# Functions

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# Important Symbols

- $P \rightarrow Q$ *:* If $P$, then $Q$
- $P \Leftrightarrow Q$ *:* $P$ if and only if $Q$
- $x \in S$ *:* $x$ belongs to $S$, $x$ *is an element/member of* $S$
- $S \subseteq T$*:* $S$ is a subset of $T$, or $S$ is *contained* in $T$
- $\forall x$: for all $x$
- $\exists x$*:* there exists $x$
- $P$ **AND** $Q$ : the **conjunction** of $P$ and $Q$
- $P$ **OR** $Q$ : the **disjunction** of $P$ and $Q$
- $\sim P$: Not $P$

# Functions

- One of the basic notions of mathematics is that of functions.
- A function can be seen as an algebraic formula involving one or more variables:

$$y = kx^2$$

Changing the numerical value will result different outputs.

- It is, however, very important to broaden the scope of function to include the relationships that could not be expressed by a simple formula and to allow variables that NOT necessarily NUMBERS.

# Exponential functions

The equation $f(x) = b^x$

defines the exponential function with base b . The domain is the set of all real numbers, while the range is the set of all positive real numbers ( $y > 0$). Note y cannot equal to zero.

$f(x) = 2^{30}$ = 1,073,741,824 This example shows how an exponential function grows extremely rapidly.

# Exponential functions

Consider a function of the form $f(x) = a^x$, where a>0. Such a function is called an exponential function. We can take 3 different cases, where $a$=1, 0< $a$ <1, and $a$ >1.

If $a$ =1, $f(x) = 1^x = 1$, it gives a constant 1.

If a >1, what happens? We use numerical cases to have a look.
Suppose a=2.

$$f(x) = 2^x$$

$$f(0) = 2^0 = 1$$

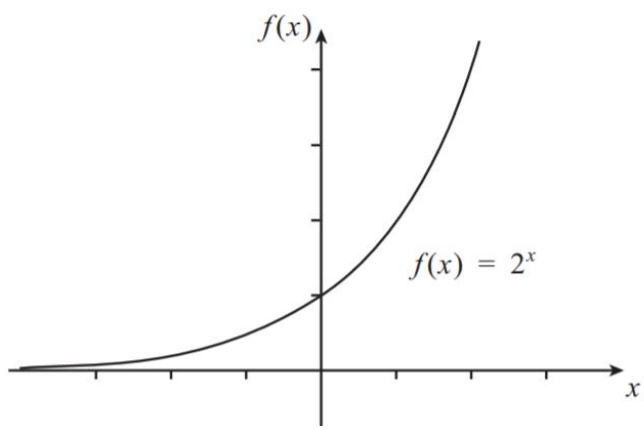
$$f(1) = 2^1 = 2 \qquad f(-1) = 2^{-1} = 1/2^1 = \tfrac{1}{2}$$

$$f(2) = 2^2 = 4 \qquad f(-2) = 2^{-2} = 1/2^2 = \tfrac{1}{4}$$

$$f(3) = 2^3 = 8 \qquad f(-3) = 2^{-3} = 1/2^3 = \tfrac{1}{8}$$

We can put these results into a table, and plot a graph of the function.

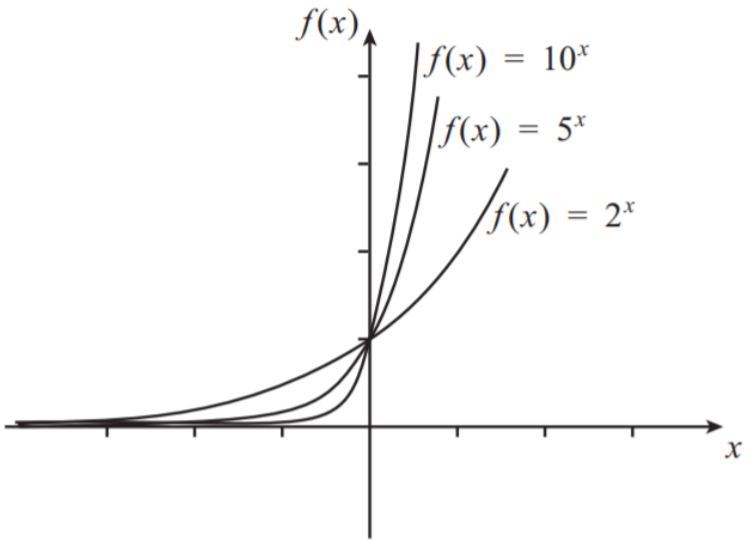
| $x$ | $f(x)$ |
|---|---|
| $-3$ | $\frac{1}{8}$ |
| $-2$ | $\frac{1}{4}$ |
| $-1$ | $\frac{1}{2}$ |
| 0 | 1 |
| 1 | 2 |
| 2 | 4 |
| 3 | 8 |




This example demonstrates the general shape for graphs of functions of the form $f(x) = a^x$, when a > 1.

We'll see the effect of varying a.

# Exponential functions (cont.)




The important properties of the graphs of these type of functions are:

- $f(0) = 1$ for all values of a, because $a^0$=1 for any value of $a$.
- $f(x) =$ >0 for all values of a, because a > 0 implies $a^x > 0$.

# Exponential functions (cont.)

What happens if $0 < a < 1$. We use $a = 1/2$ (0.5) to look at the case.

$$f(x) = \left(\frac{1}{2}\right)^x$$

$$f(0) = \left(\tfrac{1}{2}\right)^0 = 1$$

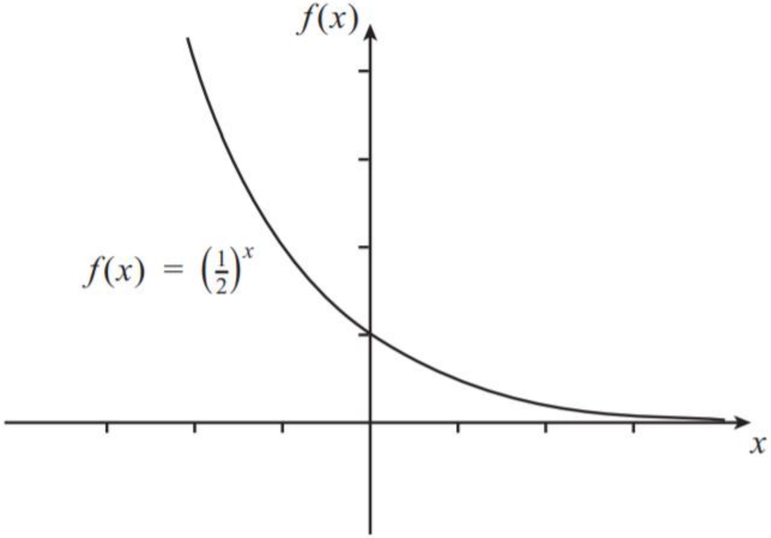
$$f(1) = \left(\tfrac{1}{2}\right)^1 = \left(\tfrac{1}{2}\right) \qquad f(-1) = \left(\tfrac{1}{2}\right)^{-1} = \left(\tfrac{2}{1}\right)^1 = 2$$

$$f(2) = \left(\tfrac{1}{2}\right)^2 = \left(\tfrac{1}{4}\right) \qquad f(-2) = \left(\tfrac{1}{2}\right)^{-2} = \left(\tfrac{2}{1}\right)^2 = 4$$

$$f(3) = \left(\tfrac{1}{2}\right)^3 = \left(\tfrac{1}{8}\right) \qquad f(-3) = \left(\tfrac{1}{2}\right)^{-3} = \left(\tfrac{2}{1}\right)^3 = 8$$

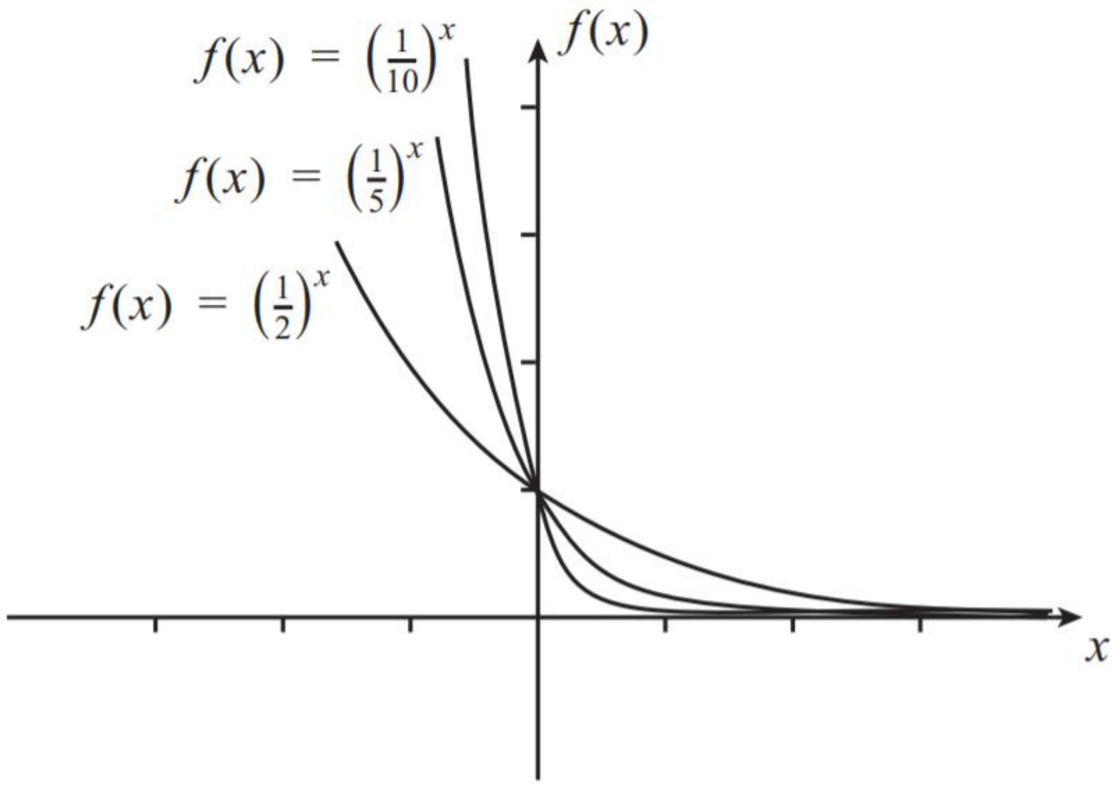
We can put these results into a table, and plot a graph of the function.

| $x$ | $f(x)$ |
|---|---|
| $-3$ | 8 |
| $-2$ | 4 |
| $-1$ | 2 |
| 0 | 1 |
| 1 | $\frac{1}{2}$ |
| 2 | $\frac{1}{4}$ |
| 3 | $\frac{1}{8}$ |



# Exponential functions (cont.)

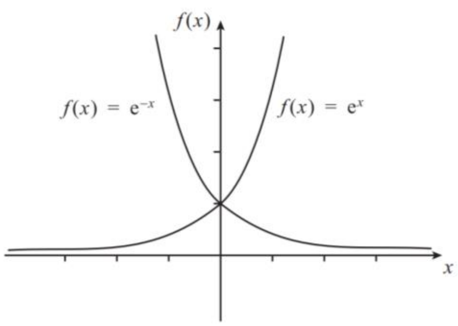
What is the effect of varying a?

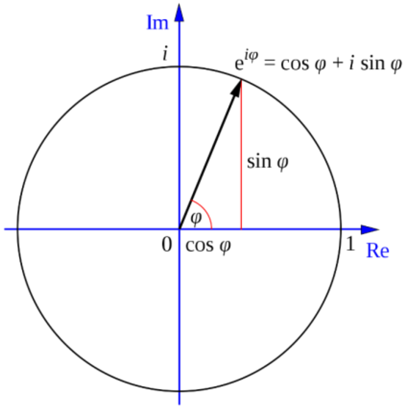


# Exponential functions (cont.)

- A particular important function arises when a = *e*, where *e* is called the **Euler's** number = 2.71828281…
- The Euler's number is extremely important especially in Electrical, electronic and computer engineering.
- To learn more about Euler's number, refer to https://www.youtube.com/watch?v=sKtloBAuP74

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{1} + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \cdots$$



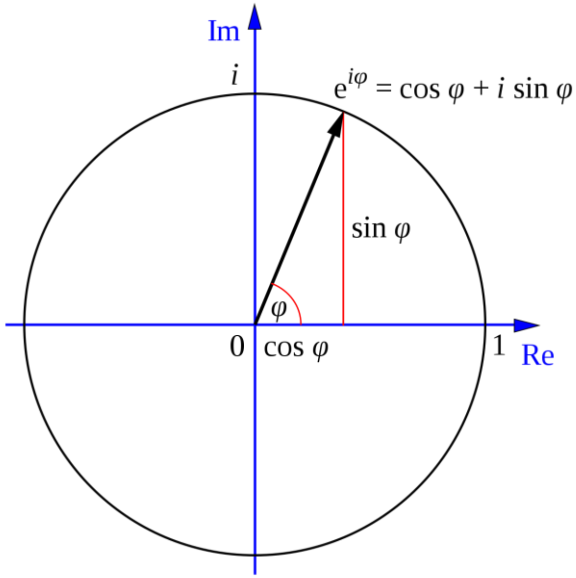

$$e^{i\theta} = cos\theta + isin\theta$$
$$e^{-i\theta} = cos\theta - isin\theta,$$

where $i$ is imaginary number, $i^2$= -1.

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta$$
$$e^{i\theta} - e^{-i\theta} = 2\sin\theta$$

Note: In engineering we usually use *j* instead of *i* because to prevent the confusion between imaginary number *i* and current *i*.

# Exponential functions (cont.)




$$e^{i\theta} = cos\theta + isin\theta$$
$$e^{-i\theta} = cos\theta - isin\theta,$$

where $i$ is imaginary number, $i^2$= -1.

**unit circle: radius = 1**

What is $e^{i(0)\pi}$ ? =1

What is $e^{i2\pi}$ ? =1

What is $e^{i\frac{\pi}{2}}$ ? $=i$

What is $e^{i\frac{5\pi}{2}}$ ? $=i$

What is $e^{i\pi}$ ? = -1

What is $e^{i3\pi}$ ? = -1

What is $e^{i\frac{3\pi}{2}}$ ? $=-i$

What is $e^{i\frac{\pi}{3}}$ ? $a+bi$

Polar form: $1.5\angle\frac{\pi}{3}$

# Logarithms function

Logarithms are another way of thinking about exponents. For example, we know $2^4 = 16$.

Now if we think on a reverse way, 2 raised to which power equals to 16? It is 4.

This is expressed by logarithm function, $log_2(16) = 4$.
Read as "log base 2 of sixteen is 4."

| Logarithmic form | | Exponential form |
|---|---|---|
| $\log_2(8) = 3$ | $\Longleftrightarrow$ | $2^3 = 8$ |
| $\log_3(81) = 4$ | $\Longleftrightarrow$ | $3^4 = 81$ |
| $\log_5(25) = 2$ | $\Longleftrightarrow$ | $5^2 = 25$ |

## Definition of a logarithm

Generalizing the examples above leads us to the formal definition of a logarithm.

$$\log_b(a) = c \quad \Longleftrightarrow \quad b^c = a$$

b is the base,
c is the exponent, and
A is called the argument.

# Logarithms function

| Logarithm | = | Exponent |
|---|---|---|
| $\log_a N = x$ | $\longleftrightarrow$ | $N = a^x$ |
| (Common Log) $\log N = x$ | $\longleftrightarrow$ | $N = 10^x$ |
| (Natural Log) $\ln N = x$ | $\longleftrightarrow$ | $N = e^x$ |

Base 10 or common log

log 0.1 =-1; $0.1 = 10^{-1}$

log 1 = 0; $1 = 10^0$

log 10 =1; $10 = 10^1$

log 100 =2; $100 = 10^2$

log 1000=3; $1000 = 10^3$

- Widely used and an important function used in every engineering area.
- It is useful to represent data that covers large range from very huge to small; i.e., cases in which one or a few points are much larger than the bulk of the data; 1 million, and others may be within 100.
- Earthquake Richter scale and EE use decibel are log scale.

# Basic Logarithms properties

Properties of Logarithms

**1. $\log_a (uv) = \log_a u + \log_a v$**

2. $\log_a (u / v) = \log_a u - \log_a v$

3. $\log_a u^{n} = n \log_a u$

Example: Earthquake Richter scale
What is the magnitude difference between Richter scale 7 and 8.5?

$$\text{RscaleA} - RscaleB = \log\left(\frac{MA}{MB}\right), \text{ base 10}$$

$$8.5\text{-}7.0 = 1.5 = \log\left(\frac{IA}{IB}\right)$$

$$1.5 = \log(31.6)$$

*IA* is 31.6 times of the magnitude of IB

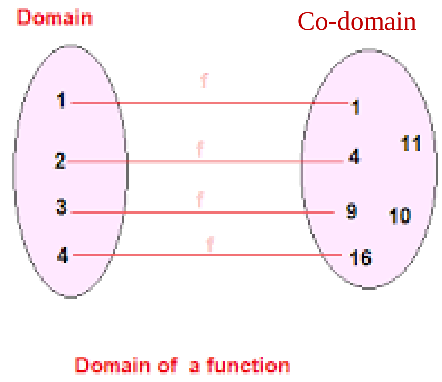
So for earthquake Richter scale

Scale 7 and 9, it means the difference in real magnitude is 100 times; 2 = log 100.

# Functions Definition

- Let $X$ and $Y$ be sets. A function, $f$, from $X$ to $Y$ is denoted by

$$f : X \to Y$$ a rule that assigns to each element $x \in X$ *a unique element* $f(x) \in Y$.




- The set $X$ is called the domain of $f$, while the set $Y$ is called the codomain of $f$.

  The $f(x) \in Y$ is called the value of the function $f$ at $x$.

- Whenever a specific element is assigned to the variable $x$, the corresponding value of $y = f(x)$ is determined by the function.

- For example if $f$ is the function from the real number, $R$, to the real numbers in which $f$ assigns to each real number its cube,
- We would say that $f(x) = x^3$.
- The value of 3 would be 27, and the value of $f$ at -3 would be -27, and so on.
- Often, the set of values of a function is not the whole codomain, but a proper subset of it.

The set of values of a function $f: X \rightarrow Y$

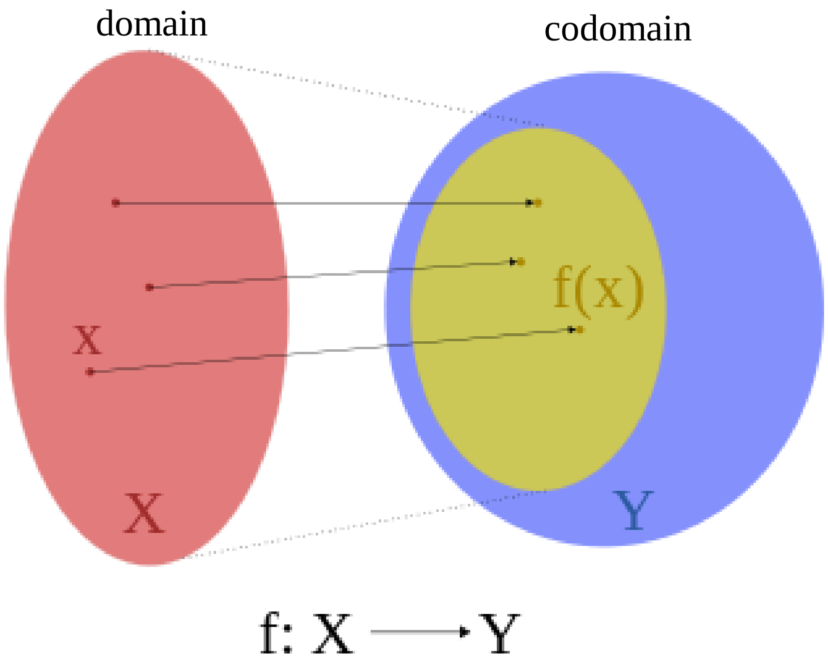
is called the **image** of $f$. It is a subset of the codomain $Y$ and

$$f(x) = \{f(x) \mid x \in X\} \subseteq Y$$

The term **range** is also used to mean image.

- A HK ID card number is assigned to an individual.
- HK ID number can be considered a function whose domain is HK citizens.
- And whose codomain is the set of possible HK ID numbers.
- The image consists of those HK ID numbers that are in actual use.

# Mapping, Domain,Codomain




Domain: a set of all possible function input values

Codomain: a set of all possible function output values

Range: a set of all **actual** output values

Codomain and range are often confusing

$f$ is a function from domain $X$ to codomain $Y$.

The smaller oval inside $Y$ is the image or range of $f$.

# Codomain vs Range

| x | f(x) |
|---|---|
| 0 | 0 |
| 1 | 2 |
| 2 | 4 |

$Example\ 1$

$f(x) = 2x$

$Domain\ (D): (-\infty, \infty)$

$Codomain\ (COD): (-\infty, \infty)$

*Range*: all even number
(not all integers)

Example 2

| x | f(x) |
|---|---|
| 0 | 3 |
| 1 | 4 |
| 2 | 7 |

$f(x) = A = x^2 + 3$

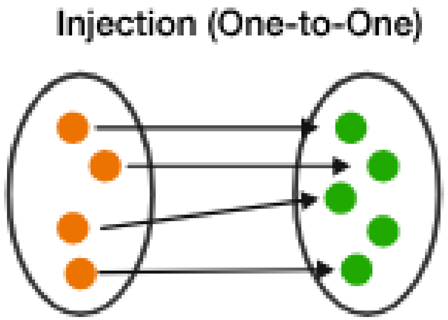
$D: (0, \infty)$

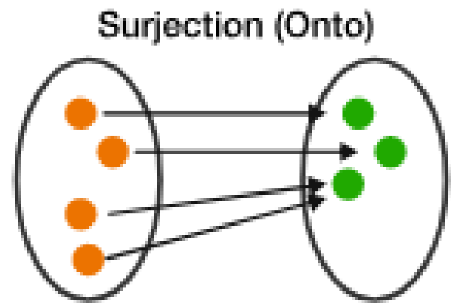
$COD: (0, \infty)$

*R*: {3, 4, 7, 12, …}
Range is a proper subset of COD

# Functions: Injective, Surjective, Bijective

3 functions that we use to define different relationship between domain and codomain

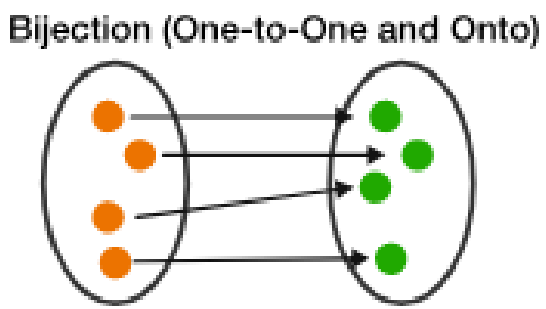



$\forall y \in Y, \exists x \in X$ such that $y = f(x)$

$\forall a. b \in X, f(a) = f(b) \Longrightarrow a = b'$

$\forall a, b \in X, a \neq b \Longrightarrow f(a) \neq f(b)$

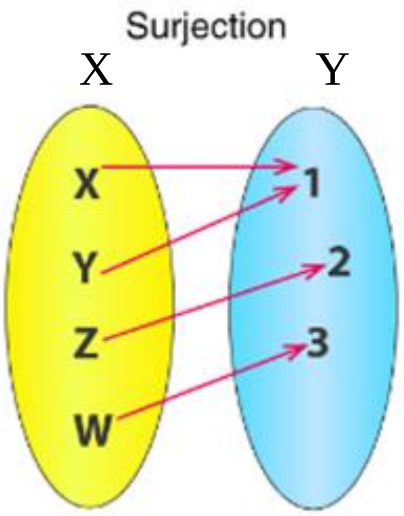


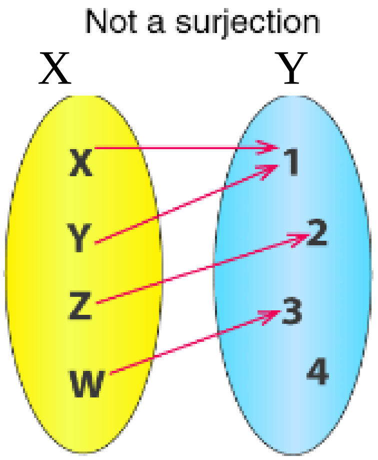

$\forall y \in Y, \exists! \, x \in X$ such that $y = f(x)$

"there exist exactly one x"

# Surjective (also called "onto")

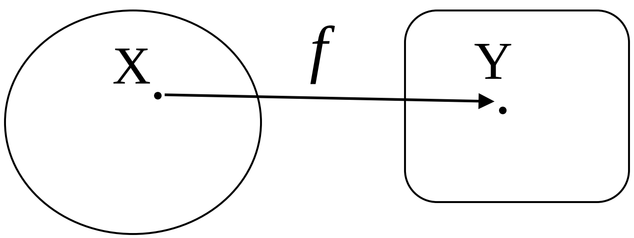
- **Surjective** means that every element in the codomain does get mapped, $\forall y \in Y, \exists x \in X$ such that $y = f(x)$
- Or the image of $f$ equals to $Y$, no element in $Y$ left out without being mapped from the domain.



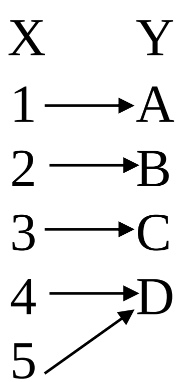
# Surjective function

Like to introduce 3 terms: surjective, injective and bijective.
Let say we have a function f, mapping from the set $X$ to the set $Y$, $f$: $X \rightarrow Y$.

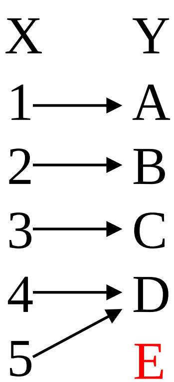



If there is an element in $Y$, there is an element x that f will map from $X$ to $Y$, that is surjective.

$\forall\, y \in Y, \exists x, x \in X$ such that $f(\mathrm{x}) = y$.
Every $y$ in the set $Y$ can be mapped to by at least one of the elements over the set $X$.




surjective

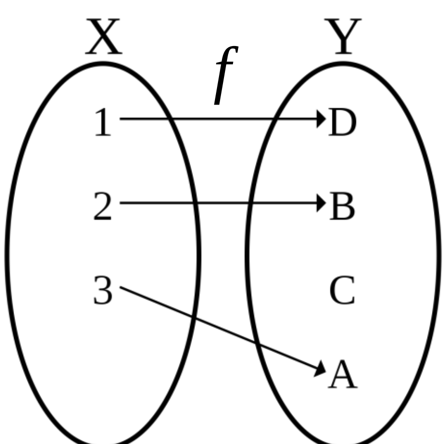



Not surjective (element E left out)

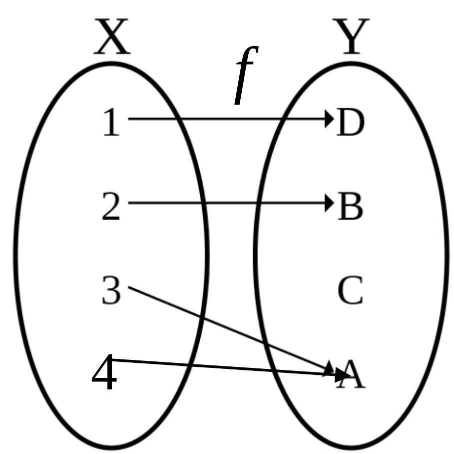
# Injective (also called one-to–one)

- **Injective** means we won't have two or more "*x*" pointing to the same "*y*" $\forall a, b \in X, f(a) = f(b) \Longrightarrow a = b$

  *or* $\forall a, b \in X, a \neq b \Longrightarrow f(a) \neq f(b)$
- Many-to-one is **NOT** injective
- One–to-many is **NOT** injective
- But we can have a *y without* being mapped from "*x*"




injective




Not injective, 4, 3 map to the same "A"

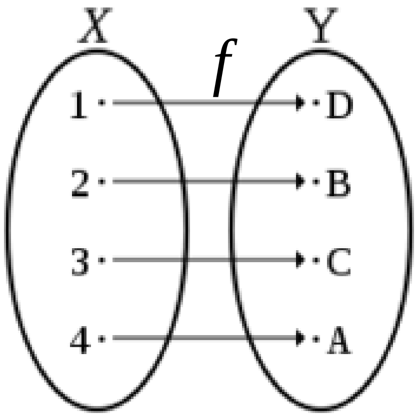
Not injective

# Bijective

- **Bijective** means both Injective and Surjective together,

$$\forall y \in Y, \exists!\, x \in X \text{ such that } y = f(x).$$

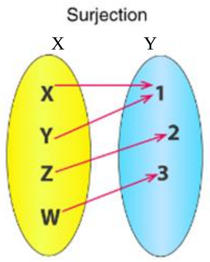
- A function that is both "one-on-one" and "onto" is called a bijection or a one-on-one correspondence.
- If every "*x*" goes to a unique "*y*" and every "*y*" has a mapping from '*x*" then we can go back and forwards without confused.
- Every bijective function, *f*, has an **inverse** function, $f^{-1}$.




Note:
∃! is read there exists only one

# Example: Prove $f$ is surjective "onto"

Prove: f: $\boldsymbol{R} \rightarrow \boldsymbol{R}$ defined by $f(x) = mx + b$, where $m$ and $b$ are also real number




Recall for surjective "onto" $\forall y \in Y, \exists x \in X$ such that $f(x) = y$

1. Approach of proof: To take any y ∈ R, show *f(x) =y*

2. we want *f(x)=y*
*or mx+b=y*
$x = \frac{y-b}{m}$

3. *y* is a real number,
*b* and *m* are real numbers,
$\therefore x = \frac{y-b}{m} \in R$ and

$f(x) = f\left(\frac{y-b}{m}\right) = m\left(\frac{y-b}{m}\right) + b$
$= y - b + b$
$f(x) = y$

4. We started from any real number in the codomain, and the above work show there exist an "*x*" in the domain such that *f(x)=y*.
Thus surjective.

# Example: Prove $f$ is injective "one-to-one"

Let $f$: $\sqsubset 0, \infty) \rightarrow R$, $f(x) = x^2$. Prove $f$ is injective.

Definition: $\forall a, b \in X, f(a) = f(b) \Longrightarrow a = b$
It means equal output are from equal inputs

Proof: For arbitrary $a$, $b$, suppose $f(a) = f(b)$, *and* $a$, $b \in \sqsubset 0, \infty)$ means they are in the interval from 0 to ∞.

So $f(a) = a^2$ and $f(b) = b^2$,
$a^2 = b^2$, because $f(a) = f(b)$,
$\sqrt{a^2} = \sqrt{b^2}$ , hence a = b because $a$, $b$ $a$, $b \in \sqsubset 0, \infty)$ , positive number
As $a$, $b$ are arbitrary number, so proved, and ***f*** is injective.

# Composition of Functions

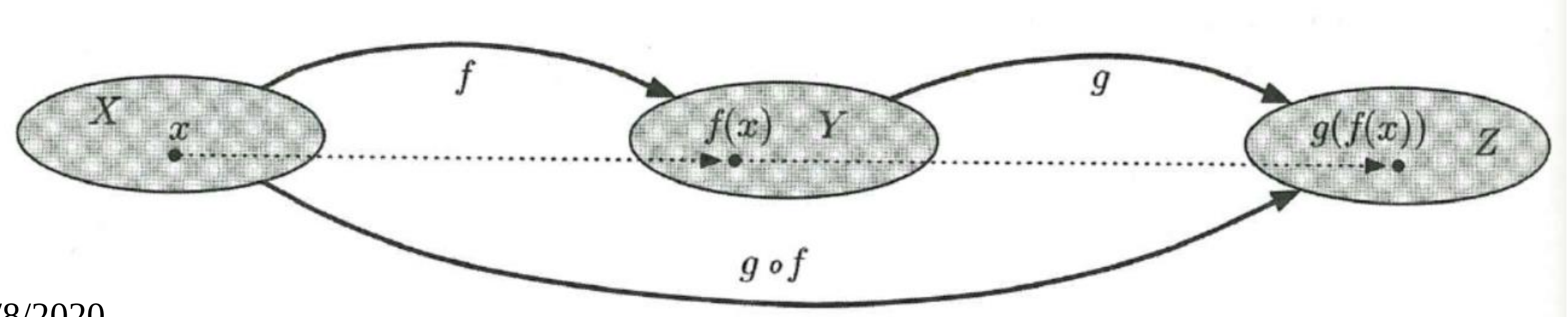
If $f: X \to Y$ and $g$: Y→ $Z$ are two functions such that the codomain of $f$ is equal to the domain of $g$, then we can form a new function

$$g \circ f : X \to Z \qquad \text{(read as } g \text{ of } f\text{)}$$

is defined by

$$(g \circ f)(x) = g\big(f(x)\big).$$

The function $f$ is applied first, and then $g$ is applied to the result of $f$.



# Simple Numerical example

Composition of functions

$(f \circ g)(x) = f\big(g(x)\big)$

Given $f(x) = 3x + 8,\ \ g(x) = 2x - 4$

Recall that function is $f(1) = (3)(1) + 8$=11

$$(f \circ g)(x) = f\big(g(x)\big)$$

$$= f(2x - 4)$$

$$= 3(2x - 4) + 8$$

$$(f \circ g)(x) = (6x - 12) + 8 = 6x - 4$$

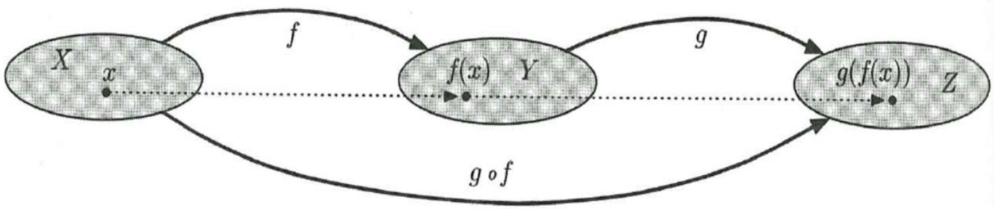
*So we can also find* $(f \circ g)(-1)$

$$(f \circ g)(-1) = 6(-1) - 4$$

= -10

# Composition of Functions

- If $f : X \to Y$ and $g : Y \to Z$
  Composition of functions
  $(g \circ f)(x) = g(f(x))$




- Two functions cannot always be composed.
- The two functions can only be composed when the codomain of the first operation, $f$, is equal to the domain of the second, $g$. operation.
- Even if $g \circ f$ is defined, where $f: X \to Y$ and $g: Y \to Z$, then the composite in the reverse order $f \circ g$, will not be defined unless $X = Z$.
- In general $g \circ f \neq f \circ g$; composition is not *commutative*.

# Example:

Let $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ and g: $\mathbb{R} \longrightarrow \mathbb{R}$ be functions defined by $f(x) = x^2$ and $\mathrm{g}(x) = 4x - 5$. Are $g \circ f$ and $f \circ g$ defined? If so, find them.

**Solution.** Since the codomain of $f$ and the domain of $g$ are both equal to $\mathbb{R}$, and the codomain of $g$ and the domain of $f$ are also equal to $\mathbb{R}$, both composites are defined and are functions from $\mathbb{R}$ to $\mathbb{R}$.

Now

$$(g \circ f)(x) = g\big(f(x)\big) = g(x^2) = 4x^2 - 5$$

While

$$(f \circ g)(x) = f\big(g(x)\big) = f(4x - 5) = (4x - 5)^2 = 16x^2 - 40x + 25$$

Note: in general $g \circ f \neq f \circ g$.
The operation of composition is not *commutative*.

END