Exam 1617 A

1\. (a)

$$\int_{-1}^{2} \sqrt{3x+5}\, dx = \int_{-1}^{2} (3x+5)^{\frac{1}{2}}\, dx$$

$$= \left[\frac{(3x+5)^{\frac{3}{2}}}{3 \cdot \frac{3}{2}} \right]_{-1}^{2}$$

$$= \frac{2}{9}\left(11^{\frac{3}{2}} - 2^{\frac{3}{2}} \right)$$

(b)

$$\int \frac{4x-1}{(2x+3)^2}\, dx = \int \frac{2(2x+3)-7}{(2x+3)^2}\, dx$$

$$= \int \frac{2}{2x+3}\, dx - 7\int (2x+3)^{-2} dx$$

$$= \frac{2}{2} \ln|2x+3| - \frac{7}{2} \frac{(2x+3)^{-1}}{-1} + c$$

$$= \ln|2x+3| - \frac{7}{2(2x+3)} + c$$

(c)

$$\int \frac{3x-2}{x^2+4x+13}\, dx$$

$$= \int \frac{\frac{3}{2}(2x+4) - 8}{x^2+4x+13}\, dx$$

$$= \frac{3}{2} \int \frac{2x+4}{x^2+4x+13}\, dx - 8 \int \frac{1}{(x+2)^2+9}\, dx$$

$$= \frac{3}{2} \ln|x^2+4x+13| - \frac{8}{9} \int \frac{1}{\left(\frac{x+2}{3}\right)^2+1}\, dx$$

$$\int \frac{f'(x)}{f(x)}\, dx = \ln|f(x)| + c$$

$$= \frac{3}{2} \ln|x^2+4x+13| - \frac{8}{9} \cdot 3 \tan^{-1}\left(\frac{x+2}{3}\right) + c$$

$$= \frac{3}{2} \ln|x^2+4x+13| - \frac{8}{3} \tan^{-1}\left(\frac{x+2}{3}\right) + c$$

2(a) $\displaystyle\int \frac{1}{(x^2+4)^2}\,dx$ Let $x = 2\tan\theta$

$\Rightarrow dx = 2\sec^2\theta \, d\theta$

Similar to test 1 Q 2(a)

(b) $\displaystyle\int \underbrace{\sin^{-1}x}_{u}\, \underbrace{dx}_{dv}$ $v = \int dx = x$

$$= x\sin^{-1}x - \int x \, d(\sin^{-1}x)$$

$$= x\sin^{-1}x - \int x \cdot \frac{1}{\sqrt{1-x^2}}\,dx$$

$\frac{d}{dx}(1-x^2) = -2x$

$\therefore -2x\,dx = d(1-x^2)$

$$= x\sin^{-1}x - \left(-\tfrac{1}{2}\right)\int \frac{-2x\,dx}{\sqrt{1-x^2}}$$

$$= x\sin^{-1}x + \tfrac{1}{2}\int (1-x^2)^{-\frac{1}{2}}\,\boxed{d(1-x^2)}$$

$$= x\sin^{-1}x + \frac{1}{2}\cdot\frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$\int u^{-\frac{1}{2}}du = \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C$ where $u = 1-x^2$

$$= x\sin^{-1}x + (1-x^2)^{\frac{1}{2}} + C$$

(c) $\displaystyle\int \frac{-13x-14}{(x+2)^3(x-1)}\,dx$

$$\frac{-13x-14}{(x+2)^3(x-1)} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)^3} + \frac{D}{x-1}$$

$$\therefore -13x-14 = A(x+2)^2(x-1) + B(x+2)(x-1) + C(x-1) + D(x+2)^3$$

Put $x=1$: $-27 = 27D \Rightarrow D = -1$

Put $x=-2$: $12 = -3C \Rightarrow C = -4$

Compare coeff. of x^3: $0 = A + D \Rightarrow A = 1$

Compare constant term: $-14 = -4A - 2B - C + 8D$

$\Rightarrow B = 3$

$$\therefore \int \frac{-13x-14}{(x+2)^3(x-1)}\,dx$$

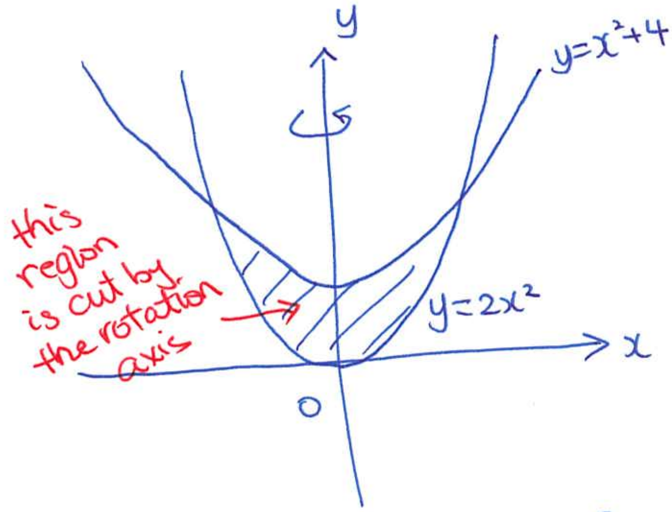
$$= \int \frac{1}{x+2}\,dx + \int \frac{3}{(x+2)^2}\,dx - \int \frac{4}{(x+2)^3}\,dx - \int \frac{1}{x-1}\,dx$$

$$= \ln|x+2| + 3\int (x+2)^{-2}dx - 4\int (x+2)^{-3} - \ln|x-1|$$

$$= \ln\left|\frac{x+2}{x-1}\right| + \frac{3(x+2)^{-1}}{-1} - 4\,\frac{(x+2)^{-2}}{-2} + C$$

$$= \ln\left|\frac{x+2}{x-1}\right| - \frac{3}{x+2} + \frac{2}{(x+2)^2} + C$$

3(a) $\begin{cases} y = 2x^2 \\ y = x^2+4 \end{cases}$



$\Rightarrow x^2+4 = 2x^2$

$\Rightarrow x^2 = 4$

$\Rightarrow x = \pm 2$

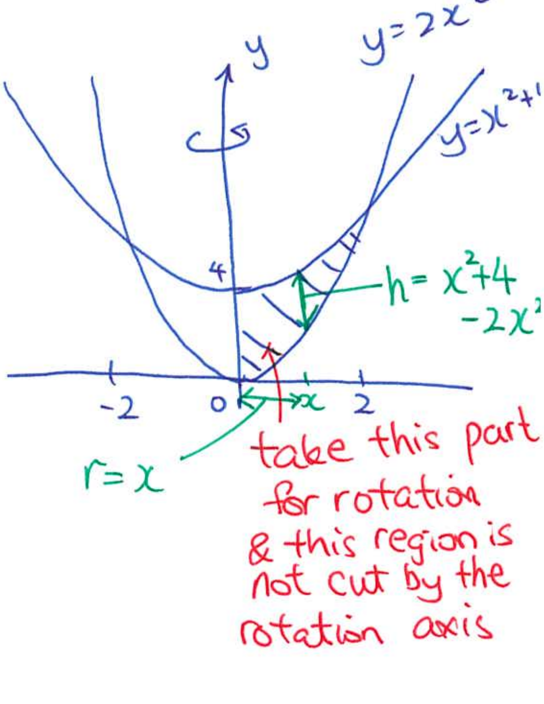
Shell method:

$$V = 2\pi \int_0^2 rh\,dx$$

$$= 2\pi \int_0^2 x(x^2+4-2x^2)\,dx$$

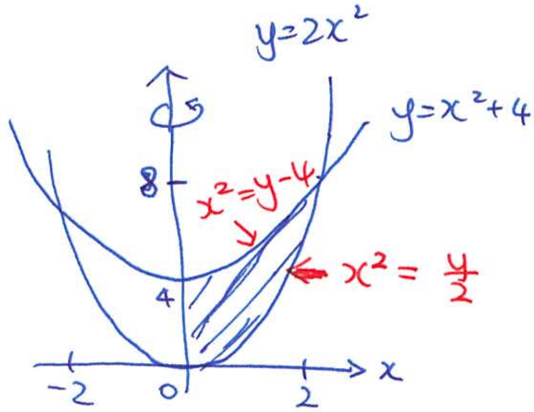
$$= 2\pi \int_0^2 (-x^3+4x)\,dx$$

$$= 2\pi \left[-\frac{x^4}{4} + 2x^2\right]_0^2$$



$= 2\pi\left[\left(-\frac{2^4}{4}+8\right)-0\right]$

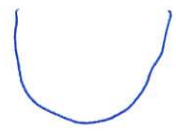
$= 8\pi$



Disk method:

When $x=2$, $y=8$.

$$V=\pi\int_0^8 \frac{y}{2}\,dy - \pi\int_4^8 (y-4)\,dy$$



$= \cdots$

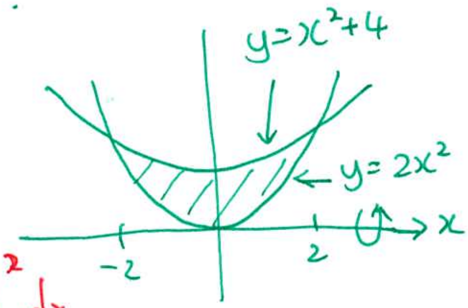
$= 8\pi$

Remarks of Q.3(a)

① If the rotation axis is x-axis:

Volume: Disk: $\pi\int \text{upper}^2 - \text{lower}^2$

$$V=\pi\int_{-2}^{2}\left[(x^2+4)^2-(2x^2)^2\right]dx$$

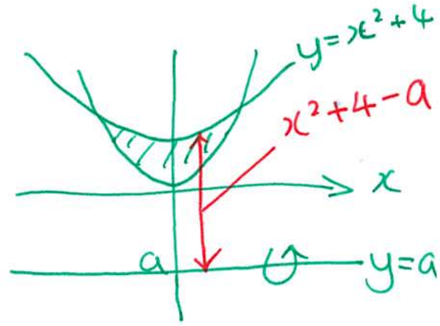


Not $\pi\int_{-2}^{2}\cancel{\left[(x^2+4)-2x^2\right]^2}dx$

② If the rotation axis is $y=a$ (where $a\leq 0$):

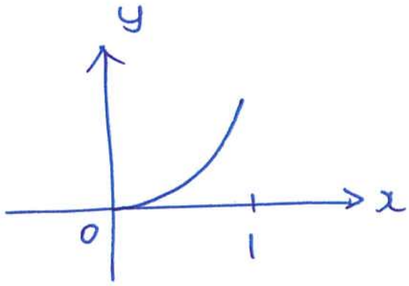
horizontal line

Volume: $$V=\pi\int_{-2}^{2}\left[(x^2+4-a)^2-(2x^2-a)^2\right]dx$$



③ Area of bounded region $= \int_{-2}^{2}\left[(x^2+4)-(2x^2)\right]dx$

3(b) Let $f(x) = x^2$



Then $f'(x) = 2x$.

$$\text{Arc length} = \int_0^1 \sqrt{1 + (f'(x))^2}\, dx$$

$$= \int_0^1 \sqrt{1 + (2x)^2}\, dx$$

$$= \int_0^1 \sqrt{1 + 4x^2}\, dx$$

$$= \int_0^{\tan^{-1}2} \sqrt{1 + \tan^2\theta}\ \tfrac{1}{2}\sec^2\theta\, d\theta$$

$$= \frac{1}{2}\int_0^{\tan^{-1}2} \sec^3\theta\, d\theta$$

Let $2x = \tan\theta$

$\Rightarrow x = \frac{1}{2}\tan\theta$

$\Rightarrow dx = \frac{1}{2}\sec^2\theta\, d\theta$

when $x = 0$,

$\theta = \tan^{-1}0 = 0$

when $x = 1$,

$\theta = \tan^{-1}2$

Consider

$$\int \sec^3\theta\, d\theta = \int \underbrace{\sec\theta}_{u}\ \underbrace{\sec^2\theta\, d\theta}_{dv}$$

$$= \int \underbrace{\sec\theta}_{u}\ d(\underbrace{\tan\theta}_{v})$$

$$= \sec\theta\tan\theta - \int \tan\theta\, d(\sec\theta)$$

$$= \sec\theta\tan\theta - \int \boxed{\tan\theta\sec\theta\tan\theta}\, d\theta$$

$= \tan^2\theta\sec\theta$

$$= \sec\theta\tan\theta - \int (\sec^2\theta - 1)\sec\theta\, d\theta$$

$$= \sec\theta\tan\theta - \int \sec^3\theta\, d\theta - \int \sec\theta\, d\theta$$

$$= \frac{1}{2}\left(\sec\theta\tan\theta - \ln|\sec\theta + \tan\theta|\right) + C$$

$$\therefore \text{ Arc length } = \frac{1}{2}\int_0^{\tan^{-1}2} \sec^3\theta \, d\theta$$

$$= \frac{1}{4}\Big[\sec\theta\tan\theta - \ln|\sec\theta + \tan\theta|\Big]_0^{\tan^{-1}2}$$

$\sec\theta = \sqrt{1+\tan^2\theta}$

$\therefore \sec(\tan^{-1}2)$

$= \sqrt{1+[\tan(\tan^{-1}2)]^2}$

$= \sqrt{5}$

Also, $\tan(\tan^{-1}2) = 2$

$$= \frac{1}{4}\Big[\big(2\sqrt{5} - \ln|\sqrt{5}+2|\big) - \big(0 - \underbrace{\ln 1}_{=0}\big)\Big]$$

$$= \frac{1}{4}\left(2\sqrt{5} - \ln|\sqrt{5}+2|\right)$$

<u>Remark of Q.3(b)</u>

If the rotation axis is x-axis,

$$\text{Surface area} = 2\pi\int_0^1 f(x)\sqrt{1+(f'(x))^2}\,dx$$

$$= 2\pi\int_0^1 x^2\sqrt{1+(2x)^2}\,dx$$

$$= 2x\int_0^1 x^2\sqrt{1+4x^2}\,dx$$

$= \cdots$ use trigo. sub. $2x = \tan\theta$

& int. by parts

4. (a) $\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} 1 & -2 & 3 \\ -3 & 1 & 2 \\ 2 & 3 & -1 \end{vmatrix} = \cdots = -42$

$$\text{Volume of parallelepiped} = |\vec{a}\cdot(\vec{b}\times\vec{c})|$$

$$= |-42|$$

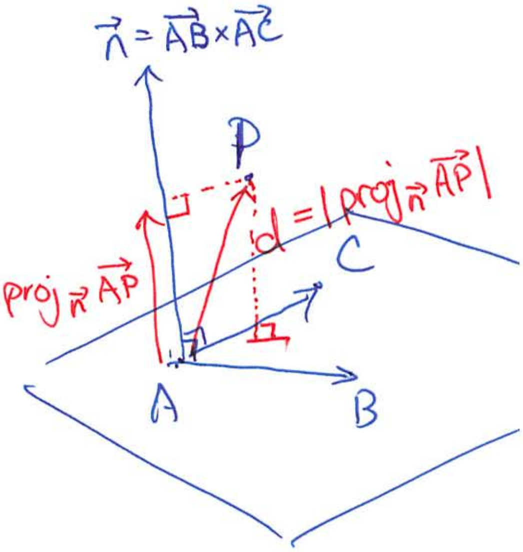
$$= 42$$

(b) $\dfrac{3-2i}{-2+i} = \cdots = -\frac{8}{5} + \frac{1}{5}i$

(c) $AB^T = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 6 & 5 & 4 \\ 3 & 2 & 1 \end{pmatrix}$

$2 \times 3 \neq 2 \times 3$

doesn't exist, since the number of columns of $A \neq$ number of rows of B^T.



5. (a) $\vec{AB} = \cdots = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$

$\vec{AC} = \cdots = \begin{pmatrix} -1 \\ 0 \\ -4 \end{pmatrix}$

$\vec{AP} = \cdots = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$

Let $\vec{n} = \vec{AB} \times \vec{AC}$

$= \cdots$

$= -4\vec{i} - 13\vec{j} + \vec{k}$

Shortest distance:

$$d = |proj_{\vec{n}} \vec{AP}| = \frac{|\vec{AP} \cdot \vec{n}|}{|\vec{n}|}$$

$$= \cdots$$

$$= \frac{|-38|}{\sqrt{186}}$$

$$= \frac{38}{\sqrt{186}}$$

(b) $z^4 + 1 = i \Rightarrow z^4 = -1 + i$

$$= \sqrt{2} e^{i\left(\frac{3\pi}{4}\right)}$$

$$\therefore z_k = (\sqrt{2})^{\frac{1}{4}} e^{i\left(\frac{2k\pi + \frac{3\pi}{4}}{4}\right)}, \quad k = 0, 1, 2, 3$$

$$\therefore z_0 = 2^{\frac{1}{8}} e^{i\left(\frac{3\pi}{16}\right)}$$

$$z_1 = 2^{\frac{1}{8}} e^{i\left(\frac{11\pi}{16}\right)}$$

$$z_2 = 2^{\frac{1}{8}} e^{i\left(\frac{19\pi}{16}\right)} \xrightarrow{-2\pi} = 2^{\frac{1}{8}} e^{i\left(-\frac{13\pi}{16}\right)}$$

$$z_3 = 2^{\frac{1}{8}} e^{i\left(\frac{27\pi}{16}\right)} \xrightarrow{-2\pi} = 2^{\frac{1}{8}} e^{i\left(\frac{-5\pi}{16}\right)}$$

6)
$$\left(\begin{array}{cccc|c} -1 & -2 & 3 & 4 & 5 \\ 2 & 3 & -4 & -5 & -6 \\ -3 & -8 & 13 & 18 & 23 \end{array}\right)$$

$$\longrightarrow \cdots$$

no pivots in col. 3 & 4

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & 2 & \boxed{-3} & \boxed{-4} & -5 \\ 0 & 1 & -2 & -3 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\therefore \begin{cases} x + 2y - 3z - 4w = -5 \\ y - 2z - 3w = -4 \end{cases}$$

Let $z = s$, $w = t$, where $s, t \in \mathbb{R}$.

Then $y = -4 + 2z + 3w = -4 + 2s + 3t$

$x = \cdots = 3 - s - 2t$

In vector form,

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -1 \\ 2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 3 \\ 0 \\ 1 \end{pmatrix}$$

where $s, t \in \mathbb{R}$.

(b) Associated homo. system:

$$\begin{cases} -x - 2y + 3z + 4w = 0 \\ 2x + 3y - 4z - 5w = 0 \\ -3x - 8y + 13z + 18w = 0 \end{cases}$$

The largest possible set of linearly indep. solutions is

$$\left\{ \begin{pmatrix} -1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \\ 0 \\ 1 \end{pmatrix} \right\}$$