MA 1200 Calc & Basic Linear Algebra I

Semester A 2017/18, Guo Luo

Supplementary Notes

Conic Sections I (page 11, Lecture 1)

- By definition, a conic section (or simply conic) is the intersection of a plane and a double-napped cone.
  (a) When the intersecting plane does not pass through the vertex of the cone, the resulting curve is one of the four basic conics.

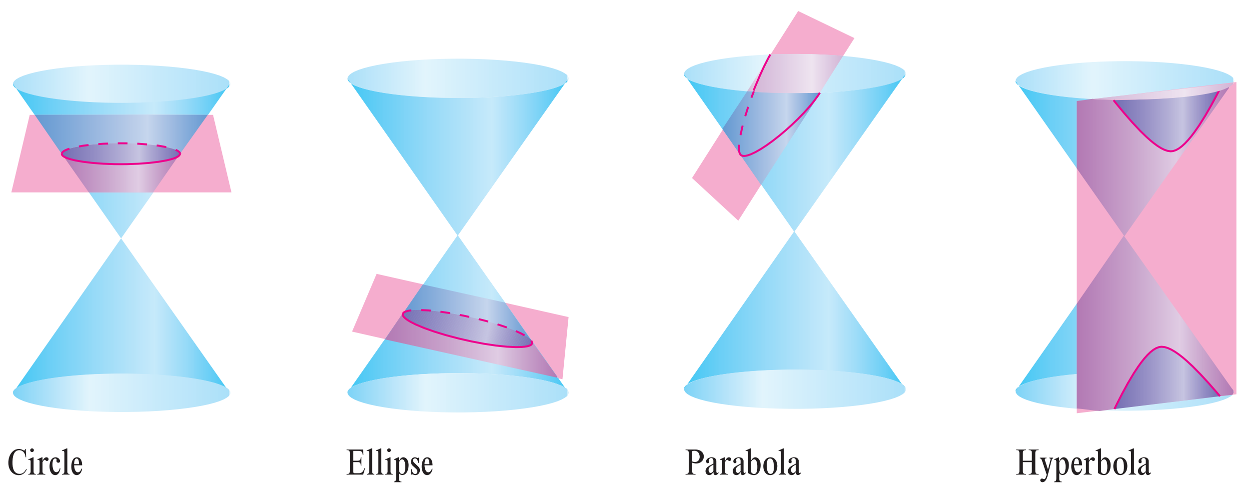


Figure 1: The four basic conics.

Conic Sections II (page 11, Lecture 1)

(b) When the plane does pass through the vertex, the resulting curve is a degenerate conic.

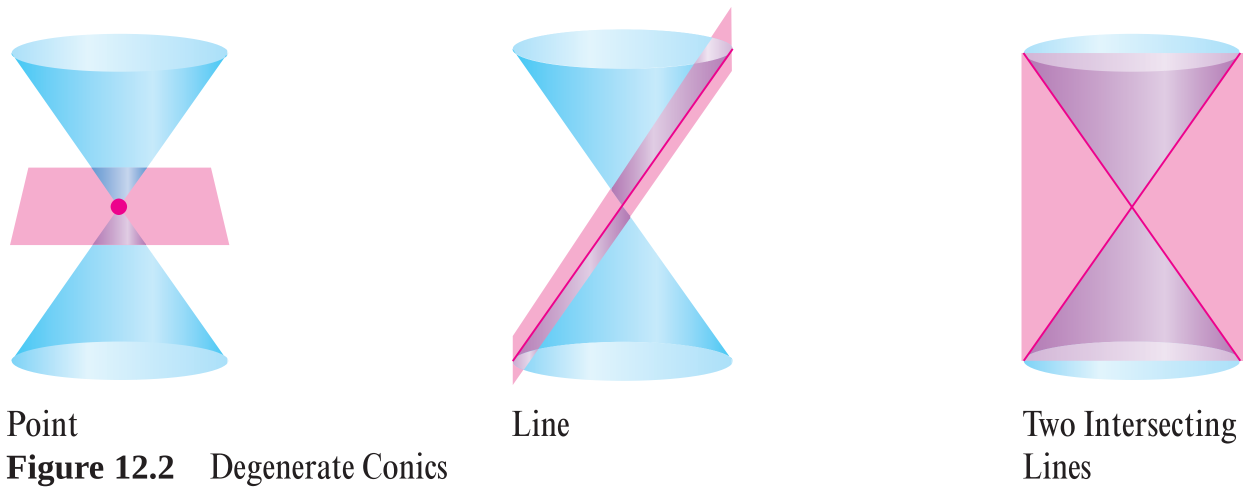


Figure 2: The three degenerate conics.

Definition of Ellipse I (page 17, Lecture 1)

- An ellipse is the set of all points in a plane, the sum of whose distances from two distinct fixed points (foci) is constant.

- The line through the foci intersects the ellipse at two points called vertices. The chord joining the vertices is the major axis, and its midpoint is the center of the ellipse. The chord perpendicular to the major axis at the center is the minor axis.

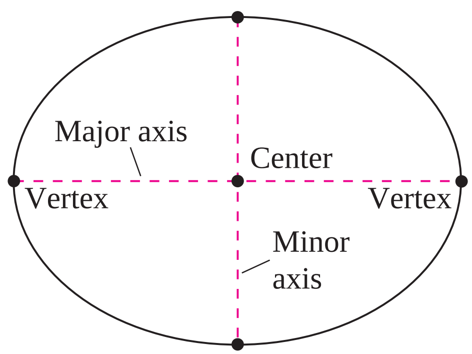


Figure 3: An ellipse and its major and minor axes.

Definition of Ellipse II (page 17, Lecture 1)

- The definition of an ellipse can be visualized by imagining two thumbtacks placed at the foci. If the ends of a string are fastened to the thumbtacks and the string is drawn taut with a pencil, the path traced by the pencil will be an ellipse.

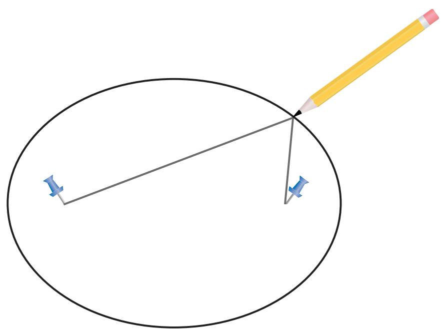


Figure 4: Visualization of the definition of an ellipse.

Standard Equation of an Ellipse I (page 22, Lecture 1)

- In the standard equation of an ellipse (assuming $0 < b < a$),

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1,$$

the relationship among the numbers a, b, and c can be understood using the following diagram:

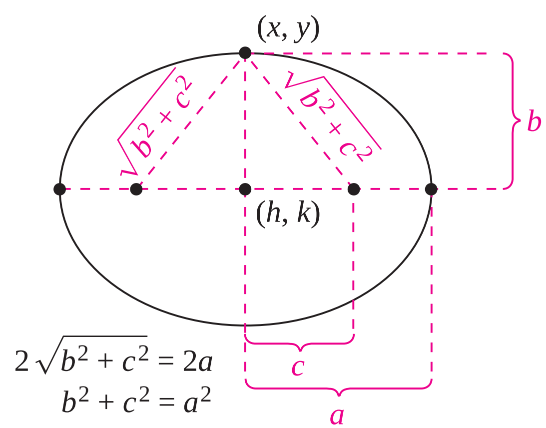


Figure 5: Understanding the standard equation of an ellipse.

Standard Equation of an Ellipse II (page 22, Lecture 1)

- Given the standard equation of an ellipse, the following procedure can be used to sketch the graph of the ellipse.
  (a) Denote by a^2 the denominator with a greater value and by b^2 the denominator with a smaller value. For example, if the equation is

  $$\frac{(x+2)^2}{4^2} + \frac{(y-1)^2}{5^2} = 1,$$

  then $a^2 = 5^2$ and $b^2 = 4^2$.

  (b) Determine the major and minor axes. If a^2 divides $(x-h)^2$, then the "x-axis" (or more precisely, the horizontal line $y = k$) passing through the center (h, k) is the major axis. If a^2 divides $(y-k)^2$, then the "y-axis" (or more precisely, the vertical line $x = h$) passing through the center (h, k) is the major axis. The minor axis can be determined in a completely similar way, by inspecting the term in the equation divided by b^2.

Standard Equation of an Ellipse III (page 22, Lecture 1)

For the above example, since $a^2 = 5^2$ divides $(y-1)^2$, the "y-axis" (or more precisely, the vertical line $x = -2$) passing through the center $(-2, 1)$ is the major axis. Likewise, since $b^2 = 4^2$ divides $(x+2)^2$, the "x-axis" (or more precisely, the horizontal line $y = 1$) passing through the center $(-2, 1)$ is the minor axis.

(c) Determine the vertices. Observe that the two vertices lying on the major axis have a distance a to the center (h, k), and the two vertices lying on the minor axis have a distance b to the center (h, k). For the above example, the two vertices lying on the major axis are given by (recall that the major axis is vertical)

$$(h, k+a) = (-2, 1+5) = (-2, 6),$$
$$(h, k-a) = (-2, 1-5) = (-2, -4).$$

Standard Equation of an Ellipse IV (page 22, Lecture 1)

The two vertices lying on the minor axis are given by (recall that the minor axis is horizontal)

$$(h+b,k) = (-2+4,1) = (2,1),$$
$$(h-b,k) = (-2-4,1) = (-6,1).$$

(d) Determine the foci (optional). These points are always located on the major axis and have a distance c to the center (h,k), where $c^2 = a^2 - b^2$. For the above example, $c^2 = a^2 - b^2 = 5^2 - 4^2 = 3^2$, so the two foci are given by

$$(h,k+c) = (-2,1+3) = (-2,4),$$
$$(h,k-c) = (-2,1-3) = (-2,-2).$$

(e) Once the center, major/minor axes, and vertices of the ellipse are determined, its graph can be easily sketched.

Eccentricity of an Ellipse I (page 17, Lecture 1)

- The eccentricity e of an ellipse is given by the ratio $e = c/a$.
- To see how this ratio is used to describe the shape of an ellipse, note first that $0 < c < a$. For an ellipse that is nearly circular, the foci are close to the center and the ratio c/a is small. On the other hand, for an elongated ellipse, the foci are close to the vertices and the ratio c/a is close to 1 (Figure 6).

Eccentricity of an Ellipse II (page 17, Lecture 1)

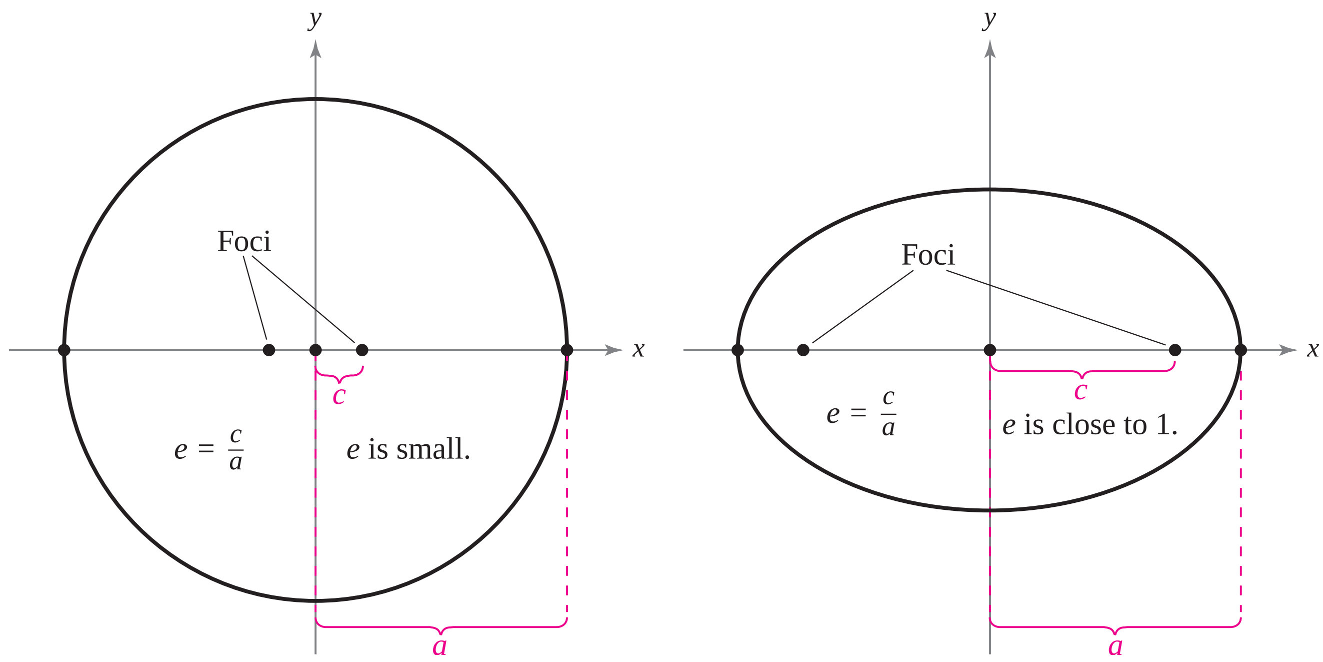


Figure 6: The eccentricity of an ellipse as a measure of ovalness.

Standard Equation of a Parabola I (page 29, Lecture 1)

- Given the standard equation of a parabola, the following procedure can be used to sketch the graph of the parabola.
  (a) Inspect the linear term in the equation. If the linear term is $\pm 4a(x-h)$, then the parabola opens along the "x-axis" (or more precisely, the horizontal line $y = k$) passing through the vertex (h, k). If the linear term is $\pm 4a(y-k)$, then the parabola opens along the "y-axis" (or more precisely, the vertical line $x = h$) passing through the vertex (h, k). For example, if the equation is

$$(x+2)^2 = -2(y-1),$$

then the parabola opens along the "y-axis" (or more precisely, the vertical line $x = -2$) passing through the vertex $(-2, 1)$. Note that the axis along which a parabola opens is also the symmetry axis of the parabola.

Standard Equation of a Parabola II (page 29, Lecture 1)

(b) Check the sign of the linear term. If the linear term has a positive sign, then the parabola opens along the positive direction of its symmetry axis. If the linear term has a negative sign, then the parabola opens along the negative direction of its symmetry axis. For the above example, the linear term $-2(y-1)$ has a negative sign, so the parabola opens along the negative direction of its symmetry axis, i.e. it opens downward along $x = -2$.

(c) Determine the focus (optional). The focus of a parabola always lies on its symmetry axis and has a distance a to the vertex (h, k). For the above example, $a = 2/4 = 1/2$, so the focus is given by (recall that the parabola opens downward)

$$(h, k-a) = (-2, 1-1/2) = (-2, 1/2).$$

(d) Once the vertex and symmetry axis of the parabola are determined, its graph can be easily sketched.

Reflective Property of a Parabola I (page 25, Lecture 1)

- Parabolas occur in a wide variety of applications.
  (a) For instance, a parabolic reflector can be obtained by revolving a parabola around its symmetry axis. The resulting surface has the property that all incoming rays parallel to the axis are reflected through the focus of the parabola.
  (b) This is the principle behind the construction of the parabolic mirrors used in reflecting telescopes.

Reflective Property of a Parabola II (page 25, Lecture 1)

(c) Conversely, the light rays emanating from the focus of a parabolic reflector used in a flashlight are all parallel to one another, as Figure 7 shows.

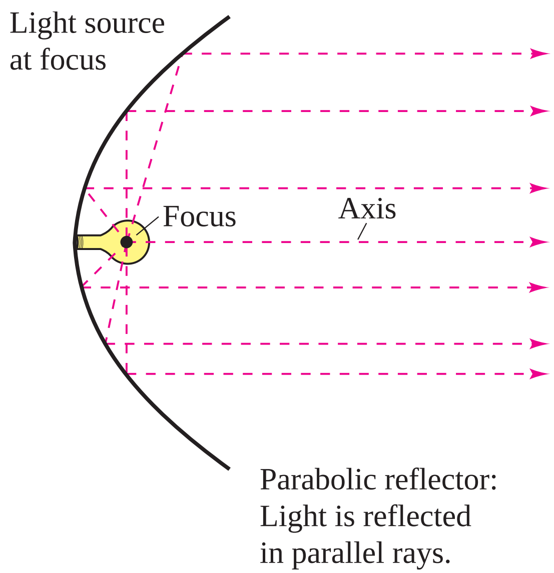


Figure 7: A parabolic reflector.

Reflective Property of a Parabola III (page 25, Lecture 1)

- Tangent lines to parabolas have special properties related to the use of parabolas in constructing reflective surfaces.
- More precisely, the tangent line to a parabola at a point P makes equal angles with the following two lines (Figure 8):
  (a) The line passing through P and the focus.
  (b) The symmetry axis of the parabola.

Reflective Property of a Parabola IV(page 25, Lecture 1)

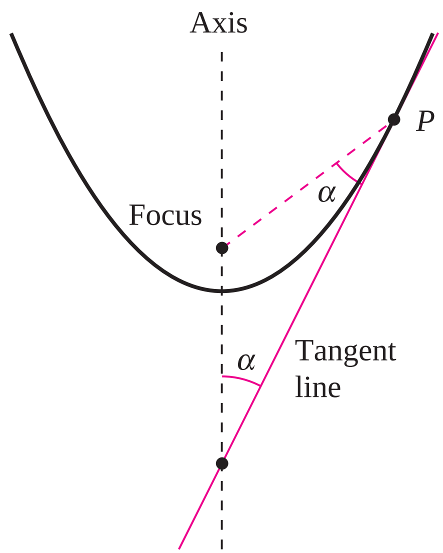


Figure 8: Properties of tangent lines to a parabola.

Definition of Hyperbola (page 30, Lecture 1)

- A hyperbola is the set of all points in a plane, the difference of whose distances from two distinct fixed points (foci) is constant.
- The graph of a hyperbola has two distinct branches. The line through the two foci intersects the hyperbola at its two vertices. The chord connecting the vertices is the transverse axis, and the midpoint of the transverse axis is the center of the hyperbola.

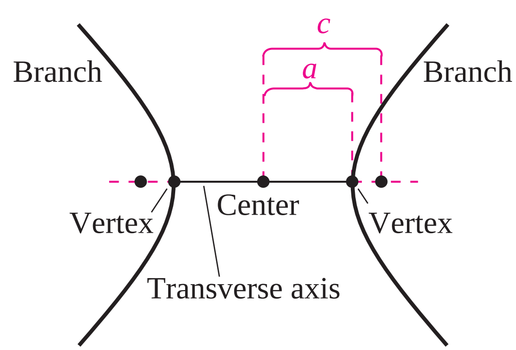


Figure 9: A hyperbola and its transverse axis.

Asymptotes of a Hyperbola (page 32, Lecture 1)

- Each hyperbola has two asymptotes that intersect at the center of the hyperbola. The asymptotes pass through the vertices of a rectangle of dimensions $2a$ by $2b$, with its center at (h, k).
- The line segment of length $2b$ joining $(h, k+b)$ and $(h, k-b)$ is the conjugate axis of the hyperbola.

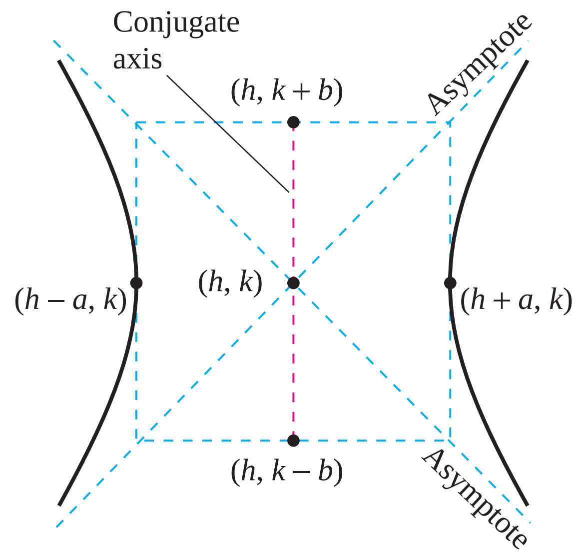


Figure 10: Asymptotes of a hyperbola.

Standard Equation of a Hyperbola I (page 34, Lecture 1)

- Given the standard equation of a hyperbola, the following procedure can be used to sketch the graph of the hyperbola.

  (a) Denote by a^2 the denominator of the term with a positive sign and by b^2 the denominator of the term with a negative sign. For example, if the equation is

  $$\frac{(y-1)^2}{3^2} - \frac{(x+2)^2}{4^2} = 1,$$

  then $a^2 = 3^2$ (even if it has a smaller value!) and $b^2 = 4^2$.

  (b) Determine the transverse and conjugate axes. If a^2 divides $(x-h)^2$, then the "x-axis" (or more precisely, the horizontal line $y = k$) passing through the center (h,k) is the transverse axis. If a^2 divides $(y-k)^2$, then the "y-axis" (or more precisely, the vertical line $x = h$) passing through the center (h,k) is the transverse axis. The conjugate axis can be determined in a completely similar way, by inspecting the term in the equation divided by b^2.

Standard Equation of a Hyperbola II (page 34, Lecture 1)

For the above example, since $a^2 = 3^2$ divides $(y-1)^2$, the "y-axis" (or more precisely, the vertical line $x = -2$) passing through the center $(-2, 1)$ is the transverse axis. Likewise, since $b^2 = 4^2$ divides $(x+2)^2$, the "x-axis" (or more precisely, the horizontal line $y = 1$) passing through the center $(-2, 1)$ is the conjugate axis. Note that a hyperbola always opens along its transverse axis.

(c) Determine the vertices. Observe that the two vertices of a hyperbola always lie on its transverse axis and have a distance a to the center (h, k). For the above example, the two vertices are given by (recall that the transverse axis is vertical)

$$(h, k+a) = (-2, 1+3) = (-2, 4),$$
$$(h, k-a) = (-2, 1-3) = (-2, -2).$$

Standard Equation of a Hyperbola III (page 34, Lecture 1)

(d) Determine the asymptotes. The two asymptotes of a hyperbola intersect at the center (h, k) and pass through the vertices of a rectangle of dimensions $2a$ by $2b$, with its center at (h, k). For the above example, the rectangle has width $2b$ (along the conjugate axis which is horizontal) and height $2a$ (along the transverse axis which is vertical), so the slopes of the two asymptotes are given by $m = \pm a/b = \pm 3/4$. The asymptotes then have the equations

$$y - k = m(x - h), \qquad \text{or} \qquad y - 1 = \pm\frac{3}{4}\,(x + 2).$$

Standard Equation of a Hyperbola IV (page 34, Lecture 1)

(e) Determine the foci (optional). These points are always located on the transverse axis and have a distance c to the center (h, k), where $c^2 = a^2 + b^2$. For the above example, $c^2 = a^2 + b^2 = 3^2 + 4^2 = 5^2$, so the two foci are given by

$$(h, k + c) = (-2, 1 + 5) = (-2, 6),$$
$$(h, k - c) = (-2, 1 - 5) = (-2, -4).$$

(f) Once the center, transverse/conjugate axes, vertices, and asymptotes of the hyperbola are known, its graph can be easily sketched.

Eccentricity of a Hyperbola I (page 30, Lecture 1)

- The eccentricity e of a hyperbola is given by the ratio $e = c/a$.
- Because $c > a$ for a hyperbola, it follows that $e > 1$. If the eccentricity is large, the branches of the hyperbola are nearly flat. If the eccentricity is close to 1, the branches of the hyperbola are more narrow (Figure 11).

Eccentricity of a Hyperbola II (page 30, Lecture 1)

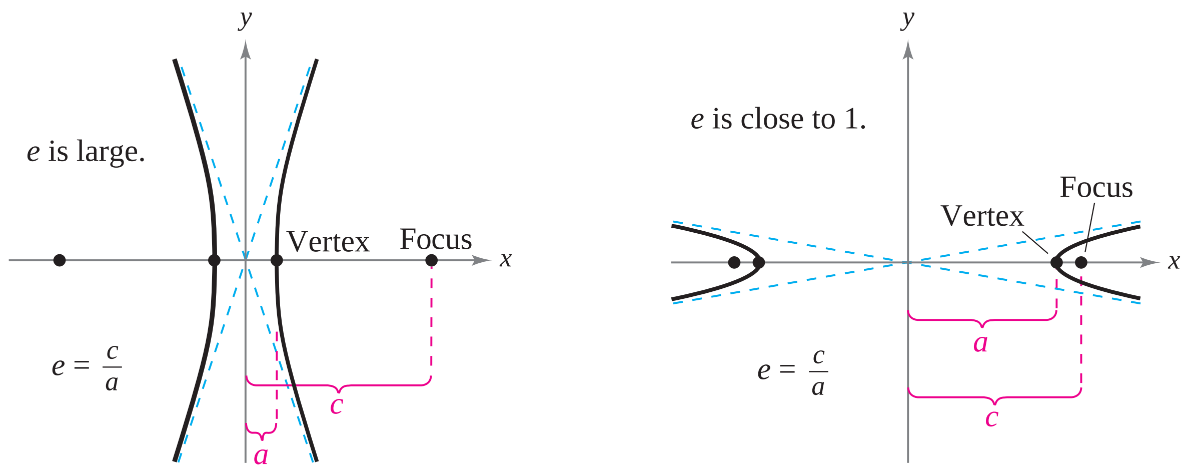


Figure 11: The eccentricity of a hyperbola as a measure of flatness.

Orbits of Comets as Conic Sections I (page 30, Lecture 1)

- An interesting application of conic sections involves the orbits of comets in our solar system.
  (a) Of the 610 comets identified prior to 1970, 245 have elliptical orbits, 295 have parabolic orbits, and 70 have hyperbolic orbits.
  (b) The center of the sun is a focus of each of these orbits, and each orbit has a vertex at the point where the comet is closest to the sun.
  (c) Surely, there have been many comets with parabolic or hyperbolic orbits that were not identified. We only get to see such comets once.
  (d) Comets with elliptical orbits, such as Halley's comet, are the only ones that remain in our solar system.

Orbits of Comets as Conic Sections II (page 30, Lecture 1)

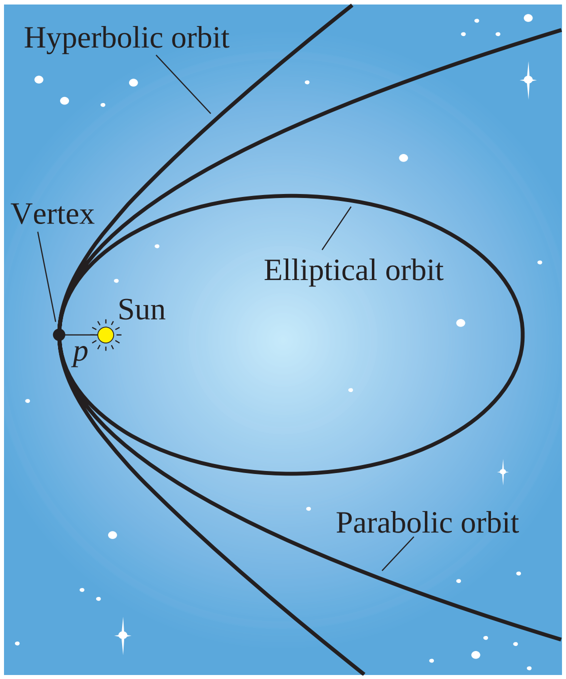


Figure 12: Orbits of comets as conic sections.

Definition of Function I (page 10, Lecture 2)

- A function f from a set A to a set B is a relation that assigns to each element x in the set A exactly one element y in the set B. The set A is the domain (or set of inputs) of the function f, and the set B contains the range (or set of outputs).

- **Example**. To help understand this definition, consider the function shown in Figure 13 that relates the time of day to the temperature.

  This function can be represented by the following set of ordered pairs, in which the first coordinate (x-value) is the input and the second coordinate (y-value) is the output:

$$\Big\{(1, 9^\circ), (2, 13^\circ), (3, 15^\circ), (4, 15^\circ), (5, 12^\circ), (6, 10^\circ)\Big\}.$$

Definition of Function II (page 10, Lecture 2)

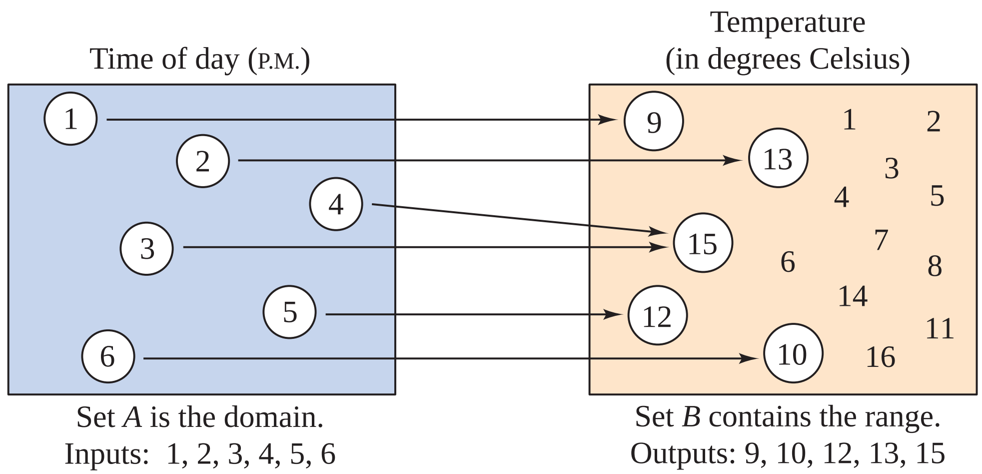


Figure 13: Temperature as a function of the time of day.

Characteristics of a Function (page 10, Lecture 2)

- Note that, for a function from a set A to a set B:
  (a) Each element in A must be matched with an element in B.
  (b) Some elements in B may not be matched with any element in A.
  (c) Two or more elements in A may be matched with the same element in B (say, in the above example, $f(3) = f(4) = 15^\circ$).
  (d) An element in A (the domain) cannot be matched with two different elements in B.

Analytical Representation of a Function (page 10, Lecture 2)

- Functions can be analytically represented by equations or formulas involving two variables.
- For instance, the equation

$$y = f(x) = x^2$$

  represents the variable y as a function of the variable x, where x is the independent variable and y is the dependent variable.
- The domain of a function is the set of all values taken on by its independent variable x, and the range of a function is the set of all values taken on by its dependent variable y.

Domain of a Function I (page 15, Lecture 2)

- The domain of a function can either be described explicitly, or be implied by the expression used to define the function.

- **Example**. For instance, the function given by

$$f(x) = \frac{1}{x^2 - 4}$$

has an implied domain that consists of all real x other than $x = \pm 2$. As another example, the function given by

$$f(x) = \sqrt{x}$$

is defined only for $x \geq 0$.

- In general, the domain of a function excludes values that would cause division by zero or that would result in the even root of a negative number.

Domain of a Function II (page 15, Lecture 2)

- In some situations, the domain of a function can also be implied by the physical context.
- **Example**. For instance, the volume V of a sphere with radius r can be represented by the function

$$V = f(r) = \frac{4}{3}\pi r^3.$$

Although the equation for f itself does not give any reason to restrict r to positive values, the physical context implies that a sphere cannot have a negative or zero radius. Thus the domain of f is the set of all real numbers r such that $r > 0$.

Graph of a Function (page 14, Lecture 2)

- The graph of a function f is the collection of all ordered pairs $(x, f(x))$ such that x lies in the domain of f.

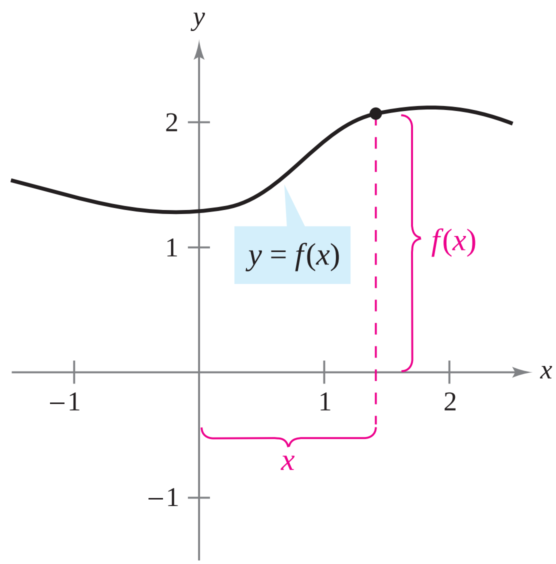


Figure 14: The graph of a function f.

Vertical Line Test for Functions I (page 13, Lecture 2)

- A set of points in a coordinate plane is the graph of y as a function of x if and only if no vertical line intersects the graph at more than one point.
- **Example**. According to the vertical line test, the graph shown below in Figure 15 (a) is not the graph of y as a function of x, while those shown in Figure 15 (b)(c)(d) are the graph of y as a function of x.

Vertical Line Test for Functions II (page 13, Lecture 2)

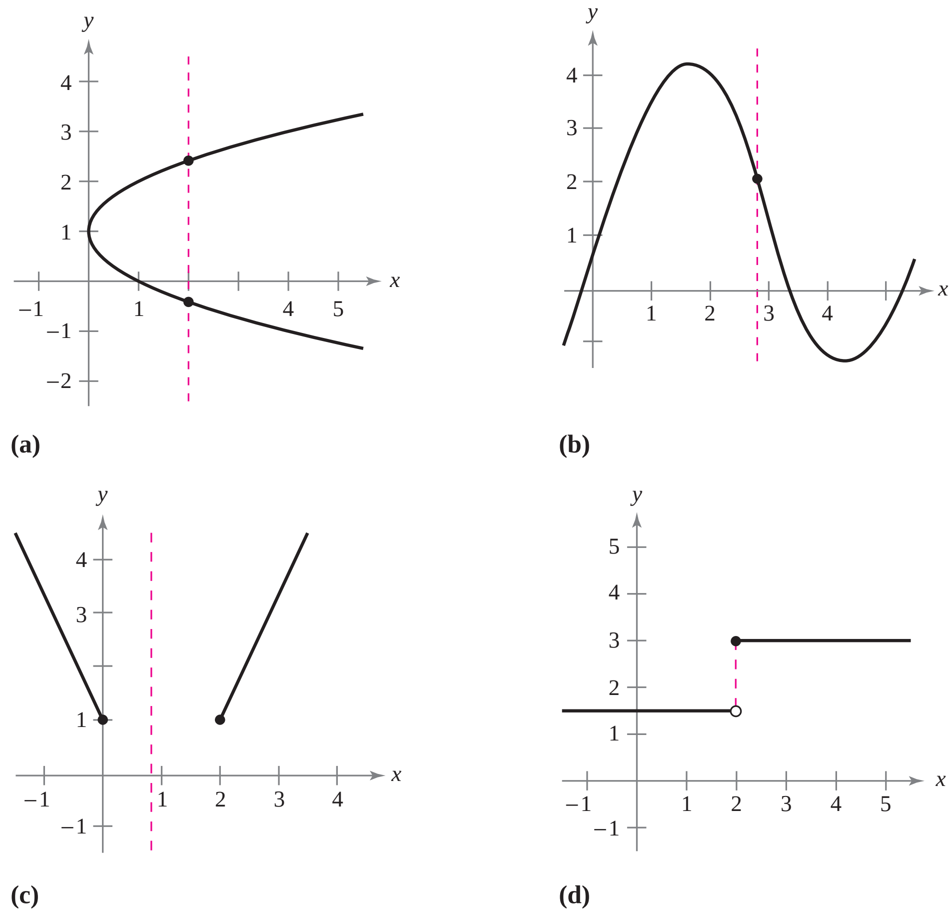


Figure 15: Applying the vertical line test.

Definition of Local Extremum I (page 26, Lecture 2)

- A function value $f(a)$ is called a local (or relative) minimum of f if there exists an interval (x_1, x_2) that contains a such that

  $$x_1 < x < x_2 \qquad \text{implies} \qquad f(a) \leq f(x).$$

  A function value $f(a)$ is called a local (or relative) maximum of f if there exists an interval (x_1, x_2) that contains a such that

  $$x_1 < x < x_2 \qquad \text{implies} \qquad f(a) \geq f(x).$$

Definition of Local Extremum II (page 26, Lecture 2)

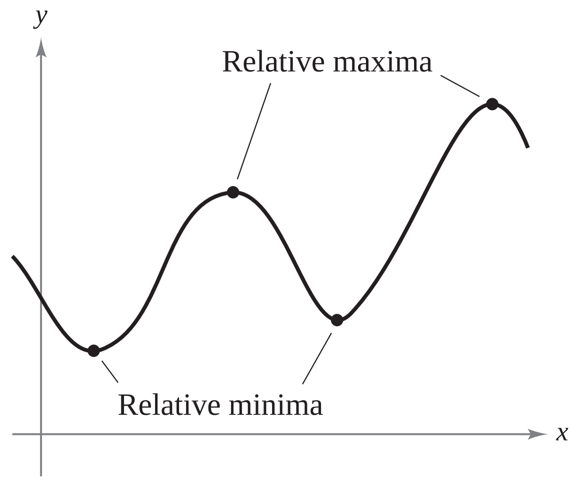


Figure 16: Local (relative) minima and maxima of a function.

Combinations of Functions I (page 17, Lecture 2)

- The domain of an arithmetic combination, namely, sum, difference, product, or quotient, of two functions f and g consists of all real numbers that are common to the domains of f and g.
- In the case of the quotient f/g, there is the further restriction that $g(x) \neq 0$.

Combinations of Functions II (page 17, Lecture 2)

- **Example**. For instance, for $f(x) = \sqrt{x}$ and $g(x) = \sqrt{4 - x^2}$, the quotient of f and g is

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x}}{\sqrt{4 - x^2}},$$

and the quotient of g and f is

$$\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)} = \frac{\sqrt{4 - x^2}}{\sqrt{x}}.$$

The domain of f is $[0, \infty)$ and the domain of g is $[-2, 2]$. The intersection of these domains is $[0, 2]$. So the domains of f/g and g/f are:

$$\text{domain of } f/g : [0, 2), \qquad \text{domain of } g/f : (0, 2].$$

Combinations of Functions III (page 17, Lecture 2)

- The composition of a function f with another function g is

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f.

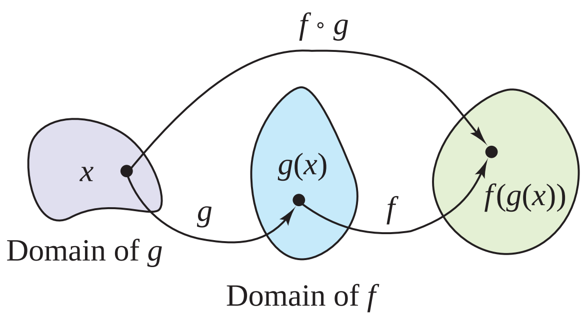


Figure 17: The composition of two function f and g.

Combinations of Functions IV (page 17, Lecture 2)

- **Example**. For instance, for $f(x) = x^2 - 9$ and $g(x) = \sqrt{9 - x^2}$, the composition $f \circ g$ is

  $$(f \circ g)(x) = f(g(x)) = f(\sqrt{9 - x^2}) = -x^2.$$

  From this, it might appear that the domain of the composition is the set of all real numbers. This, however, is not true. Because the domain of f is the set of all real numbers and the domain of g is $[-3, 3]$, the domain of $f \circ g$ is $[-3, 3]$.

Combinations of Functions V (page 17, Lecture 2)

- Besides forming the composition of two given functions, sometimes it is also important to be able to identify two functions that make up a given composite function.

- **Example**. For instance, the function h given by $h(x) = (3x-5)^3$ is the composition of f with g where $f(x) = x^3$ and $g(x) = 3x - 5$. That is,

$$h(x) = (3x-5)^3 = f(3x-5) = f(g(x)).$$

In general, to decompose a composite function, try looking for an "inner" function and an "outer" function. For the function h given above, $g(x) = 3x - 5$ is the inner function and $f(x) = x^3$ is the outer function.

Definition of Inverse Function (page 33, Lecture 2)

- Let f and g be two functions such that

$$f(g(x)) = x, \qquad \text{for every } x \text{ in the domain of } g,$$

and

$$g(f(x)) = x, \qquad \text{for every } x \text{ in the domain of } f.$$

Under these conditions, the function g is the inverse function of the function f (and vice versa). The function g is denoted by f^{-1} (read "f-inverse"), and satisfies

$$f(f^{-1}(x)) = x, \qquad \text{and} \qquad f^{-1}(f(x)) = x.$$

The domain of f must equal the range of f^{-1}, and the range of f must equal the domain of f^{-1}.

Graph of an Inverse Function (page 33, Lecture 2)

- If the point (a, b) lies on the graph of a function f, then the point (b, a) must lie on the graph of its inverse f^{-1} (assuming it exists), and vice versa. This means that the graph of f^{-1} is a reflection of the graph of f about the line $y = x$.

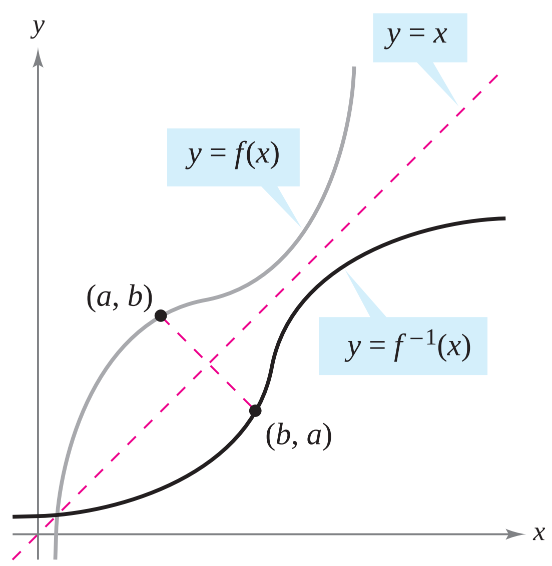


Figure 18: The graphs of a function f and its inverse function f^{-1}.

Horizontal Line Test for Inverse Functions I (page 35, Lecture 2)

- A function f has an inverse function if and only if no horizontal line intersects the graph of f at more than one point.
- If no horizontal line intersects the graph of f at more than one point, then no y-value is matched with more than one x-value. This is the essential characteristic of what are called one-to-one functions.
- In analytical terms, a function f is one-to-one if and only if

  (a) $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$, or equivalently,

  (b) $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

Horizontal Line Test for Inverse Functions II (page 35, Lecture 2)

- **Example**. According to the horizontal line test, the function given by $f(x) = x^3 - 1$ is a one-to-one function and does have an inverse function, while the function given by $f(x) = x^2 - 1$ is not a one-to-one function and does not have an inverse function.

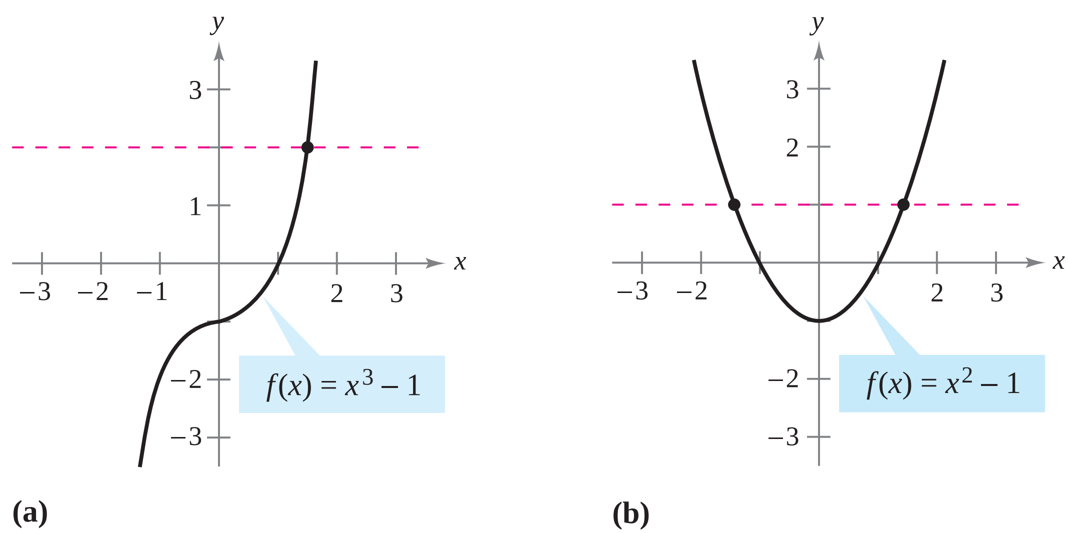


Figure 19: Applying the horizontal line test.

Finding Inverse Functions Graphically I (page 36, Lecture 2)

- The inverse function of a one-to-one function f can be obtained graphically by reflecting the graph of f about the line $y = x$.
- **Example**. For instance, for $f(x) = 2x - 3$, the graph of its inverse function $f^{-1}(x) = \frac{1}{2}(x+3)$ can be obtained by reflecting the graph of f about the line $y = x$ (Figure 20). This reflective property can be further verified by testing a few points on each graph.

Graph of f	Graph of f^{-1}
$(-1,-5)$	$(-5,-1)$
$(0,-3)$	$(-3,0)$
$(1,-1)$	$(-1,1)$
$(2,1)$	$(1,2)$
$(3,3)$	$(3,3)$

Finding Inverse Functions Graphically II (page 36, Lecture 2)

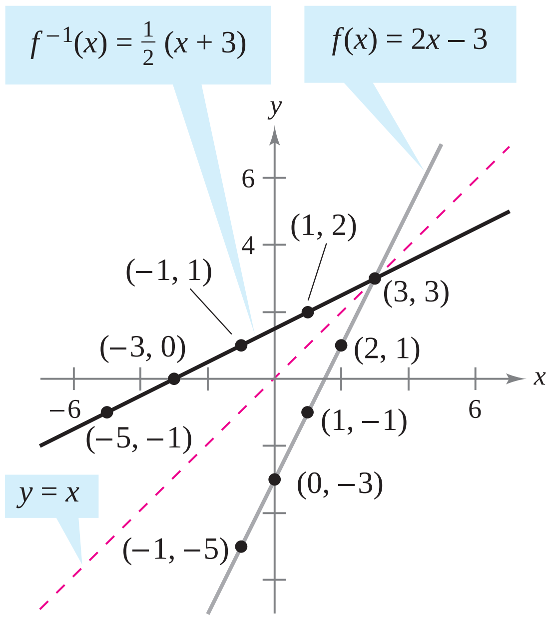


Figure 20: Finding the inverse function of $f(x) = 2x - 3$.

Finding Inverse Functions Analytically I (page 36, Lecture 2)

- The inverse function of a one-to-one function f can also be obtained analytically by using the following procedure.
  (a) If necessary, use the horizontal line test to determine whether f has an inverse function.
  (b) In the equation for $f(x)$, replace $f(x)$ by y.
  (c) Interchange the roles of x and y, and solve for y.
  (d) Replace y by $f^{-1}(x)$ in the new equation.
  (e) Verify that f and f^{-1} are inverse functions of each other by showing that the domain of f is equal to the range of f^{-1}, the range of f is equal to the domain of f^{-1}, and $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

Finding Inverse Functions Analytically II (page 36, Lecture 2)

- **Example**. For instance, for $f(x) = \sqrt{2x-3}$, the horizontal line test shows that f is one-to-one and has an inverse function. The inverse f^{-1} is then given by

$$
\begin{aligned}
f(x) &= \sqrt{2x-3}, && \text{(write original function)} \\
y &= \sqrt{2x-3}, && \text{(replace } f(x) \text{ by } y\text{)} \\
x &= \sqrt{2y-3}, && \text{(interchange } x \text{ and } y\text{)} \\
y &= \frac{1}{2}\,(x^2+3), && \text{(solve for } y\text{)} \\
f^{-1}(x) &= \frac{1}{2}\,(x^2+3), \quad x \geq 0. && \text{(replace } y \text{ by } f^{-1}(x)\text{)}
\end{aligned}
$$

Finding Inverse Functions Analytically III (page 36, Lecture 2)

Note that the range of f is the interval $[0, \infty)$, which implies that the domain of f^{-1} must also be $[0, \infty)$. Moreover, the domain of f is the interval $[3/2, \infty)$, which implies that the range of f^{-1} is $[3/2, \infty)$.

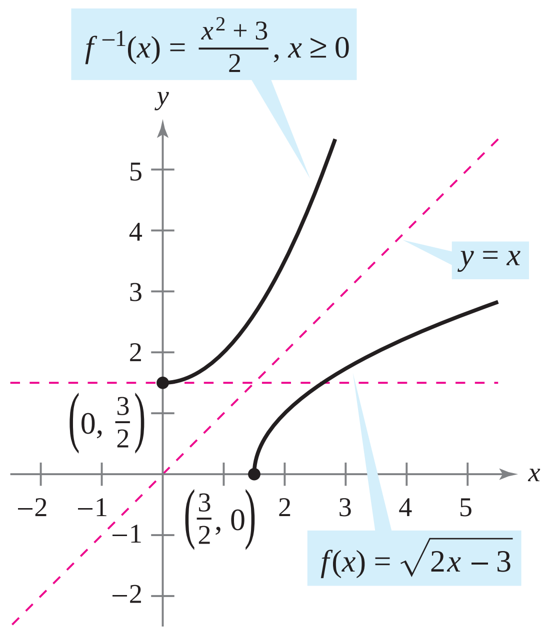


Figure 21: Finding the inverse function of $f(x) = \sqrt{2x - 3}$.

Transformations of Functions I (page 42–47, Lecture 2)

- To understand the effect of the various types of transformations of functions, consider the function

$$y = f(x) = 2x.$$

This equation can be solved for x and written in the form

$$x = \frac{1}{2}y.$$

In what follows, $c > 0$ denotes a positive real number, "equation for y" refers to the equation $y = 2x$, and "definition of f" refers to the expression $f(x) = 2x$.

Transformations of Functions II (page 42–47, Lecture 2)

(a) If, in the above definition of f, x is replaced by $x+c$, then the equation for y becomes

$$y = 2(x+c), \qquad \text{or} \qquad x = \frac{1}{2}y - c.$$

This shows that for any fixed value of y, the value of x gets reduced by c units. This corresponds to a horizontal shift of the graph of f to the left by c units.

(b) If, in the above definition of f, x is replaced by $x-c$, then the equation for y becomes

$$y = 2(x-c), \qquad \text{or} \qquad x = \frac{1}{2}y + c.$$

This shows that for any fixed value of y, the value of x gets raised by c units. This corresponds to a horizontal shift of the graph of f to the right by c units.

Transformations of Functions III (page 42–47, Lecture 2)

(c) If, in the above equation for y, y is replaced by $y + c$, then the equation for y becomes

$$y + c = 2x, \qquad \text{or} \qquad y = 2x - c = f(x) - c.$$

This shows that for any fixed value of x, the value of y gets reduced by c units. This corresponds to a vertical shift of the graph of f downward by c units. Note that replacing y by $y + c$ in the equation for y has the same effect as subtracting c from f in the definition of f.

Transformations of Functions IV (page 42–47, Lecture 2)

(d) If, in the above equation for y, y is replaced by $y - c$, then the equation for y becomes

$$y - c = 2x, \qquad \text{or} \qquad y = 2x + c = f(x) + c.$$

This shows that for any fixed value of x, the value of y gets raised by c units. This corresponds to a vertical shift of the graph of f upward by c units. Note that replacing y by $y - c$ in the equation for y has the same effect as adding c to f in the definition of f.

Transformations of Functions V (page 42–47, Lecture 2)

(e) If, in the above definition of f, x is replaced by cx, then the equation for y becomes

$$y = 2(cx), \qquad \text{or} \qquad x = \frac{1}{2c}\,y.$$

If $c > 1$, this shows that for any fixed value of y, the value of x gets shrunk by a factor of c. This corresponds to a horizontal compression of the graph of f by a factor of c. If $c < 1$, this shows that for any fixed value of y, the value of x gets amplified by a factor of $1/c$. This corresponds to a horizontal stretching of the graph of f by a factor of $1/c$.

Transformations of Functions VI (page 42–47, Lecture 2)

(f) If, in the above equation for y, y is replaced by cy, then the equation for y becomes

$$cy = 2x, \qquad \text{or} \qquad y = \frac{1}{c}(2x) = \frac{1}{c}f(x).$$

If $c > 1$ (or equivalently, $1/c < 1$), this shows that for any fixed value of x, the value of y gets shrunk by a factor of c. This corresponds to a vertical compression of the graph of f by a factor of c. If $c < 1$ (or equivalently, $1/c > 1$), this shows that for any fixed value of x, the value of y gets amplified by a factor of $1/c$. This corresponds to a vertical stretching of the graph of f by a factor of $1/c$. Note that replacing y by cy in the equation for y has the same effect as dividing f by c in the definition of f.

Transformations of Functions VII (page 42–47, Lecture 2)

(g) If, in the above definition of f, x is replaced by $-x$, then the equation for y becomes

$$y = 2(-x), \qquad \text{or} \qquad x = -\frac{1}{2}\,y.$$

This shows that for any fixed value of y, the value of x gets flipped horizontally across the y-axis. This corresponds to a horizontal reflection (or reflection about the y-axis) of the graph of f.

(h) If, in the above equation for y, y is replaced by $-y$, then the equation for y becomes

$$-y = 2x, \qquad \text{or} \qquad y = -2x = -f(x).$$

This shows that for any fixed value of x, the value of y gets flipped vertically across the x-axis. This corresponds to a vertical reflection (or reflection about the x-axis) of the graph of f.

Graphs of Polynomial Functions I (page 11, Lecture 3)

- The graph of a general polynomial function

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \qquad a_n \neq 0,$$

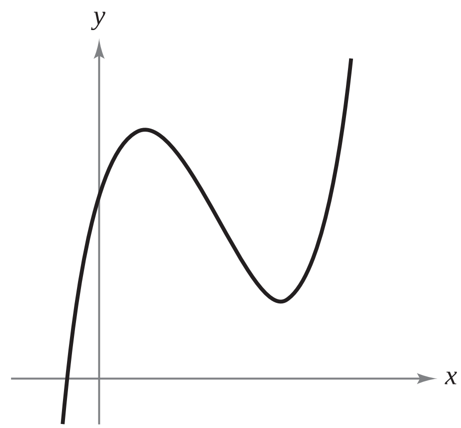
has two basic features:

(a) It is continuous[a], i.e. it has no breaks, holes, or gaps (Figure 22).

(b) It has only smooth, rounded turns (Figure 23).

[a]The precise definition of continuity will be given in Lecture 7.

Graphs of Polynomial Functions II (page 11, Lecture 3)

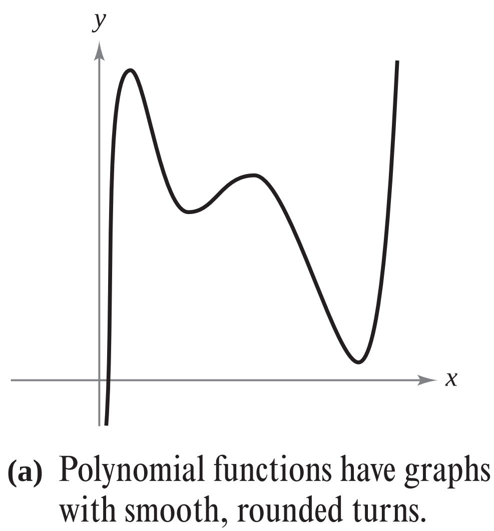


(a) Polynomial functions have continuous graphs.

(b) Functions with graphs that are not continuous are not polynomial functions.

Figure 22: The graph of a polynomial function is continuous.

Graphs of Polynomial Functions III (page 11, Lecture 3)



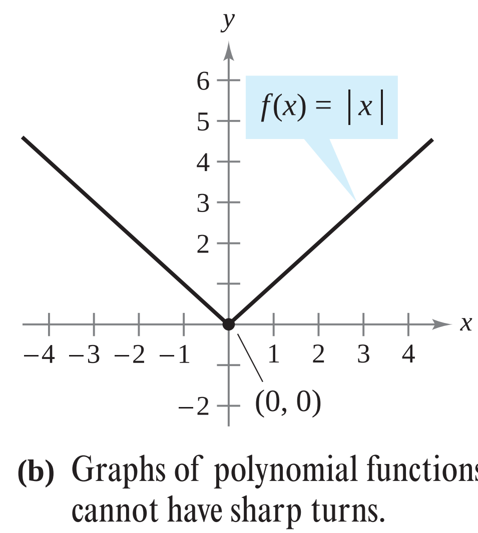


Figure 23: The graph of a polynomial function has only smooth, rounded turns.

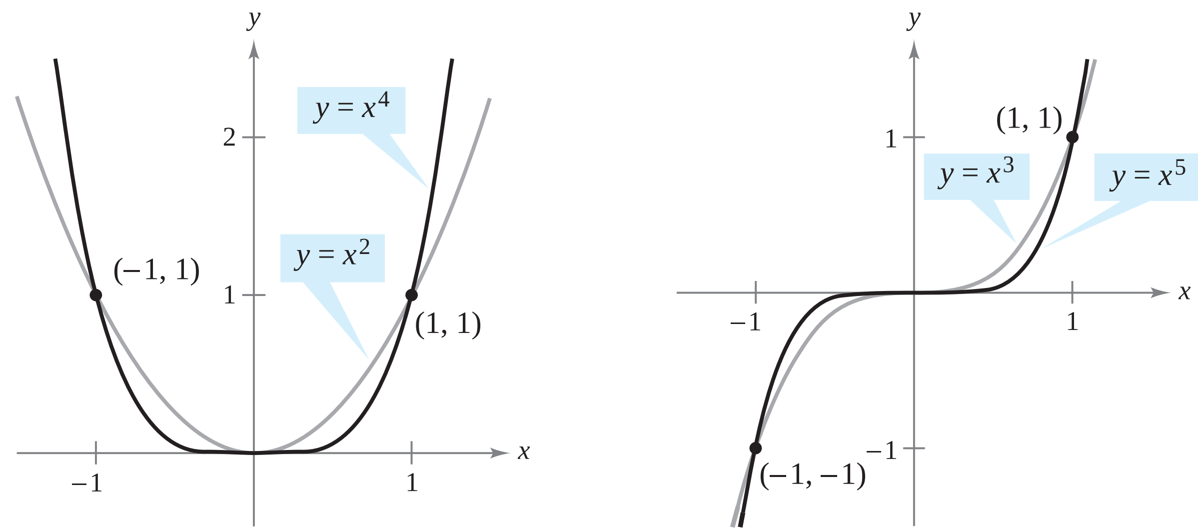
Graphs of Polynomial Functions IV (page 11, Lecture 3)

- The polynomial functions that have the simplest graphs are monomials of the form
$$p(x) = x^n,$$
where n is a positive integer.

  (a) When n is even, the graph of p is similar to that of $q(x) = x^2$, and when n is odd, the graph of p is similar to that of $q(x) = x^3$.

  (b) Moreover, the greater the value of n, the flatter the graph near the origin (Figure 24).

Graphs of Polynomial Functions V (page 11, Lecture 3)



(a) When n is even, the graph of $y = x^n$ touches the axis at the x-intercept.

(b) When n is odd, the graph of $y = x^n$ crosses the axis at the x-intercept.

Figure 24: The graph of a monomial.

The Leading Coefficient Test I (page 11, Lecture 3)

- As x moves without bound to the left or to the right, the graph of the polynomial function

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \qquad a_n \neq 0,$$

eventually rises or falls in the following manner.

(a) When n is odd: if the leading coefficient a_n is positive, the graph falls to the left and rises to the right. If the leading coefficient a_n is negative, the graph rises to the left and falls to the right (Figure 25).

(b) When n is even: if the leading coefficient a_n is positive, the graph rises to the left and right. If the leading coefficient a_n is negative, the graph falls to the left and right (Figure 26).

The Leading Coefficient Test II (page 11, Lecture 3)

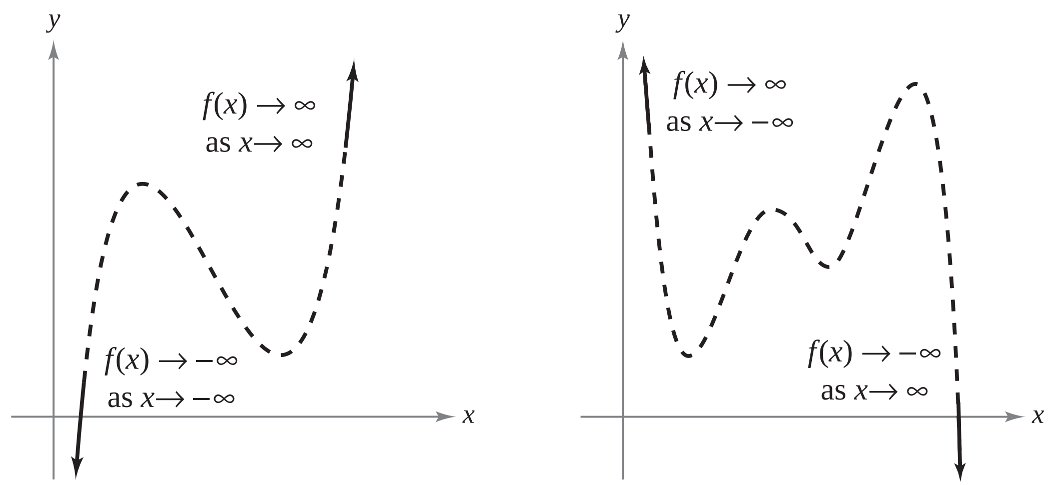


Figure 25: The large-x behavior of a polynomial function when its degree is odd.

The Leading Coefficient Test III (page 11, Lecture 3)

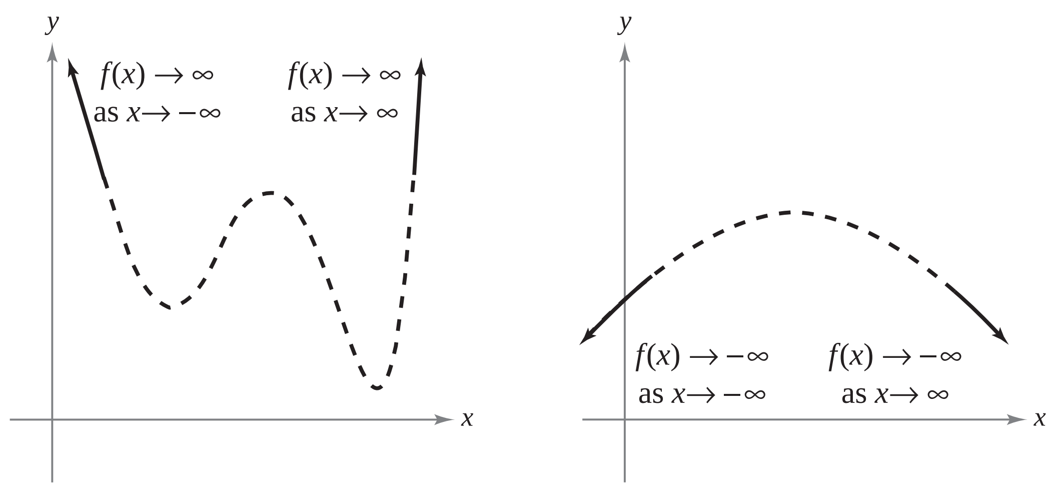


Figure 26: The large-x behavior of a polynomial function when its degree is even.

Real Zeros of Polynomial Functions I (page 12, Lecture 3)

- The zeros of a function f are the x-values for which $f(x) = 0$.
- For a polynomial function p of degree n:
  (a) The function p has at most n real zeros[a].
  (b) The graph of p has at most $n - 1$ turning points (i.e. local minima or local maxima).

[a] Based on the Fundamental Theorem of Algebra, which asserts that any polynomial of degree $n > 0$ has at least one zero in the complex domain, it can be shown that p has precisely n complex zeros, counting multiplicity.

Real Zeros of Polynomial Functions II (page 12, Lecture 3)

- When p is a polynomial function and c is a real number, the following statements are equivalent[a].
  (a) $x = c$ is a zero of the function p.
  (b) $x = c$ is a solution of the polynomial equation $p(x) = 0$.
  (c) $x - c$ is a factor of the polynomial $p(x)$.
  (d) $(c, 0)$ is an x-intercept of the graph of p.
- A factor $(x - c)^k$, $k > 1$, of a polynomial function $p(x)$ yields a repeated zero $x = c$ of multiplicity k.
  (a) When k is odd, the graph of p crosses the x-axis at $x = c$.
  (b) When k is even, the graph of p touches, but does not cross, the x-axis at $x = c$ (Figure 24).

[a] For the equivalence of (b) and (c), see Theorem 2 on page 72.

The Division Algorithm (page 15–16, Lecture 3)

- The procedure for dividing two polynomials is especially useful in factoring and finding the zeros of polynomial functions.
- When f and g are polynomials such that $g \neq 0$, and the degree of g is less than or equal to the degree of f, there exist unique polynomials q and r such that

  $$f(x) = q(x)g(x) + r(x), \qquad \text{or} \qquad \frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)},$$

  where $r = 0$ or the degree of r is less than the degree of g. When the remainder r is zero, g is said to divide evenly into f.

The Remainder Theorem (page 14, Lecture 3)

- The remainder obtained from the division algorithm has an important interpretation, as described in the following theorem.
- **Theorem 1** (The remainder theorem). *When a polynomial $f(x)$ is divided by $x - c$, the remainder is*

$$r = f(c).$$

- **Proof**. According to the division algorithm, f can be written as

$$f(x) = (x - c)q(x) + r(x),$$

for some polynomials q and r, where r is either 0 or a polynomial of degree less than 1 (the degree of $x - c$). Thus r is a constant, i.e. $r(x) \equiv r$, and evaluating f at $x = c$ yields

$$f(c) = (c - c)q(c) + r = r. \qquad \square$$

The Factor Theorem (page 14, Lecture 3)

- The remainder theorem has the following useful consequence.

- **Theorem 2** (The factor theorem)**.** *A polynomial $f(x)$ has a factor $x - c$ if and only if $f(c) = 0$.*

- **Proof**. According to the division algorithm, f can be written as

$$f(x) = (x - c)q(x) + r(x),$$

where, by the remainder theorem, $r(x) \equiv r = f(c)$. Thus

$$f(x) = (x - c)q(x) + f(c).$$

If $f(c) = 0$, then clearly $f(x) = (x - c)q(x)$, which shows that $x - c$ is a factor of $f(x)$. Conversely, if $x - c$ is a factor of $f(x)$, division of $f(x)$ by $x - c$ yields a remainder of 0. This, by the remainder theorem, shows that $f(c) = r \equiv r(x) = 0$. $\square$

The Linear Factorization Theorem I (page 14, Lecture 3)

- The factor theorem can be repeatedly applied to a given polynomial function, yielding the following useful result.
- **Theorem 3** (The linear factorization theorem)**.** *If*

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \qquad a_n \neq 0,$$

is a polynomial of degree $n > 0$, and f has n real zeros[a] $c_1, c_2, \ldots, c_n$ *(which may or may not be distinct), then f factors as*

$$f(x) = a_n(x - c_1)(x - c_2)\cdots(x - c_n).$$

[a]The theorem in fact holds true even when the c_i's are complex.

The Linear Factorization Theorem II (page 14, Lecture 3)

- **Proof**. (Sketch) It can be shown, using the factor theorem and an induction argument, that

$$f(x) = c(x - c_1)(x - c_2) \cdots (x - c_n)$$

for some constant c. Comparing the leading coefficient of f (which is a_n) and that of the polynomial on the right side (which is c) then shows that $c = a_n$. □

The Rational Zero Test I (page 17–19, Lecture 3)

- If the polynomial

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \qquad a_n \neq 0,$$

has integer coefficients, every rational zero of f has the form

$$\text{rational zero} = \frac{p}{q},$$

where p and q have no common factors other than 1, and

(a) $p =$ a factor of the constant term a_0, and

(b) $q =$ a factor of the leading coefficient a_n.

- When the leading coefficient a_n is 1, the possible rational zeros of f are simply (integer) factors of the constant term a_0.

The Rational Zero Test II (page 17–19, Lecture 3)

- To use the rational zero test:
  (a) List all rational numbers whose numerators are factors of the constant term and whose denominators are factors of the leading coefficient:
  $$\text{possible rational zeros} = \frac{\text{factors of constant term}}{\text{factors of leading coefficient}}.$$
  (b) Use a trial-and-error approach to determine which, if any, of these numbers are actual zeros of the polynomial.

The Rational Zero Test III (page 17–19, Lecture 3)

- **Example**. For instance, for the cubic polynomial

$$f(x) = 2x^3 + 3x^2 - 8x + 3,$$

the leading coefficient is $a_3 = 2$ and the constant term is $a_0 = 3$. The rational zero test then asserts that

$$\begin{aligned} &\text{possible rational zeros of } f \\ &= \frac{\text{factors of } 3}{\text{factors of } 2} = \frac{\pm 1,\ \pm 3}{\pm 1,\ \pm 2} = \pm 1,\ \pm 3,\ \pm\frac{1}{2},\ \pm\frac{3}{2}. \end{aligned}$$

Direct evaluation shows that

$$f(1) = f(1/2) = f(-3) = 0,$$

so f has zeros $x = 1,\ 1/2,\ -3$, and f factors as (see Theorem 3)

$$f(x) = 2(x-1)(x-1/2)(x+3).$$

The Rational Zero Test IV (page 17–19, Lecture 3)

Note that, if only one or two zeros of f, say $x = 1$, are detected by this procedure, then f needs to be factored using a long division, yielding

$$f(x) = (x - 1)(2x^2 + 5x - 3).$$

The zeros of the (lower-order) quadratic polynomial $2x^2 + 5x - 3$ can then be determined using the quadratic formula, which again yields $x = 1/2$ and $x = -3$.

Asymptotes of a Rational Function I (page 20, Lecture 3)

- The line $x = a$ is a vertical asymptote of the graph of a function f when[a]

  $$f(x) \to \infty \qquad \text{or} \qquad f(x) \to -\infty$$

  as $x \to a$, either from the right or from the left. The line $y = b$ is a horizontal asymptote of the graph of f when

  $$f(x) \to b$$

  as $x \to \infty$ or $x \to -\infty$.

[a]The precise definition of limit will be given in Lecture 6.

Asymptotes of a Rational Function II (page 20, Lecture 3)

- The following figure shows the vertical and horizontal asymptotes of the graphs of three rational functions.

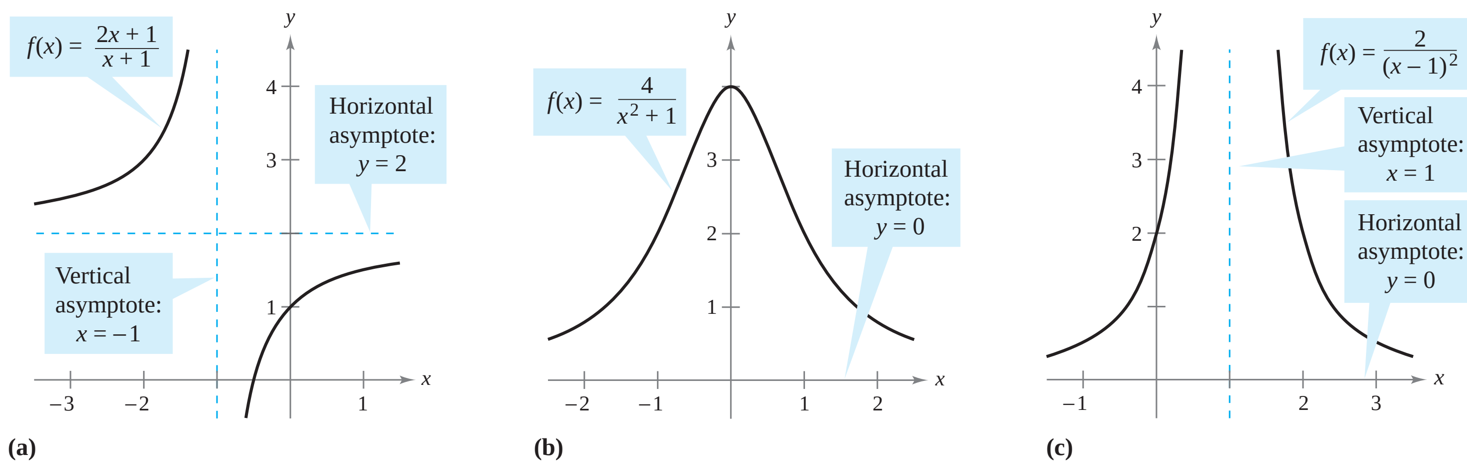


Figure 27: Vertical and horizontal asymptotes of the graphs of three rational functions.

Asymptotes of a Rational Function III (page 20, Lecture 3)

- Let f be the rational function given by

$$f(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0}, \; a_n \neq 0, \; b_m \neq 0,$$

where p and q have no common factors.

(a) The graph of f has vertical asymptotes at the zeros of q.

(b) The graph of f has one or no horizontal asymptote determined by comparing the degrees of p and q.

(i) When $n < m$, the graph of f has the line $y = 0$ (the x-axis) as a horizontal asymptote.

(ii) When $n = m$, the graph of f has the line $y = a_n / b_m$ (ratio of the leading coefficients) as a horizontal asymptote.

(iii) When $n > m$, the graph of f has no horizontal asymptote.

Asymptotes of a Rational Function IV (page 20, Lecture 3)

- **Example**. For instance, the rational function $f(x) = 2x^2/(x^2-1)$ has two vertical asymptotes $x = -1,\ x = 1$ and one horizontal asymptote $y = 2$.

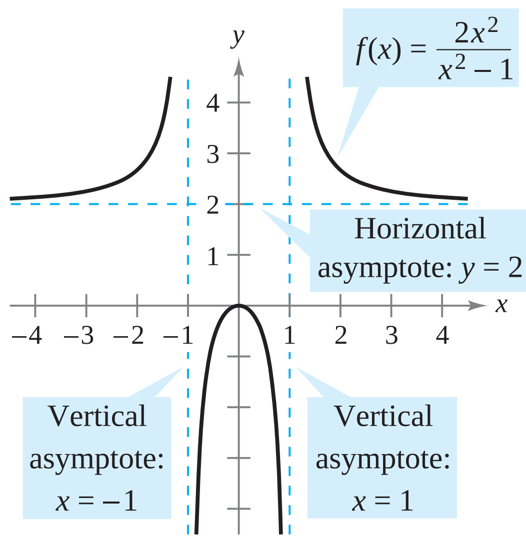


Figure 28: Vertical and horizontal asymptotes of the graph of $f(x) = 2x^2/(x^2-1)$.

Asymptotes of a Rational Function V (page 20, Lecture 3)

- Consider a rational function f. When the degree of the numerator of f is exactly one more than the degree of the denominator, the graph of f has a slant (or oblique) asymptote.

- **Example**. For instance, the graph of

$$f(x) = \frac{x^2 - x}{x + 1}$$

has a slant asymptote, as shown in Figure 29. To find the equation of the slant asymptote, use long division to obtain

$$f(x) = \frac{x^2 - x}{x + 1} = x - 2 + \frac{2}{x + 1}.$$

As x increases or decreases without bound, the remainder term $2/(x + 1)$ approaches 0, so the graph of f approaches the line $y = x - 2$, which is the slant asymptote.

Asymptotes of a Rational Function VI (page 20, Lecture 3)

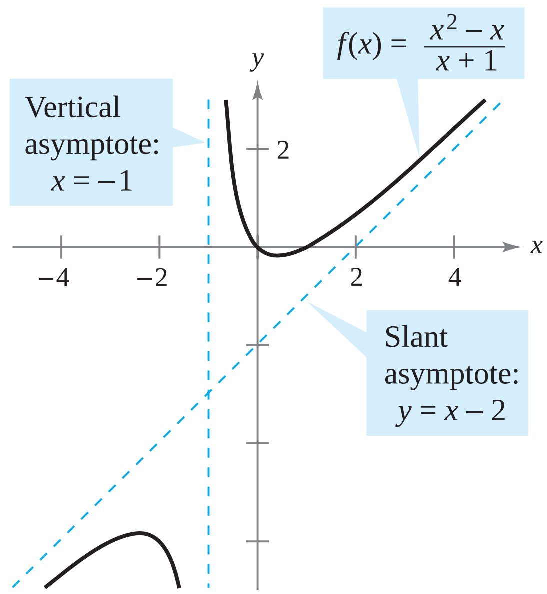


Figure 29: Slant asymptotes of the graph of $f(x) = (x^2 - x)/(x + 1)$.

Method of Partial Fractions I (page 23–24, Lecture 3)

- The method of partial fractions is a technique for rewriting rational functions as a sum of simpler fractions, the so-called partial fractions.

- The general procedure for finding the partial fractions of a rational function $f = p/q$ can be summarized as follows.

  (a) If f is improper, that is, if p is of degree at least that of q, divide p by q to obtain

  $$f(x) = \text{quotient (a polynomial)} + \frac{N(x)}{D(x)}.$$

  (b) Factor D into a product of linear and irreducible[a] quadratic factors with real coefficients.

[a] A quadratic polynomial is said to be irreducible if it cannot be written as the product of two linear factors with real coefficients.

Method of Partial Fractions II (page 23–24, Lecture 3)

(c) For each factor of the form $(ax+b)^k$, expect the decomposition to have the terms

$$\frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}.$$

(d) For each factor of the form $(ax^2+bx+c)^m$, expect the decomposition to have the terms

$$\frac{B_1 x + C_1}{(ax^2+bx+c)} + \frac{B_2 x + C_2}{(ax^2+bx+c)^2} + \cdots + \frac{B_m x + C_m}{(ax^2+bx+c)^m}.$$

Method of Partial Fractions III (page 23–24, Lecture 3)

(e) Set N/D equal to the sum of all the terms found in (c)–(d).

(f) Multiply both sides of the equation found in (e) by D and solve for the unknown constants; this can be done by either:

  (i) equating coefficients of like powers (most general);

  (ii) assigning convenient values to the variables x (most effective for simple linear factors $(ax + b)$); or

  (iii) using differentiation (most effective for repeated linear factors).

Method of Partial Fractions IV (page 29–30, Lecture 3)

- **Example**.[a] As an example, consider the rational function

$$f(x) = \frac{3x^2 - 8x + 13}{(x+3)(x-1)^2}.$$

The partial fractions of f are found as follows.

(a) Check the rational function, denoted by p/q, is proper:

$$\deg p(x) = 2 < \deg q(x) = 3.$$

(b) Make the ansatz

$$\frac{3x^2 - 8x + 13}{(x+3)(x-1)^2} = \frac{A}{x+3} + \frac{B}{x-1} + \frac{C}{(x-1)^2}.$$

[a] See Example 9 on page 29.

Method of Partial Fractions V (page 29–30, Lecture 3)

(c) Multiply both sides by $(x+3)(x-1)^2$ to obtain

$$3x^2 - 8x + 13 = A(x-1)^2 + B(x-1)(x+3) + C(x+3). \qquad (1)$$

(d) Determine the coefficients A and C by setting x to -3 and 1 (the zeros of the linear factors $x+3$ and $x-1$) in equation (1):

$$x = -3: \quad 3(-3)^2 - 8(-3) + 13 = A(-4)^2,$$
$$x = 1: \quad 3(1)^2 - 8(1) + 13 = C(4).$$

The result is $A = 4,\ C = 2$.

Method of Partial Fractions VI (page 29–30, Lecture 3)

(e) Determine the coefficient B either by setting x to a convenient value (say 0) in equation (1):

$$x = 0: \quad 3(0)^2 - 8(0) + 13 = A(-1)^2 + B(-1)(3) + C(3),$$

or by expanding both sides of (1) and comparing the coefficients of x^2:

$$3x^2 - 8x + 13 = (A + B)x^2 + (-2A + 2B + C)x + (A - 3B + 3C),$$

which yields $A + B = 3$. In either case, the result is $B = -1$ (recall $A = 4,\ C = 2$).

Method of Partial Fractions VII (page 37–39, Lecture 3)

- **Example**.[a] As another example, consider the rational function

$$f(x) = \frac{x^5 + 2x^3 - x + 1}{x^3 + 5x}.$$

The partial fractions of f are found as follows.

(a) Since the rational function, denoted by p/q, is improper:

$$\deg p(x) = 5 > \deg q(x) = 3,$$

a long division is done to decompose p/q as

$$\frac{p(x)}{q(x)} = x^2 - 3 + \frac{14x + 1}{x^3 + 5x}.$$

[a]See Example 11 on page 37.

Method of Partial Fractions VIII (page 37–39, Lecture 3)

(b) Factor the denominator as

$$x^3 + 5x = x(x^2 + 5).$$

Note that $x^2 + 5$ is an irreducible quadratic factor.

(c) For the fractional part of the rational function, make the ansatz

$$\frac{14x + 1}{x(x^2 + 5)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 5}.$$

(d) Multiply both sides by $x(x^2 + 5)$ to obtain

$$14x + 1 = A(x^2 + 5) + (Bx + C)x. \tag{2}$$

Method of Partial Fractions IX (page 37–39, Lecture 3)

(e) Determine the coefficient A by setting x to 0 in equation (2):

$$x = 0: \quad 14(0) + 1 = A(5).$$

The result is $A = 1/5$.

(f) Determine the coefficients B and C either by setting x to some convenient values (say ± 1) in equation (2):

$$\begin{aligned} x = 1: &\quad 14(1) + 1 = A[(1)^2 + 5] + [B(1) + C](1), \\ x = -1: &\quad 14(-1) + 1 = A[(-1)^2 + 5] + [B(-1) + C](-1), \end{aligned}$$

or by expanding both sides of (2) and comparing the coefficients of x^2 and x:

$$14x + 1 = (A + B)x^2 + Cx + 5A,$$

which yields $A + B = 0$, $C = 14$. In either case, the result is $B = -1/5$, $C = 14$ (recall $A = 1/5$).

*Appendix: Understanding the Method of Partial Fractions I

- The success of the method of partial fractions relies on two important theoretical observations:
  (a) Every polynomial of degree $n > 0$ with real coefficients can be factored into a product of linear and irreducible quadratic factors.
  (b) Every proper rational function f of the form
  $$f(x) = \frac{p(x)}{(x-c_1)^{k_1}(x-c_2)^{k_2}\cdots(x-c_n)^{k_n}},$$
  where $c_1, c_2, \ldots, c_n$ are distinct and $\deg p(x) < k := k_1 + k_2 + \cdots + k_n$, can be decomposed as
  $$f(x) = \frac{p_1(x)}{(x-c_1)^{k_1}} + \frac{p_2(x)}{(x-c_2)^{k_2}} + \cdots + \frac{p_n(x)}{(x-c_n)^{k_n}},$$
  where $\deg p_j(x) < k_j$, $j = 1, 2, \ldots, n$.

*Appendix: Understanding the Method of Partial Fractions II

- To verify the first statement, note that:

  (a) The Fundamental Theorem of Algebra ensures that, if f is a polynomial of degree $n > 0$, then f has precisely n complex zeros $c_1, c_2, \ldots, c_n$ (which may or may not be distinct), and f factors as

  $$f(x) = a_n(x - c_1)(x - c_2)\cdots(x - c_n),$$

  where a_n is the leading coefficient of f. This is a generalization of the linear factorization theorem (Theorem 3) which applies only to polynomials having real zeros.

  (b) If f is a polynomial that has real coefficients, and if $a + \mathrm{i}b$, where $b \neq 0$, is a zero of f, then the conjugate $a - \mathrm{i}b$ is also a zero of f. This follows from the simple observation that

  $$f(a - \mathrm{i}b) = f(\overline{a + \mathrm{i}b}) = \overline{f(a + \mathrm{i}b)} = \overline{0} = 0.$$

*Appendix: Understanding the Method of Partial Fractions III

(c) Now let f be a polynomial of degree $n > 0$ with real coefficients. Then f can be factored as

$$f(x) = a_n(x - c_1)(x - c_2) \cdots (x - c_n),$$

where a_n is the leading coefficient of f and $c_1, c_2, \ldots, c_n$ are zeros of f. With appropriate rearrangement, it may be assumed that the first k $(0 \leq k \leq n)$ zeros $c_1, c_2, \ldots, c_k$ of f are real and the last $n - k$ zeros $c_{k+1}, c_{k+2}, \ldots, c_n$ of f are complex. Since complex zeros of f occur in conjugate pairs, it may also be assumed that each pair of complex zeros c_{k+2j-1}, c_{k+2j}, $j = 1, 2, \ldots, (n-k)/2$, forms a conjugate pair:

$$c_{k+2j-1} = \alpha_j + \mathrm{i}\beta_j, \qquad c_{k+2j} = \overline{c_{k+2j-1}} = \alpha_j - \mathrm{i}\beta_j.$$

*Appendix: Understanding the Method of Partial Fractions IV

(d) Based on these observations, it follows that the product of the linear factors $(x - c_{k+2j-1})(x - c_{k+2j})$ can be rewritten as

$$\begin{aligned}&(x - c_{k+2j-1})(x - c_{k+2j})\\ &\qquad = \left[x - (\alpha_j + \mathrm{i}\beta_j)\right]\left[x - (\alpha_j - \mathrm{i}\beta_j)\right] = x^2 - 2\alpha_j x + \alpha_j^2 + \beta_j^2,\end{aligned}$$

which is an irreducible quadratic factor (with real coefficients).

*Appendix: Understanding the Method of Partial Fractions V

- To verify the second statement, note that:

  (a) The decomposition

$$\begin{aligned} f(x) &= \frac{p(x)}{(x-c_1)^{k_1}(x-c_2)^{k_2}\cdots(x-c_n)^{k_n}} \\ &= \frac{p_1(x)}{(x-c_1)^{k_1}} + \frac{p_2(x)}{(x-c_2)^{k_2}} + \cdots + \frac{p_n(x)}{(x-c_n)^{k_n}}, \end{aligned}$$

  where $\deg p(x) < k := k_1 + k_2 + \cdots + k_n$ and $\deg p_j(x) < k_j,\ j = 1, 2, \ldots, n$, can be rearranged as

$$p(x) = p_1(x)g_1(x) + p_2(x)g_2(x) + \cdots + p_n(x)g_n(x),$$

  where

$$g_j(x) = (x-c_1)^{k_1}\cdots(x-c_{j-1})^{k_{j-1}}(x-c_{j+1})^{k_{j+1}}\cdots(x-c_n)^{k_n}.$$

*Appendix: Understanding the Method of Partial Fractions VI

(b) Repeated differentiation and evaluation of the above equation at c_j yields

$$\begin{aligned} p^{(l)}(c_j) &= \frac{d^l}{dx^l}\left[p_j(x)g_j(x)\right]\Big|_{x=c_j} \\ &= \sum_{m=0}^{l}\binom{l}{m}p_j^{(m)}(c_j)g_j^{(l-m)}(c_j), \qquad l = 0, 1, \ldots, k_j - 1. \end{aligned}$$

(c) This is a linear system with nonsingular coefficient matrix (in fact, the coefficient matrix is lower triangular and has nonzero diagonal elements $g_j(c_j)$), so it uniquely determines the values of p_j and its derivatives $p_j^{(l)}$ up to order $k_j - 1$ at c_j.

*Appendix: Understanding the Method of Partial Fractions VII

(d) According to the theory of Hermite interpolation, and given the assumption that $\deg p_j(x) < k_j$, the polynomial p_j is then completely determined by the k_j data points $\{(c_j, p_j^{(l)}(c_j))\}_{l=0}^{k_j-1}$.

(e) Now set

$$q(x) = p_1(x)g_1(x) + p_2(x)g_2(x) + \cdots + p_n(x)g_n(x).$$

Since p and q both pass through the same set of $k := k_1 + k_2 + \cdots + k_n$ data points:

$$p^{(l)}(c_j) = q^{(l)}(c_j), \qquad j = 1, 2, \ldots, n,\ l = 0, 1, \ldots, k_j - 1,$$

and p and q are both polynomials of degree no greater than $k - 1$, it follows from the theory of Hermite interpolation that $p = q$, i.e.

$$p(x) = p_1(x)g_1(x) + p_2(x)g_2(x) + \cdots + p_n(x)g_n(x).$$

*Appendix: Understanding the Method of Partial Fractions VIII

- Given the above observations, it is then relatively easy to see how the method of partial fractions works.
  (a) Assume first that the denominator q of the rational function $f = p/q$ has only real zeros and hence linear factors. It then suffices to consider proper rational functions of the form

  $$\frac{p(x)}{(x-c)^k},$$

  where $\deg p(x) = n < k$.

  (b) If $n > 0$, an application of the division algorithm yields

  $$p(x) = (x-c)p_1(x) + a_0,$$

  where $\deg p_1(x) = n - 1$.

*Appendix: Understanding the Method of Partial Fractions IX

(c) If $n-1>0$, another application of the division algorithm yields

$$p_1(x) = (x-c)p_2(x) + a_1,$$

which implies that

$$\begin{aligned} p(x) &= (x-c)\big[(x-c)p_2(x) + a_1\big] + a_0 \\ &= (x-c)^2 p_2(x) + a_1(x-c) + a_0, \end{aligned}$$

where $\deg p_2(x) = n-2$.

(d) Proceeding in a similar manner, it can be shown that, after n steps, the polynomial p takes the form

$$p(x) = a_n(x-c)^n + a_{n-1}(x-c)^{n-1} + \cdots + a_1(x-c) + a_0,$$

where $a_0, a_1, \ldots, a_n$ are suitable constants.

*Appendix: Understanding the Method of Partial Fractions X

(e) It then follows that

$$\begin{aligned}\frac{p(x)}{(x-c)^k} &= \frac{a_n(x-c)^n}{(x-c)^k} + \frac{a_{n-1}(x-c)^{n-1}}{(x-c)^k} + \cdots + \frac{a_0}{(x-c)^k} \\ &= \frac{A_1}{x-c} + \frac{A_2}{(x-c)^2} + \cdots + \frac{A_k}{(x-c)^k},\end{aligned}$$

where $A_1 = \cdots = A_{k-n-1} = 0$ and

$$A_{k-n} = a_n, \quad A_{k-n+1} = a_{n-1}, \quad \cdots, \quad A_{k-1} = a_1, \quad A_k = a_0.$$

This explains how to find the partial fractions of a rational function $f = p/q$ when its denominator q has only linear factors.

*Appendix: Understanding the Method of Partial Fractions XI

(f) If, on the other hand, q has a conjugate pair of complex zeros $\alpha \pm \mathrm{i}\beta$, and hence an irreducible quadratic factor, the following sum

$$\frac{a+\mathrm{i}b}{[x-(\alpha+\mathrm{i}\beta)]^k} + \frac{a-\mathrm{i}b}{[x-(\alpha-\mathrm{i}\beta)]^k}$$

can be expected from the decomposition of $f = p/q$, which can be combined and rewritten using the division algorithm as

$$\frac{2\,\mathrm{Re}\{(a+\mathrm{i}b)[x-(\alpha-\mathrm{i}\beta)]^k\}}{(x^2-2\alpha x+\alpha^2+\beta^2)^k} = \frac{B_1 x + C_1}{x^2-2\alpha x+\alpha^2+\beta^2} + \frac{B_2 x + C_2}{(x^2-2\alpha x+\alpha^2+\beta^2)^2} + \cdots + \frac{B_k x + C_k}{(x^2-2\alpha x+\alpha^2+\beta^2)^k}.$$

This explains how to find the partial fractions of a rational function $f = p/q$ when its denominator q has irreducible quadratic factors.

Definition of Angle I (page 4, Lecture 4)

- An angle is the shape formed by two rays (half-lines) extending from the same point.
  - (a) An angle can often be obtained by rotating a single ray about its endpoint, where the starting position of the ray is the initial side of the angle, and the position after rotation is the terminal side. The endpoint of the ray is the vertex of the angle (Figure 30).
  - (b) An angle is said to be in standard position if its vertex lies at the origin and its initial side coincides with the positive x-axis.

Definition of Angle II (page 4, Lecture 4)

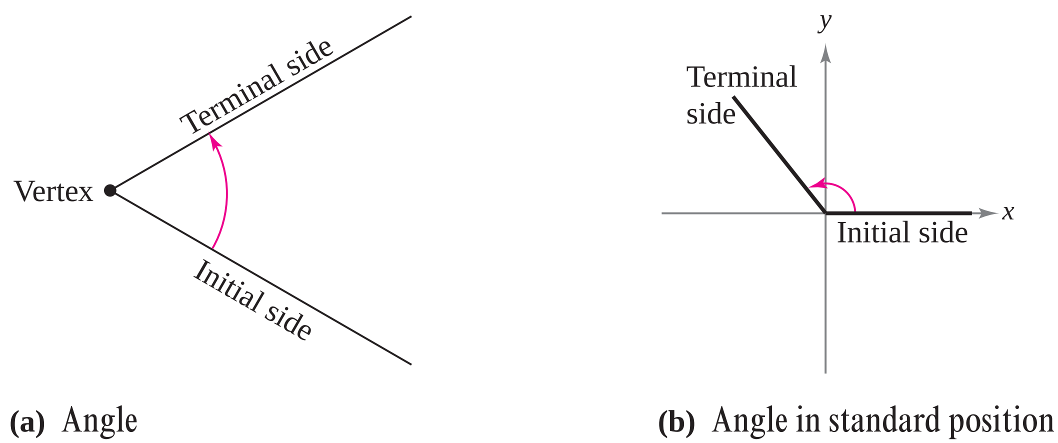


Figure 30: Illustration of an angle.

Definition of Angle III (page 4, Lecture 4)

(c) Angles generated by counterclockwise rotation are positive angles, and angles generated by clockwise rotation are negative angles.

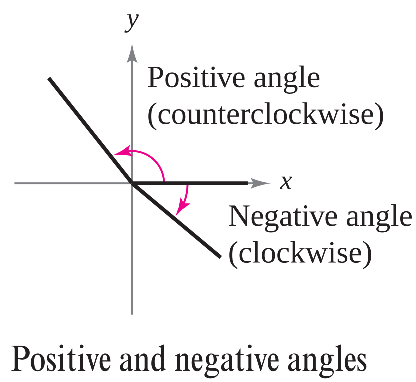


Figure 31: Positive and negative angles.

Definition of Angle IV (page 4, Lecture 4)

(d) Angles are usually labeled with Greek letters α, β, and θ, as well as uppercase letters A, B, and C. Angles having the same initial and terminal sides are said to be coterminal.

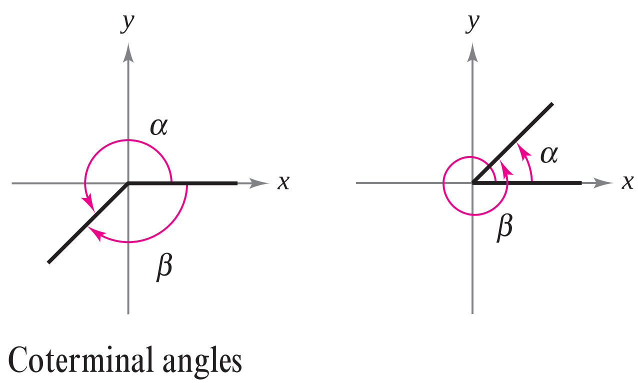


Figure 32: Coterminal angles.

Radian Measure of an Angle I (page 12, Lecture 4)

- The measure of an angle is determined by the amount of rotation from the initial side to the terminal side. Such measure can be expressed in either degrees or radians.

  (a) To define a radian, consider a central angle of a circle whose vertex lies at the center of the circle.

  (b) One radian is then the measure of a central angle θ that intercepts an arc s equal in length to the radius r of the circle (Figure 33).

  (c) In general, the radian measure of a central angle θ is obtained by dividing the arc length s by r, i.e. $\theta = s/r$. Since the units of measure for s and r are the same, the ratio s/r is dimensionless, i.e. it is a real number associated with no physical unit.

Radian Measure of an Angle II (page 12, Lecture 4)

(d) Because the circumference of a circle is $2\pi r$ units, the radian measure of an angle of one full revolution is $2\pi r/r = 2\pi$. Similarly, the radian measure of

$$\begin{aligned} \frac{1}{2} \text{ revolution} &= \frac{2\pi}{2} = \pi \text{ radians}, \\ \frac{1}{4} \text{ revolution} &= \frac{2\pi}{4} = \frac{1}{2}\pi \text{ radians}, \\ \frac{1}{6} \text{ revolution} &= \frac{2\pi}{6} = \frac{1}{3}\pi \text{ radians}. \end{aligned}$$

These and other common angles are shown in Figure 34.

Radian Measure of an Angle III (page 12, Lecture 4)

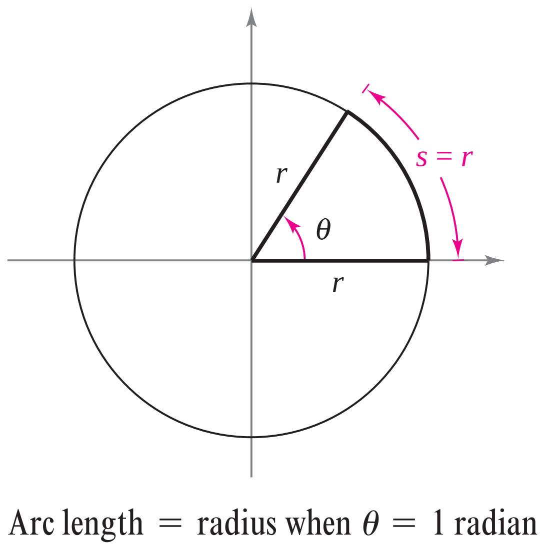


Figure 33: Definition of one radian.

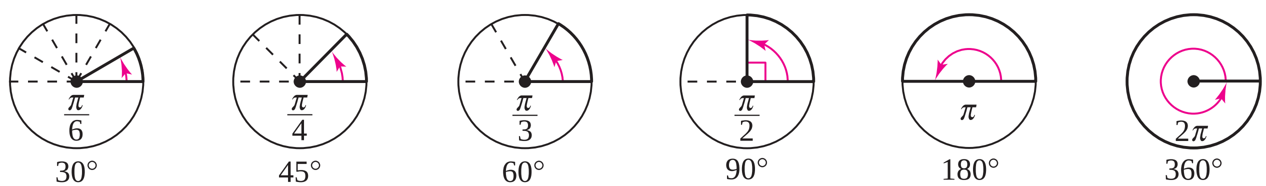


Figure 34: Common angles measured in radians.

Complementary and Supplementary Angles (page 8, Lecture 4)

- Two positive angles α and β are complementary (complements of each other) if their sum is $\pi/2$. Two positive angles are supplementary (supplements of each other) if their sum is π.

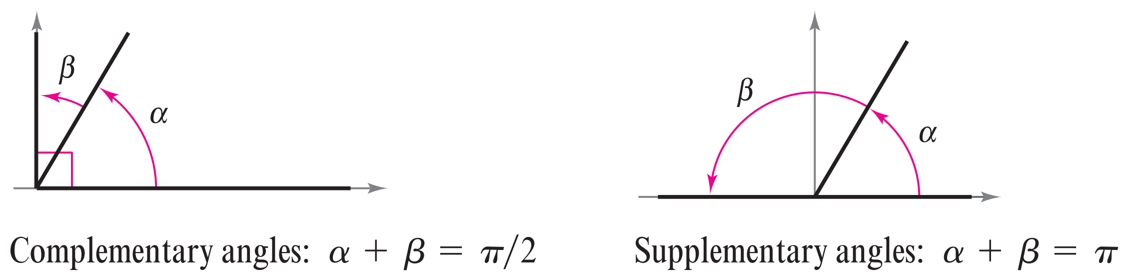


Figure 35: Complementary and supplementary angles.

Right Triangle Trigonometry I (page 5, Lecture 4)

- Consider a right triangle, with one acute angle labeled θ. Relative to the angle θ, the three sides of the triangle are the hypotenuse, the opposite side (the side opposite the angle θ), and the adjacent side (the side adjacent to the angle θ).

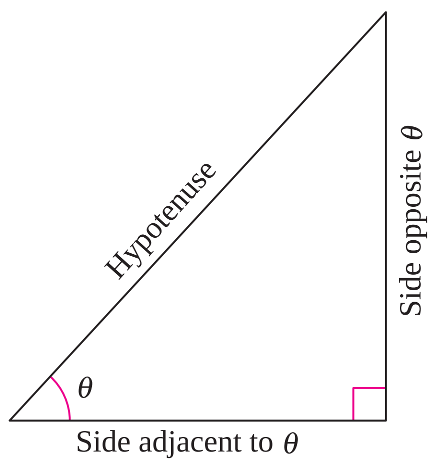


Figure 36: A right triangle and its three sides relative to the angle θ.

Right Triangle Trigonometry II (page 5, Lecture 4)

- Let θ be an acute angle of a right triangle. The six trigonometric functions of the angle θ are defined as follows.

$$\sin\theta = \frac{\text{opp}}{\text{hyp}}, \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}}, \qquad \tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{\sin\theta}{\cos\theta},$$

$$\csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{1}{\sin\theta}, \quad \sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{1}{\cos\theta}, \quad \cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{1}{\tan\theta}.$$

The abbreviations opp, adj, and hyp represent the lengths of the three sides of a right triangle:

(a) opp $=$ the length of the side opposite θ;

(b) adj $=$ the length of the side adjacent to θ;

(c) hyp $=$ the length of the hypotenuse.

Right Triangle Trigonometry III (page 5, Lecture 4)

- The above definitions of trigonometric functions for acute angles can be extended to any angle.
- Let θ be an angle in standard position with (x, y) a point on the terminal side of θ and $r = \sqrt{x^2 + y^2} \neq 0$. The six trigonometric functions of the angle θ are defined as follows.

$$\sin\theta = \frac{y}{r}, \qquad \cos\theta = \frac{x}{r}, \qquad \tan\theta = \frac{y}{x},\ x \neq 0,$$

$$\csc\theta = \frac{r}{y},\ y \neq 0, \qquad \sec\theta = \frac{r}{x},\ x \neq 0, \qquad \cot\theta = \frac{x}{y},\ y \neq 0.$$

Graph of the Cotangent Function (page 15, Lecture 4)

- The graph of the cotangent function $y = \cot x = \cos x / \sin x$, similar to that of the tangent function, has a period of π and has vertical asymptotes at $x = n\pi$, where n is an integer.

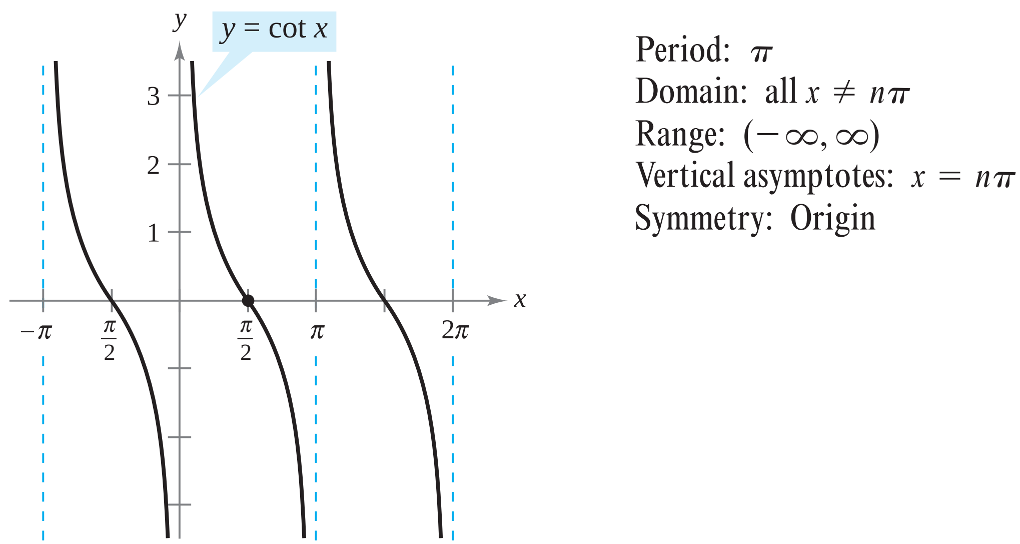


Figure 37: The graph of the cotangent function $y = \cot x$.

Graphs of the Reciprocal Functions (page 15, Lecture 4)

- The graph of the secant function $y = \sec x = 1/\cos x$ can be obtained by first sketching the graph of its reciprocal function $y = \cos x$, and then taking reciprocals of the y-coordinates. The graph of the cosecant function can be obtained in the same way.

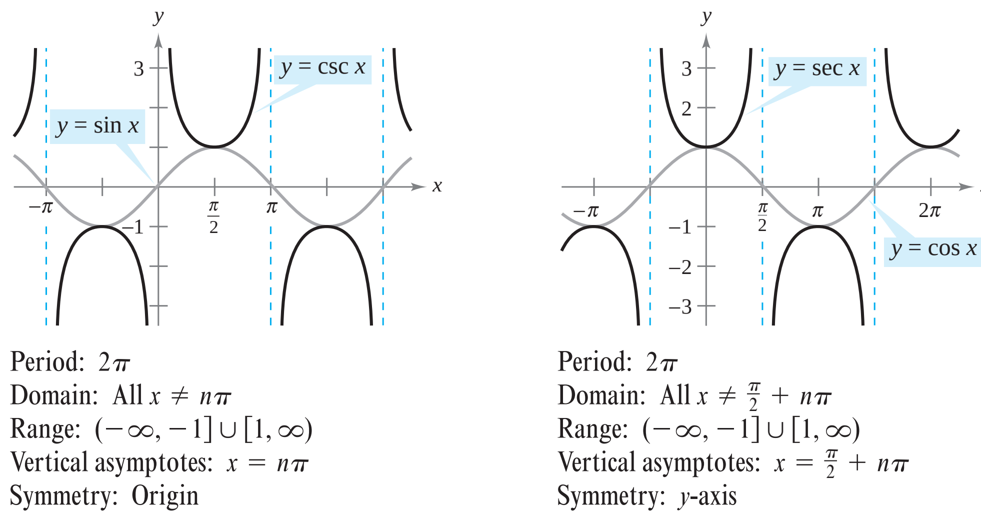


Figure 38: The graphs of the secant function $y = \sec x$ and the cosecant function $y = \csc x$.

Applying the Compound Angle Formula I (page 18, Lecture 4)

- Besides finding the exact values of certain trigonometric functions, the compound angle formula can also be used to prove a cofunction identity.

- **Example**. For instance, applying the formula for $\cos(A - B)$ with $A = \pi/2$ and $B = \theta$ yields

$$\begin{aligned}\cos\left(\frac{\pi}{2} - \theta\right) &= \cos\frac{\pi}{2}\cos\theta + \sin\frac{\pi}{2}\sin\theta \\ &= (0)\cos\theta + (1)\sin\theta = \sin\theta,\end{aligned}$$

which establishes the identity

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta.$$

Applying the Compound Angle Formula II (page 18, Lecture 4)

- With special choices of angles, the compound angle formula also leads to two other classes of trigonometric identities: the so-called double-angle formula and half-angle formula.

  (a) First, the application of the formula for $\cos(A+B)$ and $\sin(A+B)$ with $A = B = \theta$ leads to

$$\cos 2\theta = \cos(\theta+\theta) = \cos\theta\cos\theta - \sin\theta\sin\theta = \cos^2\theta - \sin^2\theta$$
$$= \begin{cases} \cos^2\theta - (1-\cos^2\theta) \\ (1-\sin^2\theta) - \sin^2\theta \end{cases} = \begin{cases} 2\cos^2\theta - 1 \\ 1 - 2\sin^2\theta \end{cases},$$
$$\sin 2\theta = \sin(\theta+\theta) = \sin\theta\cos\theta + \cos\theta\sin\theta = 2\sin\theta\cos\theta.$$

  These are the so-called double-angle formulas.

Applying the Compound Angle Formula III (page 18, Lecture 4)

(b) Next, the rearrangement of the above identities for $\cos 2\theta$ and the replacement of θ with $\theta/2$ leads to

$$\cos^2\frac{\theta}{2} = \frac{1}{2}\,(1+\cos\theta), \qquad \sin^2\frac{\theta}{2} = \frac{1}{2}\,(1-\cos\theta).$$

These are the so-called half-angle formulas.

(c) Similar formulas can also be obtained for $\tan 2\theta$ and $\tan(\theta/2)$, with

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta},$$

$$\tan^2\frac{\theta}{2} = \frac{\sin^2(\theta/2)}{\cos^2(\theta/2)} = \frac{1-\cos\theta}{1+\cos\theta}.$$

Inverse Trigonometric Functions I (page 29, Lecture 4)

- The sine function $y = \sin x$ does not have an inverse because it does not pass the horizontal line test. However, if the domain of the function is restricted to the interval $[-\pi/2, \pi/2]$, then:
  (a) On $[-\pi/2, \pi/2]$, the function $y = \sin x$ is (strictly) increasing.
  (b) On $[-\pi/2, \pi/2]$, $y = \sin x$ takes on its full range of values, $[-1, 1]$.
  (c) On $[-\pi/2, \pi/2]$, $y = \sin x$ is one-to-one.

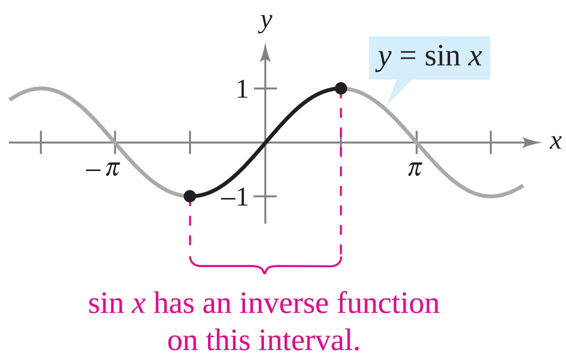


Figure 39: The sine function $y = \sin x$ restricted to $[-\pi/2, \pi/2]$.

Inverse Trigonometric Functions II (page 29, Lecture 4)

- So, on the restricted domain $[-\pi/2, \pi/2]$, $y = \sin x$ has an inverse called the inverse sine function. It is denoted by

$$y = \arcsin x, \qquad \text{or} \qquad y = \sin^{-1} x.$$

The domain of the inverse sine function $y = \sin^{-1} x$ is $[-1, 1]$, and the range is $[-\pi/2, \pi/2]$.

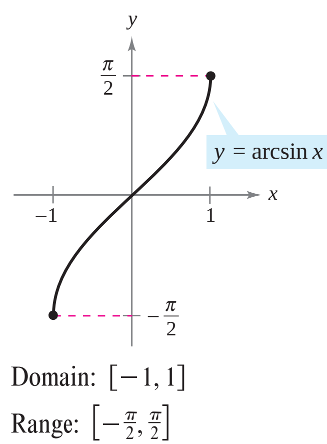


Figure 40: The inverse sine function $y = \sin^{-1} x$.

Inverse Trigonometric Functions III (page 29, Lecture 4)

- The cosine function $y = \cos x$ is (strictly) decreasing and one-to-one on the interval $[0, \pi]$.

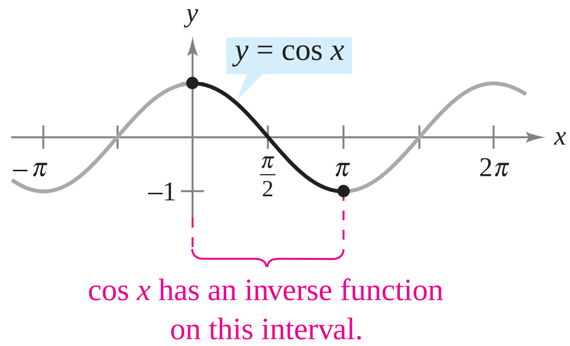


Figure 41: The cosine function $y = \cos x$ restricted to $[0, \pi]$.

Inverse Trigonometric Functions IV (page 29, Lecture 4)

- Thus, on the restricted domain $[0, \pi]$, $y = \cos x$ has an inverse called the inverse cosine function. It is denoted by

$$y = \arccos x, \qquad \text{or} \qquad y = \cos^{-1} x.$$

The domain of the inverse cosine function $y = \cos^{-1} x$ is $[-1, 1]$, and the range is $[0, \pi]$.

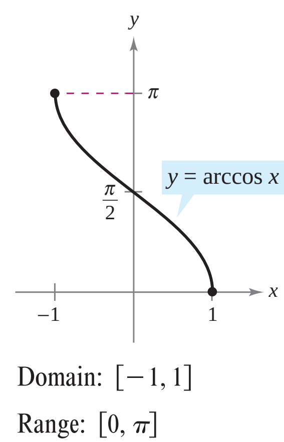


Figure 42: The inverse cosine function $y = \cos^{-1} x$.

Inverse Trigonometric Functions V (page 29, Lecture 4)

- Likewise, an inverse tangent function

$$y = \arctan x, \qquad \text{or} \qquad y = \tan^{-1} x,$$

can be defined by restricting the domain of $y = \tan x$ to the interval $(-\pi/2, \pi/2)$. The domain of the inverse tangent function $y = \tan^{-1} x$ is $(-\infty, \infty) = \mathbb{R}$, and the range is $(-\pi/2, \pi/2)$.

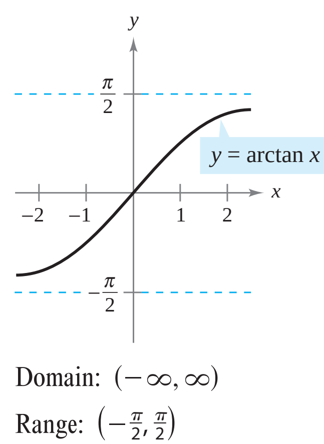


Figure 43: The inverse tangent function $y = \tan^{-1} x$.

Compositions of Trigonometric Functions I (page 31, Lecture 4)

- Recall from Lecture 2 that for all x in the domains of f and f^{-1}, inverse functions have the properties

$$f(f^{-1}(x)) = x, \qquad \text{and} \qquad f^{-1}(f(x)) = x.$$

For trigonometric functions, this implies that:

(a) If $x \in [-1, 1]$ and $y \in [-\pi/2, \pi/2]$, then

$$\sin(\sin^{-1} x) = x, \qquad \text{and} \qquad \sin^{-1}(\sin y) = y.$$

(b) If $x \in [-1, 1]$ and $y \in [0, \pi]$, then

$$\cos(\cos^{-1} x) = x, \qquad \text{and} \qquad \cos^{-1}(\cos y) = y.$$

(c) If $x \in \mathbb{R}$ and $y \in (-\pi/2, \pi/2)$, then

$$\tan(\tan^{-1} x) = x, \qquad \text{and} \qquad \tan^{-1}(\tan y) = y.$$

Compositions of Trigonometric Functions II (page 32, Lecture 4)

- The following example shows how to use right triangles to find exact values of compositions of inverse trigonometric functions.

- **Example**.[a] Consider the expressions

$$\tan\left[\cos^{-1}(2/3)\right], \qquad \text{and} \qquad \cos\left[\sin^{-1}(-3/5)\right].$$

To evaluate the first expression, define $\theta = \cos^{-1}(2/3)$ so that $\cos\theta = 2/3$. Because $\cos\theta$ is positive, θ is a first-quadrant angle, which can be sketched and labeled as shown in Figure 44 (a). Consequently,

$$\tan\left[\cos^{-1}(2/3)\right] = \tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{5}}{2}.$$

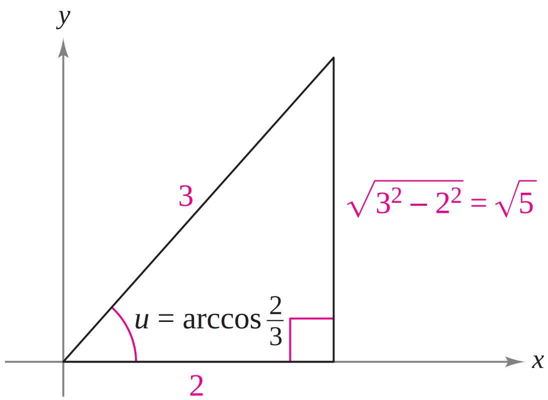
[a] See also Example 11 on page 32.

Compositions of Trigonometric Functions III (page 32, Lecture 4)

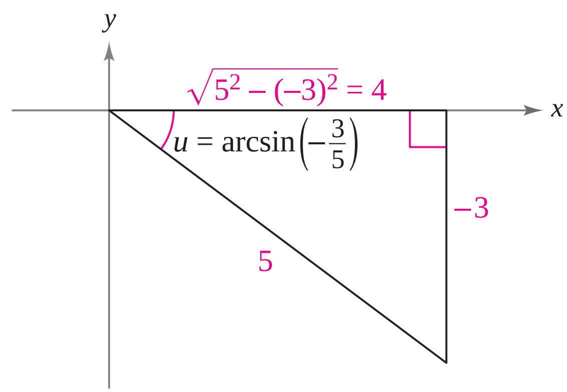
Likewise, to evaluate the second expression, define $\theta = \sin^{-1}(-3/5)$ so that $\sin\theta = -3/5$. Because $\sin\theta$ is negative, θ is a fourth-quadrant angle, which can be sketched and labeled as shown in Figure 44 (b). Consequently,

$$\cos\left[\sin^{-1}(-3/5)\right] = \cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{4}{5}.$$

Compositions of Trigonometric Functions IV (page 32, Lecture 4)



(a) Angle whose cosine is $\frac{2}{3}$



(b) Angle whose sine is $-\frac{3}{5}$

Figure 44: Evaluating compositions of inverse trigonometric functions.

Compositions of Trigonometric Functions V (page 32, Lecture 4)

- The following example shows how to use right triangles to convert a trigonometric expression into an algebraic expression.

- **Example**. Consider the expressions

$$\sin(\cos^{-1} 3x),\ x \in [0, 1/3], \qquad \text{and} \qquad \cot(\cos^{-1} 3x),\ x \in [0, 1/3).$$

  To evaluate these expression, define $\theta = \cos^{-1} 3x$ so that $\cos\theta = 3x$. Since $3x \in [0, 1]$, θ is well-defined and can be sketched and labeled as shown in Figure 45. Consequently,

$$\sin(\cos^{-1} 3x) = \sin\theta = \frac{\text{opp}}{\text{hyp}} = \sqrt{1-9x^2}, \qquad x \in [0, 1/3],$$

$$\cot(\cos^{-1} 3x) = \cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{3x}{\sqrt{1-9x^2}}, \qquad x \in [0, 1/3).$$

Compositions of Trigonometric Functions VI (page 32, Lecture 4)

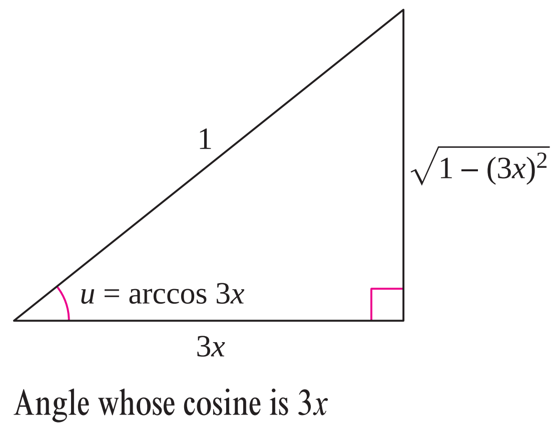
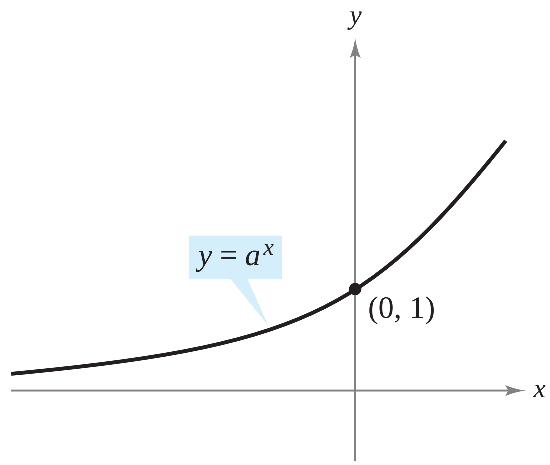


Figure 45: Converting trigonometric expressions into algebraic ones.

Graphs of Exponential Functions I (page 3, Lecture 5)

- The graphs of all exponential functions have similar characteristics, which are summarized in Figure 46 and 47.

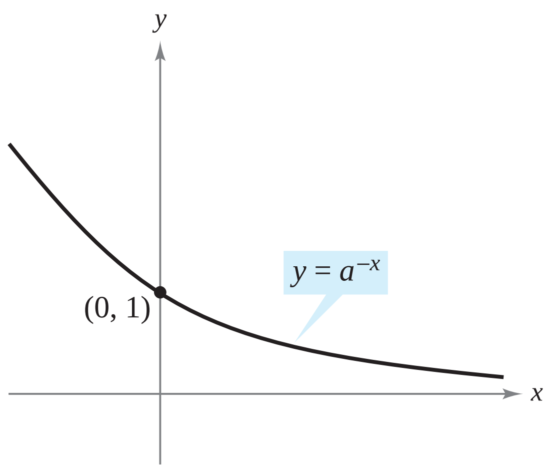


Graph of $y = a^x$, $a > 1$

- Domain: $(-\infty, \infty)$
- Range: $(0, \infty)$
- y-intercept: $(0, 1)$
- Increasing
- x-axis is a horizontal asymptote ($a^x \to 0$ as $x \to -\infty$).
- Continuous

Figure 46: The graph of $y = a^x$, $a > 1$.

Graphs of Exponential Functions II (page 3, Lecture 5)



Graph of $y = a^{-x}$, $a > 1$

- Domain: $(-\infty, \infty)$
- Range: $(0, \infty)$
- y-intercept: $(0, 1)$
- Decreasing
- x-axis is a horizontal asymptote ($a^{-x} \to 0$ as $x \to \infty$).
- Continuous

Figure 47: The graph of $y = a^{-x} = (1/a)^x$, $a > 1$.

The Natural Base I (page 5, Lecture 5)

- The definition of the natural base e as the limit

$$\mathrm{e} = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$$

can be partly understood by considering one of the most familiar examples of exponential growth, i.e. that of an investment earning continuously compounded interest.

(a) Suppose a principal P is invested at an annual interest rate r, compounded once a year. If the interest is added to the principal at the end of the year, the new balance P_1 is

$$P_1 = P + Pr = P(1+r).$$

The Natural Base II (page 5, Lecture 5)

(b) This pattern of multiplying the previous principal by $1 + r$ is then repeated each successive year, as shown below.

Year	Balance after each compounding
0	$P = P$
1	$P_1 = P(1 + r)$
2	$P_2 = P_1(1 + r) = P(1 + r)^2$
3	$P_3 = P_2(1 + r) = P(1 + r)^3$
$\vdots$	$\vdots$
t	$P_t = P(1 + r)^t$

The Natural Base III (page 5, Lecture 5)

(c) To accommodate more frequent (quarterly, monthly, or daily) compounding of interest, let n be the number of compoundings each year and let t be the number of years. Then the rate per compounding is r/n, and the account balance after t years is

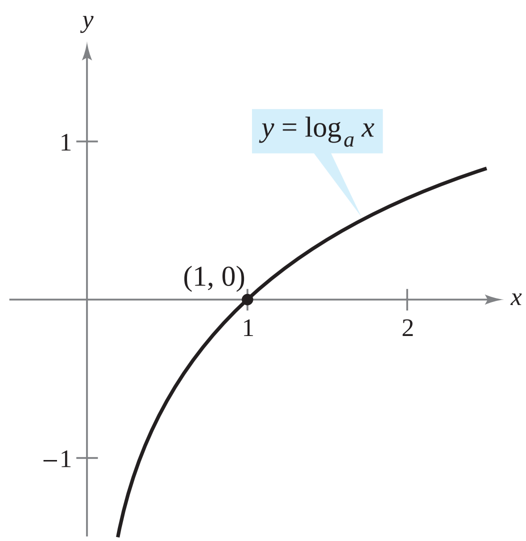
$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$

(d) If the number of compoundings increases without bound, the process approaches what is called continuous compounding. To find a formula for continuous compounding, set $n = mr$ in the formula for n compoundings per year and let $m \to \infty$. The result is

$$A = P\left(1 + \frac{r}{mr}\right)^{mrt} = P\left[\left(1 + \frac{1}{m}\right)^{m}\right]^{rt} \to P\mathrm{e}^{rt}.$$

Graphs of Logarithmic Functions (page 9, Lecture 5)

- The graphs of all logarithmic functions have similar characteristics, which are summarized in Figure 48.



Graph of $y = \log_a x$, $a > 1$

- Domain: $(0, \infty)$
- Range: $(-\infty, \infty)$
- x-intercept: $(1, 0)$
- Increasing
- One-to-one, therefore has an inverse function
- y-axis is a vertical asymptote ($\log_a x \to -\infty$ as $x \to 0^+$).
- Continuous
- Reflection of graph of $y = a^x$ about the line $y = x$
- The vertical asymptote occurs at $x = 0$, where $\log_a x$ is undefined.

Figure 48: The graph of $y = \log_a x,\ a > 1$.

Properties of Logarithmic Functions (page 10, Lecture 5)

- The following basic properties of logarithmic functions follow directly from the definition (assume $a > 0,\ a \neq 1$).
  (a) $\log_a 1 = 0$ because $a^0 = 1$.
  (b) $\log_a a = 1$ because $a^1 = a$.
  (c) In general, $\log_a a^x = x$ for all $x \in \mathbb{R}$ and $a^{\log_a x} = x$ for all $x > 0$ (inverse properties).
  (d) If $\log_a x = \log_a y$, then $x = y$ (one-to-one property).
  (e) $\log_a x = \log_b x / \log_b a$ for all $x > 0$ and $b > 0,\ b \neq 1$ (change-of-base formula).

Notion of Limit I (page 2, Lecture 6)

- The notion of a limit is fundamental to the study of calculus.
- To get an idea of the way limits are used in calculus, consider the following two classic problems in calculus, the tangent line problem and the area problem.

  (a) In the tangent line problem, a function f and a point P on its graph are given, and the goal is to find an equation of the tangent line to the graph at point P (Figure 49).

  (b) Except for cases involving a vertical tangent line, the problem of finding the tangent line at a point P is equivalent to that of finding the slope of the tangent line at P.

  (c) This slope, on the other hand, can be approximated by using a line through the point of tangency and a second point on the curve (Figure 50 (a)). Such a line is called a secant line.

Notion of Limit II (page 2, Lecture 6)

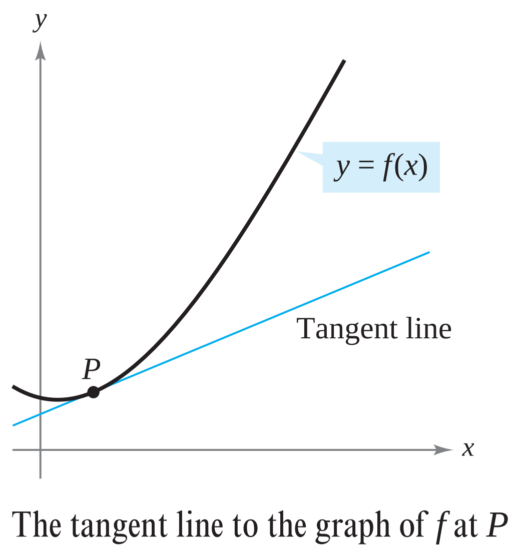
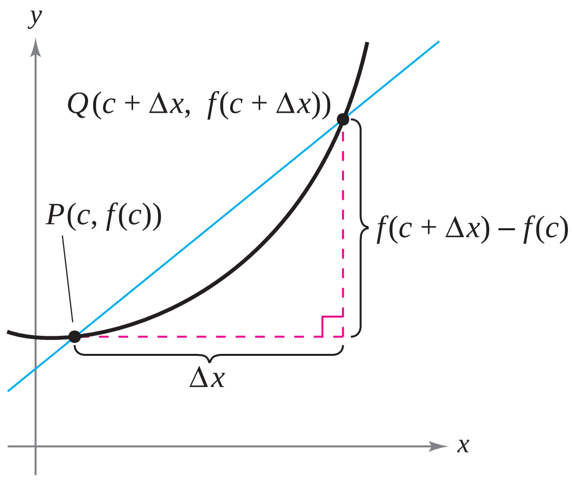
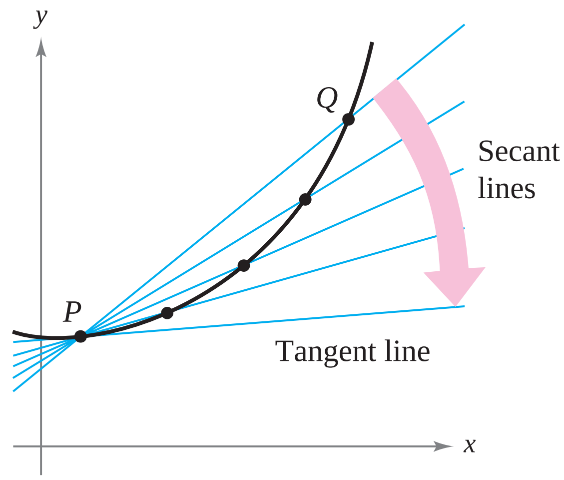


Figure 49: The tangent line to the graph of f at P.

Notion of Limit III (page 2, Lecture 6)



(a) The secant line through $(c, f(c))$ and $(c + \Delta x, f(c + \Delta x))$



(b) As Q approaches P, the secant lines approach the tangent line.

Figure 50: Approximating a tangent line by secant lines.

Notion of Limit IV (page 2, Lecture 6)

(d) If $P = (c, f(c))$ is the point of tangency and $Q = (c + \Delta x, f(c + \Delta x))$ (where $\Delta x \neq 0$) is a second point on the graph of f, the slope of the secant line through these two points is given by

$$m_{\text{sec}} = \frac{f(c + \Delta x) - f(c)}{(c + \Delta x) - c} = \frac{f(c + \Delta x) - f(c)}{\Delta x}.$$

(e) As point Q approaches point P, the slopes of the secant lines approach the slope of the tangent line (Figure 50 (b)). When such a "limiting position" exists, the slope of the tangent line is said to be the limit of the slopes of the secant lines.

Notion of Limit V (page 2, Lecture 6)

(f) In the area problem, on the other hand, the goal is to find the area of the region bounded by the graph of a given function f, the x-axis, and the vertical lines $x = a$ and $x = b$ (Figure 51).

(g) The area of the region can be approximated by rectangles, as indicated in Figure 52. As the number of rectangles increases, the approximation tends to become better and better because the amount of area missed by the rectangles becomes smaller and smaller.

(h) The goal is then to determine the limit of the sum of the areas of the rectangles as the number of rectangles increases without bound, which, when it exists, will be used to define the area of the region.

Notion of Limit VI (page 2, Lecture 6)

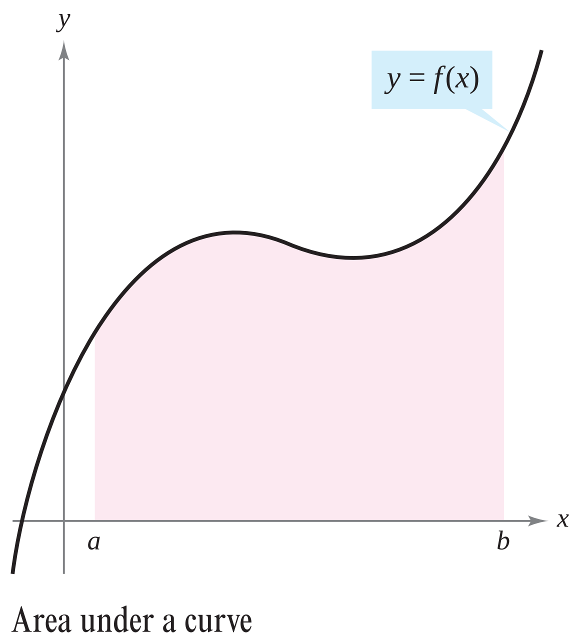


Figure 51: Area under the graph of a function f.

Notion of Limit VII (page 2, Lecture 6)

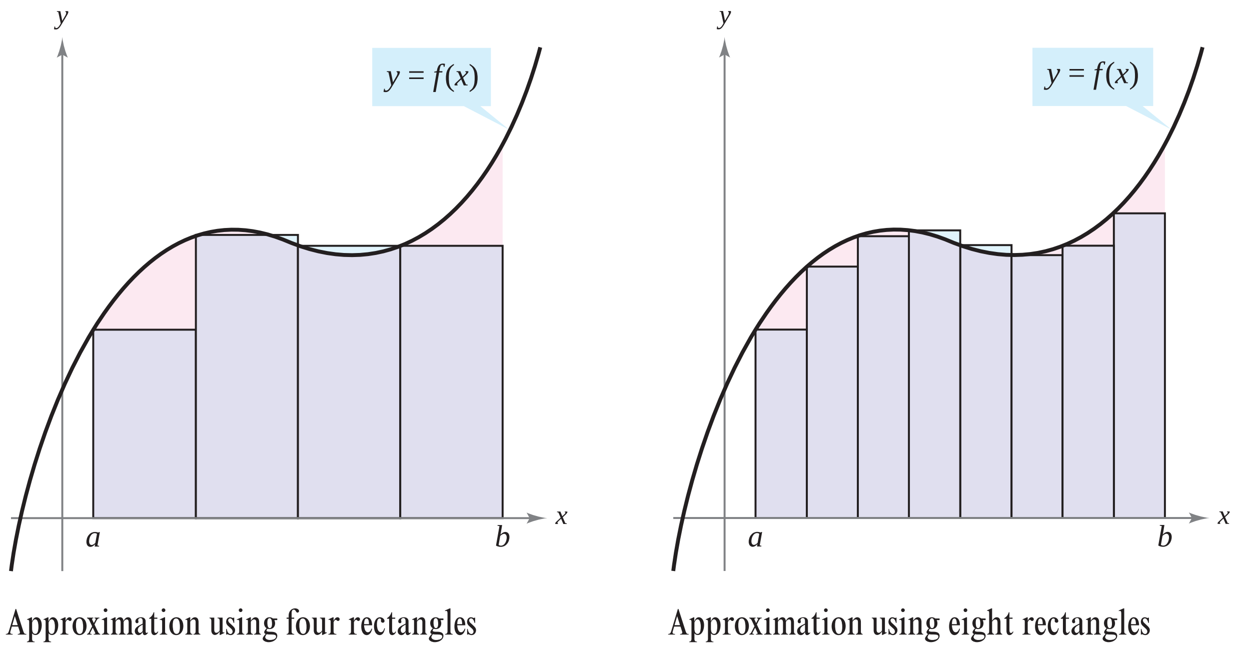


Figure 52: Approximating the area under the graph of f by rectangles.

Definition of Limit I (page 3, Lecture 6)

- Let c be a real number and f be a function defined everywhere on an open interval containing c, except possibly at c, and let L be a real number. If for each $\epsilon > 0$, there exists a $\delta > 0$ such that

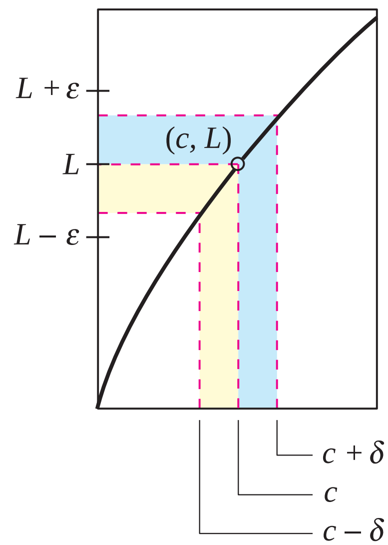
$$|f(x) - L| < \epsilon \qquad \text{whenever} \qquad 0 < |x - c| < \delta,$$

then the limit of $f(x)$ as x approaches c exists (and equals L), and is denoted by

$$\lim_{x \to c} f(x) = L.$$

Definition of Limit II (page 3, Lecture 6)

- The following figure illustrates the above definition of limit.



The ε-δ definition of the limit of $f(x)$ as x approaches c

Figure 53: The ϵ-δ definition of the limit of $f(x)$ as x approaches c.

Definition of Infinite Limit I (page 8, Lecture 6)

- Let c be a real number and f be a function defined everywhere on an open interval containing c, except possibly at c. If for each $M > 0$, there exists a $\delta > 0$ such that

$$f(x) > M \qquad \text{whenever} \qquad 0 < |x - c| < \delta,$$

then the limit of $f(x)$ as x approaches c becomes infinite, and is denoted by[a]

$$\lim_{x \to c} f(x) = \infty.$$

Similar definition can be made for the statement

$$\lim_{x \to c} f(x) = -\infty.$$

[a] Carefully note that the equal sign in the statement $\lim_{x \to c} f(x) = \infty$ does not mean that the limit exists.

Definition of Infinite Limit II (page 8, Lecture 6)

- The following figure illustrates the above definition of infinite limits.

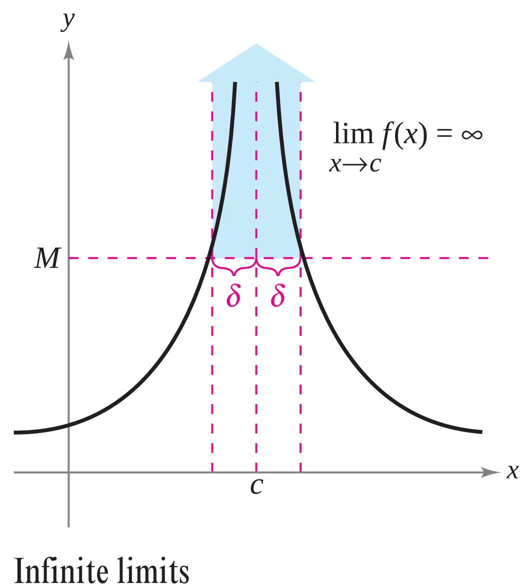


Figure 54: The definition of infinite limits of $f(x)$ as x approaches c.

Definition of Limit at Infinity I (page 44, Lecture 6)

- Let c be a real number and f be a function defined everywhere on the open interval (c, ∞), and let L be a real number. If for each $\epsilon > 0$, there exists an $R > c$ such that

$$|f(x) - L| < \epsilon \qquad \text{whenever} \qquad x > R,$$

then the limit of $f(x)$ as x approaches ∞ exists (and equals L), and is denoted by

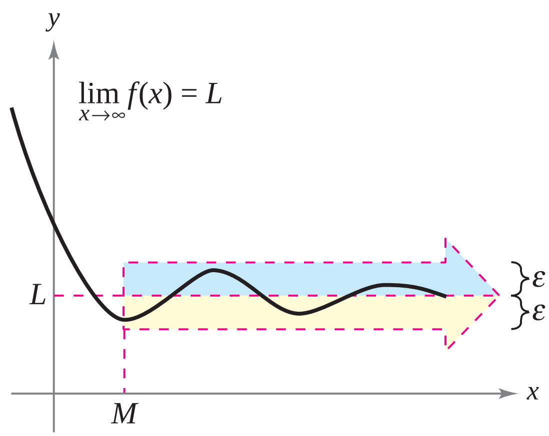
$$\lim_{x \to \infty} f(x) = L.$$

Similar definition can be made for the statement

$$\lim_{x \to -\infty} f(x) = L.$$

Definition of Limit at Infinity II (page 44, Lecture 6)

- The following figure illustrates the above definition of limits at infinity.



$f(x)$ is within ε units of L as $x \to \infty$.

Figure 55: The definition of the limit of $f(x)$ as x approaches ∞.

Definition of Infinite Limit at Infinity (page 44, Lecture 6)

- Let c be a real number and f be a function defined everywhere on the open interval (c, ∞). If for each $M > 0$, there exists an $R > c$ such that

$$f(x) > M \qquad \text{whenever} \qquad x > R,$$

then the limit of $f(x)$ as x approaches ∞ becomes infinite, and is denoted by

$$\lim_{x \to \infty} f(x) = \infty.$$

Similar definition can be made for the statements

$$\lim_{x \to \infty} f(x) = -\infty, \qquad \text{and} \qquad \lim_{x \to -\infty} f(x) = \pm\infty.$$

Evaluating Limits by Direct Substitution I (page 5, Lecture 6)

- By definition, the limit of a function $f(x)$ as x approaches c does not depend on the value of f at $x = c$.
- It may happen, however, that the limit is precisely $f(c)$. In such cases, the limit can be evaluated by direct substitution, i.e.

$$\lim_{x \to c} f(x) = f(c).$$

  Functions whose limits can be evaluated by direct substitution at $x = c$ are said to be continuous[a] at c.

[a]The precise definition of continuity will be given in Lecture 7.

Evaluating Limits by Direct Substitution II (page 5, Lecture 6)

(a) For example, if b and c are real numbers and n is a positive integer, then

$$\lim_{x\to c} b = b, \qquad \lim_{x\to c} x = c, \qquad \lim_{x\to c} x^n = c^n.$$

(b) More generally, if p is a polynomial and c is a real number, then

$$\lim_{x\to c} p(x) = p(c).$$

(c) Likewise, if r is a rational function given by $r = p/q$ and c is a real number such that $q(c) \neq 0$, then

$$\lim_{x\to c} r(x) = r(c) = \frac{p(c)}{q(c)}.$$

(d) In general, the substitution rule applies to other elementary functions including trigonometric, exponential, and logarithmic functions, which are all continuous on their domains.

Limits of Composite Functions I (page 5, Lecture 6)

- If f and g are functions such that

$$\lim_{x \to c} g(x) = L, \qquad \text{and} \qquad \lim_{x \to L} f(x) = f(L),$$

then

$$\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right) = f(L).$$

Limits of Composite Functions II (page 5, Lecture 6)

- **Example**. For instance, for the limit

$$\lim_{x\to 0}\sqrt{x^2+4},$$

the choice $f(x)=\sqrt{x}$ and $g(x)=x^2+4$ leads to

$$\lim_{x\to 0} g(x) = \lim_{x\to 0}(x^2+4) = 4,$$
$$\lim_{x\to 4} f(x) = \lim_{x\to 4}\sqrt{x} = 2.$$

Thus

$$\lim_{x\to 0}\sqrt{x^2+4} = \sqrt{\lim_{x\to 0}(x^2+4)} = \sqrt{4} = 2.$$

Strategy for Finding Limits (page 24, Lecture 6)

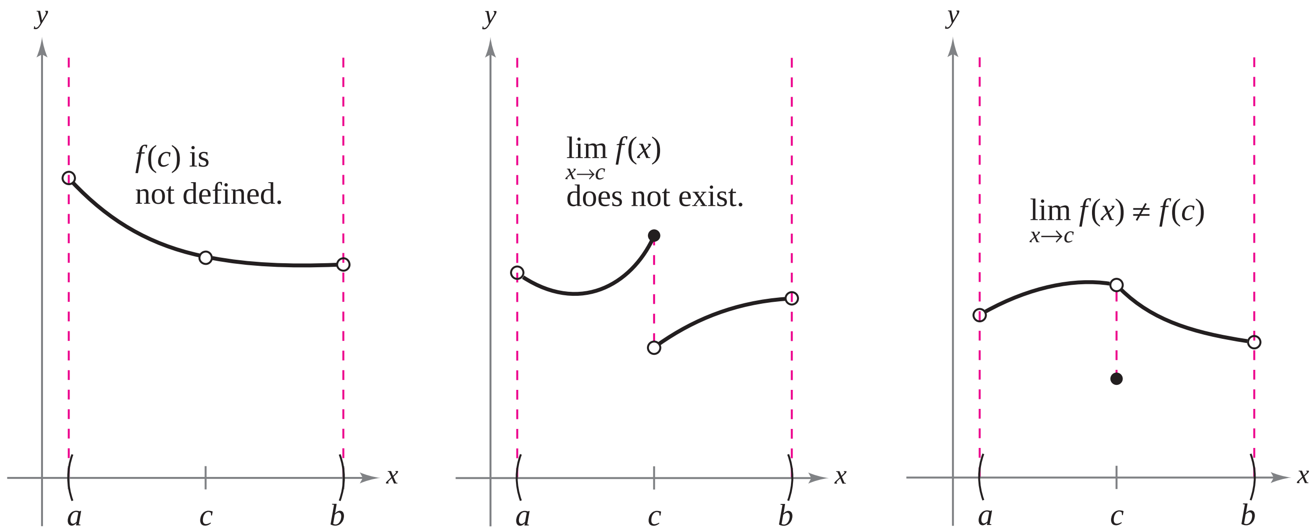
- To find the limit of a function f at a point c at which f may not be defined, the following strategy could be used.
  (a) Learn to recognize the limits that can be evaluated by direct substitution (for example, the limits of all elementary functions which are continuous on their domains).
  (b) If the limit of $f(x)$ as x approaches c cannot be evaluated by direct substitution, try to find a function g such that g agrees with f for all x other than $x = c$, and that the limit of g can be evaluated by direct substitution.
  (c) Use the definition of limit to conclude that

$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = g(c).$$

Definition of Continuity I (page 4, Lecture 7)

- Let f be a function defined everywhere on an open interval containing c, except possibly at c. Then f is continuous at c if
  (a) $f(c)$ is defined;
  (b) $\lim_{x\to c} f(x)$ exists; and
  (c) $\lim_{x\to c} f(x) = f(c)$.
- The following figure identifies three situations in which f is not continuous.

Definition of Continuity II (page 4, Lecture 7)



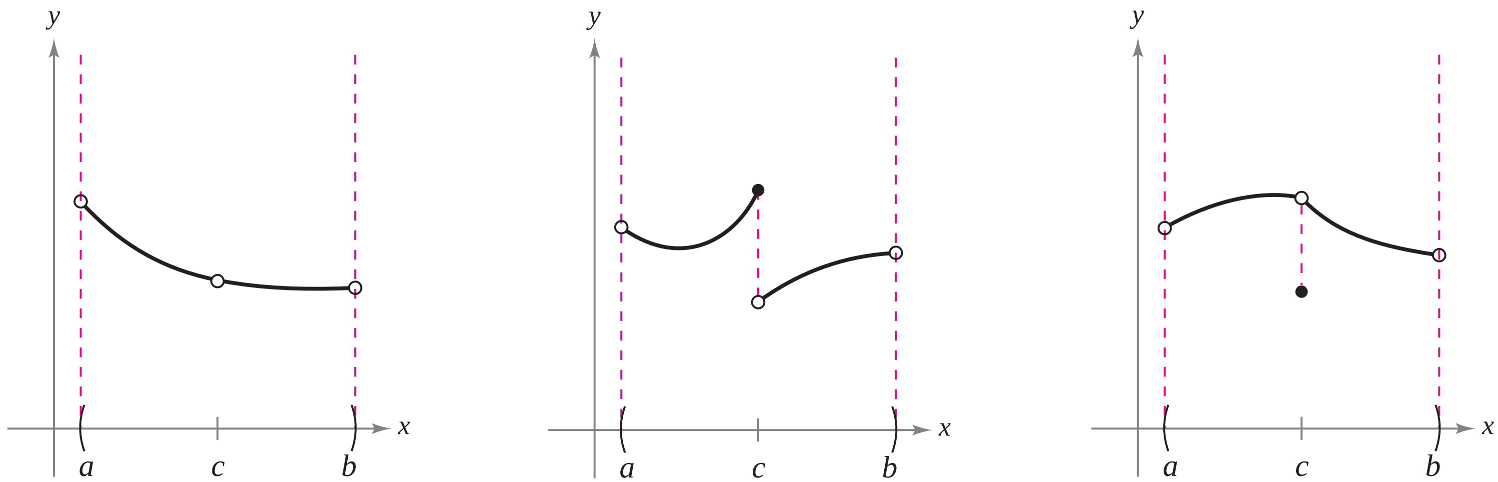
Three conditions exist for which the graph of f is not continuous at $x = c$.

Figure 56: Three situations in which f is not continuous.

Definition of Continuity III (page 4, Lecture 7)

- If f is not continuous at c, then c is called a discontinuity of f.
- Discontinuities fall into two categories: removable and nonremovable. A discontinuity at c is called removable if f can be made continuous by appropriately defining or redefining $f(c)$.
- **Example**. For instance, the functions shown below in Figure 57 (a)(c) have removable discontinuities at c, while the one shown in Figure 57 (b) has a nonremovable discontinuity at c.

Definition of Continuity IV (page 4, Lecture 7)



(a) Removable discontinuity **(b)** Nonremovable discontinuity **(c)** Removable discontinuity

Figure 57: Removable and nonremovable discontinuities.

Differentiation of Exponential Functions I (page 8, Lecture 8)

- To investigate the differentiability and find the derivative of the natural exponential function $f(x) = \mathrm{e}^x$, note that:
  (a) For any fixed $x \in \mathbb{R}$ and any $h \neq 0$,
  $$\frac{f(x+h) - f(x)}{h} = \frac{\mathrm{e}^{x+h} - \mathrm{e}^x}{h} = \frac{\mathrm{e}^x \cdot \mathrm{e}^h - \mathrm{e}^x}{h} = \mathrm{e}^x \, \frac{\mathrm{e}^h - 1}{h}.$$
  (b) For all $h \in (-1, 1)$ with $h \neq 0$, there holds[a]
  $$1 + h \leq \mathrm{e}^h \leq \frac{1}{1-h}.$$
  This implies
  $$h \leq \mathrm{e}^h - 1 \leq \frac{1}{1-h} - 1 = \frac{h}{1-h}.$$

---

[a]This follows from the monotonicity of $(1+h)^{1/h}$ for $h \neq 0$ and the definition of e as $\lim_{h\to 0}(1+h)^{1/h}$.

Differentiation of Exponential Functions II (page 8, Lecture 8)

(c) It then follows that

$$1 \le \frac{e^h - 1}{h} \le \frac{1}{1-h}, \qquad \forall h \in (0,1),$$

$$\frac{1}{1-h} \le \frac{e^h - 1}{h} \le 1, \qquad \forall h \in (-1,0).$$

and thus

$$\lim_{h\to 0^+} \frac{e^h - 1}{h} = \lim_{h\to 0^-} \frac{e^h - 1}{h} = 1.$$

(d) This shows that the derivative of e^x exists, and it equals

$$\lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} e^x \cdot \lim_{h\to 0} \frac{e^h - 1}{h} = e^x \cdot 1 = e^x.$$

Differentiability and Continuity I (page 18, Lecture 8)

- **Theorem 4** (Differentiability implies continuity)**.** *If a function f is differentiable at $x = c$, then f is also continuous at $x = c$.*

- **Proof**. The idea is to show $f(x)$ approaches $f(c)$ as x approaches c. To do this, observe that

$$f(x) - f(c) = (x - c) \cdot \frac{f(x) - f(c)}{x - c}.$$

Since the limit

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} \overset{x=c+h}{=} \lim_{h \to 0} \frac{f(c+h) - f(c)}{(c+h) - c} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

exists and equals $f'(c)$ (by assumption), the limit $\lim_{x \to c} \big[f(x) - f(c)\big]$ also exists and equals

$$\lim_{x \to c} \big[f(x) - f(c)\big] = \lim_{x \to c} (x - c) \cdot \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = 0 \cdot f'(c) = 0.$$

Differentiability and Continuity II (page 18, Lecture 8)

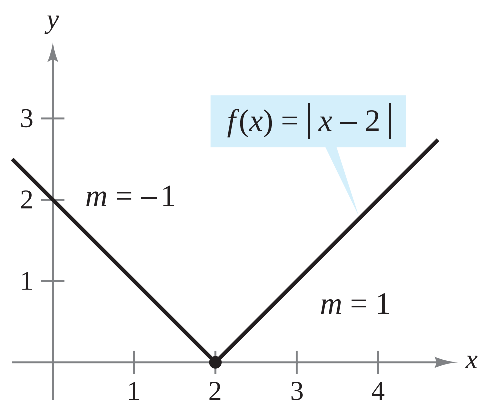
It then follows that

$$\begin{aligned}\lim_{x\to c} f(x) &= \lim_{x\to c}\big[f(x) - f(c) + f(c)\big]\\ &= \lim_{x\to c}\big[f(x) - f(c)\big] + \lim_{x\to c} f(c) = f(c),\end{aligned}$$

which shows that $\lim_{x\to c} f(x)$ exists and equals $f(c)$, i.e. f is continuous at $x = c$. $\square$

Differentiability and Continuity III (page 18, Lecture 8)

- Note that it is possible for a function f to be continuous at $x = c$ but not be differentiable at $x = c$. This happens if the graph of f has a sharp turn or a vertical tangent line at $x = c$.



f is not differentiable at $x = 2$, because the derivatives from the left and from the right are not equal.

Figure 58: The function $f(x) = |x - 2|$ is continuous but not differentiable at $x = 2$.

Differentiability and Continuity IV (page 18, Lecture 8)

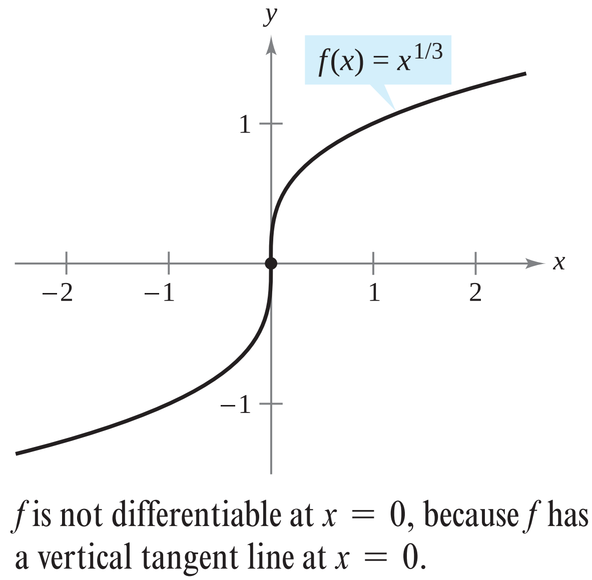


Figure 59: The function $f(x) = \sqrt[3]{x}$ is continuous but not differentiable at $x = 0$.

Rates of Change I (page 20, Lecture 8)

- Besides determining slopes of tangent lines, the derivative of a function can also be used to determine the rate of change of one variable with respect to another.
- Applications involving rates of change occur in a wide variety of fields. A few examples are:
  - population growth rates
  - production rates
  - water flow rates
  - velocity, and
  - acceleration

Rates of Change II (page 20, Lecture 8)

- A common use for rate of change is to describe the motion of an object moving in a straight line.
  (a) In such problems, it is customary to use either a horizontal or a vertical line with a designated origin to represent the line of motion.
  (b) On such lines, movement to the right (or upward) is considered to be in the positive direction, and movement to the left (or downward) is considered to be in the negative direction.

Rates of Change III (page 20, Lecture 8)

(c) The function s that gives the position (relative to the origin) of an object as a function of time t is called a position function. If, over a period of time Δt, the object changes its position by the amount $\Delta s = s(t + \Delta t) - s(t)$, then, by the familiar formula

$$\text{rate} = \frac{\text{distance}}{\text{time}},$$

the average velocity is

$$\frac{\text{change in distance}}{\text{change in time}} = \frac{\Delta s}{\Delta t}.$$

Rates of Change IV (page 20, Lecture 8)

(d) For any fixed time t, the instantaneous velocity (or simply the velocity) of the object at t can be approximated by the average velocity $\Delta s/\Delta t$ over a small interval $[t, t + \Delta t]$. The velocity of the object at time t is then obtained by taking the limit as Δt approaches zero:

$$v(t) = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} \frac{s(t + \Delta t) - s(t)}{\Delta t} = s'(t),$$

provided that the limit exists. In other words, the velocity function is the derivative of the position function.

(e) Velocity can be negative, zero, or positive. The speed of an object is the absolute value of its velocity. Speed cannot be negative.

Rates of Change V (page 20, Lecture 8)

(f) For example, the position of a free-falling object (neglecting air resistance) under the influence of gravity can be represented by the equation

$$s(t) = \frac{1}{2}gt^2 + v_0 t + s_0,$$

where s_0 is the initial height of the object, v_0 is the initial velocity of the object, and g is the acceleration due to gravity. On Earth, the value of g is approximately -32 feet per second per second or -9.8 meters per second per second.

The Product Rule I (page 25, Lecture 8)

- The product of two differentiable functions f and g is itself differentiable. Moreover,

$$\big[f(x)g(x)\big]' = f'(x)g(x) + f(x)g'(x).$$

- **Proof**. Let x be any point of differentiability of f and g and let $h \neq 0$ be sufficiently small (so that $f(x+h),\ g(x+h)$ are well defined). Then

$$\begin{aligned}
&\frac{1}{h}\big[f(x+h)g(x+h) - f(x)g(x)\big] \\
&\quad = \frac{1}{h}\big[f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)\big] \\
&\quad = \frac{1}{h}\big[f(x+h) - f(x)\big]g(x+h) + f(x)\cdot\frac{1}{h}\big[g(x+h) - g(x)\big] \\
&\quad =: F(x,h)g(x+h) + f(x)G(x,h),
\end{aligned}$$

The Product Rule II (page 25, Lecture 8)

where

$$F(x,h) = \frac{1}{h}\big[f(x+h) - f(x)\big], \qquad G(x,h) = \frac{1}{h}\big[g(x+h) - g(x)\big].$$

Since f and g are differentiable and hence continuous at x (by assumption), the derivative of fg exists at x, and it equals

$$\begin{aligned}
&\lim_{h\to 0}\frac{1}{h}\big[f(x+h)g(x+h) - f(x)g(x)\big] \\
&\qquad = \lim_{h\to 0} F(x,h)\cdot\lim_{h\to 0} g(x+h) + \lim_{h\to 0} f(x)\cdot\lim_{h\to 0} G(x,h) \\
&\qquad = f'(x)g(x) + f(x)g'(x).
\end{aligned}$$

$\square$

The Product Rule III (page 25, Lecture 8)

- **Example**. Not every product needs to be differentiated by the product rule. For example, while $f_1(x) = \sqrt{x}\, g(x)$ needs to be differentiated by the product rule:

$$f_1'(x) = \left[\sqrt{x}\, g(x)\right]' = (\sqrt{x})' g(x) + \sqrt{x}\left[g(x)\right]' = \frac{g(x)}{2\sqrt{x}} + \sqrt{x}\, g'(x),$$

the function $f_2(x) = \sqrt{2}\, g(x)$ doesn't:

$$f_2'(x) = \left[\sqrt{2}\, g(x)\right]' = \sqrt{2}\left[g(x)\right]' = \sqrt{2}\, g'(x).$$

The Product Rule IV (page 25, Lecture 8)

- The product rule can be extended to products involving more than two factors. For example, if f, g, and h are differentiable functions of x, then

$$[f(x)g(x)h(x)]' = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x).$$

- **Example**. For instance, the derivative of $f(x) = x^2(x+1)(2x-3)$ is

$$\begin{aligned} f'(x) &= (2x)(x+1)(2x-3) \\ &\quad + x^2(1)(2x-3) + x^2(x+1)(2) = x(8x^2-3x-6). \end{aligned}$$

The Quotient Rule I (page 25, Lecture 8)

- The quotient f/g of two differentiable functions f and g is itself differentiable at all values of x for which $g(x) \neq 0$. Moreover,

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}.$$

- **Proof**. Let x be any point of differentiability of f and g such that $g(x) \neq 0$ and let $h \neq 0$ be sufficiently small. Then

$$\begin{aligned}
\frac{1}{h}\left[\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}\right] &= \frac{f(x+h)g(x) - f(x)g(x+h)}{hg(x)g(x+h)} \\
&= \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{hg(x)g(x+h)} \\
&= \frac{h^{-1}[f(x+h) - f(x)]g(x) - f(x) \cdot h^{-1}[g(x+h) - g(x)]}{g(x)g(x+h)} \\
&= \frac{F(x,h)g(x) - f(x)G(x,h)}{g(x)g(x+h)},
\end{aligned}$$

The Quotient Rule II (page 25, Lecture 8)

where

$$F(x,h) = \frac{1}{h}\big[f(x+h) - f(x)\big], \qquad G(x,h) = \frac{1}{h}\big[g(x+h) - g(x)\big].$$

Since f and g are differentiable and hence continuous at x (by assumption), and $g(x) \neq 0$, the derivative of f/g exists at x, and it equals

$$\lim_{h\to 0} \frac{1}{h}\left[\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}\right]$$
$$= \frac{\lim_{h\to 0} F(x,h) \cdot \lim_{h\to 0} g(x) - \lim_{h\to 0} f(x) \cdot \lim_{h\to 0} G(x,h)}{\lim_{h\to 0} g(x) \cdot \lim_{h\to 0} g(x+h)}$$
$$= \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}.$$

$\square$

The Quotient Rule III (page 25, Lecture 8)

- **Example**. Not every quotient needs to be differentiated by the quotient rule. For example, each quotient in the following table can be considered as the product of a constant times a function of x. In such cases, it is more convenient to use the constant multiple rule.

$f(x)$	Rewrite	$f'(x)$	Simplify
$\frac{x^2+3x}{6}$	$\frac{1}{6}\left(x^2+3x\right)$	$\frac{1}{6}(2x+3)$	$\frac{2x+3}{6}$
$\frac{5x^4}{8}$	$\frac{5}{8}x^4$	$\frac{5}{8}\left(4x^3\right)$	$\frac{5x^3}{2}$
$\frac{-3(3x-2x^2)}{7x}$	$-\frac{3}{7}(3-2x)$	$-\frac{3}{7}(-2)$	$\frac{6}{7}$
$\frac{9}{5x^2}$	$\frac{9}{5}\left(x^{-2}\right)$	$\frac{9}{5}\left(-2x^{-3}\right)$	$-\frac{18}{5x^3}$

The Chain Rule I (page 29, Lecture 8)

- If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then $y = f(g(x))$ is a differentiable function of x. Moreover,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}, \qquad \text{or} \qquad \big[f(g(x))\big]' = f'(g(x)) \cdot g'(x).$$

- **Proof**. Let c be any point of differentiability of g such that $f'(g(c))$ exists, and assume first that there exists a $\delta > 0$ such that

$$g(x) \neq g(c) \qquad \text{whenever} \qquad 0 < |x - c| < \delta.$$

Then for all x such that $0 < |x - c| < \delta$,

$$\frac{f(g(x)) - f(g(c))}{x - c} = \frac{f(g(x)) - f(g(c))}{g(x) - g(c)} \cdot \frac{g(x) - g(c)}{x - c}.$$

The Chain Rule II (page 29, Lecture 8)

Since f and g are differentiable and hence continuous at $g(c)$ and c, respectively, the derivative of $f(g(x))$ exists at c, and it equals

$$\lim_{x\to c}\frac{f(g(x))-f(g(c))}{x-c}$$
$$=\lim_{x\to c}\frac{f(g(x))-f(g(c))}{g(x)-g(c)}\cdot\lim_{x\to c}\frac{g(x)-g(c)}{x-c}=f'(g(c))\cdot g'(c).$$

On the other hand, if there exists no $\delta>0$ such that

$$g(x)\neq g(c)\qquad\text{whenever}\qquad 0<|x-c|<\delta,$$

then for each $\delta>0$, there exists at least one x with $0<|x-c|<\delta$ such that $g(x)=g(c)$. In particular, for each $\delta=1/n$, $n=1,2,\ldots$, there exists an x_n such that $0<|x_n-c|<1/n$ and $g(x_n)=g(c)$.

The Chain Rule III (page 29, Lecture 8)

Clearly,

$$\lim_{n\to\infty} \frac{g(x_n) - g(c)}{x_n - c} = \lim_{n\to\infty} \frac{g(c) - g(c)}{x_n - c} = 0.$$

Since g is differentiable at c, this shows that[a]

$$g'(c) = \lim_{x\to c} \frac{g(x) - g(c)}{x - c} = \lim_{n\to\infty} \frac{g(x_n) - g(c)}{x_n - c} = 0.$$

Thus for any $\epsilon > 0$, there exists a $\delta_1 > 0$ such that

$$\left|\frac{g(x) - g(c)}{x - c}\right| < \epsilon \qquad \text{whenever} \qquad 0 < |x - c| < \delta_1.$$

[a]That

$$\lim_{x\to c} \frac{g(x) - g(c)}{x - c} = \lim_{n\to\infty} \frac{g(x_n) - g(c)}{x_n - c}$$

can be established using a contradiction argument.

The Chain Rule IV (page 29, Lecture 8)

Given that f is differentiable and hence continuous at $g(c)$, there exists an $\epsilon_0 > 0$ such that

$$\left|\frac{f(u) - f(g(c))}{u - g(c)} - f'(g(c))\right| < 1 \quad \text{whenever} \quad 0 < |u - g(c)| < \epsilon_0.$$

This implies that

$$|f(u) - f(g(c))| < \big[|f'(g(c))| + 1\big] \cdot |u - g(c)|,$$

whenever $0 < |u - g(c)| < \epsilon_0$. Since g is continuous at c, there exists a $\delta_2 > 0$ such that

$$|g(x) - g(c)| < \epsilon_0 \qquad \text{whenever} \qquad 0 < |x - c| < \delta_2.$$

The Chain Rule V (page 29, Lecture 8)

Thus for all x such that $0 < |x - c| < \delta := \min\{\delta_1, \delta_2\}$,

$$|f(g(x)) - f(g(c))| < \big[|f'(g(c))| + 1\big] \cdot |g(x) - g(c)|,$$

from which it follows that

$$\left|\frac{f(g(x)) - f(g(c))}{x - c}\right| \\ < \big[|f'(g(c))| + 1\big] \cdot \left|\frac{g(x) - g(c)}{x - c}\right| < \epsilon\big[|f'(g(c))| + 1\big],$$

for all x such that $0 < |x - c| < \delta$. Since $\epsilon > 0$ is arbitrary, this shows that the derivative of $f(g(x))$ at c exists and equals 0. Consequently,

$$\big[f(g(x))\big]'\Big|_{x=c} = 0 = f'(g(c)) \cdot 0 = f'(g(c)) \cdot g'(c). \qquad \square$$

The Chain Rule VI (page 29, Lecture 8)

- **Example**. A set of gears is constructed, as shown in Figure 60, such that the second and third gears are on the same axle. As the first axle revolves, it drives the second axle, which in turn drives the third axle. Let y, u, and x represent the numbers of revolutions per minute of the first, second, and third axles, respectively.

The Chain Rule VII (page 29, Lecture 8)

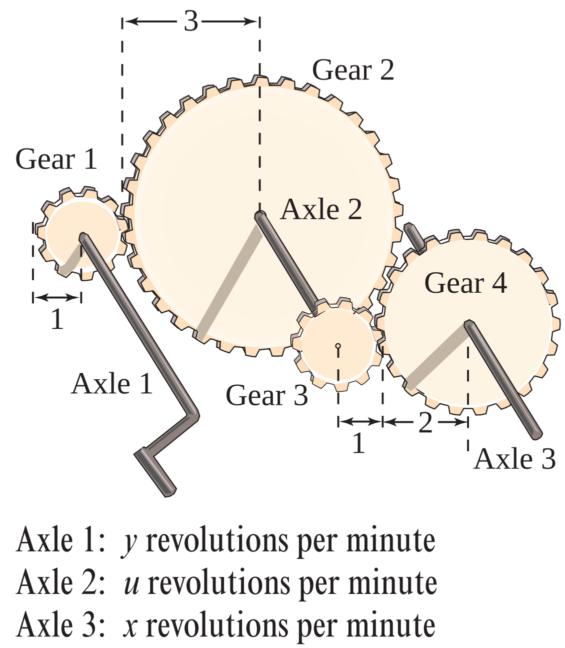


Figure 60: Illustration of the chain rule.

The Chain Rule VIII (page 29, Lecture 8)

Because the circumference of the second gear is three times that of the first, the first axle must make three revolutions to turn the second axle once. Similarly, the second axle must make two revolutions to turn the third axle once. Therefore,

$$\frac{dy}{du} = 3, \qquad \text{and} \qquad \frac{du}{dx} = 2.$$

Combining these two results, it can be deduced that the first axle must make six revolutions to turn the third axle once. Thus

$$\frac{dy}{dx} = 6 = 3 \cdot 2 = \frac{dy}{du} \cdot \frac{du}{dx},$$

which is consistent with the chain rule.

The Chain Rule IX (page 29, Lecture 8)

- When applying the chain rule:
  (a) It is helpful to think of the composite function $f(g(x))$ as having two parts, an inner function $g(x)$ and an outer function $f(u)$.
  (b) The derivative of $f(g(x))$ is then the derivative of the outer function $f(u)$ (at the inner function u) times the derivative of the inner function $g(x)$.

- **Example**. For instance, the derivative of the function $\sin(e^{x^3-3x+1})$ can be computed as

$$\begin{aligned}\left[\sin(e^{x^3-3x+1})\right]' &= \cos(e^{x^3-3x+1})\cdot(e^{x^3-3x+1})' \\ &= \cos(e^{x^3-3x+1})\cdot e^{x^3-3x+1}\cdot(x^3-3x+1)' \\ &= \cos(e^{x^3-3x+1})\cdot e^{x^3-3x+1}\cdot(3x^2-3).\end{aligned}$$

The Chain Rule X (page 29, Lecture 8)

- **Example**. As another application of the chain rule, let's verify the quotient rule
$$\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$
by converting the quotient f/g into a product $f \cdot (1/g)$ and then applying the product rule, followed by the chain rule:
$$\begin{aligned}\left[\frac{f(x)}{g(x)}\right]' &= \left[f(x) \cdot \frac{1}{g(x)}\right]' = f'(x) \cdot \frac{1}{g(x)} + f(x) \cdot \left[\frac{1}{g(x)}\right]' \\ &= \frac{f'(x)}{g(x)} - f(x) \cdot \frac{1}{g^2(x)} \cdot [g(x)]' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}.\end{aligned}$$

The Implicit Function Theorem (page 9, Lecture 9)

- **Theorem 5** (The implicit function theorem[a]). *Let $f\colon D \subset \mathbb{R}^2 \to \mathbb{R}$ be a continuously differentiable function such that $f(x_0, y_0) = 0$ for some $(x_0, y_0) \in D$. If $\partial f/\partial y(x_0, y_0) \neq 0$, then there exist open sets $U \subset \mathbb{R}^2$ and $V \subset \mathbb{R}$, with $(x_0, y_0) \in U$ and $x_0 \in V$, such that:*

  (a) To every $x \in V$ corresponds a unique y such that

  $$(x, y) \in U, \qquad \text{and} \qquad f(x, y) = 0.$$

  (b) If this y is defined to be $g(x)$, then $g\colon V \to \mathbb{R}$ is continuously differentiable, $g(x_0) = y_0$, and

  $$f(x, g(x)) = 0, \quad g'(x) = -\frac{\partial f/\partial x}{\partial f/\partial y}(x, g(x)), \qquad \forall x \in V.$$

[a] This is a special case of a more general theorem, which applies to continuously differentiable functions mapping $\mathbb{R}^{n+m}$ into $\mathbb{R}^m$.

Guidelines for Implicit Differentiation (page 3–4, Lecture 9)

- Equations implicitly defining y as a function of x can be differentiated using the following procedure.
  (a) Differentiate both sides of the equation with respect to x.
  (b) Collect all terms involving $y'(x)$ on the left side of the equation and move all other terms to the right side of the equation.
  (c) Factor $y'(x)$ out of the left side of the equation.
  (d) Solve for $y'(x)$.

The Inverse Function Theorem (page 10, Lecture 9)

- **Theorem 6** (The inverse function theorem[a]). *Let $f: D \subset \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function such that $f(x_0) = y_0$ for some $x_0 \in D$. If $f'(x_0) \neq 0$, then:*

  (a) There exist open sets U and V in $\mathbb{R}$ such that $x_0 \in U$, $y_0 \in V$, f is one-to-one on U, and $f(U) = V$.

  (b) If g is the inverse of f (which exists, by (a)), defined on V by

  $$g(f(x)) = x, \qquad \forall x \in U,$$

  then $g: V \to \mathbb{R}$ is continuously differentiable, and

  $$\frac{dx}{dy} = \frac{1}{dy/dx}, \quad \text{or} \quad g'(y) = \frac{1}{f'(x)} = \frac{1}{f'(g(y))}, \qquad \forall y \in V.$$

---

[a] This is a special case of a more general theorem, which applies to continuously differentiable functions mapping $\mathbb{R}^n$ into $\mathbb{R}^n$.

Related Rates I (page 3, Lecture 9)

- An important application of implicit differentiation is finding the rates of change of two or more related variables.
- **Example**. For example, when water is drained out of a conical tank, the volume V, the radius r, and the height h of the water level are all functions of time t (Figure 61). Knowing that these variables are related by the equation

$$V = \frac{1}{3}\pi r^2 h,$$

one can differentiate implicitly with respect to t to obtain

$$V' = \left(\frac{1}{3}\pi r^2 h\right)' = \frac{1}{3}\pi(2rr'h + r^2h').$$

It can be seen from this related-rate equation that the rate of change of V is related to the rates of change of both h and r.

Related Rates II (page 3, Lecture 9)

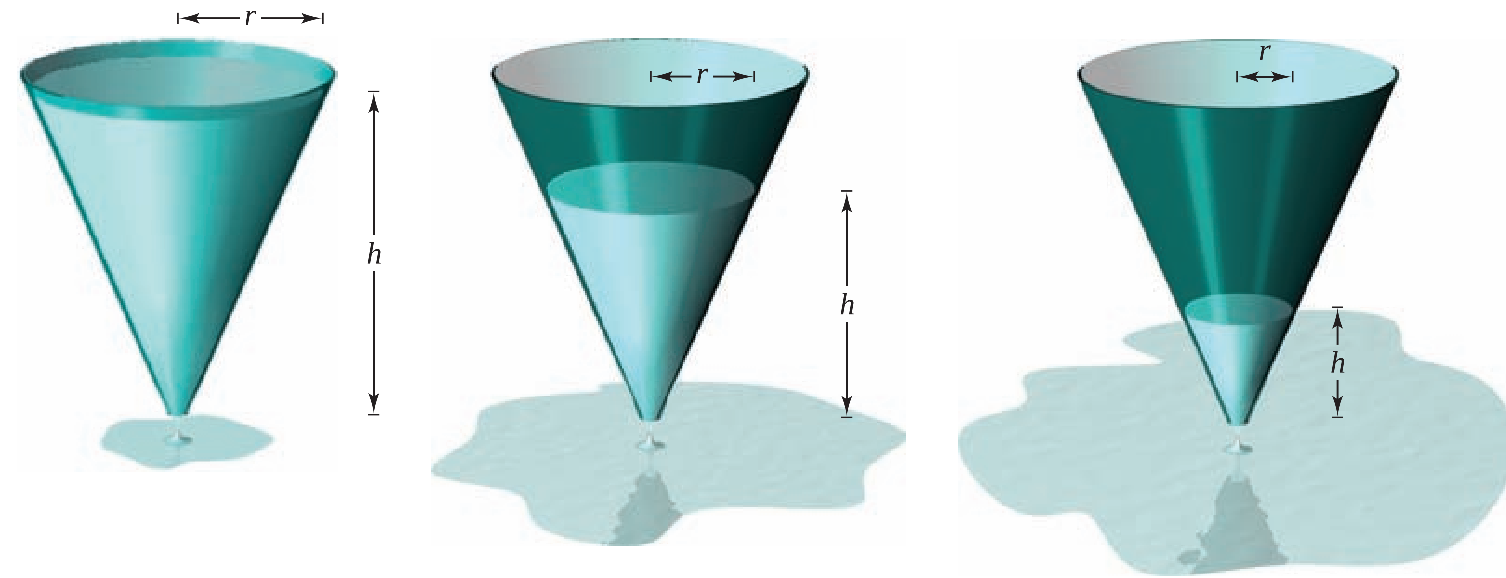


Figure 61: Related rates for a draining tank.

Related Rates III (page 3, Lecture 9)

- To solve problems involving related rates, the following procedure can be used.
  (a) Identify all given quantities[a] and quantities to be determined.
  (b) Find an equation involving the variables whose rates of change either are given or are to be determined.
  (c) Using the chain rule, implicitly differentiate both sides of the equation with respect to time t.
  (d) Substitute into the resulting equation all known values for the variables and their rates of change. Solve for the required quantity.

[a]Here the word "quantity" refers to either a variable, such as volume, or a rate of change, such as the rate at which a volume expands.

Related Rates IV (page 3, Lecture 9)

- **Example**. As an example, suppose air is being pumped into a spherical balloon at a rate of 4.5 cubic feet per minute (Figure 62), and we would like to find the rate of change of the radius when the radius is 2 feet.

  (a) Let V be the volume of the balloon and r be its radius. The problem can be stated as:

  $$\begin{aligned} \text{given}: &\qquad V' = \frac{9}{2} \text{ (constant rate)},\\ \text{find}: &\qquad r' \text{ (variable rate) when } r = 2. \end{aligned}$$

  (b) The radius r is related to the volume V through the equation

  $$V = \frac{4}{3}\pi r^3.$$

Related Rates V (page 3, Lecture 9)

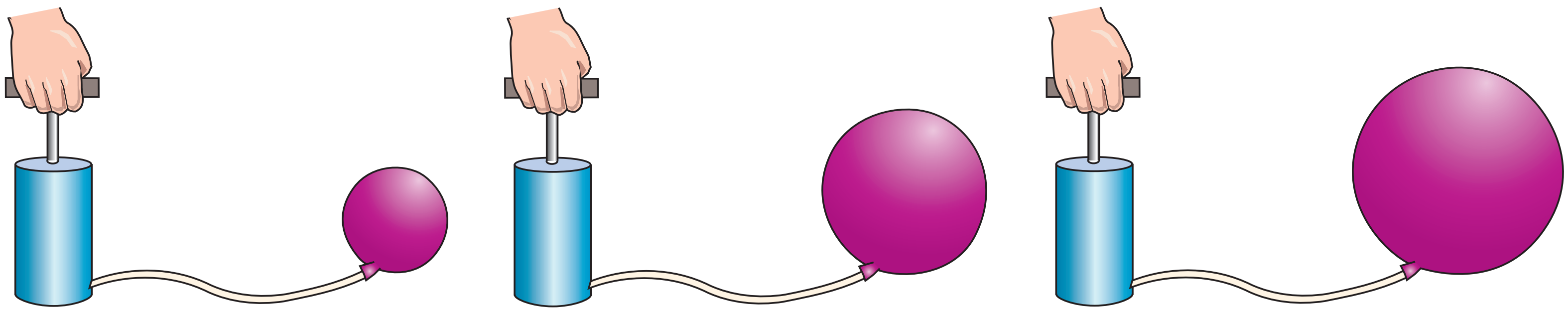


Figure 62: Related rates for an inflating balloon.

Related Rates VI (page 3, Lecture 9)

(c) Differentiating both sides of the equation with respect to t yields

$$V' = 4\pi r^2 r'.$$

(d) When $r = 2$, the equation becomes

$$\frac{9}{2} = 4\pi(2)^2 r'.$$

So $r' = 9/(32\pi)$, i.e. the radius of the balloon is growing at a rate of $9/(32\pi) \approx 0.09$ foot per minute when $r = 2$ feet.

Related Rates VII (page 3, Lecture 9)

- **Example**. As another example, suppose an airplane is flying at an altitude of 6 miles and is on a flight path that will take it directly over a radar station (Figure 63). We want to find the speed of the plane when s, the distance between the plane and the station, is 10 miles, given that at that moment s is decreasing at a rate of 400 miles per hour.

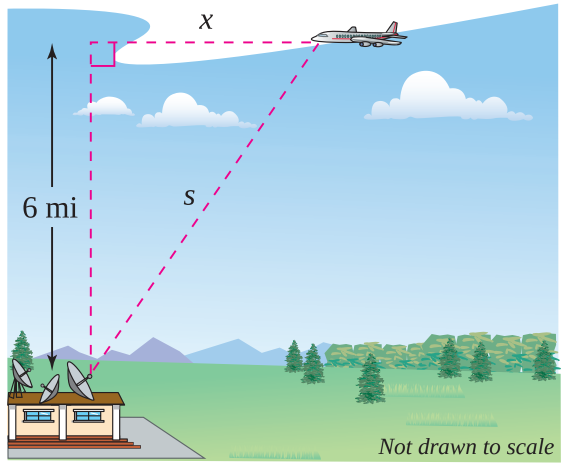
  (a) Let x be the horizontal distance between the plane and the station. The problem can be stated as:

$$\begin{aligned} \text{given :} \quad & s' = -400 \text{ when } s = 10, \\ \text{find :} \quad & x' \text{ when } s = 10. \end{aligned}$$

  (b) The horizontal distance x is related to the distance s through the equation

$$x^2 + 6^2 = s^2.$$

Related Rates VIII (page 3, Lecture 9)



An airplane is flying at an altitude of 6 miles, *s* miles from the station.

Figure 63: Related rates for an airplane tracked by radar.

Related Rates IX (page 3, Lecture 9)

(c) Differentiating both sides of the equation with respect to t yields

$$2xx' = 2ss'.$$

(d) When $s = 10$, the equation becomes

$$xx' = (10)(-400),$$

where

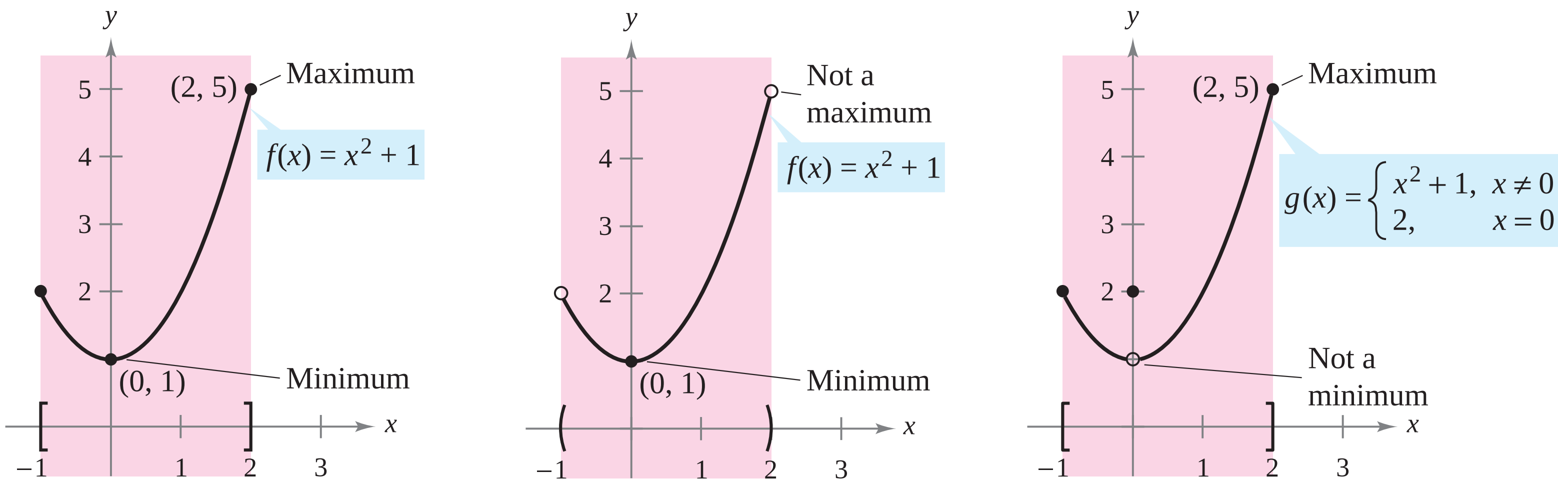
$$x = \sqrt{(10)^2 - 6^2} = 8.$$

Thus $x' = -500$, i.e. the plane is traveling (to the left) at a speed of 500 miles per hour when $s = 10$ miles.

Definition of Global Extremum I (page 29, Lecture 10)

- Let c be a real number and f be a function defined on an interval I containing c.
  (a) $f(c)$ is called the global (or absolute) minimum, or simply the minimum, of f on I if $f(c) \leq f(x)$ for all $x \in I$.
  (b) $f(c)$ is called the global (or absolute) maximum, or simply the maximum, of f on I if $f(c) \geq f(x)$ for all $x \in I$.
- **Example**. Note that a function need not have a minimum or a maximum on an interval. For instance, the function $f(x) = x^2 + 1$ shown below in Figure 64 (a)(b) has both a minimum and a maximum on the closed interval $[-1, 2]$, but does not have a maximum on the open interval $(-1, 2)$. Moreover, as Figure 64 (c) shows, the continuity (or the lack of it) of a function can also affect the existence of an extremum.

Definition of Global Extremum II (page 29, Lecture 10)



(a) f is continuous, $[-1, 2]$ is closed. **(b)** f is continuous, $(-1, 2)$ is open. **(c)** g is not continuous, $[-1, 2]$ is closed.

Figure 64: Global (absolute) minima and maxima of a function.

The Extreme Value Theorem I (page 29, Lecture 10)

- **Theorem 7** (The extreme value theorem). *Let f be a continuous function defined on a closed interval $[a, b]$. Then f has both a minimum and a maximum on the interval.*

- **Proof**. (Sketch) Since f is continuous on $[a, b]$, it is bounded, in view of the compactness[a] of $[a, b]$. Thus both the greatest lower bound m and the least upper bound M of f, defined by

$$m = \inf_{x \in [a,b]} f(x), \qquad M = \sup_{x \in [a,b]} f(x),$$

exist and are finite.

[a] A set K in a metric space is called compact if every sequence $\{x_n\}$ contained in K has a convergent subsequence in K.

The Extreme Value Theorem II (page 29, Lecture 10)

For each $n = 1, 2, \ldots,$ let $x_n,\ y_n \in [a, b]$ be chosen such that

$$m \le f(x_n) < m + \frac{1}{n}, \qquad M - \frac{1}{n} < f(y_n) \le M.$$

In view of the compactness of $[a, b]$, there exist subsequences $\{x_{n_k}\}_{k=1}^{\infty}$, $\{y_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ such that

$$x_{n_k} \to x_0, \quad y_{n_k} \to y_0, \qquad \text{as } k \to \infty,$$

for some $x_0,\ y_0 \in [a, b]$. It then follows from the continuity of f that

$$f(x_0) = \lim_{k\to\infty} f(x_{n_k}) = m, \qquad f(y_0) = \lim_{k\to\infty} f(y_{n_k}) = M. \qquad \square$$

Global and Local Extremum I (page 31, Lecture 10)

- Recall that a function value $f(a)$ is called a local minimum of f if there exists an interval (x_1, x_2) that contains a such that
$$x_1 < x < x_2 \qquad \text{implies} \qquad f(a) \leq f(x).$$
Likewise, a function value $f(a)$ is called a local maximum of f if there exists an interval (x_1, x_2) that contains a such that
$$x_1 < x < x_2 \qquad \text{implies} \qquad f(a) \geq f(x).$$
- Clearly, if c is an interior point of I[a] and $f(c)$ is the global minimum (or maximum) of f on I, then $f(c)$ is also a local minimum (or maximum) of f.

[a]Meaning that there exists an open interval $(a, b) \subseteq I$ that contains c.

Global and Local Extremum II (page 31, Lecture 10)

- Note that:

  (a) If $f(c)$ is a local minimum (maximum) of a function f, then $f(c)$ needs to be less (greater) than or equal to $f(x)$ for all x to the immediate left and immediate right of c. In particular, $f(x)$ needs to be defined for all x to the immediate left and immediate right of c.

  (b) A local minimum (maximum) $f(c)$ need not be a global minimum (maximum), since there may exist points c_1 away from c such that $f(c_1)$ is less (greater) than $f(c)$. For example, the function g shown in Figure 64 (c) has a local maximum at $c = 0$, but $g(0) = 2$ is not the global maximum of g on $[-1, 2]$, which is given by $g(2) = 5$.

Global and Local Extremum III (page 31, Lecture 10)

(c) A global minimum (maximum) $f(c)$ need not be a local minimum (maximum), since $f(x)$ may not be defined (or available) for all x to the immediate left and immediate right of c. For example, the function f shown in Figure 64 (a) has a global maximum on $[-1, 2]$ at $c = 2$, but $f(2) = 5$ is not a local maximum of f, since $f(x)$ is not available to the immediate right of $c = 2$.

(d) If $f(c)$ is the global minimum (maximum) of a function f on some interval I, then either $f(c)$ is a local minimum (maximum) of f (in case c lies in the interior of I), or c is an endpoint of I, in which case $f(c)$ is not a local minimum (maximum) of f.

Local Extremum at Critical Point I (page 31, Lecture 10)

- It turns out that the local minima and local maxima of a function f occur only at the critical points of f, namely, points c at which either $f'(c) = 0$ or $f'(c)$ is not defined (Figure 65).

- **Proof**. Suppose first that f has a local minimum at the point $x = c$. If f is not differentiable at c, then by definition c is a critical point of f. If f is differentiable at c, on the other hand, then $f'(c)$ exists and equals

$$f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}.$$

Since $f(c)$ is a local minimum of f, there exists a $\delta > 0$ such that

$$f(x) - f(c) \geq 0 \qquad \text{whenever} \qquad |x - c| < \delta.$$

Local Extremum at Critical Point II (page 31, Lecture 10)

In particular, this implies that

$$\frac{f(x)-f(c)}{x-c} \le 0, \qquad \text{whenever} \qquad -\delta < x-c < 0,$$
$$\frac{f(x)-f(c)}{x-c} \ge 0, \qquad \text{whenever} \qquad 0 < x-c < \delta.$$

It then follows that

$$f'(c) = \lim_{x\to c^-} \frac{f(x)-f(c)}{x-c} \le 0,$$
$$f'(c) = \lim_{x\to c^+} \frac{f(x)-f(c)}{x-c} \ge 0,$$

and thus $f'(c)=0$, i.e. c is a critical point of f. The case where f has a local maximum at $x=c$ can be handled in the same way. $\square$

Local Extremum at Critical Point III (page 31, Lecture 10)

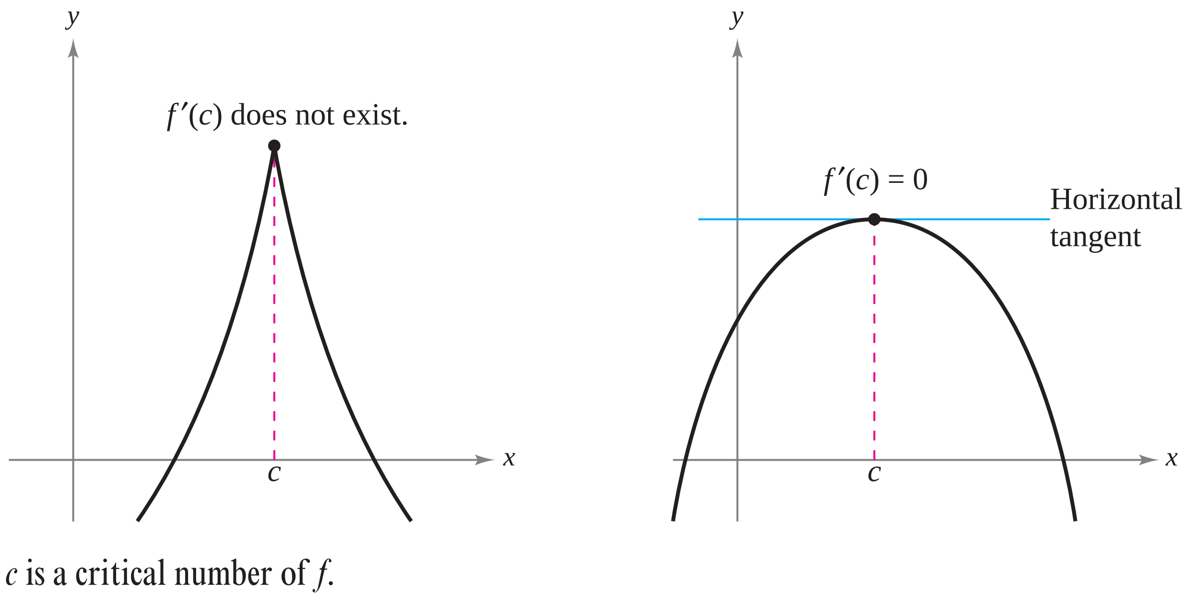


Figure 65: The two types of critical points of a function f.

Finding Extrema on a Closed Interval I (page 38, Lecture 10)

- The (global) extrema of a continuous function f on a closed interval $[a, b]$ can be determined using the following procedure.
  (a) Find the critical points of f in (a, b).
  (b) Evaluate f at each critical point in (a, b).
  (c) Evaluate f at each endpoint of $[a, b]$.
  (d) The least of these values is the minimum. The greatest is the maximum. (In particular, it is not necessary to determine whether f has a local minimum or a local maximum at each critical point.)

Finding Extrema on a Closed Interval II (page 38, Lecture 10)

- **Example**. As an example, consider the function

$$f(x) = 3x^4 - 4x^3$$

defined on the interval $[-1, 2]$. The extrema of f are found as follows.

(a) Compute

$$f'(x) = 12x^3 - 12x^2 = 12x^2(x-1),$$

and find the critical points of f in $(-1, 2)$, which are $x = 0$ and $x = 1$ since $f'(0) = f'(1) = 0$.

(b) Evaluate f at each critical point:

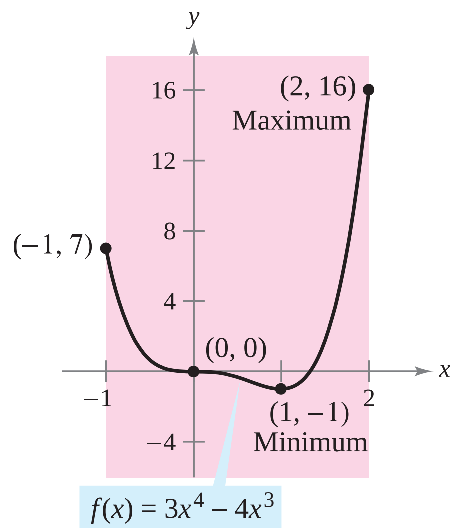
$$f(0) = 0, \qquad f(1) = -1.$$

(c) Evaluate f at each endpoint of $[-1, 2]$:

$$f(-1) = 7, \qquad f(2) = 16.$$

Finding Extrema on a Closed Interval III (page 38, Lecture 10)

(d) Comparing these values shows that $f(1) = -1$ is the minimum of f on $[-1,2]$ and $f(2) = 16$ is the maximum of f on $[-1,2]$. Note that the critical point $x = 0$ does not yield a local extremum of f.



On the closed interval $[-1, 2]$, f has a minimum at $(1, -1)$ and a maximum at $(2, 16)$.

Figure 66: Finding the extrema of $f(x) = 3x^4 - 4x^3$ on $[-1,2]$.

Finding Extrema on a Closed Interval IV (page 38, Lecture 10)

- **Example**. As another example, consider the function

$$f(x) = 2x - 3x^{2/3}$$

defined on the interval $[-1, 3]$. The extrema of f are found as follows.

(a) Compute

$$f'(x) = 2 - 2x^{-1/3},$$

and find the critical points of f in $(-1, 3)$, which are $x = 0$ since $f'(0)$ is undefined and $x = 1$ since $f'(1) = 0$.

(b) Evaluate f at each critical point:

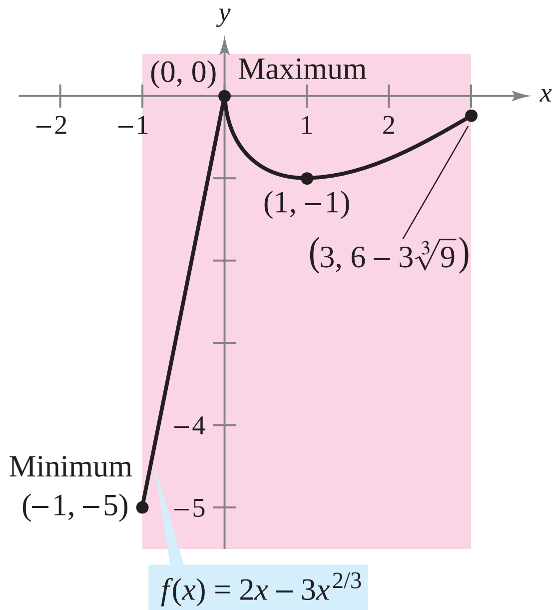
$$f(0) = 0, \qquad f(1) = -1.$$

(c) Evaluate f at each endpoint of $[-1, 3]$:

$$f(-1) = -5, \qquad f(3) = 6 - 3^{5/3} \approx -0.24.$$

Finding Extrema on a Closed Interval V (page 38, Lecture 10)

(d) Comparing these values shows that $f(-1) = -5$ is the minimum of f on $[-1, 3]$ and $f(0) = 0$ is the maximum of f on $[-1, 3]$.



On the closed interval $[-1, 3]$, f has a minimum at $(-1, -5)$ and a maximum at $(0, 0)$.

Figure 67: Finding the extrema of $f(x) = 2x - 3x^{2/3}$ on $[-1, 3]$.

Applied Optimization Problems I (page 38, Lecture 10)

- The techniques introduced in the previous section can be used to solve problems that seek to maximize or minimize a certain function (such as profit or cost). The procedure is as follows.

  (a) Identify all given quantities and all quantities to be determined.

  (b) Find a primary equation for the quantity that is to be maximized or minimized.

  (c) Reduce the primary equation to one having a single independent variable. This may involve the use of secondary equations relating the independent variables of the primary equation.

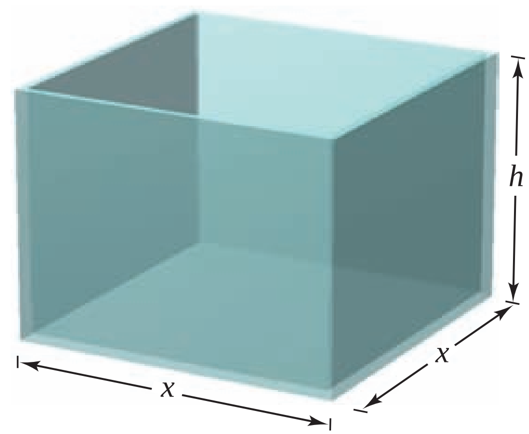
  (d) Determine the domain of the primary equation.

  (e) Determine the desired maximum or minimum value by the techniques introduced in the previous section.

Applied Optimization Problems II (page 38, Lecture 10)

- **Example**. As an example, suppose a manufacturer wants to design an open box having a square base and a surface area of 108 square inches (Figure 68), and we would like to find the dimensions that produce a box with maximum volume.
  (a) Let V be the volume of the box, x be the side of its (square) base, and h be its height.
  (b) The quantity to be maximized is the volume V, and it is related to the side x and height h through the primary equation
  $$V = x^2h.$$

Applied Optimization Problems III (page 38, Lecture 10)



Open box with square base:
$S = x^2 + 4xh = 108$

Figure 68: Maximizing the volume of an open box.

Applied Optimization Problems IV (page 38, Lecture 10)

(c) To write V as a function of a single independent variable, observe that the surface area of the box is

$$S = x^2 + 4xh = 108.$$

It follows from this secondary equation that $h = (108 - x^2)/(4x)$, so

$$V = x^2 h = x^2 \left(\frac{108 - x^2}{4x} \right) = 27x - \frac{1}{4}\, x^3.$$

(d) For the above expression to be physically meaningful, x must be nonnegative and must be such that $x^2 \leq S = 108$. Thus the domain of V (as a function of x) is the closed interval $[0, \sqrt{108}] = [0, 6\sqrt{3}]$.

Applied Optimization Problems V (page 38, Lecture 10)

(e) To maximize V, compute

$$V'(x) = 27 - \frac{3}{4}\,x^2,$$

and find the critical points of V in $(0, 6\sqrt{3})$, which is $x = 6$ since $V'(6) = 0$ (there is another critical point $x = -6$ but it lies outside the domain). Evaluating V at the critical point and at the endpoint of the domain $[0, 6\sqrt{3}]$ yields

$$V(6) = 108, \qquad V(0) = 0, \qquad V(6\sqrt{3}) = 0.$$

Thus V is maximum when $x = 6$, and the corresponding dimensions of the box are $6 \times 6 \times 3$ inches.

Applied Optimization Problems VI (page 38, Lecture 10)

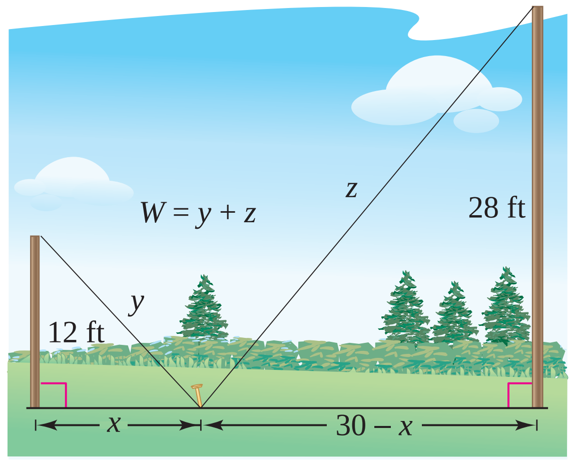
- **Example**. As another example, suppose two posts, one 12 feet high and the other 28 feet high, stand 30 feet apart. They are to be secured upright by two wires, attached to a single stake, running from ground level to the top of each post (Figure 69). We want to find the location of the stake that minimizes the amount of wire used.

  (a) Let W be the length of the wire, y be the distance from the stake to the top of the left post, and z be the distance from the stake to the top of the right post.

  (b) The quantity to be minimized is the wire length W, and it is related to the distances y and z through the primary equation

$$W = y + z.$$

Applied Optimization Problems VII (page 38, Lecture 10)



The quantity to be minimized is length. From the diagram, you can see that x varies between 0 and 30.

Figure 69: Minimizing the length of a wire securing two posts.

Applied Optimization Problems VIII (page 38, Lecture 10)

(c) To write W as a function of a single independent variable, observe that if x denotes the distance from the stake to the base of the left post, then the distance from the stake to the base of the right post is $30 - x$, and

$$x^2 + 12^2 = y^2, \qquad (30 - x)^2 + 28^2 = z^2.$$

It follows from these secondary equations that

$$y = \sqrt{x^2 + 144}, \qquad z = \sqrt{x^2 - 60x + 1684},$$

so

$$W = y + z = \sqrt{x^2 + 144} + \sqrt{x^2 - 60x + 1684}.$$

(d) For the above expression to be physically meaningful, x must be nonnegative and must not exceed 30. Thus the domain of W (as a function of x) is the closed interval $[0, 30]$.

Applied Optimization Problems IX (page 38, Lecture 10)

(e) To minimize W, compute

$$W'(x) = \frac{x}{\sqrt{x^2+144}} + \frac{x-30}{\sqrt{x^2-60x+1684}},$$

and find the critical points of W in $(0, 30)$, which is $x = 9$ since $W'(9) = 0$ (there is another critical point $x = -45/2$ but it lies outside the domain). Evaluating W at the critical point and at the endpoints of the domain $[0, 30]$ yields

$$W(9) = 50, \qquad W(0) \approx 53.04, \qquad W(30) \approx 60.31.$$

Thus W is minimum when $x = 9$, i.e. when the wire is staked at 9 feet from the left post.

Rolle’s Theorem I (page 38, Lecture 10)

- The extreme value theorem (Theorem 7) asserts that a continuous function on a closed interval must have both a minimum and a maximum on the interval.
- Both of these values, however, can occur at the endpoints.
- The following theorem gives conditions that guarantee the existence of an extreme value in the interior of a closed interval.

Rolle's Theorem II (page 38, Lecture 10)

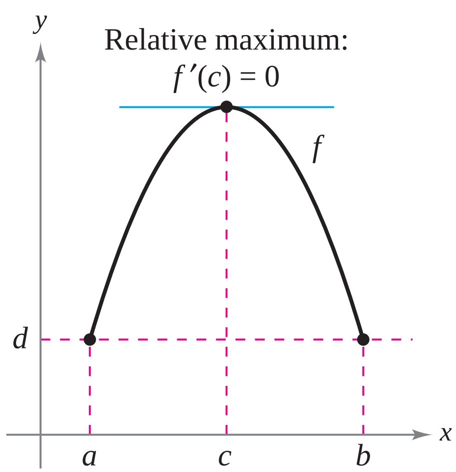
- **Theorem 8** (Rolle's theorem)**.** *Let f be continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b). If*

$$f(a) = f(b),$$

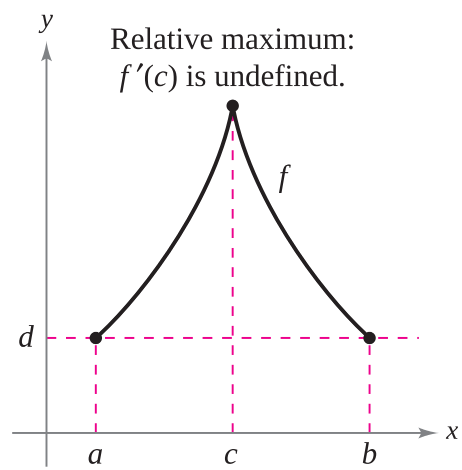
then there exists at least one point c in (a,b) such that $f'(c) = 0$.

- **Proof**. According to the extreme value theorem, f has a minimum and a maximum at some point c_1 and c_2 in the closed interval $[a,b]$. If one of these points lies in the open interval (a,b), then f has a local minimum (in case of c_1) or a local maximum (in case of c_2) at this point as well. Since f is differentiable on (a,b), f' must vanish at this local extremum, which establishes the theorem. If, on the other hand, both c_1 and c_2 are located at the endpoints, then f must be constant on $[a,b]$ since $f(a) = f(b)$. It then follows that $f'(c) = 0$ for any c in (a,b). $\square$

Rolle's Theorem III (page 38, Lecture 10)



(a) f is continuous on $[a, b]$ and differentiable on (a, b).



(b) f is continuous on $[a, b]$, but not differentiable on (a, b).

Figure 70: Illustration of Rolle's theorem.

The Mean Value Theorem I (page 38, Lecture 10)

- Rolle's theorem can be generalized to functions where $f(a)$ and $f(b)$ are not necessarily equal.

- **Theorem 9** (The mean value theorem)**.** *Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then there exists at least one point c in (a, b) such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The Mean Value Theorem II (page 38, Lecture 10)

- **Proof**. Consider the secant line that passes through the points $(a, f(a))$ and $(b, f(b))$, whose equation is given by (Figure 71)

$$y = \frac{f(b) - f(a)}{b - a}(x - a) + f(a).$$

Let g denote the difference between f and y, so that

$$g(x) = f(x) - y = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

It is readily verified that g is continuous on the closed interval $[a, b]$, differentiable on the open interval (a, b), and $g(a) = g(b) = 0$. Thus by Rolle's theorem, there exists at least one point c in (a, b) such that

$$0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}. \qquad \square$$

The Mean Value Theorem III (page 38, Lecture 10)

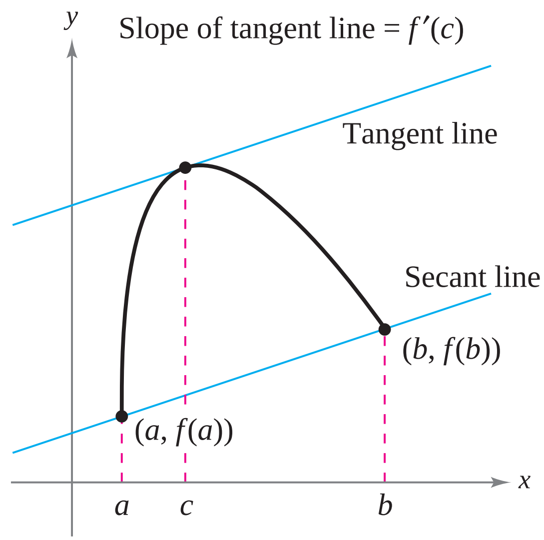


Figure 71: Illustration of the mean value theorem.

The Mean Value Theorem IV (page 38, Lecture 10)

- The mean value theorem has the following implications.
  (a) Geometrically, the theorem guarantees the existence of a tangent line that is parallel to the secant line through the points $(a, f(a))$ and $(b, f(b))$, as shown in Figure 71.
  (b) In terms of rates of change, the mean value theorem implies that there must be a point in the open interval (a, b) at which the instantaneous rate of change is equal to the average rate of change over the interval $[a, b]$. This is illustrated in the following example.

The Mean Value Theorem V (page 38, Lecture 10)

- **Example**. Two stationary patrol cars equipped with radar are 5 miles apart on a highway, as shown in Figure 72. As a truck passes the first patrol car, its speed is clocked at 55 miles per hour. Four minutes later, when the truck passes the second patrol car, its speed is clocked at 50 miles per hour. The truck must have exceeded the speed limit (of 55 miles per hour) at some time during the 4 minutes, as shown below.

The Mean Value Theorem VI (page 38, Lecture 10)

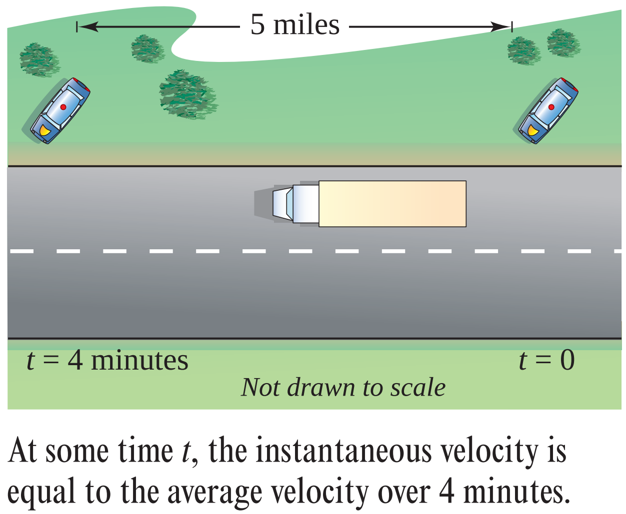


Figure 72: Finding an instantaneous rate of change using the mean value theorem.

The Mean Value Theorem VII (page 38, Lecture 10)

(a) Let $t = 0$ be the time when the truck passes the first patrol car. The time (in hours) when the truck passes the second patrol car is

$$t = \frac{4}{60} = \frac{1}{15}\,\text{hour.}$$

(b) Let $s(t)$ be the distance (in miles) traveled by the truck since $t = 0$. Then $s(0) = 0$, $s(1/15) = 5$, and the average velocity of the truck over the five-mile stretch of highway is

$$v_{\text{avg}} = \frac{s(1/15) - s(0)}{1/15 - 0} = \frac{5}{1/15} = 75\,\text{miles per hour.}$$

(c) Assume that s is differentiable. Then the mean value theorem (applied to s on $[0, 1/15]$) implies that the truck must have been traveling at a rate of 75 miles per hour sometime during the 4 minutes.

Taylor's Theorem I (page 6, Lecture 10)

- **Theorem 10** (Taylor's theorem)**.** *Let n be a positive integer and f be a function defined on a closed interval $[a, b]$. If f is $(n-1)$-times continuously differentiable on the closed interval $[a, b]$ and n-times differentiable on the open interval (a, b), then for each fixed x_0, $x \in [a, b]$ with $x \neq x_0$, there exists ξ between x_0 and x such that*

$$f(x) = \sum_{k=0}^{n-1} \frac{1}{k!} f^{(k)}(x_0)(x - x_0)^k + \frac{1}{n!} f^{(n)}(\xi)(x - x_0)^n.$$

- Note that for $n = 1$, Taylor's theorem reduces to the mean value theorem, which asserts the existence of ξ between x_0 and x such that

$$f(x) = f(x_0) + f'(\xi)(x - x_0).$$

Taylor's Theorem II (page 6, Lecture 10)

- **Proof**. Define the Taylor polynomial p and the number M by

$$p(t) = \sum_{k=0}^{n-1} \frac{1}{k!} f^{(k)}(x_0)(t - x_0)^k, \qquad M = \frac{f(x) - p(x)}{(x - x_0)^n},$$

and consider the function

$$g(t) = f(t) - p(t) - M(t - x_0)^n.$$

To complete the proof of the theorem, it suffices to show that

$$f(x) - p(x) - \frac{1}{n!} f^{(n)}(\xi)(x - x_0)^n = 0, \qquad \text{or} \qquad f^{(n)}(\xi) = n!M,$$

for some ξ between x_0 and x. Since

$$g^{(n)}(t) = f^{(n)}(t) - n!M,$$

this amounts to showing $g^{(n)}(\xi) = 0$ for some ξ between x_0 and x.

Taylor's Theorem III (page 6, Lecture 10)

To this end, observe that

$$p^{(k)}(x_0) = f^{(k)}(x_0), \qquad k = 0, 1, \ldots, n-1,$$

so that

$$g(x_0) = g'(x_0) = \cdots = g^{(n-1)}(x_0) = 0.$$

Since $g(x) = 0$ (by the choice of M), the mean value theorem (applied to g) implies the existence of x_1 between x_0 and x such that $g'(x_1) = 0$. Since $g'(x_0) = 0$, the mean value theorem (applied now to g') implies the existence of x_2 between x_0 and x_1 such that $g''(x_2) = 0$. Proceeding in a similar fashion, the mean value theorem (applied to $g'', g''', \ldots,$ and finally to $g^{(n-1)}$) implies the existence of x_n between x_0 and x_{n-1} such that $g^{(n)}(x_n) = 0$. Setting $\xi = x_n$ and observing that ξ lies between x_0 and x then completes the proof of the theorem. $\square$

Testing for Monotonicity I (page 33, Lecture 10)

- The derivatives of a function f can be used to classify the local extrema of f as either local minima or local maxima.
- To see how this can be done, it is instructive to investigate first the relation between f' and the monotonicity of f.
- To this end, let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b).
  (a) If $f'(x) > 0$ for all x in (a, b), then f is (strictly) increasing on $[a, b]$.
  (b) If $f'(x) < 0$ for all x in (a, b), then f is (strictly) decreasing on $[a, b]$.
  (c) If $f'(x) = 0$ for all x in (a, b), then f is constant on $[a, b]$.

Testing for Monotonicity II (page 33, Lecture 10)

- **Proof**. Assume first that $f'(x) > 0$ for all x in (a, b) and let $x_1 < x_2$ be any two points in $[a, b]$. By the mean value theorem (applied to f on $[x_1, x_2]$), there exists a point c in (x_1, x_2) such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Since $f'(c) > 0$ and $x_2 - x_1 > 0$, it follows that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1) > 0,$$

which shows that f is (strictly) increasing on $[a, b]$. The other two cases (where $f'(x) < 0$ or $f'(x) = 0$ for all x in (a, b)) can be handled in the same way. $\square$

Testing for Monotonicity III (page 33, Lecture 10)

- Let f be continuous on the interval (a, b). The open intervals on which f is monotone (i.e. either increasing or decreasing) can be determined using the following procedure.
  (a) Locate the critical points of f in (a, b), and use these points to determine test intervals.
  (b) Determine the sign of f' at one test value in each of these intervals.
  (c) Use the relation between f' and the monotonicity of f to determine whether f is increasing or decreasing on each interval.
- These guidelines are also valid if the interval (a, b) is replaced by an interval of the form $(-\infty, b)$, (a, ∞), or $(-\infty, \infty)$.

Testing for Monotonicity IV (page 33, Lecture 10)

- **Example**. As an example, consider the function

$$f(x) = x^3 - \frac{3}{2}x^2$$

defined on the entire real line $(-\infty, \infty)$. The open intervals on which f is monotone are found as follows.

(a) Compute

$$f'(x) = 3x^2 - 3x = 3x(x-1),$$

and find the critical points of f in $(-\infty, \infty)$, which are $x = 0$ and $x = 1$ since $f'(0) = f'(1) = 0$. These two critical points divide $(-\infty, \infty)$ into three test intervals: $(-\infty, 0)$, $(0, 1)$, and $(1, \infty)$.

Testing for Monotonicity V (page 33, Lecture 10)

(b) Evaluate f' at one test value in each of these intervals.

Interval	Test value x	$f'(x)$	Monotonicity of f
$(-\infty, 0)$	-1	$+6$	Increasing
$(0, 1)$	$1/2$	$-3/4$	Decreasing
$(1, \infty)$	2	$+6$	Increasing

Testing for Monotonicity VI (page 33, Lecture 10)

(c) This shows that f is increasing on the intervals $(-\infty, 0)$, $(1, \infty)$ and decreasing on the interval $(0, 1)$.

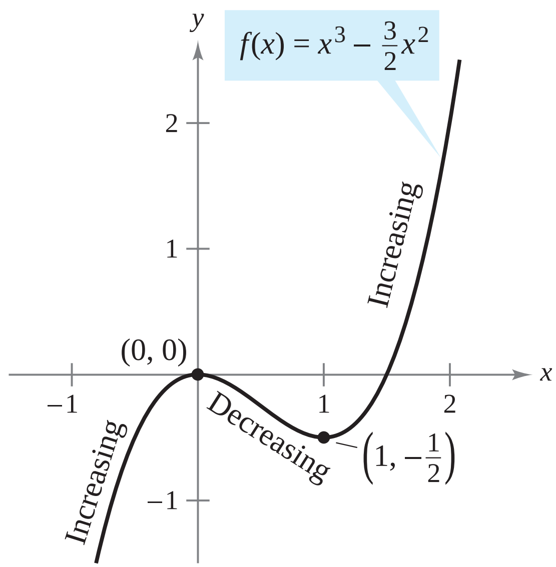


Figure 73: Finding open intervals on which $f(x) = x^3 - (3/2)x^2$ is monotone.

The First Derivative Test I (page 34, Lecture 10)

- Let c be a critical point of a function f that is continuous on an open interval containing c. If f is differentiable on the interval, except possibly at c, then $f(c)$ can be classified as follows.
  (a) If f' changes from negative to positive at c, then f has a local minimum at $(c, f(c))$.
  (b) If f' changes from positive to negative at c, then f has a local maximum at $(c, f(c))$.
  (c) If f' is positive on both sides of c or negative on both sides of c, then $f(c)$ is neither a local minimum nor a local maximum.
- The above method for locating the local minima and local maxima of a function f is known as the first derivative test.

The First Derivative Test II (page 34, Lecture 10)

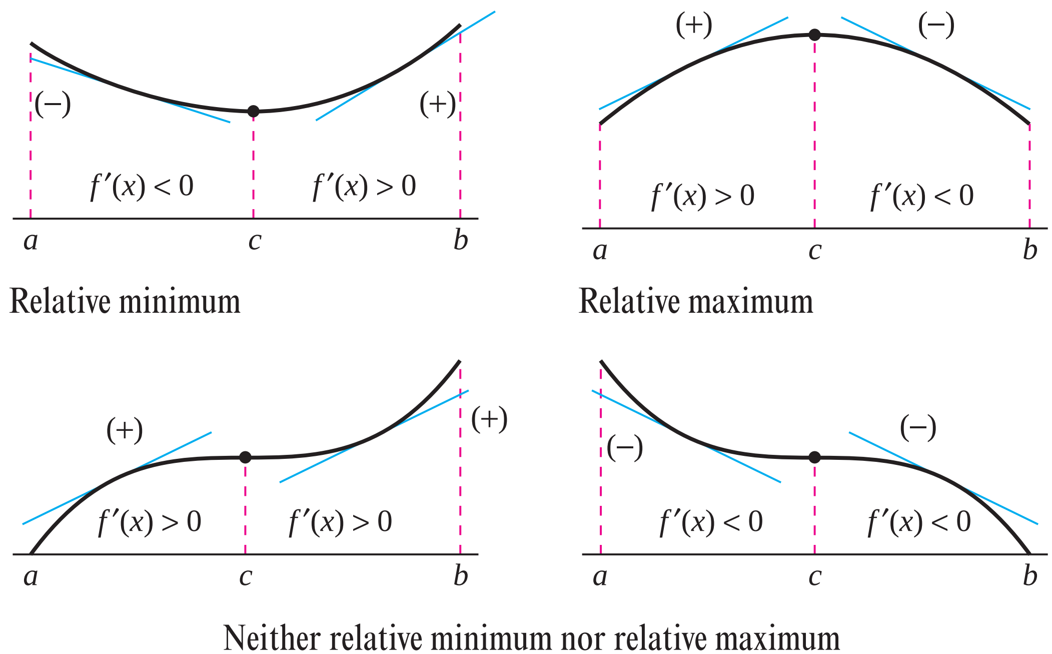


Figure 74: The first derivative test.

The First Derivative Test III (page 34, Lecture 10)

- **Example**. As an example, consider the function

$$f(x) = (x^2 - 4)^{2/3}$$

defined on the entire real line $(-\infty, \infty)$. The local extrema of f can be found as follows.

(a) Compute

$$f'(x) = \frac{2}{3}(x^2 - 4)^{-1/3}(2x) = \frac{4x}{3(x^2 - 4)^{1/3}},$$

and find the critical points of f in $(-\infty, \infty)$, which are $x = \pm 2$ since $f'(\pm 2)$ is undefined and $x = 0$ since $f'(0) = 0$. These three critical points divide $(-\infty, \infty)$ into four test intervals: $(-\infty, -2)$, $(-2, 0)$, $(0, 2)$, and $(2, \infty)$.

The First Derivative Test IV (page 34, Lecture 10)

(b) Evaluate f' at one test value in each of these intervals.

Interval	Test value x	$f'(x)$	Monotonicity of f
$(-\infty, -2)$	-3	$-4/5^{1/3}$	Decreasing
$(-2, 0)$	-1	$+4/3^{4/3}$	Increasing
$(0, 2)$	1	$-4/3^{4/3}$	Decreasing
$(2, \infty)$	3	$+4/5^{1/3}$	Increasing

The First Derivative Test V (page 34, Lecture 10)

(c) According to the first derivative test, f has a local minimum at the point $(-2, 0)$, a local maximum at the point $(0, 2^{4/3})$, and another local minimum at the point $(2, 0)$.

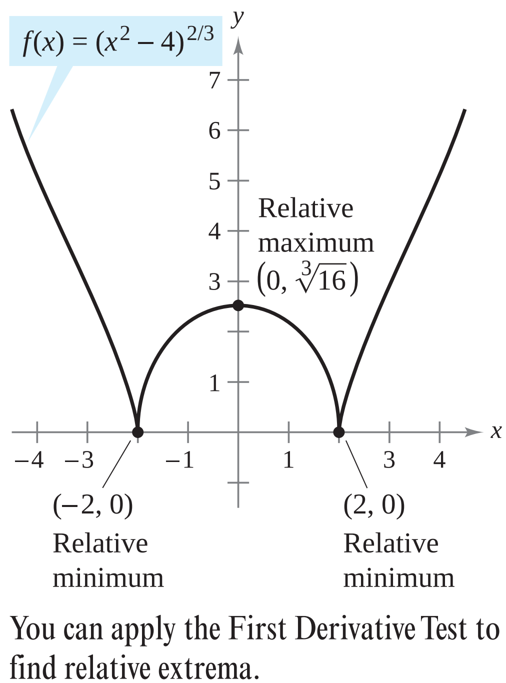


Figure 75: Finding the local extrema of $f(x) = (x^2 - 4)^{2/3}$.

The First Derivative Test VI (page 34, Lecture 10)

- **Example**. As another example, consider the function

$$f(x) = \frac{x^4+1}{x^2}$$

defined on the entire real line $(-\infty, \infty)$ except at $x = 0$ (i.e. defined on $(-\infty, \infty) \setminus \{0\}$). The local extrema of f can be found as follows.

(a) Compute

$$f'(x) = 2x - \frac{2}{x^3} = \frac{2}{x^3}(x^2+1)(x+1)(x-1),$$

and find the critical points of f in $(-\infty, \infty) \setminus \{0\}$, which are $x = \pm 1$ since $f'(\pm 1) = 0$. These two critical points divide $(-\infty, \infty) \setminus \{0\}$ into four test intervals: $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$, and $(1, \infty)$.

The First Derivative Test VII (page 34, Lecture 10)

(b) Evaluate f' at one test value in each of these intervals.

Interval	Test value x	$f'(x)$	Monotonicity of f
$(-\infty, -1)$	-2	$-15/4$	Decreasing
$(-1, 0)$	$-1/2$	$+15$	Increasing
$(0, 1)$	$1/2$	-15	Decreasing
$(1, \infty)$	2	$+15/4$	Increasing

The First Derivative Test VIII (page 34, Lecture 10)

(c) According to the first derivative test, f has a local minimum at the point $(-1, 2)$ and another at the point $(1, 2)$. Note that $x = 0$ does not yield a local maximum of f since f is not defined there.

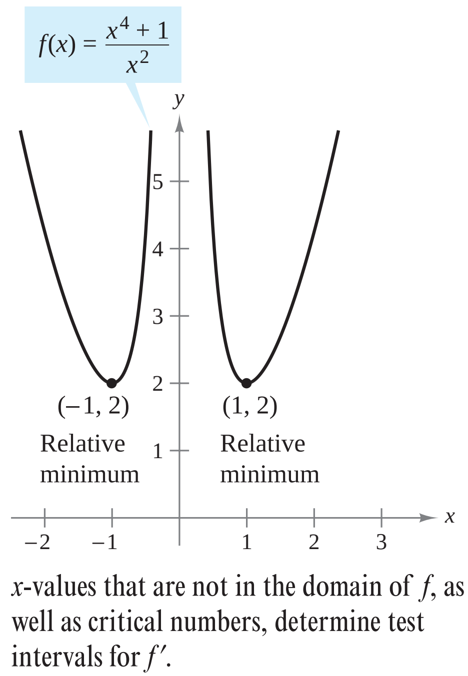
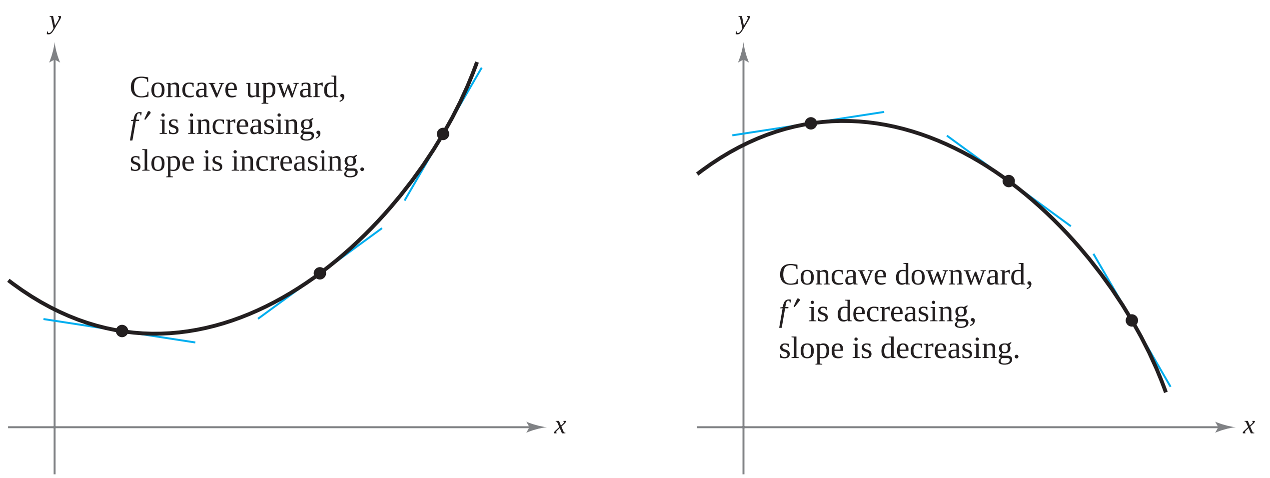


Figure 76: Finding the local extrema of $f(x) = (x^4 + 1)/x^2$.

Concavity I (page 36, Lecture 10)

- Let f be differentiable on an open interval I.
  (a) The graph of f is concave up on I if f' is increasing on the interval, and concave down on I if f' is decreasing on the interval.
  (b) Geometrically, the graph of f is concave up on I if it lies above all of its tangent lines on I, and the graph of f is concave down on I if it lies below all of its tangent lines on I (Figure 77).

Concavity II (page 36, Lecture 10)



(a) The graph of f lies above its tangent lines. **(b)** The graph of f lies below its tangent lines.

Figure 77: The geometric interpretation of concavity.

Testing for Concavity I (page 36, Lecture 10)

- Let f be a function whose second derivative exists on an open interval I. Given the relation between f'' and the monotonicity of f', it is easily seen that:
  (a) If $f''(x) > 0$ for all x in I, then the graph of f is concave up on I.
  (b) If $f''(x) < 0$ for all x in I, then the graph of f is concave down on I.
- A point $(c, f(c))$ on the graph of f at which f has a tangent line and the concavity of f changes is called a point of inflection.
- Clearly, if $(c, f(c))$ is a point of inflection of the graph of f, then c must be a critical point of f'.

Testing for Concavity II (page 36, Lecture 10)

- **Example**. As an example, consider the function

$$f(x) = \frac{x^2+1}{x^2-4}$$

defined on the entire real line $(-\infty, \infty)$ except at $x = \pm 2$. The open intervals on which the graph of f is concave are found as follows.

(a) Compute

$$f'(x) = \frac{-10x}{(x^2-4)^2}, \qquad f''(x) = \frac{10(3x^2+4)}{(x^2-4)^3},$$

and observe that f' has no critical points on $(-\infty, \infty) \setminus \{\pm 2\}$ since f'' is defined everywhere on this set and is never zero there. The test intervals are then simply $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$.

Testing for Concavity III (page 36, Lecture 10)

(b) Evaluate f'' at one test value in each of these intervals.

Interval	Test value x	$f''(x)$	Concavity of f
$(-\infty, -2)$	-3	$+62/25$	Concave up
$(-2, 2)$	0	$-5/8$	Concave down
$(2, \infty)$	3	$+62/25$	Concave up

Testing for Concavity IV (page 36, Lecture 10)

(c) This shows that f is concave up on the intervals $(-\infty, -2)$, $(2, \infty)$ and concave down on the interval $(-2, 2)$.

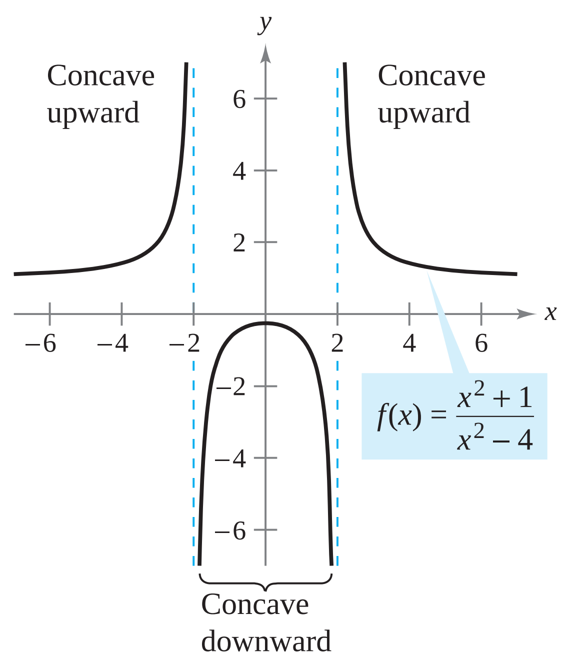


Figure 78: Finding open intervals on which $f(x) = (x^2+1)/(x^2-4)$ is concave.

The Second Derivative Test I (page 36, Lecture 10)

- Let f be a function such that $f'(c) = 0$ and the second derivative of f exists on an open interval containing c.
  (a) If $f''(c) > 0$, then f has a local minimum at $(c, f(c))$.
  (b) If $f''(c) < 0$, then f has a local maximum at $(c, f(c))$.
  (c) If $f''(c) = 0$, then f may have a local minimum at $(c, f(c))$, a local maximum at $(c, f(c))$, or neither.
- The above method for locating the local minima and local maxima of a function f is known as the second derivative test.

The Second Derivative Test II (page 36, Lecture 10)

- **Proof**. Suppose first that $f'(c) = 0$ and $f''(c) > 0.$ Then by the definition of $f''(c)$, there exists a $\delta > 0$ such that

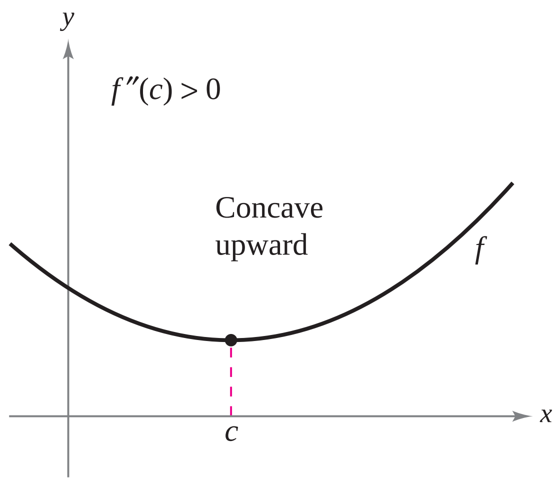
$$\frac{f'(x) - f'(c)}{x - c} = \frac{f'(x)}{x - c} > 0 \qquad \text{whenever} \qquad 0 < |x - c| < \delta.$$

In particular, this implies that

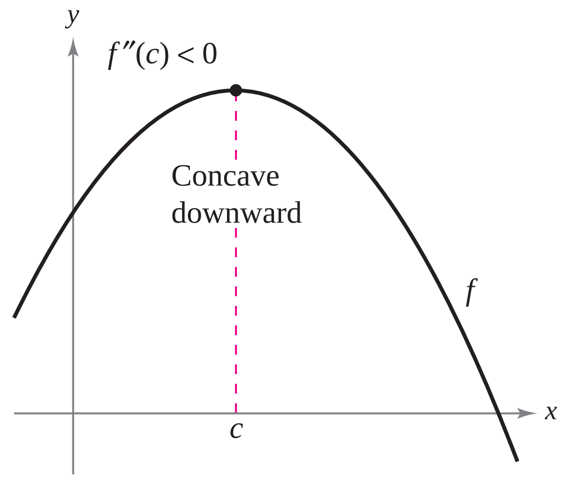
$$\begin{aligned} f'(x) &< 0, \qquad \text{whenever} \qquad -\delta < x - c < 0, \\ f'(x) &> 0, \qquad \text{whenever} \qquad 0 < x - c < \delta, \end{aligned}$$

so f' changes from negative to positive at c. By the first derivative test, f then has a local minimum at $(c, f(c))$. The case where $f'(c) = 0$ and $f''(c) > 0$ can be handled in the same way. $\square$

The Second Derivative Test III (page 36, Lecture 10)



If $f'(c) = 0$ and $f''(c) > 0$, $f(c)$ is a relative minimum.



If $f'(c) = 0$ and $f''(c) < 0$, $f(c)$ is a relative maximum.

Figure 79: The second derivative test.

The Second Derivative Test IV (page 36, Lecture 10)

- **Example**. As an example, consider the function

$$f(x) = -3x^5 + 5x^3$$

defined on the entire real line $(-\infty, \infty)$. The local extrema of f can be found as follows.

(a) Compute

$$f'(x) = -15x^4 + 15x^2 = 15x^2(1 - x^2),$$

and find the zeros of f', which are $x = 0$ and $x = \pm 1$.

The Second Derivative Test V (page 36, Lecture 10)

(b) Compute

$$f''(x) = -60x^3 + 30x,$$

and evaluate f'' at each of the zeros of f'.

Zero x of f'	$f''(x)$	Concavity of f
-1	$+30$	Concave up
0	0	–
1	-30	Concave down

The Second Derivative Test VI (page 36, Lecture 10)

(c) For the point $x = 0$ at which $f''(x) = 0$, choose the test intervals $(-1, 0)$, $(0, 1)$ and evaluate f' at one test value in each of these intervals.

Interval	Test value x	$f'(x)$	Monotonicity of f
$(-1, 0)$	$-1/2$	$+45/16$	Increasing
$(0, 1)$	$1/2$	$+45/16$	Increasing

(d) According to the second derivative test, f has a local minimum at the point $(-1, -2)$ and a local maximum at the point $(1, 2)$. The point $(0, 0)$ is neither a local minimum nor a local maximum, according to the first derivative test (Figure 80).

The Second Derivative Test VII (page 36, Lecture 10)

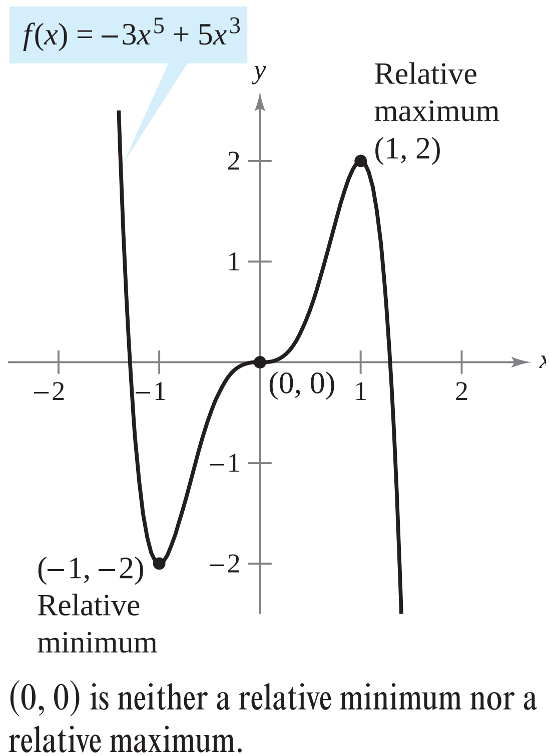


Figure 80: Finding the local extrema of $f(x) = -3x^5 + 5x^3$.

Indeterminate Forms I (page 60, Lecture 10)

- Limits of one of the following types

$$\frac{0}{0}, \quad \frac{\pm\infty}{\pm\infty}, \quad 0 \cdot (\pm\infty), \quad \infty - \infty, \quad (+0)^0, \quad 1^{\pm\infty}, \quad \infty^0,$$

  are known as indeterminate forms. Their values cannot be inferred from the expressions $0/0$, $\pm\infty/\pm\infty$, etc. and need to be determined on a case-by-case basis.

- Note that the following limits are not indeterminate forms:

$$\frac{0}{\pm\infty} = 0, \quad \frac{\pm\infty}{0} = \pm\infty, \quad 0 \cdot 0 = 0, \quad (\pm\infty) \cdot (\pm\infty) = \pm\infty,$$

$$\infty + \infty = \infty, \quad (+0)^{\infty} = 0, \quad (+0)^{-\infty} = \frac{1}{(+0)^{\infty}} = \frac{1}{+0} = \infty,$$

$$1^0 = 1, \quad \infty^{\infty} = \infty, \quad \infty^{-\infty} = \frac{1}{\infty^{\infty}} = \frac{1}{\infty} = 0.$$

Indeterminate Forms II (page 60, Lecture 10)

(a) $0/0$: the value of the limit depends on the rate at which the numerator and the denominator approach 0. If the numerator approaches 0 at a faster rate, then the limit is 0; for example:

$$\lim_{x\to 0} \frac{x^2}{x} = \lim_{x\to 0} x = 0.$$

If, on the other hand, the denominator approaches 0 at a faster rate, then the limit is ∞ or $-\infty$; for example:

$$\lim_{x\to 0^+} \frac{x}{x^2} = \lim_{x\to 0^+} \frac{1}{x} = \infty, \qquad \lim_{x\to 0^-} \frac{x}{x^2} = \lim_{x\to 0^-} \frac{1}{x} = -\infty.$$

If the numerator and the denominator approach 0 at the same rate, then the limit can be any real number; for example:

$$\lim_{x\to 0} \frac{cx}{x} = \lim_{x\to 0} c = c, \qquad \forall c \in \mathbb{R}.$$

Indeterminate Forms III (page 60, Lecture 10)

(b) $\pm\infty/\pm\infty$: the value of the limit depends on the rate at which the numerator and the denominator approach $\pm\infty$. If the numerator approaches $\pm\infty$ at a faster rate, then the limit is $\pm\infty$; for example:

$$\lim_{x\to\infty} \frac{x^2}{x} = \lim_{x\to\infty} x = \infty.$$

If, on the other hand, the denominator approaches $\pm\infty$ at a faster rate, then the limit is 0; for example:

$$\lim_{x\to\pm\infty} \frac{x}{x^2} = \lim_{x\to\pm\infty} \frac{1}{x} = 0.$$

If the numerator and the denominator approach $\pm\infty$ at the same rate, then the limit can be any real number; for example:

$$\lim_{x\to\pm\infty} \frac{cx}{x} = \lim_{x\to\pm\infty} c = c, \qquad \forall c \in \mathbb{R}.$$

Indeterminate Forms IV (page 60, Lecture 10)

(c) $0 \cdot (\pm\infty)$: this limit can be rewritten as either $0/0$ or $\pm\infty/\pm\infty$:

$$0 \cdot (\pm\infty) = 0 \cdot \frac{1}{0} = \frac{0}{0},$$
$$0 \cdot (\pm\infty) = \frac{1}{\pm\infty} \cdot (\pm\infty) = \frac{\pm\infty}{\pm\infty}.$$

It can take any real value or $\pm\infty$, depending on the rate at which the two factors approach 0 and $\pm\infty$, respectively. Some examples:

$$\lim_{x\to 0} x^2 \cdot \frac{1}{x} = \lim_{x\to 0} x = 0,$$
$$\lim_{x\to 0^+} x \cdot \frac{1}{x^2} = \lim_{x\to 0^+} \frac{1}{x} = \infty,$$
$$\lim_{x\to 0} cx \cdot \frac{1}{x} = \lim_{x\to 0} c = c, \qquad \forall c \in \mathbb{R}.$$

Indeterminate Forms V (page 60, Lecture 10)

(d) $\infty - \infty$: the value of the limit depends on the rate at which the two terms approach ∞. If the first term approaches ∞ at a faster rate, then the limit is ∞; for example:

$$\lim_{x\to\infty}(x^2 - x) = \lim_{x\to\infty} x(x-1) = \infty \cdot \infty = \infty.$$

If, on the other hand, the second term approaches ∞ at a faster rate, then the limit is $-\infty$; for example:

$$\lim_{x\to\infty}(x - x^2) = \lim_{x\to\infty} x(1-x) = \infty \cdot (-\infty) = -\infty.$$

If the two terms approach ∞ at the same rate, then the limit can be any real number; for example:

$$\lim_{x\to\infty}\big[(x+c) - x\big] = \lim_{x\to\infty} c = c, \qquad \forall c \in \mathbb{R}.$$

Indeterminate Forms VI (page 60, Lecture 10)

(e) $(+0)^0$: this limit can be rewritten as $0 \cdot (\pm\infty)$:

$$(+0)^0 = \mathrm{e}^{\log(+0)^0} = \mathrm{e}^{0\cdot\log(+0)} = \mathrm{e}^{0\cdot(-\infty)}.$$

It can take any real value or $\pm\infty$, depending on the rate at which the base and the exponent approach 0. Some examples:

$$\lim_{x\to 0^+} (\mathrm{e}^{-1/x^2})^x = \lim_{x\to 0^+} \mathrm{e}^{-1/x} = \mathrm{e}^{-\infty} = 0,$$
$$\lim_{x\to 0^-} (\mathrm{e}^{-1/x^2})^x = \lim_{x\to 0^-} \mathrm{e}^{-1/x} = \mathrm{e}^{\infty} = \infty,$$
$$\lim_{x\to 0^+} (\mathrm{e}^{-1/x})^{x^2} = \lim_{x\to 0^+} \mathrm{e}^{-x} = \mathrm{e}^{0} = 1,$$
$$\lim_{x\to 0^+} (\mathrm{e}^{-1/x})^{cx} = \lim_{x\to 0^+} \mathrm{e}^{-c} = \mathrm{e}^{-c}, \qquad \forall c \in \mathbb{R}.$$

Indeterminate Forms VII (page 60, Lecture 10)

(f) $1^{\pm\infty}$: this limit can be rewritten as $0 \cdot (\pm\infty)$:

$$1^{\pm\infty} = \mathrm{e}^{\log(1^{\pm\infty})} = \mathrm{e}^{\pm\infty\cdot\log 1} = \mathrm{e}^{\pm\infty\cdot 0}.$$

It can take any real value or $\pm\infty$, depending on the rate at which the base and the exponent approach 1 and $\pm\infty$, respectively. Some examples:

$$\begin{aligned}
\lim_{x\to 0^+} (\mathrm{e}^{-x})^{1/x^2} &= \lim_{x\to 0^+} \mathrm{e}^{-1/x} = \mathrm{e}^{-\infty} = 0,\\
\lim_{x\to 0^-} (\mathrm{e}^{-x})^{1/x^2} &= \lim_{x\to 0^-} \mathrm{e}^{-1/x} = \mathrm{e}^{\infty} = \infty,\\
\lim_{x\to 0^+} (\mathrm{e}^{-x^2})^{1/x} &= \lim_{x\to 0^+} \mathrm{e}^{-x} = \mathrm{e}^{0} = 1,\\
\lim_{x\to 0^+} (\mathrm{e}^{-cx})^{1/x} &= \lim_{x\to 0^+} \mathrm{e}^{-c} = \mathrm{e}^{-c}, \qquad \forall c \in \mathbb{R}.
\end{aligned}$$

Indeterminate Forms VIII (page 60, Lecture 10)

(g) ∞^0: this limit can be rewritten as $0 \cdot (\pm\infty)$:

$$\infty^0 = \mathrm{e}^{\log(\infty^0)} = \mathrm{e}^{0 \cdot \log \infty} = \mathrm{e}^{0 \cdot \infty}.$$

It can take any real value or $\pm\infty$, depending on the rate at which the base and the exponent approach ∞ and 0, respectively. Some examples:

$$\begin{aligned}
\lim_{x \to 0^+} (\mathrm{e}^{1/x^2})^{-x} &= \lim_{x \to 0^+} \mathrm{e}^{-1/x} = \mathrm{e}^{-\infty} = 0, \\
\lim_{x \to 0^-} (\mathrm{e}^{1/x^2})^{-x} &= \lim_{x \to 0^-} \mathrm{e}^{-1/x} = \mathrm{e}^{\infty} = \infty, \\
\lim_{x \to 0^+} (\mathrm{e}^{1/x})^{-x^2} &= \lim_{x \to 0^+} \mathrm{e}^{-x} = \mathrm{e}^{0} = 1, \\
\lim_{x \to 0^+} (\mathrm{e}^{1/x})^{-cx} &= \lim_{x \to 0^+} \mathrm{e}^{-c} = \mathrm{e}^{-c}, \qquad \forall c \in \mathbb{R}.
\end{aligned}$$

L'Hôpital's Rule I (page 52–53, Lecture 10)

- **Theorem 11** (L'Hôpital's rule). *Let f and g be differentiable functions defined on an open interval (a, b) where $-\infty \leq a < b \leq \infty$. Suppose that $g'(x) \neq 0$ for all x in (a, b), and that*

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = L,$$

where $-\infty \leq L \leq \infty$. If

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0,$$

or if

$$\lim_{x \to a} g(x) = \infty,$$

then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = L.$$

L’Hôpital’s Rule II (page 52–53, Lecture 10)

- **Proof**. Suppose first that $|a| < \infty$, $f'(a)$ and $g'(a)$ both exist and are finite, $g'(a) \neq 0$, and $f(a) = g(a) = 0$. Then by the definition of $f'(a)$ and $g'(a)$,

$$\begin{aligned}\lim_{x\to a}\frac{f(x)}{g(x)} &= \lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)}\\ &= \lim_{x\to a}\frac{\dfrac{f(x)-f(a)}{x-a}}{\dfrac{g(x)-g(a)}{x-a}} = \frac{\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}}{\lim\limits_{x\to a}\dfrac{g(x)-g(a)}{x-a}} = \frac{f'(a)}{g'(a)} = L.\end{aligned}$$

The proof of the more general case requires the use of Cauchy’s mean value theorem and is omitted here for the sake of brevity. □

L’Hôpital’s Rule III (page 52–53, Lecture 10)

- To find
$$\lim_{x \to a} \frac{f(x)}{g(x)}$$
by L’Hôpital’s rule:
  (a) Continue to differentiate f and g, so long as $f^{(k)}/g^{(k)}$ still evaluates to $0/0$ or ∞/∞ at $x = a$.
  (b) Stop differentiating as soon as one or the other of these derivatives is different from zero or infinity at $x = a$.
  (c) L’Hôpital’s rule does not apply when either the numerator or the denominator has a finite, nonzero limit.