Block B Unit 3 Outline

1) First-order circuits

- Methodology [Section 7.1]
- RC and RL circuits [Section 7.2, 7.3]

2) Step response

- RC circuit [Section 7.5]
- RL circuit [Section 7.6]

Alexander & Sadiku,
“Fundamentals of Electric Circuits”
7th Edition Chapter 7

First-order Circuits

1) Apply Kirchhoff's laws to purely resistive circuits, resulting in algebraic equations

2) Apply Kirchhoff's laws to RC and RL circuits, produces first order differential equations

- A first-order circuit is characterized by a first-order differential equation
- Two ways to excite the circuits
  - 1) Source-free circuits, initial conditions of the storage elements
  - 2) By independent sources, i.e. dc sources

1.1 Source-Free RC Circuit

- A source-free RC circuit occurs when its dc source is suddenly disconnected.
- The energy that is initially stored in the capacitor is eventually dissipated in the resistor
- Capacitor is initially charged

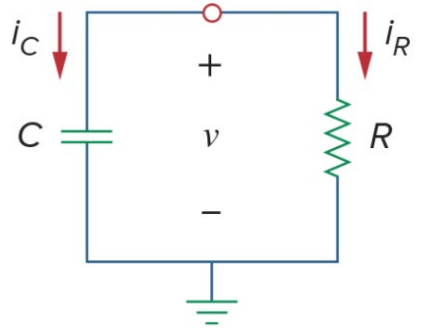
Initial voltage: $v(0) = V_0$

- Initial value of energy stored

$$w(0) = \frac{1}{2} C {V_0}^2$$

- Apply KCL, $i_C + i_R = 0$
- By definition, $i_C = C\, dv/dt$, $i_R = v/R$
- Circuit response, the circuit reacts to an excitation.

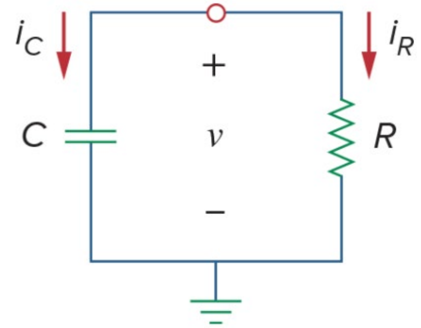
$$v(t) = V_0 e^{-t/RC}$$



A source-free RC circuit

1.1 Source-Free RC Circuit

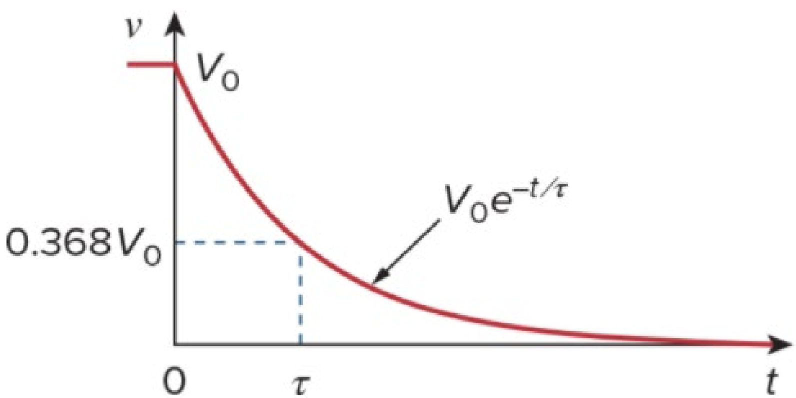
- Derivation of RC circuit response: $v(t) = V_0 e^{-t/RC}$



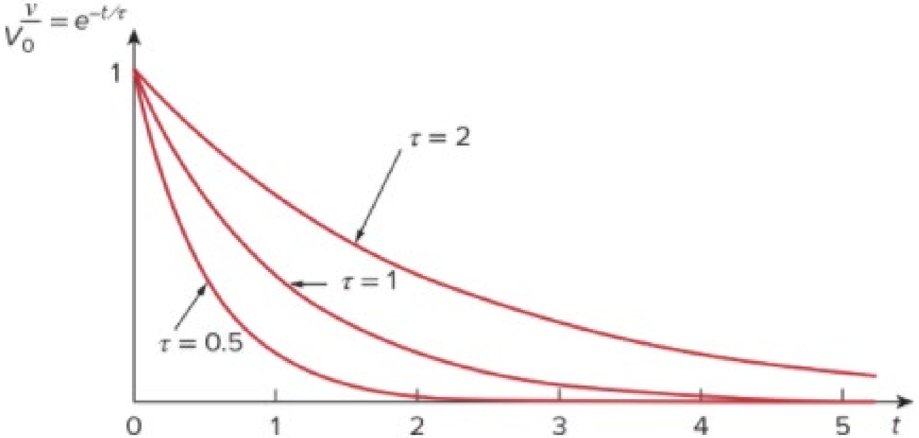
A source-free RC circuit

Natural Response of RC Circuit

- Voltage response of the RC circuit is an exponential decay of the initial voltage
- The voltage response is due to the initial energy stored, and not due to external voltage or current source
- Time constant ($\tau = RC$), is the same regardless of what the output is defined to be. $v(t) = V_0 e^{-t/\tau}$



Voltage response of RC circuit



Small time constant gives a fast response

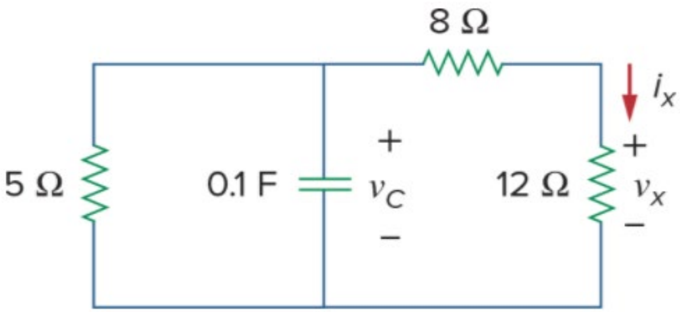
Thevenin's Theorem in RC Circuit

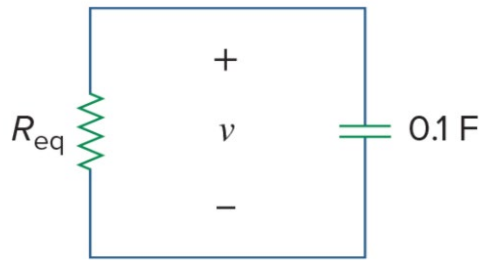
- Once $v(t)$ is obtained, i_C , v_R, and i_R can be determined.
- R, the Thevenin equivalent resistance at the terminals of the capacitor
- Take out the capacitor C, find $R = R_{th}$ at terminals
- Method:
  - Single capacitor, find Thevenin equivalent at the terminals of the capacitor
  - Several capacitors, combine to form a single equivalent capacitor, find Thevenin equivalent
  - Obtain capacitor voltage, v_C
  - Then determine v_x and i_x

Example 1.1.1

- Let $v_C(0) = 15\ V$. Find v_C, v_x and i_x for t > 0.

(Ans. 15e$^{-2.5t}$ V, 9e$^{-2.5t}$ V, 0.75e$^{-2.5t}$ A)





<u>General steps:</u>
Step 1, find Req
Step 2 find τ
Step 3, find v_C
Step 4, find v_x and i_x

$$R_{eq} = \frac{20x5}{20+5} = 4\Omega$$

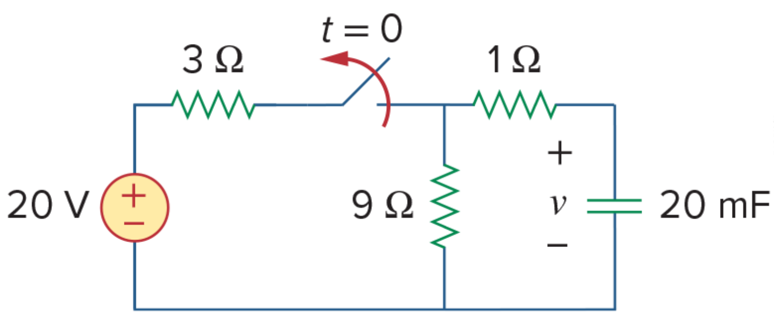
$$v_C = v = 15e^{-2.5t}\ \text{V}$$

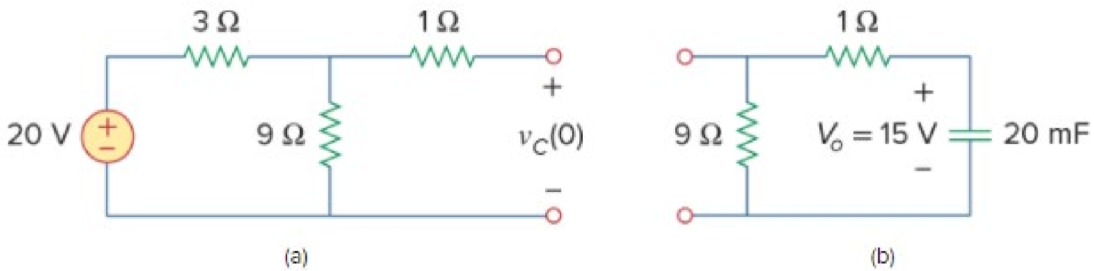
By voltage divider, $v_x =$

By ohm's law, $i_x =$

Example 1.1.2

The switch has been closed for a long time, and it is opened at t = 0. Find $v(t)$, the voltage across the capacitor, for t ≥ 0. ($15e^{-5t}$ V)





(a) $t < 0$, (b) $t > 0$.

- Fig. (a), for t<0, switch is closed, capacitor is open (at DC)..

At t<0, $v_C(t) =$

At t=0, $v_C(0) = V_0 = 15V$

- Fig. (b), for t>0, switch is open.

<u>Step 1:</u> Req = <u>Step 2:</u> Time constant (τ) =

<u>Step 3:</u> At t>0, voltage across the capacitor, $v_c(t)$ =

1.2 Source-Free RL Circuit

- Circuit response, the current $i(t)$ through the inductor
- For inductor, current cannot change instantaneously
- Initial current: $i(0) = I_0$

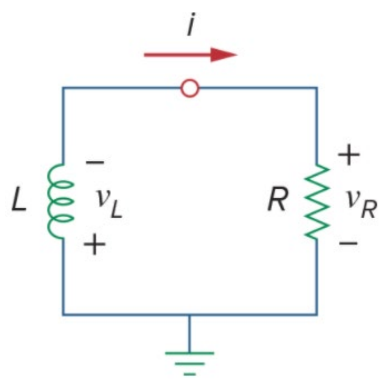
Apply KVL, $v_L + v_R = 0$

By definition, $v_L = L\,di/dt$

$$L\frac{di}{dt} + v_R = 0$$

- Initial value of energy stored

$$w(0) = \frac{1}{2}L{I_0}^2$$

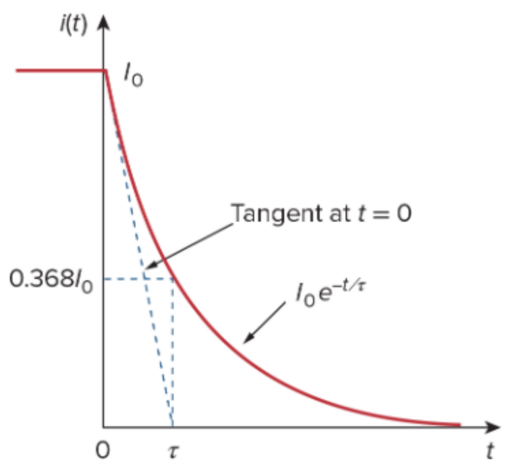


A source-free RL circuit

- Circuit response, $i(t) = I_0 e^{-Rt/L}$

Natural Response of RL Circuit

- Natural response of the RL circuit is an exponential decay of the initial current
- Time constant, $\tau = \frac{L}{R}$. Current response, $i(t) = I_0 e^{-t/\tau}$
- Voltage across resistor, $v_R(t) = iR =$



Current response of RL circuit

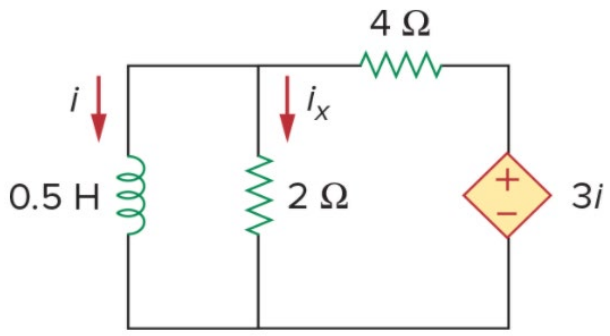
- Similar to RC circuit, the smaller the τ of a circuit, the faster the rate of decay of the response.
- The larger the τ, the slower the rate of decay of the response.

Thevenin's Theorem in RL Circuit

- Once $i_L(t)$ is obtained, v_L , v_R, and i_R can be determined.
- R, the Thevenin equivalent resistance at the terminals of the inductor
- Take out the inductor L, find $R = R_{th}$ at terminals
- Method:
  - Single inductor, find Thevenin equivalent at the terminals of the inductor
  - Several inductors, combine to form a single equivalent inductor, find Thevenin equivalent
  - Obtain inductor voltage, v_L
  - Then determine v_x and i_x

Example 1.2.1

Assuming that $i(0) = 10\ A$, calculate $i(t)$, and $i_x(t)$.

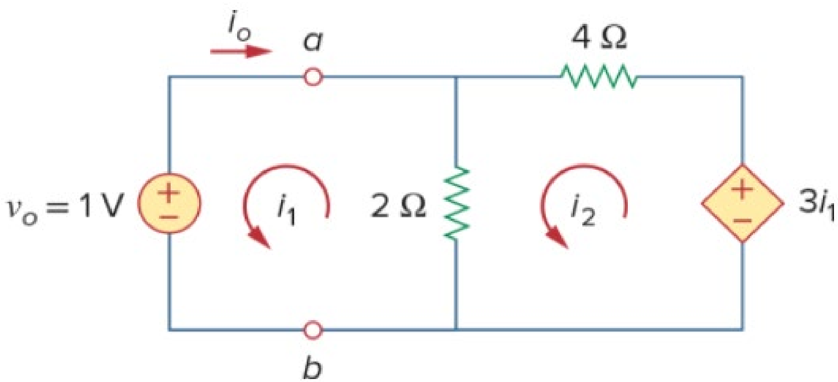


Two approach:

1. Obtain equivalent resistance at the inductor terminals, apply current response
2. Use KVL

* First obtain the inductor current

- Method 1: Because of the dependent source, insert a voltage source with $v_0 = 1\ V$ at the inductor terminals a-b

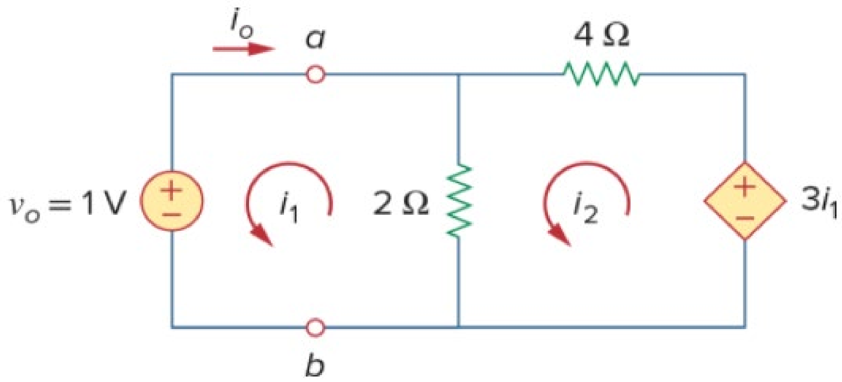


Apply KVL, using Mesh Analysis:

Left loop: $1 = 2(i_2 - i_1)$

Right loop: $3i_1 = 4i_2 + 2(i_2 - i_1)$

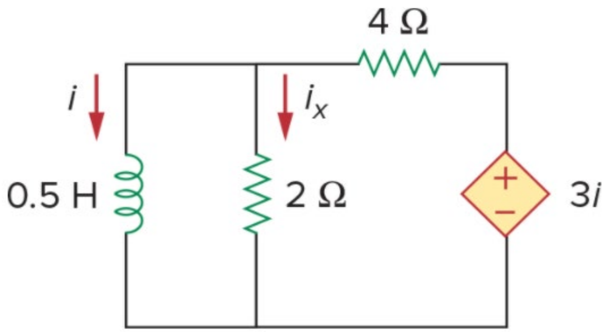
Example 1.2.1 (continued)

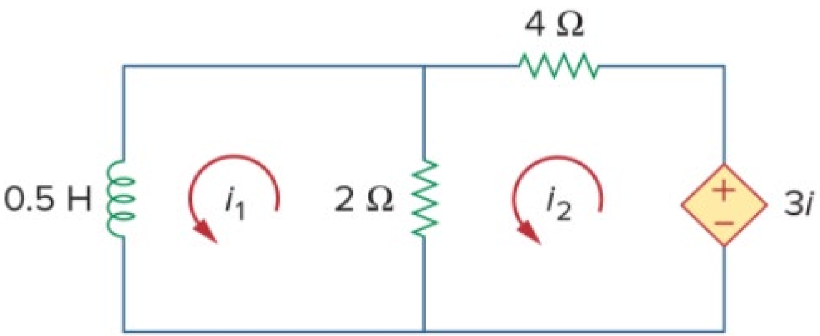


- $i_1 = -3A, i_2 = -2.5A$
- $i_0 = -i_1 = 3A$
  - Therefore, R_{eq} (at the terminals of inductor) = $R_{Th} = \frac{v_o}{i_0} = \frac{1}{3}\Omega$
  - Time constant $\tau = \frac{L}{R_{eq}} = 1.5\ s$
  - Circuit response, $i(t) = i(0)e^{-\frac{t}{\tau}} = 10e^{-\left(\frac{2}{3}\right)t}A$, t > 0

Example 1.2.1 (continued)

Method 2: Direct apply KVL to the circuit



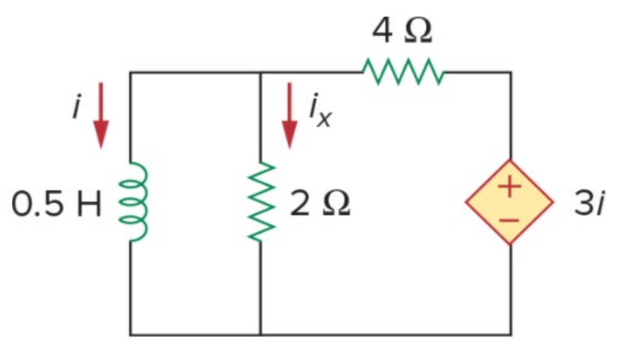


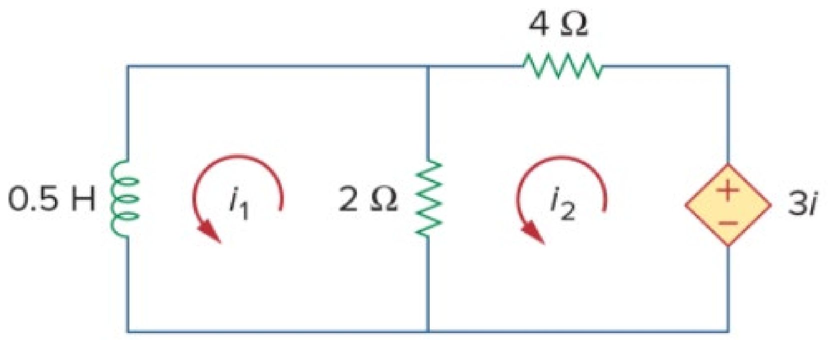
$$v_L(t) = L\frac{di}{dt}$$

Left loop: $v_L(t) = 0.5\frac{di_1}{dt} = 2(i_2 - i_1)$

Right loop: $3i_1 = 4i_2 + 2(i_2 - i_1)$

Example 1.2.1 (continued)



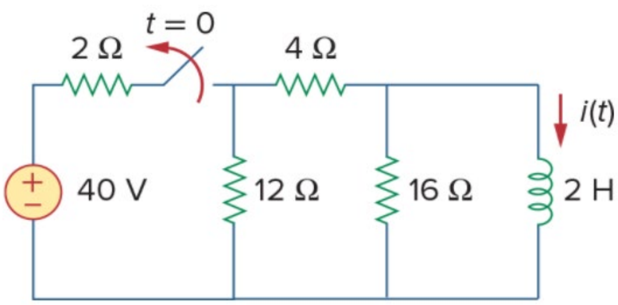


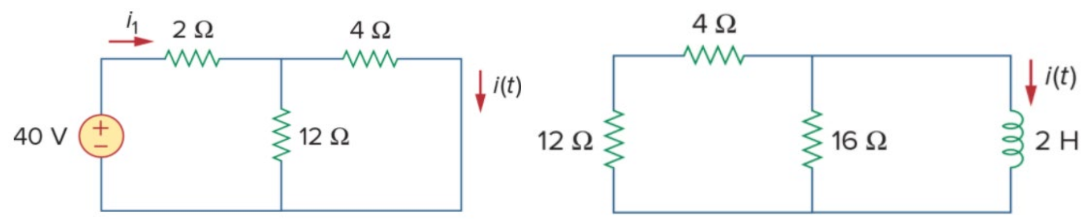
- Circuit response, $i(t) = i(0)e^{-\frac{t}{\tau}} = 10e^{-\left(\frac{2}{3}\right)t}A$, t > 0
- Voltage across the inductor, $v_L(t) = L\frac{di}{dt} = -\frac{10}{3}e^{-\left(\frac{2}{3}\right)t}V$
- Inductor and the 2-Ω resistor are in parallel,

$$i_x(t) = \frac{v_L}{2} = -\frac{5}{3}e^{-\left(\frac{2}{3}\right)t}A,\ t > 0$$

Example 1.2.2

The switch in the circuit below has been closed for a long time. At t = 0, the switch is opened. Calculate $i(t)$ for t > 0. (6e^{-4t} A)





(a) $t < 0$, (b) $t > 0$.

- Fig. (a), for t<0, switch is closed, inductor is shorted.

At t<0, $i_1 = \frac{40}{2+3} = 8A$. By current division, $i(t) = \frac{12}{12+4} i_1 = 6A$

At t=0, $i(0) = 6A$

- Fig. (b), for t>0, switch is open. $R_{eq} = (12+4)||16 = 8\Omega$

Time constant (τ) = $\qquad$ $i(t) = i(0)e^{-\frac{t}{\tau}} =$

Transient and Steady-State Response

$$\text{Complete response} = \underbrace{\text{natural response}}_{\text{stored energy}} + \underbrace{\text{forced response}}_{\text{independent source}}$$

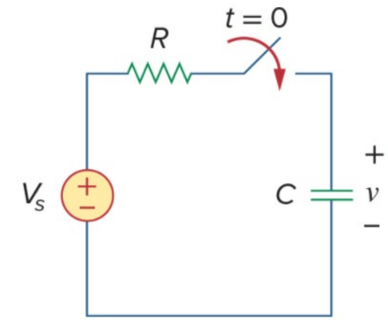
$$\text{Complete response} = \underbrace{\text{transient response}}_{\text{temporary part}} + \underbrace{\text{steady-state response}}_{\text{permanent part}}$$

- **Transient response**, the circuit's temporary response that will die out with time
- **Steady-state response**, the behavior of the circuit a long time after an external excitation is applied
- The complete response is the sum of the transient response and the steady-state response

2.1 Step Response of RC Circuit

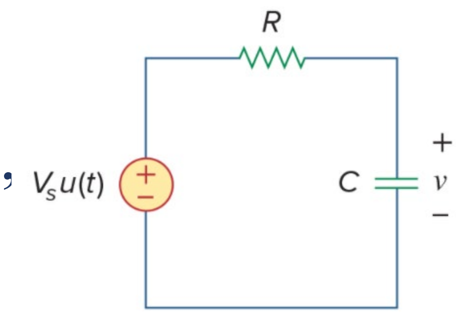
- Step response, the response of the circuit due to a sudden application of a dc voltage or current source
- Let the response be the sum of the transient response and the steady-state response

$$v = v_t + v_{SS}$$



- $v_t = Ae^{-t/\tau}$, a decaying exponential, $\tau = RC$

  , where A is a constant



- $v_{SS} = V_S$, after switch is closed for a long time, capacitor is opened
- $v = Ae^{-t/\tau} + V_S$

An RC circuit with voltage step input

Complete Response of RC Circuit

- Determine A from V_0, the initial voltage across the capacitor
- At t = 0, $V_0 = A + V_S$
- Substitute A in the voltage response equation,

$$\mathrm{v(t)} = (V_0 - V_S)e^{-t/\tau} + V_S$$

- Complete response:

$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}$$

- Summary to find step response of an RC circuit:

  (1) Initial capacitor voltage, v(0) at t = 0

  (2) Final capacitor voltage, v(∞)

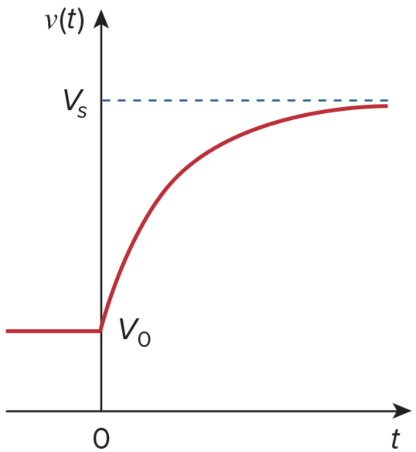
  (3) Time constant, $\tau = RC$

With and Without Initial Voltage

Complete response: $$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}$$

Capacitor is initially charged:

$$v(t) = \begin{cases} V_0, & t < 0 \\ V_s + (V_0 - V_s)e^{-t/\tau}, & t > 0 \end{cases}$$

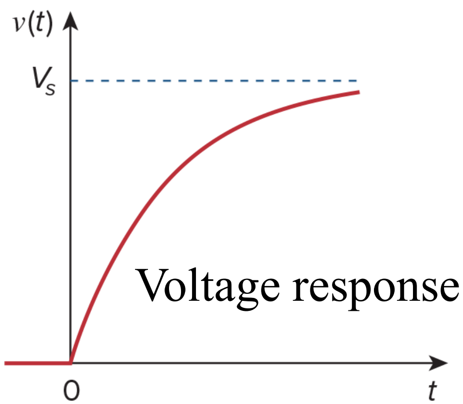


Response of an RC circuit with initially charged capacitor

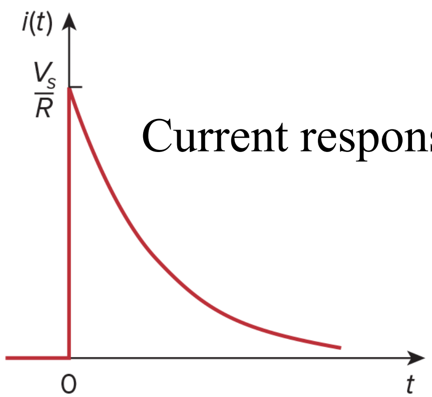
Capacitor is uncharged initially ($V_0 = 0$):

$$v(t) = \begin{cases} 0, & t < 0 \\ V_s\left(1 - e^{-t/\tau}\right), & t > 0 \end{cases}$$

$$i(t) = C\frac{dv}{dt} = \frac{C}{\tau}V_s e^{-t/\tau}, \quad \tau = RC, \quad t > 0$$



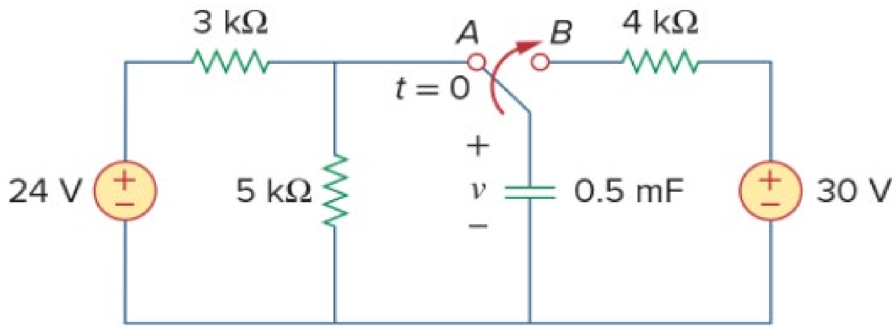
Voltage response



Current response

Example 2.1.1

The switch in the circuit has been in position A for a long time. At t = 0, the switch moves to B. Determine v(t) for t > 0 and calculate its value at t = 1 and 4 s. (Ans. v(1)=20.9 V, v(4)=27.97 V)



- For t < 0, capacitor acts like an open circuit.

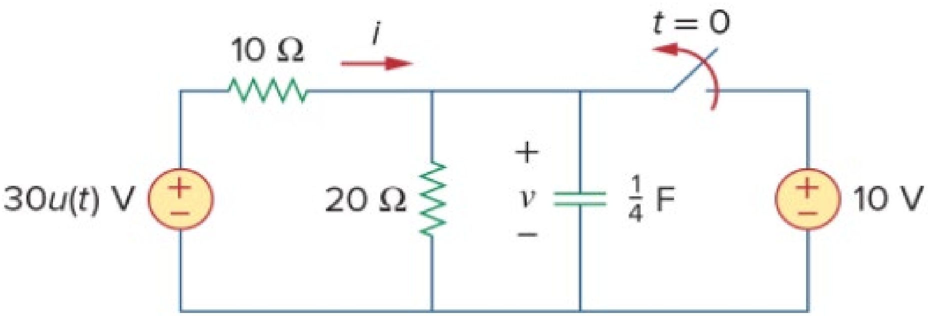
By voltage division, v(0) = 5/(5+3) * 24 = 15 V

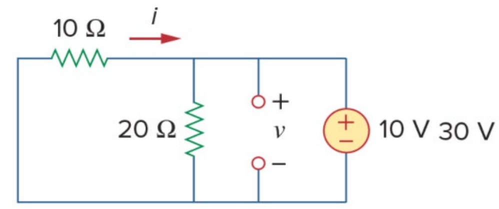
- For t > 0, R_{th} (terminals at capacitor) = 4 kΩ, $\tau = R_{Th}C = 2\ s$

At dc steady state, v(∞) = 30 V, $v(\mathrm{t}) = v(\infty) + (V_0 - v(\infty))e^{-\frac{t}{\tau}}$

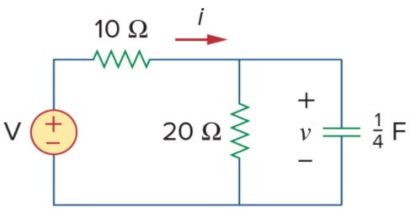
Example 2.1.2

The switch has been closed for a long time and is opened at t = 0. Find i and v for all time.





t < 0



t > 0

- For $t < 0$, switch is closed, capacitor → open, $30u(t) = 0$ V,

$v = v(0) = 10$ V, $I = -v/10 = -1$ A

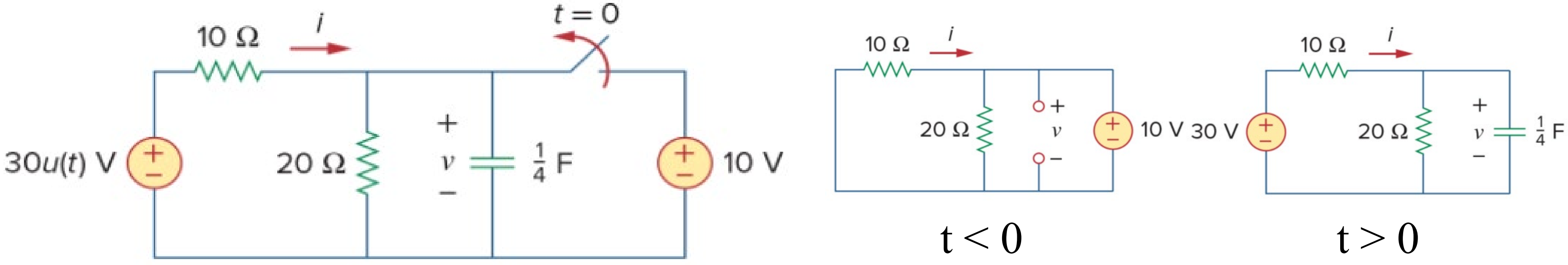
- For $t > 0$, switch is open, $R_{th} = 10||20\ \Omega$, $\tau = R_{Th}C = \frac{5}{3}s$

After long time, circuit reaches steady-state, capacitor → open

$v(\infty) = 20/(20+10) * 30 = 20$ V

Example 2.1.2 (continued)

- $v(\mathrm{t}) = v(\infty) + \big(V_0 - v(\infty)\big)e^{-\frac{t}{\tau}} = (20 - 10e^{-0.6t})V$



t < 0 t > 0

- To obtain i at t > 0, apply KCL,

$$i = \frac{v}{20} + C\frac{dv}{dt} = (1 + e^{-0.6t})\ A$$

- Check, apply KVL, $v + 10i = 30$

$$v + 10i = (20 - 10e^{-0.6t}) + 10(1 + e^{-0.6t}) = 30$$

Therefore, the result is confirmed.

2.2 Step Response of RL Circuit

- To find the inductor current I as the circuit response
- Let the response be the sum of the transient response and the steady-state response

$$\mathrm{i} = i_t + i_{SS}$$

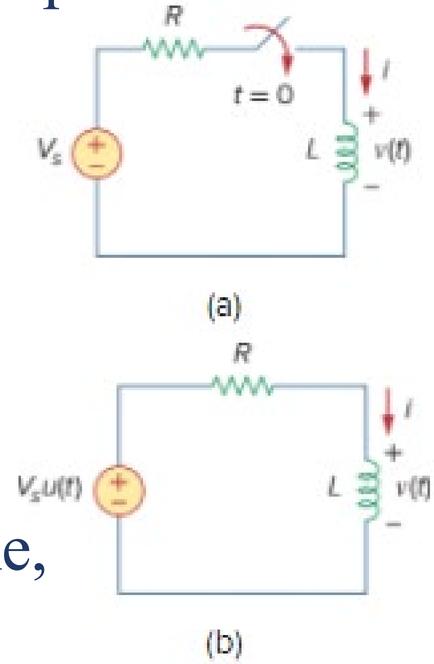
- $i_t = Ae^{-t/\tau}$, a decaying exponential, $\tau = \frac{L}{R}$

  , where A is a constant

- $i_{SS} = \frac{V_S}{R}$, after switch is closed for a long time,

  inductor becomes shorted

- $\mathrm{i} = Ae^{-t/\tau} + \frac{V_S}{R}$



An RL circuit with voltage step input

Complete Response of RL Circuit

- Determine A from i_0, the initial current through the inductor
- At t = 0, $I_0 = A + \frac{V_S}{R}$
- Substitute A in the circuit response equation,

$$i(t) = (I_0 - \frac{V_S}{R})e^{-t/\tau} + \frac{V_S}{R}$$

- Complete response:

$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau}$$

- Summary to find step response of an RL circuit:
  (1) Initial inductor current, i(0) at t = 0
  (2) Final inductor current, i(∞)
  (3) Time constant, $\tau = L/R$

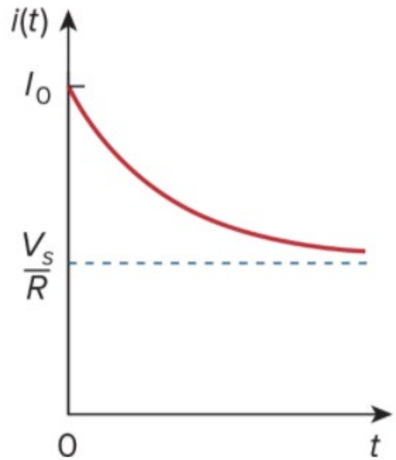
With and Without Initial Current

Complete response:

$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau}$$

With initial inductor current:

$$\mathrm{i(t)} = \frac{V_S}{R} + (I_0 - \frac{V_S}{R})e^{-t/\tau}$$

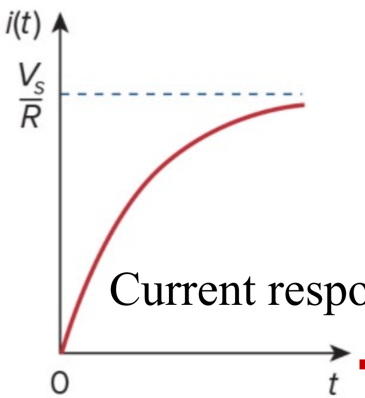


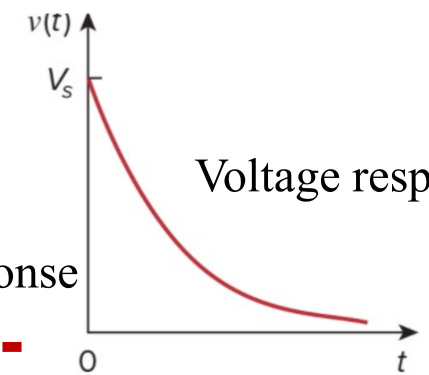
Complete response of RL circuit with initial inductor current I_0

Without initial inductor current ($I_0 = 0$):

$$i(t) = \begin{cases} 0, & t < 0 \\ \frac{V_s}{R}\left(1 - e^{-t/\tau}\right), & t > 0 \end{cases}$$

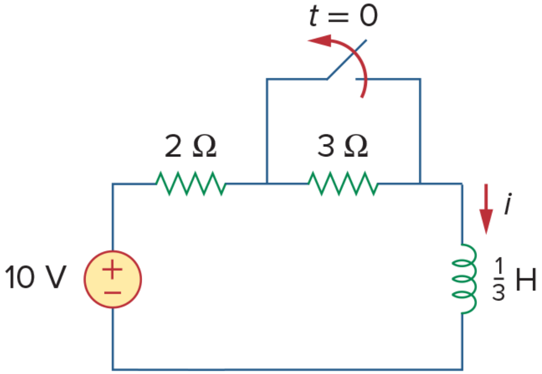
$$v(t) = L\frac{di}{dt} = V_s \frac{L}{\tau R} e^{-t/\tau}, \quad \tau = \frac{L}{R}, \quad t > 0$$





Example 2.2.1

Find i(t) in the circuit for t > 0. Assume that the switch has been closed for a long time



<u>Solutions:</u>

- When t < 0, 3-Ω is shorted, inductor acts like shorted

Current through inductor, i(0) = 10/2 = 5 A

- When t > 0, switch is open, after a long time, inductor acts like shorted

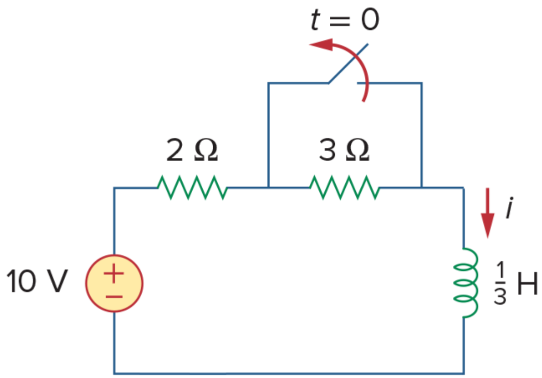
Current through inductor, i(∞) = 10/(2+3) = 2 A

- Transient response: consider Thevenin resistance across inductor terminals

R_{Th} = 2+3 = 5 Ω

- Time constant, $\tau = \mathrm{L}/R_{Th}$=1/15 s

Example 2.2.1 (continued)



- Using the complete current response

$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau}$

Therefore,

$i(t) = 2 + 3e^{-t/\tau} A, t > 0$

- Check:

For t > 0, KVL must be satisfied

$$10 = 5i + L\frac{di}{dt}$$

$$5i + L\frac{di}{dt} =$$

- This confirms the result.