MPEC notes

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thanks to Che-Lin Su

Grad IO

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Unconstrained Optimization

Basic idea in many estimation problems is to use Newton-type methods to solve FOCs of equilibrium or estimating equations

$$\min f(x) : x \in \mathbb{R}^n$$

- $f : R^n \to R, c : R^n \to R^m$ smooth (typically $\mathcal{C}^2$)
- $x \in \mathbb{R}^n$ finite dimensional (may be large)

Want to find a local minimizer

$$\nabla f(x^*) = 0$$

Optimization Algorithms generate a sequence $x^{(k)}$ such that the gradient test

$$||\nabla f(x^{(k)})|| \leq \tau$$

is satisfied for some tolerance $\tau = 1e - 6$ or so. Warning!

General NLP Problem

A Nonlinear Programming (NLP) problem is defined by:

$$\begin{cases} \min_x f(x) \quad \text{objective} \\ \text{subject to} \quad c(x) = 0 \quad \text{constraints} \\ x \geq 0 \quad \text{variables} \end{cases}$$

Typical assumptions

- $f : R^n \to R, c : R^n \to R^m$ smooth (typically $\mathcal{C}^2$)
- $x \in R^n$ finite dimensional (perhaps large)
- more general $l \leq c(x) \leq u$ is possible

Optimality Conditions for NLP

Constraint Qualifications (CQ)

Linearization of $c(x) = 0$ characterizes all feasible permutations, x^* local minimizer & CQ holds $\exists$ multipliers λ^*, γ^*:

$$\begin{array}{rcl} \nabla f(x^*) - \nabla c(x^*)\lambda^* - \gamma^* & = & 0 \\ c(x^*) = & = & 0 \\ X^*\lambda^* & = & 0 \\ x^* \geq 0, \gamma^* & \geq & 0 \end{array}$$

Where $X^* = diag(x^*)$, thus $X^*\lambda^* = 0 \Leftrightarrow x_i^*\gamma_i^* = 0$

NLP Solvers

$F(w) = 0$ where $w = (x, \lambda, \gamma)$ with $x \geq 0, z \geq 0$. Optimization Algorithms generate a sequence $w^{(k)}$ such that the gradient test

$$||\nabla f(w^{(k)})|| \leq \tau$$

is satisfied for some tolerance $\tau = 1e - 6$ or so. (Same warning).

Optimization

“Folk Theory” of Optimization in Economics

- Unconstrained Optimization is easier than Constrained Optimization
- More parameters are harder
- Quasi-Newton Methods are unreliable

Consequences of Folk Theory

- Rewrite all problems as unconstrained optimization
- Use fixed points and multi-step procedures to reduce parameter space
- Use Nelder-Mead/Simplex methods for optimization

Optimization

Thanks to recent advances in optimization:

More Accurate Description of Optimizaiton

1. *Shape* of the problem is what matters – convexity is really important
2. Constrained Problems are not much more difficult
3. More parameters can make the problem easier (or harder)

Consequences of State of the Art Optimization

- Tested stable Newton-routines are very reliable.
- Good Solvers handle 10,000+ parameters
- Computational burden are Jacobian and Hessian (and storage)

Recent Advances in Optimization Literature

Large Scale Algorithms

- Much focus has been on very large convex optimization problems – these have gotten really good.
- Most of these rely on first and second derivatives and quadratic approximations.
- Ways to do derivatives: analytic, numeric, symbolic and automatic (new!)
- Easy to solve 10,000+ parameter constrained problems often in less than 20 major iterations.
- Lots of industrial strength software packages.
- When in doubt express your problem as a convex one.
- Algorithm is polynomial $\approx O(k^3)$

Convexity

An optimization problem is convex if

$$\min_x f(\mathbf{x}) \quad s.t. \quad h(\mathbf{x}) \leq 0 \quad A\mathbf{x} = 0$$

- $f(\mathbf{x}), h(\mathbf{x})$ are convex (PSD second derivative matrix)
- Equality Constraint is affine

Some helpful identities about convexity

- Compositions and sums of convex functions are convex.
- Norms $||$ are convex, max is convex, log is convex
- $\log(\sum_{i=1}^{n} \exp(x_i))$ is convex.
- Fixed Points can introduce non-convexities.
- Globally convex problems have a unique optimum

Nested Logit Model

FIML Nested Logit Model is Non-Convex

$$\min_{\theta} \sum_j q_j \ln P_j(\theta) \quad \text{s.t.} \quad P_j(\theta) = \frac{e^{x_j \beta/\lambda} (\sum_{k \in g_l} e^{x_j \beta/\lambda})^{\lambda - 1}}{\sum_{\forall l'} (\sum_{k \in g_l'} e^{x_j \beta/\lambda})^{\lambda}}$$

This is a pain to show but the problem is with the cross term $\frac{\partial^2 P_j}{\partial \beta \partial \lambda}$ because $\exp[x_j \beta/\lambda]$ is not convex.

A Simple Substitution Saves the Day: let $\gamma = \beta/\lambda$

$$\min_{\theta} \sum_j q_j \ln P_j(\theta) \quad \text{s.t.} \quad P_j(\theta) = \frac{e^{x_j \gamma} (\sum_{k \in g_l} e^{x_j \gamma})^{\lambda - 1}}{\sum_{\forall l'} (\sum_{k \in g_l'} e^{x_j \gamma})^{\lambda}}$$

This is much better behaved and easier to optimize.

Nested Logit Model (Conlon Mortimer AEJ 2013)

	Original[1]	**Substitution**[2]	**No Derivatives**[3]
Parameters	49	49	49
Nonlinear λ	5	5	5
Neg LL	2.279448	2.279448	2.27972
Iterations	197	146	352
Time	59.0 s	10.7 s	192s

Discuss Simplex, Sparsity.

Extremum Estimators

Often faced with extremum estimator problems in econometrics (ML, GMM, MD, etc.) that look like:

$$\hat{\theta} = \arg\max_{\theta} Q_n(\theta), \quad \theta \in \Theta \tag{1}$$

Many economic problems contain constraints, such as: market clearing (supply equals demand), consumer's consume their entire budget set, or firm's first order conditions are satisfied. A natural way to represent these problems is as constrained optimization.

Constrained Problems

MPEC

$$\hat{\theta} = \arg\max_{\theta, P} Q_n(\theta, P), \quad \text{s .t.} \quad \Psi(P, \theta) = 0, \quad \theta \in \Theta$$

Fixed Point / Implicit Solution

In much of the literature the tradition has been to express the solutions $\Psi(P, \theta) = 0$ implicitly as $P(\theta)$:

$$\hat{\theta} = \arg\max_{\theta} Q_n(\theta, P(\theta)), \quad \theta \in \Theta$$

Constrained Problems

MPEC

$$\hat{\theta} = \arg\max_{\theta, P} Q_n(\theta, P), \quad \text{s .t.} \quad \Psi(P, \theta) = 0, \quad \theta \in \Theta$$

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Rust Problem

- Bus repairman sees mileage x_t at time t since last overhaul
- Repairman chooses between overhaul and normal maintenance

$$u(x_t, d_t, \theta^c, RC) = \begin{cases} -c(x_t, \theta^c) & \text{if} \quad d_t = 0 \\ -(RC + c(0, \theta^c) \quad) & \text{if} \quad d_t = 1 \end{cases}$$

- Repairman solves DP:

$$V_\theta(x_t) = \sum_{f_t, f_{t+1}, \dots} E\left\{ \sum_{j=t}^{\infty} \beta^{j-t} [u(x_j, f_j, \theta) + \varepsilon_j(f_j)] | x_t \right\}$$

- Econometrician
  - Observes mileage x_t and decision d_t but not cost.
  - Assumes extreme value distribution for $\varepsilon_t(d_t)$
- Structural parameters to be estimated $\theta = (\theta^c, RC, \theta^p)$.
  - Coefficients of cost function $c(x, \theta^c) = \theta_1^c x + \theta_2^c x^2$
  - Overhaul cost RC
  - Transition probabilities in mileages $p(x_{t+1} | x_t, d_t, \theta^p)$

Rust Problem

- Data: time series $(x_t, d_t)_{t=1}^{T}$
- Likelihood function

$$
\begin{aligned}
\mathcal{L}(\theta) &= \prod_{t=2}^{T} P(d_t|x_t, \theta^c, RC) p(x_t|x_{t-1}, d_{t-1}, \theta^p) \\
\text{with } P(d|x, \theta^c, RC) &= \frac{\exp[u(x, d, \theta^c, RC) + \beta EV_\theta(x, d)}{\sum_{d' \in \{0,1\}} \exp[u(x, d', \theta^c, RC) + \beta EV_\theta(x', d)} \\
EV_\theta(x, d) &= T_\theta(EV_\theta)(x, d)
\end{aligned}
$$

$$
\equiv \int_{x'=0}^{\infty} \log \left[\sum_{d' \in \{0,1\}} \exp[u(x, d', \theta^c, RC) + \beta EV_\theta(x', d)] \right] p(dx'|x, d, \theta^p)
$$

Rust Problem

- Outer Loop: Solve Likelihood

$$\max_{\theta \geq 0} \mathcal{L}(\theta) \quad = \quad \prod_{t=2}^{T} P(d_t | x_t, \theta^c, RC) p(x_t | x_{t-1}, d_{t-1}, \theta^p)$$

- Convergence test: $\|\nabla_\theta \mathcal{L}(\theta)\| \leq \epsilon_{out}$
- Inner Loop: Compute expected value function EV_θ for a given θ
- EV_θ is the implicit expected value function defined by the Bellman equation or the fixed point function

$$EV_\theta = T_\theta(EV_\theta)$$

- Convergence test: $\|EV_\theta^{(k+1)} - EV_\theta^{(k)}\| \leq \epsilon_{in}$
- Start with contraction iterations and polish with Newton Steps

NFXP Concerns

- Inner-loop error propagates into outer-loop function and derivatives
- NFXP needs to solve inner-loop exactly each stage of parameter search
  - to accurately compute the search direction for the outer loop
  - to accurately evaluate derivatives for the outer loop
  - for outer loop to converge!
- Stopping rules: choosing inner-loop and outer-loop tolerance
  - inner loop can be slow: contraction mapping is linearly convergent
  - tempting to loosen inner loop tolerance ϵ_{in} (such as $1e-6$ or larger!).
  - Outer loop may not converge with loose inner loop tolerance.
    - check solver output message
    - tempting to loosen outer loop tolerance ϵ_{out} to promote convergence ($1e-3$ or larger!).

Stopping Rules

- $\mathcal{L}(EV(\theta, \epsilon_{in}), \theta)$ the programmed outer loop objective function
- L: the Lipschitz constant (like modulus) of inner-loop contraction mapping
- Analytic derivatives $\nabla_\theta \mathcal{L}(EV(\theta, \epsilon_{in}), \theta)$ is provided: $\epsilon_{out} = O(\frac{L}{1-L}\epsilon_{in})$
- Finite-difference derivatives are used: $\epsilon_{out} = O(\sqrt{\frac{L}{1-L}\epsilon_{in}})$

Stopping Rules

- Form the augmented likelihood function for data $X = (x_t, d_t)_{t=1}^{T}$

$$\begin{aligned} \mathcal{L}(EV, \theta; X) &= \prod_{t=2}^{T} P(d_t | x_t, \theta^c, RC) p(x_t | x_{t-1}, d_{t-1}, \theta^p) \\ \text{with } P(d|x, \theta^c, RC) &= \frac{\exp[u(x, d, \theta^c, RC) + \beta EV(x, d)}{\sum_{d' \in \{0,1\}} \exp[u(x, d', \theta^c, RC) + \beta EV(x', d)} \end{aligned}$$

- Rationality and Bellman equation imposes a relationship between θ and EV

$$EV = T(EV, \theta)$$

- Solve constrained optimization problem

$$\begin{aligned} \max_{(\theta, EV)} \mathcal{L}(EV, \theta; X) \\ \text{subject to } EV = T(EV, \theta) \end{aligned}$$

Results

β	Imple.	Parameters						MSE
		RC	θ_{11}	θ_{31}	θ_{32}	θ_{33}	θ_{34}	
	true	11.726	2.457	0.0937	0.4475	0.4459	0.0127	
0.975	MPEC1	12.212	2.607	0.0943	0.4473	0.4454	0.0127	3.111
		(1.613)	(0.500)	(0.0036)	(0.0057)	(0.0060)	(0.0015)	–
	MPEC2	12.212	2.607	0.0943	0.4473	0.4454	0.0127	3.111
		(1.613)	(0.500)	(0.0036)	(0.0057)	(0.0060)	(0.0015)	–
	NFXP	12.213	2.606	0.0943	0.4473	0.4445	0.0127	3.123
		(1.617)	(0.500)	(0.0036)	(0.0057)	(0.0060)	(0.0015)	–
0.980	MPEC1	12.134	2.578	0.0943	0.4473	0.4455	0.0127	2.857
		(1.570)	(0.458)	(0.0037)	(0.0057)	(0.0060)	(0.0015)	–
	MPEC2	12.134	2.578	0.0943	0.4473	0.4455	0.0127	2.857
		(1.570)	(0.458)	(0.0037)	(0.0057)	(0.0060)	(0.0015)	–
	NFXP	12.139	2.579	0.0943	0.4473	0.4455	0.0127	2.866
		(1.571)	(0.459)	(0.0037)	(0.0057)	(0.0060)	(0.0015)	–

Results

β	Imple.	Parameters						MSE
		RC	θ_{11}	θ_{31}	θ_{32}	θ_{33}	θ_{34}	
	true	11.726	2.457	0.0937	0.4475	0.4459	0.0127	
0.985	MPEC1	12.013	2.541	0.0943	0.4473	0.4455	0.0127	2.140
		(1.371)	(0.413)	(0.0037)	(0.0057)	(0.0060)	(0.0015)	–
	MPEC2	12.013	2.541	0.0943	0.4473	0.4455	0.0127	2.140
		(1.371)	(0.413)	(0.0037)	(0.0057)	(0.0060)	(0.0015)	–
	NFXP	12.021	2.544	0.0943	0.4473	0.4455	0.0127	2.136
		(1.368)	(0.411)	(0.0037)	(0.0057)	(0.0060)	(0.0015)	–
0.990	MPEC1	11.830	2.486	0.0943	0.4473	0.4455	0.0127	1.880
		(1.305)	(0.407)	(0.0036)	(0.0057)	(0.0060)	(0.0015)	–
	MPEC2	11.830	2.486	0.0943	0.4473	0.4455	0.0127	1.880
		(1.305)	(0.407)	(0.0036)	(0.0057)	(0.0060)	(0.0015)	–
	NFXP	11.830	2.486	0.0943	0.4473	0.4455	0.0127	1.880
		(1.305)	(0.407)	(0.0036)	(0.0057)	(0.0060)	(0.0015)	–

Results

β	Imple.	Parameters						MSE
		RC	θ_{11}	θ_{31}	θ_{32}	θ_{33}	θ_{34}	
	true	11.726	2.457	0.0937	0.4475	0.4459	0.0127	
0.995	MPEC1	11.819	2.492	0.0942	0.4473	0.4455	0.0127	1.892
		(1.308)	(0.414)	(0.0036)	(0.0057)	(0.0060)	(0.0015)	–
	MPEC2	11.819	2.492	0.0942	0.4473	0.4455	0.0127	1.892
		(1.308)	(0.414)	(0.0036)	(0.0057)	(0.0060)	(0.0015)	–
	NFXP	11.819	2.492	0.0942	0.4473	0.4455	0.0127	1.892
		(1.308)	(0.414)	(0.0036)	(0.0057)	(0.0060)	(0.0015)	–

Results

β	Imple.	Runs Conv.	CPU Time (in sec.)	# of Major Iter.	# of Func. Eval.	# of Contrac. Mapping Iter.
0.975	MPEC1	1240	0.13	12.8	17.6	–
	MPEC2	1247	7.9	53.0	62.0	–
	NFXP	998	24.6	55.9	189.4	$1.348e+5$
0.980	MPEC1	1236	0.15	14.5	21.8	–
	MPEC2	1241	8.1	57.4	70.6	–
	NFXP	1000	27.9	55.0	183.8	$1.625e+5$
0.985	MPEC1	1235	0.13	13.2	19.7	–
	MPEC2	1250	7.5	55.0	62.3	–
	NFXP	952	42.2	61.7	227.3	$2.658e+5$
0.990	MPEC1	1161	0.19	18.3	42.2	–
	MPEC2	1248	7.5	56.5	65.8	–
	NFXP	935	70.1	66.9	253.8	$4.524e+5$
0.995	MPEC1	965	0.14	13.4	21.3	–
	MPEC2	1246	7.9	59.6	70.7	–
	NFXP	950	111.6	58.8	214.7	$7.485e+5$

Results

```
KNITRO 5.2.0: alg=1
opttol=1.0e-6
feastol=1.0e-6

Problem Characteristics
-----------------------
Objective goal:  Maximize
Number of variables:                    207
    bounded below:                        6
    bounded above:                      201
    bounded below and above:              0
    fixed:                                0
    free:                                 0
Number of constraints:                  202
    linear equalities:                    1
    nonlinear equalities:               201
    linear inequalities:                  0
    nonlinear inequalities:               0
    range:                                0
Number of nonzeros in Jacobian:        2785
Number of nonzeros in Hessian:         1620
```

Results

```
Final Statistics
----------------
Final objective value                     =  -2.35221126396447e+03
Final feasibility error (abs / rel)       =   1.33e-15 / 1.33e-15
Final optimality error  (abs / rel)       =   1.00e-08 / 6.71e-10
# of iterations                           =         12
# of CG iterations                        =          0
# of function evaluations                 =         13
# of gradient evaluations                 =         13
# of Hessian evaluations                  =         12
Total program time (secs)                 =       0.10326 (     0.097 CPU time)
Time spent in evaluations (secs)          =       0.05323

=================================================================

KNITRO 5.2.0: Locally optimal solution.
objective -2352.211264; feasibility error 1.33e-15
12 major iterations; 13 function evaluations
```

BLP Demand Example

BLP 1995

The estimator solves the following mathematical program:

$$
\begin{aligned}
\min_{\theta_2} \quad & g(\xi(\theta_2))' W g(\xi(\theta_2)) \quad \text{s.t.} \\
g(\xi(\theta_2)) &= \frac{1}{N} \sum_{\forall j,t} \xi_{jt}(\theta_2)' z_{jt} \\
\xi_{jt}(\theta_2) &= \delta_j(\theta_2) - x_{jt}\beta - \alpha p_{jt} \\
s_{jt}(\delta(\theta_2), \theta_2) &= \int \frac{\exp[\delta_j(\theta_2) + \mu_{ij}]}{1 + \sum_k \exp[\delta_j(\theta_2) + \mu_{ik}]} f(\mu|\theta_2) \\
\log(S_{jt}) &= \log(s_{jt}(\delta(\theta_2), \theta_2)) \quad \forall j, t
\end{aligned}
$$

BLP Algorithm

The estimation algorithm is generally as follows:

1. Guess a value of nonlinear parameters θ_2
2. Compute $s_{jt}(\delta, \theta_2)$ via integration
3. Iterate on $\delta_{jt}^{h+1} = \delta_{jt}^{h} + \log(S_{jt}) - \log(s_{jt}(\delta^h, \theta_2))$ to find the δ that satisfies the share equation
4. IV Regression δ on observable X and instruments Z to get residual ξ.
5. Use ξ to construct $g(\xi(\theta_2))$.
6. Possibly construct other errors/instruments from supply side.
7. Construct GMM Objective

The idea is that $\delta(\theta_2)$ is an implicit function of the nonlinear parameters θ_2. And for each guess we find that implicit solution for reduce the parameter space of the problem.

Dube Fox Su 2009

BLP-MPEC

The estimator solves the following mathematical program:

$$\begin{aligned}
\min_{\sigma,\alpha,\beta,\xi} \quad & g(\xi)' W g(\xi) \quad \text{s.t.} \\
g(\xi) &= \frac{1}{N} \sum_{\forall j,t} \xi'_{jt} z_{jt} \\
s_{jt}(\sigma,\alpha,\beta,\xi) &= \sum_i w_i \frac{\exp[x_{jt}\beta + \xi_{jt} - \alpha p_{jt} + \sum_l \nu_{il} x^l_{jt} \sigma_l]}{1 + \sum_k \exp[x_{kt}\beta + \xi_{kt} - \alpha p_{kt} + \sum_l \nu_{il} x^l_{kt} \sigma_l]} \\
\log(S_{jt}) &= \log s_{jt}(\sigma,\alpha,\beta,\xi) \quad \forall j,t
\end{aligned}$$

- Expand the parameter space of the nonlinear search to include α, β, ξ
- Don't have to solve for ξ except at the end.
- No implicit functions of θ_2
- Sparsity!

Empirical Likelihood Methods

Empirical Likelihood often statistically better than GMM

- Higher order efficiency (gets to semi-parametric efficiency bound faster) (Kitamura 2001, 2006)
- No problems in estimating the weight matrix (Altonji Segal 1995, Newey Smith 2004).
- Likelihood based units for testing
- No problems with scaling of parameters / instruments.
- GMM prone to non-identification in finite-sample (Dominguez Lobato 2004)

EL as NPMLE (Owen 1990, Kitamura 2006)

Nonparametric MLE

$$l_{NP}(p_1, \ldots, p_n) = \sum_{i=1}^{n} \log p_i \quad (p_1, \ldots, p_n) \in \Delta \tag{2}$$

- Observed z_i are IID with measure μ
- Simplex defined as $(p_1, p_2, \ldots, p_n) \in \Delta$ such that $\sum_{i=1}^{n} p_i = 1$ and $p_i \geq 0$
- Trivial max at $p_i = \frac{1}{n}$ and $\mu_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{z_i}$ and $l_{NP} = -n \log n$

Adding moment conditions

(Owen 1990) Shows we can extend NPMLE to moment condition models.

$$E[g(z_i,\theta)] = \int g(z,\theta)d\mu = 0 \in \mathbb{R}^q, \quad \theta \in \Theta \in \mathbb{R}^k$$

This exactly follows our definition of an MPEC problem:

Empirical Likelihood Estimator

$$\arg\max_{\theta,p} l_{NP}$$

$$l_{NP} = \sum_{i=1}^{n} \log p_i \qquad \text{s.t.} \quad \sum_{i=1}^{n} p_i g(z_i,\theta) = 0 \quad \text{and} \quad \sum_{i=1}^{n} p_i = 1$$

Alternative-Generalized Minimum Contrast

Idea is to look for a statistical model $\mathcal{P}$ close to the true measure μ

$$\begin{aligned}\mathcal{P}(\theta) &= \{P \in M : \int g(z,\theta)dP = 0\} \\ \mathcal{P} &= \cup_{\theta\in\Theta}\mathcal{P}(\theta)\end{aligned}$$

We do this by minimizing the contrast function $D(P,U) = \int \phi(p)d\mu$ with $p = \frac{dP}{d\mu}$.

$$\inf_{\theta\in\Theta} \rho(\theta,\mu) \text{ where } \rho(\theta,\mu) = \inf_{P\in\mathcal{P}} D(P,\mu)$$

Produces (infinite dimensional) constrained problem:

$$v(\theta) = \inf_p \int \phi(p)d\mu \quad \int g(z,\theta)pd\mu = 0 \quad \int pd\mu = 1$$

Choose a contrast function $\phi(x) = \log(x)$ (EL) or $\phi(x) = \frac{1}{2}(x^2 - 1)$ (CUE)

Interpreting EL

Simple Alternate Explanation

- GMM asks what are the parameters θ that minimize the quadratic distance under some metric A between my model applied to the observed data, and my model in the "ideal" case. (Model generally doesn't hold exactly – overidentified).
- EL asks how different a distribution of data would I need to observe in order to meet the implications of my model.
- EL as solutions to systems of equations (which one do we pick?)
- Now I have "likelihood" of different models and can compare structural assumptions.

Standard Algorithm

Lagrangian

Construct the dual by differentiating the Lagrangian:

$$
\begin{aligned}
\mathcal{L} &= \sum_{i=1}^{n} \log p_i + \lambda(1 - \sum_{i=1}^{n} p_i) - n\gamma' \sum_{i=1}^{n} p_i g(z_i, \theta) \\
\hat{\gamma}(\theta) &= \arg\min_{\gamma \in \mathbb{R}^q} - \sum_{i=1}^{n} \log(1 + \gamma' g(z_i, \theta)) \\
\hat{p}_i(\theta) &= \frac{1}{n(1 + \hat{\gamma}(\theta)' g(z_i, \theta))} \qquad \hat{\lambda} = n \\
\hat{\theta}_{EL} &= \arg\max_{\theta \in \Theta} l_{NP}(\theta) = \arg\max_{\theta \in \Theta} \min_{\gamma \in \mathbb{R}^q} - \sum_{i=1}^{n} \log(1 + \gamma' g(z_i, \theta))
\end{aligned}
$$

Computational Challenges

$$\hat{\theta}_{EL} = \arg\max_{\theta \in \Theta} l_{NP}(\theta) = \arg\max_{\theta \in \Theta} \min_{\gamma \in \mathbb{R}^q} -\sum_{i=1}^{n} \log(1 + \gamma' g(z_i, \theta))$$

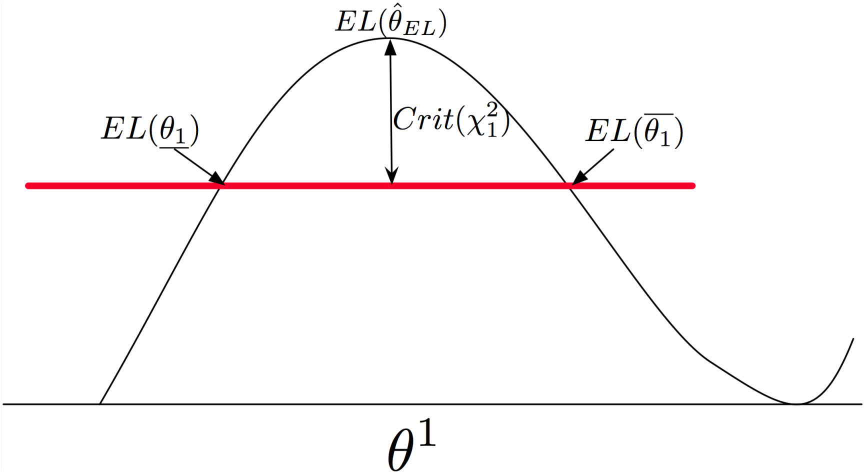
The dual approach presents a number of computational challenges:

- For each guess of θ we must find the optimal p (actually γ), but it may be that $\nexists p$ s.t $\sum_{i=1}^{n} p_i g(z_i, \theta) = 0$ at some θ.[4]
- At some z_i we may find that $\gamma' g(z_i, \theta) \leq -1$.
- Not much in terms of max min solvers (stuck with nested
- max of a convex function (or the min of a concave function).
- $\nabla_\theta \cdot l_{NP}(\theta) = \nabla_\theta \left[\min_{\gamma \in \mathbb{R}^q} -\sum_{i=1}^{n} \log(1 + \gamma' g(z_i, \theta))\right]$ hard

EL Inference

- We can construct likelihood based testing units via Empirical Likelihood Ratio test statistic: $\frac{EL(\hat{\theta}_r)}{EL(\hat{\theta})} \sim \chi^2_r$
- We can invert the test statistic to construct confidence intervals
- MPEC lets us impose the ELR test statistic as an additional restriction
- Solve problems such as $\max \theta_1 : EL(\theta) > EL(\hat{\theta_0}) - crit$
- Nice for nonlinear predictions such as elasticities to test directly without delta method (still have higher order efficiency).

EL Inference



Altonji Segal 1996 / Abowd Card 1989

The dataset (PSID) tracks 1536 individuals over ten years, and 210 moment conditions are constructed out of all of the possible permutations of variances, covariances, and autocovariances for different lags. A stationary model is estimated with 45 parameters.

	MPEC-EL[5]	**Dual**[6]	**ELike-M**[7]
Moments	210	210	210
Parameters	1591	255 (45)	255(45)
Iterations	65	1023	10,000+
# Converged	50	46	42
Time	$\approx 25s$	$\approx$ 20 m	$\approx$ 45 m

Dube Fox Su 2009

An MPEC Algorithm for computing the same estimator as BLP has been suggested:

BLP-MPEC

The estimator solves the following mathematical program:

$$\begin{aligned} \min_{\theta,\xi,s,g} \quad & g(\xi)' W g(\xi) \quad \text{s.t.} \\ g(\xi) &= \frac{1}{N} \sum_{\forall j,t} \xi'_{jt} z_{jt} \\ s_{jt}(\theta) &= \int \frac{\exp[x_{jt}\beta_i + \xi_{jt} - \alpha_i p_{jt}]}{1 + \sum_k \exp[x_{kt}\beta_i + \xi_{kt} - \alpha_i p_{kt}]} f(\beta_i|\theta) \\ \log(S_{jt}) &= \log(s_{jt}(\beta, \alpha, \xi, \theta)) \quad \forall j, t \end{aligned}$$

(Knittel Metaxoglou 2008)

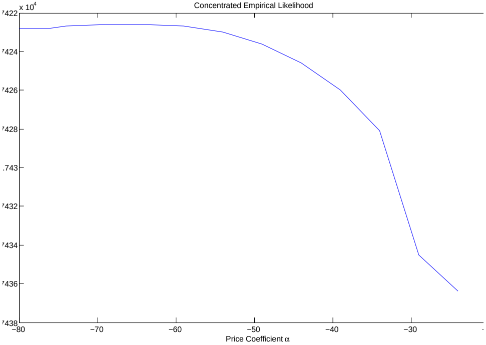
Recent paper (just updated) takes 10 algorithms, 50 starting values and uncovers 100+ parameter estimates and Nevo code/data:

- *a local minimum may yield parameter values that are close to the true values but have an objective function value that is very different. Therefore we focus on the economic meaning of the variation in parameter estimates...*
- *... researchers will need to use multiple starting values, at least 50 and multiple algorithms*
- Mistakes abound
- What weight matrix is used?

Nevo Results

	Nevo	BLP-MPEC	EL
Price	-28.189	-62.726	-61.433
σ_p	0.330	0.558	0.524
σ_{const}	2.453	3.313	3.143
σ_{sugar}	0.016	-0.006	0
σ_{mushy}	0.244	0.093	0.085
$\pi_{p,inc}$	15.894	588.206	564.262
$\pi_{p,inc2}$	-1.200	-30.185	-28.930
$\pi_{p,kid}$	2.634	11.058	11.700
$\pi_{c,inc}$	5.482	2.29084	2.246
$\pi_{c,age}$	0.2037	1.284	1.37873
GMM	29.3611	4.564	
EL			-17422
Time	28 s	12s	19s

Profile Empirical Likelihood



Conditional Empirical Likelihood

- Looks a lot like weak instruments
- CEL does slightly better
- Theory tells us that $E[\xi_{jt}|Z_{jt}] = 0$ but we usually estimate $E[\xi_{jt} Z_{jt}] = 0$ which is the weaker exogeneity restriction.
- The CEL curve looks a bit less flat and better identified.
- If you thought n extra parameters was bad – imagine n^2.