EE1001
Foundations of Digital Techniques

Sequences and Series

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Outline

1. Sequences and series
2. Recurrence relations of sequences
3. Arithmetic series and geometric series

Class Intended Learning Outcomes (CILO)

- Understanding the definitions of sequences and series
- Identifying recurrence relations and solving them with iteration method
- Designing and formulating arithmetic and geometric sequences and series

1. Sequences and Series

Sequence and Series

- Sequence: A list of numbers, called **terms**, in a special order.
  - For example,
    - 5, 7, 9, 11, 13
    - 32, 64, 128, 256

List with commas

- Series: The **sum** of all the **terms** of a sequence.
  - For example,
    - 5 + 7 + 9 + 11 + 13
    - 32 + 64+ 128 + 256

"Indicated sum"

Finite and Infinite

- The sequence and series can be **finite** or **infinite**

Finite Sequence	Infinite Sequence
Example: 32, 64, 128, 256	Example: 32, 64, 128, 256, …
General Form: $a_1, a_2, a_3, \ldots, a_n$ (n number of terms)	General Form: $a_1, a_2, a_3, \ldots$ (infinite number of terms)

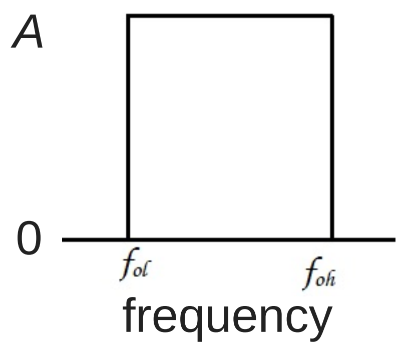
Finite Series	Infinite Series
Example: 32 + 64 + 128 + 256	Example: 32 + 64 + 128 + 256, …
General Form: $a_1 + a_2 + a_3 + \ldots + a_n$ (n number of terms)	General Form: $a_1, a_2, a_3, \ldots$ (infinite number of terms)

Why do we learn both finite and infinite calculations?

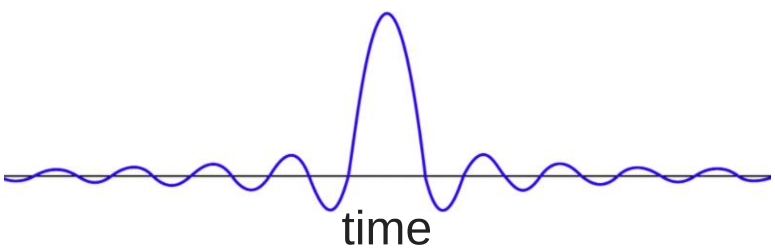
Here gives a real-world engineering example,

Bandpass filters are essential in various engineering applications, such as filtering unwanted signals, limit the transmission bandwidth of the output RF signals, …

Below is an ideal bandpass filter in frequency domain

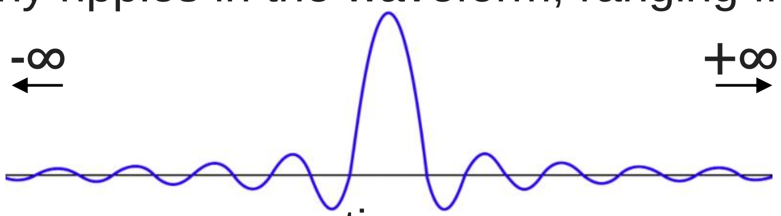


When we implement the filter practically, we need to convert it to time-domain



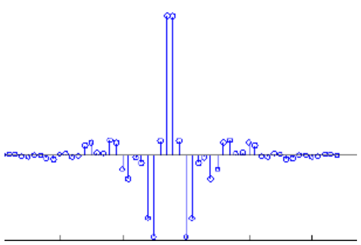
Why do we learn both finite and infinite calculations?

There are many ripples in the waveform, ranging from –inf. to +inf.

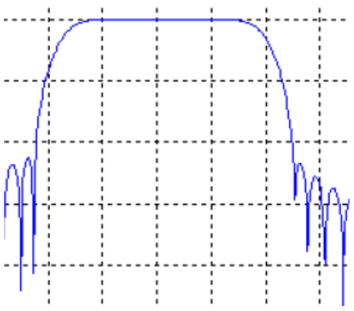


Practically, there is impossible to implement something from –inf. time to +inf. time. Therefore, we need to set a **finite** range and make the practical filter similar to the ideal one.

Finite bandpass filter using Kaiser window



time



frequency

On the other hand, assuming the ideal cases using **infinite** condition can **ease calculations** and predict general performance.

General Formula of the n^{th} term

- For sequences and series, we can denote the n^{th} term (or so-called the general term) using superscript notation, i.e., a_n.

    The 1^{st} term of a sequence/series is then called a_1

    The 2^{nd} term is a_2

    The 3^{rd} term is a_3

    The $(k+1)^{th}$ term is a_{k+1}, and so on.

Any term in a sequence/series can be found by <u>substituting</u> its position number into a <u>given formula</u> for a_n.

General Formula of the n^{th} term

- For instant, the formula for the k^{th} term of a sequence is given by $m_k = 5k + 2$.

Then,	$m_1 = 5\times1 + 2 = 7$	The 1st term
	$m_2 = 5\times2 + 2 = 12$	The 2nd term
	$m_3 = 5\times3 + 2 = 17$	The 3rd term
	…	…
	$m_{(t+2)} = 5\times(t+2) + 2 = 5t + 12$	The $(t+2)^{th}$ term

- ❑ The first three terms in the sequence are: 7, 12, 17
- ❑ The first m+2 terms in the sequence are: 7, 12, 17, …, $5m$+12

Summation Notation (Σ)

- We can denote a series using the general term (a_n) and summation notation (Σ).
  - The Greek capital letter Σ (sigma) is a summation symbol to represent the sum and abbreviate a series.
  - The summation notation can be expressed as

N = last value of n

n = index of summation

$$\sum_{n=k}^{N} a_n = a_k + a_{k+1} + a_{k+2} + \cdots + a_{N-1} + a_N, \forall k \in Z \text{ and } k \le N$$

k = first value of n

For example, the series 2 + 4 + 6 + 8 + 10 or a general term $a_n = 2n$ can be written as

$$\sum_{n=1}^{5} a_n \text{ or } \sum_{n=1}^{5} 2n$$

and it is read as "the sum of the terms a_n or $2n$, as n varies from 1 to 5."

Product Notation (Π)

- We can indicate repeated multiplication using the general term (a_n) and product notation (Π).
  - The Greek capital letter Π (pi) is a product symbol to the product of sequences.
  - The product notation can be expressed as

N = last value of n

n = index of summation

$$\prod_{n=k}^{N} a_n = a_k \times a_{k+1} \times a_{k+2} \times \cdots \times a_{N-1} \times a_N, \forall k \in \mathrm{Z} \text{ and } k \le N$$

k = first value of n

For example, the series 2 × 4 × 6 × 8 × 10 or a general term $a_n = 2n$ can be written as

$$\prod_{n=1}^{5} a_n \text{ or } \prod_{n=1}^{5} 2n = 2(1)\times 2(2)\times 2(3)\times 2(4)\times 2(5)$$

and it is read as "the product of the terms a_n or $2n$, as n varies from 1 to 5."

Factorial Notation (n!)

- Factorial notation (n!)

$$n! = \prod_{k=1}^{n} k = 1 \times 2 \times 3 \times \cdots \times (n-1) \times n \,, \forall n \in \mathrm{Z} \text{ and } n > 0$$

- It is also a kind of production notation with the general term of $a_k = k$

- Factorial is frequently used in mathematics, science, and engineering, especially when determining the number of permutations and combinations.
  - We will go into details in the section of "Counting".

Permutations and Combinations

Number of permutations (order matters) of n things taken r at a time:

$$P(n, r) = \frac{n!}{(n-r)!}$$

Number of combinations (order does not matter) of n things taken r at a time:

$$C(n, r) = \frac{n!}{(n-r)!r!}$$

Number of different permutations of n objects where there are n_1 repeated items, n_2 repeated items, ...n_k repeated items

$$\frac{n!}{n_1!n_2!...n_k!}$$

Properties of Summation and Product

- If "a_n, a_{n+1}, a_{n+2}, …" and "b_n, b_{n+1}, b_{n+2}, …" are sequences of real numbers and c is a constant, then the following equations are true.

1. $$\sum_{n=k}^{N} a_n + \sum_{n=k}^{N} b_n = \sum_{n=k}^{N} (a_n + b_n)$$

2. $$c \times \sum_{n=k}^{N} a_n = \sum_{n=k}^{N} c \times a_n$$

3. $$\left(\prod_{n=k}^{N} a_n\right)\left(\prod_{n=k}^{N} b_n\right) = \prod_{n=k}^{N} a_n b_n$$

The summation and product notations can range from –inf. to +inf., i.e.,

$$\sum_{n=-\infty}^{\infty} a_n = \cdots + a_{-1} + a_0 + a_1 + \cdots$$

$$\prod_{n=-\infty}^{\infty} a_n = \cdots \times a_{-1} \times a_0 \times a_1 \times \cdots$$

2. Recurrence Relation

Recurrence Relation

- A ***recurrence relation*** for the sequence $\{a_n\}$ is a function that defines a_n in terms of one or more of the previous terms of the sequence, i.e., a_0, a_1, …, a_{n-1}, for all integers n with $n \geq n_0$, where n_0 is a non-negative integer.

e.g., $a_n = a_{n-1} + 2a_{n-2} - 3\, a_{n-3}$

- To define a sequence using a recurrence relation, we need
  - the function relating each term to a previous term, and
  - the **initial conditions**, which is a finite set of start-up values.

For example, express a sequence of {-5, -1, 3, 7, …} using a recurrence relation

The initial condition is a_1 = -5, and the function is $a_{n+1} = a_n + 4$.

a_1 = -5 (Initial condition)

$a_2 = a_1 + 4 = -5 + 4 = -1$

$a_3 = a_2 + 4 = -1 + 4 = 3$

$a_4 = a_3 + 4 = 3 + 4 = 7$,

and so on

In-class Exercises

Express the following sequences using recurrence relation:

a) {1, 3, 7, 15, 31, 63, …}

b) {108, 52, 24, 10, 3, -0.5, …}

Find out the first five terms of the following sequence.

Initial condition: $a_1 = 1$, $a_2 = 3$

$$a_{n+2} = 1 + 2a_n - a_{n+1}$$

In-class Exercises 1

Express the following sequences using recurrence relation:

a) {1, 3, 7, 15, 31, 63, …}

b) {108, 52, 24, 10, 3, -0.5, …}

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Iteration Method for Solving Recurrence Relations

- *Iteration* methods (backtracking) is one of the methods of solving recurrence relations
  - Given a recursive expression with initial conditions a_1, some recurrence relations of a_n can be derived to become an <u>explicit formula</u>
  - For example, $a_n = 2a_{n-1}$, with the initial condition of $a_1=1$, can be derived as follows.

$$
\begin{aligned}
a_n &= 2a_{n-1} \\
&= 2 \times (2a_{n-2}) = 2^2 \times a_{n-2} \\
&= 2^2 \times (2a_{n-3}) = 2^3 \times a_{n-3} \\
&= \ldots \\
&= 2^{n-3} \times (2a_2) = 2^{n-2} \times a_2 \\
&= 2^{n-2} \times (2a_2) = 2^{n-1} \times a_1 \qquad (a_1 = 1) \\
&= 2^{n-1}
\end{aligned}
$$

We can then find the k^{th} term easily by substituting k into n, e.g.,

$a_2 = 2^{2-1} = 2$

$a_3 = 2^{3-1} = 4$

$a_{10} = 2^{10-1} = 512$

In-class Exercises 2

Find the explicit formula for the following sequence.

$a_n = a_{n-1} + 3$, for n = 2, 3, 4, …, and the initial condition is $a_1 = 2$

In-class Exercises 2

a) Find the explicit formula for the following sequence.

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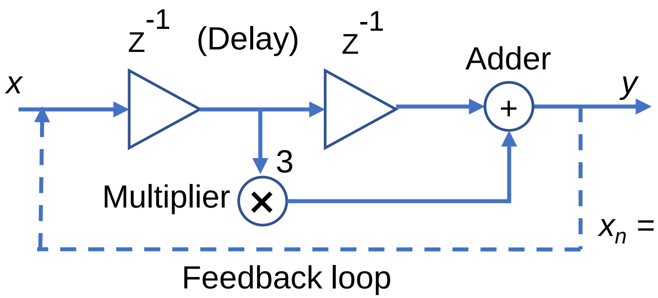
Recurrence Relations in Engineering

Delay and Feedback in Filter Design



$x_n = x_{n-2} + 3x_{n-1}$

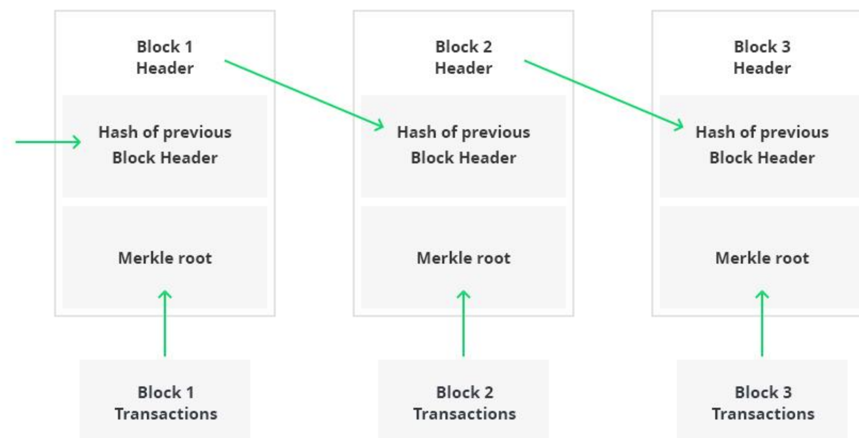
Programming

```
Int a, b;

a = 10;
b = 20;

For (I = 1; i<=10; i++)
{
        a = a + 2;
}
```

Initial condition: $a_1 = 10$

$a_{n+1} = a_n + 2$

Blockchains



$\text{Block\_header}_{k+1} = \text{Block\_header}_k + \text{transaction}_k$

3. Arithmetic Series & Geometric Series

Arithmetic Sequence

- An <u>arithmetic sequence</u> is a sequence in which there is a **common difference *d*** between two consecutive terms, i.e.,
  - The common difference $d = (a_n - a_{n-1}) = (a_{n-1} - a_{n-2}) = \ldots = (a_2 - a_1)$
  - Therefore:
    - 1st term: a_1
    - 2nd term: $a_2 = a_1 + d$
    - 3rd term: $a_3 = a_1 + 2d$
    - …
- The arithmetic formula of the n^{th} term is as follows.

$$a_n = a_1 + (n-1)d$$

a_1 = the 1st term
n = the term number
d = the common difference

Geometric Sequence

- A geometric sequence is a sequence in which there is a **common ratio *r*** between two consecutive terms, i.e.,
  - The common ratio $r = (a_n - a_{n-1}) = (a_{n-1} - a_{n-2}) = \ldots = (a_2 - a_1)$
  - Therefore,
    - 1^{st} term: a_1
    - 2^{nd} term: $a_2 = a_1 \times r$
    - 3^{rd} term: $a_3 = a_1 \times r^2$
    - …

- The geometric formula of the n^{th} term is as follows.

$$a_n = a_1 \times r^{n-1}$$

a_1 = the 1^{st} term
n = the term number
r = the common difference

Simple Exercise

- Determine the following sequences whether they are arithmetic sequence or geometric sequence
  1) {11.5, 25, 38.5, 52}
  2) {10, 0.1, 10, 0.1}
  3) {5, 10, 20, 60}

- Given {79, 75, 71, 67, 63, …}, what term number is -169?

Simple Exercise

- Determine the following sequences whether they are arithmetic sequence or geometric sequence
  1) {11.5, 25, 38.5, 52}
  2) {10, 0.1, 10, 0.1}
  3) {5, 10, 20, 60}

- Given {79, 75, 71, 67, 63, …}, what term number is -169?

Sum of Arithmetic Series

- *Arithmetic series* is an expression formed by adding the terms of an **arithmetic sequence**.

- The sum of a series containing N terms can be denoted by S_N.
  - For example, $S_5 = 1 + 3 + 5 + 7 + 9 = 25$

How about $S_{100} = 1 + 2 + 3 + \ldots + 99 + 100$?

Sum of Arithmetic Series

- To find the sum of the first n terms, we can adopt the Gauss' method as follows.

symmetric

$$\begin{cases} S_{100} = 1 + 2 + 3 + \ldots + 99 + 100 \\ S_{100} = 100 + 99 + 98 + \ldots + 2 + 1 \end{cases}$$

From 1st to 100th term

From 100th to 1st term

$$2S_{100} = \underbrace{101 + 101 + 101 + \ldots + 101 + 101}_{\text{100 terms}}$$

Therefore, we have

$$2S_{100} = 100 \times 101$$

$$S_{100} = 100 \times 101 / 2 = 5050$$

Sum of Arithmetic Series

- Considering the general form of an arithmetic series having N terms and a common difference of d,

$$S_N = a + (a + d) + (a + 2d) + \dots + (a + (N - 2)d) + (a + (N - 1)d)$$

$$S_N = a + (a + d) + (a + 2d) + \dots + (f - d) + f$$
$$S_N = f + (f - d) + (f - 2d) + \dots + (a + d) + a$$

$$2S_N = \underbrace{(a + f) + (a + f) + (a + f) + \dots + (a + f)}_{N \text{ terms}}$$

(a notes the first term)
(f denotes the last term)

Therefore, we have $2S_N = N(a + f)$, and

The sum of arithmetic series is $\boxed{S_N = \frac{N}{2}(a + f)}$ or $\boxed{S_N = \frac{N}{2}(2a + (N - 1)d)}$

N = number of terms, a = initial term, f = last term, d = common difference

Example

- Find the sum of the first 23 terms of the arithmetic series

-5 + -11 + -17 + -23 + …

Since we don't know the last term, we adopt the following formula

$$\boxed{S_N = \frac{N}{2}(2a + (N-1)d)}$$

d = -11 - (-5) = -6, *a* = -5, and *N* = 23

$$S_{23} = \frac{23}{2}[2(-5) + (23-1)(-6)] = -1633$$

Sum of Geometric Series

- *Geometric series* is an expression formed by <u>adding</u> the terms of a **geometric sequence**.

- The sum of a series containing N terms can be denoted by S_N.
  - For example, $S_5 = 1 + 2 + 4 + 8 + 16 = 56$

How about $\{a, ar, ar^2, ar^3, ..., ar^N\}$?

Sum of Geometric Series

- Given a finite geometric series with *n* terms, a common ratio *r* and the first term *a*,

$$S_N = a + ar^1 + ar^2 + \cdots + ar^{N-2} + ar^{N-1}$$

$$S_N = a(1 + r^1 + r^2 + \cdots + r^{N-2} + r^{N-1})$$

$$S_N = \frac{a(1 + r^1 + r^2 + \cdots + r^{N-2} + r^{N-1})(1-r)}{1-r}$$

As a result, the general formula of a finite geometric series is

$$\boxed{S_N = \frac{a(1-r^N)}{1-r}}$$

N = number of terms, *a* = initial term, *r* = common ratio

Sum of Infinite Geometric Series

- If the common ratio $|\, r \,| < 1$, the infinite geometric series has the sum:

$$\boxed{S = \frac{a}{1-r}}$$

a = initial term
r = common ratio

For example, $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

$$S = \frac{1/2}{1-\frac{1}{2}} = 1$$

Exercise

- Given the following geometric series,

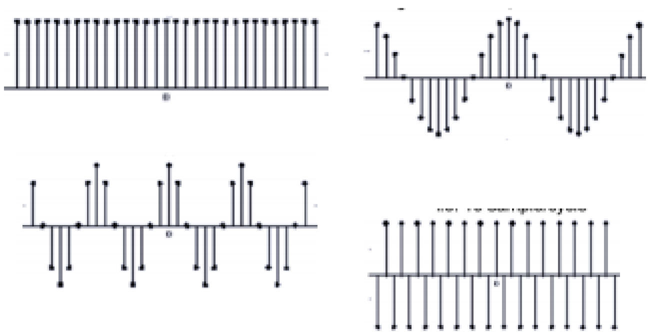
$$\{1, -0.25, 0.0625, -0.015625, \ldots\}$$

- a) Find the sum of the first 9 terms

- b) Find the sum of all terms

Arithmetic and Geometric in Engineering

Calculate the energy and power of a discrete time signal



Energy and Power

- The energy of a discrete time signal is defined as

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

Example: $\{x[n]\}=\{1,2,2,2,1\}$

The energy of $x[n]$ is $E_x=1+4+4+4+1=14$

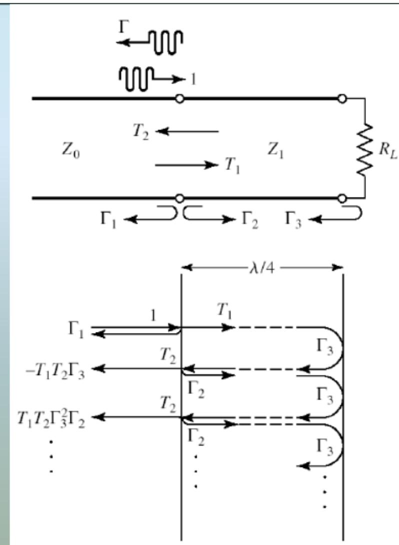
Example: $x[n] = \exp\{(-0.2 + j0.1\pi)n\}\, u[n]$

$$E_x = \sum_{n=0}^{\infty} |\exp\{(-0.2 + j0.1\pi)n\}\,|^2 = \sum_{n=0}^{\infty} e^{-0.4n}$$

Using geometric progression, $E_x = \frac{1}{1-e^{-0.4}} = 3.033$

RF Calculation

(e.g., multiple reflection)



$$\Gamma_1 = \frac{Z_1 - Z_0}{Z_1 + Z_0}, \Gamma_2 = \frac{Z_0 - Z_1}{Z_0 + Z_1} = -\Gamma_1,$$

$$\Gamma_3 = \frac{R_L - Z_1}{R_L + Z_1}$$

$$T_1 = \frac{2Z_1}{Z_1 + Z_0}, T_2 = \frac{2Z_0}{Z_1 + Z_0}$$

$$\Gamma = \Gamma_1 - T_1T_2\Gamma_3 + T_1T_2\Gamma_2\Gamma_3^2 - T_1T_2^2\Gamma_2^2\Gamma_3^2 + \cdots$$

$$= \Gamma_1 - T_1T_2\Gamma_3\sum_{n=0}^{\infty}(-\Gamma_2\Gamma_3)^n$$

$$= \Gamma_1 - \frac{T_1T_2\Gamma_3}{1+\Gamma_2\Gamma_3} = \frac{\Gamma_1 + \Gamma_1\Gamma_2\Gamma_3 - T_1T_2\Gamma_3}{1+\Gamma_2\Gamma_3}$$

$$\left(\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}\right)$$

- END -