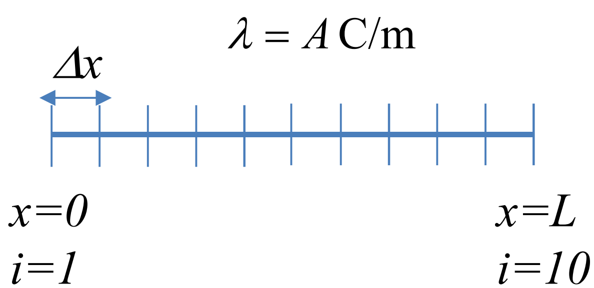
Integration

In PHY1202, we often asked to determine physical quantities, such as force, field, potential … etc., that is due to a particular charge or current distribution, such as from a line, ring, disk, sphere, box, ...etc.

Integration is a method that allows us to determine the quantity due to the whole object by summing up the contributions from its individual parts.

For example, if we have a line charge with constant charge density $\lambda=A$ C/m where A is a constant and if we want to determine the amount of charge Q contains in a line with length L, then obviously the answer is just Q=LA C. However, we can also come up with the same answer by chopping up the line charge into 10 small pieces each of length Δx=0.1L with each segment containing $\Delta q=0.1LA$ and sum them all up.



$$Q = A\Delta x_1 + Ax_2 + \cdots + A\Delta x_9 + A\Delta x_{10}$$

$$= A\sum_{i=1}^{10}\Delta x_i = LA$$

Δx_i *is the* i^{th} *segment*

Integration

In this simple example, the line charge density is uniform, so it really does not matter how short or long we make Δx and we will still get the same answer, however what if the charge density is a function of position? Say,

$$\lambda = \lambda(x) = Bx \text{ C/m}$$

$x=0$ $x=L$

then the answer will depends on location of the section, also the size of Δx will now influence our result. It is obvious that we can obtain a more accurate result by reducing the size of Δx. Here we define the integration as when in the limit of Δx approaches zero, thus allowing us to obtain the most accurate result.

$$\boxed{Q = \int_{x=0}^{x=L} \lambda(x)dx}$$

$\int$	*Integral sign*
$\lambda(x)$	*Integrand*
dx	*Differential element*
$x = 0$	*Lower limit of the integration*
$x = L$	*Upper limit of the integration.*

The expression is called a <u>definite</u> integral and it means we are taking the integration of the integrand $\lambda(x)$ along the x direction with the <u>defined</u> end points from $x=0$ to L. The line from $x=0$ to L is the path of the integration.

Integration

Integration can be very complicated and challenging and I will only provide you with the most basics information you need for PHY1202 here. In this course, we will only use the simplest examples, if at all, and save the more challenging problems for the upper year EM classes; after you had your proper calculus training in your math courses.

Before we go further, let's introduce the concept of antiderivatives and indefinite integration.

Definition of Antiderivative: A function F is called an antiderivative of the function f if for every x in the domain of f

$$\frac{dF(x)}{dx} = f(x)$$

$F(x)$ is the antiderivative of $f(x)$;

$f(x)$ is the first derivative of $F(x)$

$$y = \int f(x)dx = F(x) + C$$

y	*Integral function*
$f(x)$	*Integrand*
dx	*Differential element*
C	*Constant of integration*

This is an indefinite integral because we have not defined the limit of integration.

Integration

$$y(x) = \int f(x)dx = F(x) + C$$

The above expression relates the integrand $f(x)$ with the value of the integral $y(x)$, so using our line charge example, $y(x)$ represents the sum of all charges from some reference location to the location x, in a way this is similar to how we define potential, and the same reasoning for the need of the constant C, because we do not know the initial integral value y at the reference point.

With this understanding we can now calculate the definite integral value by substituting the end points into the expression:

$$y(x) = \int_{x_1}^{x_2} f(x)dx = [F(x) + C]\Big|_{x_1}^{x_2} = \left[F(x_2) + C\right] - \left[F(x_1) + C\right] = F(x_2) - F(x_1)$$

Obviously the key to solve the integral problem is to find the relationship between the integrand and its antiderivative, now this can be extremely difficult, but fortunately a lot of the relationships are available and we will only use some of the simplest as mentioned earlier. Furthermore, you will be provided with the integration table in your exam if they are needed to answer the questions.

Some basic integration formulas

$$\int \cos x dx = \sin x + C$$

$$\int \sin x dx = -\cos x + C$$

$$\int \sec^2 x dx = \tan x + C$$

$$\int \sec x \tan x dx = \sec x + C$$

$$\int \csc^2 x dx = -\cot x + C$$

$$\int \csc x \cot x dx = -\csc x + C$$

$$\int 0 dx = C$$

$$\int A dx = Ax + C$$

$$\int x dx = \frac{x^2}{2} + C$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

$$\int \frac{1}{x^2} dx = -\frac{1}{x} + C$$

$$\int \frac{1}{x} dx = \ln x + C$$

$$\int e^x dx = e^x + C$$

Integration

$$Q = \int_{x=0}^{x=L} \lambda(x)dx$$

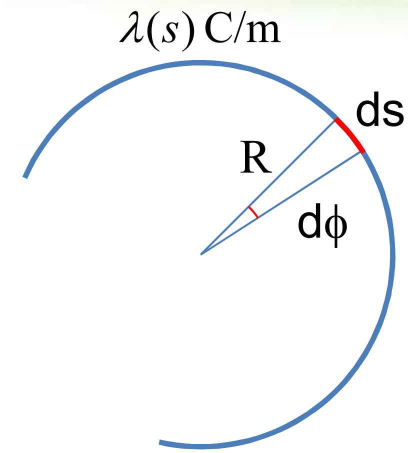
Consider the simple case of $\lambda = A$ C/m in our example, with A constant that is not a function of x, we can take it outside of the integral and the integrand is now just 1. The antiderivative of 1 is simply x, as $dx/_{dx} = 1$, thus we have

$$Q = A\int_{x=0}^{x=L} dx = A[x + C]\Big|_0^L = A[L - 0] = AL \quad \mathrm{C}$$

In the case of $\lambda = Bx$ C/m, with B constant that is not a function of x, we can still take it outside of the integral but we cannot take the variable x out of the integral because its value changes along the path of the integration; so the integrand is now x. The antiderivative of x is $x^2/2$, as $\frac{d}{dx}\frac{x^2}{2} = x$, thus we have

$$Q = B\int_{x=0}^{x=L} xdx = B[\frac{x^2}{2} + C]\Bigg|_0^L = B[\frac{L^2}{2} - 0] = \frac{BL^2}{2} \quad \mathrm{C}$$

Integration



Let's consider a slightly more complicated situation where the line of charge follows an arc of radius R, as shown. Here we can no longer express the line charge as a function of x only and need to express it as a function of some variable that can better represent the path of the line charge.

To calculate the total charge, we can once again chop up the circle into small segments, Δs, and sum them all up.

Doing this by integration, we found that the integral can be expressed as

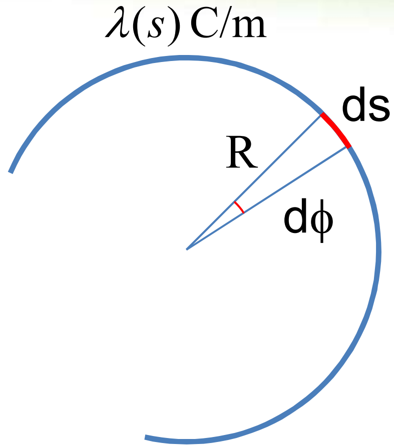
$$Q = \int_{s_1}^{s_2} \lambda(s)ds$$

To solve the integral, we will need to convert ds into a coordinate system that we can manage. From the diagram, we observe that ds is proportional to $d\phi$, and in fact $ds = Rd\phi$. Note that if $d\phi=2\pi$, then $ds=2\pi R$, or the circumference of the circle. Substituting ds into the original integral with $d\phi$, then expressing λ as a function of ϕ and reset the integral limits, we have

$$Q = \int_{\phi_1}^{\phi_2} \lambda(\phi)Rd\phi$$

Integration

$$Q = \int_{\phi_1}^{\phi_2} \lambda(\phi) R d\phi$$



Let say we want to determine the total charge of the line charge that covers ¾ of the circle from ϕ=0 to $3\pi/2$ with radius R and constant charge density $\lambda(\phi)$=A C/m, we can write down the expression as

$$Q = A \int_0^{3\pi/2} d\phi = A\phi\Big|_0^{3\pi/2} = A\left(\frac{3\pi}{2} - 0\right) = A\frac{3\pi}{2} \quad \text{C}$$

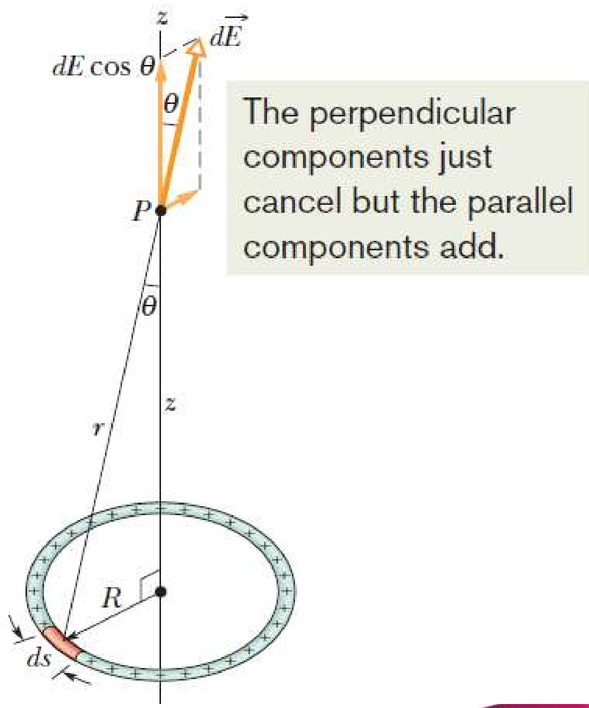
We can use the same method to calculate the total charge as long as we know the beginning and ending angles of the arc, and the angle does not need to be limited to within 0 and 2π, because the arc can be wrap around the circle multiple times.

Integration

Of course it is unlikely that you will find questions that ask you to calculate the total charge of some line charge distribution in a physics exam. In this course, we are more interested in finding the electric field, force, potential … etc., near the charge distribution.

For example, we may need to calculate the electric field due to a ring of uniform charge with constant line charge density λ C/m, as shown in the figure. We know that the electric field due to a point charge is simply

$$\vec{E} = k\frac{q}{r^2}\hat{r}$$

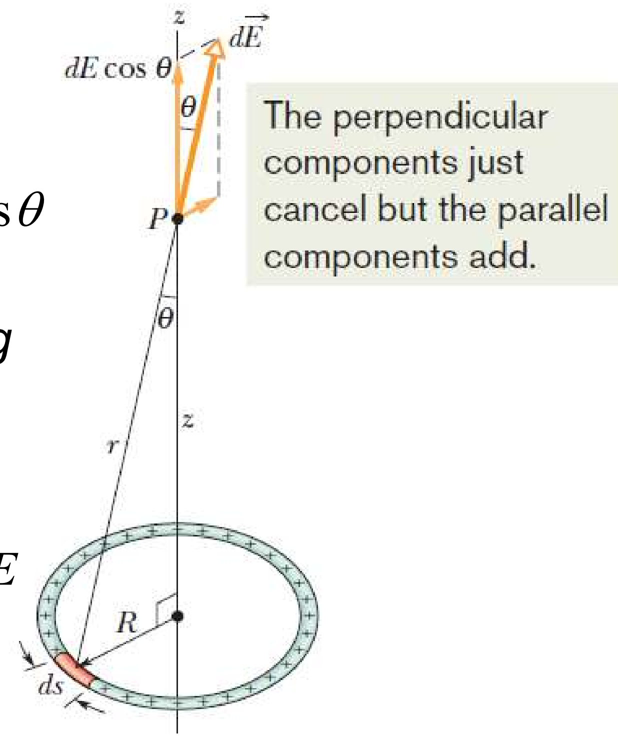


We can divide the ring into tiny segments and treat them as individual point charges and sum up their contributions to determine the total electric field. For a small ring segment ds, the amount of charge it contains is $dq=\lambda ds$ and the electric field from dq is

$$d\vec{E} = k\frac{dq}{r^2}\hat{r}$$

Integration

$$d\vec{E} = k\frac{dq}{r^2}\hat{r} = k\frac{\lambda ds}{r^2}\hat{r} = k\frac{\lambda ds}{z^2+R^2}\hat{r}$$



Note that the $d\vec{E}$ vectors have components that are parallel to and perpendicular to z and they are $dE\cos\theta$ and $dE\sin\theta$, respectively.

Because we are considering the complete ring, using symmetry we observe that any perpendicular component of $d\vec{E}$ to the z direction will be cancelled out by $d\vec{E}$ from the dq element that is located directly across from the ring. So that only the only non-zero E component is parallel to z.

The parallel components are

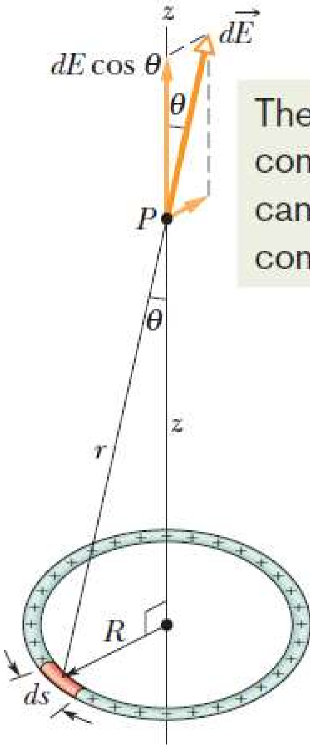
$$dE\cos\theta = k\frac{dq}{r^2}\cos\theta = k\frac{\lambda ds}{z^2+R^2}\frac{z}{\left(z^2+R^2\right)}$$

$$= k\lambda\frac{z}{\left(z^2+R^2\right)^{3/2}}ds$$

Integration

We can now express the integral as

$$E_z = \int_0^{2\pi R} \boxed{k \frac{z\lambda}{\left(z^2+R^2\right)^{3/2}}} ds = k \frac{z\lambda}{\left(z^2+R^2\right)^{3/2}} \int_0^{2\pi R} ds$$



The perpendicular components just cancel but the parallel components add.

Here, the parameters inside the dotted box are all independent of s*, that can be moved outside of the integral. We can also rewrite the integral using the expression in slides 7 and 8 and obtain the same result.*

$$E_z = k \frac{z\lambda}{\left(z^2+R^2\right)^{3/2}} \int_0^{2\pi R} ds = k \frac{z\lambda}{\left(z^2+R^2\right)^{3/2}} \int_0^{2\pi} \phi d\phi = k \frac{z\lambda 2\pi R}{\left(z^2+R^2\right)^{3/2}}$$

Note that if we consider only a portion of the ring instead of the whole ring, then there will be a non zero perpendicular element that we need to consider.

Integration

We can use the same method to determine the contribution due to a surface A or volume V distribution by dividing the surface and volume into small differential area, dA, and dV, perform the integration in 2D and 3D.

Below are some examples of integrals in other dimensions:

$\int f(x)dx$	*Line integral – integration along a path.*
$\iint f(x,y)dxdy$	*Surface integral – integration over a surface*
$\iiint f(x,y,z)dxdydz$	*Volume integral – integration of a volume.*
$\oint f(s)ds$	*Closed line integral – integration along a closed loop. Such as the loop encloses a surface.*
$\oiint f(a)da$	*Closed surface integral – integration over closed surface that encloses a volume.*