EXE 3

**Question 1**

Suppose $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$ are $n$-vectors. Determine whether each expression below makes sense (i.e., uses valid notation). If the expression does make sense, give its dimensions.

(a) $\begin{bmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \vdots \\ \boldsymbol{a}_n \end{bmatrix}$

(b) $\begin{bmatrix} \boldsymbol{a}_1^{\mathrm{T}} \\ \boldsymbol{a}_2^{\mathrm{T}} \\ \vdots \\ \boldsymbol{a}_n^{\mathrm{T}} \end{bmatrix}$

(c) $[\boldsymbol{a}_1 \quad \boldsymbol{a}_2 \quad \cdots \quad \boldsymbol{a}_n]$

(d) $[\boldsymbol{a}_1^{\mathrm{T}} \quad \boldsymbol{a}_2^{\mathrm{T}} \quad \cdots \quad \boldsymbol{a}_n^{\mathrm{T}}]$

(e) $[\boldsymbol{a}_1^{\mathrm{T}} \quad \boldsymbol{a}_2 \quad \cdots \quad \boldsymbol{a}_n^{\mathrm{T}}]$

**Question 2**

Suppose the block matrix

$$\begin{bmatrix} \boldsymbol{A} & \boldsymbol{I} \\ \boldsymbol{I} & \boldsymbol{C} \end{bmatrix}$$

makes sense, where $\boldsymbol{A}$ is a $p \times q$ matrix. What are the dimensions of $\boldsymbol{C}$?

**Question 3**

Assuming the matrix

$$\boldsymbol{K} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{A}^{\mathrm{T}} \\ \boldsymbol{A} & \emptyset \end{bmatrix}$$

makes sense, which of the following statements must be true? (`Must be true' means that it follows with no additional assumptions.)

(a) $\boldsymbol{K}$ is square.
(b) $\boldsymbol{A}$ is square or wide.
(c) $\boldsymbol{K}$ is symmetric.
(d) The identity and zero submatrices in $\boldsymbol{K}$ have the same dimensions.
(e) The zero submatrix is square.

**Question 4**

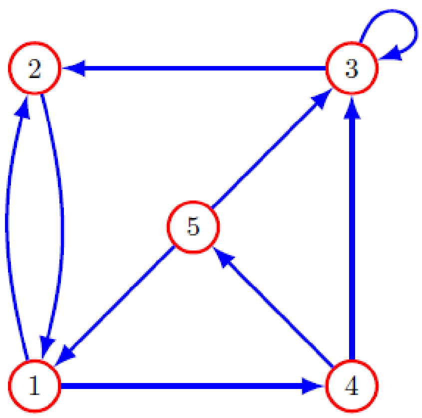
Let $\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} 1 & 0 & 2 \\ -2 & 3 & -1 \end{bmatrix}$

(a) Find $\boldsymbol{AB}^{\mathrm{T}}$ and $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{B}$

(b) For each of the $\boldsymbol{AB}^{\mathrm{T}}$ and $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{B}$**,** determine whether it is invertible, and find its inverse if it is invertible?

**Question 5**

Suppose $\boldsymbol{A}$ is the adjacency matrix of a directed graph.




What are the entries of the vector $\boldsymbol{A}\mathbf{1}$? What are the entries of the vector $\boldsymbol{A}^{\mathrm{T}}\mathbf{1}$?

**Question 6**

For each of the following matrixes, describe in words how in words how $\boldsymbol{x}$ and $\boldsymbol{y} = \boldsymbol{Ax}$ are related. In each case $\boldsymbol{x}$ and $\boldsymbol{y}$ are $n$ vectors, with $n = 3k$.

(a) $\boldsymbol{A} = \begin{bmatrix} \emptyset & \emptyset & \boldsymbol{I}_k \\ \emptyset & \boldsymbol{I}_k & \emptyset \\ \boldsymbol{I}_k & \emptyset & \emptyset \end{bmatrix}$

(b) $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{E} & \emptyset & \emptyset \\ \emptyset & \boldsymbol{E} & \emptyset \\ \emptyset & \emptyset & \boldsymbol{E} \end{bmatrix}$, where is a $k \times k$ matrix with all entries equal to $\frac{1}{k}$.

**Question 7**

We consider a set of $n$ currencies, labeled $1, \ldots, n$. (These might correspond to USD, RMB, EUR, and so on.). At a particular time the exchange or conversion rates among the n currencies are given by an $n \times n$ (exchange rate) matrix $\boldsymbol{R}$, where $r_{ij}$ is the amount of currency $i$ that you can buy for one unit of currency $j$. The exchange rates include commission charges, so we have $r_{ij}r_{ji} < 1$ for all i 6= j. You can assume that $r_{ii} = 1$.

Suppose $\boldsymbol{y} = \boldsymbol{Rx}$, where $\boldsymbol{x}$ is a vector (with nonnegative entries) that represents the amounts of the currencies that we hold. What is $y_i$? Your answer should be in English.

**Question 8**

(a) Work out the complexity of computing the $m$ -vector $\boldsymbol{Ax}$, where $\boldsymbol{A}$ is an $m \times n$ matrix and $\boldsymbol{x}$ is an $n$ –vector. (The number of additions and the number of multiplications)

(b) Consider that we would like to compute computing the $\boldsymbol{y} = \boldsymbol{ABx}$, where $\boldsymbol{A}$ and $\boldsymbol{B}$ are $n \times n$ matrices and $\boldsymbol{x}$ is an $n$ –vector. There are two methods to compute $\boldsymbol{y}$**.**
The first method is to comput**e** $\boldsymbol{Bx}$ first, while the second method is to compute $\boldsymbol{AB}$ first. Which method is better? Explain.

**Question 9**

Patients and symptoms. Each of a set of $N$ patients can exhibit any number of a set of $n$ symptoms. We express this as an $N \times n$ matrix $\boldsymbol{S}$, with

$$s_{ij} = \begin{cases} 1 & \text{patient } i \text{ exhibits sumptom } j \\ 0 & \text{patient } i \text{ does not exhibits sumptom } j \end{cases}$$

Give simple English descriptions of the following expressions. Include the dimensions, and describe the entries.

(a) $\boldsymbol{S1}$.

(b) $\boldsymbol{S}^{\mathrm{T}}\boldsymbol{1}$.

**Question 10**

We consider $m$ students, $n$ classes, and $p$ majors. Each student can be in any number of the classes , and can have any number of the majors (although the common values would be 0, 1, or 2). The data about the students' classes and majors are given by an $m \times n$ matrix $\boldsymbol{C}$ and an $m \times p$ matrix $\boldsymbol{M}$, where

$$C_{ij} = \begin{cases} 1 & \text{student } i \text{ is class } j \\ 0 & \text{student } i \text{ is not class } j \end{cases}$$

$$M_{ik} = \begin{cases} 1 & \text{student } i \text{ is in major } k \\ 0 & \text{student } i \text{ is not in major } k \end{cases}$$

(a) Let $\boldsymbol{e}$ be the $n$ -vector with $e_j$ being the enrollment in class $j$. Express $\boldsymbol{e}$ using matrix notation, in terms of the matrices $\boldsymbol{C}$ and/or $\boldsymbol{M}$.

(b) Define the $n \times p$ matrix $\boldsymbol{S}$ where $C_{jk}$ is the total number of students in class $j$ with major $k$. Express $\boldsymbol{S}$ using matrix notation, in terms of the matrices $\boldsymbol{C}$ and $\boldsymbol{M}$.

**Question 11**

A student says that for any square matrix $\boldsymbol{A}$,

$(\boldsymbol{A} + \boldsymbol{I})^3 = \boldsymbol{A}^3 + 3\boldsymbol{A}^2 + 3\boldsymbol{A} + \boldsymbol{I}$

Is she right? If she is, explain why; if she is wrong, give a counterexample, i.e., a square matrix $\boldsymbol{A}$ for which it does not hold.

**Question 12**
A student says that for any square matrices $\boldsymbol{A}$ and $\boldsymbol{B}$
$(\boldsymbol{A}+\boldsymbol{B})^3 = \boldsymbol{A}^3 + 3\boldsymbol{A}^2\boldsymbol{B} + 3\boldsymbol{A}\boldsymbol{B}^2 + \boldsymbol{I}$
Is she right? If she is, explain why; if she is wrong, explain.

**Question 13**
True or false, For square matrix, with a reason if true or a counterexample if false:
(a) The determinant of $\boldsymbol{I}+\boldsymbol{A}$ is $1 + |\boldsymbol{A}|$.
(b) The determinant of $\boldsymbol{ABC}$ is $|\boldsymbol{A}||\boldsymbol{B}||\boldsymbol{C}|$.
(c) The determinant of $4\boldsymbol{A}$ is $\boldsymbol{4}|\boldsymbol{A}|$.
(d) The determinant **of** $\boldsymbol{AB}-\boldsymbol{BA}$ is zero. Tray an example with $\boldsymbol{A} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$

**Question 14**
Do these matrices have determinant 0, 1, 2, or 3"

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \boldsymbol{B} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad \boldsymbol{C} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} \end{bmatrix}$$

**Question 15**
The inverse of 2-by-2 matrix seems to have determinant =1:

$$|\boldsymbol{A}^{-1}| = \left|\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right| = \frac{ad-bc}{ad-bc} = 1$$

What is wrong with this calculation? What is the correct $|\boldsymbol{A}^{-1}|$**?**

**Question 16**
Find the determinants of $\boldsymbol{A}$**,** $\boldsymbol{A}^{-1}$**,** and $\boldsymbol{A}-\lambda\boldsymbol{I}$

$$\boldsymbol{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},\ \boldsymbol{A}^{-1} = \frac{\mathbf{1}}{\mathbf{3}}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix},\ \boldsymbol{A}-\lambda\boldsymbol{I} = \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}$$

Which number of $\lambda$ lead to $|\boldsymbol{A}-\lambda\boldsymbol{I}|$=0?

**Question 17**
Let $\boldsymbol{A}$ be the $5 \times 5$ matrix:

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

(a) How is $\boldsymbol{Ax}$ related to $\boldsymbol{x}$? Your answer should be in English.
(b) What is $\boldsymbol{A}^5$? Hint. The answer should make sense, given your answer to part (a).

**Question 18**

Find the inverses of the matrices

$\boldsymbol{A} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$, $\boldsymbol{A} = \begin{pmatrix} 6 & 4 \\ 2 & 3 \end{pmatrix}$,

**Question 19**

Prove that if $\boldsymbol{A}$ and $\boldsymbol{B}$ are same-sized invertible matrices , then $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$.

**Question 20**

Prove that if a matrix is invertible, then its inverse is unique.

**Question 21**

$\boldsymbol{A}$ and $\boldsymbol{B}$ are matrices. Prove that $(\boldsymbol{AB})^T = \boldsymbol{B}^T\boldsymbol{A}^T$.

**Question 22**

Prove that if $\boldsymbol{A}$ and $\boldsymbol{A}^T$ are invertible, then $(\boldsymbol{A}^T)^{-1} = (\boldsymbol{A}^{-1})^{\boldsymbol{T}}$.

**Question 23**

**Let** $\boldsymbol{A}$ and $\boldsymbol{B}$ are $2 \times 2$ matrix. If $\boldsymbol{A}\,\boldsymbol{B} = \boldsymbol{2I},$ show that

$|\boldsymbol{A}| = \frac{4}{|\boldsymbol{B}|}$

**Question 24**

For the following matrix find $\boldsymbol{A}^{-1}$ **by trail and error (with 1's and 0's in entries).**

(a) $\boldsymbol{A} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ (b) $\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

**Question 25**

(a) If $\boldsymbol{A}$ is invertible and $\boldsymbol{AB} = \boldsymbol{AC}$, prove that $\boldsymbol{B} = \boldsymbol{C}$

(b) If $\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, find two different matrices such that $\boldsymbol{AB} = \boldsymbol{AC}$**.**

**Question 26**

Suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ are square matrix.

True or false (given a reason if true or a 2-by2 example if false

(a) If $\boldsymbol{A}$ is not invertible, then $\boldsymbol{AB}$ is not invertible.
(b) $|\boldsymbol{A}-\boldsymbol{B}|$=$|\boldsymbol{A}|-|\boldsymbol{B}|$
(c) $\boldsymbol{AB}$ and $\boldsymbol{BA}$ has the same determinant.

(a) True $|\boldsymbol{AB}| = |\boldsymbol{A}||\boldsymbol{B}|$ => $|\boldsymbol{AB}| = 0$, inverse does not exist.
(b) False use the definition of determinant of 2-by-2
(c) True

**Question 27**

What is wrong with this proof that $|\boldsymbol{P}| = 1$?

$$\boldsymbol{P} = \boldsymbol{A}(\boldsymbol{A}^{\boldsymbol{T}}\boldsymbol{A})^{-\boldsymbol{1}}\boldsymbol{A}^{\boldsymbol{T}} \text{ so } |\boldsymbol{P}| = |\boldsymbol{A}|\frac{\boldsymbol{1}}{|\boldsymbol{A}^{\boldsymbol{T}}||\boldsymbol{A}|}|\boldsymbol{A}^{\boldsymbol{T}}|$$

Note that A may not be a square matrix.

**Question 28**

Apply the elimination method to solve

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix} \boldsymbol{x} = \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$$

How do you know this system has no solution?

Change the number 6 so there is(are) solution(s)

**Question 29**

**Solve the following system with elimination method.**

$$\begin{aligned} 2x_1 + 4x_2 - 2x_3 &= 2 \\ 4x_1 + 9x_2 - 3x_3 &= 8 \\ -2x_1 - 3x_2 + 7x_3 &= 10 \end{aligned}$$

**Question 30**

Choose the number of $a, b, c, d$ in the following system so that there is
(a) No solution
(b) Infinitely many solution

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & d \end{bmatrix} \boldsymbol{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

**Question 31**

Find $\boldsymbol{A^{-1}}$ **and** $\boldsymbol{B^{-1}}$ (if exists) by elimination

$$\boldsymbol{A} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

**Question 32**

Find $\boldsymbol{A^{-1}}$ **and** $\boldsymbol{B^{-1}}$ (if exists) by elimination

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

**Question 33**

True or false

(a) A 4-by-4 matrix with a row of zeros is not invertible.
(b) Every matrix with 1's in the main diagonal is invertible.
(c) If $\boldsymbol{A}$ **is** invertible then $\boldsymbol{A^{-1}}$ and $\boldsymbol{A^2}$ are invertible.

(a) True if If $\boldsymbol{A}$ is with a row of zeros**,** then $\boldsymbol{AB}$ is with a row of zeros too. Not possible to have $\boldsymbol{AB} = \boldsymbol{I}$
(b) False a one matrix is not invertible.
(c) True . The inverse of $\boldsymbol{A^{-1}}$ is $\boldsymbol{A}$**.** The inverse of $\boldsymbol{A^2}$ is $(\boldsymbol{AA})^{-1} = (\boldsymbol{A^{-1}})^2$

**Question 34**

Let $\boldsymbol{U}$ and $\boldsymbol{V}$ are $n \times n$ orthogonal. Show that $\boldsymbol{UV}$ and the $2n \times 2n$ matrix:

$\frac{1}{\sqrt{2}}\begin{bmatrix} \boldsymbol{U} & \boldsymbol{U} \\ \boldsymbol{V} & -\boldsymbol{V} \end{bmatrix}$ are orthogonal.

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**Question 35**

Suppose $\boldsymbol{A}$ is an $n \times n$ matrix and $\boldsymbol{x}$ is an $n$-vector. The product $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{Ax}$ is a scalar.

(a) Verify that $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{Ax} = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j a_{ij}$.
(b) Show that $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{x} = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{Ax}$
(c) Show that $\frac{1}{2}\boldsymbol{x}^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}} + \boldsymbol{A})\boldsymbol{x} = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{Ax}$
(d) Express $2x_1^2 - 3x_1x_2 - 3x_2^2$ as the product form $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{Ax}$

**Question 36**

Find the eigenvalues and eigenvectors of these two matrices

$$\boldsymbol{A} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}, \boldsymbol{A} + \boldsymbol{I} = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}$$

**Question 37**

(a) Find the eigenvalues and eigenvectors of

$$\boldsymbol{A} = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix},$$

(b) Evaluate $\boldsymbol{A}^4$

**Question 38**

In the lecture notes, given a directed graph, we discuss the way to get the number of paths with lengths $\boldsymbol{l}$ from node i to node j, by examining $\boldsymbol{A}^{\boldsymbol{l}}$, where $\boldsymbol{A}$ **is** the adjacency matrix

(a) Suppose we are only interested in the number of paths starting node-*i* only. Your answer should be based on the adjacency matrix rather than visual inspection. Discuss a better way to examine the number of paths with length $l$ from node i to all other nodes.
(b) Discuss a way to determine there is no path from node i to node *j*

**Question 39**

Suppose we have a data matrix

$$\boldsymbol{X} = (\boldsymbol{x_1} \quad \boldsymbol{x_2} \quad \boldsymbol{x_3} \quad \boldsymbol{x_4}) = \begin{pmatrix} 1 & 2.2 & 2.8 & 4 \\ 2 & 1.8 & 2.2 & 4 \end{pmatrix}$$

(a) What is the mean vector for this data matrix ?
(b) What is the covariance matrix?
(c) What are the eigenvectors and eigenvalues of the covariance matrix?
(d) Suppose we only keep one PCA component. What is the forward transform matrix to encode the data vectors?
(e) Using $\boldsymbol{x_2}$ to show the PC encoding process.
(f) Suppose we only keep one PCA component. What is the error to encode $\boldsymbol{x_2}$?

**Question 40**

Suppose we have a triangle on a 2D plane. There vertices are (0,0), (1,0), and (1,2).

(a) Draw the triangle
(b) What is the rotation matrix to rotate this triangle around (0,) with counter clockwise $\theta$
(c) If we apply a counter clockwise rotation with $\frac{\pi}{3}$ around (0,0), what are the new coordinates of the three vertices.
(d) Discuss the procedure to rotate the triangle around the vertex (1,0)

**Question 41**

Let $\boldsymbol{A} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{-2} \\ \mathbf{1} & \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{3} \end{bmatrix}$

(a) Evaluate $\boldsymbol{A}^3 - 5\boldsymbol{A}^2 + 8\boldsymbol{A} - 4\boldsymbol{I}$
(b) Find $\boldsymbol{A}^{-1}$

**Question 42**

**(a)** Show that $(\boldsymbol{A}^2 - \boldsymbol{A} + \boldsymbol{I})(\boldsymbol{A} + \boldsymbol{I}) = \boldsymbol{A}^3 + \boldsymbol{I}$

**(b)** Let $\boldsymbol{A} = \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}$. Evaluate $\boldsymbol{A}^2$ and $\boldsymbol{A}^3$ .

(c) Find the inverse of $\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$