# In-Class Exercise 10

1. Determine the Laplace transform of

$$x(t) = e^{2t}u(-t+2)$$

   Specify the region of convergence (ROC) and find all poles of $X(s)$.

2. Determine the Laplace transform of

$$x(t) = \begin{cases} e^{t}\sin(2t), & t \le 0 \\ 0, & t > 0 \end{cases}$$

   Specify its ROC. Find all the pole(s).

3. Determine the Laplace transform of

$$x(t) = e^{-t}u(t) \otimes \sin(3\pi t)u(t)$$

Specify its ROC. Find all the pole(s).

4. Consider an absolutely integrable signal $x(t)$. Its Laplace transform $X(s)$ is a rational function and is known to have a pole at $s = 2$. $X(s)$ may have other poles. Answer the following questions:

(a) Can $x(t)$ be of finite duration? Why?
(b) Can $x(t)$ be left-sided? Why?
(c) Can $x(t)$ be right-sided? Why?
(d) Can $x(t)$ be two-sided? Why?

5. Prove the convolution property of Laplace transform:

$$x(t) \otimes y(t) \leftrightarrow X(s)Y(s)$$

6. Let

$$g(t) = x(t) + \alpha x(-t)$$

where

$$x(t) = \beta e^{-t} u(t)$$

It is known that the Laplace transform of $g(t)$ is:

$$G(s) = \frac{s}{s^2 - 1}, \quad -1 < \Re\{s\} < 1$$

Determine the values of $\alpha$ and $\beta$.

7. Consider a signal $y(t)$ which is related to two signals $x_1(t)$ and $x_2(t)$ by

$$y(t) = x_1(t-2) \otimes x_2(-t+3)$$

where

$$x_1(t) = e^{-2t}u(t) \quad \text{and} \quad x_2(t) = e^{-3t}u(t)$$

Determine the Laplace transform of $y(t)$.

8. Determine the inverse Laplace transform of

$$X(s) = \frac{2(s+2)}{s^2+7s+12}, \quad \Re\{s\} > -3$$

## Solution

1.

There are two directions to find $X(s)$. One is to directly apply (9.1) and the second is to make use of relevant properties. Here we consider the latter only.

Consider:

$$y(t)=e^{-2t}u(t) \leftrightarrow Y(s)=\frac{1}{s+2}, \quad \Re\{s\}>-2$$

According to the time shifting property, we have:

$$y(t+2)=e^{-2(t+2)}u(t+2) \leftrightarrow \frac{e^{2s}}{s+2}, \quad \Re\{s\}>-2$$

According to the time reversal property, we have:

$$y(-t+2) = e^{-2(-t+2)}u(-t+2) \leftrightarrow \frac{e^{-2s}}{-s+2}, \quad \Re\{s\} < 2$$

Hence

$$e^{-2(-t+2)}u(-t+2) = e^{-4}e^{2t}u(-t+2) \leftrightarrow \frac{e^{-2s}}{-s+2}$$
$$\Rightarrow e^{2t}u(-t+2) \leftrightarrow \frac{e^{4}e^{-2s}}{-s+2}, \quad \Re\{s\} < 2$$

The pole is at $s = 2$.

2.

Using the Euler formula, we have:

$$e^{t}\sin(2t)=\frac{1}{2j}\left[e^{(1+j2)t}-e^{(1-j2)t}\right]$$

For the first component, we have:

$$\begin{aligned}X_1(s) &= \int_{-\infty}^{0} e^{(1+j2)t}e^{-st}dt=\int_{-\infty}^{0} e^{(1+j2-s)t}dt\\ &= \frac{1}{1+j2-s}e^{(1+j2-s)t}\Big|_{-\infty}^{0}\\ &= \frac{1}{1+j2-s}, \quad \Re\{s\}<1\end{aligned}$$

Similarly, the second component is:

$$X_2(s) = -\frac{1}{1-j2-s}, \quad \Re\{s\} < 1$$

Combining the results yields:

$$\begin{aligned} X(z) &= \frac{1}{2j}\left[\frac{1}{1+j2-s} + \frac{1}{1-j2-s}\right] \\ &= \frac{-2}{(1-s)^2+4}, \quad \Re\{s\} < 1 \end{aligned}$$

The poles are $1-j2$ and $1+j2$.

3.

Using Table 9.1, we have:

$$e^{-t}u(t) \leftrightarrow \frac{1}{s+1}, \quad \Re\{s\} > -1$$

$$\sin(3\pi t)u(t) \leftrightarrow \frac{3\pi}{s^2+(3\pi)^2}, \quad \Re\{s\} > 0$$

According to the convolution property, $X(s)$ is:

$$X(s) = \frac{3\pi}{(s+1)(s^2+9\pi^2)}, \quad \Re\{s\} > 0$$

The poles are $-1$, $3\pi j$ and $-3\pi j$.

4.(a)

No. For a finite-duration signal, the ROC is the entire $s$-plane. On the other hand, the ROC cannot contain pole at $s=2$.

4.(b)

Yes. Since the signal is absolutely integrable, the ROC must include the $j\Omega$-axis. Furthermore, $X(s)$ has a pole at $s=2$. Therefore, one valid ROC is $\Re\{s\}<2$, which corresponds to a left-sided signal.

4.(c)

No. Since the signal is absolutely integrable, the ROC must include the $j\Omega$-axis. Furthermore, $X(s)$ has a pole at $s=2$. If it is right-sided, then the ROC should be $\Re\{s\} > \alpha$ where $\alpha < 0$ is a real number. But $\Re\{s\} > \alpha$ will include the pole of $s=2$, which is not valid.

4.(d)

Yes. Since the signal is absolutely integrable, the ROC must include the $j\Omega$-axis. Furthermore, $X(s)$ has a pole at $s=2$. Therefore, one valid ROC is $\alpha < \Re\{s\} < 2$, where $\alpha$ is a real number, which corresponds to a two-sided signal.

5.

$$\int_{-\infty}^{\infty} x(t) \otimes y(t) e^{-st} dt = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau) y(t-\tau) e^{-st} d\tau dt$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau) y(u) e^{-s\tau} e^{-su} d\tau du, \quad u = t - \tau$$

$$= \left[ \int_{-\infty}^{\infty} x(\tau) e^{-s\tau} d\tau \right] \cdot \left[ \int_{-\infty}^{\infty} y(u) e^{-su} du \right]$$

$$= X(s) \cdot Y(s)$$

6.

Using the time reversal property, we have:

$$g(t) = x(t) + \alpha x(-t) \leftrightarrow G(s) = X(s) + \alpha X(-s)$$

The Laplace transform of $x(t)$ is:

$$X(s) = \beta \cdot \frac{1}{s+1}, \quad \Re\{s\} > -1$$

We then have:

$$X(s) + \alpha X(-s) = \frac{\beta}{s+1} + \frac{\alpha\beta}{-s+1} = \beta \left[ \frac{s(1-\alpha) - (1+\alpha)}{s^2 - 1} \right] = \frac{s}{s^2-1}$$

Hence we get $\alpha = -1$ and $\beta = 0.5$.

7.

The Laplace transforms of $x_1(t)$ and $x_2(t)$ are

$$X_1(s) = \frac{1}{s+2}, \quad \Re\{s\} > -2$$
$$X_2(s) = \frac{1}{s+3}, \quad \Re\{s\} > -3$$

Using the time shifting and time reversal properties of Laplace transform, we obtain

$$x_1(t-2) \leftrightarrow e^{-2s}X_1(s) = \frac{e^{-2s}}{s+2}, \quad \Re\{s\} > -2$$
$$x_2(t+3) \leftrightarrow e^{3s}X_2(s)$$
$$\Rightarrow x_2(-t+3) \leftrightarrow e^{3(-s)}X_2(-s) = \frac{e^{-3s}}{-s+3}, \quad \Re\{s\} < 3$$

Finally, we have:

$$Y(s) = -\frac{e^{-5s}}{(s+2)(s-3)}, \quad 3 > \Re\{s\} > -2$$

8.

By means of partial fraction expansion, we get:

$$X(s) = \frac{2(s+2)}{s^2+7s+12} = \frac{4}{s+4} - \frac{2}{s+3}, \quad \Re\{s\} > -3$$

Taking the inverse Laplace transform, we have

$$x(t) = 4e^{-4t}u(t) - 2e^{-3t}u(t)$$