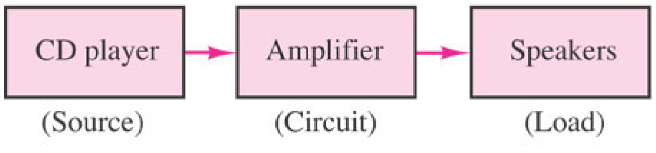
Frequency response

1) Changing the frequency affects the currents and voltages in a circuit
2) This is due to changes in the impedances of the various components in a circuit
3) This affects the working frequency range of a particular device or circuit
4) Hence it is important to find out the frequency response of a circuit
5) The frequency response of a circuit is a measure of the variation of a load-related voltage or current in relation to the input frequency
6) We typically express this in terms of variation in output voltage over the source:

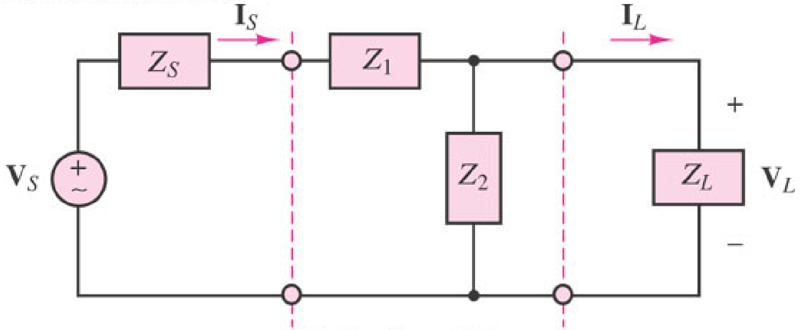
$$H_V(j\omega) = \frac{V_L(j\omega)}{V_S(j\omega)}$$

How does V_L change relative to V_S for different frequencies?
How does V_L change with respect to <u>phase</u> and <u>magnitude</u>?

Voltage divider rule used almost always to derive expression of output/input



A physical system



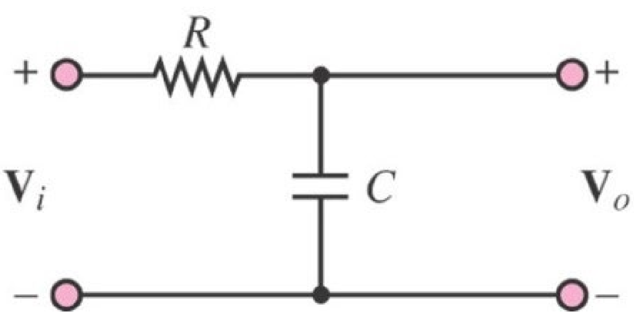
A circuit model

Low pass filter

Let us consider the response of the output V_o in relation to the input V_i. We keep the amplitude of V_i constant but vary its frequency ω.

By voltage divider rule: $$\frac{V_o}{V_i}(j\omega) = \frac{1/j\omega C}{R + 1/j\omega C}$$

$$= \frac{1}{1 + j\omega CR}$$



Note that CR is a constant based on the circuit values unlike ω, which is a variable.
Re-write CR as a constant with the same unit as angular frequency:

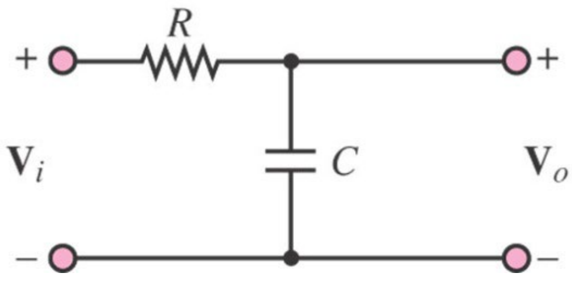
$$\omega_c = \frac{1}{RC}$$

This frequency is called the <u>cutoff</u> radian frequency (ω_c) and is a CONSTANT

Sub back into above equation: $$\frac{V_o}{V_i}(j\omega) = \frac{1}{1 + j(\omega/\omega_c)}$$

Analyze the response of low pass filter

$$\frac{V_o}{V_i}(j\omega) = \frac{1}{1 + j(\omega/\omega_c)}$$



Both the phase and magnitude of V_o/V_i will change when ω is allowed to vary.

Magnitude

When $\omega \to 0$:

$V_o/V_i \to$ ________ ⇨ $|V_o/V_i| \to 1$

When $\omega \to$ Infinity:

$V_o/V_i \to$ ________ ⇨ $|V_o/V_i| \to 0$

Phase

When $\omega \to 0$:

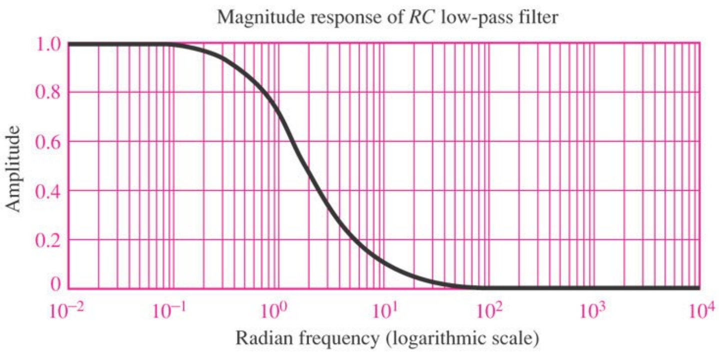
$V_o/V_i \to$ ________ ⇨ $\angle(V_o/V_i) \to 0°$

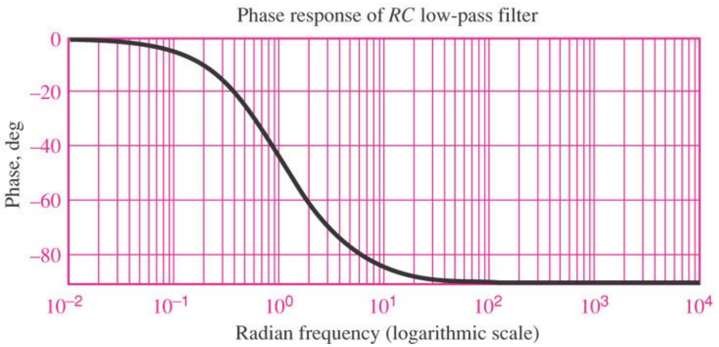
When $\omega \to$ Infinity:

$V_o/V_i \to$ ________ ⇨ $\angle(V_o/V_i) \to -90°$

Sketch the response of low pass filter

Allows lower frequency signals to pass and filters off higher frequency signals





Observations:

When ω approaches zero, magnitude of V_o/V_i approaches 1 and its phase is close to zero

When ω becomes large, magnitude of V_o/V_i approaches zero and its phase is close to $-\pi/2$

Allows lower frequency signal to pass and filters off higher frequency signals

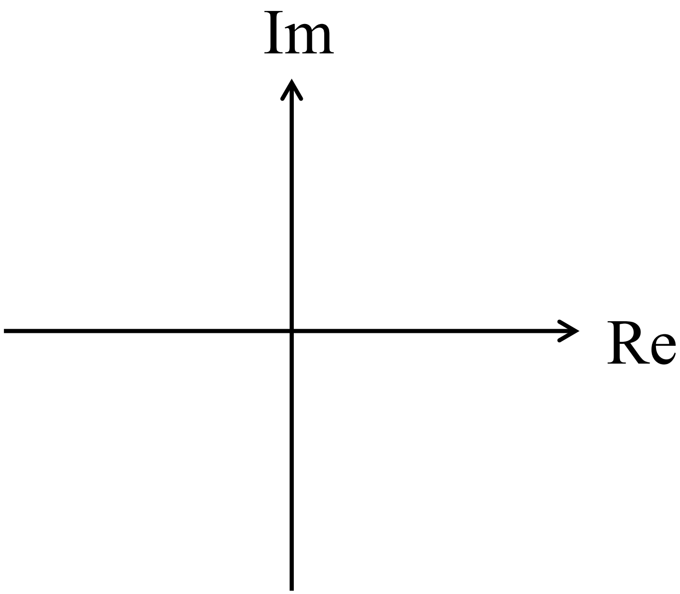
What about in between these two extremes, around ω_c?

The above graphs are presentations of semi-log plots

Semi-log plots: y-axis follows a linear scale, x-axis follows a logarithmic scale

Logarithmic scale (base 10): Between each interval on axis, we increase/decrease by a factor of 10 (it is the power/index that changes)

At the cut off frequency



Plot of denominator for V_o/V_i

$$\frac{V_o}{V_i}(j\omega) = \frac{1}{1 + j(\omega/\omega_c)}$$

At $\omega = \omega_c$:

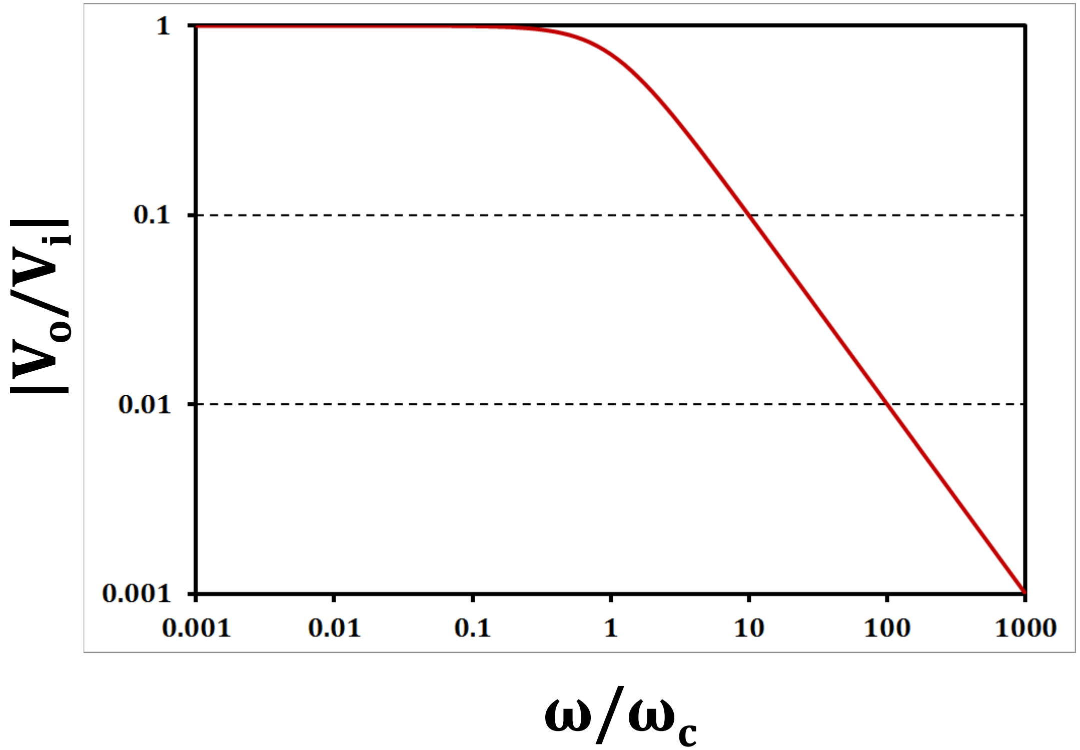
Denominator:

Denominator as a phasor: ___________

$$\frac{V_o}{V_i} =$$

At $\omega = \omega_c$, V_o/V_i drops to $1/\sqrt{2}$ of the maximum and has a phase of -45^o

Log-Log Plot for Magnitude (Low Pass)



Two parts of the curve:

When $\omega << \omega_c$:
$|V_o/V_i|$ stays flat close to 1

When $\omega >> \omega_c$:
$|V_o/V_i|$ decreases with ω;
x10 time reduction for every x10 increase in ω

Change of $|V_o/V_i|$ with ω seen as a linear slope on the log-log plot

Change between the 2 parts occurs at ω_c

Log-log plot: **Both** the y-axis and x-axis are on logarithmic scales (base 10). This means moving by 1 interval on either axis, the value increases or decreases by a factor of 10.
Note that on a log scale, one never arrives at zero/infinity.

Bode plot for Low Pass

Bode plot typically comes as a **pair of graphs:**

(1) Log-log plot of magnitude ratio of V_o/V_i vs. frequency

(Log-log plot: Both x and y axes are on logarithmic scales)

(2) Semi-log plot of the phase of V_o/V_i vs. frequency

(Semi-log plot: Linear scale for y-axis (Phase) and log scale for x-axis)

Magnitude

$$\boxed{\frac{V_o}{V_i}(j\omega) = \frac{1}{1 + j\,\omega/\omega_c}}$$

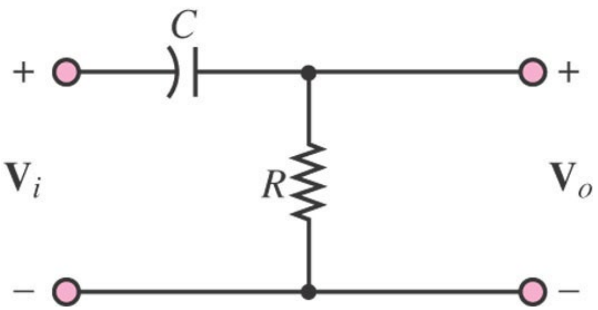
Phase

High pass filter

Let us consider the response of the output V_o in relation to the input V_i. We keep the amplitude of V_i constant but vary its frequency ω.

By voltage divider rule: $$\frac{V_o}{V_i}(j\omega)=\frac{R}{R+1/j\omega C}$$

$$=\frac{1}{1+1/j\omega CR}$$

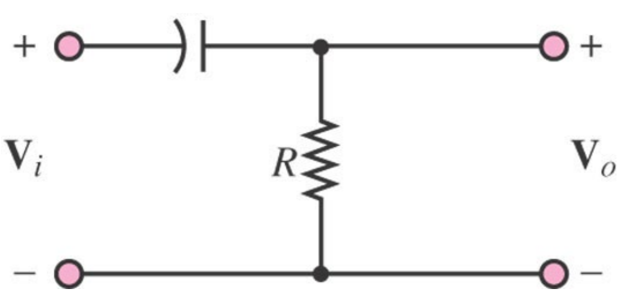


Once again we re-write RC to define the cut off radian frequency, ω_c whereby $\omega_c = 1/RC$:

Sub back into above equation: $$\frac{V_o}{V_i}(j\omega)=\frac{1}{1+(\omega_c/j\omega)}$$

Analyze response of high pass filter

$$\frac{V_o}{V_i}(j\omega) = \frac{j(\omega/\omega_c)}{1 + j(\omega/\omega_c)}$$



Both the phase and magnitude of V_o/V_i will change when ω is allowed to vary.

<u>Magnitude</u>

When $\omega \to 0$:

$V_o/V_i \to$ ____________ ⇨ $|V_o/V_i| \to 0$

When $\omega \to$ Infinity:

$V_o/V_i \to$ ____________ ⇨ $|V_o/V_i| \to 1$

<u>Phase</u>

When $\omega \to 0$:

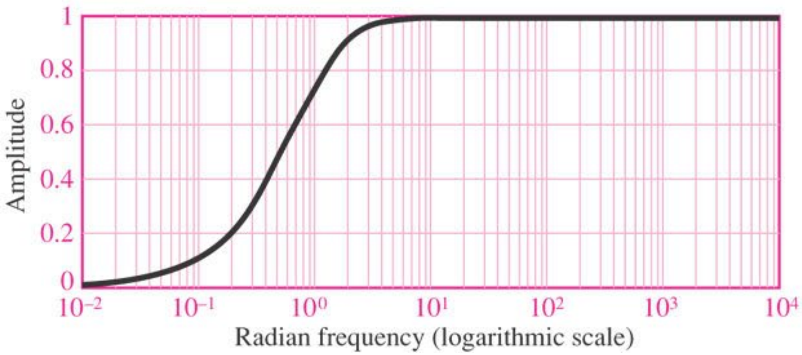
$V_o/V_i \to$ ____________ ⇨ $\angle(V_o/V_i) \to 90°$

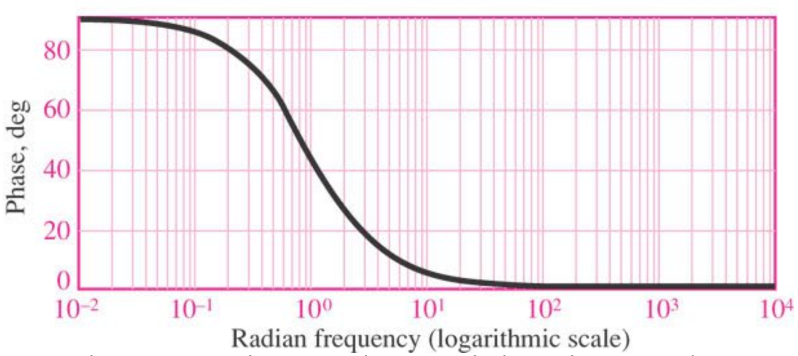
When $\omega \to$ Infinity:

$V_o/V_i \to$ ____________ ⇨ $\angle(V_o/V_i) \to 0°$

Sketch response of high pass filter

Allows higher frequency signals to pass and filters off lower frequency signals





Semi-log plots: y-axis on linear scale, x-axis on logarithmic scale

Observations:

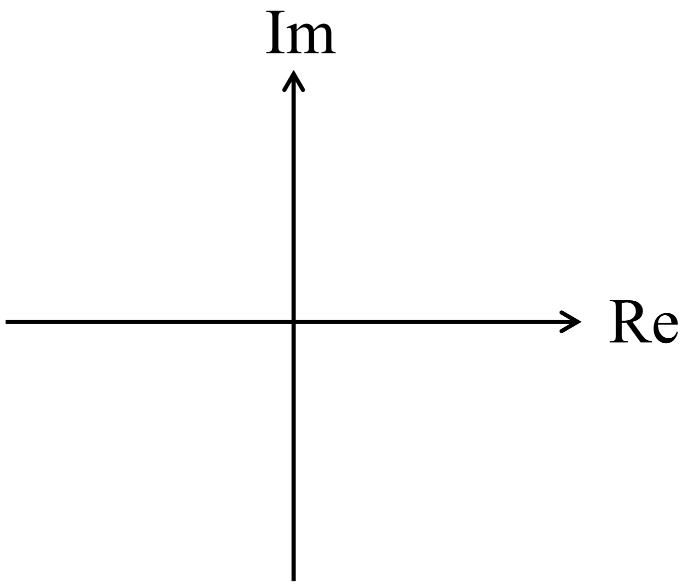
When ω approaches zero, V_o/V_i approaches zero and phase is close to $\pi/2$

When ω becomes large, V_o/V_i approaches 1 and phase is close to 0

Allows higher frequency signals to pass and filters off lower frequency signals

What about in between these two extremes, around ω_c?

At the cut off frequency



Plot of denominator for V_o/V_i

$$\frac{V_o}{V_i}(j\omega) = \frac{1}{1+(\omega_c / j\omega)}$$

At $\omega = \omega_c$:

Denominator:

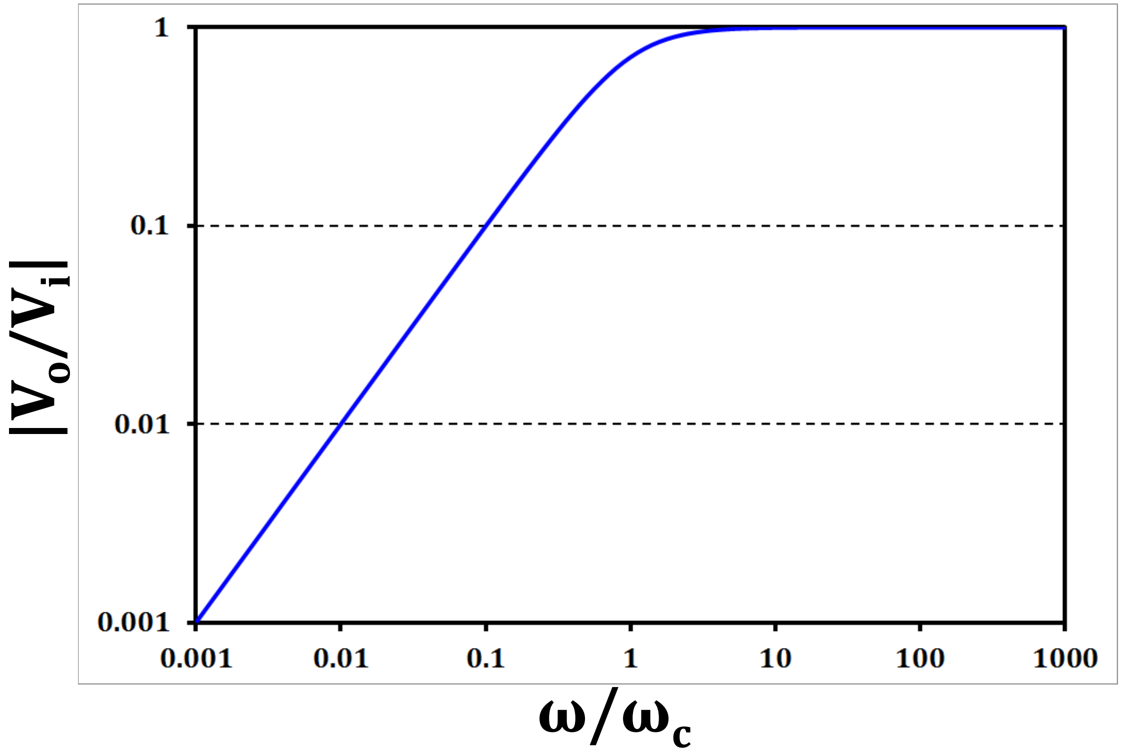
Denominator as a phasor: ___________

Note that numerator adds a phase shift of 90° from j

$$\frac{V_o}{V_i} =$$

At $\omega = \omega_c$, V_o/V_i is once again $1/\sqrt{2}$ of the maximum and has a phase of 45^o

Log-Log Plot for $|V_o/V_i|$ (High Pass)



When $\omega >> \omega_c$:
$|V_o/V_i|$ stays flat close to 1

When $\omega << \omega_c$:
$|V_o/V_i|$ decreases with ω;
x10 time reduction for every x10 reduction in ω

Seen as a linear slope on the log-log plot

Bode plot for High Pass

Bode plot typically comes as a **pair of graphs:**

(1) Log-log plot of magnitude ratio of V_o/V_i vs. frequency

(Log-log plot: Both x and y axes are on logarithmic scales)

(2) Semi-log plot of the phase of V_o/V_i vs. frequency

(Semi-log plot: Linear scale for y-axis (Phase) and log scale for x-axis)

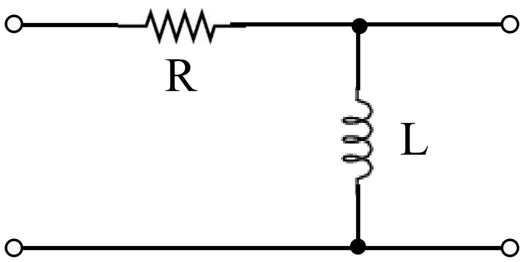
Magnitude

$$\boxed{\frac{V_o}{V_i}(j\omega) = \frac{1}{1 + \omega_c / j\omega}}$$

Phase

Other examples on filters 1

Determine the frequency response characteristics (low pass or high pass) for the following filter circuits. Hence draw the bode plot of the filter.



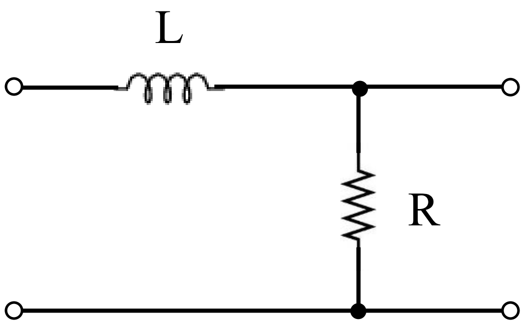
Bode Plots

Magnitude

Phase

Other examples on filters 2

Determine the frequency response characteristics (low pass or high pass) for the following filter circuits. Hence draw the bode plot of the filter.



Bode Plots

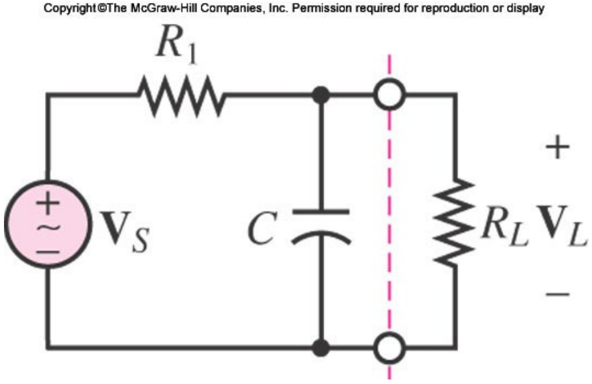
Magnitude

Phase

Frequency response example 1

Compute the frequency response of V_L/V_s for the following circuit:

$R_1 = 1\text{k}\Omega$; $C = 10\mu\text{F}$; $R_L = 10\text{k}\Omega$



First find the combined impedance of the capacitor C and resistor R_L in parallel:

$$Z_{RC} = \frac{R_L}{(1 + j\omega R_L C)} = \frac{10}{(1 + j0.1\omega)}\ \text{k}\Omega$$

Now apply voltage divider rule:

$$\frac{V_L}{V_S}(j\omega) = \frac{Z_{RC}}{Z_{RC} + R_1} = \frac{10/(1 + j0.1\omega)}{[10/(1 + j0.1\omega)] + 1}$$

$$= \frac{10}{11 + j0.1\omega} = \frac{100}{110 + j\omega}$$

Frequency response example 1

$$\frac{V_L}{V_S}(j\omega)=\frac{100}{110+j\omega}=\left(\frac{100}{110}\right)\left(\frac{1}{1+j\omega/110}\right)$$

When $\omega \to 0$:

$V_o/V_i \to$

$|V_o/V_i| \to$

$\angle(V_o/V_i) \to$

When $\omega \to$ Infinity:

$V_o/V_i \to$

$|V_o/V_i| \to$

$\angle(V_o/V_i) \to$

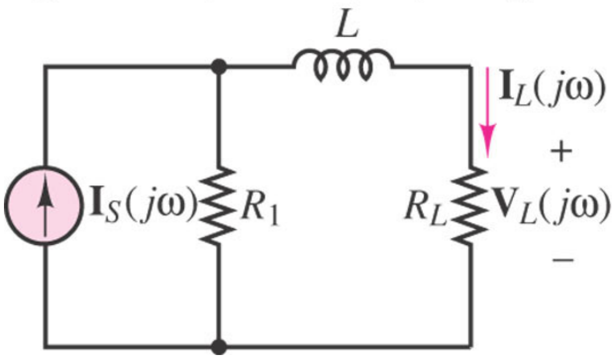
Magnitude

Phase

Frequency response example 2

Compute the frequency response of V_L/I_S for the following circuit:

R_1 = 1kΩ; L = 2mH; R_L = 4kΩ



First find the combined impedance of the inductor and resistor in series:

$$Z_{RL} = R_L + j\omega L = 4000 + j\left(2\times10^{-3}\right)\omega\ \Omega$$

Now apply current divider rule:

$$\frac{V_L}{I_S}(j\omega) = \left(\frac{I_L}{I_S}\right)R_L = \left(\frac{R_1}{R_1 + Z_{RL}}\right)R_L$$

$$= \frac{(1000)(4000)}{1000 + 4000 + j\left(2\times10^{-3}\right)\omega} = \frac{4\times10^{6}}{5000 + j\left(2\times10^{-3}\right)\omega}$$

$$= \frac{800}{1 + j\left(4\times10^{-7}\right)\omega}$$

Frequency response example 2

$$\frac{V_L}{I_S}(j\omega) = \frac{800}{1 + j\left(4\times10^{-7}\right)\omega}$$

When $\omega \to 0$:

$V_o/V_i \to$

$|V_o/V_i| \to$

$\angle(V_o/V_i) \to$

When $\omega \to$ Infinity:

$V_o/V_i \to$

$|V_o/V_i| \to$

$\angle(V_o/V_i) \to$

Magnitude

Phase

General form for low pass filter

$$\frac{V_o}{V_i}(j\omega) = A\left[\frac{1}{1 + j(\omega/\omega_c)}\right]$$

A is a constant and real number

A defines the magnitude of V_o/V_i in the **pass band**

Magnitude

Phase

General form for high pass filter

$$\frac{V_o}{V_i}(j\omega) = A\left[\frac{j(\omega/\omega_c)}{1 + j(\omega/\omega_c)}\right]$$

A is a constant and real number

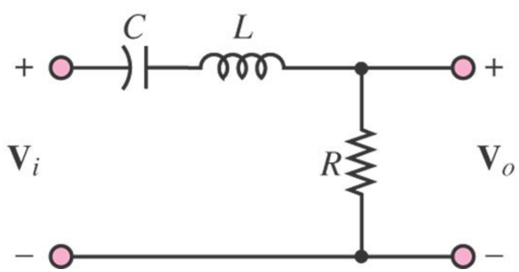
A defines the magnitude of V_o/V_i in the **pass band**

Magnitude

Phase

RLC Series Resonator (Bandpass filter)

RLC bandpass filter. The circuit preserves frequencies within a band.

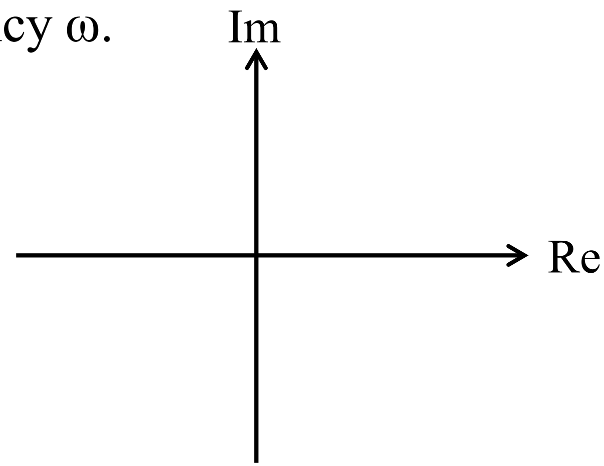


Let us consider the response of the output V_o in relation to the input V_i. We keep the amplitude of V_i constant but vary its frequency ω.

By voltage divider rule:

$$\frac{V_o}{V_i}(j\omega) = \frac{R}{R + j\omega L + 1/j\omega C}$$

$$= \frac{1}{1 + j\left(\frac{\omega L}{R} - \frac{1}{\omega RC}\right)}$$



Plot of denominator for V_o/V_i

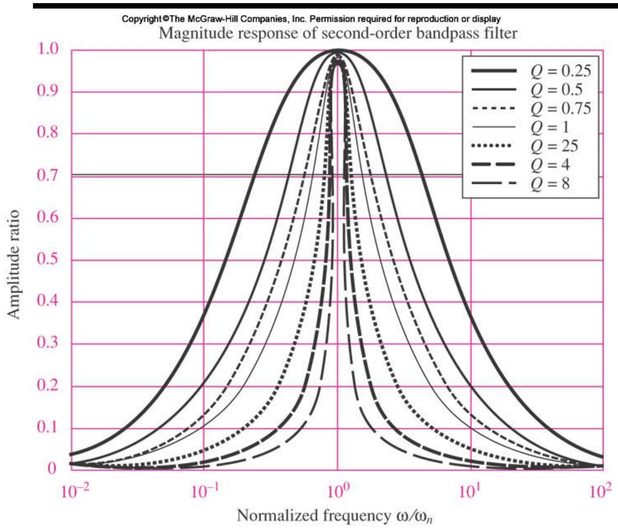
$|V_o/V_i|$ is max when denominator is minimized

We can see that the output will be at its maximum when the imaginary part of the denominator is zero:

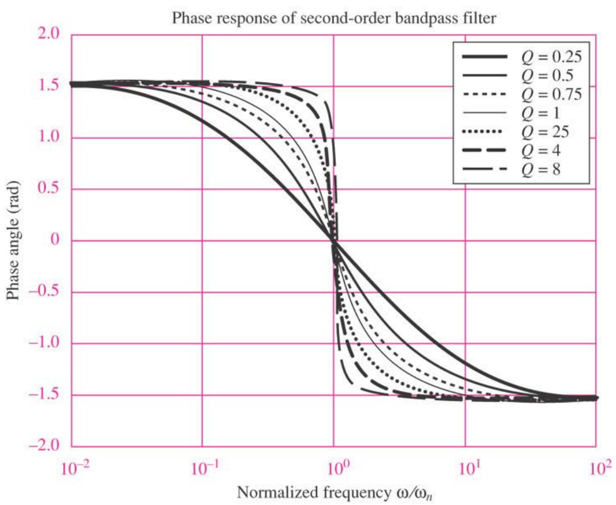
$(\omega L/R) – [1/(\omega RC)] = 0 \rightarrow \omega^2 = 1/(LC)$

When this happens, the impedances of the capacitor and inductor are equal and opposite. This is known as resonance. Max value of V_o/V_i for all frequencies is 1 in this case.

Quality factor



(a)



(b)

(1) There is no "flat" part in the frequency response curve

(2) Response peaks at one frequency: $\omega = 1/\sqrt{(LC)}$, this is known at the resonance frequency, ω_0

(3) For frequencies move further away from ω_0 (whether higher or lower), $|V_o/V_i|$ gets increasingly smaller

$$\frac{V_o}{V_i}(j\omega) = \frac{j\omega CR}{1-\omega^2 LC + j\omega CR}$$

Make the subst. using: $\omega_0 = \dfrac{1}{\sqrt{LC}}$

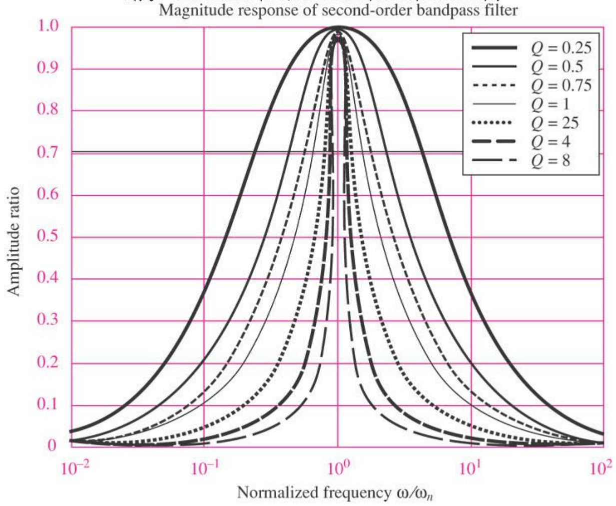
$$\frac{V_o}{V_i}(j\omega) = \frac{j\left(\dfrac{\omega}{\omega_0}\right)\dfrac{R}{\sqrt{L/C}}}{1-\left(\dfrac{\omega}{\omega_0}\right)^2 + j\left(\dfrac{\omega}{\omega_0}\right)\dfrac{R}{\sqrt{L/C}}}$$

If we define: $Q = \dfrac{\sqrt{L/C}}{R}$

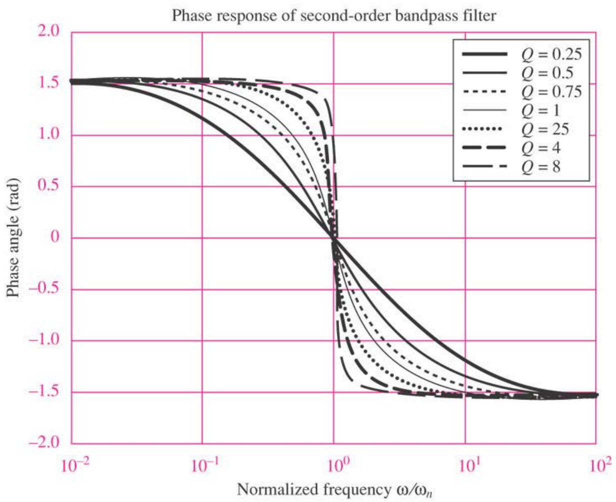
$$\frac{V_o}{V_i}(j\omega) = \frac{j(\omega/\omega_0)(1/Q)}{1-(\omega/\omega_0)^2 + j(\omega/\omega_0)(1/Q)}$$

Q is known as the **Quality Factor** and it describes the sharpness or width of the peak relative to ω_0

Frequency response



(a)



(b)

As Q increases, the resonance peak becomes sharper

In this sense, the resonator only responds at the resonance frequency, ω_0

Frequencies outside ω_0 are filtered out

The resonator works like frequency selector

$$\frac{V_o}{V_i}(j\omega) = \frac{j(\omega/\omega_0)(1/Q)}{1-(\omega/\omega_0)^2 + j(\omega/\omega_0)(1/Q)} \qquad \frac{V_o}{V_i}(j\omega) = \frac{R}{R + j\omega L + 1/j\omega C}$$

Magnitude

When $\omega \to 0$ or $\to$ Infinity: $|V_o/V_i| \to 0$

When $\omega = \omega_0$: $|V_o/V_i| = 1$

Phase

When $\omega \to 0$: $\angle(V_o/V_i) = 90°$ since $V_o/V_i \to j\omega CR$

When $\omega \to$ Infinity: $\angle(V_o/V_i) = -90°$ since $V_o/V_i \to R/j\omega L$

When $\omega = \omega_0$: $\angle(V_o/V_i) = 0$ since $V_o/V_i \to 1$