In-Class Exercise 5

1. Use the definition of the moment generating function $\phi(t)$ in (2.28) to compute $\mathbb{E}\{X\}$ of the exponential random variable with parameter λ, whose probability density function (PDF) is:

$$p(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

2. Let Y be a random variable transformed from $X \sim \mathcal{U}(0,1)$ via $Y = e^X$. Find the cumulative density function (CDF) and PDF of Y.

3. Radar detects a flying object by measuring the power reflected from it. The reflected power of an aircraft can be modelled by a RV Y with PDF:

$$p(y) = \begin{cases} \frac{1}{P_0} e^{-y/P_0}, & y \geq 0 \\ 0, & y < 0 \end{cases}$$

where $P_0 > 0$. Suppose that the aircraft is correctly identified by the radar if its reflected power is larger than the expected value of Y or $\mathbb{E}\{Y\}$. Find the probability that the aircraft is correctly identified.

4. A discrete random variable X has the probability mass function (PMF) $p(x) = 1/5$, $x = 0, 1, 2, 3, 4$. Define another random variable as $Y = \sin(\pi/2 \cdot X)$. Compute $\mathbb{E}\{Y\}$.

5. Suppose R is a Poisson random variable with PMF:

$$p(r) = P(R = r) = e^{-\lambda}\frac{\lambda^r}{r!}, \quad r = 0, 1, 2, \cdots$$

Find the PMF of $S = 2R$.

6. Let X be a continuous random variable with PDF:

$$p(x) = \begin{cases} \dfrac{3}{x^4}, & x \geq 1 \\ 0, & x < 1 \end{cases}$$

Determine $\mathbb{E}\{X\}$ and $\mathrm{var}(X)$.

7. Suppose a random variable X has mean μ_x and variance σ_x^2. Let $Y = aX + b$ where a and b are constants. Determine the mean and variance of Y in terms of μ_x and σ_x^2.

   Then write down the MATLAB command to generate a Gaussian random variable $Y \sim \mathcal{N}(1,2)$ with the use of `randn`.

8. Let X be a continuous random variable with PDF:

$$p(x) = \begin{cases} x^2\left(2x + \dfrac{3}{2}\right), & 0 < x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

   If $Y = \dfrac{2}{X} + 3$, determine $\mathrm{var}(Y)$.

9. Compute $\mathbb{E}\{R\}$ of the geometric random variable R with parameter p, whose PMF is:

$$p(r) = (1-p)^{r-1}p, \quad 1 \leq r < \infty$$

Solution

1.

$$\begin{aligned}\phi(t) = \mathbb{E}\{e^{tX}\} &= \int_{-\infty}^{\infty} e^{tx} p(x) dx = \int_0^{\infty} e^{tx} \lambda e^{-\lambda x} dx = \int_0^{\infty} \lambda e^{(t-\lambda)x} dx \\ &= \lambda \frac{1}{t-\lambda} e^{(t-\lambda)x} \Big|_0^{\infty} \\ &= \frac{\lambda}{\lambda - t}, \quad \lambda > t\end{aligned}$$

Note that $\phi(t)$ is only defined for $\lambda > t$ but this does not affect our calculation because $\lambda > 0$ and we evaluate at $t = 0$.

$$\phi'(t) = \frac{\lambda}{(\lambda - t)^2} \Rightarrow \phi'(0) = \mathbb{E}\{X\} = \frac{\lambda}{(\lambda)^2} = \frac{1}{\lambda}$$

Note that when using the moment generating function, it is simpler than using (2.20) because there is no need to deal with the integration of $x\lambda e^{-\lambda x}$ which requires integrating by parts.

2.

Following Example 2.26:

$$F_X(x) = \begin{cases} 0, & x \leq 0 \\ x, & 0 < x < 1 \\ 1, & x \geq 1 \end{cases}$$

That is, $F_X(x) = P(X \leq x) = x$ for $0 < x < 1$.

Let $Y = e^X$. As $X \in (0,1)$, we have $Y \in (1,e)$. Hence we know $F_Y(1) = P(Y \leq 1) = 0$ and $F_Y(e) = P(Y \leq e) = 1$, and we only investigate the range in $(1,e)$.

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(e^X \leq y) = P(X \leq \ln y) \\ &= F_X(\ln y), \quad 0 < \ln y < 1 \\ &= \ln y \end{aligned}$$

Combining the results, we have:

$$F_Y(y) = \begin{cases} 0, & y \leq 1 \\ \ln y, & 1 < y < e \\ 1, & y \geq e \end{cases}$$

Applying (2.10), we get:

$$P_Y(y) = \frac{d}{dy} \ln y = \frac{1}{y}, \quad 1 < y < e$$

Hence:

$$P_Y(y) = \begin{cases} \dfrac{1}{y}, & 1 < y < e \\ 0, & \text{otherwise} \end{cases}$$

Writing $Y = g(X)$, the PDF can also be obtained as:

$$P_Y(y) = P_X(g^{-1}(y)) \left| \frac{dg^{-1}(y)}{dy} \right|$$

Now $Y = g(X) = e^X \Rightarrow g^{-1}(y) = X = \ln y$. Again, we know that Y only has values between $(1, e)$. We then compute:

$$P_X(g^{-1}(y)) = P_X(\ln y) = 1, \quad 0 < \ln y < 1$$

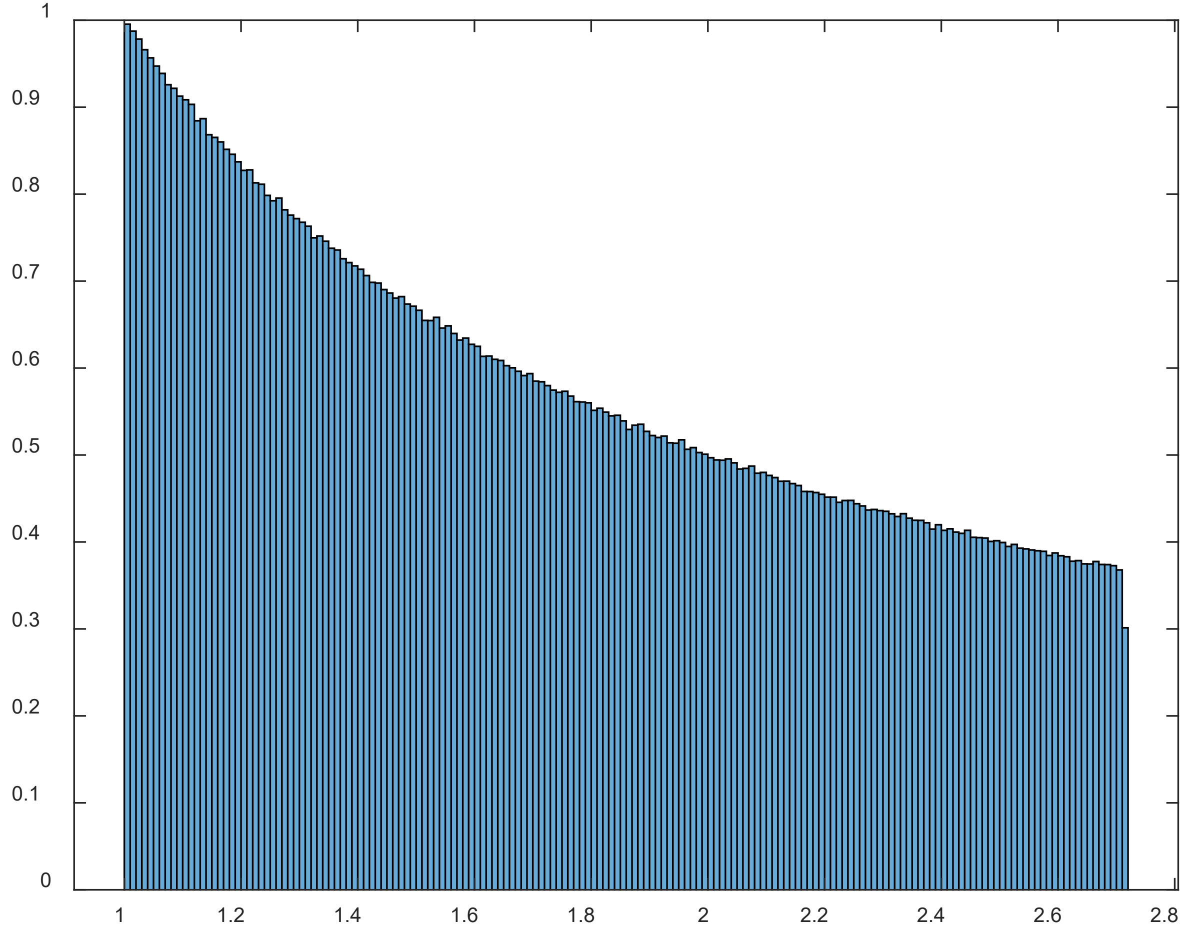
$$\frac{dg^{-1}(y)}{dy} = \frac{d \ln y}{dy} = \frac{1}{y}$$

Combining the results, we get:

$$P_Y(y) = 1 \cdot \frac{1}{y} = \frac{1}{y}, \quad 1 < y < e$$

Integrating with respect to $P_Y(y)$ yields the same CDF $F_Y(y)$.

```
x=rand([1,10000000]);
y=exp(x);
histogram(y,'Normalization','pdf')
```



3.

Using the result of Question 1, we know that $\mathbb{E}\{Y\} = 1/\lambda = P_0$.

With the use of (2.18), the probability is determined as:

$$P(Y > P_0) = \int_{P_0}^{\infty} \frac{1}{P_0} e^{-y/P_0} dy = 1 - P(Y \leq P_0)$$
$$= 1 - \left(1 - e^{-\frac{1}{P_0} \cdot P_0}\right) = e^{-1} = 0.3679$$

4.

Applying (2.24), we have:

$$\mathbb{E}\{g(X)\} = \sum_{x=0}^{4} g(x)p(x) = \sum_{x=0}^{4} \sin(\pi x/2) \cdot \frac{1}{5} = \frac{1}{5}(0+1+0-1+0) = 0$$

Alternatively, we can find the PMF of y using $Y = \sin(\pi/2 \cdot X)$:

$$\begin{gathered} x = 0 \Rightarrow y = \sin(\pi/2 \cdot 0) = 0 \\ x = 1 \Rightarrow y = \sin(\pi/2 \cdot 1) = 1 \\ x = 2 \Rightarrow y = \sin(\pi/2 \cdot 2) = 0 \\ x = 3 \Rightarrow y = \sin(\pi/2 \cdot 3) = -1 \\ x = 4 \Rightarrow y = \sin(\pi/2 \cdot 4) = 0 \end{gathered}$$

Hence $P_Y(-1) = 1/5$, $P_Y(0) = 3/5$, and $P_Y(1) = 1/5$

$$\mathbb{E}\{Y\} = \sum_{y=-1}^{1} y P_Y(y) = 0$$

5.

Since $S = 2R$, we then know that the admissible values of S are 0, 2, 4, ….

The PMF of S can be determined as:

$$P(S/2 = r) = e^{-\lambda}\frac{\lambda^r}{r!}, \quad r = 0, 1, 2, \cdots$$

$$\Rightarrow P(S = 2r) = e^{-\lambda}\frac{\lambda^r}{r!}, \quad r = 0, 1, 2, \cdots$$

$$\Rightarrow P(S = s) = e^{-\lambda}\frac{\lambda^{s/2}}{(s/2)!}, \quad s = 0, 2, 4, \cdots$$

6.
According to (2.21):

$$\mathbb{E}\{X\} = \int_{-\infty}^{\infty} xp(x)dx = \int_{1}^{\infty} x \cdot \frac{3}{x^4}dx = \int_{1}^{\infty} \frac{3}{x^3}dx = -\frac{3}{2}x^{-2}\bigg|_{1}^{\infty} = \frac{3}{2}$$

$$\mathbb{E}\{X^2\} = \int_{-\infty}^{\infty} x^2p(x)dx = \int_{1}^{\infty} x^2 \cdot \frac{3}{x^4}dx = \int_{1}^{\infty} \frac{3}{x^2}dx = -3x^{-1}\big|_{1}^{\infty} = 3$$

Applying (2.23), we have:

$$\mathrm{var}(X) = \mathbb{E}\{X^2\} - (\mathbb{E}\{X\})^2 = 3 - \left(\frac{3}{2}\right)^2 = \frac{3}{4}$$

7.

$$\mathbb{E}\{Y\} = \mu_y = \mathbb{E}\{aX + b\} = \mathbb{E}\{aX\} + \mathbb{E}\{b\} = a\mathbb{E}\{X\} + b = a\mu_x + b$$

The same result can be obtained by following Example 2.28 or applying directly (2.27).

$$\begin{aligned}
\mathrm{var}(Y) = \sigma_y^2 &= \mathbb{E}\{(Y - \mu_y)^2\} \\
&= \mathbb{E}\{(aX + b - (a\mu_x + b))^2\} \\
&= \mathbb{E}\{(aX - a\mu_x))^2\} \\
&= a^2\mathbb{E}\{(X - \mu_x))^2\} \\
&= a^2\mathrm{var}(X) \\
&= a^2\sigma_x^2
\end{aligned}$$

From the above results, if $X \sim \mathcal{N}(\mu_x, \sigma_x^2)$, then

$$Y \sim \mathcal{N}(a\mu_x + b, a^2\sigma_x^2)$$

The MATLAB command `randn` generates $X \sim \mathcal{N}(0,1)$. To produce $Y \sim \mathcal{N}(1,2)$, a and b are computed as:

$$a\mu_x + b = 1 \Rightarrow a \cdot 0 + b = 1 \Rightarrow b = 1$$

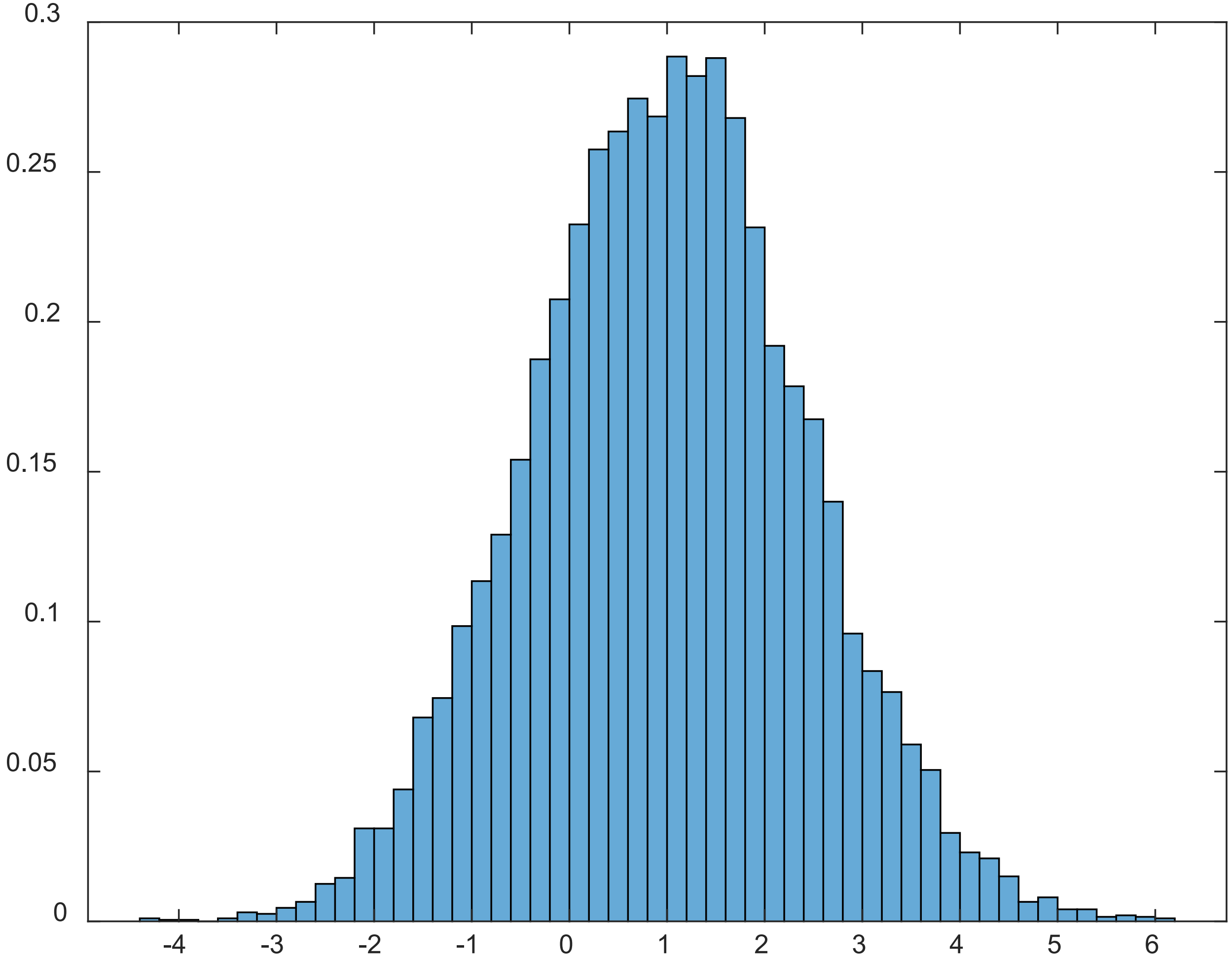
$$a^2\sigma_x^2 = 2 \Rightarrow a^2 \cdot 1 = 2 \Rightarrow a = \sqrt{2}$$

Hence the MATLAB command is `sqrt(2)*randn+1`

Following Example 2.25:

```
Y= sqrt(2)*randn(1,10000)+1;
mean(Y)
= 1.0023
mean((Y-mean(Y)).*(Y-mean(Y)))
= 1.9659
```

```
histogram(Y,'Normalization','pdf')
```



We see the mean is shifted to 1 and there is a wider spread.

8.

According to the results in Question 7, we have:

$$\mathrm{var}(Y) = 2^2 \mathrm{var}\left(\frac{1}{X}\right) = 4\mathrm{var}\left(\frac{1}{X}\right)$$

Then we apply (2.23):

$$\mathrm{var}\left(\frac{1}{X}\right) = \mathbb{E}\left\{\frac{1}{X^2}\right\} - \left(\mathbb{E}\left\{\frac{1}{X}\right\}\right)^2$$

Considering $g(X) = 1/X$ and $g(X) = 1/X^2$, and applying (2.25):

$$\mathbb{E}\left\{\frac{1}{X}\right\} = \int_{-\infty}^{\infty} \frac{1}{x} p(x) dx = \int_0^1 x\left(2x + \frac{3}{2}\right) dx = \int_0^1 \left(2x^2 + \frac{3}{2}x\right) dx = \frac{17}{12}$$

$$\mathbb{E}\left\{\frac{1}{X^2}\right\} = \int_{-\infty}^{\infty} \frac{1}{x^2} p(x) dx = \int_0^1 \left(2x + \frac{3}{2}\right) dx = \int_0^1 \left(2x + \frac{3}{2}\right) dx = \frac{5}{2}$$

Combining the results yields:

$$\mathrm{var}(Y) = \frac{71}{36}$$

9.

We can use (2.19):

$$
\begin{aligned}
\mathbb{E}\{R\} = \sum_{r=1}^{\infty} rp(r) &= \sum_{r=1}^{\infty} rp(1-p)^{r-1} \\
&= p\sum_{r=1}^{\infty} rq^{r-1}, \quad q = 1-p \\
&= p\sum_{r=1}^{\infty} \frac{d}{dq}q^r = p\frac{d}{dq}\sum_{r=1}^{\infty} q^r \\
&= p\frac{d}{dq}\frac{q}{1-q} = p\frac{(1-q)-q(-1)}{(1-q)^2} = p\frac{1}{(1-q)^2} = \frac{1}{p}
\end{aligned}
$$

That is, the expected number of independent trials we need to perform until the first success equals the reciprocal of the probability of a success in a trial, e.g., if $p=0.1$, then on average it takes $1/p=10$ trials for a success.

We can also use (2.31):

$$\phi(t) = \sum_{r=1}^{\infty} e^{tr} p(1-p)^{r-1} = pe^t \sum_{r=1}^{\infty} [e^t(1-p)]^{r-1} = pe^t \sum_{k=0}^{\infty} [e^t(1-p)]^k$$

$$= \frac{pe^t}{1-e^t(1-p)}, \quad |e^t(1-p)| < 1$$

$$\phi'(t) = \frac{[1-e^t(1-p)]pe^t - (pe^t)[-e^t(1-p)]}{[1-e^t(1-p)]^2} = \frac{[1-e^t(1-p)]pe^t + p(1-p)e^{2t}}{[1-e^t(1-p)]^2}$$

Hence

$$\Rightarrow \phi'(0) = \mathbb{E}\{R\} = \frac{[1-(1-p)]p + p(1-p)}{[1-(1-p)]^2} = \frac{1}{p}$$