Assignment 1: Solution

Q1. (10%) Simplify $\frac{2\sqrt{7}-\sqrt{5}}{\sqrt{7}+\sqrt{5}}$ such that the denominator consists of an integer only.

Solution:

$$\frac{2\sqrt{7}-\sqrt{5}}{\sqrt{7}+\sqrt{5}} = \frac{(2\sqrt{7}-\sqrt{5})(\sqrt{7}-\sqrt{5})}{(\sqrt{7}+\sqrt{5})(\sqrt{7}-\sqrt{5})} = \frac{14-3\sqrt{35}+5}{7-5} = \frac{19-3\sqrt{35}}{2}$$

Q2. (15%) $A = \{red, green, blue\}$, $B = \{red, yellow, orange\}$,

$C = \{red, orange, yellow, green, blue, purple\}$. Find the following:

a. (5%) $A \cup B$
b. (5%) $A \cap B$
c. (5%) $A^C \cap C$

Solution:

a. $A \cup B = \{red, green, blue, yellow, orange\}$
b. $A \cap B = \{red\}$
c. $A^C \cap C = \{orange, yellow, purple\}$

Q3. (10%) Suppose $A = \{cow, horse\}$, $B = \{egg, juice\}$, $H = \{cat, dog, rabbit, mouse\}$, $F = \{dog, cow, duck, pig, rabbit\}$, $W = \{duck, rabbit, deer, frog, mouse\}$.

a. (5%) Find cartesian product: $A \times B$.

b. (5%) Use Venn diagram to illustrate $(H \cap F)^C \cap W$.

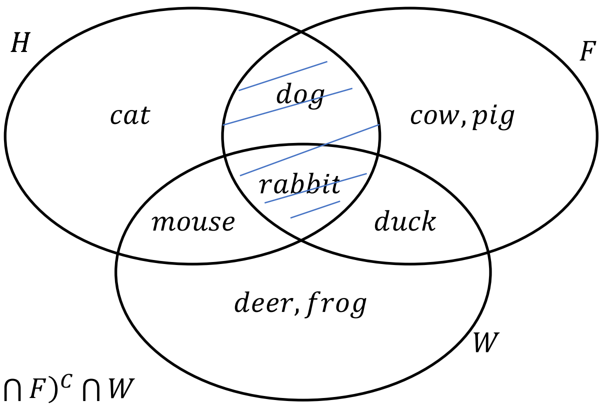
Solution:

(a)

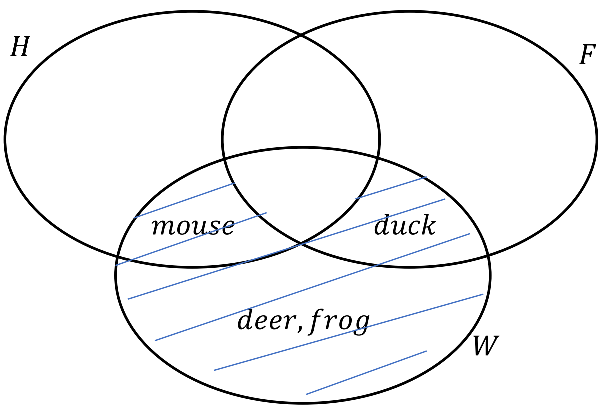
	egg	$jucie$
cow	(cow, egg)	$(cow, jucie)$
$horse$	$(horse, egg)$	$(horse, juice)$

$$A \times B = \{(cow, egg), (horse, juice), (cow, jucie), (horse, egg)\}$$

(b)



$(H \cap F)^C \cap W$



Q4.

a. (5%) Write the following sets in the set-builder form:

$$A = \{3,15,35,63,99,143,195,255\}$$

Solution:

$$\mathrm{A} = \{(2x)^2 - 1 | \mathrm{x} \in \boldsymbol{Z}, 1 \leq x \leq 8\}$$

$\boldsymbol{Z}$ is the set of Integer

b. (5%) Find the set A, $A = \{x \in \boldsymbol{R} | x = x^2\}$.

Solution:

$$A = \{0,1\}$$

Q5. (15%) Determine if the follow functions are injective, surjective, or bijective.

a. (5%) $f: R \to R, f(x) = x^2$
b. (5%) $f: N \to N, f(x) = x + 2$
c. (5%) $f: R \to R, f(x) = 2x - 3$

Solution:

a. Not injective nor surjective.

Counterexample of Injective, when $y = x^2 = 1$ is in codomain R, $x = 1$ or $x = -1$ is in domain R, where x is not distinct.

Counterexample of Surjective, when $y = x^2 = -1$ is in codomain R, $x = i$ or $x = -i$ is not in domain R.

b. Injective.

Counterexample of Surjective, when $y = x + 2 = 1$ is in codomain N, $x = -1$ is not in domain N.

c. Bijective

Proof c:

If $f(x_1) = f(x_2)$ then $2x_1 - 3 = 2x_2 - 3 \Rightarrow \mathrm{x}_1 = \mathrm{x}_2$. Hence injective.

$2x - 3 = y$, so $x = \frac{y+5}{3}$, which belongs to R and $f(x) = y$. Hence surjective.

$$\therefore \text{Injective \& Surjective}$$

$$\therefore Bijective$$

Q6. (20%)

a. (10%) $f(x) = 2x + 3, g(x) = -x^2 + 5$. Find $(g \circ f)(x)$.
b. (10%)$f(x) = \frac{3}{5}x + 4, g(x) = 2x^2 - 5x + 9$. Find $(f \circ g)\left(\frac{1}{2}\right)$.

Solution:

a. $(g \circ f)(x) = -(2x + 3)^2 + 5$

$$= -4x^2 - 12x - 9 + 5$$
$$= -4x^2 - 12x - 4$$

b. $(f \circ g)(x) = \frac{3}{5}(2x^2 - 5x + 9) + 4$

$(f \circ g)(x) = \frac{6x^2}{5} - 3x + \frac{27}{5} + 4$

$(f \circ g)\left(\frac{1}{2}\right) = \frac{6}{5\times4} - \frac{3}{2} + \frac{27}{5} + 4 = \frac{6}{20} - \frac{3}{2} + \frac{27}{5} + 4 = \frac{6-30+108+80}{20}$

$(f \circ g)\left(\frac{1}{2}\right) = \frac{164}{20} = \frac{41}{5}$

Q7. (10%) Define $f, g: R \to R, f(x) = 3^x, g(x) = x^3$. Prove g is surjective and f is not surjective.

("onto") $\forall y \in Y, \exists x \in X,$ such that $y = f(x)$.

Proof:

Since $x \in R$, then 3^x is always positive.

But there are some $b \leq 0$, when b is the co-domain of f.

$\therefore$ f is not surjective.

On the other hand, for any $b \in R$, the $b = g(x)$ has a solution (namely x $= \sqrt[3]{b}$), so b has a preimage under g.

$\therefore$ g is surjective.

Q8. (10%) Use contrapositive proof to prove: If x and $y \in Z$, $x + y$ is even, then x and y have the same parity (either both are even, or both are odd).

Proof:

Contrapositive.

Prove If both x and y do not have the same parity, then $x + y$ is odd.

Assume: x is odd and y is even.

Then $\exists m \in Z$, such that $x = 2m + 1$

$\exists n \in Z$, such that $y = 2n$

$\therefore x + y = (2m + 1) + 2n = 2(m + n) + 1$

$\therefore x + y$ must be odd.

Proved. ■