EXE 3

Question 1

Suppose $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$ are n-vectors. Determine whether each expression below makes sense (i.e., uses valid notation). If the expression does make sense, give its dimensions.

(a) $\begin{bmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \vdots \\ \boldsymbol{a}_n \end{bmatrix}$

(b) $\begin{bmatrix} \boldsymbol{a}_1^{\mathrm{T}} \\ \boldsymbol{a}_2^{\mathrm{T}} \\ \vdots \\ \boldsymbol{a}_n^{\mathrm{T}} \end{bmatrix}$

(c) $[\boldsymbol{a}_1 \quad \boldsymbol{a}_2 \quad \cdots \quad \boldsymbol{a}_n]$

(d) $[\boldsymbol{a}_1^{\mathrm{T}} \quad \boldsymbol{a}_2^{\mathrm{T}} \quad \cdots \quad \boldsymbol{a}_n^{\mathrm{T}}]$

(e) $[\boldsymbol{a}_1^{\mathrm{T}} \quad \boldsymbol{a}_2 \quad \cdots \quad \boldsymbol{a}_n^{\mathrm{T}}]$

Ans:

(a) Make sense, the matrix is $nm \times 1$
(b) Make sense, the matrix is $n \times m$
(c) Make sense, the matrix is $m \times n$
(d) Make sense, the matrix is $1 \times nm$
(e) Not make sense

Question 2

Suppose the block matrix

$$\begin{bmatrix} \boldsymbol{A} & \boldsymbol{I} \\ \boldsymbol{I} & \boldsymbol{C} \end{bmatrix}$$

makes sense, where $\boldsymbol{A}$ is a $p \times q$ matrix. What are the dimensions of $\boldsymbol{C}$?

Ans: The upper right identity matrix is $p \times p$

The lower left identity matrix is $q \times q$

Hence $\boldsymbol{C}$ is $q \times p$

The matrix is $(p + q) \times (p + q)$

Question 3

Assuming the matrix

$$\boldsymbol{K} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{A}^{\mathrm{T}} \\ \boldsymbol{A} & \emptyset \end{bmatrix}$$

makes sense, which of the following statements must be true? (`Must be true' means that it follows with no additional assumptions.)
(a) $\boldsymbol{K}$ is square.
(b) $\boldsymbol{A}$ is square or wide.
(c) $\boldsymbol{K}$ is symmetric.
(d) The identity and zero submatrices in $\boldsymbol{K}$ have the same dimensions.
(e) The zero submatrix is square.

(a) Yes

(b) No

(c) Y

(d) N

(e) Y

Question 4

Let $\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} 1 & 0 & 2 \\ -2 & 3 & -1 \end{bmatrix}$

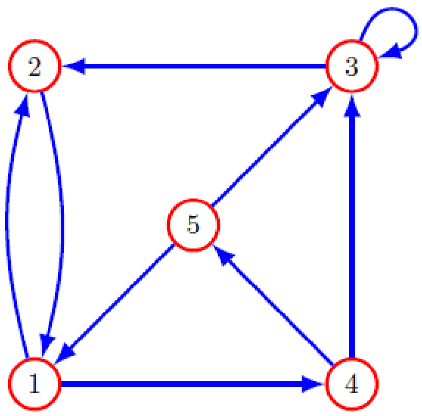
(a) Find $\boldsymbol{AB}^{\mathrm{T}}$ and $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{B}$
(b) For each of the $\boldsymbol{AB}^{\mathrm{T}}$ and $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{B}$**,** determine whether it is invertible, and find its inverse if it is invertible?

(a) $\boldsymbol{AB}^{\mathrm{T}} = \begin{bmatrix} 3 & -3 \\ 0 & 3 \end{bmatrix}$, $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{B} = \begin{bmatrix} 1 & 0 & 2 \\ -2 & 3 & -1 \\ 1 & 0 & 2 \end{bmatrix}$

(b) The inverse of $\boldsymbol{AB}^{\mathrm{T}} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

The det of $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{B} = 0$ => no inverse

Question 5
Suppose $\boldsymbol{A}$ is the adjacency matrix of a directed graph.



What are the entries of the vector $\boldsymbol{A}\mathbf{1}$? What are the entries of the vector $\boldsymbol{A}^{\mathrm{T}}\mathbf{1}$?

Ans:

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$[\boldsymbol{A}\mathbf{1}]_i$ is the number of edges leaving node i (out degree)

$[\boldsymbol{A}^{\mathrm{T}}\mathbf{1}]_i$ is the number of edges entering node i (in degree)

Question 6
For each of the following matrixes, describe in words how in words how $\boldsymbol{x}$ and $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$ are related. In each case $\boldsymbol{x}$ and $\boldsymbol{y}$ are n vectors, with $n = 3k$.

(a) $\boldsymbol{A} = \begin{bmatrix} \emptyset & \emptyset & \boldsymbol{I}_k \\ \emptyset & \boldsymbol{I}_k & \emptyset \\ \boldsymbol{I}_k & \emptyset & \emptyset \end{bmatrix}$

(b) $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{E} & \emptyset & \emptyset \\ \emptyset & \boldsymbol{E} & \emptyset \\ \emptyset & \emptyset & \boldsymbol{E} \end{bmatrix}$, where is a $k \times k$ matrix with all entries equal to $\frac{1}{k}$.

(a) $\boldsymbol{y}$ is a swap version of $\boldsymbol{x}$. The overall operation is to swap the first 1/3 part of $\boldsymbol{x}$ **with** the last 1/3 part of $\boldsymbol{x}$**.**

(b) Entries of the first 1/3 part of $\boldsymbol{y}$ are the mean **of** the first 1/3 part of $\boldsymbol{x}$**. …. ….**

Question 7
We consider a set of n currencies, labeled $1, \ldots, n$. (These might correspond to USD, RMB, EUR, and so on.). At a particular time the exchange or conversion rates among the n currencies are given by an $n \times n$ (exchange rate) matrix $\boldsymbol{R}$, where r_{ij} is the amount of currency i that you can buy for one unit of currency j. The exchange rates include commission charges, so we have $r_{ij}r_{ji}$ < 1 for all i 6= j. You can assume that $r_{ii} = 1$.
Suppose $\boldsymbol{y} = \boldsymbol{Rx}$, where $\boldsymbol{x}$ is a vector (with nonnegative entries) that represents the amounts of the currencies that we hold. What is y_i? Your answer should be in English.

Ans: This is the total amount of equivalent currency i that we hold.

Question 8

(a) Work out the complexity of computing the m -vector $\boldsymbol{Ax}$, where $\boldsymbol{A}$ is an $m \times n$ matrix and $\boldsymbol{x}$ is an n –vector. (The number of additions and the number of multiplications)

(b) Consider that we would like to compute computing the $\boldsymbol{y} = \boldsymbol{ABx}$, where $\boldsymbol{A}$ and $\boldsymbol{B}$ are $n \times n$ matrices and $\boldsymbol{x}$ is an n –vector. There are two methods to compute $\boldsymbol{y}$**.**
The first method is to comput**e** $\boldsymbol{Bx}$ first, while the second method is to compute $\boldsymbol{AB}$ first. Which method is better? Explain.

(a)
Each entry of **:** n multiplications and $n - 1$ additions
Total **:** mn multiplications and $m(n-1)$ additions

(b) The second method.

Question 9

Patients and symptoms. Each of a set of N patients can exhibit any number of a set of n symptoms. We express this as an $N \times n$ matrix $\boldsymbol{S}$, with

$$s_{ij} = \begin{cases} 1 & \text{patient } i \text{ exhibits sumptom } j \\ 0 & \text{patient } i \text{ does not exhibits sumptom } j \end{cases}$$

Give simple English descriptions of the following expressions. Include the dimensions, and describe the entries.
(a) $\boldsymbol{S1}$.
(b) $\boldsymbol{S}^{\mathrm{T}}\boldsymbol{1}$.

Ans; (a) the i element => the number of symptoms that patient i has.
(b) The j element => the number of patients having symptom j.

Question 10

We consider m students, n classes, and p majors. Each student can be in any number of the classes , and can have any number of the majors (although the common values would be 0, 1, or 2). The data about the students' classes and majors are given by an $m \times n$ matrix $\boldsymbol{C}$ and an $m \times p$ matrix $\boldsymbol{M}$, where

$$C_{ij} = \begin{cases} 1 & \text{student } i \text{ is class } j \\ 0 & \text{student } i \text{ is not class } j \end{cases}$$

$$M_{ik} = \begin{cases} 1 & \text{student } i \text{ is in major } k \\ 0 & \text{student } i \text{ is not in major } k \end{cases}$$

(a) Let $\boldsymbol{e}$ be the n -vector with e_j being the enrollment in class j. Express $\boldsymbol{e}$ using matrix notation, in terms of the matrices $\boldsymbol{C}$ and/or $\boldsymbol{M}$.

(b) Define the $n \times p$ matrix $\boldsymbol{S}$ where C_{jk} is the total number of students in class j with major k. Express $\boldsymbol{S}$ using matrix notation, in terms of the matrices $\boldsymbol{C}$ and $\boldsymbol{M}$.

Ans:

(a) $e_j = \sum_{i=1}^{m} C_{ij}$ => $\boldsymbol{e} = \boldsymbol{C}^T \boldsymbol{1}$

(b) $S_{jk} = \sum_{i=1}^{m} C_{ij} M_{ik}$=> $\boldsymbol{S} = \boldsymbol{C}^T \boldsymbol{M}.$

Question 11

A student says that for any square matrix $\boldsymbol{A}$,

$(\boldsymbol{A} + \boldsymbol{I})^3 = \boldsymbol{A}^3 + 3\boldsymbol{A}^2 + 3\boldsymbol{A} + \boldsymbol{I}$

Is she right? If she is, explain why; if she is wrong, give a counterexample, i.e., a square matrix $\boldsymbol{A}$ for which it does not hold.

Ans: yes , expand the result

Question 12

A student says that for any square matrices $\boldsymbol{A}$ and $\boldsymbol{B}$

$(\boldsymbol{A} + \boldsymbol{B})^3 = \boldsymbol{A}^3 + 3\boldsymbol{A}^2\boldsymbol{B} + 3\boldsymbol{A}\boldsymbol{B}^2 + \boldsymbol{I}$

Is she right? If she is, explain why; if she is wrong, explain.

Ans: No, in general $\boldsymbol{AB} \neq \boldsymbol{BA}$

The expression has eight terms

Question 13

True or false, For square matrix, with a reason if true or a counterexample if false:

(a) The determinant of $\boldsymbol{I}+\boldsymbol{A}$ is $1+\ |\boldsymbol{A}|$**.**

(b) The determinant of $\boldsymbol{ABC}$ is $|\boldsymbol{A}||\boldsymbol{B}||\boldsymbol{C}|$**.**

(c) The determinant of $4\boldsymbol{A}$ is $\boldsymbol{4}|\boldsymbol{A}|$**.**

(d) The determinant **of** $\boldsymbol{AB}-\boldsymbol{BA}$ is zero. Tray an example with $\boldsymbol{A}=\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{1}\end{bmatrix}$

(a) False

(b) True

(c) $|4\boldsymbol{A}|$ is $4^n|\boldsymbol{A}|$

(d) False: $\boldsymbol{A}=\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{1}\end{bmatrix}$, $\boldsymbol{A}=\begin{bmatrix}\mathbf{0} & \mathbf{1}\\ \mathbf{1} & \mathbf{0}\end{bmatrix}$, $\boldsymbol{AB}-\boldsymbol{BA}=\begin{bmatrix}\mathbf{0} & \mathbf{1}\\ \mathbf{1} & \mathbf{0}\end{bmatrix}$ **is invertible.**

Question 14

Do these matrices have determinant 0, 1, 2, or 3"

$$\boldsymbol{A}=\begin{bmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}\quad \boldsymbol{B}=\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{bmatrix}\quad \boldsymbol{C}=\begin{bmatrix}\mathbf{1} & \mathbf{1} & \mathbf{1}\\ \mathbf{1} & \mathbf{1} & \mathbf{1}\\ \mathbf{1} & \mathbf{1} & \mathbf{1}\end{bmatrix}$$

(a) $|\boldsymbol{A}|$ **is 1. (if we multiply a vector to A, change the position of the vector. Permutation matrix)**

(b) $|\boldsymbol{B}|$ **is 2**

(c) **Zero. Note that if the row vectors (column vectors) are linearly dependent => determinant is zero.**

Question 15

The inverse of 2-by-2 matrix seems to have determinant =1:

$$|\boldsymbol{A}^{-1}|=\left|\frac{1}{ad-bc}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}\right|=\frac{ad-bc}{ad-bc}=1$$

What is wrong with this calculation? What is the correct $|\boldsymbol{A}^{-1}|$**?**

$$|\boldsymbol{A}^{-1}|=\left|\frac{1}{ad-bc}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}\right|=\frac{1}{(ad-bc)^2}\left|\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}\right|=\frac{1}{ad-bc}$$

Question 16
Find the determinants of $\boldsymbol{A}$, $\boldsymbol{A^{-1}}$, and $\boldsymbol{A}-\lambda\boldsymbol{I}$

$$\boldsymbol{A}=\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix},\ \boldsymbol{A^{-1}}=\frac{\boldsymbol{1}}{\boldsymbol{3}}\begin{bmatrix}2 & -1\\-1 & 2\end{bmatrix},\ \boldsymbol{A}-\lambda\boldsymbol{I}=\begin{bmatrix}2-\lambda & 1\\1 & 2-\lambda\end{bmatrix}$$

Which number of λ lead to $|\boldsymbol{A}-\lambda\boldsymbol{I}|$=0?

3
1/3
1 and 3.
$$\boldsymbol{A}-\boldsymbol{I}=\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}, \boldsymbol{A}-\boldsymbol{3I}=\begin{bmatrix}-1 & 1\\1 & -1\end{bmatrix}$$

Question 17
Let $\boldsymbol{A}$ be the 5×5 matrix:

$$\boldsymbol{A}=\begin{bmatrix}0 & 0 & 0 & 0 & 1\\1 & 0 & 0 & 0 & 0\\0 & 1 & 0 & 0 & 0\\0 & 0 & 1 & 0 & 0\\0 & 0 & 0 & 1 & 0\end{bmatrix}$$

(a) How is $\boldsymbol{Ax}$ related to $\boldsymbol{x}$? Your answer should be in English.
(b) What is $\boldsymbol{A}^5$? Hint. The answer should make sense, given your answer to part (a).

Ans:

(a) moving the final entry to the first position, while shifting all other entries down one position,
(b) The identity matrix

Question 18
Find the inverses of the matrices
$$\boldsymbol{A}=\begin{pmatrix}2 & 3\\1 & 2\end{pmatrix}, \boldsymbol{A}=\begin{pmatrix}6 & 4\\2 & 3\end{pmatrix},$$

Ans:
$$\boldsymbol{A}^{-1}=\begin{pmatrix}2 & -3\\-1 & 2\end{pmatrix}$$

$$\boldsymbol{A}^{-1}=\frac{\boldsymbol{1}}{\boldsymbol{12}}\begin{pmatrix}3 & -4\\-2 & 6\end{pmatrix}$$

Question 19
Prove that if $\boldsymbol{A}$ and $\boldsymbol{B}$ are same-sized invertible matrices , then $(\boldsymbol{AB})^{-1}$ = $\boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$.

= $\boldsymbol{B}^{-1}\boldsymbol{A}^{-1}\boldsymbol{AB} = \boldsymbol{I}$

That means AB is inverse of $\boldsymbol{B}^{-1}\boldsymbol{A}^{-1} => (\boldsymbol{AB})^{-1}$ = $\boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$

Question 20
Prove that if a matrix is invertible, then its inverse is unique.

Let A be invertible, suppose B and C are its inverse,
B=BI=B(AC)=(BA)C=C
That means B=C => unique.

Question 21
$\boldsymbol{A}$ and $\boldsymbol{B}$ are matrices. Prove that $(\boldsymbol{AB})^T$ = $\boldsymbol{B}^T\boldsymbol{A}^T$.

$(\boldsymbol{AB})_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$

Note that $(\boldsymbol{A})_{ik} = a_{ik}$

$(\boldsymbol{A}^T)_{ik} = a_{ki}$

Take transpose first

$((\boldsymbol{AB})^T)_{ij} = (\boldsymbol{AB})_{ji} = \sum_{k=1}^{n} a_{jk}b_{ki}$

$(\boldsymbol{B}^T\boldsymbol{A}^T)_{ij} = \sum_{k=1}^{n} \ (\boldsymbol{B}^T)_{ik}(\boldsymbol{A}^T)_{kj} = \sum_{k=1}^{n} \ b_{ki}a_{jk}$

Question 22
Prove that if $\boldsymbol{A}$ and $\boldsymbol{A}^T$ are invertible, then $(\boldsymbol{A}^T)^{-1} = (\boldsymbol{A}^{-1})^{\boldsymbol{T}}$.

$(\boldsymbol{AA}^{-1})^{\boldsymbol{T}} = (\boldsymbol{A}^{-1})^{\boldsymbol{T}}\boldsymbol{A}^{\boldsymbol{T}} = \boldsymbol{I}$

That means $(\boldsymbol{A}^{-1})^{\boldsymbol{T}}$ is inverse of $\boldsymbol{A}^{\boldsymbol{T}}$

=>$(\boldsymbol{A}^T)^{-1} = \ (\boldsymbol{A}^{-1})^{\boldsymbol{T}}$

Question 23
Let $\boldsymbol{A}$ and $\boldsymbol{B}$ are 2×2 matrix. If $\boldsymbol{A}\ \boldsymbol{B} = \boldsymbol{2I}$**,** show that
$|\boldsymbol{A}| = \frac{4}{|\boldsymbol{B}|}$

Ans: $|\boldsymbol{A}\ \boldsymbol{B}| = 4 \Rightarrow |\boldsymbol{A}||\boldsymbol{B}| = 4$

Question 24

For the following matrix find $\boldsymbol{A}^{-1}$ **by trail and error (with 1's and 0's in entries).**

(a) $\boldsymbol{A} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ (b) $\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

Ans:

(a) $\boldsymbol{A}^{-1} = \boldsymbol{A}$
(b) $\boldsymbol{A}^{-1} = \boldsymbol{A}^{\boldsymbol{T}}$

Is it always $\boldsymbol{A}^{-1} = \boldsymbol{A}^{\boldsymbol{T}}$ **?**

Question 25

(a) If $\boldsymbol{A}$ is invertible and $\boldsymbol{AB} = \boldsymbol{AC}$, prove that $\boldsymbol{B} = \boldsymbol{C}$
(b) If $\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, find two different matrices such that $\boldsymbol{AB} = \boldsymbol{AC}$**.**

Ans: $\boldsymbol{A}^{-1}\boldsymbol{AB} = \boldsymbol{A}^{-1}\boldsymbol{AC} \Rightarrow \boldsymbol{B} = \boldsymbol{C}$

$\boldsymbol{AB} = \boldsymbol{AC} \Leftrightarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} (\boldsymbol{B} - \boldsymbol{C}) = \emptyset$,

$(\boldsymbol{B} - \boldsymbol{C})$=$\begin{pmatrix} -x & -y \\ x & y \end{pmatrix}$

Question 26

Suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ are square matrix.

True or false (given a reason if true or a 2-by2 example if false

(a) If $\boldsymbol{A}$ is not invertible, then $\boldsymbol{AB}$ is not invertible.
(b) $|\boldsymbol{A} - \boldsymbol{B}|$=$|\boldsymbol{A}| - |\boldsymbol{B}|$
(c) $\boldsymbol{AB}$ and $\boldsymbol{BA}$ has the same determinant.

(a) True $|\boldsymbol{AB}| = |\boldsymbol{A}||\boldsymbol{B}| \Rightarrow |\boldsymbol{AB}| = 0$, inverse does not exist.
(b) False use the definition of determinant of 2-by-2
(c) True

Question 27
What is wrong with this proof that $|\boldsymbol{P}| = 1$?
$\boldsymbol{P} = \boldsymbol{A}(\boldsymbol{A^T}\boldsymbol{A})^{-\mathbf{1}}\boldsymbol{A^T}$ so $|\boldsymbol{P}| = |\boldsymbol{A}|\frac{\mathbf{1}}{|\boldsymbol{A^T}||\boldsymbol{A}|}|\boldsymbol{A^T}|$

Note that A may not be a square matrix.

Question 28
Apply the elimination method to solve

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix} \boldsymbol{x} = \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$$

How do you know this system has no solution?
Change the number 6 so there is(are) solution(s)

After the elimination, the last row is 0=3 ($0x_3 = 3$), contradiction. No solution.
Change 6 to zero. Then infinite solutions.

Question 29
Solve the following system with elimination method.

$$\begin{aligned} 2x_1 + 4x_2 - 2x_3 &= 2 \\ 4x_1 + 9x_2 - 3x_3 &= 8 \\ -2x_1 - 3x_2 + 7x_3 &= 10 \end{aligned}$$

Ans

$$\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} \boldsymbol{x} = \begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix}$$

$$\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ 0 & 1 & 5 \end{bmatrix} \boldsymbol{x} = \begin{pmatrix} 2 \\ 8 \\ 12 \end{pmatrix}$$

$$\begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 1 & 5 \end{bmatrix} \boldsymbol{x} = \begin{pmatrix} 2 \\ 4 \\ 12 \end{pmatrix}$$

$$\begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix} \boldsymbol{x} = \begin{pmatrix} 2 \\ 4 \\ 8 \end{pmatrix}$$

$$\boldsymbol{x} = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$$

Question 30

Choose the number of a, b, c, d in the following system so that there is
(a) No solution
(b) Infinitely many solution

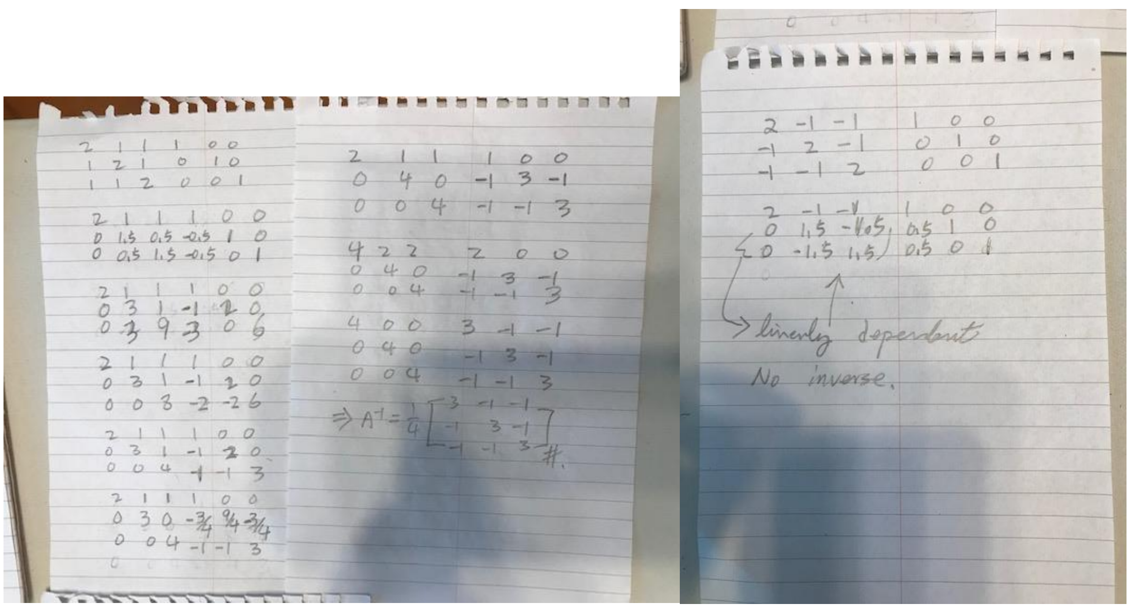
$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & d \end{bmatrix} \boldsymbol{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

(a) No solution if $d = 0$ and $c \neq 0$
(b) Both of them equal to zero

Question 31

Find $\boldsymbol{A^{-1}}$ **and** $\boldsymbol{B^{-1}}$ (if exists) by elimination

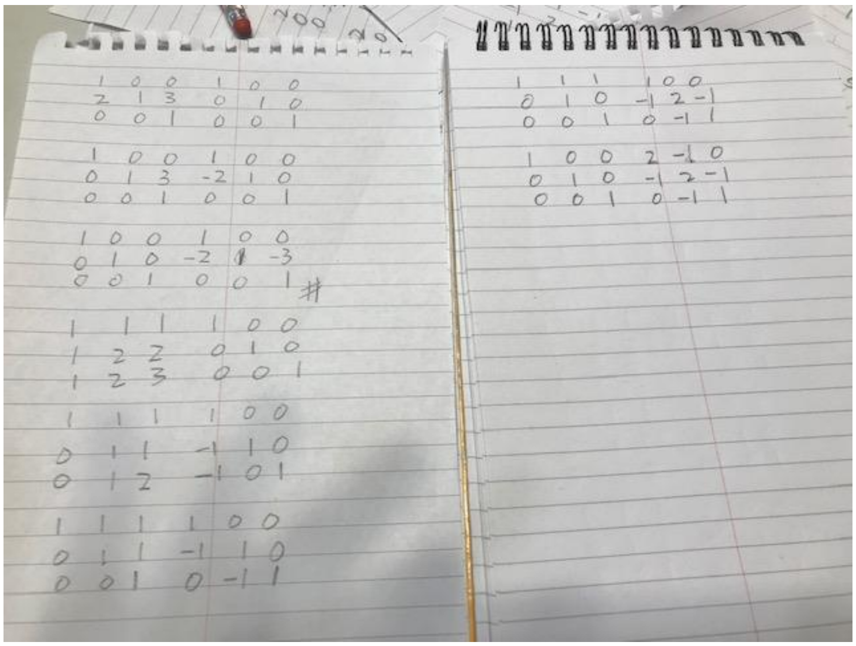
$$\boldsymbol{A} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$



Question 32

Find $\boldsymbol{A}^{-1}$ **and** $\boldsymbol{B}^{-1}$ (if exists) by elimination

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$



Question 33

True or false

(a) A 4-by-4 matrix with a row of zeros is not invertible.
(b) Every matrix with 1's in the main diagonal is invertible.
(c) If $\boldsymbol{A}$ **is** invertible then $\boldsymbol{A}^{-1}$ and $\boldsymbol{A}^{2}$ are invertible.

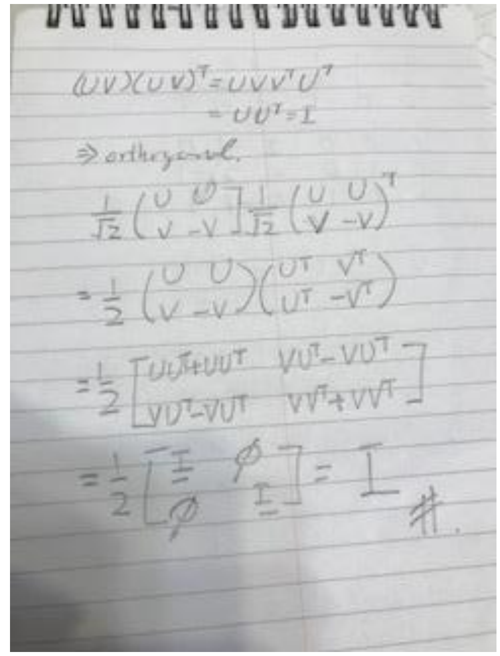
(a) True if If $\boldsymbol{A}$ is with a row of zeros**,** then $\boldsymbol{AB}$ is with a row of zeros too. Not possible to have $\boldsymbol{AB} = \boldsymbol{I}$
(b) False a one matrix is not invertible.
(c) True . The inverse of $\boldsymbol{A}^{-1}$ is $\boldsymbol{A}$**.** The inverse of $\boldsymbol{A}^{2}$ is $(\boldsymbol{AA})^{-1} = (\boldsymbol{A}^{-1})^{2}$

Question 34

Let $\boldsymbol{U}$ and $\boldsymbol{V}$ are $n \times n$ orthogonal. Show that $\boldsymbol{UV}$ and the $2n \times 2n$ matrix:

$\frac{1}{\sqrt{2}}\begin{bmatrix}\boldsymbol{U} & \boldsymbol{U}\\ \boldsymbol{V} & -\boldsymbol{V}\end{bmatrix}$ are orthogonal.

1



$(UV)(UV)^T = UVV^TU^T$

$= UU^T = I$

$\Rightarrow$ orthogonal.

$\frac{1}{\sqrt{2}}\begin{pmatrix}U & U\\ V & -V\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}U & U\\ V & -V\end{pmatrix}^T$

$= \frac{1}{2}\begin{pmatrix}U & U\\ V & -V\end{pmatrix}\begin{pmatrix}U^T & V^T\\ U^T & -V^T\end{pmatrix}$

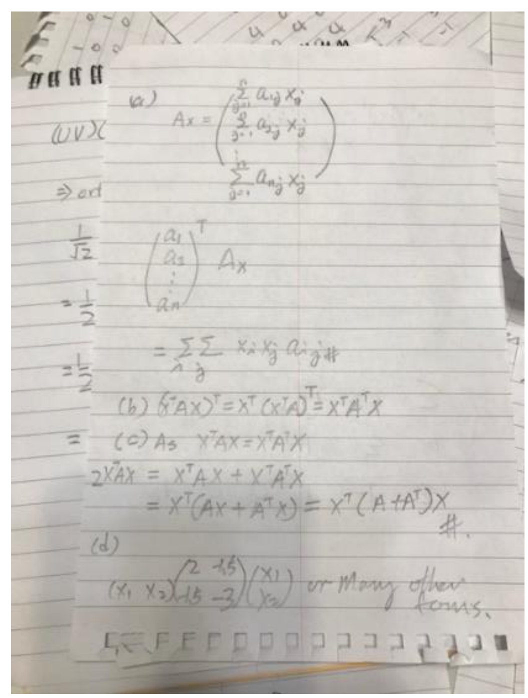
$= \frac{1}{2}\begin{bmatrix}UU^T+UU^T & VU^T-VU^T\\ VU^T-VU^T & VV^T+VV^T\end{bmatrix}$

$= \frac{1}{2}\begin{bmatrix}I & \emptyset\\ \emptyset & I\end{bmatrix} = I$ #.

Question 35

Suppose $\boldsymbol{A}$ is an $n \times n$ matrix and $\boldsymbol{x}$ is an n-vector. The product $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x}$ is a scalar.

(a) Verify that $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x} = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j a_{ij}$.
(b) Show that $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{x} = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x}$
(c) Show that $\frac{1}{2}\boldsymbol{x}^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}} + \boldsymbol{A})\boldsymbol{x} = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x}$
(d) Express $2x_1^2 - 3x_1x_2 - 3x_2^2$ as the product form $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x}$



(a) $Ax = \begin{pmatrix}\sum_{j=1}^{n} a_{1j}x_j\\ \sum_{j=1}^{n} a_{2j}x_j\\ \vdots\\ \sum_{j=1}^{n} a_{nj}x_j\end{pmatrix}$

$\begin{pmatrix}a_1\\ a_2\\ \vdots\\ a_n\end{pmatrix}^T Ax$

$= \sum_i\sum_j x_i x_j a_{ij}$ #

(b) $(x^TAx)^T = x^T(x^TA)^T = x^TA^TX$

(c) As $X^TAX = X^TA^TX$

$2X^TAX = X^TAX + X^TA^TX$

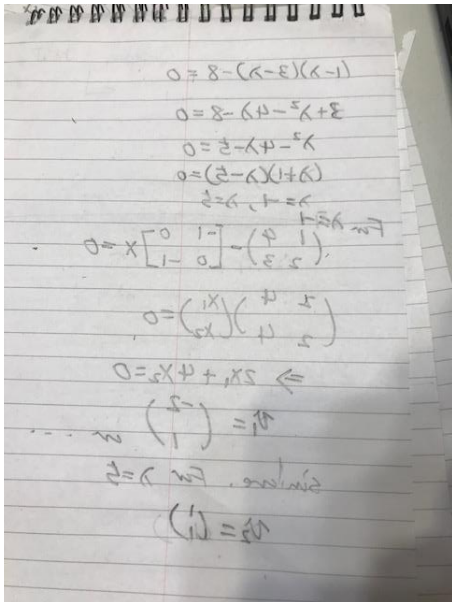
$= X^T(AX + A^TX) = X^T(A+A^T)X$ #.

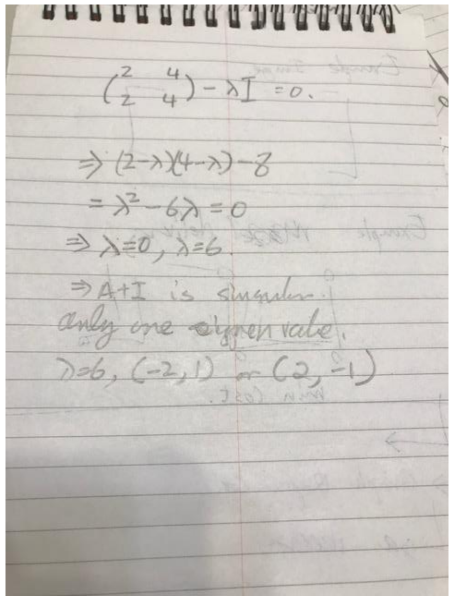
(d) $(X_1\ X_2)\begin{pmatrix}2 & -1.5\\ -1.5 & -3\end{pmatrix}\begin{pmatrix}X_1\\ X_2\end{pmatrix}$ or Many other forms.

Question 36

Find the eigenvalues and eigenvectors of these two matrices

$$\boldsymbol{A} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}, \boldsymbol{A} + \boldsymbol{I} = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}$$




$\begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix} - \lambda I = 0.$

$\Rightarrow (2-\lambda)(4-\lambda) - 8$

$= \lambda^2 - 6\lambda = 0$

$\Rightarrow \lambda = 0, \lambda = 6.$

$\Rightarrow A + I$ is singular

only one eigen vale.

$\lambda = 6, (-2, 1)$ or $(2, -1)$

Question 37

(a) Find the eigenvalues and eigenvectors of

$$\boldsymbol{A} = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix},$$

(b) Evaluate $\boldsymbol{A}^4$

(a) 4, -1

For 4,

$$\begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix} \boldsymbol{x} = \boldsymbol{0}$$

$\boldsymbol{x} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (I normalize it)

For -1,

$$\begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} \boldsymbol{x} = \boldsymbol{0}$$

$\boldsymbol{x} = \frac{1}{\sqrt{13}} \begin{pmatrix} -3 \\ 2 \end{pmatrix}$ (I normalize it)

(b)

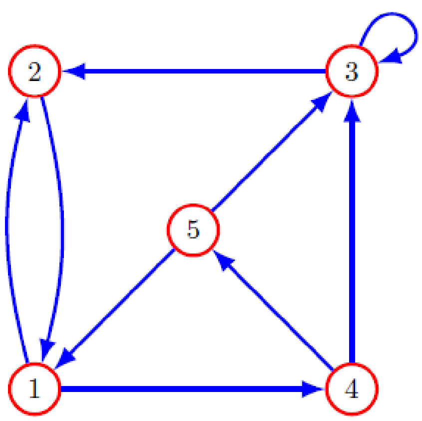
$$\boldsymbol{A} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-3}{\sqrt{13}} \\ \frac{1}{\sqrt{2}}0 & \frac{2}{\sqrt{13}} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-3}{\sqrt{13}} \\ \frac{1}{\sqrt{2}}0 & \frac{2}{\sqrt{13}} \end{bmatrix}^{-1}$$

$$\boldsymbol{A}^4 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-3}{\sqrt{13}} \\ \frac{1}{\sqrt{2}}0 & \frac{2}{\sqrt{13}} \end{bmatrix} \begin{bmatrix} -1^4 & 0 \\ 0 & 4^4 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-3}{\sqrt{13}} \\ \frac{1}{\sqrt{2}}0 & \frac{2}{\sqrt{13}} \end{bmatrix}^{-1}$$

$$\boldsymbol{A}^4 = \begin{bmatrix} 103 & 153 \\ 102 & 154 \end{bmatrix}$$

Question 38

In the lecture notes, given a directed graph, we discuss the way to get the number of paths with lengths $\boldsymbol{l}$ from node i to node j, by examining $\boldsymbol{A}^{\boldsymbol{l}}$**,** where $\boldsymbol{A}$ **is** the adjacency matrix



Suppose we are only interested in the number of paths starting node-*i* only. Your answer should be based on the adjacency matrix rather than visual inspection

(a) Discuss a better way to examine the number of paths with length l from node i to all other nodes.
(b) Discuss a way to determine there is no path from node i to node *j*

(a) $\boldsymbol{x}^{T}\boldsymbol{AAAAAAA}$**,** where the *i* th elements of $\boldsymbol{x}$ is 1. Others are equal to zero.
We multiply the vector one-by-one to each matrix, rather than using precomputation of $\boldsymbol{A}^{\boldsymbol{l}}$
The jth element of the resultant row vector indicate the number of the paths from i to j.

(b) Examine $\boldsymbol{x}^{T}\boldsymbol{A}^{\boldsymbol{m}}$**,** where *m* is equal to the number of nodes**.**

If the jth element the resultant row vector is zero, there is no path from node i to node j.

Question 39

Suppose we have a data matrix

$$\boldsymbol{X} = (\boldsymbol{x_1} \quad \boldsymbol{x_2} \quad \boldsymbol{x_3} \quad \boldsymbol{x_4}) = \begin{pmatrix} 1 & 2.2 & 2.8 & 4 \\ 2 & 1.8 & 2.2 & 4 \end{pmatrix}$$

(a) What is the mean vector for this data matrix ?
(b) What is the covariance matrix?
(c) What are the eigenvectors and eigenvalues of the covariance matrix?
(d) Suppose we only keep one PCA component. What is the forward transform matrix to encode the data vectors?
(e) Using $\boldsymbol{x_2}$ to show the PC encoding process.
(f) Suppose we only keep one PCA component. What is the error to encode $\boldsymbol{x_2}$?

(a) $\begin{pmatrix} 2.5 \\ 2.5 \end{pmatrix}$

(b) $\ddot{\boldsymbol{X}} = \begin{pmatrix} 1 & 2.2 & 2.8 & 4 \\ 2 & 1.8 & 2.2 & 4 \end{pmatrix} - 2.5 \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$

$$\frac{1}{3}\ddot{\boldsymbol{X}}(\ddot{\boldsymbol{X}})^T = \begin{pmatrix} 1.5600 & 1.0400 \\ 1.0400 & 1.0267 \end{pmatrix}$$

(c) 2.3670, $\begin{pmatrix} -0.7901 \\ -0.6130 \end{pmatrix}$ and 0.2197, $\begin{pmatrix} 0.6130 \\ -0.7901 \end{pmatrix}$

(d) $(-0.7901 \quad -0.6130)$

(e) $y = (-0.7901 \quad -0.6130)(\boldsymbol{x_2} - \boldsymbol{mean\ vector}\ )$=0.6661

(f) $\boldsymbol{x_2} - \left(\begin{pmatrix} -0.7901 \\ -0.6130 \end{pmatrix} 0.6661 + mean\ vector \right) = \begin{pmatrix} 2.2 \\ 1.8 \end{pmatrix} - \begin{pmatrix} 1.9737 \\ 2.0916 \end{pmatrix} = \begin{pmatrix} 0.2263 \\ -0.2916 \end{pmatrix}$

Question 40

Suppose we have a triangle on a 2D plane. There vertices are (0,0), (1,0), and (1,2).

(a) Draw the triangle
(b) What is the rotation matrix to rotate this triangle around (0,) with counter clockwise θ
(c) If we apply a counter clockwise rotation with $\frac{\pi}{3}$ around (0,0), what are the new coordinates of the three vertices.

(d) Discuss the procedure to rotate the triangle around the vertex (1,0)

(a) Easy

(b) $\boldsymbol{A} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, given a vector $\boldsymbol{a}$

If we want to rotation $\boldsymbol{a}$ with counter clockwise

$\boldsymbol{b} = \boldsymbol{A}^T\boldsymbol{a}$

Or saying $\boldsymbol{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

(c)

(0,0)->(0,0),
(1,0)->(0.5, 0.866)
(1,2)->(-1.232,1.866)

(d) move the coordinate to point (1,0)

0,0)->(-1,0),
(1,0)->(0,0)
(1,2)->(0,2)

Apply rotation,
Add back the displacement.

Question 41

Let $\boldsymbol{A} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{-2} \\ \mathbf{1} & \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{3} \end{bmatrix}$

(a) Evaluate $\boldsymbol{A}^3 - 5\boldsymbol{A}^2 + 8\boldsymbol{A} - 4\boldsymbol{I}$

(b) Find $\boldsymbol{A}^{-1}$

(a) A zero matrix

(b) $\boldsymbol{A}^3 - 5\boldsymbol{A}^2 + 8\boldsymbol{A} = 4\boldsymbol{I} \Rightarrow \boldsymbol{A}(\boldsymbol{A}^2 - 5\boldsymbol{A} + 8\boldsymbol{I}) = 4\boldsymbol{I}$

$$\boldsymbol{A}^{-1} = \frac{1}{4}(\boldsymbol{A}^2 - 5\boldsymbol{A} + 8\boldsymbol{I})$$

Question 42

(a) Show that $(\boldsymbol{A}^2 - \boldsymbol{A} + \boldsymbol{I})(\boldsymbol{A} + \boldsymbol{I}) = \boldsymbol{A}^3 + \boldsymbol{I}$

(b) Let $\boldsymbol{A} = \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}$. Evaluate $\boldsymbol{A}^2$ and $\boldsymbol{A}^3$.

(c) Find the inverse of $\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$

(a) Easy

$$\boldsymbol{A}^2 = \begin{bmatrix} 0 & 0 & ac \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \boldsymbol{A}^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(c)

$(\boldsymbol{A}^2 - \boldsymbol{A} + \boldsymbol{I})$, put a, b, c into A