Tutorial 1
Deterministic Signal Analysis

Problem 1 (Fourier Spectrum)

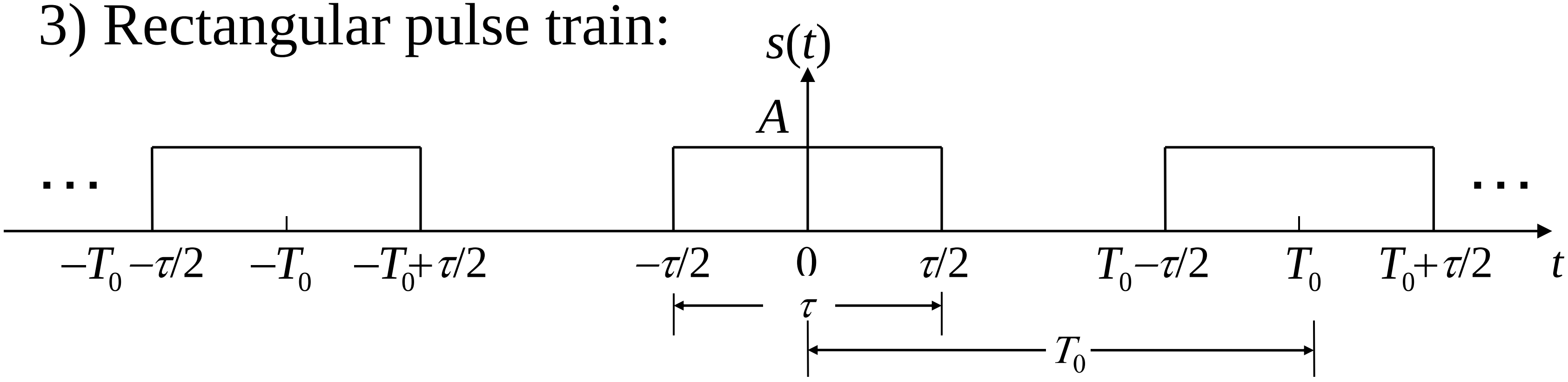
Derive the Fourier spectrum of the following signals:

1) Truncated sinusoidal signal: $s(t)=\begin{cases} A\cos 2\pi f_0 t & |t|\le \tau/2 \\ 0 & \text{elsewhere} \end{cases}$

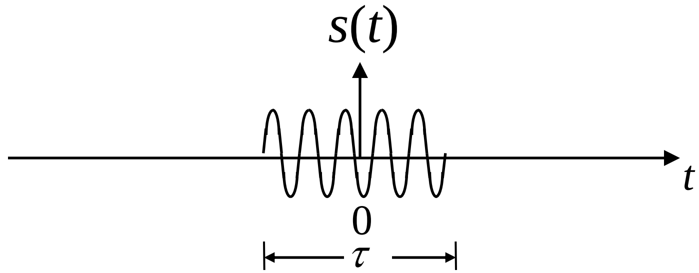
where f_0 is an integer multiple of $1/\tau$.

2) Sinc-shaped pulse: $s(t)=A\operatorname{sinc}(t/\tau)$

3) Rectangular pulse train:



Solution: Spectrum of Truncated Sinusoidal Signal



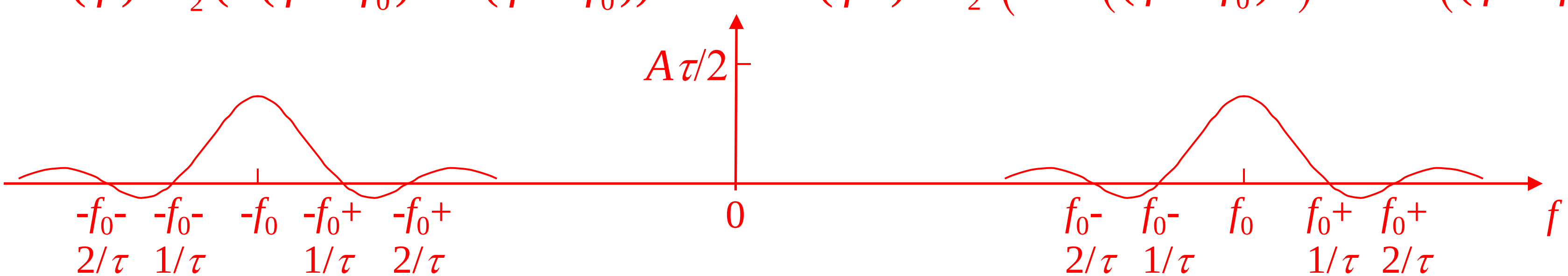
$$s(t) = \begin{cases} A\cos 2\pi f_0 t & |t| \le \tau/2 \\ 0 & \text{elsewhere} \end{cases}$$

- $s(t)$ can be written as $s(t) = \cos 2\pi f_0 t \cdot x(t)$

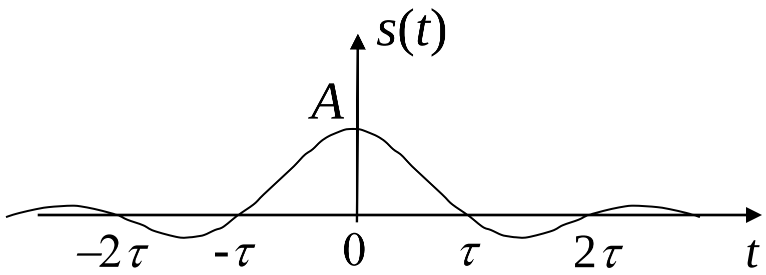
  where $x(t)$ is a single rectangular pulse: $x(t) = \begin{cases} A & |t| \le \tau/2 \\ 0 & \text{otherwise} \end{cases}$

- According to $\cos 2\pi f_0 t \Leftrightarrow \frac{1}{2}(\delta(f - f_0) + \delta(f + f_0))$ and $x(t) \Leftrightarrow A\tau \text{sinc}(f\tau)$

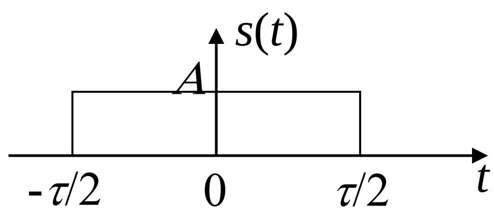
$$S(f) = \tfrac{1}{2}(\delta(f - f_0) + \delta(f + f_0)) * A\tau \text{sinc}(f\tau) = \tfrac{A\tau}{2}\Big(\text{sinc}\big((f - f_0)\tau\big) + \text{sinc}\big((f + f_0)\tau\big)\Big)$$



Solution: Spectrum of Sinc-shaped Pulse



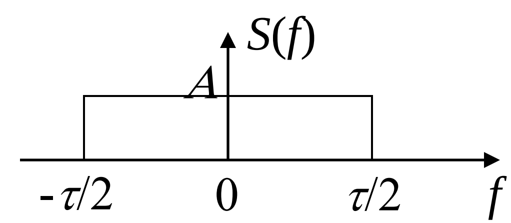
$$s(t) = A\mathrm{sinc}(t/\tau)$$



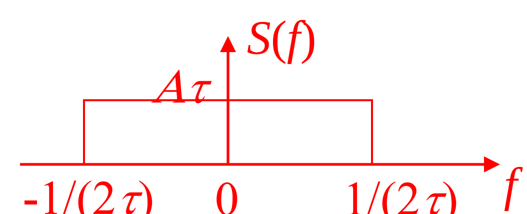
$$\Leftrightarrow \quad S(f) = A\tau\mathrm{sinc}(f\tau)$$

Duality: $S(t) \Leftrightarrow s(-f)$

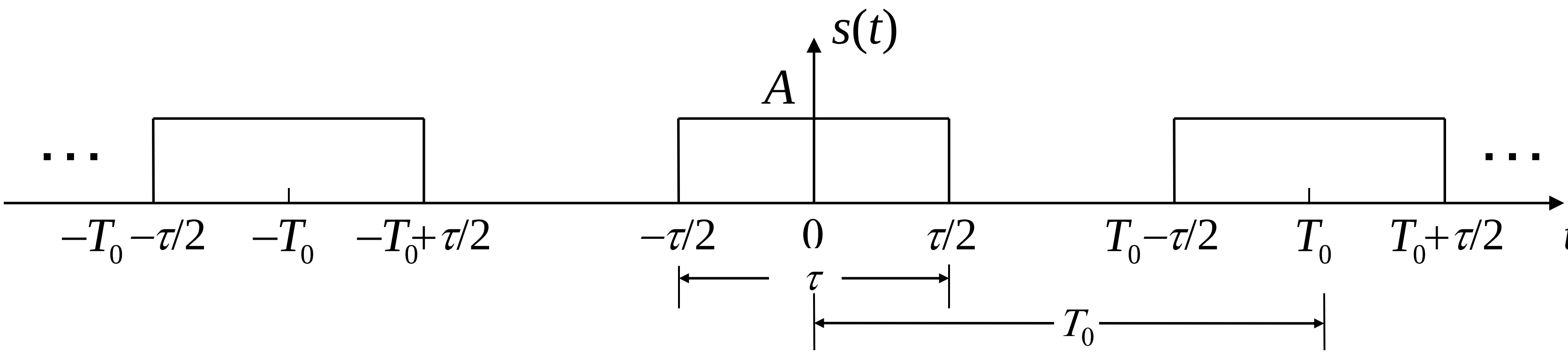
$$s(t) = A\tau\mathrm{sinc}(t\tau) \quad \Leftrightarrow$$



$$s(t) = A\mathrm{sinc}(t/\tau) = A\tau \cdot \frac{1}{\tau}\mathrm{sinc}\left(t \cdot \frac{1}{\tau}\right) \quad \Leftrightarrow$$



Solution: Spectrum of Rectangular Pulse Train

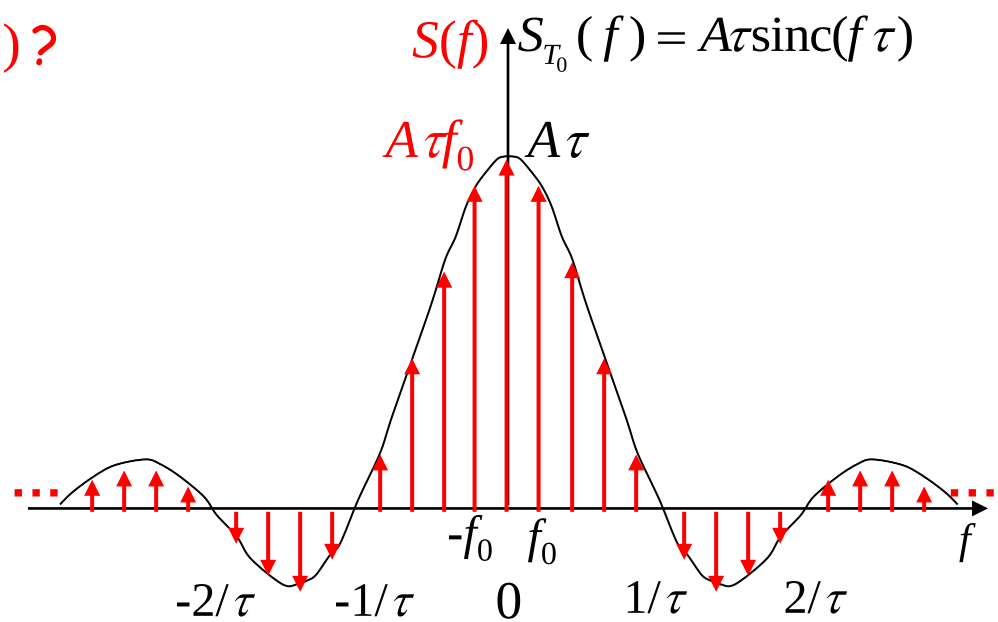


- What is the spectrum of $s_{T_0}(t)$?

$$S_{T_0}(f) = A\tau \operatorname{sinc}(f\tau)$$

- $$S(f) = f_0 \sum_{n=-\infty}^{\infty} S_{T_0}(nf_0)\delta(f - nf_0)$$

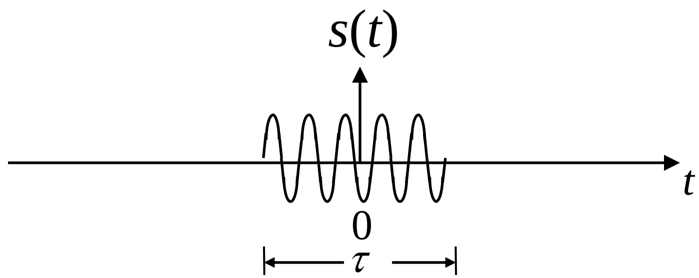
$$= A\tau f_0 \sum_{n=-\infty}^{\infty} \operatorname{sinc}(\tau n f_0)\delta(f - nf_0)$$



Problem 2 (Energy/Power Spectrum)

Determine whether the signals given in Problem 1 are power-type or energy-type signals. For energy-type signals, determine the signal energy and the energy spectrum. For power-type signals, determine the signal power and the power spectrum.

Solution: Energy Spectrum of Truncated Sinusoidal Signal



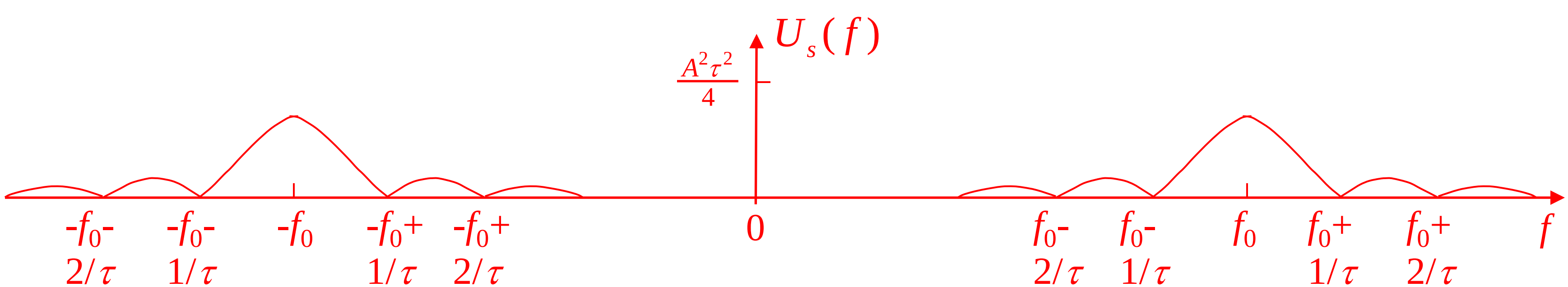
$$s(t) = \begin{cases} A\cos 2\pi f_0 t & |t| < \tau/2 \\ 0 & \text{elsewhere} \end{cases}$$

- Signal energy: $E_s = \int_{-\infty}^{\infty} |s(t)|^2\, dt = \int_{-\tau/2}^{\tau/2} A^2 \cos^2 2\pi f_0 t dt = A^2\tau/2 < \infty$

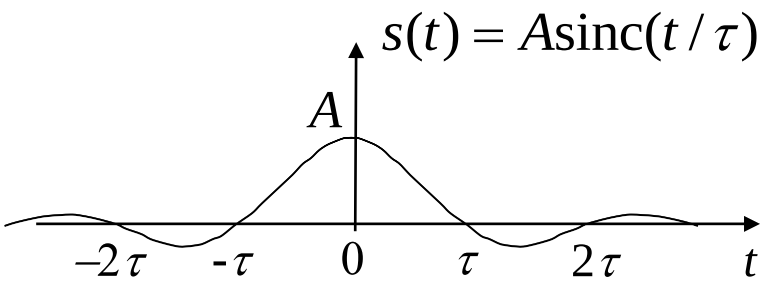
  f_0 is an integer multiple of $1/\tau$

- $s(t)$ is an energy-type signal.

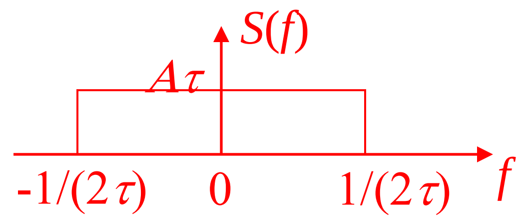
- Energy spectrum: $U_s(f) = |S(f)|^2 = \frac{A^2\tau^2}{4}\left(\text{sinc}\left((f-f_0)\tau\right) + \text{sinc}\left((f+f_0)\tau\right)\right)^2$



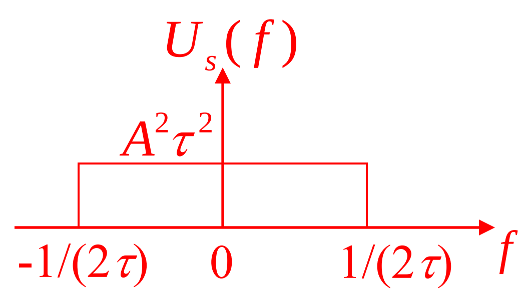
Solution: Energy Spectrum of Sinc-shaped Pulse



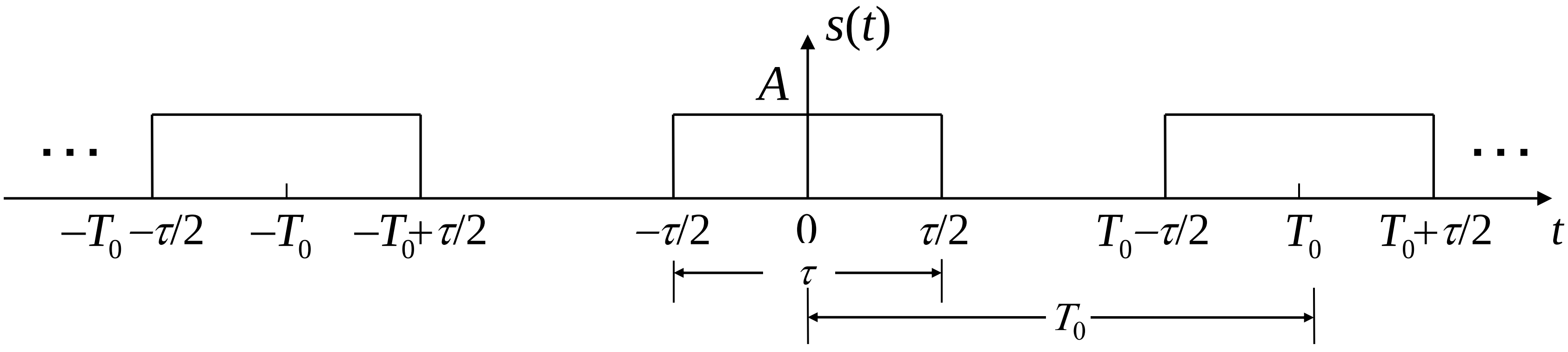
$\Leftrightarrow$



- Signal energy: $E_s = \int_{-\infty}^{\infty} |S(f)|^2 \, df = \int_{-\frac{1}{2\tau}}^{\frac{1}{2\tau}} A^2 \tau^2 df = A^2 \tau < \infty$
- $s(t)$ is an energy-type signal.
- Energy spectrum: $U_s(f) = |S(f)|^2$



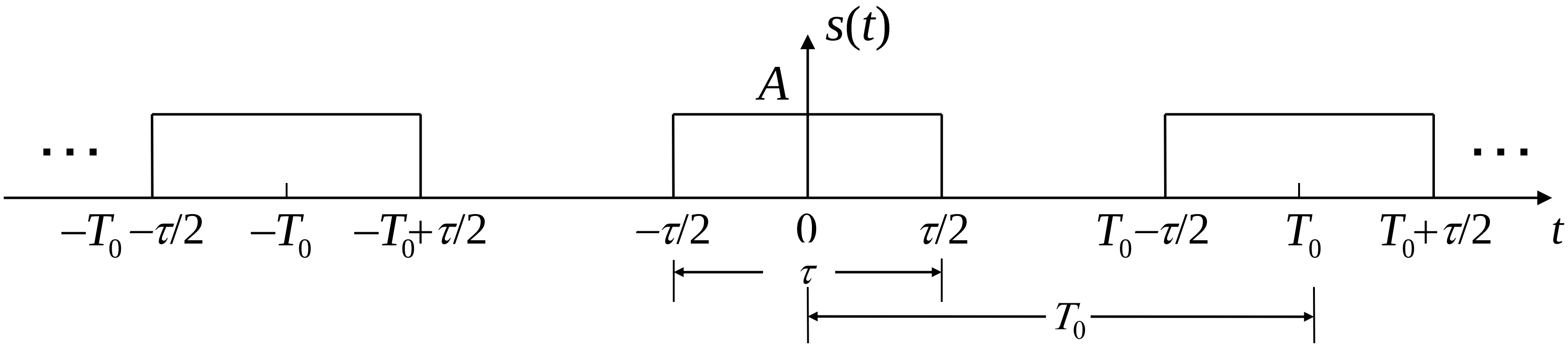
Solution: Power Spectrum of Rectangular Pulse Train



- $s(t)$ is a periodic signal.
- $s(t)$ is a power-type signal.
- Power of $s(t)$:

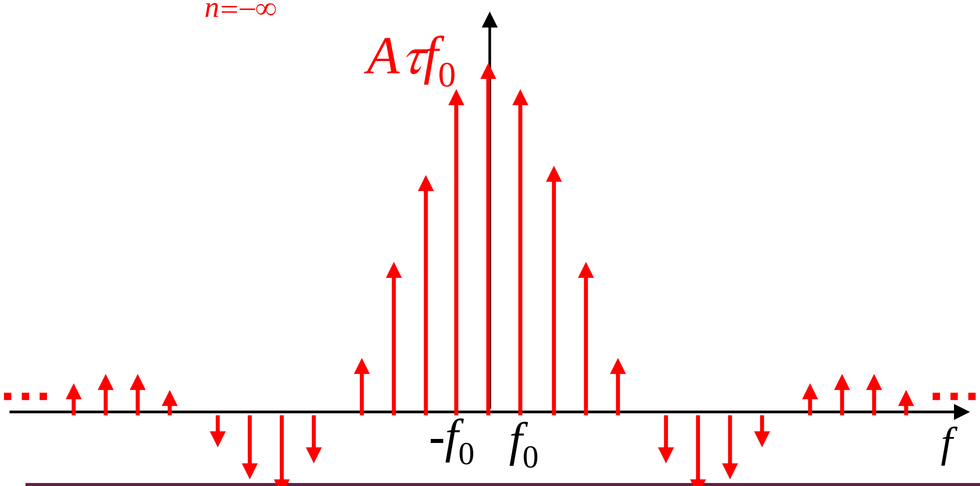
$$P_s = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}|s(t)|^2\,dt = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2}|s(t)|^2\,dt = \frac{1}{T_0}\int_{-\tau/2}^{\tau/2}A^2dt = A^2\tau/T_0$$

Solution: Power Spectrum of Rectangular Pulse Train



- Fourier Spectrum:

$$S(f)=\sum_{n=-\infty}^{\infty} A\tau f_0 \operatorname{sinc}(\tau n f_0)\delta(f-nf_0)$$



- Power Spectrum:

$$G_s(f)=\sum_{n=-\infty}^{\infty} A^2\tau^2 f_0^2 \operatorname{sinc}^2(\tau n f_0)\delta(f-nf_0)$$