MA2001 Test …023

Name: Ng Chung Wah

Student ID:

Answer ALL questions. (Full marks: 100)

---

**Question 1 (15 marks)**

Evaluate the following.

$\int_0^{\pi}\int_x^{\pi} \frac{\sin y}{y}\,dydx$ by change of order of integration.

**Question 2 (30 marks)**

$\begin{cases} x^2 + y^2 = 2z^2 \\ x + y + z = 1 \end{cases}$ for $\begin{cases} x = 1 \\ y = -1 \\ z = 1 \end{cases}$.

Find $\frac{dx}{dz}$, $\frac{dy}{dz}$, $\frac{d^2x}{dz^2}$ and $\frac{d^2y}{dz^2}$.

**Question 3 (40 marks)**

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$

(a) find the eigenvalues of $A$ and one of them is 14;

(b) find the eigenvector $\underset{\sim}{x}_1$ with respect to the eigenvalue 14;

(c) show that $\underset{\sim}{x}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ is one of the eigenvector of $A$;

(d) find the third eigenvector $\underset{\sim}{x}_3$ which is orthogonal to $\underset{\sim}{x}_1$, and $\underset{\sim}{x}_2$;

(e) find a matrix $P$ having its columns to be the normalized eigenvectors of $A$, and show that $P^T P = I$.

(f) show that $A$ is diagonalizable.

**Question 4 (15 marks)**

Let $f(x,y) = \cos(x^2 + y^2)$. Find

(a) the linear approximation of the function near the origin;
(b) the quadratic approximation of the function near the origin.

**MA2001** **Test** **Semester B, 2022/2023**

Name: Ng Chung Wah Student ID: 57147463

Q1: 3 Q2: 7 Q3: 20 Q4: 15 Total Marks: 38

3(a) Given $\lambda = 14$

$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & 4-\lambda & 6 \\ 3 & 6 & 9-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda)(9-\lambda) - 2(18-2\lambda-18) + 3(12-12+3\lambda)$

$= (1-\lambda)(36-13\lambda+\lambda^2) - 36+4\lambda+36+36-36+9\lambda$

$= (36-13\lambda+\lambda^2-36\lambda+13\lambda^2-\lambda^3 - 36+4\lambda+36+36-36+9\lambda$

$= -\lambda^3+14\lambda^2-36\lambda+36$

$=$

3(a) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$ $A-\lambda I = \begin{bmatrix} 1-\lambda & 2 & 3 \\ 2 & 4-\lambda & 6 \\ 3 & 6 & 9-\lambda \end{bmatrix}$

$|A-\lambda I| = (1-\lambda)(4-\lambda)(9-\lambda) - 2[2(9-\lambda)-(3\times 6)] + 3[12-(3\times(4-\lambda))]$

$= (1-\lambda)(36-13\lambda+\lambda^2) - 2[18-2\lambda-18] + 3[12-12+3\lambda]$

$= (36-13\lambda+\lambda^2-36\lambda+13\lambda^2-\lambda^3) + 4\lambda + 9\lambda$

$= -\lambda^3+14\lambda^2-36\lambda+36$

$= (\lambda-14)$

① $x^2+y^2 = 2z^2 \Rightarrow x^2+y^2 = 2(1-x-y)^2$

② $x+y+z=1 \Rightarrow z = 1-x-y$

①: $z^2 = \frac{x^2+y^2}{2}$

$z = \frac{1}{\sqrt{2}}\sqrt{x^2+y^2} = 2^{-\frac{1}{2}}(x^2+y^2)^{\frac{1}{2}}$

②: $z = 1-x-y$

$\frac{dx}{dz} =$

Q2) Eq 1: $x^2 = 2z^2 - y^2 \Rightarrow 2x\frac{dx}{dz} = 4z \Rightarrow 2\frac{d^2x}{dz^2} = 4 \Rightarrow \frac{d^2x}{dz^2} = 2$

Eq 2: $x = 1-y-z \Rightarrow \frac{dx}{dz} = -1$

Eq 2': $y = 1-x-z \Rightarrow \frac{dy}{dz} = -1$

Eq 1': $y^2 = 2z^2 - x^2 \Rightarrow 2y\frac{dy}{dz} = 4z \Rightarrow 2\frac{d^2y}{dz^2} = 4 \Rightarrow \frac{d^2y}{dz^2} = 2$

2) $x^2+y^2-2z^2-1-x-y-z=0$

$2x\frac{dx}{dz} - 4z - \frac{dx}{dz} - 1 = 0$

$\frac{dx}{dz} = \frac{1+4z}{2x-1}$

$\left.\frac{dx}{dz}\right|_{1,-1,1} = \frac{1+4}{2-1} = 5$

$\frac{d^2x}{dz^2} = (1+4z)(2x-1)^{-1}$

$= -(1+4z)(2\frac{dx}{dz})(2x-1)^{-2} + (2x-1)^{-1}(4)$

$\left.\frac{d^2x}{dz^2}\right|_{1,-1,1} = -50+4 = -46$

$2y\frac{dy}{dz} - 4z - \frac{dy}{dz} - 1 = 0$

$\frac{dy}{dz} = \frac{1+4z}{2y-1}$

$\left.\frac{dy}{dz}\right|_{1,-1,1} = \frac{5}{-3}$

$\left.\frac{d^2y}{dz^2}\right| = (1+4z)(2y-1)^{-1}$

$= (1+4z)(-1)(2y-1)^{-2}(2\frac{dy}{dz})$

$+ (2y-1)^{-1}(1+4)$

$\left.\frac{d^2y}{dz^2}\right|_{1,-1,1} = \frac{-5}{9} + \left(-\frac{5}{3}\right)$

$= \frac{-20}{9}$

4) $f(x,y) = \cos(x^2+y^2)$

$f'(x) = -2x \sin(x^2+y^2) \qquad f'(y) = -2y \sin(x^2+y^2)$

$L = f(0,0) + f_x'(0,0)x + f_y(0,0)y$

$= 1 + 0 + 0$

$= 1$

(b) $f_{xx} = f_x\left[-2x\sin(x^2+y^2)\right] \qquad f_{xx}(0,0) = 0$

$= -2\sin(x^2+y^2) + (-2x)(2x\cos(x^2+y^2))$

$f_{yy} = -2\sin(x^2+y^2) + (-2y)(2y\cos(x^2+y^2)) \qquad f_{yy}(0,0) = 0$

$f_{xy} + f_{yx} = \quad 2y\cos(x^2+y^2) + \sin(x^2+y^2) \qquad f_{xy}(0,0) = 0$

$L = f(0,0) + f_x(0,0)x + f_y(0,0)y + \frac{1}{2!}\left[f_{xx}(0,0)(x^2) + f_{xy}(0,0)xy + f_{yy}(0,0)y^2\right]$

$= 1 + 0$

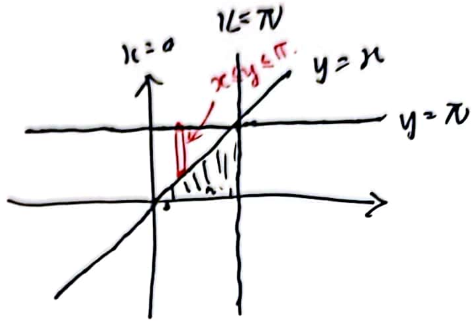
$= 1$

$x \le y \le \pi$

$$\int_0^\pi \int_x^\pi \frac{\sin y}{y}\, dy\, dx$$

$y = \pi$
$y = x$

$x = \pi$
$x = 0$




$$= \int_0^\pi \int_0^y \frac{\sin y}{y}\, dx\, dy$$

$$= \int_0^\pi \left[ \frac{x \sin y}{y} \right]_y^\pi dy$$

$$= \int_0^\pi \left[ \frac{\pi \sin y}{y} - \frac{y \sin y}{y} \right] dy = \pi \int_0^\pi \frac{\sin y}{y}\, dy - \int_0^\pi \sin y\, dy$$

$$= \int_0^\pi \left[ \sin y \left( \frac{\pi}{y} - 1 \right) \right] dy$$

$$= \int_0^\pi \sin y\, dy.$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} \qquad A - \lambda I = \begin{bmatrix} 1-\lambda & 2 & 3 \\ 2 & 4-\lambda & 6 \\ 3 & 6 & 9-\lambda \end{bmatrix}$$

$$|A - \lambda I| = (1-\lambda)\left[(4-\lambda)(9-\lambda) - 36\right] - 2\left[2(9-\lambda) - 18\right] + 3\left[12 - 12 + 3\lambda\right]$$

$$= (1-\lambda)\left[36 - 13\lambda + \lambda^2 - 36\right] - 2\left[-2\lambda\right] + 9\lambda$$

$$= (1-\lambda)\left[\lambda^2 - 13\lambda\right) + 4\lambda + 9\lambda$$

$$= \lambda^2 - \cancel{13\lambda} - \lambda^3 + 13\lambda^2 + \cancel{13\lambda}$$

$$= -\lambda^3 + 14\lambda^2$$

$$= \lambda^2(14 - \lambda)$$

$\therefore\ \lambda = 14,\ \lambda = 0$ (repeated)

(b) when $\lambda = 14$

$$A = \begin{bmatrix} -13 & 2 & 3 \\ 2 & -10 & 6 \\ 3 & 6 & -5 \end{bmatrix} \sim \left[\begin{array}{ccc|c} 1 & \frac{2}{-13} & \frac{3}{-13} & 0 \\ 0 & \frac{126}{13} & \frac{-84}{13} & 0 \\ 0 & -\frac{84}{13} & \frac{56}{13} & 0 \end{array}\right] \overset{\times 1.5}{\sim} \left[\begin{array}{ccc|c} 1 & \frac{2}{-13} & \frac{3}{-13} & 0 \\ 0 & \frac{126}{13} & \frac{-84}{13} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -1.5 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\times -2 \sim \left[\begin{array}{ccc|c} 1 & 0 & \frac{13}{3} & 0 \\ 0 & -1.5 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 + \frac{13}{3} x_3 = 0 \\ -1.5 x_2 + x_3 = 0 \end{array} \quad \overset{t = x_3}{\Rightarrow} \quad \vec{x}_1 = t\begin{bmatrix} -\frac{3}{13} \\ 1.5 \\ 1 \end{bmatrix}$$

(c) $$A\vec{x}_2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\because\ A x_2 = 0$

$\therefore$ is eigenvector

(d) $a\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + b\Big[$

$\therefore \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ same $\lambda = 0$

(e) $$P = \begin{bmatrix} -2 & -2 & -\frac{3}{13} \\ 1 & 1 & 1.5 \\ 0 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 14 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$