Test 1

Q1. (10%) Use contrapositive method to prove: If $x^3 - 1$ is even, then x is odd, $x \in \boldsymbol{Z}$.

Proof:

Contrapositive method, we prove: If x is even, then $x^3 - 1$ is odd, for $x \in \boldsymbol{Z}$.

Suppose x is even and $x \in \boldsymbol{Z}$, then x can be written as $2k, k \in \boldsymbol{Z}$.

$$x^3 - 1 = (2k)^3 - 1 = 8k^3 - 1 = 8k^3 - 2 + 1 = 2(4k^3 - 1) + 1 = 2b + 1$$

$\therefore x^3 - 1$ must be ODD. ■

Q2. (10%) Richter earthquake scale is determined by logarithm function: $= \log_{10} \frac{I_c}{I_n}$, where I_c is the intensity of quake (100km from the epicenter), and I_n is the intensity of a standard day. It is given that an earthquake event A is 8,000,000 stronger than another earthquake event B. How much larger is its magnitude on the Richter scale?

Solution:

Let the I_{c1} be the intensity of earthquake event A.

Let the I_{c2} be the intensity of earthquake event B.

Hence,

$$\frac{I_{c1}}{I_{c2}} = 8000000$$

$$I_{c1} = 8000000 I_{c2}$$

Let the R_1 be the intensity of earthquake event A on the Richter scale

Let the R_2 be the intensity of earthquake event B on the Richter scale

$$R_1 - R_2 = \log_{10} \frac{I_{c1}}{I_n} - \log_{10} \frac{I_{c2}}{I_n}$$

$$R_1 - R_2 = \log_{10} \frac{8000000 I_{c2}}{I_n} - \log_{10} \frac{I_{c2}}{I_n}$$

$$R_1 - R_2 = (\log_{10} 8000000 I_{c2} - \log_{10} I_n) - (\log_{10} I_{c2} - \log_{10} I_n)$$
$$R_1 - R_2 = (\log_{10} 8000000 I_{c2}) - (\log_{10} I_{c2})$$
$$R_1 - R_2 = \log_{10} \frac{8000000 I_{c2}}{I_{c2}}$$
$$R_1 - R_2 = \log_{10} 8000000$$
$$R_1 - R_2 = 6.9$$

Q3. (10%)

a. (5%) Use direct proof to show: If $a|b$, then $a^2|b^2$.
b. (5%) Prove: If x and y are even integers, then $x + y$ is also even.

Proof:

a. $a|b$, it means $b = (k)(a)$, where k is an integer.
Note: we target to show $b^2 = \omega a^2$
Now $b^2 = (ka)^2 = (k)^2 a^2$, where k^2 is an integer.
So, $a^2|b^2$. ■

b. Suppose $x + y$ are even integers.
It means $x = 2k$ and $y = 2n$, where $k, n \in \boldsymbol{Z}$.
$x + y = 2k + 2n = 2(k + n) = 2\omega$, where $\omega \in \boldsymbol{Z}$.
$\therefore x + y$ must be even. ■

Q4. (10%)

a. (5%) Perform subtraction of the below and your answer must be in HEX:
(2 numbers in HEX) $FD9 - 8AC$
b. (5%) Find the 2's complement of $a = 89$, and $b = -110$.
Find $a + b$ in 2's complement.

Solution:

a. $(FD9)_{16} = (1111\ 1101\ 1001)_2$ and $(8AC)_{16} = (1000\ 1010\ 1100)_2$
Convert $(1000\ 1010\ 1100)_2$ to the 2's complement: $(0111\ 0101\ 0100)_2$

	1111 1101 1001
+	0111 0101 0100
1	0111 0010 1101

MSB 1 is the overflow bit.
Convert $(0111\ 0010\ 1101)_2$ to equivalent hexadecimal number: $(72D)_{16}$

b. $a = (89)_{10}$, 2's complement is in 8-bit form $(0101\ 1001)_2$.

$b = (-110)_{10}$, 2's complement is in 8-bit form $(1001\ 0010)_2$.
$a + b$ is

	0101 1001
+	1001 0010
	1110 1011

$a + b$ in 2's complement is $(1110\ 1011)_2$.

Q5. (10%) $A = \{-377, -194, -83, -26, -5, -2, 1, 22, 79, 190, 373\}$. Use set builder to write it.

Solution:

$$A = \{3x^3 - 2 \mid x \in \boldsymbol{Z}, -5 \leq x \leq 5\}$$

Q6. (10%)

a. (5%) Find the 5th minimum value of the IEEE 754 32bits floating point notation,
b. (5%) Then find its corresponding value in decimal.

Solution:

a. Minimum value:

$$1\ [1111\ 1110][1111\ 1111\ 1111\ 1111\ 1111\ 111]$$

5th minimum value:

$1\ [1111\ 1110][1111\ 1111\ 1111\ 1111\ 1111\ 011]$ 5%

b. $(-1)^{(1)} \times 2^{(254-127)} \times \left(1 + (1 - 2^{-21} - 2^{-23})\right) = -2^{127} \times (2 - 2^{-21} - 2^{-23}) \approx -3.403 \times 10^{38}$ 5%

Q7. (10%) The following 2 numbers are in IEEE 754 floating point format:

A [0][1010 1001][1011 0011 0100 1001 0000 000]

B [0][1010 0011][1111 0101 0001 0000 0000 000]

Find $A + B$ in IEEE 754 format.

Solution:

$$A = [0][1010\ 1001][1011\ 0011\ 0100\ 0000\ 0000\ 000]$$

B exponent is $1001\ 0011$, which is 6 less than A.

B exponent change to $1010\ 1001$

So, mantissa shift by 6 from 1.1111 0101 0001 0000 0000 000

$$0.00\ 0001\ 1111\ 0101\ 0001\ \cdots$$

B becomes $[0][1010\ 1001][0.0000\ 0111\ 1101\ 0100\ 01 \cdots]$

A $[0][1010\ 1001][1.1011\ 0011\ 0100\ 1001\ 00 \cdots]$

$A + B$ $[0][1010\ 1001][1.1011\ 1011\ 0001\ 1101\ 01 \cdots]$

$A + B =$ $[0][1010\ 1001][1011\ 1011\ 0001\ 1101\ 0100\ 000]$

Q8. (20%)

a. (6%) Are the functions, surjective, injective and bijective invertible. Use plain English language to explain why they are, and they are not.

b. (14%)

   i. (4%) Let f: $\{a, b, c, d, e\} \to \{1, 2, 3, 4\}$, $f(a) = 3, f(b) = 2, f(c) = 1, f(d) = 4$, $f(e) = 2$. Is f a surjective, injective, bijective or else? Explain your answer in few words.

   ii. (4%) Is the function f: $\boldsymbol{Z} \to \boldsymbol{Z}$, $f(x) = x^2$, injective, surjective or else? Explain your answer in few words.

   iii. (4%) Is the function f: $\boldsymbol{Z} \to \boldsymbol{Z}$, $f(x) = x + 1$, surjective or injective? Explain your answer in few words.

   iv. (2%) The CityU student ID number is co-domain, and the CityU students are elements of the domain. Is this mapping process injective, surjective, or bijective? Explain your answer in few words.

Solution:

a. (6%) **surjective is NOT invertible**, because several elements of domain can map into the same element of co-domain 2%
   Injective is NOT invertible, it is one-to-one but there are elements not mapped from the domain. No inverse can be found for those elements. 2%
   Bijective is invertible, also one-to-to 2%
   no explanation only gets 1% for correct answer.

b. (14%) Just correct answer without explanation will get 1% only. Need wordings to get full 4 %.

   i. It is surjective because 4 elements of co-domain are images of elements of domain. But they overlapped so not one-to-one, so not injective. 4%

ii. The function is not injective, surjective, or bijective, because there is no integer x, $x^2 = -1$. 4%

iii. **The function is onto, and one-to-one, so bijective.** It is because for every integer y in the co-domain, there is a unique integer x such that f(x) =y= x+1. 4%

iv. Bijective, all students have a unique ID number. 2%

Q9. (10%) Let $g: \{a, b, c\} \to \{a, b, c\}$, $g(a) = b, g(b) = c, g(c) = a$. Let $f: \{a, b, c\} \to \{1,2,3\}$, $f(a) = 3, f(b) = 2, f(c) = 1$. What is the composition function *f* of *g*, $f \circ g$, and what is the composition function $g \circ f$.

Solution:

$(f \circ g)(a) = f\big(g(a)\big) = f(b) = 2.$ 3%

$(f \circ g)(b) = f\big(g(b)\big) = f(c) = 1.$ 3%

$(f \circ g)(c) = f\big(g(c)\big) = f(a) = 3.$ 3%

$g \circ f$ is undefined as the range of *f* is not a subset of the domain of *g*. *1%*