Exercise questions for Chapter 2

Question 1

Periodic energy usage. The 168-vector $\boldsymbol{w}$ gives the hourly electricity consumption of a manufacturing plant, starting on Sunday midnight to 1AM, over one week, in MWh (megawatt-hours). The consumption pattern is the same each day, i.e., it is 24-periodic, which means that $w_{t+24} = w_t$ for $t = 1, \ldots, 144$. Let $\boldsymbol{d}$ be the 24-vector that gives the energy consumption over one day, starting at midnight.

(a) Use vector notation to express $\boldsymbol{w}$ in terms of $\boldsymbol{d}$.
(b) Use vector notation to express $\boldsymbol{d}$ in terms of $\boldsymbol{w}$.

Ans:
(a)

$$\boldsymbol{w} = \begin{bmatrix} \boldsymbol{d} \\ \boldsymbol{d} \\ \boldsymbol{d} \\ \boldsymbol{d} \\ \boldsymbol{d} \\ \boldsymbol{d} \\ \boldsymbol{d} \end{bmatrix}$$

(b) $\boldsymbol{d} = w_{1:24}$

Question 2

Suppose the n -vector $\boldsymbol{x}$ is sparse, i.e., has only a few nonzero entries. Give a short sentence or two explaining what this means in each of the following contexts.
(a) $\boldsymbol{x}$ represents the daily cash flow of some business over n days.
(b) $\boldsymbol{x}$ represents the annual dollar value purchases by a customer of n products or services.
(c) $\boldsymbol{x}$ represents a portfolio, say, the dollar value holdings of n stocks.
(d) $\boldsymbol{x}$ represents a bill of materials for a project, i.e., the amounts of n materials needed.
(e) $\boldsymbol{x}$ represents a monochrome image, i.e., the brightness values of n pixels.
(f) $\boldsymbol{x}$ is the daily rainfall in a location over one year.

Ans:

(a) The business do not have operated in most of days.
(b) The customer only buys a few product per year.
(c) The person only hold a few stocks.
(d) The project only involve a few materials
(e) The image very dark.
(f) The location only has a few raining day for a year.

Question 3

In a computer game, let $\boldsymbol{a} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ be the initial coordinates of a jet fighter and $\boldsymbol{b} = \begin{pmatrix} 4 \\ 3 \\ 6 \end{pmatrix}$ be its final coordinates. What is the displacement vector?

Ans: $\boldsymbol{d} = \begin{pmatrix} 4 \\ 3 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$.

Question 4

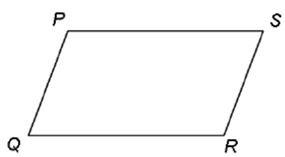
Suppose a car is at the position $\boldsymbol{a} = \begin{pmatrix} 6 \\ 7 \end{pmatrix}$. The car moves along the direction $\boldsymbol{d} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ per hour. After 3 hours, what is the final position of the car?

Ans:

$\boldsymbol{b} = \begin{pmatrix} 6 \\ 7 \end{pmatrix} + 3\begin{pmatrix} 3 \\ -2 \end{pmatrix}$

Question 5

In the figure, *PQRS* is a parallelogram and the coordinates of the four points are denoted as $\boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{R},$ and $\boldsymbol{S}$. Denote $\boldsymbol{d}_{P1,P2}$ be the displacement from point P1 to point P2.



Suppose $\boldsymbol{d}_{Pc} = \boldsymbol{u}$ and $\boldsymbol{d}_{PS} = \boldsymbol{v}$. Express each of the following vectors in terms of $\boldsymbol{u}$ and $\boldsymbol{v}$.

(a) $\boldsymbol{d}_{RQ}$, (b) $\boldsymbol{d}_{RS}$, (c) $\boldsymbol{d}_{PR}$, , (d) $\boldsymbol{d}_{QS}$

Ans:

(a) $\boldsymbol{d}_{RQ} = -\boldsymbol{d}_{PS} = -\boldsymbol{v}$,

(b) $\boldsymbol{d}_{RS} = -\boldsymbol{d}_{PQ} = -\boldsymbol{u}$,

(c) $\boldsymbol{d}_{PR} = \boldsymbol{d}_{PS} + \boldsymbol{d}_{SR} = \boldsymbol{d}_{PS} - \boldsymbol{d}_{RS} = \boldsymbol{u} + \boldsymbol{v}$,

(d) $\boldsymbol{d}_{QS} = \boldsymbol{d}_{QR} + \boldsymbol{d}_{RS} = \boldsymbol{v} - \boldsymbol{u}$

Question 6

Let $\boldsymbol{a} = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix}$ bet the coordinates of two points. Let $\boldsymbol{p}$ be a point on line segment between $\boldsymbol{a}$ and . Find $\boldsymbol{p}$ in each of the following cases.

(a) $\boldsymbol{p}$ be a point middle point between $\boldsymbol{a}$ and $\boldsymbol{b}$.

(b) $\|\boldsymbol{p} - \boldsymbol{a}\| : \|\boldsymbol{b} - \boldsymbol{p}\| = 2:3$

(a) $\boldsymbol{p} = (\frac{1}{2})(\boldsymbol{a} + \boldsymbol{b})$

(b) Note that $\boldsymbol{p} - \boldsymbol{a}$ and **$\boldsymbol{p} - \boldsymbol{a}$ are with same direction.**

=>$\|\boldsymbol{b} - \boldsymbol{a}\|$=$\|\boldsymbol{p} - \boldsymbol{a}\| + \|\boldsymbol{b} - \boldsymbol{p}\|$

Also,=>

=> $\frac{\|\boldsymbol{p}-\boldsymbol{a}\|}{\|\boldsymbol{b}-\boldsymbol{a}\|} = \mathbf{2}$

=>$\frac{\|\boldsymbol{b}-\boldsymbol{p}\|}{\|\boldsymbol{b}-\boldsymbol{a}\|} = \mathbf{3}$

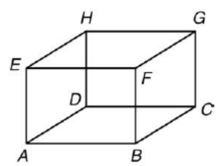
$\boldsymbol{p} = \boldsymbol{a} + (\frac{\|\boldsymbol{p}-\boldsymbol{a}\|}{\|\boldsymbol{b}-\boldsymbol{a}\|})(\boldsymbol{b} - \boldsymbol{a})$

Note that p is equal to $\boldsymbol{a}$ +displacement. Displacement = the length of $\|\boldsymbol{p} - \boldsymbol{a}\|$ times the unit vector $(\boldsymbol{b} - \boldsymbol{a})$. $(\frac{1}{\|\boldsymbol{b}-\boldsymbol{a}\|})(\boldsymbol{b} - \boldsymbol{a})$ is the unit vector**.**

$\boldsymbol{p} = \boldsymbol{a} + \left(\frac{2}{5}\right)(\boldsymbol{b} - \boldsymbol{a})$

Question 7

In the figure, *ABCDHEFG* is a right prism with parallelogram *ABCD* as its base. $\boldsymbol{d}_{P1,P2}$ be the displacement from point P1 to point P2.



(a) Show that $\boldsymbol{d}_{BA} + \boldsymbol{d}_{BC} + \boldsymbol{d}_{BF} = \boldsymbol{d}_{BH}$.

(b) The coordinates of the vertices are *A*=(4, 0, 3), *B*=(5, 3, 4), *C*=(1, 4, 5) and *F*=(4, 2, 8).

(i) Find the coordinates of *H*.

(ii) Find the lengths of *BH* and *OH*, where *O* is the origin.

Ans: $\boldsymbol{d}_{BH} = \boldsymbol{d}_{BC} + \boldsymbol{d}_{CG} + \boldsymbol{d}_{GH} = \boldsymbol{d}_{BC} + \boldsymbol{d}_{CG} + \boldsymbol{d}_{BA} = \boldsymbol{d}_{BC} + \boldsymbol{d}_{BF} + \boldsymbol{d}_{BA}$

Note that $\boldsymbol{d}_{GH} = \boldsymbol{d}_{BA}$ and $\boldsymbol{d}_{CG} = \boldsymbol{d}_{BF}$

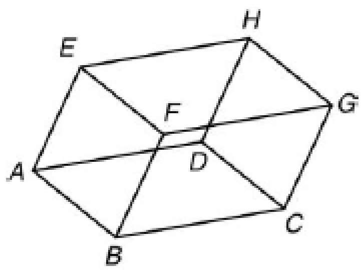
$$H = B + \boldsymbol{d}_{BH} = B + \boldsymbol{d}_{BC} + \boldsymbol{d}_{CG} + \boldsymbol{d}_{BA} = B + (C - B) + (G - C) + (A - B)$$

$$\boldsymbol{d}_{BH} = (C - B) + (G - C) + (A - B)$$

⇨ $\|\boldsymbol{d}_{BH}\| = \|(C - B) + (G - C) + (A - B)\|$

$$H = B + (C - B) + (G - C) + (A - B)$$
$$\|H\| = \|B + (C - B) + (G - C) + (A - B)\|$$

Question 8
In the figure, *ABCDHEFG* is a prism with parallelogram *ABCD* as its base.



(a) Show that $\boldsymbol{d}_{CB} + \boldsymbol{d}_{CD} + \boldsymbol{d}_{CG} = \boldsymbol{d}_{CE}$
(b) It is given that the coordinates of the vertices are B=(4, 6, 0), C=(7, 12,1), D=(5, 8, 3) and G=(11, 12, 0).
(i) Find CE .
(ii) Find the coordinates of E and a unit vector in the same direction of OE .

Ans: $\boldsymbol{d}_{CE} = \boldsymbol{d}_{CB} + \boldsymbol{d}_{BF} + \boldsymbol{d}_{FE} = \boldsymbol{d}_{CB} + \boldsymbol{d}_{CG} + \boldsymbol{d}_{FE} = \boldsymbol{d}_{BC} + \boldsymbol{d}_{CG} + \boldsymbol{d}_{CD}$

Note that $\boldsymbol{d}_{BF} = \boldsymbol{d}_{CG}$ and $\boldsymbol{d}_{FE} = \boldsymbol{d}_{CD}$

$$\boldsymbol{d}_{CE} = \boldsymbol{d}_{CB} + \boldsymbol{d}_{CD} + \boldsymbol{d}_{CG} = (B - C) + (D - C) + (G - C)$$

$$E = C + \boldsymbol{d}_{CB} + \boldsymbol{d}_{CD} + \boldsymbol{d}_{CG} = C + (B - C) + (D - C) + (G - C)$$

The unit vector is $\frac{E}{\|E\|}$

Question 9
Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two non-parallel vectors in 3D space. Let $\boldsymbol{c} = \alpha\boldsymbol{a} + \beta\boldsymbol{b}$ for some real constants α, β such that $\boldsymbol{c}^T\boldsymbol{a} = 0$ and $\boldsymbol{c}^T\boldsymbol{b} = 1$.
(a) Find α and β in terms of $\boldsymbol{a}^T\boldsymbol{a}$, $\boldsymbol{a}^T\boldsymbol{b}$, $\boldsymbol{a}^T\boldsymbol{a}$
(b) For any 3D vector $\boldsymbol{x}$ such that $\boldsymbol{x}^T\boldsymbol{a} = 0$ and $\boldsymbol{x}^T\boldsymbol{b} = 1$, prove that $\boldsymbol{x} - \boldsymbol{c}$ is perpendicular to **a** and **b**.

Ans:
$\boldsymbol{c}^T\boldsymbol{a} = 0$ => $\alpha\boldsymbol{a}^T\boldsymbol{a} + \beta\boldsymbol{b}^T\boldsymbol{a} = 0$
$\boldsymbol{c}^T\boldsymbol{b} = 1$ => $\alpha\boldsymbol{a}^T\boldsymbol{b} + \beta\boldsymbol{b}^T\boldsymbol{b} = 1$
The you can solve α, β

$(\boldsymbol{x} - \boldsymbol{c})^T\boldsymbol{a} = 0 - 0 = 0$
$(\boldsymbol{x} - \boldsymbol{c})^T\boldsymbol{b} = 1 - 1 = 0$

Question 10

Calculate the norms of the following vectors:

(a) $\begin{pmatrix}1\\2\end{pmatrix}$, (b) $\begin{pmatrix}-1\\3\\6\end{pmatrix}$, (c) $\begin{pmatrix}1\\1\\2\\8\end{pmatrix}$,

Ans: (a) sqrt(5), (b) sqrt(46), (c) sqrt(90)

Question 11

Find all unit vectors that are parallel to the vector $\begin{pmatrix}1\\2\\-4\end{pmatrix}$

Ans :

If $\boldsymbol{x} = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$ is parallel to $\begin{pmatrix}1\\2\\-4\end{pmatrix}$, then $\boldsymbol{x} = \lambda\begin{pmatrix}1\\2\\-4\end{pmatrix}$. The norm of $\begin{pmatrix}1\\2\\-4\end{pmatrix}$ is $\sqrt{21}$.

⇨ $\lambda = \frac{\pm 1}{\sqrt{21}}$. Hence $\boldsymbol{x} = \frac{\pm 1}{\sqrt{21}}\begin{pmatrix}1\\2\\-4\end{pmatrix}$

Question 12

Compute the inner product each of the following pairs of vectors:

(a) $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}1\\0\end{pmatrix}$ (b)$\begin{pmatrix}a\\b\end{pmatrix}$ and $\begin{pmatrix}b\\a\end{pmatrix}$, (c) $\begin{pmatrix}1\\2\\1\end{pmatrix}$ and $\begin{pmatrix}3\\-6\\2\end{pmatrix}$.

Ans: (a) 0, (b) 2ab, (c) -7.

Question 13

Compute the inner product each of the following pairs of vectors:

(a) $\begin{pmatrix}1\\2\end{pmatrix}$ and $\begin{pmatrix}3\\-1\end{pmatrix}$ (b) $\begin{pmatrix}1\\0\\-4\end{pmatrix}$ and $\begin{pmatrix}6\\1\\2\end{pmatrix}$, (c) $\begin{pmatrix}-2\\3\\0\\1\end{pmatrix}$, and $\begin{pmatrix}1\\2\\8\\-2\end{pmatrix}$

Ans:

(a) 3-2, .. … …. …

Question 14
Let **a, b** and **c** be three **unit vectors** in 3D space. The angles between **b** and **c**, **c** and **a**, **a** and **b** are α, β, γ respectively. Let **u** and **v** be two vectors defined by $\boldsymbol{u} = (\boldsymbol{a}^T\boldsymbol{b})\boldsymbol{c} - (\boldsymbol{c}^T\boldsymbol{a})\boldsymbol{b}$ and $\boldsymbol{v} = \boldsymbol{b} - (\boldsymbol{a}^T\boldsymbol{b})\boldsymbol{a}$

(i) Determine, in terms of α, β, γ, the magnitudes of $\boldsymbol{u}$ and $\boldsymbol{v}$.
(ii) Prove that both $\boldsymbol{u}$ and $\boldsymbol{v}$ are perpendicular to **a**.
(iii) Show that, if $\boldsymbol{u}$ and $\boldsymbol{v}$ are perpendicular to each other, then $\cos\beta = \cos\alpha\cos\ \gamma$

Ans:
(i)
$((\boldsymbol{a}^T\boldsymbol{b})\boldsymbol{c} - (\boldsymbol{c}^T\boldsymbol{a})\boldsymbol{b})^T((\boldsymbol{a}^T\boldsymbol{b})\boldsymbol{c} - (\boldsymbol{c}^T\boldsymbol{a})\boldsymbol{b}) = (\boldsymbol{a}^T\boldsymbol{b})^2 + (\boldsymbol{c}^T\boldsymbol{b})^2 - \boldsymbol{2}(\boldsymbol{a}^T\boldsymbol{b})(\boldsymbol{c}^T\boldsymbol{b})\ \boldsymbol{b}^T\boldsymbol{c}$
$= \cos^2\gamma + \cos^2\alpha - 2\cos\alpha\cos\beta\cos\gamma$
=> $\|\boldsymbol{u}\| = \sqrt{\cos^2\gamma + \cos^2\alpha - 2\cos\alpha\cos\beta\cos\gamma}$

Note that for two units vector a and b.
$\boldsymbol{a}^T\boldsymbol{b} = \cos\gamma\ \|\boldsymbol{b}\|\|\boldsymbol{a}\|$ $=\boldsymbol{a}^T\boldsymbol{b} = \cos\gamma$

Using the similar method,

=> $\|\boldsymbol{v}\| = \sin\gamma$.

(ii) Try $\boldsymbol{u}^T\boldsymbol{a}$ and $\boldsymbol{v}^T\boldsymbol{a}$
(iii) Try $\boldsymbol{u}^T\boldsymbol{v} = 0$, i.e., $((\boldsymbol{a}^T\boldsymbol{b})\boldsymbol{c} - (\boldsymbol{c}^T\boldsymbol{a})\boldsymbol{b})^T(\boldsymbol{b} - (\boldsymbol{a}^T\boldsymbol{b})\boldsymbol{a}) = 0$

Question 15
Given 3 points, *A*, *B*, *C*, such that $A = \binom{1}{2}, B = \binom{-6}{-2}, \boldsymbol{d}_{AB} - \boldsymbol{d}_{BC} = \binom{-12}{6}$

(a) Find $\boldsymbol{d}_{AB}$ and *C*.
(b) If *X* is a point such that $\boldsymbol{d}_{AX} = kX$
(i) What is *X* (in terms of k)
(ii) $\boldsymbol{d}_{OX}$ is $\perp$ $\boldsymbol{d}_{BX}$, find the value of *k* and hence find $\boldsymbol{d}_{AX} + \boldsymbol{d}_{BX} + \boldsymbol{d}_{CX}$
(iii) Furthermore, if *M* is the mid-point of *BC*, find $\boldsymbol{d}_{BC}$.

Ans:
(a) $\boldsymbol{d}_{AB}$=(-7,4),

$$C = B + \boldsymbol{d}_{BC} = B + \boldsymbol{d}_{AB} - \binom{-12}{6} = \binom{-1}{-12}$$

(b) $A + \boldsymbol{d}_{AX} = X$=>
$A + kX = X \Rightarrow X = \frac{1}{1-k}A$

Try $B + \boldsymbol{d}_{BX} = X$=>$\boldsymbol{d}_{BX} = X - B$, now try $X^T\boldsymbol{d}_{BX} = 0$
⇨ k=3/2

- ⇨ a zero vector
- ⇨ $\begin{pmatrix} -\frac{9}{2} \\ -9 \end{pmatrix}$

Question 16
Distance between Boolean vectors. Suppose that $\boldsymbol{x}$ and $\boldsymbol{y}$ are Boolean n-vectors, which means that each of their entries is either 0 or 1. What is their distance $\|\boldsymbol{x}-\boldsymbol{y}\|$?

Ans: squared root of the number of different elements in $\boldsymbol{x}$ and $\boldsymbol{y}$

Question 17
Norm identities. Verify that the following identities hold for any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ of the same size.

(a) $(\boldsymbol{a}-\boldsymbol{b})^{\boldsymbol{T}}(\boldsymbol{a}+\boldsymbol{b}) = \|\boldsymbol{a}\|^2 - \|\boldsymbol{b}\|^2$
(b) $\|\boldsymbol{a}+\boldsymbol{b}\|^2 + \|\boldsymbol{a}-\boldsymbol{b}\|^2 = 2\|\boldsymbol{a}\|^2 + 2\|\boldsymbol{b}\|^2$

This is called the parallelogram law.
Ans: from the basic definition of norm

Question 18
Show that

(a) $\boldsymbol{a} \perp \boldsymbol{b}$ if and only $\|\boldsymbol{a}+\boldsymbol{b}\|^2 = \|\boldsymbol{a}\|^2 + \|\boldsymbol{b}\|^2$
(b) Nonzero vectors a and b make an acute angle if and only $\|\boldsymbol{a}+\boldsymbol{b}\| > \sqrt{\|\boldsymbol{a}\|^2 + \|\boldsymbol{b}\|^2}$
(c) Nonzero vectors a and b make an obtuse angle if and only if if and only $\|\boldsymbol{a}+\boldsymbol{b}\| < \sqrt{\|\boldsymbol{a}\|^2 + \|\boldsymbol{b}\|^2}$

(a)
$\|\boldsymbol{a}+\boldsymbol{b}\|^2 = (\boldsymbol{a}+\boldsymbol{b})^{\boldsymbol{T}}(\boldsymbol{a}+\boldsymbol{b}) = \|\boldsymbol{a}\|^2 + \|\boldsymbol{b}\|^2 + \boldsymbol{2a^Tb}$

Forward If $\boldsymbol{a} \perp \boldsymbol{b}$ =>$\boldsymbol{a^Tb} = 0$=>$\|\boldsymbol{a}+\boldsymbol{b}\|^2 = (\boldsymbol{a}+\boldsymbol{b})^{\boldsymbol{T}}(\boldsymbol{a}+\boldsymbol{b}) = \|\boldsymbol{a}\|^2 + \|\boldsymbol{b}\|^2$

backward
As $\|\boldsymbol{a}+\boldsymbol{b}\|^2 = (\boldsymbol{a}+\boldsymbol{b})^{\boldsymbol{T}}(\boldsymbol{a}+\boldsymbol{b}) = \|\boldsymbol{a}\|^2 + \|\boldsymbol{b}\|^2 + \boldsymbol{2a^Tb}$**,** $\|\boldsymbol{a}+\boldsymbol{b}\|^2 = \|\boldsymbol{a}\|^2 + \|\boldsymbol{b}\|^2$ **=>**

$\boldsymbol{a^Tb} = 0$ **, =>** $\boldsymbol{a} \perp \boldsymbol{b}$.
(b)

$$\|\boldsymbol{a}+\boldsymbol{b}\|\boldsymbol{2} = \|\boldsymbol{a}\|^2 + \|\boldsymbol{b}\|^2 + \boldsymbol{2a^Tb}$$

If acute angle, $\boldsymbol{a^Tb} > 0$, => $\|\boldsymbol{a}+\boldsymbol{b}\|^2 > \|\boldsymbol{a}\|^2 + \|\boldsymbol{b}\|^2$ **=>** $\|\boldsymbol{a}+\boldsymbol{b}\| > \sqrt{\|\boldsymbol{a}\|^2 + \|\boldsymbol{b}\|^2}$

Now

$$\|\boldsymbol{a}+\boldsymbol{b}\|^2 = \|\boldsymbol{a}\|^2 + \|\boldsymbol{b}\|^2 + \boldsymbol{2a^Tb} => \boldsymbol{2a^Tb} = \|\boldsymbol{a}+\boldsymbol{b}\|^2 - \|\boldsymbol{a}\|^2 + \|\boldsymbol{b}\|^2$$

If $\|\boldsymbol{a}+\boldsymbol{b}\| > \sqrt{\|\boldsymbol{a}\|^2 + \|\boldsymbol{b}\|^2}$ **=>** $\|\boldsymbol{a}+\boldsymbol{b}\|^2 > \|\boldsymbol{a}\|^2 + \|\boldsymbol{b}\|^2$

=> $\boldsymbol{2a^Tb} = \|\boldsymbol{a}+\boldsymbol{b}\|^2 - \|\boldsymbol{a}\|^2 + \|\boldsymbol{b}\|^2 > 0$, =>$\boldsymbol{a^Tb} > 0$
(c) easy

Question 19
A Boolean n-vector $\boldsymbol{a}$ is one for which all entries are either 0 or 1. Such vectors are used to encode whether each of n conditions holds, with a_i = 1 meaning that condition i holds. Another common encoding of the same information uses the two values -1 and +1 for the entries. For example the Boolean vector (0, 1, 1, 0) would be written using this alternative encoding as (-1,+1,+1,-1). Suppose that $\boldsymbol{x}$ is a Boolean vector with entries that are 0 or 1, and $\boldsymbol{y}$ is a vector encoding the same information using the values -1 and +1. Express $\boldsymbol{y}$ in terms of $\boldsymbol{x}$ using vector notation. Also, express $\boldsymbol{x}$ in terms of $\boldsymbol{y}$ using vector notation.

$\boldsymbol{y} = 2\boldsymbol{x} - \boldsymbol{1}$
$\boldsymbol{x} = (\frac{1}{2})(\boldsymbol{y} + \boldsymbol{1})$

where $\boldsymbol{1}$ is an one vector

Modern CPU and GPU are suitable to handle vector operations, rather than write a for loop to perform the operation on each element one-by-one.
Writing in vector form let you to easily code the algorithm in modern CPU or GPU.

Question 20
Profit and sales vectors. A company sells n different products or items. The n -vector $\boldsymbol{p}$ gives the profit, in dollars per unit, for each of the n items. The n-vector $\boldsymbol{s}$ gives the total sales of each of the items, over some period (such as a month), i.e., s_i is the total number of units of item i sold. What is the total profit in terms of $\boldsymbol{p}$ and $\boldsymbol{s}$ using vector notation.

Ans: $\boldsymbol{p}^T\boldsymbol{s}$

Question 21
Symptoms vector. A 20-vector $\boldsymbol{s}$ records whether each of 20 different symptoms is present in a medical patient, with s_i = 1 meaning the patient has the symptom i and s_i = 0 meaning she does not. Express the following using vector notation. The total number of symptoms the patient has.

Ans: $\boldsymbol{1}^T\boldsymbol{s}$

Question 22
Total score from course record. The record for each student in a class is given as a 10-vector $\boldsymbol{r}$, where $r_1, \ldots, r_8$ are the grades for the 8 homework assignments, each on a 0-10 scale, r_9 is the midterm exam grade on a 0-120 scale, and r_{10} is final exam score on a 0-160 scale. The student's total course score s, on a 0-100 scale, is based 25% on the homework, 35% on the midterm exam, and 40% on the final exam. Express s in the form $\boldsymbol{s} = \boldsymbol{w}^T\boldsymbol{r}$. (That is, determine the 10-vector w.)

Ans: $w_1 = (\frac{100}{10}\frac{25}{8})$,... $w_8 = \left(\frac{100}{10}\frac{1}{8}\frac{25}{100}\right), w_9 = (\frac{100}{120}\frac{35}{100})$, $w_{10} = (\frac{40}{100})$

Question 23

Word count and word count histogram vectors. Suppose the n-vector $\boldsymbol{w}$ is the word count vector associated with a document and a dictionary of n words. For simplicity we will assume that all words in the document appear in the dictionary.

(a) What is $\mathbf{1}^T\boldsymbol{w}$?

(b) What does $w_{282} = 0$ mean?

(c) Let $\boldsymbol{h}$ be the n-vector that gives the histogram of the word counts, i.e., h_i is the fraction of the words in the document that are word i. Use vector notation to express $\boldsymbol{h}$ in terms of $\boldsymbol{w}$. (You can assume that the document contains at least one word.)

Ans:

(a) The total number of words in the document.
(b) The word count of the ith keyword is zero, or saying there is no the ith keyword in the document.
(c) $\frac{1}{(\mathbf{1}^T\boldsymbol{w})}\boldsymbol{w}$

Question 24

Total cash value. An international company holds cash in five currencies: USD (US dollar), RMB (Chinese yuan), EUR (euro), GBP (British pound), and JPY (Japanese yen), in amounts given by the 5-vector $\boldsymbol{c}$. For example, c_2 gives the number of RMB held. Express the total (net) value of the cash in USD, using vector notation. Be sure to give the size and define the entries of any vectors that you introduce in your solution. Your solution can refer to currency exchange rates.

r_i is exchange rate: it means that it takes r_i U.S. **dollars to** buy 1 currency i.

Total value=$\boldsymbol{r}^T\boldsymbol{c}$

Question 25

Linear combinations of linear combinations. Suppose that each of the vectors $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_k$ is a linear combination of the vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$, and $\boldsymbol{c}$ is a linear combination of $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_k$. Then c is a linear combination of $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$. Show this for the case with $k = m = 2$.

$\boldsymbol{c} = \alpha_1\boldsymbol{b}_1 + \alpha_2\boldsymbol{b}_2$
$\boldsymbol{b}_1 = \beta_{11}\boldsymbol{a}_1 + \beta_{12}\boldsymbol{a}_2$
$\boldsymbol{b}_2 = \beta_{21}\boldsymbol{a}_1 + \beta_{22}\boldsymbol{a}_2$

$\boldsymbol{c} = \alpha_1\beta_{11}\boldsymbol{a}_1 + \alpha_1\beta_{12}\boldsymbol{a}_2 + \alpha_2\beta_{21}\boldsymbol{a}_1 + \alpha_2\beta_{22}\boldsymbol{a}_2 = (\alpha_1\beta_{11} + \alpha_2\beta_{21})\boldsymbol{a}_1 + (\alpha_1\beta_{12} + \alpha_2\beta_{22})\boldsymbol{a}_2$

Quesiton 26
Nearest neighbor document. Consider the 5 Wikipedia pages.
What is the nearest neighbor of (the word count histogram vector of) `Veterans Day' among the others?

	Veterans Day	Memorial Day	Academy Awards	Golden Globe Awards	Super Bowl
Veterans Day	0	0.095	0.130	0.153	0.170
Memorial Day	0.095	0	0.122	0.147	0.164
Academy A.	0.130	0.122	0	0.108	0.164
Golden Globe A.	0.153	0.147	0.108	0	0.181
Super Bowl	0.170	0.164	0.164	0.181	0

Ans: Memiroal day

Question 27
Neighboring electronic health records. Let $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N$ be n-vectors that contain n features extracted from a set of N electronic health records (EHRs), for a population of N patients. (The features might involve patient attributes and current and past symptoms, diagnoses, test results, hospitalizations, procedures, and medications.) Briefly describe in words a practical use for identifying the 10 nearest neighbors of a given EHR (as measured by their associated feature vectors), among the other EHRs.

Ans: Let $\boldsymbol{x}_i$ be a given EHR, compute the $N-1$ distances, $\|\boldsymbol{x}_i - \boldsymbol{x}_{i'}\|$'s for all $i' \neq i$. Afterwards, find out the 10 smallest values from the $N-1$ distances.

Question 28
Angle between two nonnegative vectors. Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be two nonzero n-vectors with nonnegative entries. Show that the angle between x and y lies between 0 and $\frac{\pi}{2}$. When are x and y orthogonal?

Ans:

$$\cos\theta = \frac{\boldsymbol{x}^T\boldsymbol{y}}{\|\boldsymbol{x}\|\|\boldsymbol{y}\|}$$

Since $\boldsymbol{x}$ and $\boldsymbol{y}$ be two nonzero n-vectors with nonnegative entries, $\boldsymbol{x}^T\boldsymbol{y} > 0$.

$\boldsymbol{x}^T\boldsymbol{y} > 0$=>$\cos\theta$=> $\theta < \frac{\pi}{2}$.

Let A be the set of indices of x_i with nonzero value.
Let B be the set of indices of y_i with nonzero value

If A and B are with intersection, x and y orthogonal.

Question 29

Distance versus angle nearest neighbor. Suppose $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k$ is a collection of n-vectors, and $\boldsymbol{b}$ is another n-vector.

The vector $\boldsymbol{a}_j$ is the (distance) nearest neighbor of $\boldsymbol{b}$ (among the given vectors), if $\|\boldsymbol{b}-\boldsymbol{a}_j\| \leq \|\boldsymbol{b}-\boldsymbol{a}_{j'}\|$ for all j.

The vector $\boldsymbol{a}_j$ is the (angle) nearest neighbor of $\boldsymbol{b}$ (among the given vectors), if $\angle(\boldsymbol{a}_j, \boldsymbol{b}) \leq \angle(\boldsymbol{a}_{j'}, \boldsymbol{b})$ for all j.

(a) Give a simple numerical example where the (distance) nearest neighbor is not the same as the angle nearest neighbor.

(b) Now suppose that the vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k$ are normalized. Show that in this case the distance nearest neighbor and the angle nearest neighbor are always the same.

(a) Consider three vectors which norms are with large variation,

(b) $\|\boldsymbol{b}-\boldsymbol{a}_j\| = \sqrt{constant - 2\boldsymbol{b}^T\boldsymbol{a}_j}$

$\angle(\boldsymbol{a}_j, \boldsymbol{b}) = \cos^{-1}(\frac{\boldsymbol{a}_j{}^T\boldsymbol{b}}{\|\boldsymbol{b}\|})$

Since $\cos^{-1}$ is decreasing function, for any $\angle(\boldsymbol{a}_j, \boldsymbol{b})$ and $\angle(\boldsymbol{a}_{j'}, \boldsymbol{b})$,

If $\angle(\boldsymbol{a}_j, \boldsymbol{b}) < \angle(\boldsymbol{a}_{j'}, \boldsymbol{b})$ => $\boldsymbol{a}_j{}^T\boldsymbol{b} > \boldsymbol{a}_{j'}{}^T\boldsymbol{b}$ => $\|\boldsymbol{b}-\boldsymbol{a}_j\| < \|\boldsymbol{b}-\boldsymbol{a}_{j'}\|$

Complete by yourself ....

Question 30

Let $\boldsymbol{a} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

(a) Find all vectors $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t_1\boldsymbol{a} + t_2\boldsymbol{b}$

(b) Is the vector $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ one of the vectors your find in (a).

(a) $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} t_1 \\ t - t_1 \\ 2t_1 \end{pmatrix}$ for all t_1 and t_2

(b) $\boldsymbol{No}$

Question 31

Linear independence of stacked vectors. Consider the stacked vectors

$$\boldsymbol{c}_1 = \begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{b}_1 \end{pmatrix}, \dots, \boldsymbol{c}_k = \begin{pmatrix} \boldsymbol{a}_k \\ \boldsymbol{b}_k \end{pmatrix}$$

where $\boldsymbol{a}_1, \dots, \boldsymbol{a}_k$ are n-vectors, and where $\boldsymbol{b}, \dots, \boldsymbol{b}_k$ are m-vectors

(a) Suppose $\boldsymbol{a}_1, \dots, \boldsymbol{a}_k$ are linearly independent. Can we conclude that the stacked vectors $\boldsymbol{c}_1, \dots, \boldsymbol{c}_k$ are linearly independent?

(b) Now suppose that $\boldsymbol{a}_1, \dots, \boldsymbol{a}_k$ are linearly dependent. Can we conclude that the stacked vectors$\boldsymbol{c}_1, \dots, \boldsymbol{c}_k$ are linearly dependent?

Ans

(a) $\boldsymbol{a}_1, \dots, \boldsymbol{a}_k$ are linearly independent ⇔ any $\boldsymbol{a}_i$ cannot be a linearly weighted sum of $\boldsymbol{a}_1, \dots, \boldsymbol{a}_k$. Hence any $\boldsymbol{c}_i$ cannot be a linearly weighted sum of $\boldsymbol{c}_1, \dots, \boldsymbol{c}_k$.

(b) Cannot because $\boldsymbol{b}, \dots, \boldsymbol{b}_k$ can be linearly independent. Note that

Question 32

Norm of linear combination of orthonormal vectors. Suppose $\boldsymbol{a}_1, \dots, \boldsymbol{a}_k$ are orthonormal n-vectors, and $\_\boldsymbol{x} = \beta_1 \boldsymbol{a}_1 +, \dots, \beta_k \boldsymbol{a}_k$, where $\beta_1, \dots, \beta_k$ are scalars. Express $\|\boldsymbol{x}\|$ in terms of $\beta_1, \dots, \beta_k$

Ans: Consider $(\boldsymbol{x}^T \boldsymbol{x}) = \|\boldsymbol{x}\|^2$, and the fact $(\boldsymbol{a}_i^T \boldsymbol{a}_{i'}) = 0 \; for \; i \neq i'$ and $(\boldsymbol{a}_i^T \boldsymbol{a}_i) = 1$

........=>$\|\boldsymbol{x}\| = \sqrt{\sum_{i=1}^{k} \beta_i^2}$ **(Have important meaning in many applications)**

Question 33

Orthogonalizing vectors. Suppose that $\boldsymbol{a}$ and $\boldsymbol{b}$ are any n-vectors. Show that we can always find a scalar β that $(\boldsymbol{a} - \beta \boldsymbol{b}) \perp \boldsymbol{b}$, and that β is unique if $\boldsymbol{b} \neq \boldsymbol{0}$.

Ans: The condition for $\perp$ is $(\boldsymbol{a} - \beta \boldsymbol{b})^T \boldsymbol{b} = 0 \Rightarrow \boldsymbol{a}^T \boldsymbol{b} - \beta \|\boldsymbol{b}\|^2 = 0$

$\beta = \frac{\boldsymbol{a}^T \boldsymbol{b}}{\|\boldsymbol{b}\|^2}$ Thus, if $\boldsymbol{b} \neq \boldsymbol{0}$, we can always be able to a β.

Question 34

Show that the following vectors are linear independent?

$$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\-1\\0\end{pmatrix}$$

Question 35

Which of the following set of vectors are orthonormal ?

(a) $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\\0\end{pmatrix}$

(b) $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix}1\\1\\-2\end{pmatrix}$

(c) $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\end{pmatrix}$

(d) $\frac{1}{\sqrt{6}}\begin{pmatrix}2\\-1\\-1\end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix}-1\\2\\-1\end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix}-1\\-1\\2\end{pmatrix}$

Ans:

(a) No
(b) yes
(c) No
(d) No

Question 36

Give three vectors: $\boldsymbol{x}_1 = \begin{pmatrix}2\\-3\\1\end{pmatrix}, \boldsymbol{x}_2 = \begin{pmatrix}2\\3\\5\end{pmatrix}, \boldsymbol{x}_3 = \begin{pmatrix}-9\\4\\6\end{pmatrix}$,

(a) show that they are orthogonal

(b) Let $\boldsymbol{v} = \begin{pmatrix}2\\-3\\1\end{pmatrix}$. Compute the projection of $\boldsymbol{v}$ onto the directions $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$, respectively.

(c) Is it $\boldsymbol{v}$ equal to th sum of the three projected vectors?

Ans:

Easy