EE1001 Test, Q+A, 2020-21, Sem B, KF Tsang, ver 2

Q1) (10 marks)

Determine the validity of the following arguments and brief the reasons.

(i) If it rains today, we would not go swimming. We go swimming today. Therefore, it did not rain today.

(ii) Alex is a history major. Thus, Alex is either an engineering major or a history major.

(iii) If Smith plays football, then he will stay outside too long. If Smith stays outside too long, then he will get heatstroke. Therefore, if Smith plays football, he will get heatstroke.

(iv) If I am tired and if I do not spend time on the mobile phone, then I will sleep early. I didn't sleep early. Therefore, I was not tired and I spent time on the mobile phone.

(v) If Kate doesn't work hard, then she will fail in this course. If Kate doesn't fail in this course, then she will get the scholarship. Kate gets the scholarship. Therefore, Kate worked hard.

Solution of Q1 (10 marks)

(i) **Valid**. (Inference Rules – Modus Tollens)

(ii) **Valid**. (Inference Rules – Addition)

(iii) **Valid**. (Inference Rules – Hypothetical Syllogism)

(iv) **Invalid**.

Let p = "I am tired"; q = "I spend time on the mobile phone"; r = "I sleep early".

1. $p \wedge \sim q \rightarrow r$ (premise)
2. $\sim r$ (premise)
3. $\sim(p \wedge \sim q)$ (MT 1,2)
4. $\sim p \vee q$ (De Morgan's law)

Therefore, to make the argument valid, the conclusion should be "I was not tired **or** I spent time on the mobile phone"

(v) **Invalid**.

Let a = "Kate works hard"; b = "Kate fails in this course"; c = "Kate gets the scholarship".

1. $\sim a \rightarrow b$ (premise)
2. $\sim b \rightarrow c$ (premise)
3. c (premise)

These 3 premises have no logical relation with each other.

Alternative proof: using a truth table:

Variables			Premise 1	Premise 2	Premise 3	Conclusion
a	b	c	$\sim a \rightarrow b$	$\sim b \rightarrow c$	c	a
T	T	T	**T**	**T**	**T**	**T**
T	T	F	T	T	F	T
T	F	T	**T**	**T**	**T**	**T**
T	F	F	T	F	F	T
F	T	T	**T**	**T**	**T**	**F**
F	T	F	T	T	F	F
F	F	T	F	T	T	F
F	F	F	F	F	F	F

Rows 1, 3, and 5 are critical rows (i.e., all premises are true). It is observed that in row 5, all premises are true, but the conclusion is false. Therefore, this argument is invalid.

Q2) (10 marks)

Without using a truth table, determine whether $\sim(a \vee \sim b) \wedge (\sim b \vee c) \rightarrow (a \vee c)$ is a tautology. State the reason for each step.

Solution of Q2 (10 marks)

$\sim(a \vee \sim b) \wedge (\sim b \vee c) \rightarrow (a \vee c)$

$\equiv (\sim a \wedge b) \wedge (\sim b \vee c) \rightarrow (a \vee c)$ (De Morgan's laws)

$\equiv \sim a \wedge [(b \wedge \sim b) \vee (b \wedge c)] \rightarrow (a \vee c)$ (Distributive laws)

$\equiv \sim a \wedge [\mathbf{c} \vee (b \wedge c)] \rightarrow (a \vee c)$ (Negation laws)

$\equiv \sim a \wedge (b \wedge c) \rightarrow (a \vee c)$ (Identity laws)

$\equiv \sim(\sim a \wedge b \wedge c) \vee (a \vee c)$ (Definition of $\rightarrow$)

$\equiv a \vee \sim b \vee \sim c \vee a \vee c$ (De Morgan's laws)

$\equiv a \vee \sim b \vee a \vee \mathbf{t}$ (Negation laws)

$\equiv \mathbf{t}$ (Universal bound laws)

Yes, it's a tautology.

Q3) (10 marks)

Annie, Ben, and Charis are suspected of a crime. Their statements are as follows:
Annie: “At least one of us is guilty.”
Ben: “I’m innocent, and at least one of the others is innocent.”
Charis: “If Ben is guilty, then Annie is also guilty.”

Let A, B, and C be
A = “Annie is innocent”
B = “Ben is innocent”
C = “Charis is innocent”

(i) Formulate the statements of Annie, Ben, and Charis.

(ii) Given that the innocent told the truth and the guilty lied, who is innocent and who is guilty?

Solution of Q3 (10 marks)

Q3(i)
Annie’s statement = $\boldsymbol{\sim A \vee \sim B \vee \sim C}$
Ben’s statement = $\boldsymbol{B \wedge (A \vee C)}$
Charis’s statement = $\boldsymbol{\sim B \to \sim A}$

Q3(ii)
Given that the innocent told the truth and the guilty lied, the relation between the suspects’ roles and their statements can be formulated as follows:
Annie: $A \leftrightarrow (\sim A \vee \sim B \vee \sim C)$
Ben: $B \leftrightarrow [B \wedge (A \vee C)]$
Charis: $C \leftrightarrow (\sim B \to \sim A)$

The condition is satisfied when $[A \leftrightarrow (\sim A \vee \sim B \vee \sim C)] \wedge \{B \leftrightarrow [B \wedge (A \vee C)]\} \wedge [C \leftrightarrow (\sim B \to \sim A)]$ is true.

A	B	C	$\sim A \vee \sim B \vee \sim C$	$B \wedge (A \vee C)$	$\sim B \to \sim A$	$A \leftrightarrow (\sim A \vee \sim B \vee \sim C)$	$B \leftrightarrow [B \wedge (A \vee C)]$	$C \leftrightarrow (\sim B \to \sim A)$	$[A \leftrightarrow (\sim A \vee \sim B \vee \sim C)] \wedge \{B \leftrightarrow [B \wedge (A \vee C)]\} \wedge [C \leftrightarrow (\sim B \to \sim A)]$
T	T	T	F	T	T	F	T	T	F
T	T	F	T	T	T	T	T	F	F
T	F	T	T	F	F	T	T	F	F
T	**F**	**F**	**T**	**F**	**F**	**T**	**T**	**T**	**T**
F	T	T	T	T	T	F	T	T	F
F	T	F	T	F	T	F	F	F	F
F	F	T	T	F	T	F	T	T	F
F	F	F	T	F	T	F	T	F	F

Therefore, Annie is innocent, and Ben and Charis are guilty.

Q4) (10 marks)

Given the initial condition of $a_1 = -5$ and the general formula of $a_n = -2a_{n-1} - 3$

(i) Express the general formula using an explicit formula.

(ii) Determine the value of the 13th term, i.e., a_{13}.

Solution of Q4 (10 marks)

Q4(i)

$$
\begin{aligned}
a_n &= -2a_{n-1} - 3 \\
&= -2(-2a_{n-2} - 3) - 3 = (-2)^2 a_{n-2} - 3 - 3(-2) \\
&= (-2)^2(-2a_{n-3} - 3) - 3 - 3(-2) = (-2)^3 a_{n-3} - 3 - 3(-2) - 3(-2)^2 \\
&= \ldots \\
&= (-2)^{n-1}a_1 - 3 - 3(-2) - 3(-2)^2 - \ldots - 3(-2)^{n-2}, \text{ for } n>1 \\
&= (-5)(-2)^{n-1} - 3[\,1+(-2) + (-2)^2 + \ldots + (-2)^{n-2}\,] \\
&= (-5)(-2)^{n-1} - 3\sum_{k=1}^{n-1}(-2)^{k-1} \quad \text{or} \quad (-5)(-2)^{n-1} - [1-(-2)^{n-1}]
\end{aligned}
$$

Q4(ii)

$$a_n = (-5)(-2)^{n-1} - 3\sum_{k=1}^{n-1}(-2)^{k-1} = (-5)(-2)^{n-1} - 3\frac{(-2)^0[1-(-2)^{n-1}]}{1-(-2)} = (-5)(-2)^{n-1} - \left[1-(-2)^{n-1}\right]$$

$$a_{13} = (-5)(-2)^{13-1} - [1-(-2)^{13-1}]$$

$a_{13} = -16385$

Q5) (10 marks)

Given the following sequence of positive integers:

{1, 3, 5, …, 2017, 2019, 2021}

(i) Determine the sum of the integers.

(ii) The powers of 3 (i.e., 3^0, 3^1, 3^2, …) are now removed from the above sequence. Determine the sum of the remaining integers.

Solution of Q5 (10 marks)

Q5(i)

$a_1 = 1$; $d = 2$

$a_n = a_1 + (n\text{-}1)d$

$1 + (n\text{-}1)\times 2 = 2021$

$n = 1011$

Sum of the sequence:

$$S_n = \frac{n}{2}(a_1 + a_n)$$

$$S_{1011} = \frac{1011}{2}(1 + 2021)$$

$S_{1011} = 1022121$

Q5(ii)

Given the power of 3 and upper limit of 2021,

$3^k \le 2021$

$k \le \log(2021)/\log(3)$

$k \le 6.93$

k must be an integer, and thus, $k = 6$.

Sequence of the power of 3 = $\{3^0, 3^1, 3^2, \dots, 3^6\}$ that is a geometric sequence.

Therefore, sum of the sequence of the power of 3 can be calculated as follows.

$a_1 = 1$; $r = 3$; $n = 7$

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

$$S_7 = \frac{1 - 3^7}{1 - 3} = 1093$$

Thus, after removing power of 3, the sum of remaining integers is:

$S = S_{1011} - S_7 = 1022121 - 1093 = 1021028$

Q6 (8 marks)

Consider the word *BUILDN*. With or without meaning, to form words with length of 6, considering each of the following circumstance:

(i) If the letters can be reused (repeat), how many arrangements can be made using the letters?

(ii) If the letters cannot be reused, how many arrangements can be made using all the letters?

(iii) If the letters cannot be reused, how many arrangements begin with the letter N?

(iv) If the letters cannot be reused, how many arrangements begin with the letter N and end with D?

(v) If the letters cannot be reused, how many arrangements with DIN?

(vi) If the letters cannot be reused, how many arrangements end with DIN?

(vii) If the letters cannot be reused, how many arrangements begin with N or end in D?

Solution of Q6 (8 marks)

(i) $6^6=46656$ (1 MARK)

(ii) ${}_6P_6=720$ (1 MARK)

(iii) ${}_5P_5=120$ (1 MARK)

(iv) ${}_4P_4=24$ (1 MARK)

(v) ${}_4P_4=24$ (1 MARK)

(vi) ${}_3P_3=6$ (1 MARK)

(vii) $2{}_5P_5-{}_4P_4=2*120-24=216$ (2 MARKS)

Q7 (10 marks)

(i) Find $(2x-1)^6-(2x+1)^6$ with binomial theorem.

(ii) Compute $(\sqrt{3}-1)^6-(\sqrt{3}+1)^6$.

Solution of Q7 (10 marks)

(i)

For $(2x-1)^6 = {}_6C_0(2x)^6- {}_6C_1(2x)^5+ {}_6C_2(2x)^4- {}_6C_3(2x)^3+ {}_6C_4(2x)^2- {}_6C_5(2x)^1 + {}_6C_6(2x)^0$

For $(2x+1)^6 = {}_6C_0(2x)^6+ {}_6C_1(2x)^5+ {}_6C_2(2x)^4+ {}_6C_3(2x)^3+ {}_6C_4(2x)^2+ {}_6C_5(2x)^1 + {}_6C_6(2x)^0$

Thus,

$(2x-1)^6-(2x+1)^6=-2*{}_6C_1(2x)^5-2*{}_6C_3(2x)^3 - 2*{}_6C_5(2x)^1= -384x^5- 320x^3-24x$

(ii) Let $x= \sqrt{3}/2$

We have $(\sqrt{3}-1)^6-(\sqrt{3}+1)^6 = -240\sqrt{3}$

Q8 (8 marks)

In a standard deck of cards, there are 52 cards.
- There are 4 of each card (4 Aces, 4 Kings, 4 Queens, etc.)
- There are 4 suits (Clubs, Hearts, Diamonds, and Spades) and there are 13 cards in each suit (Clubs/Spades are black, Hearts/Diamonds are red)

You are playing a game with one of your friends.

(i) You and your friend are required to pick 5 cards in total. In how many ways can 5 cards be picked so that 3 are with you and 2 with your friend?
For instance:
you pick 2 cards → your friend pick 2 cards → you pick 1 card again
you pick 3 cards → your friend pick 2 cards

(ii) You need to pick two cards from the 52 cards. What is the probability that your picked two cards are a pair (such as heart 9 and diamond 9)?

(iii) Both of your friends need to pick two cards from the 40 cards with only numbers (No King, queen…). Your friends first pick an 8 and a 7 and you need to pick 2 cards from the rest. What is the probability that the sum of your cards is larger than your friends?

Solution of Q8 (8 marks)

(i) Set your card A, your friend's card as B, it is three A and two B to form different words.
Which is: ${}_5P_5/(3!*2!) = 10$ (2 marks)

(ii) For the total combinations of 2 cards: ${}_{52}C_2 = 1326$;
For the total combinations of 2 cards is a pair: ${}_{13}C_1*{}_4C_2 = 78$
The probability is: 78/1326 = 5.9% (2 marks)

Case analysis:

A	B	Combination	Result
6	10	4C1*4C1	16
7	9	3C1*4C1	12
7	10	3C1*4C1	12
8	8	3C2	3
8	9	3C1*4C1	12
8	10	3C1*4C1	12
9	9	4C2	6
9	10	4C1*4C1	16
10	10	4C2	6

Answer:95

The answer can be solved by analysing 3 colored portions

A	B	Combination	Result
6	10	4C1*4C1	16
7	9/10	3C1*8C1	24
8/9/10	8/9/10	11C2	55

Answer:16 + 24 + 55 = 95
For 6 and 7, only 10 and 9/10 can be used.
For 8, 9 and 10, the combination is free.

Alternatively:

1. Only 6 – 10 can be used in combination, so 18C2 = 153
2. 6, 7-9 is wrong, so - 4C1*10C1 = -40
3. 7,8 is wrong, so - 3C1*3C1 = - 9
4. 6, 6 and 7, 7 is wrong, so – 4C2 – 3C2 = - 9

Answer: 153 – 40 – 9 – 9 = 95

Q9 (4 marks)

Let A={x|x<12}, B={x|x≤8}, C={x|x=2k, k=1,2,3,4,5…}. Identify the following sets with A, B, and C.

(i) {10};

(ii) {x| x is odd integer that are larger than 12};

Solution of Q9 (4 marks)

(i) $A \cap \bar{B} \cap C$ (2 marks)

(ii) $\bar{A} \cap \bar{C}$ (2 marks)

Q10 (10 marks)

Suppose there was a wooden stick with a length of 200cm. First, Leo drew marks on the stick every 3cm from the left end to the right end. After that, he passed the stick to Bob. Second, Bob drew marks on the stick every 4cm from the left end to the right end. Thirdly, Tom draws marked on the stick every 5cm from the left end to the right end. (Noticed that marks from different students could be overlapped). Finally, Jack obtained this stick and cut the stick according to these marks. How many short sticks did Jack obtain?

Solution of Q10 (10 marks)

$|A| = 66$

$|B| = 50-1 = 49$

$|C| = 40-1 = 39$

$|A\cap B| = 16$

$|A\cap C| = 13$

$|B\cap C| = 10-1=9$

$|A\cap B \cap C| = 3$

The number of marks:

$|A\cup B \cup C|=|A|+|B|+|C|-|A\cap B|-|A\cap C|-|A\cap C|+|A\cap B \cap C|=66+49+39-16-13-9+3=119$

Since there are 119 marks, it will have 119+1=120 short sticks.

Q11 (10 marks)

In the plane, there are n non-parallel straight lines. These lines will form several angles (Fig. Q11 a). To ensure there is at least one angle less than 26 degrees, what is the minimal number of lines? Why?

(Tip: The translation of the straight line does not change the angle, shown in Fig. Q11 b)

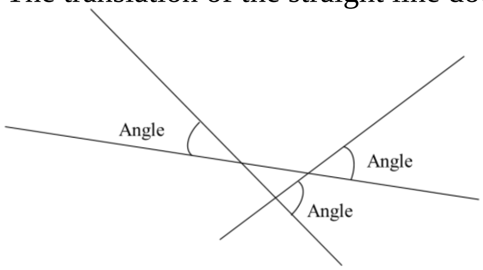

Fig. Q11 **a**

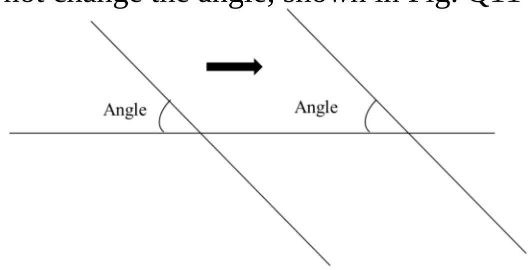

Fig. Q11 **b**

Solution of Q11 (10 marks)

7 lines are required. The reason is:
Since the translation of the straight line does not change the angle,
we can move these straight lines and let them have only one intersection point.
Therefore, n straight lines will form $2n$ angles.
To let there have at least one angle less than 26 degrees, we need $\lceil 360/26 \rceil + 1 = 14$
Therefore, 14 angles are needed, and we need 7 lines.

-- END --