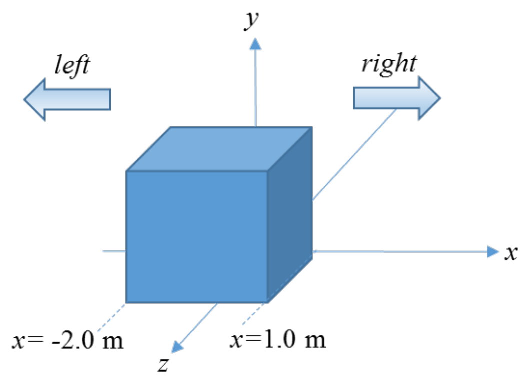
AP1202
Assignment 2
Due Date: 11:59 pm Sunday, October 18th, 2020
Please submit your assignment online via Canvas
(Total 100 marks)

Lecture 04: Gauss's Law

L04-01 (10 marks) A non-uniform electric field $\vec{E} = 3.0x\hat{i} + 4.0\hat{j}$ N/C pierces the Gaussian cube shown in the below figure. (a) What is the electric flux through the right face, the left face, and the top face (b) Find the charge q_{enc} enclosed by the cube.



Ans: This question is identical to the example in L04, slide 15, except that the cube is shifted to the left by 3 m and the linear dimension of the cube is now 3 m.

a) The right face at x=1.0 m, $\vec{A} = 9\text{m}\,\hat{i}$

$$\Phi_{right} = \int \vec{E} \cdot d\vec{A} = \int (3.0x)(dA)\hat{i} \cdot \hat{i}$$
$$= 3.0\int xdA = 3.0\int (1.0)dA = (3.0\text{N/C})(9.0\,\text{m}^2)$$
$$= 27\,\text{N} \cdot \text{m}^2/\text{C}$$

The left face at x=-2.0 m, $\vec{A} = -9\text{m}\,\hat{i}$

$$\Phi_{left} = \int \vec{E} \cdot d\vec{A} = \int (3.0x)(dA)\hat{i} \cdot -\hat{i}$$
$$= -3.0\int xdA = 3.0\int (-2.0)dA = (-6.0\text{N/C})(-9.0\,\text{m}^2)$$
$$= 54\,\text{N} \cdot \text{m}^2/\text{C}$$

A common mistake here is for not realizing that the area vector is pointing in the $-x$ direction.

The top face at y=3.0 m, $\vec{A} = 9\text{m}\,\hat{j}$

$$\Phi_{top} = \int \vec{E} \cdot d\vec{A} = \int (4.0)(dA)\hat{j} \cdot \hat{j}$$
$$= 4.0\int dA = 4.0\int dA = (4.0\text{N/C})(9.0\,\text{m}^2)$$
$$= 36\,\text{N} \cdot \text{m}^2/\text{C}$$

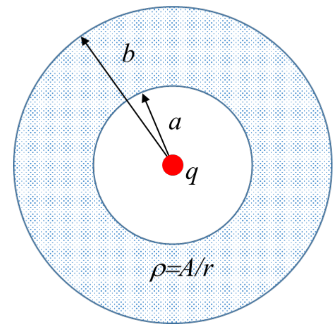
b) Since E is independent of y and z, the net flux in the y and z direction are zero, and only the left and right faces contribute to the enclosed charges

$$q_{enc} = \varepsilon_0(\Phi_{left} + \Phi_{right}) = (8.85 \times 10^{12}\,\text{C}^2/\text{N} \cdot \text{m}^2)(27 + 54)\text{N} \cdot \text{m}^2/\text{C}$$
$$= 7.17 \times 10^{-10}\,\text{C}$$

L04-02

(10 marks) A non-conducting spherical shell of inner radius *a=2.00 cm* and outer radius *b=2.40 cm* has a positive volume charge density *ρ=A/r*, where A is a constant and r is he distance from the center of the shell. In addition, a small point charge *q=45.0* fC is located at the center. What value should *A* have if the electric field in the shell $(a \le r \le b)$ is to be uniform?

(Note: $1\,\mathrm{fC} = 1\times 10^{-15}\,\mathrm{C}$)



Ans:

To find an expression for the electric field inside the shell in terms of *A* and the distance from the center of the shell, select *A* so the field does not depend on the distance. We use a Gaussian surface in the form of a sphere with radius r_g, concentric with the spherical shell and within it ($a < r_g < b$). Gauss' law will be used to find the magnitude of the electric field a distance r_g from the shell center. The charge that is both in the shell and within the Gaussian sphere is given by the integral $q_s = \int \rho\, dV$ over the portion of the shell within the Gaussian surface. Since the charge distribution has spherical symmetry, we may take dV to be the volume of a spherical shell with radius *r* and infinitesimal thickness *dr*: $dV = 4\pi r^2\, dr$. Thus,

$$q_s = 4\pi\int_a^{r_g} \rho r^2 dr = 4\pi\int_a^{r_g}\frac{A}{r}r^2 dr = 4\pi A\int_a^{r_g} r dr = 4\pi A\left[\frac{r^2}{2}\right]_a^{r_g} = 2\pi A(r_g^2 - a^2).$$

The total charge inside the Gaussian surface is

$$q + q_s = q + 2\pi A(r_g^2 - a^2).$$

The electric field is radial, so the flux through the Gaussian surface is $\Phi = 4\pi r_g^2 E$, where *E* is the magnitude of the field. Gauss' law yields

$$4\pi\varepsilon_0 E r_g^2 = q + 2\pi A(r_g^2 - a^2)$$

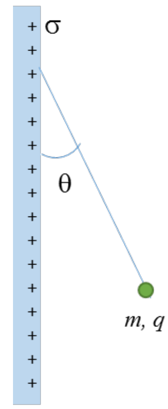
We solve for *E*:

$$E = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{r_g^2} + 2\pi A a^2 - \frac{2\pi A a^2}{r_g^2}\right].$$

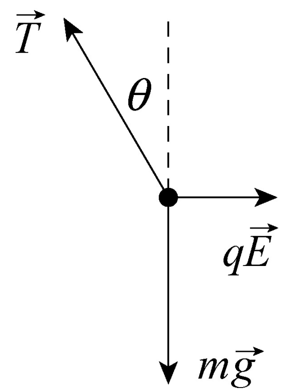
Note that the first and last terms in the bracket varies as a function of $1/r_g^2$, so their values will change depends on the value of r_g, while the second term is a constant provided both *A* and *a* are constants. For the field to be uniform, the first and last terms in the brackets must cancel. They do if $q - 2\pi A a^2 = 0$ or $A = q/2\pi a^2$. With $a = 2.00 \times 10^{-2}$ m and $q = 45.0 \times 10^{-15}$ C, we have

$A = 1.79 \times 10^{-11} \text{C/m}^2$.

L04-03 (10 marks) A small, nonconducting ball of mass $m = 1.0$ mg and charge $q = 2.0 \times 10^{-8}$ C (distributed uniformly through its volume) hangs from an insulating thread that makes an angle $\theta = 30°$ with a vertical, uniformly charged nonconducting sheet, as shown in the figure. Consider the gravitational force on the ball and assuming the sheet extends far vertically and into and out of the page, calculate the surface charge σ of the sheet.



Ans: The forces acting on the ball are shown in the diagram below. The gravitational force has magnitude *mg*, where *m* is the mass of the ball; the electrical force has magnitude *qE*, where *q* is the charge on the ball and *E* is the magnitude of the electric field at the position of the ball; and the tension in the thread is denoted by *T*.



The electric field produced by the plate is normal to the plate and points to the right. Since the ball is positively charged, the electric force on it also points to the right. The tension in the thread makes the angle θ (= 30°) with the vertical.

Since the ball is in equilibrium the net force on it vanishes. The sum of the horizontal components yields

$$qE - T\sin\theta = 0$$

and the sum of the vertical components yields

$$T\cos\theta - mg = 0 .$$

The expression $T = qE/\sin\theta$, from the first equation, is substituted into the second to obtain $qE = mg\tan\theta$. The electric field produced by a large uniform plane of charge is given by $E = \sigma/2\varepsilon_0$, where σ is the surface charge density. Thus,

$$\frac{q\sigma}{2\varepsilon_0} = mg\tan\theta$$

and

$$\sigma = \frac{2\varepsilon_0 mg\tan\theta}{q} = \frac{2\left(8.85\times10^{-12}\ \mathrm{C^2/N.m^2}\right)\left(1.0\times10^{-6}\ \mathrm{kg}\right)\left(9.8\ \mathrm{m/s^2}\right)\tan 30°}{2.0\times10^{-8}\ \mathrm{C}}$$
$$= 5.0\times10^{-9}\ \mathrm{C/m^2}.$$

Lecture 05: Electric Potential

L05-01 (10 marks) Two very large, parallel, conducting plates are 12 cm apart and have charges of equal magnitude and opposite sign on their facing surfaces. An electrostatic force of 3.3×10^{-15} N acts on an electron placed in between the plates and away from the edges of the plates. (a) Find the electric field at the position of the electron. (b) What is the potential difference between the two plates?

Ans:

(a)
$$E = F/e = \left(3.3\times10^{-15}\ \mathrm{N}\right)/\left(1.60\times10^{-19}\ \mathrm{C}\right) = 2.1\times10^{4}\ \mathrm{N/C} = 2.1\times10^{4}\ \mathrm{V/m}.$$

(b) $\Delta V = E\Delta s = (2.1\times10^{4}\ \mathrm{N/C})(0.12\mathrm{m}) = 2.5\times10^{3}\ \mathrm{V}$

L05-02 (10 marks) In the rectangle of the below Figure, the sides have lengths 5.0 cm and 15 cm, q_1 = -5.0 μC, and q_2 = +2.0 μC. With $V = 0$ at infinity, what are the electric potentials (a) at corner A and (b) at corner B? (c) How much work is required to move a third charge q_3 = +3.0 μC from B to A along a diagonal of the rectangle? (d) Does this work increase or decrease the electric energy of the three-charge system? Is more, less, or the same work required if q_3 is moved along paths that are (e) inside the rectangle but not on a diagonal and (f) outside the rectangle?

a) Let l=0.15 m be the length of the rectangle and w=0.050 m is its width. Charge q_1 is a distance l from point A and charge q_2 is a distance w, so the electric potential at A is

$$V_A = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1}{l}+\frac{q_2}{w}\right]$$
$$= (8.99\times10^9\ \mathrm{N\cdot m^2/C^2})\left[\frac{-5.0\times10^{-6}\,\mathrm{C}}{0.15\mathrm{m}}+\frac{2.0\times10^{-6}\,\mathrm{C}}{0.050\mathrm{m}}\right]$$
$$= 6.0\times10^4\,\mathrm{V}.$$

b) Charge q_1 is a distance w from point B and charge q_2 is a distance l, so the electric potential at B is

$$V_B = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1}{w}+\frac{q_2}{l}\right]$$
$$= (8.99\times10^9\ \mathrm{N\cdot m^2/C^2})\left[\frac{-5.0\times10^{-6}\,\mathrm{C}}{0.050\mathrm{m}}+\frac{2.0\times10^{-6}\,\mathrm{C}}{0.15\mathrm{m}}\right]$$
$$= -7.8\times10^5\,\mathrm{V}.$$

c) Let U_A and U_B are the potential energies when q_3 is placed at points A and B, respectively, where U_A and U_B corresponds to the negative of the work done by the system to bring q_3 from infinity to points A and B. (L05 Slide 15 $U_f = -W_\infty$). Here, $U_A = q_3 V_A$ and $U_B = q_3 V_B$, or

$$U_A = qV_A = 3.0\times10^{-6}\,\mathrm{C}\times6.0\times10^4\,\mathrm{V} = 0.18\,\mathrm{J}$$
$$U_B = qV_B = 3.0\times10^{-6}\,\mathrm{C}\times(-7.8\times10^5\,\mathrm{V}) = -2.34\,\mathrm{J}$$

From L05 slide 15, we have $\Delta U = U_f - U_i = -W$, *it is important to note that W in this expression refers to the work done by the system* that is required to change the system from the initial i state to the final f state. Looking at the potential energies at A and B, with the initial state at infinity,

$$-W = U_A - U(\infty) = 0.18\,\mathrm{J} - 0 \quad\rightarrow\quad W = -0.18\,\mathrm{J}$$
$$-W = U_B - U(\infty) = -2.34\,\mathrm{J} - 0 \quad\rightarrow\quad W = 2.34\,\mathrm{J}$$

Therefore, the system did negative work (or gained energy) to bring q_3

to A, that's why the $U_A > 0$. Similarly, the system did work (released energy) to bring q_3 to B. See L05 slide 17, second paragraph from the bottom for a better explanation.

The amount of work that is needed to bring q_3 from point B to A is simply

$$-W = U_A - U_B = 0.18\text{ J} - (-2.34)\text{J} = 2.52\text{ J} \rightarrow W = -2.52\text{ J}$$

Therefore, the amount of work needed to bring q_3 from B to A is 2.52 J. The negative sign is just to let us know who did the work which will be answered in d).

d) From c), where the work W refers to the work done by the system and with $W = -2.52\text{ J} < 0$; the system did negative work, in another word the external agent added energy to the system and <u>the system has more energy when q_3 is moved from B to A</u>. We can also argue that because q_3 is moved from a lower electric potential location at B to a higher potential location at A, the external agent added the necessary energy to increase the potential.

e) and f), the electrostatic force is a conservative force and it is path independent, so the work is the same no matter which path it took.

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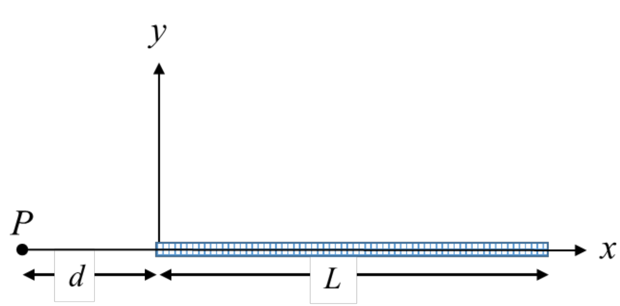
L05-03 (10 marks) A thin plastic rod of length L=12.0 cm and uniform positive charge Q=47.9 fC lying on an x axis. With $V=0$ at infinity, find the electric potential at point P on the axis at distance d=2.50 cm from one end of the rod.



[*Hints*: $\int \frac{1}{a+x}dx = \ln(a+x) + C$]

In the question, I gave the wrong hints of $1\,fC = 1\times10^{-12}\,C$ instead is should be $1\,fC = 1\times10^{-15}C$. I will accept both answer with $Q = 47.9\times10^{-15}C$ or $Q = 47.9\times10^{-12}C$

Ans: Consider an infinitesimal segment of the rod, located between x and $x + dx$. It has length dx and contains charge $dq = \lambda\, dx$, where $\lambda = Q/L$ is the linear charge density of the rod. Its distance from P_1 is $d + x$ and the potential it creates at P_1 is

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{d+x} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda dx}{d+x}.$$

To find the total potential at P_1, we integrate over the length of the rod and obtain:

$$V = \frac{\lambda}{4\pi\varepsilon_0}\int_0^L \frac{dx}{d+x} = \frac{\lambda}{4\pi\varepsilon_0}\left[\ln(d+x)\right]_0^L = \frac{Q}{4\pi\varepsilon_0 L}\ln\left(1+\frac{L}{d}\right)$$

$$= \frac{(8.99\times10^9\,\mathrm{N\cdot m^2/C})(47.9\times10^{-15}\mathrm{C})}{0.12\mathrm{m}}\ln\left(1+\frac{0.12\mathrm{m}}{0.025\mathrm{m}}\right) = 6.31\times10^{-3}\ \mathrm{V}$$

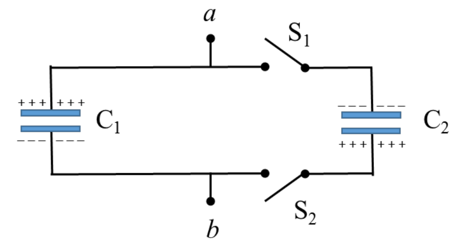
or

$$V = \frac{\lambda}{4\pi\varepsilon_0}\int_0^L \frac{dx}{d+x} = \frac{\lambda}{4\pi\varepsilon_0}\left[\ln(d+x)\right]_0^L = \frac{Q}{4\pi\varepsilon_0 L}\ln\left(1+\frac{L}{d}\right)$$

$$= \frac{(8.99\times10^9\,\mathrm{N\cdot m^2/C})(47.9\times10^{-12}\mathrm{C})}{0.12\mathrm{m}}\ln\left(1+\frac{0.12\mathrm{m}}{0.025\mathrm{m}}\right) = 6.31\ \mathrm{V}$$

Lecture 06: Capacitance

L06-01 (10 marks) In the Figure, the capacitances are $C_1 = 1.0\ \mu\text{F}$ and $C_2 = 3.0\ \mu\text{F}$, and both are charged to a potential difference of $V = 100\text{ V}$ but with opposite polarity as shown. Switches S_1 and S_2 are now closed. (a) What is now the potential difference between points a and b? (b) What are the charges in capacitors C_1 and C_2 after the switches are closed for a long time?



Ans:

(a) After the switches are closed, the potential differences across the capacitors are the same and the two capacitors are in parallel. The potential difference from a to b is given by $V_{ab} = Q/C_{eq}$, where Q is the net charge on the combination and C_{eq} is the equivalent capacitance. The equivalent capacitance is $C_{eq} = C_1 + C_2 = 4.0 \times 10^{-6}$ F. The total charge on the combination is the net charge on either pair of connected plates. The charge on capacitor 1 is

$$q_1 = C_1 V = \left(1.0\times10^{-6}\text{ F}\right)\left(100\text{ V}\right) = 1.0\times10^{-4}\text{C}$$

and the charge on capacitor 2 is

$$q_2 = C_2 V = (3.0\times10^{-6}\text{F})(100\text{V}) = 3.0\times10^{-4}\text{VC}.$$

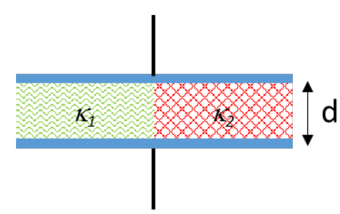
so the net charge on the combination is 3.0×10^{-4} C – 1.0×10^{-4} C = 2.0×10^{-4} C. The potential difference is

$$V_{ab} = \frac{2.0\times10^{-4}\text{C}}{4.0\times10^{-6}\text{F}} = 50\text{V}.$$

(b) The charge on capacitor 1 is now $q_1 = C_1 V_{ab} = (1.0 \times 10^{-6}\text{ F})(50\text{ V}) = 5.0 \times 10^{-5}$ C.

(c) The charge on capacitor 2 is now $q_2 = C_2 V_{ab} = (3.0 \times 10^{-6}\text{ F})(50\text{ V}) = 1.5 \times 10^{-4}$ C.

L06-02 (10 marks) The Figure shows a parallel-plate capacitor with a plate area of $A = 5.56\text{ cm}^2$ and separation $d = 5.56\text{ mm}$. The left half of the gap is filled with materials of dielectric constant $\kappa_1 = 7.00$; the right half is filled with material of dielectric constant $\kappa_2 = 10.0$. What is the capacitance?



Ans: The capacitor can be viewed as two capacitors C_1 and C_2 in parallel, each with surface area $A/2$ and plate separation d, filled with dielectric materials with dielectric constants κ_1 and κ_2, respectively. Thus, (in SI units),

$$C = C_1 + C_2 = \frac{\varepsilon_0 (A/2)\kappa_1}{d} + \frac{\varepsilon_0 (A/2)\kappa_2}{d} = \frac{\varepsilon_0 A}{d}\left(\frac{\kappa_1 + \kappa_2}{2}\right)$$

$$= \frac{(8.85\times10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2)(5.56\times10^{-4}\ \text{m}^2)}{5.56\times10^{-3}\ \text{m}}\left(\frac{7.00+10.00}{2}\right) = 7.52\times10^{-12}\ \text{F.}$$

L06-03

(10 marks) A parallel-plate air-filled capacitor having area 40 cm^2 and plate spacing 1.0 mm is charged to a potential difference of 600 V. Find (a) the capacitance, (b) the magnitude of the charge on each plate, (c) the stored energy, (d) the electric field between the plates, and (e) the energy density between the plates.

Ans:

a) The capacitance is

$$C = \frac{\varepsilon_0 A}{d} = \frac{(8.85\times10^{-12})(40\times10^{-4})}{1.0\times10^{-3}} = 3.5\times10^{-11}\ \text{F} = 35\ \text{pF}.$$

b) $q = CV = (35\ \text{pF})(600\ \text{V}) = 2.1\times10^{8}\ \text{C} = 21\ \text{nC}.$

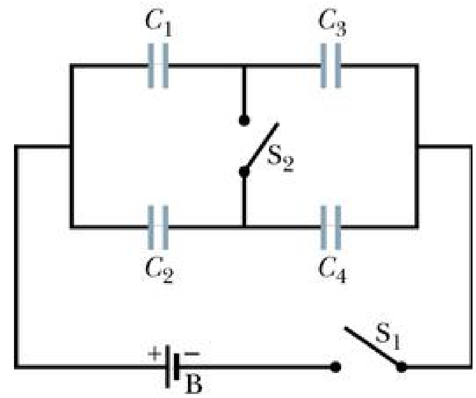
c) $U = \dfrac{CV^2}{2} = \dfrac{1}{2}(35\ \text{pF})(600\ \text{V})^2 = 6.3\times10^{-6}\ \text{J} = 6.3\ \mu\text{J}.$

d) $E = \dfrac{V}{d} = \dfrac{600\ \text{V}}{1.0\times10^{-3}\ \text{m}} = 6.0\times10^{5}\ \text{V/m}.$

e) The energy density is $u = \dfrac{U}{Ad} = \dfrac{6.3\times10^{-6}}{(40\times10^{-4})(1.0\times10^{-3})} = 1.58\ \text{J/m}^3.$

L06-04

(10 marks) In the given Figure, battery B supplies 12 V. Find the charge on each capacitor (a) first when only switch S_1 is closed and (b) later when switch S_2 is also closed. Take $C_1 = 1.0\ \mu F$, $C_2 = 2.0\ \mu F$, $C_3 = 3.0\ \mu F$, and $C_4 = 4.0\ \mu F$.



Ans:

a) In this situation, capacitor 1 and 3 are in series, which means their charges are the same:

$$q_1 = q_3 = C_{13}V = \frac{C_1C_3V}{C_1+C_3} = \frac{(1.0\mu F)(3.0\mu F)(12V)}{1.0\mu F + 3.0\mu F} = 9.0\ \mu C\,.$$

Similarly, for capacitors 2 and 4,

$$q_2 = q_4 = C_{24}V = \frac{C_2C_4V}{C_2+C_4} = \frac{(2.0\mu F)(4.0\mu F)(12V)}{2.0\mu F + 4.0\mu F} = 16\ \mu C\,.$$

b) With switch 2 also closed, the potential difference V_1 across C_1 must equal the potential difference across C_2 and the potential difference across C_3 and C_4, V_2 must also be the same. Furthermore, $V = V_1 + V_2$

$$V_1 == \frac{C_3+C_4}{C_1+C_2+C_3+C_4}V = \frac{(3.0\mu F + 4.0\mu F)(12V)}{1.0\mu F + 2.0\mu F + 3.0\mu F + 4.0\mu F} = 8.4\ V.$$

$$V_2 = V - V_1 = (12-8.4)\ V = 3.6\ V.$$

Thus

$$q_1 = C_1V_1 = (1.0\ \mu F)(8.4\ V) = 8.4\ \mu C\,, q_2 = C_2V_1 = (2.0\ \mu F)(8.4\ V) = 16.8\ \mu C\,,$$
$$q_3 = C_3V_2 = (3.0\ \mu F)(3.6\ V) = 10.8\ \mu C\,, q_4 = C_4V_2 = (4.0\ \mu F)(3.6\ V) = 14.4\ \mu C\,.$$