Recursion

- In C/C++, a function may call itself either directly or indirectly.
- When a function calls itself recursively, each invocation gets a fresh set of all explicit parameters and automatic (local) variables, independent of the previous set.

Example: Compute the greatest common divisor (GCD) of two numbers.

Let M > N > 0, and M = QN + R such that 0 <= R < N

- if R = 0, then M = QN, and gcd(M, N) = N
- if R > 0, then N > R, and gcd(M, N) = gcd(N, R)

Non-recursive algorithm to compute the GCD of 2 numbers.

```
unsigned gcd(unsigned M, unsigned N)
/* given M > 0 and N > 0, compute gcd of M & N */
{
   unsigned R;
   while (R = M % N)   //R = M % N and R != 0
   {  M = N;
      N = R;
   }
   return N;
}
```

Recursive algorithm to compute the GCD of 2 numbers

```
unsigned gcd(unsigned M, unsigned N)
/* given M > 0 and N > 0, compute gcd of M & N */
{
   unsigned R = M % N;
   if (R == 0)           // base case
      return N;
   else
      return gcd(N, R);  // recursion
}
```

Two fundamental rules of recursion:

1. **Base cases**
You must have some base cases, which can be solved without recursion.

2. **Making progress (recursion)**
For the cases that are to be solved recursively, the recursive call must always be to a case that makes progress toward a base case.

Example: Fibonacci number

$f(0) = 0;$
$f(1) = 1;$
$f(n) = f(n-2) + f(n-1)$ for $n \geq 2$;

```
unsigned fib(unsigned n)
{  //  precondition: given n >= 0

   if (n == 0)      // base case #1
      return 0;

   if (n == 1)      // base case #2
      return 1;

   return fib(n-2) + fib(n-1);   // recursion

} // this program works but is unacceptably slow
```

Usually, looping (iteration) is more efficient than recursion.
Computing the Fibonacci number using iteration is much more efficient.

Recursion is preferred when the problem is recursively defined, or when the data structure that the algorithm operates on is recursively defined.

Example: Ackermann's function

$$A(M,N) = \begin{cases} N+1 & \text{if } M = 0 \\ A(M-1, 1) & \text{if } N = 0 \\ A(M-1,\ A(M,\ N-1)) & \text{otherwise} \end{cases}$$

```
unsigned Ack(unsgined m, unsigned n)
//  precondition: given m >= 0 and n >= 0
{
   if (m == 0)
      return n+1;

   if (n == 0)
      return Ack(m-1, 1);

   return Ack(m-1, Ack(m, n-1));
}
```

Non-Recursive binary search

```
// Precondition: A[] is sorted in ascending order
int binSearch(int A[], int N, int key)
{
  int low = 0;
  int high = N-1;

  while (low <= high)
  {
     int mid = (low + high) / 2;

     if (A[mid] < key)
       low = mid+1;
     else if (A[mid] > key)
       high = mid-1;
     else
       return mid; // key == A[mid]
  }

  return -1;
}
```

Recursive binary search

```
// Precondition: A[] is sorted in ascending order
// A[low..high] represents the search range

int binSearch_R(int A[], int low, int high, int key)
{
   if (low > high) // search range is empty
      return -1;   // key is not found

   int mid = (low + high) / 2; // probe the mid-point of
   if (A[mid] == key)          // the search range
      return mid;

   if (A[mid] < key) // recursion with reduced search range
      return binSearch_R(A, mid+1, high, key);
   else
      return binSearch_R(A, low, mid-1, key);
}
```

Recursive quicksort algorithm

```
// Refer to the notes in the chapter for sorting
// for the details of the function partition()

void simpleQuicksort(int x[], int start, int end)
{
   if (start < end)
   {
      int j = partition(x, start, end);
      simpleQuicksort(x, start, j-1);
      simpleQuicksort(x, j+1, end);
   }
}
```

Non-recursive quicksort algorithm using stack

```
void nonRecursive_quicksort(int x[], int n)
{
   stack<int> S;

   S.push(0);    // start index
   S.push(n-1); // end index

   while (!S.empty()) // some sublist not yet sorted
   {
      int end =   S.top();  S.pop(); // retrieve end before
      int start = S.top();  S.pop(); // start from stack

      if (start < end)
      {
         int j = partition(x, start, end);

         // processing order of left and right sublist
         // is not important in this case

         S.push(start); // left sublist
         S.push(j-1);

         S.push(j+1);    // right sublist
         S.push(end);
      }
   }
}
```

Example: Towers of Hanoi

- There are 3 towers, labeled L, M and R
- Initially, tower L contains a stack of N disks of graded sizes, stacked in order of size with the smallest disk at the top.
- A move consists of taking the top disk from one tower and placing it on another tower, subject to the constraint that a disk may never be placed above a disk smaller than itself.
- The problem is to move the N disks from tower L to tower R in the smallest number of moves.

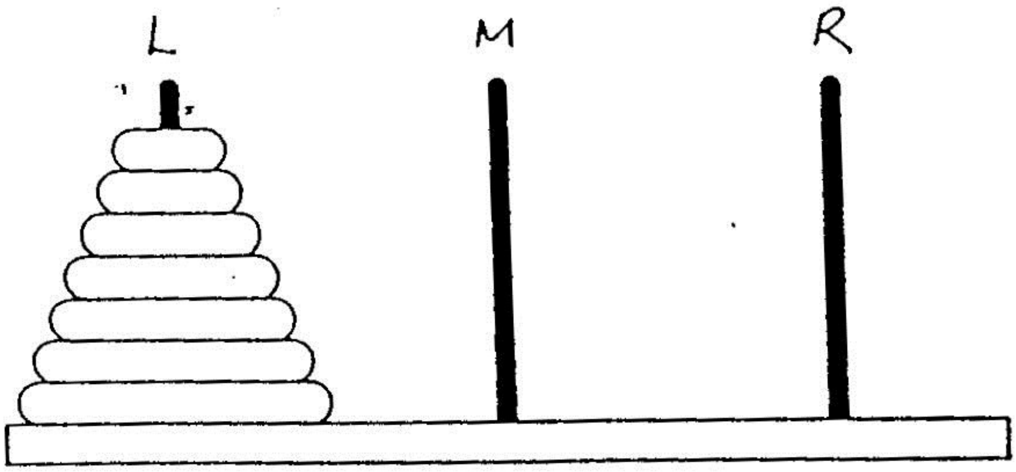


Figure 3.8. The Towers of Hanoi

Sequence of moves:

N = 1	N = 2	N = 3
L → R	L → M L → R M → R	L → R L → M R → M L → R M → L M → R L → R

The problem can be solved in 3 "steps" using recursion:

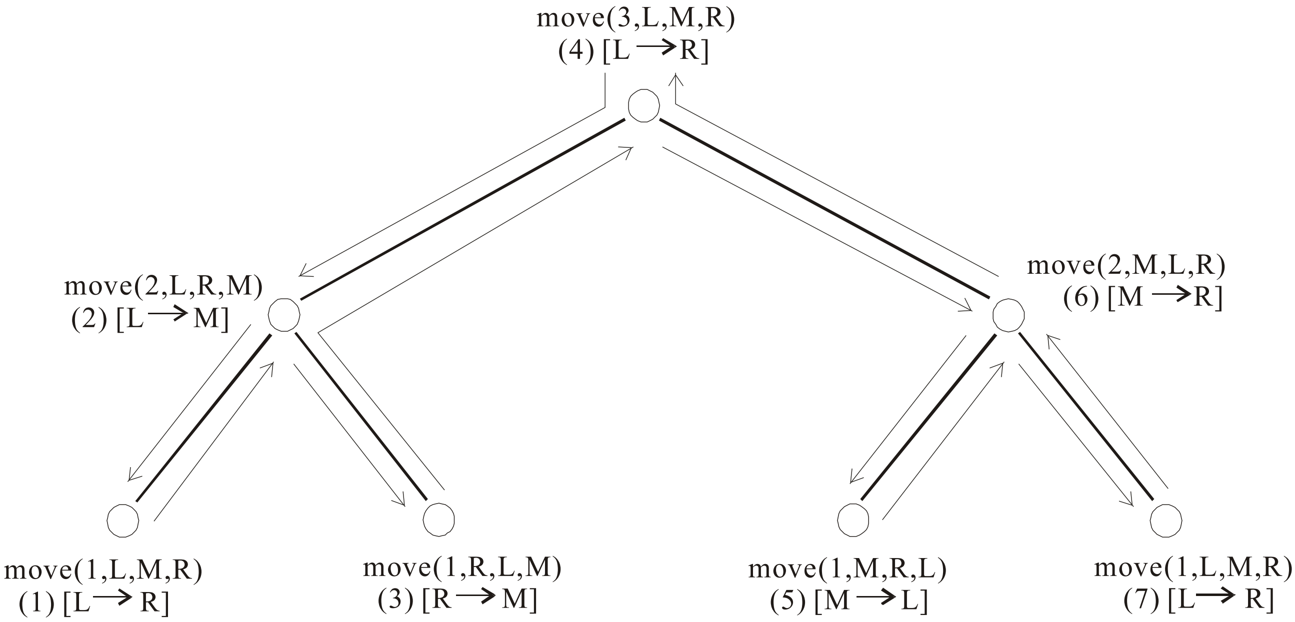
1. Move the top N-1 disks from L to M using R as a temporary working space.
2. Move the bottom disk from L to R.
3. Move the remaining N-1 disks from M to R using L as a temporary working space.

Let `move(N, F, U, T)` denotes the operation of moving `N` disks from tower `F` to tower `T` using tower `U`.

Recursive function to print out the sequence of moves:

```
void move(int N, char F, char U, char T)
{
   if (N == 1)
      cout << F << " -> " << T << endl;
   else if (N > 1)
   {
      move(N-1, F, T, U);
      cout << F << " -> " << T << endl;
      move(N-1, U, F, T);
   }
}
```

Recursion tree for moving 3 disks



depth of the recursion = number of disks to be moved = N

number of moves = number of nodes in the recursion tree

$$= 1 + 2 + 4 + \ldots + 2^{N-1} = 2^N - 1$$

```
//version 2, without the else statement

void move_v2(int N, char F, char U, char T)
{
   if (N > 0)
   {
      move_v2(N-1, F, T, U);
      cout << F << " -> " << T << endl;
      move_v2(N-1, U, F, T);
   }
}
```

Recursive function to print the permutations of an array of symbols

```
//Consider the case for N=3

char x[3] = {'a', 'b', 'c'};

Permutations of the 3 symbols are:
a b c
a c b
b a c
b c a
c b a
c a b
```

Pseudo code:

```
void permute(char x[], int s, int e)
// Enumerate the permutations of the symbols in x[s..e].
// The symbols in x[0..s-1] have already been fixed.
{
   if (s == e) // base case
      // only 1 permutation is possible
      print the contents of x[];
   else // recursion
   {
      // x[s] can have e-s+1 choices
      for each choice of x[s]
         find the permutations of x[s+1..e] by recursion;
   }
}
```

```
//Detailed codes

void swap(char x[], int i, int j) // swap x[i] with x[j]
{
   char t = x[i];
   x[i] = x[j];
   x[j] = t;
}

void permute(char x[], int s, int e)
// Enumerate the permutations of the symbols in x[s..e]
{
   if (s == e) // base case
   {
      for (int k = 0; k <= e; k++)
         cout << x[k] << " ";
      cout << endl;
   }
   else  //recursion
   {
      for (int k = s; k <= e; k++)
      {
         //1. there are e-s+1 choices for the first symbol
         swap(x, s, k);

         //2. once we fix the first symbol,
         //   find the permutations of the remaining e-s
         //   symbols by recursion
         permute(x, s+1, e);

         //3. restore the original order of the array
         swap(x, s, k);
      }
   }
}

void testPermute()
{
   char x[4] = {'a', 'b', 'c', 'd'};

   permute(x, 0, 3); //print permutations of 4 symbols
}
```

Recursion and Backtracking

Search space is a set of possible right answers to be explored.

Backtracking is a technique whereby

- the *search space* is explored by systematically trying each possible path,
- back up to try another path whenever a dead end is encountered.

The Eight-Queens Problem:
Put eight queens on the chessboard such that no queen is being attacked by the others.

		Q					
					Q		
			Q				
	Q						
							Q
				Q			
						Q	
Q							

Rather than viewing the chessboard as consisting of 64 squares, it can be seen as being comprised of eight columns, each with eight rows.

- To solve the problem, we need to place one Queen in each column.
- In each column we will place a Queen in one of the eight rows, and then take a step forward by making a recursive call.
- This call will return having either **succeeded** in finding a solution, or having **failed** (meaning that there is no solution given the current placement of Queens).
- If the recursive call fails, we move the Queen to another row and try again.

Outline of the solution in pseudo code

```
bool EightQueen(ChessBoard square, int col)
{
    bool success = false;

    if (col >= 8)
        success = true;
    else
    {
        for (int row = 0; row < 8 and not success; row++)
        {
            if (square[row, col] is safe)
            {
                put a queen in square[row, col];
                success = EightQueen(col+1);
                if (not success)
                    remove queue from square[row, col];
            }
        }
    }
    return success;
}
```

```
#define NumRows 8
#define NumCols 8
typedef int ChessBoard[NumRows][NumCols];

bool safePos(ChessBoard square, int row, int col)
/* The poistion is safe if there is not another Queen in
   the given row and diagonals. */
{
    bool conflict = false;
    for (int j = col-1; j >= 0 && !conflict; j--)
    {
        if (square[row][j] == 1) // check the row
            conflict = true;

        int i = row + col - j; // check upper diagonal
        if (i < NumRows && square[i][j] == 1)
            conflict = true;

        i = row - (col - j); // check lower diagonal
        if (i >= 0 && square[i][j] == 1)
            conflict = true;
    }
    return !conflict;
}


bool EightQueen(ChessBoard square, int col)
{
    bool success = false;
    if (col >= 8)  //base case: solution found
        success = true;
    else
    {
        for (int row = 0; row < NumRows && !success; row++)
        {
            if (safePos(square, row, col))
            {
                //put a queue at square[row][col]
                square[row][col] = 1;
                success = EightQueen(square, col+1);
                if (!success) //remove the queue
                    square[row][col] = 0;
            }
        }
    }
    return success;
}
```

```
void solveEightQueen(ChessBoard square)
{
    // initialize the chessboard
    for (int i = 0; i < NumRows; i++)
        for (int j = 0; j < NumCols; j++)
            square[i][j] = 0;

    if (EightQueen(square, 0))
    {
         // print solution
         for (int i = NumRows-1; i >= 0; i--)
         {
             for (int j = 0; j < NumCols; j++)
                 cout << square[i][j];
             cout << endl;
         }
    }
    else
        cout << "no solution found" << endl;
}
```