Basics of Set Theory

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EE

August 2020

CityU

Set

- A set is a collection of objects.
- Suppose the set A contains an object called x.
  - x is an element (or member) of A.
  - x is denoted by $x \in A$.
- The roster notation of a set simply lists all members of the set inside braces { }.
- Example:

  $$C = \{10¢, 20¢, 50¢, \$1, \$2, \$5, \$10\}$$

- The order is not important.
- The same element needs not appear more than once. (Duplicate elements are redundant and can be removed.)

Elements of a Set

- The object used to form a set are called its element or its members.
- In set theory notation, the elements of a set are written inside a pair of curly braces { } and are represented by commas ",".
  {3, t, 6, m, A}
- The name of the set is always written in capital letter
  A = {3, t, 6, m, A}

- We use a mathematical notation, ∈, to imply an element of .
  3 ∈ A, m ∈ A.

Two Basic Properties od Sets

1. The change in order of writing the elements does not make any changes in the set

   {a, b, c} can also be written as {c, a, b}

2. If one or many elements of a set are repeated, the set remains the same

   Set A = {1, 2, 3, 3, 4, 4, 4} is the same as {1, 2, 2, 3, 4}

The set of letters in the word "GOOGLE" = {G, O, L, E}

Identity

- Two sets are identical iff (if and only if) they have exactly the same members.

- A=B iff for every x, $x \in A \Leftrightarrow x \in B$.

- Example:

  $\{0,2,4\} = \{x \mid x \text{ is an even natural number less than } 5\}$

(Note: ‘iff’ is the abbreviation of ‘if an only if’,

$\in$ is read as is an element or a member of)

Cardinality

- The cardinality of a set A is defined as the number of elements in the set.
- It is denoted by $|A|$.

- Example:

  $C = \{10¢, 20¢, 50¢, \$1, \$2, \$5, \$10\}$

  $|C| = 7$.

Subset and Proper subset

- A is a subset of B, written as $A \subseteq B$, if every member of A is also a member of B.
- A is a proper subset of B, $A \subset B$, B contains some elements that are not in A.
  - i.e., A is not the same as B.
- Example:
  - The set of all women is a proper subset of the set of all human beings.

The Empty Set (Null set)

- A set is empty if it contains no elements at all.
- There is only one empty set.
- We denote it by Ø and is also denoted by { }.
- A set of elements with certain properties turns out to be { }, i.e., the set of all positive integers that are greater than the squares is { }.
- Confusion: the empty set Ø with the set {Ø}.
- {Ø} is a **singleton** set with 1 element, Ø. Consider the empty set { } as an empty folder, whereas {Ø} is a folder with exactly one folder inside, namely, empty folder.

Power set

- Many CS problems involve testing all combinations of elements of a set to see if they satisfy certain property. To consider all such combinations of elements of a set S, we build a new set that has its members all the subsets of S
- The set of all subsets of a set A is called the *power set* of A.
- It is denoted as $P(A)$
- Example: What is the power set {0, 1, 2}?
  - $P(\{0, 1, 2\}) is\ the\ set\ of\ all\ subsets\ of\ \{0, 1, 2\}$.
  - $P(\{0, 1, 2\}) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$

Example and simple proof

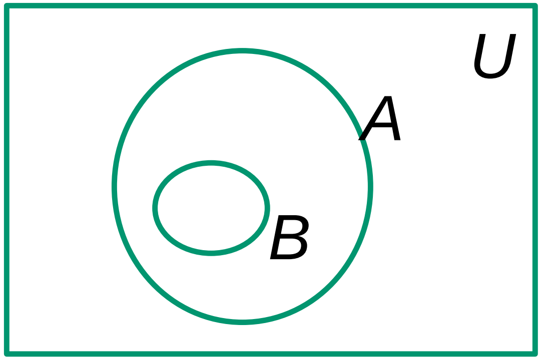
For an arbitrary sets A and C
show that if $A \subseteq C$, then $P(A) \subseteq P(A \cup C)$

If mathematical format
If $X \in P(A)$, *then* $X \in P(A \cup C)$,
$X \in P(A)$ given
$X \subset A$ *defn of power set*
$A \subset A \cup C$ defn of $\cup$
$X \subset A \cup C$
$X \in P(A \cup C)$
Thus $P(A) \subseteq P(A \cup C)$

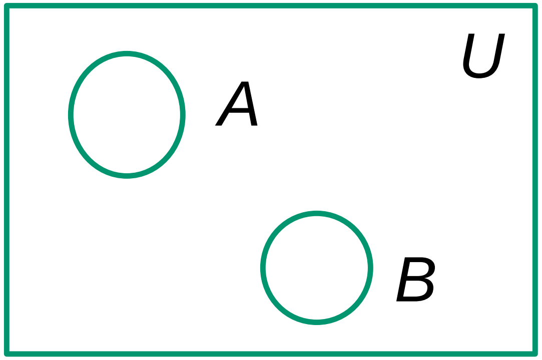
- If $A \subseteq C$, *then* $A \cup C = C$
- *So* $P(A \cup C) = P(C)$
- So we just need to show $P(A) \subseteq P(C)$
- Consider any element from $P(A)$. It must be a subset with elements from A.
- But this element from *P(A)* MUST also be contained in $P(C)$, since this set, $P(C)$, contains all subsets of elements of C.
- One of these subsets will thus contain exactly the desired elements from A, since $A \subseteq C$
- So $P(A) \subseteq P(C)$, Thus $P(A) \subseteq P(A \cup C)$.

Venn Diagram and Relationship between Sets

- A universal set U is a set containing everything that we are considering.
- Venn diagram
  - U is represented by a rectangular box.
  - Subsets of U (e. g. A and B) are represented by circles (more precisely, regions inside closed curves).
- A and B are disjoint if they have no elements in common.



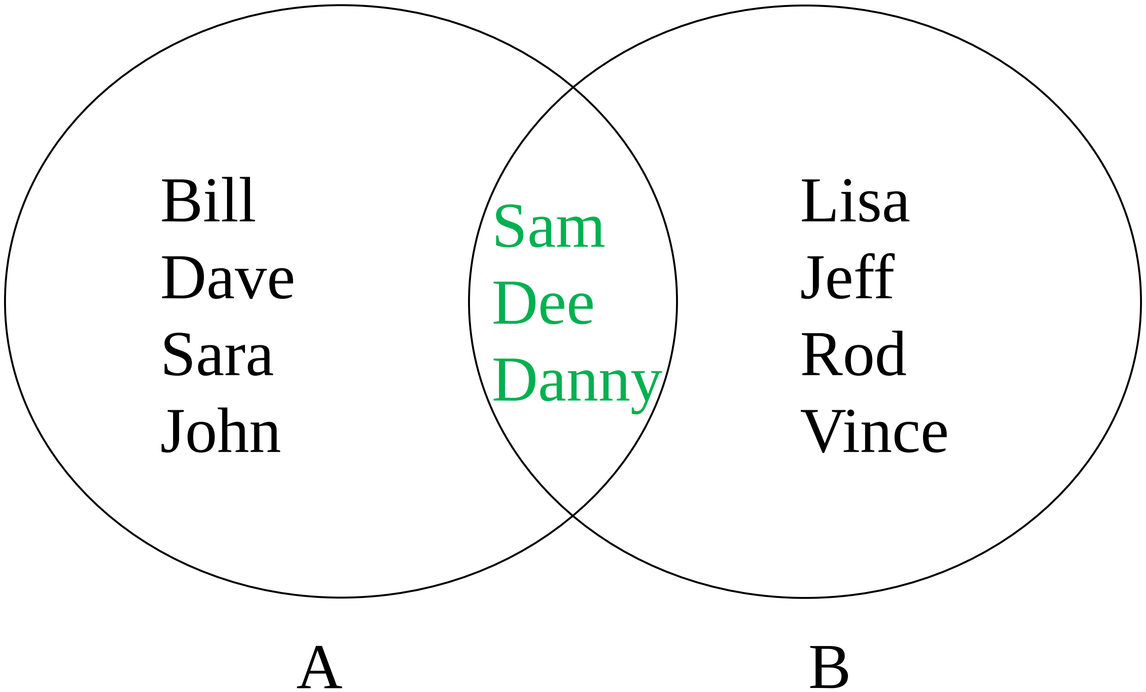
B is a subset of A.



A and B are disjoint.

Venn Diagram Examples

My friends A:{like baseball} B:{like football}

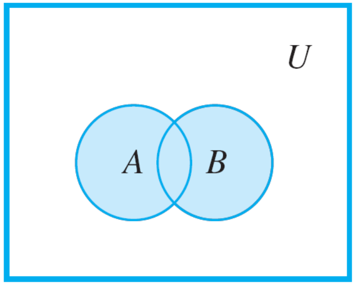


A = {Bill, Dave, Sara, John, Sam, Dee, Danny}

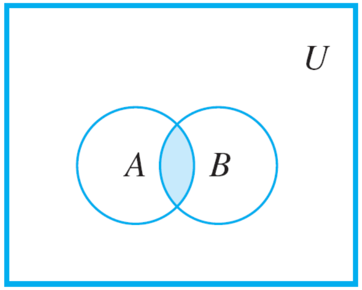
B = {Lisa, Jeff, Rod, Vince, Sam, Dee, Danny}

Four Fundamental Operations

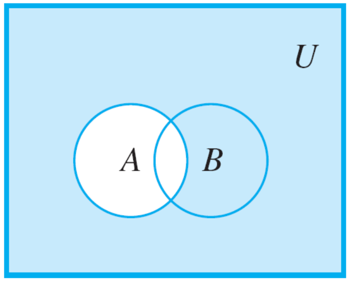
- The **union** of A and B, denoted by $A \cup B$, is the set of all elements that belong to either A or B, or in both.



- The **intersection** of A and B, denoted by $A \cap B$, is the set of all elements that are in both A and B.



- The **complement** of A, denoted by A^c, is the set of all elements in U that do not belong to A.



- Cartesian Product of sets

Cartesian Product of sets

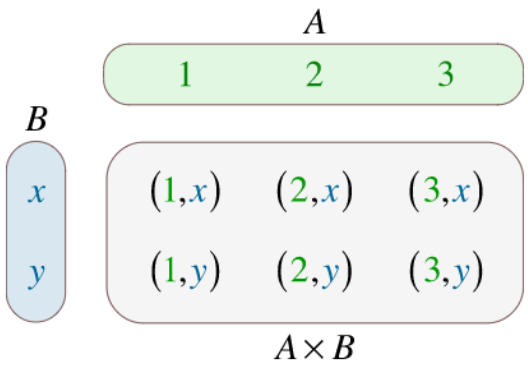
- A Cartesian product is defined as
  - A X B = {(a,b) | a ∈ A and b ∈ B}
  - This is a set of ordered pairs, hence order is essential

A = {a, b, c}, B={b, e}

A X B = {(a, b), (a, e), {b, b), (b, e), (c, b), (c, e)}

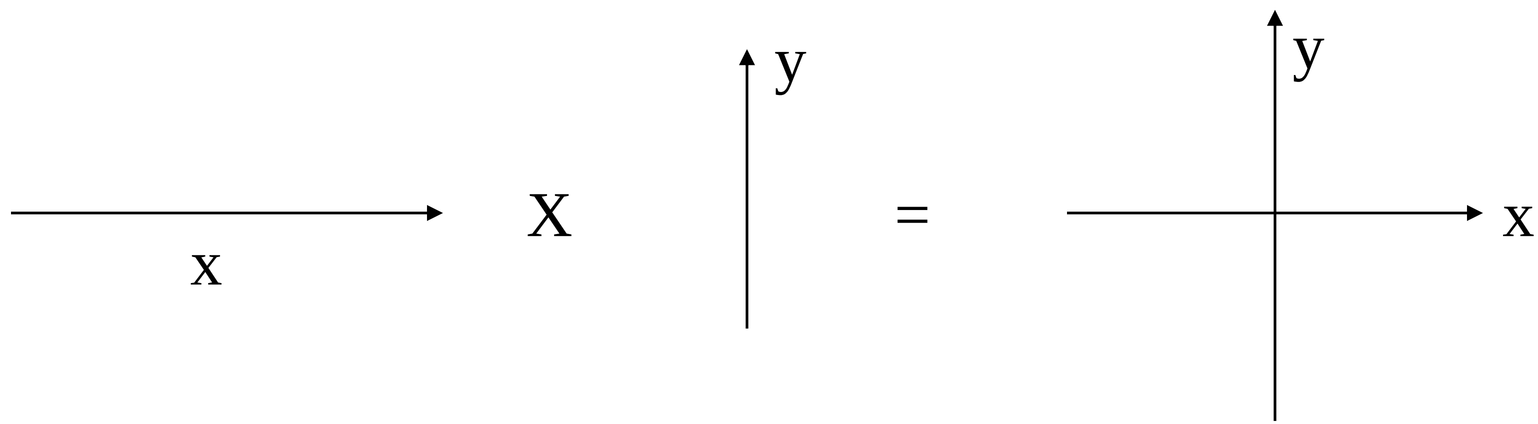
In Cartesian Product or people also call it cross product, the order is important

A X B ≠ B X A



Example of Cartesian Product

X = {x ∈ all x coordinates of any point on the XY-plane}
Y = {y ∈ all y coordinates of any point on the XY-plane}



$$X \text{ X } Y = \{(x, y) |\, x \in X \;\&\; y \in Y\}$$

Set-Builder Notation

- Use a structural mathematical language to describe sets instead of listing all elements of a set
- Most of the time we cannot list the long items list of all elements

"equal"

"such that"

"Set A" → $A = \{\, x \mid \text{conditions} \,\}.$

"the set of all x"

varies with sets

Read as:
A equals the set of all x such that x …

- Example:
  - Let $C = \{10¢, 20¢, 50¢, \$1, \$2, \$5, \$10\}$.
  - A is the set whose elements, x, are elements of C, $x \in C$, such that x is completely bronze in color.
  - $A = \{x \mid x \in C$, is completely bronze in color.$\}$
  - $= \{10¢, 20¢, 50¢\}$

Set-Builder Notation (cont.)

Example1: $A = \{1,2,3,4,\ldots\}$

$$A = \{\, x \mid x \in N \,\}$$

Example 3: $A = \{2, 4, 6,8, \ldots\}$

$A = \{\, 2x \mid x \in N \,\}$

we use lower case x
N is the natural number
Read as: *A* equals the set of
all *x* such that *x* is an element of *N*

Example 2 : $A = \{1, 2, 3, 4,\ldots, 53\}$

and

$$A = \{\, x \mid x \in N \,,\ 1 \leq x \leq 53 \,\}$$

Read as: A equals the set of
all x such that x is an element of N and
x is equal or bigger than 1 and
smaller and equal than 53

Set-Builder Notation (cont.)

Example 1: $B = \{1, 4, 9, 16, \ldots, 100\}$

Class 1 minute exercise

Set-Builder Notation (cont.)

Home exercise

Exercise 1: $A = \{\ldots, -5, -4, -3, -2, -1\}$

Exercise 2: $A = \{1, 2, 3 ,5 ,7, 11, 13\}$

Exercise 3: $A = \{5, 17, 37, 65, 101\}$

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