MA1201 Past paper 1213B

Section A

1 (a) $\int \tan 2x \, dx = \int \frac{\sin 2x}{\cos 2x} \, dx$

Let $y = \cos 2x$

$\frac{dy}{dx} = -2 \sin 2x$

$\Rightarrow dx = \frac{1}{-2 \sin 2x} dy$

$= \int \frac{\sin 2x}{\cos 2x} \cdot \frac{1}{-2 \sin 2x} \, dy$

$= -\frac{1}{2} \int \frac{1}{y} \, dy$

$= -\frac{1}{2} \ln|y| + C$

$= -\frac{1}{2} \ln|\cos 2x| + C$

(b) $\int \frac{x^3}{3+x^2} \, dx = \int \frac{x(x^2+3-3)}{3+x^2} \, dx$

$= \int \left(x - \frac{3x}{3+x^2}\right) dx$

$= \int x \, dx - 3 \int \frac{x}{3+x^2} \, dx$

Let $y = 3+x^2$

$\frac{dy}{dx} = 2x$

$\Rightarrow dx = \frac{1}{2x} dy$

$= \frac{x^2}{2} - 3 \int \frac{x}{3+x^2} \cdot \frac{1}{2x} \, dy$

$= \frac{x^2}{2} - \frac{3}{2} \int \frac{1}{y} \, dy$

$= \frac{x^2}{2} - \frac{3}{2} \ln|y| + C$

$= \frac{x^2}{2} - \frac{3}{2} \ln|3+x^2| + C$ //

1(c) $\int e^{-3x}\sin(2x)\,dx$

$= \int \sin 2x \; d(-\frac{1}{3}e^{-3x})$

$= \sin 2x \cdot (-\frac{1}{3}e^{-3x})$

$- \int -\frac{1}{3}e^{-3x}\, d(\sin 2x)$

Take $u = \sin 2x$

$dv = e^{-3x}$

$\Rightarrow v = \int e^{-3x}dx$

$= -\frac{1}{3}e^{-3x}$

$= -\frac{1}{3}e^{-3x}\sin 2x + \frac{1}{3}\int e^{-3x}\cdot 2\cos 2x\,dx$

$= -\frac{1}{3}e^{-3x}\sin 2x + \frac{2}{3}\int e^{-3x}\cos 2x\,dx$

$= -\frac{1}{3}e^{-3x}\sin 2x + \frac{2}{3}\int \cos 2x \; d(-\frac{1}{3}e^{-3x})$

$= -\frac{1}{3}e^{-3x}\sin 2x + \frac{2}{3}\left[\cos 2x\cdot(-\frac{1}{3}e^{-3x}) - \int -\frac{1}{3}e^{-3x}\,d(\cos 2x)\right]$

$= -\frac{1}{3}e^{-3x}\sin 2x - \frac{2}{9}e^{-3x}\cos 2x + \frac{2}{9}\int e^{-3x}\cdot(-2\sin 2x)\;dx$

$= -\frac{1}{3}e^{-3x}\sin 2x - \frac{2}{9}e^{-3x}\cos 2x - \frac{4}{9}\int e^{-3x}\sin 2x\,dx$

$\Rightarrow \left(1+\frac{4}{9}\right)\int e^{-3x}\sin 2x\,dx = -\frac{1}{3}e^{-3x}\sin 2x - \frac{2}{9}e^{-3x}\cos 2x$

$\Rightarrow \int e^{-3x}\sin 2x\;dx = \frac{9}{13}\left(-\frac{1}{3}e^{-3x}\sin 2x - \frac{2}{9}e^{-3x}\cos 2x\right)$

$+ C$ //

1 (d) $\int \sqrt{9-16x^2}\, dx$

Let $4x = 3\sin\theta$

$\Rightarrow x = \frac{3}{4}\sin\theta$

$dx = \frac{3}{4}\cos\theta\, d\theta$

$$= \int \sqrt{9 - 9\sin^2\theta} \cdot \frac{3}{4}\cos\theta\, d\theta$$

$$= \frac{9}{4}\int \cos^2\theta\, d\theta$$

$$= \frac{9}{4}\int \frac{1}{2}(1+\cos 2\theta)\, d\theta$$

$$= \frac{9}{8}\left[\theta + \frac{\sin 2\theta}{2}\right] + C$$

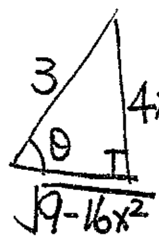
$x = \frac{3}{4}\sin\theta$

$\theta = \sin^{-1}\left(\frac{4x}{3}\right)$

$$= \frac{9}{8}\left[\sin^{-1}\left(\frac{4x}{3}\right) + \frac{2\sin\theta\cos\theta}{2}\right] + C$$

$$= \frac{9}{8}\left[\sin^{-1}\left(\frac{4x}{3}\right) + \frac{4x}{3}\cdot\frac{\sqrt{9-16x^2}}{3}\right] + C$$

//



$\because \sin\theta = \frac{4x}{3}$

$\therefore \cos\theta = \frac{\sqrt{9-16x^2}}{3}$

1. (e) $\int \frac{9x-7}{(x+2)(x^2-4x+13)}\,dx$

$$\frac{9x-7}{(x+2)(x^2-4x+13)} = \frac{A}{x+2} + \frac{Bx+C}{x^2-4x+13}$$

$$\therefore 9x-7 = A(x^2-4x+13) + (Bx+C)(x+2)$$

Put $x=-2$: $-25 = 25A \Rightarrow A=-1$

Equate coeff. of x^2: $0 = A+B \Rightarrow B=-A=1$

Equate constant terms: $-7 = 13A+2C \Rightarrow C=3$

$$\therefore \int \frac{9x-7}{(x+2)(x^2-4x+13)}\,dx = \int \frac{-1}{x+2}\,dx + \int \frac{x+3}{x^2-4x+13}\,dx$$

$$= -\ln|x+2| + \int \frac{\frac{1}{2}(2x-4)+5}{x^2-4x+13}\,dx$$

$$= -\ln|x+2| + \frac{1}{2}\int \frac{2x-4}{x^2-4x+13}\,dx + 5\int \frac{1}{(x-2)^2+9}\,dx$$

$$= -\ln|x+2| + \frac{1}{2}\ln|x^2-4x+13| + \frac{5}{9}\int \frac{1}{\left(\frac{x-2}{3}\right)^2+1}\,dx$$

$$= -\ln|x+2| + \frac{1}{2}\ln|x^2-4x+13| + \frac{5}{9}\cdot\frac{1}{\frac{1}{3}}\tan^{-1}\left(\frac{x-2}{3}\right) + C$$

$$= -\ln|x+2| + \frac{1}{2}\ln|x^2-4x+13| + \frac{5}{3}\tan^{-1}\left(\frac{x-2}{3}\right) + C$$

1 (f) $$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x^2 \sin^5 x}{1+\cos^3 x}\, dx$$

Let $f(x) = \dfrac{x^2 \sin^5 x}{1+\cos^3 x}$

Then $$f(-x) = \frac{(-x)^2 \sin^5(-x)}{1+\cos^3(-x)}$$

$$= \frac{x^2 [-\sin(x)]^5}{1+(\cos x)^3}$$

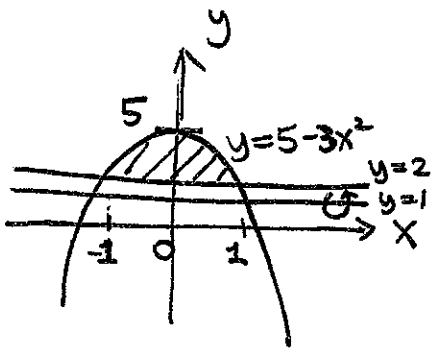
$$= \frac{-x^2 \sin^5 x}{1+\cos^3 x}$$

$$= -f(x)$$

$\therefore$ $f(x)$ is odd fn.

$$\therefore \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x^2 \sin^5 x}{1+\cos^3 x}\, dx = 0$$ //

2 (a) $\begin{cases} y = 5 - 3x^2 \\ y = 2 \end{cases}$



$\Rightarrow \quad 2 = 5 - 3x^2$

$\Rightarrow \quad 3x^2 = 3$

$\Rightarrow \quad x^2 = 1$

$\Rightarrow \quad x = \pm 1$

$$\text{Volume} = \int_{-1}^{1} \pi \left[(5-3x^2) - 1\right]^2 dx - \int_{-1}^{1} \pi (2-1)^2 dx$$

$$= \pi \int_{-1}^{1} (16 - 24x^2 + 9x^4 - 1)\, dx$$

$$= \pi \int_{-1}^{1} (15 - 24x^2 + 9x^4)\, dx$$

$$= \pi \left[15x - 24\frac{x^3}{3} + 9\frac{x^5}{5}\right]_{-1}^{1}$$

$$= \pi \left[\left(15 - 8 + \frac{9}{5}\right) - \left(-15 + 8 - \frac{9}{5}\right)\right]$$

$$= \frac{88}{5}\pi \text{ cubic units}$$

//

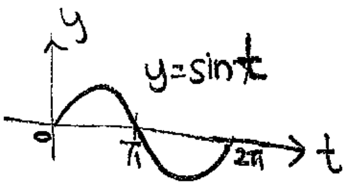
2(b) $\begin{cases} x = t - \sin t \\ y = 1 - \cos t \end{cases}$, $0 \leq t \leq 2\pi$

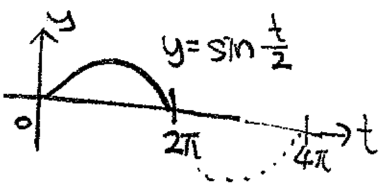
$$x'(t) = 1 - \cos t$$
$$y'(t) = \sin t$$

$$\text{Arc length} = \int_{t=0}^{2\pi} \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$
$$= \int_0^{2\pi} \sqrt{(1-\cos t)^2 + \sin^2 t}\, dt$$
$$= \int_0^{2\pi} \sqrt{1 - 2\cos t + \underbrace{\cos^2 t + \sin^2 t}_{=1}}\, dt$$
$$= \int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt$$
$$= \int_0^{2\pi} \sqrt{4\sin^2\left(\tfrac{t}{2}\right)}\, dt$$
$$= 2\int_0^{2\pi} \left|\sin\left(\tfrac{t}{2}\right)\right| dt$$
$$= 2\int_0^{2\pi} \sin\left(\tfrac{t}{2}\right) dt$$
$$= 2\left[\frac{-\cos\left(\frac{t}{2}\right)}{\frac{1}{2}}\right]_0^{2\pi}$$
$$= -4(\cos\pi - \cos 0)$$
$$= -4[(-1) - 1] = 8 \text{ units}$$

$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$

$\Rightarrow 1 - \cos t = 2\sin^2\left(\frac{t}{2}\right)$





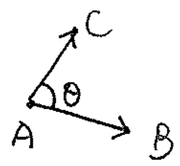
$\sin\left(\frac{t}{2}\right) \geq 0$ for $t \in [0, 2\pi]$

Section B

Q3. (a) $\vec{AB} = \vec{OB} - \vec{OA}$

$= (\vec{i} - 3\vec{j} + 2\vec{k}) - (3\vec{i} - 2\vec{j} + \vec{k})$

$= -2\vec{i} - \vec{j} + \vec{k}$



$\vec{AC} = \vec{OC} - \vec{OA}$

$= (2\vec{i} - \vec{j} - 3\vec{k}) - (3\vec{i} - 2\vec{j} + \vec{k})$

$= -\vec{i} + \vec{j} - 4\vec{k}$

$$\cos\theta = \frac{\vec{AB}\cdot\vec{AC}}{|\vec{AB}|\,|\vec{AC}|} = \frac{(-2)(-1) + (-1)(1) + (1)(-4)}{\sqrt{(-2)^2 + (-1)^2 + 1^2}\cdot\sqrt{(-1)^2 + 1^2 + (-4)^2}}$$

$$= \frac{-3}{\sqrt{6}\cdot\sqrt{18}}$$

$\therefore \angle BAC = \theta = 106.78°$

(b)

$$\vec{AB}\times\vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & -1 & 1 \\ -1 & 1 & -4 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} -1 & 1 \\ 1 & -4 \end{vmatrix} - \vec{j}\begin{vmatrix} -2 & 1 \\ -1 & -4 \end{vmatrix} + \vec{k}\begin{vmatrix} -2 & -1 \\ -1 & 1 \end{vmatrix}$$

$$= \vec{i}(4-1) - \vec{j}(8-(-1)) + \vec{k}(-2-1)$$

$$= 3\vec{i} - 9\vec{j} - 3\vec{k}$$

$$|\vec{AB}\times\vec{AC}| = \sqrt{3^2 + (-9)^2 + (-3)^2} = \sqrt{99}$$

$\therefore$ A unit vector perpendicular to $\vec{AB}$ and $\vec{AC}$ is

$$\frac{\vec{AB}\times\vec{AC}}{|\vec{AB}\times\vec{AC}|} = \frac{1}{\sqrt{11}}\vec{i} - \frac{3}{\sqrt{11}}\vec{j} - \frac{1}{\sqrt{11}}\vec{k}$$

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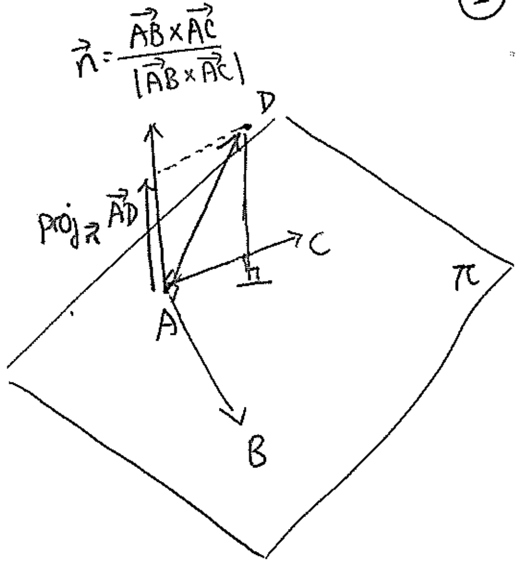
Q3 (c) From (b), let $\vec{n} = \dfrac{\vec{AB} \times \vec{AC}}{|\vec{AB} \times \vec{AC}|}$

$$= \frac{1}{\sqrt{11}}\vec{i} - \frac{3}{\sqrt{11}}\vec{j} - \frac{1}{\sqrt{11}}\vec{k}$$



$$\vec{AD} = \vec{OD} - \vec{OA}$$

$$= (-4\vec{i} - \vec{j} + 2\vec{k}) - (3\vec{i} - 2\vec{j} + \vec{k})$$

$$= -7\vec{i} + \vec{j} + \vec{k}$$

shortest distance $= |proj_{\vec{n}} \vec{AD}|$

$$= \frac{|\vec{AD} \cdot \vec{n}|}{|\vec{n}|}$$

$$= \frac{\left|-\frac{7}{\sqrt{11}} - \frac{3}{\sqrt{11}} - \frac{1}{\sqrt{11}}\right|}{1}$$

$$= \left|\frac{-11}{\sqrt{11}}\right|$$

$$= \sqrt{11} \;//$$

Q4 (a) $|2-\sqrt{2}i| = \sqrt{2^2 + (-\sqrt{2})^2} = \sqrt{6}$

$\arg(2-\sqrt{2}i) = \tan^{-1}\left(\frac{-\sqrt{2}}{2}\right) = -0.61548$

$\therefore 2-\sqrt{2}i = \sqrt{6}\, e^{i(-0.61548)}$

$$(2-\sqrt{2}i)^5 = \left[\sqrt{6}\, e^{i(-0.61548)}\right]^5$$

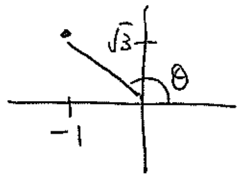
$$= 6^{\frac{5}{2}} e^{i(-0.61548 \times 5)}$$

$$= 6^{\frac{5}{2}} e^{-3.077i}$$

Q4 (b) $z^3 + 1 = \sqrt{3}\,i$

$$\Rightarrow z^3 = \sqrt{3}\,i - 1$$

$$= 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$



$$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right) + \pi$$

$$= -\frac{\pi}{3} + \pi = \frac{2\pi}{3}$$

$$r = \sqrt{(\sqrt{3})^2 + (-1)^2} = 2$$

$$\therefore z_k = 2^{\frac{1}{3}}\left[\cos\left(\frac{2k\pi + \frac{2\pi}{3}}{3}\right) + i\sin\left(\frac{2k\pi + \frac{2\pi}{3}}{3}\right)\right]$$

for $k = 0, 1, 2$

$\therefore$ The solutions are

$$z_0 = 2^{\frac{1}{3}}\left(\cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}\right)$$

$$z_1 = 2^{\frac{1}{3}}\left(\cos\frac{8\pi}{9} + i\sin\frac{8\pi}{9}\right)$$

$$z_2 = 2^{\frac{1}{3}}\left(\cos\frac{14\pi}{9} + i\sin\frac{14\pi}{9}\right)$$

$$= 2^{\frac{1}{3}}\left[\cos\left(\frac{-4\pi}{9}\right) + i\sin\left(\frac{-4\pi}{9}\right)\right]$$

Q5(a)

$$\left(\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 2 & 1 & -1 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 & 0 & 1 \end{array}\right) \underset{\substack{R_3 + 3\times R_1 \\ R_1 \times (-1)}}{\overset{R_2 + 2\times R_1}{\sim}} \left(\begin{array}{ccc|ccc} 1 & -1 & -2 & -1 & 0 & 0 \\ 0 & 3 & 3 & 2 & 1 & 0 \\ 0 & 5 & 6 & 3 & 0 & 1 \end{array}\right)$$

$$\underset{R_3 - \frac{5}{3}\times R_2}{\overset{R_2 \times \frac{1}{3}}{\sim}} \left(\begin{array}{ccc|ccc} 1 & -1 & -2 & -1 & 0 & 0 \\ 0 & 1 & 1 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{5}{3} & 1 \end{array}\right)$$

$$\underset{R_2 - R_3}{\overset{R_1 + R_2}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & -\frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{5}{3} & 1 \end{array}\right)$$

$$\overset{R_1+R_3}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{2}{3} & -\frac{4}{3} & 1 \\ 0 & 1 & 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{5}{3} & 1 \end{array}\right)$$

$$\therefore \quad A^{-1} = \begin{pmatrix} -\frac{2}{3} & -\frac{4}{3} & 1 \\ 1 & 2 & -1 \\ -\frac{1}{3} & -\frac{5}{3} & 1 \end{pmatrix}$$

5 (b)

$$|A| = \begin{vmatrix} -1 & 1 & 2 \\ 2 & 1 & -1 \\ 3 & 2 & 0 \end{vmatrix}$$

$$= 3 \times \begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} - 2 \times \begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix} + 0 \times \begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix}$$

$$= 3 \times (-1-2) - 2 \times (1-4) + 0$$

$$= -3$$

$$|A^T A| = |A^T|\,|A| = |A|\,|A| = (-3)^2 = 9$$

$$|A^5| = |A|^5 = (-3)^5 = -243$$

$$|A^{-1}| = \frac{1}{|A|} = -\frac{1}{3}$$

Q5 (c) $\left(\begin{array}{cccc|c} 1 & 3 & -2 & -1 & 1 \\ 2 & 5 & -1 & 3 & 2 \\ -1 & -1 & -3 & 2 & -3 \end{array}\right)$

$$\underset{R_3+R_1}{\overset{R_2-2\times R_1}{\sim}} \left(\begin{array}{cccc|c} 1 & 3 & -2 & -1 & 1 \\ 0 & -1 & 3 & 5 & 0 \\ 0 & 2 & -5 & 1 & -2 \end{array}\right)$$

$$\underset{R_3+2\times R_2}{\overset{R_2\times(-1)}{\sim}} \left(\begin{array}{cccc|c} 1 & 3 & -2 & -1 & 1 \\ 0 & 1 & -3 & -5 & 0 \\ 0 & 0 & 1 & 11 & -2 \end{array}\right)$$

$$\therefore \begin{cases} x_1 + 3x_2 - 2x_3 - x_4 = 1 \\ x_2 - 3x_3 - 5x_4 = 0 \\ x_3 + 11x_4 = -2 \end{cases}$$

Let $x_4 = t$, $t \in \mathbb{R}$.

Then $x_3 = -2 - 11t$

$x_2 = 3x_3 + 5x_4 = 3(-2-11t) + 5t$

$= -6 - 28t$

$x_1 = 1 - 3x_2 + 2x_3 + x_4$

$= 1 - 3(-6-28t) + 2(-2-11t) + t$

$= 15 + 63t$

$$\therefore \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 15+63t \\ -6-28t \\ -2-11t \\ t \end{pmatrix}, \text{ where } t \in \mathbb{R}.$$