EE1001 Test 3, 18 November 2020, 9:05am -10:20am, 1hr 15min

<u>TEST 3</u>

1) Find $(x+1)^6 + (x-1)^6$. Hence or otherwise evaluate $((\sqrt{2})+1)^6 + ((\sqrt{2})-1)^6$.

2) In a blockchain system, blocks of memory are chained in an arithmetic progression manner. The sum of the first ten memory blocks is 400 and the sum of the next ten terms is 1000. Find the common difference and the first memory block of the blockchain system.

3) A geometric progression has first term a, common ratio r and sum to infinity 6. A second geometric progression has first term 2a, common ratio r^2 and sum to infinity 7. Find the values of a and r.

4) A series of experiments is performed which involve the use of increasing amounts of a chemical. 6g of the chemical was used in the first experiment and 7.8g of the chemical was used in the second experiment.

4-1. If the chemical used obeys an arithmetic progression, find the total amount of chemical used in the first 30 experiments.

4-2. If, instead of obeying an arithmetic progress, the chemical used indeed obeys an geometric progression. Now a total of 1800g of chemical is available. Show that the maximum number of permissible experiment is N such that $1.3^N \leq 91$.

5) Company A predicts a yearly profit of HKD1.2million in 2020. Company A predicts that the yearly profit will increase by 5% annually. The predicted yearly profit obeys a geometric sequence with r=1.05.

5-1) Show that the predicted profit in 2023 will be HKD1,389,150.

5-2) Find the first year in which the yearly predicted profit exceeds HKD2million.

5-3) Find the total predicted profit for the years 2020 to 2030 inclusive (answer to nearest HKD)

6) Four players are playing bridge (cards). Suppose you are sitting at S (south) and your partner is facing you at N. The opposing players are sitting on either side of you at W (west) and E (east). In how many ways can 5 cards be divided so that 3 are with E and 2 with W?

7) You know that between them the opposing players have four cards in the heart suit. It is important to know the relative likelihood of various distributions of these four cards between W and E. Each of the four cards can be in one of two places.

7-1. How many of these arrangements are involved all four hearts being with W and none with E?

7-2. How many of them involve E having only one heart?

7-3. Bridge players sometimes claim that in these circumstances a '1-3 split' (either way) is more likely than an even split. Is this true? Explain.

7-4. Extend (repeat) this to the case where S knows that W and E have 5 hearts between them.

8) Consider the expansion of $[(2)^{1/4}+1/(3)^{1/4}]^n$. If the ratio of the 5^{th} term from the beginning to the 5^{th} term from the end is $(6)^{1/2}:1$, find n.

9) In the expansion of $(a+b)^n$, if the first three terms of the expansion are 729, 7290 and 30375 respectively, find a, b and n.

10) Consider a row of 8 LED lights, each light has two states: red light and blue light. These LED lights are used to transmit information: when all lights are blue, it represents 00000000; when all lights are red, it represents 11111111.

(1) How many combinations of information can be transmitted by using the 8 lights? (6 points)
(2) Now if only THREE (3) of the LEDs can be used to transmit information, how many combinations of information can be transmitted? (8 points)
(3) If the THREE (3) selected LEDs cannot be adjacent, how many combinations of information can be transmitted? (8 points)

Solution guideline for Test 3

1) Find $(x+1)^6+(x-1)^6$. Hence or otherwise evaluate $((\sqrt{2})+1)^6+((\sqrt{2})-1)^6$. **(8 points)**

Ans Q1)

Using Binomial Theorem, the expressions, $(x+1)^6$ and $(x-1)^6$, can be expanded as:

$$(x+1)^6={}^6C_0x^6+{}^6C_1x^5+{}^6C_2x^4+{}^6C_3x^3+{}^6C_4x^2+{}^6C_5x+{}^6C_6$$

$$(x-1)^6={}^6C_0x^6-{}^6C_1x^5+{}^6C_2x^4-{}^6C_3x^3+{}^6C_4x^2-{}^6C_5x+{}^6C_6$$

Hence,

$$(x+1)^6+(x-1)^6=2[{}^6C_0x^6+{}^6C_2x^4+{}^6C_4x^2+{}^6C_6]$$

$$=2[x^6+15x^4+15x^2+1]$$

By putting $x=\sqrt{2}$, we obtain

$(\sqrt{2}+1)^6+(\sqrt{2}-1)^6=2[(\sqrt{2})^6+15(\sqrt{2})^4+15(\sqrt{2})^2+1]$

$=2(8+15\times4+15\times2+1)$

$=2(8+60+30+1)$

$=2(99)=198$

2) In a blockchain system, blocks of memory are chained in an arithmetic progression manner. The sum of the first ten memory blocks is 400 and the sum of the next ten terms is 1000. Find the common difference and the first memory block of the blockchain system. **(8 points)**

Ans Q2)

$$S_n=\frac{n}{2}\left[2a+(n-1)d\right]$$

Given $S_{10}=400$

$$\therefore \frac{10}{2}\left[2a+9d\right]=400$$

$$\therefore 10a+45d=400 \qquad (1)$$

And $S_{20}=1400$

$$\therefore \frac{10}{2}\left[2a+19d\right]=1400$$

$$\therefore 20a+190d=1400 \qquad (2)$$

Multiply Eqt. (1) by 2: $20a+90d=800$ (3)

(2) – (3): $100d=600$

$$\therefore d=6$$

Sub. $d=6$ into (1)

$$\therefore 10a+45(6)=400$$

$$\therefore 10a+270=400$$

$$\therefore 10a=130$$

$$\therefore a=13$$

$$\therefore a=13,\ d=6$$

3) A geometric progression has first term a, common ratio r and sum to infinity 6. A second geometric progression has first term 2a, common ratio r^2 and sum to infinity 7. Find the values of a and r. **(8 points)**

Ans Q3)

Details:

$$S_\infty = \frac{a}{1-r}, \ -1 < r < 1$$

1st series:

$$\frac{a}{1-r} = 6 \implies a = 6(1-r) \qquad \textbf{(1)}$$

2nd series:

$$\frac{2a}{1-r^2} = 7 \implies 2a = 7(1-r^2) \qquad \textbf{(2)}$$

(2)/(1) given

$$\frac{2a}{a} = \frac{7(1-r^2)}{6(1-r)}$$

Hence,

$$2 = \frac{7(1+r)(1-r)}{6(1-r)} = \frac{7(1+r)}{6}$$

Hence,

$$12 = 7 + 7r$$

$$7r = 5$$

$$r = \frac{5}{7} \qquad \textbf{(3)}$$

Sub (3) in (1)

$$a = 6\left(1 - \frac{5}{7}\right) = \frac{12}{7}$$

As a result,

$a = \dfrac{12}{7}$**, and** $r = \dfrac{5}{7}$

4) A series of experiments is performed which involve the use of increasing amounts of a chemical. 6g of the chemical was used in the first experiment and 7.8g of the chemical was used in the second experiment.

4-1. If the chemical used obeys an arithmetic progression, find the total amount of chemical used in the first 30 experiments. **(8 points)**

4-2. If, instead of obeying an arithmetic progress, the chemical used indeed obeys an geometric progression. Now a total of 1800g of chemical is available. Show that the maximum number of permissible experiment is N such that $1.3^N \leq 91$. **(8 points)**

ANS Q4)

1^{st} experiment: 6g

2^{nd} experiment: 7.8g

Amounts form an arithmetic progression

$$a\ ,\ a+d\ ,\ a+2d\ ,\ a+3d\ ,\ a+4d\ ,\ \dots$$

Where the n^{th} term, $u_n = a+(n-1)d$

$$S_n = \frac{n}{2}\left[2a+\left(n-1\right)d\right]$$

Total amount of chemical for 30 experiments:

$$S_{30} = u_1 + u_2 + \dots + u_{30}$$

$$S_{30} = 6 + 7.8 + \dots + u_{30}$$

$$d = u_2 - u_1 = 1.8$$

$$S_{30} = \frac{30}{2}[2(6)+29(1.8)]$$

$$S_{30} = 963$$

Ans Q4-2)

4-2. If, instead of obeying an arithmetic progress, the chemical used indeed obeys an geometric progression. Now a total of 1800g of chemical is available. Show that the maximum number of permissible experiment is N such that $1.3^N \leq 91$.

1^{st} experiment: 6g

2^{nd} experiment: 7.8g

Amounts form a geometric progression,

$$a\,,\; ar\,,\; ar^2\,,\; ar^3\,,\,\ldots$$

$$S_N = \frac{a(r^N - 1)}{r-1}$$

Given total amount of chemical = 1800g

Therefore, for N experiments:

$$S_N \le 1800$$

But

$$S_N = 6 + 7.8 + \ldots$$
$$\therefore a = 6$$
$$\therefore r = \frac{7.8}{6} = 1.3$$

$$\therefore \frac{6(1.3^N - 1)}{1.3 - 1} \le 1800$$

$$\therefore 20(1.3^N - 1) \le 1800$$
$$\therefore (1.3^N - 1) \le 90$$
$$\therefore 1.3^N \le 91$$
$$\therefore \log 1.3^N \le \log 91$$
$$\therefore N \log 1.3 \le \log 91$$
$$\therefore N \le \frac{\log 91}{\log 1.3}$$
$$\therefore N \le 17.193$$
$$\therefore \max N = 17$$

Q5) Company A predicts a yearly profit of HKD1.2million in 2020. Company A predicts that the yearly profit will increase by 5% annually. The predicted yearly profit obeys a geometric sequence with r=1.05.

5-1) Show that the predicted profit in 2023 will be HKD1,389,150. (2 points)

Ans 5-1)

Details:
Predicted Profit = 1200000*(1.05)^3 =1,389,150

5-2) Find the first year in which the yearly predicted profit exceeds HKD2million. (6 points)

2020+11=2031

Ans 5-2)

Details:

$$u_n > 2000000$$

$$1200000(1.05)^{n-1} > 2000000$$

Hence,

$$(1.05)^{n-1} > \frac{5}{3}$$

$$\log(1.05)^{n-1} > \log\left(\frac{5}{3}\right)$$

$$(n-1)\log 1.05 > \log\left(\frac{5}{3}\right)$$

$$(n-1) > \frac{\log\left(\frac{5}{3}\right)}{\log 1.05}$$

$$n > \frac{\log\left(\frac{5}{3}\right)}{\log 1.05} + 1$$

$$n > 11.46...$$

At least, **n = 12**

Hence, **Year = 2020 + 11 = 2031**

5-3) Find the total predicted profit for the years 2020 to 2030 inclusive (answer to nearest HKD) (4 points)

Ans 5-3)

Details:

2013, 2014, … 2023

$$\begin{aligned} Total\ Profit &= S_{11} \\ &= \frac{1200000((1.05)^{11}-1)}{1.05-1} \\ &= £17048144.59 \\ &= £17048145\,(to\ the\ nearest\ £) \end{aligned}$$

Total Profit = S_{11} = [1200000 ((1.05)^{11}-1)]/(1.05-1) =17048145 HKD

6) Four players are playing bridge (cards).

6A) Suppose you are sitting at S (south) and your partner is facing you at N. The opposing players are sitting on either side of you at W (west) and E (east). In how many ways can 5 cards be divided so that 3 are with E and 2 with W? (3 points)

Ans 6A)

Imagine all five cards in a row. Each can be marked either E or W. There must be 3 Es and 2 Ws in any order. The answer to the question is, therefore, the number of different combinations possible with 3 Es and 2 Ws.

EEEWW, EWWEE, EEWEW, WEEEW, EEWWE, WEEWE, EWEWE, WEWEE, EWEEW, WWEEE

As the list opposite shows, that number is 10.

6B) You know that between them the opposing players have four cards in the heart suit. It is important to know the relative likelihood of various distributions of these four cards between W and E. Each of the four cards can be in one of two places.

Four hearts:

6B-1. How many of these arrangements are involved all four hearts being with W and none with E?

(2 points)

6B-2. How many of them involve E having only one heart? (3 points)

6B-3. Bridge players believe that a '1-3 split' (either way) is more likely than a '2-2 split'. Justify whether this is a correct statement. (3 points)

Ans 6B)

(6B-1) 1 (no hearts)

(6B-2) 4 (could be any of the 4 hearts)

(6B-3) This is true as a 1-3 split or a 3-1 split has probability of 8 out of 16 while a 2-2 split has probability 6 out of 16 .

6C) Now, between the players, the opposing players have five cards in the heart suit.

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Five hearts:

Replace four hearts by five hearts in 6B, and repeat 6B-1 to 6B-3. (8 points)

Ans 6C)

(6C-1) 1 (no hearts)

(6C-2) Number of situations that E has only one heart: 5

(6C-3) The statement of "1-4 split" is more likely than "2-3 split" is FALSE.

Probability of 1-4 split or 4-1 split = 10/32

Probability of 2-3 split or 3-2 split = 20/32

7) Consider the expansion of $[(2)^{1/4} + 1/(3)^{1/4}]^n$. If the ratio of the 5^{th} term from the beginning to the 5^{th} term from the end is $(6)^{1/2}:1$, find n. (10 marks)

Ans Q7)

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In the expansion, $(a+b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 + \dots + {}^nC_{n-1} a b^{n-1} + {}^nC_n b^n$,

Fifth term from the beginning $= {}^nC_4 a^{n-4} b^4$

Fifth term from the end $= {}^nC_{n-4} a^4 b^{n-4}$

Therefore, it is evident that in the expansion of $\left(\sqrt[4]{2} + \frac{1}{\sqrt[4]{3}}\right)^n$, the fifth term from the beginning is ${}^nC_4 \left(\sqrt[4]{2}\right)^{n-4} \left(\frac{1}{\sqrt[4]{3}}\right)^4$ and the fifth term from the end is ${}^nC_{n-4} \left(\sqrt[4]{2}\right)^4 \left(\frac{1}{\sqrt[4]{3}}\right)^{n-4}$

.

$$ {}^nC_4 \left(\sqrt[4]{2}\right)^{n-4} \left(\frac{1}{\sqrt[4]{3}}\right)^4 = {}^nC_4 \frac{\left(\sqrt[4]{2}\right)^n}{\left(\sqrt[4]{2}\right)^4} \cdot \frac{1}{3} = {}^nC_4 \frac{\left(\sqrt[4]{2}\right)^n}{2} \cdot \frac{1}{3} = \frac{n!}{6.4!(n-4)!} \left(\sqrt[4]{2}\right)^n \qquad \dots(1) $$

$$ {}^nC_{n-4} \left(\sqrt[4]{2}\right)^4 \left(\frac{1}{\sqrt[4]{3}}\right)^{n-4} = {}^nC_{n-4} . 2 \cdot \frac{\left(\sqrt[4]{3}\right)^4}{\left(\sqrt[4]{3}\right)^n} = {}^nC_{n-4} . 2 \cdot \frac{3}{\left(\sqrt[4]{3}\right)^n} = \frac{6n!}{(n-4)!4!} \cdot \frac{1}{\left(\sqrt[4]{3}\right)^n} \qquad \dots(2) $$

It is given that the ratio of the fifth term from the beginning to the fifth term from the end is $\sqrt{6}:1$. Therefore, from (1) and (2), we obtain

$$\frac{n!}{6.4!(n-4)!}\left(\sqrt[4]{2}\right)^n : \frac{6n!}{(n-4)!4!}\cdot\frac{1}{\left(\sqrt[4]{3}\right)^n} = \sqrt{6}:1$$

$$\Rightarrow \frac{\left(\sqrt[4]{2}\right)^n}{6} : \frac{6}{\left(\sqrt[4]{3}\right)^n} = \sqrt{6}:1$$

$$\Rightarrow \frac{\left(\sqrt[4]{2}\right)^n}{6} \times \frac{\left(\sqrt[4]{3}\right)^n}{6} = \sqrt{6}$$

$$\Rightarrow \left(\sqrt[4]{6}\right)^n = 36\sqrt{6}$$

$$\Rightarrow 6^{\frac{n}{4}} = 6^{\frac{5}{2}}$$

$$\Rightarrow \frac{n}{4} = \frac{5}{2}$$

$$\Rightarrow n = 4\times\frac{5}{2} = 10$$

Thus, the value of n is 10.

8) In the expansion of $(a+b)^n$, if the first three terms of the expansion are 729, 7290 and 30375 respectively, find a, b and n. (10 points)

Ans Q8)

It is known that $(r + 1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a + b)^n$ is given by $T_{r+1} = {}^nC_r a^{n-r} b^r$.

The first three terms of the expansion are given as 729, 7290, and 30375 respectively.

Therefore, we obtain

$$T_1 = {}^nC_0 a^{n-0} b^0 = a^n = 729 \qquad \ldots(1)$$

$$T_2 = {}^nC_1 a^{n-1} b^1 = na^{n-1}b = 7290 \qquad \ldots(2)$$

$$T_3 = {}^nC_2 a^{n-2} b^2 = \frac{n(n-1)}{2} a^{n-2} b^2 = 30375 \qquad \ldots(3)$$

Dividing (2) by (1), we obtain

$$\frac{na^{n-1}b}{a^n} = \frac{7290}{729}$$

$$\Rightarrow \frac{nb}{a} = 10 \qquad \ldots(4)$$

Dividing (3) by (2), we obtain

$$\frac{n(n-1)a^{n-2}b^2}{2na^{n-1}b} = \frac{30375}{7290}$$

$$\Rightarrow \frac{(n-1)b}{2a} = \frac{30375}{7290}$$

$$\Rightarrow \frac{(n-1)b}{a} = \frac{30375 \times 2}{7290} = \frac{25}{3}$$

$$\Rightarrow \frac{nb}{a} - \frac{b}{a} = \frac{25}{3}$$

$$\Rightarrow 10 - \frac{b}{a} = \frac{25}{3} \qquad [\text{Using } (4)]$$

$$\Rightarrow \frac{b}{a} = 10 - \frac{25}{3} = \frac{5}{3} \qquad ...(5)$$

From (4) and (5), we obtain

$$n \cdot \frac{5}{3} = 10$$

$$\Rightarrow n = 6$$

Substituting $n = 6$ in equation (1), we obtain

$a^6 = 729$

$$\Rightarrow a = \sqrt[6]{729} = 3$$

From (5), we obtain

$$\frac{b}{3} = \frac{5}{3} \Rightarrow b = 5$$

Thus, $a = 3$, $b = 5$, and $n = 6$.

Q9) Consider a row of 8 LED lights, each light has two states: red light and blue light. These LED lights are used to transmit information: when all lights are blue, it represents 00000000; when all lights are red, it represents 11111111.

(4) How many combinations of information can be transmitted by using the 8 lights? (3 points)
(5) Now if only THREE (3) of the LEDs can be used to transmit information, how many combinations of information can be transmitted? (3 points)
(6) If the THREE (3) selected LEDs cannot be adjacent, how many combinations of information can be transmitted? (3 points)

Ans Q9)

(1) $2^8 = 256$
(2) ${}_8C_3 * 2^3 = 56*8 = 448$
(3) Since the selected light cannot be adjacent. It can be solved by inserting 3 selected lights into the intervals among the 5 unselected lights. Since there are 6 intervals, the number of types of

information is computed as： ${}_6C_3 * 2^3 = 160$