Single-agent dynamic optimization models

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Grad IO

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- The long-run demand elasticity for laundry detergent might be zero (or very close)
- If detergent goes on sale periodically, we might see a nonzero short-run elasticity (perhaps even a large one) as customers might purchase during the sales and store the detergent.
- Dynamic estimation typically involves estimating the primitives of decision makers' objective functions. We might estimate the model using short-run variation, but once we know the decision maker's objective function, we could simulate a response to long-run variation.

Why are dynamics difficult?

- The computational burden of solving dynamic problems blows up as the state space gets large. With standard dynamic estimation techniques, this is especially problematic, for estimation may involve solving the dynamic problem many times.
- Serially correlated unobservables and unobserved heterogeneity (easy to confuse with state dependence)
- Modeling expectations
- Solving for equilibria, multiplicity (dynamic games)

Outline

- Introduction to dynamic estimation: Rust (1987)
- MPEC: an alternative algorithm (Su and Judd 2012)
- Conditional choice probabilities: Hotz and Miller (1993)
- Euler equation estimation: Scott (2014)

Rust: Theory of Dynamic Discrete Choice (DDC)

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Setup in Rust:

- Harold Zurcher manages a bus depot in Madison, WI
- Each week he must decide to continue with the current engine $i_t = 0$ and pay $c(x_t)$ or overhaul the engine $i_t = 1$ and pay fixed cost RC.
- His goal is to minimize long run average cost (discounted).
- Buses are all independent of one another.

Rust (1987)

The agent makes a series of decisions $i_t, i_{t+1}, \ldots \in \{0, 1\}$.

$$\max_{\{i_1, i_2, i_3, \cdots, i_t, i_{t+1}, \cdots\}} \mathbb{E}_t \sum_{t=1}^{\infty} \beta^{t-1} \pi(x_t, i_t)$$

$$\pi\left(x_t, i_t\right) = \begin{cases} -c\left(x_t\right) & \text{if } i_t = 0 \\ -c\left(0\right) - RC & \text{if } i_t = 1 \end{cases}$$

- When costs are increasing in mileage (x_{it}) then this is an optimal stopping problem.
- We know from Stokey Lucas Prescott (1989) that solutions to optimal stopping problems are characterized by critical value or cutoff rule which we call x^*.

Rust (1987)

- For those who took first year macro, this problem is trivial.
- But Rust had a different goal in mind
  - Can we use the observed decisions of the agent to identify the primitives of $\pi(x_t, i_t)$?
  - Instead of just optimizing Bellman's equation, can we actually estimate parameters?
- Need to make two modifications:
  - Parameters for $c(x_t, \theta_1)$ and RC
  - Full support errors ε_t, otherwise can't justify why $i(x) = 1$ and $i(x') = 0$ if $x' > x$ at some point in the data.

Model: Modified Payoff Function

- per-period profit function:

$$\pi\left(x_t, i_t; \theta_1\right) = \begin{cases} -c\left(x_t, \theta_1\right) + \varepsilon_t\left(0\right) & \text{if } i_t = 0 \\ -\left(RC - c\left(0, \theta_1\right)\right) + \varepsilon_t\left(1\right) & \text{if } i_t = 1 \end{cases}$$

  where

  - $c\left(x_t, \theta_1\right)$ - regular maintenance costs (including expected breakdown costs),
  - RC - the net costs of replacing an engine,
  - ε - payoff shocks.

- x_t is the observed state variable known to both agent and econometrician
- ε is the unobserved state variable known only to agent

Model: Bellman

It is helpful to define the choice-specific value function:

$$\tilde{V}(x_t, i_t) = \begin{cases} \pi(x_t, 1; \theta_1) + \beta \mathbb{E}\, V(0, \varepsilon') & \text{if } i_t = 1 \\ \pi(x_t, 0; \theta_1) + \beta \mathbb{E}_{x', \varepsilon' | x_t, \varepsilon_t, i_t} V(x', \varepsilon') & \text{if } i_t = 0 \end{cases}$$

$$V(x_t, \varepsilon_t) = \max \left\{ \tilde{V}(x_t, 1) + \varepsilon_t(1), \tilde{V}(x_t, 0) + \varepsilon_t(0) \right\}$$

And the ex-ante value function

$$V_\theta\left(x_t, \varepsilon_t\right) = \max_i \left[\pi\left(i, x_t, \theta_1\right) + \beta E V_\theta\left(x_t, \varepsilon_t, i\right)\right]$$

$$E V_\theta\left(x_t, \varepsilon_t, i_t\right) = \int V_\theta\left(x_{t+1}, \varepsilon_{t+1}\right) p\left(x_{t+1}, \varepsilon_{t+1} | x_t, \varepsilon_t, i_t, \theta_2, \theta_3\right)$$

This is the period t expectation of the $t+1$ period continuation value.

Parameter Definitions

- θ_1 - parameters of cost function
- θ_2 - parameters of distribution of ε (these will be assumed/normalized away)
- θ_3 - parameters of x-state transition function
- RC - replacement cost
- discount factor β will be imputed (more on this later)

Assumptions

First Order Markov
Conditional Independence Assumption

The transition density of the controlled process $\{x_t, \varepsilon_t\}$ factors as:

$$p\left(x_{t+1}, \varepsilon_{t+1} | x_t, \varepsilon_t, i_t, \theta_2, \theta_3\right) = q\left(\varepsilon_{t+1} | x_{t+1}, \theta_2\right) p\left(x_{t+1} | x_t, i_t, \theta_3\right)$$

- CI assumption is very powerful: it means we don't have to treat ε_t as a state variable, which would be very difficult since it's unobserved.

IID Type I EV Assumption

For now we will also assume that $q\left(\varepsilon_{t+1} | x_{t+1}, \theta_2\right) = q\left(\varepsilon_{t+1}\right)$ and q is an IID Type I Extreme Value (logit) distribution.

- This is not required for identification but commonly employed to simplify estimation.
- Rust assumes that mean is 0 and variance is $\pi^2/6$.

Implications

Given the assumptions:

$$V_\theta(x_t, \varepsilon_t) = \max\left\{\tilde{V}_\theta(x_t, \varepsilon_t, 1) + \varepsilon_t(1), \tilde{V}_\theta(x_t, \varepsilon_t, 0) + \varepsilon_t(0)\right\}$$

$$\begin{aligned} Pr(i_t = 1 | x_t, \varepsilon_t, \theta) &= Pr\left(\varepsilon_t(1) - \varepsilon_t(0) \geq \tilde{V}_\theta(x_t, \varepsilon_t, 0) - \tilde{V}_\theta(x_t, \varepsilon_t, 1)\right) \\ &= \frac{\exp[\tilde{V}_\theta(x_t, \varepsilon_t, 1)]}{\exp[\tilde{V}_\theta(x_t, \varepsilon_t, 0)] + \exp[\tilde{V}_\theta(x_t, \varepsilon_t, 1)]} \end{aligned}$$

This expression is logit-like. Recall the static logit:

$$Pr(i_t = 1 | x_t, \varepsilon_t, \theta) = \frac{\exp[u_\theta(x_t, 1)]}{\exp[u_\theta(x_t, 0)] + \exp[u_\theta(x_t, 1)]}$$

Theorem 1 preview

- Assumption CI has two powerful implications:
  - We can write $EV_\theta(x_t, i_t)$ instead of $EV_\theta(x_t, \varepsilon_t, i_t)$,
  - We can consider a Bellman equation for $V_\theta(x_t)$, which is computationally simpler than the Bellman equation for $V_\theta(x_t, \varepsilon_t)$.

Theorem 1
Given CI,

$$P\left(i|x,\theta\right)=\frac{\partial}{\partial\pi\left(x,i,\theta_1\right)}W\left(\pi\left(x,\theta_1\right)+\beta EV_\theta\left(x\right)|x,\theta_2\right)$$

and EV_θ is the unique fixed point of the contraction mapping:

$$EV_\theta\left(x,i\right)=\int_y W\left(\pi\left(y,\theta_1\right)+\beta EV_\theta\left(y\right)|y,\theta_2\right)p\left(dy|x,i,\theta_3\right)$$

- $P\left(i|x,\theta\right)$ is the probability of action i conditional on state x
- $W\left(\cdot|x,\theta_2\right)$ is the (ex-ante) surplus function:

$$W\left(v|x,\theta_2\right)\equiv\int_\varepsilon \max_i\left[v\left(i\right)+\varepsilon\left(i\right)\right]q\left(d\varepsilon|x,\theta_2\right)$$

Theorem 1 example: logit shocks

- $v_\theta(x,i) \equiv \pi(x,i,\theta_1) + \beta EV_\theta(x,i)$ – the choice-specific value function.
- Suppose that $\varepsilon(i)$ is distributed independenly across i with $Pr(\varepsilon(i) \leq \varepsilon_0) = e^{-e^{-\varepsilon_0}}$ – logit shocks. Then,

$$W(v(x)) = \int \max_i [v(x,i) + \varepsilon(i)] \prod_i e^{-\varepsilon(i)} e^{-e^{-\varepsilon(i)}} d\varepsilon$$

$$= \ln\left(\sum_i \exp(v(x,i))\right) + \gamma$$

where $\gamma \approx .577216$ is Euler's gamma.

- It is then easy to derive expressions for conditional choice probabilities:

$$P(i|x,\theta) = \frac{\exp(v_\theta(x,i))}{\sum_{i'} \exp(v_\theta(x,i'))}$$

- The conditional value function plays the same role as a static utility function when computing choice probabilities.

Is this testable?

Problem: How do we do know true model is dynamic and not static logit?

- The general problem is that we can often write an observationally equivalent myopic model
- In general without additional (parametric) restrictions we are under identified (especially w.r.t β).
- May be able to determine "reasonableness" of parameters (how much could it cost to replace a bus engine?)
- Magnac Thesmar (Ecma 2002): we need exclusion restrictions:
    - Variables that move $u_\theta(x)$ but not $EV_\theta(x,i)$.
    - Variables that move $EV_\theta(x,i)$ but not $u_\theta(x)$.