Tutorial – Week4

Q1. Given Euler's formula: $e^{i\theta} = cos\,\theta + \mathrm{i}\,sin\,\theta$, use Euler's formula to prove

$$cos(\alpha + \beta) = cos\,\alpha\,cos\,\beta - sin\,\alpha\,sin\,\beta$$

$$sin(\alpha + \beta) = cos\,\alpha\,sin\,\beta + sin\,\alpha\,cos\,\beta$$

Solution:

$$e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta} = (cos\,\alpha + i\,sin\,\alpha)(cos\,\beta + i\,sin\,\beta)$$
$$= cos\,\alpha\,cos\,\beta - sin\,\alpha\,sin\,\beta + i(sin\,\alpha\,cos\,\beta + sin\,\beta\,cos\,\alpha)$$

$$\because e^{i(\alpha+\beta)} = cos(\alpha + \beta) + i\,sin(\alpha + \beta)$$

We compare the real part and imagining part,

We have

$cos(\alpha + \beta) = cos\,\alpha\,cos\,\beta - sin\,\alpha\,cos\,\beta$ (Real part)

$sin(\alpha + \beta) = sin\,\beta\,cos\,\alpha + sin\,\alpha\,cos\,\beta$ (Imagining part)

Q2. Solve $2\,log_3\,x - log_3(x + 6) = 1.$

Solution:

$$2\,log_3\,x - log_3(x + 6) = 1$$

$$log_3\,x^2 - log_3(x + 6) = 1$$

$$log_3\left(\frac{x^2}{x + 6}\right) = 1$$

$$log_3\frac{x^2}{x + 6} = log_3\,3$$

$$\therefore 3 = \frac{x^2}{x + 6}$$

$$x^2 = 3x + 18$$

$$x^2 - 3x - 18 = 0$$

$$(x - 6)(x + 3) = 0$$

$$\therefore x = 6, -3$$

It is a 1^{st} order problem but somehow we made it x^2 to create 1 more possible solution.

$\therefore$ Check if $x = -3$

$2\,log_3(-3) - \log_3(3) = 1$ Wrong (because $log_3(-3)$)

Check if $x = 6$

$2\,log_3(6) - log_3(12) = 1$ Right

$\therefore x = 6$

Q3. The ratio of power P (measured power) to a reference power P_0 is represented as L_p

$$\mathrm{L_p} = 10\log_{10}\left(\frac{\mathrm{P}}{\mathrm{P_0}}\right)\mathrm{dB} \text{ ----- decibels}$$

a. Find P which equals to half of the reference power P_0.

b. If a circuit with constant R_1, the gain in power is the ratio between $\frac{V^2}{R_1}$ and $\frac{V_2^2}{R_1}$ (reference). Find the power gain expression in dB.

Solution:

a. $L_p = 10\,log_{10}\frac{\frac{1}{2}P_0}{P_0} = 10\,log_{10}\frac{1}{2} = 10(-0.3) = -3\,dB$

b. $Gain\ dB = 10\,log_{10}\frac{\frac{V_1^2}{R_1}}{\frac{V_2^2}{R_1}} = 10\,log_{10}\left(\frac{V_1}{V_2}\right)^2 = 20\,log_{10}\frac{V_1}{V_2}$

Q4. Injective proof: Let $f: Z \to Z$, $f(x) = 3x + 7$. Prove f is injective. Recall the definition of injective

$$\forall a, b \in X, f(a) = f(b) \Rightarrow a = b.$$

Solution:

We need to show that for every integer a and b, $f(a) = f(b) \Rightarrow a = b$.

So, let a and b be integers (this is an important step to fulfills the above $f: Z \to Z$).

And suppose that $f(a) \ = f(b)$, we need to show $a = b$.

As our assumption, we know $f(a) = f(b)$, so substituting in the given formula for the function $f(x)$,

It means $3a + 7 = 3b + 7$.

So $3a = 3b$, $a = b$. This is what we need to show.

Thus $f(x)$ is injective one-to-one.