

Tutorial 2

Question 1

Periodic energy usage. The 168-vector \mathbf{w} gives the hourly electricity consumption of a manufacturing plant, starting on Sunday midnight to 1AM, over one week, in MWh (megawatt-hours). The consumption pattern is the same each day, i.e., it is 24-periodic, which means that $w_{t+24} = w_t$ for $t = 1, \dots, 144$. Let \mathbf{d} be the 24-vector that gives the energy consumption over one day, starting at midnight.

- (a) Use vector notation to express \mathbf{w} in terms of \mathbf{d} .
- (b) Use vector notation to express \mathbf{d} in terms of \mathbf{w} .

Question 2

Suppose the n -vector \mathbf{x} is sparse, i.e., has only a few nonzero entries. Give a short sentence or two explaining what this means in each of the following contexts.

- (a) \mathbf{x} represents the daily cash flow of some business over n days.
- (b) \mathbf{x} represents the annual dollar value purchases by a customer of n products or services.
- (c) \mathbf{x} represents a portfolio, say, the dollar value holdings of n stocks.
- (d) \mathbf{x} represents a bill of materials for a project, i.e., the amounts of n materials needed.
- (e) \mathbf{x} represents a monochrome image, i.e., the brightness values of n pixels.
- (f) \mathbf{x} is the daily rainfall in a location over one year.

Question 3

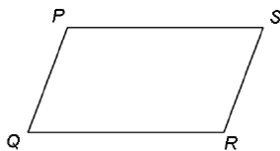
In a computer game, let $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ be the initial coordinates of a jet fighter and $\mathbf{b} = \begin{pmatrix} 4 \\ 3 \\ 6 \end{pmatrix}$ be its final coordinates. What is the displacement vector?

Question 4

Suppose a car is at the position $\mathbf{a} = \begin{pmatrix} 6 \\ 7 \end{pmatrix}$. The car moves along the direction $\mathbf{d} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ per hour. After 3 hours, what is the final position of the car?

Question 5

In the figure, $PQRS$ is a parallelogram and the coordinates of the four points are denoted as \mathbf{P} , \mathbf{Q} , \mathbf{R} , and \mathbf{S} . Denote $\mathbf{d}_{P1,P2}$ be the displacement from point P1 to point P2.



Suppose $\mathbf{d}_{PQ} = \mathbf{u}$ and $\mathbf{d}_{PS} = \mathbf{v}$. Express each of the following vectors in terms of \mathbf{u} and \mathbf{v} .

- (a) \mathbf{d}_{RQ} , (b) \mathbf{d}_{RS} , (c) \mathbf{d}_{PR} , (d) \mathbf{d}_{QS}

Question 6

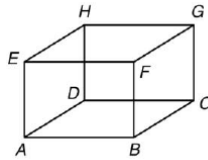
Let $\mathbf{a} = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix}$ be the coordinates of two points. Let \mathbf{p} be a point on line segment between \mathbf{a} and \mathbf{b} . Find \mathbf{p} in each of the following cases.

(a) \mathbf{p} be a point middle point between \mathbf{a} and \mathbf{b} .

(b) $\|\mathbf{p} - \mathbf{a}\| : \|\mathbf{b} - \mathbf{p}\| = 2 : 3$

Question 7

In the figure, $ABCDHEFG$ is a right prism with parallelogram $ABCD$ as its base. \mathbf{d}_{P_1, P_2} be the displacement from point P_1 to point P_2 .



(a) Show that $\mathbf{d}_{BA} + \mathbf{d}_{BC} + \mathbf{d}_{BF} = \mathbf{d}_{BH}$.

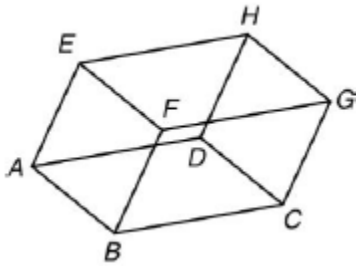
(b) The coordinates of the vertices are $A=(4, 0, 3)$, $B=(5, 3, 4)$, $C=(1, 4, 5)$ and $F=(4, 2, 8)$.

(i) Find the coordinates of H .

(ii) Find the lengths of BH and OH , where O is the origin.

Question 8

In the figure, $ABCDHEFG$ is a prism with parallelogram $ABCD$ as its base.



(a) Show that $\mathbf{d}_{CB} + \mathbf{d}_{CD} + \mathbf{d}_{CG} = \mathbf{d}_{CE}$

(b) It is given that the coordinates of the vertices are $B=(4, 6, 0)$, $C=(7, 12, 1)$, $D=(5, 8, 3)$ and $G=(11, 12, 0)$.

(i) Find \mathbf{CE} .

(ii) Find the coordinates of E and a unit vector in the same direction of \mathbf{OE} .

Question 9

Let \mathbf{a} and \mathbf{b} be two non-parallel vectors in 3D space. Let $\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b}$ for some real constants α, β such that $\mathbf{c}^T \mathbf{a} = 0$ and $\mathbf{c}^T \mathbf{b} = 1$.

(a) Find α and β in terms of $\mathbf{a}^T \mathbf{a}$, $\mathbf{a}^T \mathbf{b}$, $\mathbf{a}^T \mathbf{a}$

(b) For any 3D vector \mathbf{x} such that $\mathbf{x}^T \mathbf{a} = 0$ and $\mathbf{x}^T \mathbf{b} = 1$, prove that $\mathbf{x} - \mathbf{c}$ is perpendicular to \mathbf{a} and \mathbf{b} .

Question 10

Calculate the norms of the following vectors:

(a) $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, (b) $\begin{pmatrix} -1 \\ 3 \\ 6 \end{pmatrix}$, (c) $\begin{pmatrix} 1 \\ 1 \\ 2 \\ 8 \end{pmatrix}$,

Question 11

Find all unit vectors that are parallel to the vector $\begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$

Question 12

Compute the inner product each of the following pairs of vectors:

(a) $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (b) $\begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} b \\ a \end{pmatrix}$, (c) $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix}$.

Question 13

Compute the inner product each of the following pairs of vectors:

(a) $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ 1 \\ 2 \end{pmatrix}$, (c) $\begin{pmatrix} -2 \\ 3 \\ 0 \\ 1 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ 2 \\ 8 \\ -2 \end{pmatrix}$

Question 14

Let \mathbf{a} , \mathbf{b} and \mathbf{c} be three **unit vectors** in 3D space. The angles between \mathbf{b} and \mathbf{c} , \mathbf{c} and \mathbf{a} , \mathbf{a} and \mathbf{b} are α, β, γ respectively. Let \mathbf{u} and \mathbf{v} be two vectors defined by $\mathbf{u} = (\mathbf{a}^T \mathbf{b})\mathbf{c} - (\mathbf{c}^T \mathbf{a})\mathbf{b}$ and $\mathbf{v} = \mathbf{b} - (\mathbf{a}^T \mathbf{b})\mathbf{a}$

(i) Determine, in terms of α, β, γ , the magnitudes of \mathbf{u} and \mathbf{v} .

(ii) Prove that both \mathbf{u} and \mathbf{v} are perpendicular to \mathbf{a} .

(iii) Show that, if \mathbf{u} and \mathbf{v} are perpendicular to each other, then $\cos \beta = \cos \alpha \cos \gamma$

Question 15

Given 3 points, A, B, C , such that $A = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, B = \begin{pmatrix} -6 \\ -2 \end{pmatrix}, \mathbf{d}_{AB} - \mathbf{d}_{BC} = \begin{pmatrix} -12 \\ 6 \end{pmatrix}$

- (a) Find \mathbf{d}_{AB} and C .
- (b) If X is a point such that $\mathbf{d}_{AX} = kX$
 - (i) What is X (in terms of k)
 - (ii) \mathbf{d}_{OX} is $\perp \mathbf{d}_{BX}$, find the value of k and hence find $\mathbf{d}_{AX} + \mathbf{d}_{BX} + \mathbf{d}_{CX}$
 - (iii) Furthermore, if M is the mid-point of BC , find \mathbf{d}_{BC} .

Question 16

Distance between Boolean vectors. Suppose that \mathbf{x} and \mathbf{y} are Boolean n -vectors, which means that each of their entries is either 0 or 1. What is their distance $\|\mathbf{x} - \mathbf{y}\|$?

Question 17

Norm identities. Verify that the following identities hold for any two vectors \mathbf{a} and \mathbf{b} of the same size.

- (a) $(\mathbf{a} - \mathbf{b})^T(\mathbf{a} + \mathbf{b}) = \|\mathbf{a}\|^2 - \|\mathbf{b}\|^2$
- (b) $\|\mathbf{a} + \mathbf{b}\|^2 + \|\mathbf{a} - \mathbf{b}\|^2 = 2\|\mathbf{a}\|^2 + 2\|\mathbf{b}\|^2$

This is called the parallelogram law.

Question 18

Show that

- (a) $\mathbf{a} \perp \mathbf{b}$ if and only $\|\mathbf{a} + \mathbf{b}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2$
- (b) Nonzero vectors \mathbf{a} and \mathbf{b} make an acute angle if and only $\|\mathbf{a} + \mathbf{b}\| > \sqrt{\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2}$
- (c) Nonzero vectors \mathbf{a} and \mathbf{b} make an obtuse angle if and only if $\|\mathbf{a} + \mathbf{b}\| < \sqrt{\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2}$

Question 19

A Boolean n -vector \mathbf{a} is one for which all entries are either 0 or 1. Such vectors are used to encode whether each of n conditions holds, with $a_i = 1$ meaning that condition i holds. Another common encoding of the same information uses the two values -1 and +1 for the entries. For example the Boolean vector (0, 1, 1, 0) would be written using this alternative encoding as (-1, +1, +1, -1). Suppose that \mathbf{x} is a Boolean vector with entries that are 0 or 1, and \mathbf{y} is a vector encoding the same information using the values -1 and +1. Express \mathbf{y} in terms of \mathbf{x} using vector notation. Also, express \mathbf{x} in terms of \mathbf{y} using vector notation.

Question 20

Profit and sales vectors. A company sells n different products or items. The n -vector \mathbf{p} gives the profit, in dollars per unit, for each of the n items. The n -vector \mathbf{s} gives the total sales of each of the items, over some period (such as a month), i.e., s_i is the total number of units of item i sold. What is the total profit in terms of \mathbf{p} and \mathbf{s} using vector notation.

Question 21

Symptoms vector. A 20-vector \mathbf{s} records whether each of 20 different symptoms is present in a medical patient, with $s_i = 1$ meaning the patient has the symptom i and $s_i = 0$ meaning she does not. Express the following using vector notation. The total number of symptoms the patient has.

Question 22

Total score from course record. The record for each student in a class is given as a 10-vector \mathbf{r} , where r_1, \dots, r_8 are the grades for the 8 homework assignments, each on a 0-10 scale, r_9 is the midterm exam grade on a 0-120 scale, and r_{10} is final exam score on a 0-160 scale. The student's total course score s , on a 0-100 scale, is based 25% on the homework, 35% on the midterm exam, and 40% on the final exam. Express s in the form $s = \mathbf{w}^T \mathbf{r}$. (That is, determine the 10-vector \mathbf{w} .)

Question 23

Word count and word count histogram vectors. Suppose the n -vector \mathbf{w} is the word count vector associated with a document and a dictionary of n words. For simplicity we will assume that all words in the document appear in the dictionary.

(a) What is $\mathbf{1}^T \mathbf{w}$?

(b) What does $w_{282} = 0$ mean?

(c) Let \mathbf{h} be the n -vector that gives the histogram of the word counts, i.e., h_i is the fraction of the words in the document that are word i . Use vector notation to express \mathbf{h} in terms of \mathbf{w} . (You can assume that the document contains at least one word.)

Question 24

Total cash value. An international company holds cash in five currencies: USD (US dollar), RMB (Chinese yuan), EUR (euro), GBP (British pound), and JPY (Japanese yen), in amounts given by the 5-vector \mathbf{c} . For example, c_2 gives the number of RMB held. Express the total (net) value of the cash in USD, using vector notation. Be sure to give the size and define the entries of any vectors that you introduce in your solution. Your solution can refer to currency exchange rates.

Question 25

Linear combinations of linear combinations. Suppose that each of the vectors $\mathbf{b}_1, \dots, \mathbf{b}_k$ is a linear combination of the vectors $\mathbf{a}_1, \dots, \mathbf{a}_m$, and \mathbf{c} is a linear combination of $\mathbf{b}_1, \dots, \mathbf{b}_k$. Then \mathbf{c} is a linear combination of $\mathbf{a}_1, \dots, \mathbf{a}_m$. Show this for the case with $k = m = 2$.

Question 26

Nearest neighbor document. Consider the 5 Wikipedia pages.

What is the nearest neighbor of (the word count histogram vector of) 'Veterans Day' among the others?

	Veterans Day	Memorial Day	Academy Awards	Golden Globe Awards	Super Bowl
Veterans Day	0	0.095	0.130	0.153	0.170
Memorial Day	0.095	0	0.122	0.147	0.164
Academy A.	0.130	0.122	0	0.108	0.164
Golden Globe A.	0.153	0.147	0.108	0	0.181
Super Bowl	0.170	0.164	0.164	0.181	0

Question 27

Neighboring electronic health records. Let $\mathbf{x}_1, \dots, \mathbf{x}_N$ be n -vectors that contain n features extracted from a set of N electronic health records (EHRs), for a population of N patients. (The features might involve patient attributes and current and past symptoms, diagnoses, test results, hospitalizations, procedures, and medications.) Briefly describe in words a practical use for identifying the 10 nearest neighbors of a given EHR (as measured by their associated feature vectors), among the other EHRs.

Question 28

Angle between two nonnegative vectors. Let \mathbf{x} and \mathbf{y} be two nonzero n -vectors with nonnegative entries. Show that the angle between \mathbf{x} and \mathbf{y} lies between 0 and $\frac{\pi}{2}$. When are \mathbf{x} and \mathbf{y} orthogonal?

Question 29

Distance versus angle nearest neighbor. Suppose $\mathbf{a}_1, \dots, \mathbf{a}_k$ is a collection of n -vectors, and \mathbf{b} is another n -vector.

The vector \mathbf{a}_j is the (distance) nearest neighbor of \mathbf{b} (among the given vectors), if $\|\mathbf{b} - \mathbf{a}_j\| \leq \|\mathbf{b} - \mathbf{a}_{j'}\|$ for all j .

The vector \mathbf{a}_j is the (angle) nearest neighbor of \mathbf{b} (among the given vectors), if $\angle(\mathbf{a}_j, \mathbf{b}) \leq \angle(\mathbf{a}_{j'}, \mathbf{b})$ for all j .

Question 30

Let $\mathbf{a} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

(a) Find all vectors $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t_1 \mathbf{a} + t_2 \mathbf{b}$

(b) Is the vector $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ one of the vectors you find in (a).

Question 31

Linear independence of stacked vectors. Consider the stacked vectors

$$\mathbf{c}_1 = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{b}_1 \end{pmatrix}, \dots, \mathbf{c}_k = \begin{pmatrix} \mathbf{a}_k \\ \mathbf{b}_k \end{pmatrix}$$

where $\mathbf{a}_1, \dots, \mathbf{a}_k$ are n-vectors, and where $\mathbf{b}_1, \dots, \mathbf{b}_k$ are m-vectors

(a) Suppose $\mathbf{a}_1, \dots, \mathbf{a}_k$ are linearly independent. Can we conclude that the stacked vectors $\mathbf{c}_1, \dots, \mathbf{c}_k$ are linearly independent?

(b) Now suppose that $\mathbf{a}_1, \dots, \mathbf{a}_k$ are linearly dependent. Can we conclude that the stacked vectors $\mathbf{c}_1, \dots, \mathbf{c}_k$ are linearly dependent?

Question 32

Norm of linear combination of orthonormal vectors. Suppose $\mathbf{a}_1, \dots, \mathbf{a}_k$ are orthonormal n-vectors, and $\mathbf{x} = \beta_1 \mathbf{a}_1 + \dots + \beta_k \mathbf{a}_k$, where β_1, \dots, β_k are scalars. Express $\|\mathbf{x}\|$ in terms of β_1, \dots, β_k .

Question 33

Orthogonalizing vectors. Suppose that \mathbf{a} and \mathbf{b} are any n-vectors. Show that we can always find a scalar β that $(\mathbf{a} - \beta \mathbf{b}) \perp \mathbf{b}$, and that β is unique if $\mathbf{b} \neq \mathbf{0}$.

Question 34

Show that the following vectors are linear independent?

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

Question 35

Which of the following set of vectors are orthonormal ?

$$(a) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$(b) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

$$(c) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$(d) \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$$

Question 36

Give three vectors: $\mathbf{x}_1 = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}, \mathbf{x}_3 = \begin{pmatrix} -9 \\ 4 \\ 6 \end{pmatrix},$

(a) show that they are orthogonal

(b) Let $\mathbf{v} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$. Compute the projection of \mathbf{v} onto the directions $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$, respectively.

(c) Is it \mathbf{v} equal to the sum of the three projected vectors?