MA 2001 Assignment 3

1. Boundaries : $\begin{cases} x=0 \\ y=0 \\ y+z=2 \\ x=4-y^2 \end{cases}$

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$$\iiint_V f(x,y,z)\,dV = \int_0^2 \int_0^{2-z} \int_0^{4-y^2} f(x,y,z)\,dx\,dy\,dz$$

$$= \int_0^2 \int_0^{4-y^2} \int_0^{2-y} f(x,y,z)\,dz\,dx\,dy$$

$$= \int_0^1 \int_0^{4-(2-z)^2} \int_0^{2-z} f(x,y,z)\,dy\,dx\,dz +$$

$$\int_0^1 \int_{4-(2-z)^2}^{4} \int_0^{\sqrt{4-x}} f(x,y,z)\,dy\,dx\,dz$$

2. (a) $u + v = 3x$

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$v - 2u = 3y$ $\Rightarrow$ $x = \frac{1}{3}(u+v)$

$y = \frac{1}{3}(v-2u)$

$\frac{\partial x}{\partial u} = \frac{1}{3}$ $\frac{\partial x}{\partial v} = \frac{1}{3}$ $\frac{\partial y}{\partial u} = -\frac{2}{3}$ $\frac{\partial y}{\partial v} = \frac{1}{3}$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \det\begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix} = \frac{1}{3}\times\frac{1}{3} - (-\frac{2}{3})\times\frac{1}{3}$$

$$= \frac{1}{3}$$

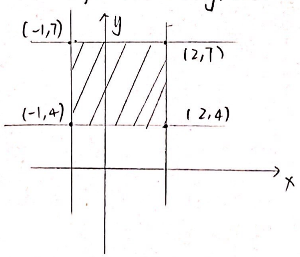
(b) $y = -2x + 4$ $\Rightarrow$ $\frac{1}{3}(v-2u) = -\frac{2}{3}(u+v)+4$ $\rightarrow$ $v = 4$

$y = -2x + 7$ $\Rightarrow$ $\frac{1}{3}(v-2u) = -\frac{2}{3}(u+v)+7$ $\rightarrow$ $v = 7$

$y = x - 2$ $\Rightarrow$ $\frac{1}{3}(v-2u) = \frac{1}{3}(u+v)-2$ $\rightarrow$ $u = 2$

$y = x + 1$ $\Rightarrow$ $\frac{1}{3}(v-2u) = \frac{1}{3}(u+v)+1$ $\rightarrow$ $u = -1$

Transformed region:



(c) $\iint\limits_R (2x^2-xy-y^2)\,dx\,dy = \iint\limits_R (2x+y)(x-y)\,dx\,dy$

$$= \int_4^7 \int_{-1}^2 \frac{1}{3} uv \, du\, dv$$

$$= \frac{1}{3} \int_4^7 \left(\frac{1}{2} vu^2\right)\Big|_{-1}^2 \, dv$$

$$= \frac{1}{3} \int_4^7 \frac{3}{2} v \, dv$$

$$= \frac{1}{3} \cdot \frac{3}{2} v^2 \Big|_4^7$$

$$= \frac{33}{4}$$

3. (a).

$x = r\cos\theta$

$y = r\sin\theta$

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The region R is $\begin{cases} r \leq 2 \\ -\frac{\pi}{3} \leq \theta \leq \frac{\pi}{3} \\ r\cos\theta \geq 1 \end{cases}$

(b). $J = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = \det\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = r$

$$\iint_R \frac{1}{\sqrt{x^2+y^2}}\, dx\,dy = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \int_{\frac{1}{\cos\theta}}^{2} \frac{1}{r}\cdot r\, dr\, d\theta$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (2 - \sec\theta)\, d\theta$$

$$= \left(2\theta - \ln|\sec\theta + \tan\theta|\right)\Big|_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$$

$$= \frac{4\pi}{3} - \ln(\sqrt{3}+2) + \ln(2-\sqrt{3})$$

4. Boundaries: $\begin{cases} z=x \\ z=0 \\ x=\sqrt{1-y^2} \\ x=0 \\ y=1 \\ y=-1 \end{cases}$ We set $\begin{cases} x=r\cos\theta \\ y=r\sin\theta \\ z=z \end{cases}$

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Then boundaries can be transferred into $\begin{cases} z=r\cos\theta \\ z=0 \\ r=1 \\ r\cos\theta=0 \\ r\sin\theta=1 \\ r\sin\theta=-1 \end{cases}$

The region is $0\le z\le r\cos\theta$

$0\le x\le\sqrt{1-y^2} \Rightarrow -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$ $\quad 0\le r\le 1$

$$J=\frac{\partial(x,y,z)}{\partial(r,\theta,z)}=\det\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial z} \end{pmatrix}=\begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}=r$$

Hence $\int_{-1}^{1}\int_0^{\sqrt{1-y^2}}\int_0^x (x^2+y^2)\,dz\,dx\,dy$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^1\int_0^{r\cos\theta} r^2\cdot r\,dz\,dr\,d\theta$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^1 r^4\cos\theta\,dr\,d\theta$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{5}r^5\cos\theta\Big|_0^1 d\theta=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{5}\cos\theta\,d\theta=\frac{1}{5}\sin\theta\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=\frac{2}{5}$$

5.

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(a) $V=\int_0^1\int_{-1}^1\int_0^{y^2} 1\, dz\, dy\, dx$

(b) $x^2+y^2=1 \quad z=-y \quad z=0$

Let $x=r\cos\theta$, $y=r\sin\theta$ $\quad x^2+y^2=r^2$ $\quad 0\le r\le 1 \quad 0\le\theta\le\pi$

$V=\int_0^{\pi}\int_0^1\int_{-r\sin\theta}^{0} r\, dz\, dr\, d\theta$

(c) Let $x=r\cos\theta \quad y=r\sin\theta \quad z=z$

$1-x^2-y^2=-\sqrt{1-x^2-y^2} \Rightarrow z=1-r^2 \quad z=-\sqrt{1-r^2} \quad 0\le r\le 1 \quad 0\le\theta\le 2\pi$

$V=\int_0^{2\pi}\int_0^1\int_{-\sqrt{1-r^2}}^{1-r^2} r\, dz\, dr\, d\theta$

(d) $z\ge 0 \Rightarrow \rho\cos\phi \ge 0 \quad -\frac{\pi}{2}\le\phi\le\frac{\pi}{2}$

Transformed region: $\cos\phi\le\rho\le 2$

$-\frac{\pi}{2}\le\phi\le\frac{\pi}{2}$

$0\le\theta\le \cancel{\frac{\pi}{2}} 2\pi$

$V=\cancel{\iint}$ $\quad V=\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_{\cos\phi}^{2} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$

6.

(a) $\operatorname{curl}(\vec{F}) = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial x & \partial y & \partial z \\ f_1 & f_2 & f_3 \end{pmatrix}$ where $f_1 = y+z$, $f_2 = z+x$, $f_3 = x+y$

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$= (\partial_y f_3 - \partial_z f_2)\vec{i} + (\partial_z f_1 - \partial_x f_3)\vec{j} + (\partial_x f_2 - \partial_y f_1)\vec{k}$

$= (1-1)\vec{i} + (1-1)\vec{j} + \cancel{(1 \times k}(1-1)\vec{k} = \vec{0}$

Hence $\vec{F}$ is conservative. Suppose $\phi(x,y,z)$ is the potential function.

$\because \phi_x = y+z \Rightarrow \phi = x(y+z) + f(y,z)$

$\phi_y = x + f_y = x+z \Rightarrow f_y = z \quad f(y,z) = yz + g(z)$

$\therefore \phi = x(y+z) + yz + g(z)$

$\phi_z = x+y+g'(z) = x+y \quad \therefore g(z) = c$ where c is constant

$\therefore \phi(x,y,z) = xy + xz + yz + c$, where c is arbitrary constant

(b) $\operatorname{div}(\vec{F}) = \dfrac{\partial(y+z)}{\partial x} + \dfrac{\partial(z+x)}{\partial y} + \dfrac{\partial(x+y)}{\partial z} = 0$

Hence $\operatorname{div}\vec{F}$ at $(1,2,3)$ is 0

(c) From (a), we get $\operatorname{Curl}(\vec{F}) = \vec{0}$

7. (a)

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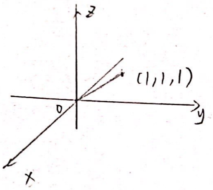

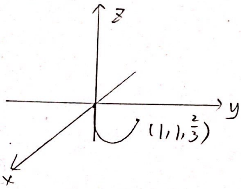

(b) $\vec{r}'(t) = \vec{i} + 2t\,\vec{j} + 2t^2\,\vec{k}$

$|\vec{r}'(t)| = \sqrt{1^2 + (2t)^2 + (2t^2)^2} = 2t^2 + 1$

$\int_{C_2} xz \, dC = \int_0^1 t \cdot \frac{2}{3}t^2 \cdot (2t^2+1)\,dt$

$= \left(\frac{2}{9}t^6 + \frac{1}{6}t^4\right)\Big|_0^1 = \frac{7}{18}$

(c) $\vec{F} = (y+z)\,\vec{i} + (z+x)\,\vec{j} + (x+y)\,\vec{k}$

$= 2t\,\vec{i} + 2t\,\vec{j} + 2t\,\vec{k}$

$\vec{r}'(t) = \vec{i} + \vec{j} + \vec{k}$

$\int_{C_1} \vec{F}\, dC = \int_0^1 (2t, 2t, 2t)\cdot(1,1,1)\,dt$

$= 3t^2\Big|_0^1 = 3$

8. (a) Set $\begin{cases} x = x & 0 \le x \le 2 \\ y = y & -2 \le y \le 2 \\ z = 4-y^2 \end{cases}$

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(b) $\vec{r}(x,y) = x\vec{i} + y\vec{j} + (4-y^2)\vec{k}$

$\vec{r}_x = \vec{i} \quad \vec{r}_y = \vec{j} - 2y\vec{k}$

$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & -2y \end{vmatrix} = 2y\vec{j} + \vec{k}$

$|\vec{r}_x \times \vec{r}_y| = \sqrt{4y^2+1}$

$\iint_S xz\, dS = \int_{-2}^{2}\int_0^2 x(4-y^2)\sqrt{4y^2+1}\, dx\, dy$

$= \int_{-2}^{2} (4-y^2)\sqrt{4y^2+1}\, dy \cdot \int_0^2 x\, dx$

We consider $\int (4-y^2)\sqrt{4y^2+1}\, dy$ part. Let $y = \frac{1}{2}\tan\theta$

$\int (4-y^2)\sqrt{4y^2+1}\, dy = \int (4 - \frac{1}{4}\tan^2\theta)\sec\theta\, d(\frac{1}{2}\tan\theta)$

$= \int [4 - \frac{1}{4}(\sec^2\theta - 1)]\, \frac{1}{2}\sec^3\theta\, d\theta$

$= \frac{17}{8}\int \sec^3\theta\, d\theta - \frac{1}{8}\int \sec^5\theta\, d\theta$

$\int \sec^3\theta\, d\theta = \int \sec\theta\, d(\tan\theta) = \sec\theta \times \tan\theta - \int \tan^2\theta \sec\theta\, d\theta$

$= \sec\theta\tan\theta - \int (\sec^3\theta - \sec\theta)\, d\theta$

$= \sec\theta\tan\theta - \int \sec^3\theta\, d\theta + \ln|\tan\theta + \sec\theta| + C$

$\therefore \int \sec^3\theta\, d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\tan\theta + \sec\theta| + C$

Similarly $\int \sec^5\theta\, d\theta = \frac{1}{4}\sec^3\theta\tan\theta + \frac{3}{8}\sec\theta\tan\theta + \frac{3}{8}\ln|\sec\theta + \tan\theta| + C$

$$\therefore \int (4-y^2)\sqrt{4y^2+1}\,dy = \frac{17}{8}\left(\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta|\right) - \frac{1}{8}\times\left(\frac{1}{4}\sec^3\theta\tan\theta + \frac{3}{8}\sec\theta\tan\theta + \frac{3}{8}\ln|\sec\theta+\tan\theta|\right) + C$$

$$= \frac{65}{64}\sec\theta\tan\theta + \frac{65}{64}\ln|\sec\theta+\tan\theta| - \frac{1}{32}\sec^3\theta\tan\theta + C$$

Hence $\int_{-2}^{2}(4-y^2)\sqrt{4y^2+1}\,dy = \int_{-\tan^{-1}(4)}^{\tan^{-1}(4)}\left(4-\frac{1}{4}(\sec^2\theta-1)\right)\frac{1}{2}\sec^3\theta\,d\theta$

$$= \left(\frac{65}{64}\sec\theta\tan\theta + \frac{65}{64}\ln|\sec\theta+\tan\theta| - \frac{1}{32}\sec^3\theta\tan\theta\right)\Bigg|_{-\tan^{-1}(4)}^{\tan^{-1}(4)}$$

$$= \frac{65}{8}\sqrt{17} - \frac{17}{4}\sqrt{17} + \frac{65}{32}\ln(\sqrt{17}+4)$$

$$\therefore \int_0^2\int_{-2}^{2}(4-y^2)\sqrt{4y^2+1}\,dy\,dx = \frac{31}{4}\sqrt{17} + \frac{65}{16}\ln(\sqrt{17}+4)$$

(c) $\iint_S \vec{F}\,dS = \int_{-2}^{2}\int_0^2 (4+y-y^2,\ 4+x-y^2,\ x+y)\cdot(0,2y,1)\,dx\,dy$

$$= \int_{-2}^{2}\int_0^2 (9y+x+2xy-2y^3)\,dx\,dy$$

$$= \int_{-2}^{2}\left((9y-2y^3)x + (2y+1)\frac{x^2}{2}\Big|_0^2\right)dy$$

$$= 2\int_{-2}^{2}(1+11y-2y^3)\,dy$$

$$= 2\left(y+\frac{11}{2}y^2-\frac{1}{2}y^4\Big|_{-2}^{2}\right)$$

$$= 2\times 4 = 8$$