,k)$ pair).

2. Analytic: Use the retailer's FOC and apply the implicit function theorem.

$$f(\mathbf{p^r}, \mathbf{mc^r}) \equiv \mathbf{p^r} - \mathbf{mc^r} - (\mathcal{H}_r \odot \Delta(\mathbf{p^r}))^{-1}\mathbf{s}(\mathbf{p^r}) = 0 \qquad \text{(retailer FOC)}$$

   See Jaffe Weyl (AEJM 2013) or Miller Weinberg (2017 Appendix E) or Conlon Rao (2022). This is what PyBLP does.

# Multivariate IFT: Easy Part

The multivariate IFT says that for some system of $J$ nonlinear equations

$$f(\mathbf{p^r}, \mathbf{p^w}) \equiv [F_1(\mathbf{p^r}, \mathbf{p^w}), \dots, F_J(\mathbf{p^r}, \mathbf{p^w})] = [0, \dots, 0]$$

with $J$ endogenous variables $\mathbf{p^r}$ and $J$ exogenous parameters $\mathbf{p^w}$.

$$\frac{\partial \mathbf{p^r}}{\partial \mathbf{p^w}} = -\begin{pmatrix} \frac{\partial F_1}{\partial p_1^r} & \dots & \frac{\partial F_1}{\partial p_J^r} \\ \dots & \dots & \dots \\ \frac{\partial F_J}{\partial p_1^r} & \dots & \frac{\partial F_J}{\partial p_J^r} \end{pmatrix}^{-1} \cdot \underbrace{\begin{pmatrix} \frac{\partial F_1}{\partial p_k^w} \\ \dots \\ \frac{\partial F_J}{\partial p_k^w} \end{pmatrix}}_{=-\mathbb{I}_J} \tag{PTR}$$

Because the system of equations is additive in $\mathbf{mc^r} = \mathbf{c^r} + \mathbf{p^w}$ this simplifies dramatically.

# Multivariate IFT: Hard Part

Use the substitution $\Omega(\mathbf{p^r}) \equiv \mathcal{H}_r \odot \Delta_r(\mathbf{p^r})$, and differentiate the wholesalers' system of FOC's with respect to $p_l$, to get the $J \times J$ matrix with columns $l$ given by:

$$\frac{\partial f(\mathbf{p^r}, \mathbf{p^w})}{\partial p_l^r} \equiv e_l - \Omega^{-1}(\mathbf{p^r}) \left[ \mathcal{H}_r \odot \frac{\partial \Delta(\mathbf{p^r})}{\partial p_l^r} \right] \Omega^{-1}(\mathbf{p^r})\, \mathbf{s}(\mathbf{p^r}) - \Omega^{-1}(\mathbf{p^r}) \frac{\partial \mathbf{s}(\mathbf{p^r})}{\partial p_l^r}. \tag{1}$$

The complicated piece is the demand Hessian: a $J \times J \times J$ tensor with elements $(j, k, l)$, $\frac{\partial^2 s_j}{\partial p_k^r \partial p_l^r} = \frac{\partial^2 \mathbf{s}}{\partial \mathbf{p^r} \partial p_l^r} = \frac{\partial \Delta(\mathbf{p^r})}{\partial p_l^r}$.

This also shows a key relationship between pass through and demand curvature (2nd derivatives).

# Pass-through Counterfactuals?

How do we solve for $p^w$ under a counterfactual pass-through matrix?

- Idea: pass-through only augments the matrix $\Delta_r(\mathbf{p^r})$.
- Example: a constant sales tax rate $P \equiv \frac{\partial \mathbf{p^r}}{\partial \mathbf{p^w}} = \text{diag}(1+\tau_r)$

$$\mathbf{p^w} = \mathbf{mc^w} + \left(\mathcal{H}_w \odot \left(\frac{\partial \mathbf{p^r}}{\partial \mathbf{p^w}} \cdot \Delta_r(\mathbf{p^r})\right)\right)^{-1} \mathbf{s}(\mathbf{p^r})$$

Adapt the Morrow Skerlos $\zeta$ fixed point where $P\Delta(\boldsymbol{p}_t) = P\Lambda_t(\boldsymbol{p}_t) - P\Gamma_t(\boldsymbol{p}_t)$

$$\boldsymbol{p}_t \leftrightarrow \boldsymbol{c}_t + \boldsymbol{\zeta}_t(\boldsymbol{p}_t) \qquad \text{where}$$
$$\boldsymbol{\zeta}_t(\boldsymbol{p}_t) = \Lambda_t(\boldsymbol{p}_t)^{-1} P^{-1} \left[\mathcal{H}_t^* \odot P\,\Gamma_t(\boldsymbol{p}_t)\right](\boldsymbol{p}_t - \boldsymbol{c}_t) - \Lambda_t(\boldsymbol{p}_t)^{-1} P^{-1} \boldsymbol{s}_t(\boldsymbol{p}_t)$$

For diagonal $P$ (not sure about general case with Hadamard product):

$$\boldsymbol{\zeta}_t(\boldsymbol{p}_t) = \Lambda_t(\boldsymbol{p}_t)^{-1} \left[\mathcal{H}_t^* \odot \Gamma_t(\boldsymbol{p}_t)\right](\boldsymbol{p}_t - \boldsymbol{c}_t) - \Lambda_t(\boldsymbol{p}_t)^{-1} P^{-1} \boldsymbol{s}_t(\boldsymbol{p}_t)$$