**EE2302 Foundations of Information Engineering** Semester A, 2022/23

Solutions to Test 3

1. $f$ is a linear function, since it satisfies
$$f(ax+b\tilde{x}) = (ax^T + b\tilde{x}^T)\begin{pmatrix} y_2 \\ -y_1 \end{pmatrix} = ax^T\begin{pmatrix} y_2 \\ -y_1 \end{pmatrix} + b\tilde{x}^T\begin{pmatrix} y_2 \\ -y_1 \end{pmatrix} = af(x) + bf(\tilde{x}),$$
where $x, \tilde{x} \in \mathbb{R}^2$.

2. Let $x = (b_1 - 1, b_2 - 1, \dots, b_M - 1)$ and $y = (1,1,\dots,1) \in \mathbb{R}^M$ . By Cauchy-Schwarz inequality, we have $\sum_{m=1}^{M}(b_m - 1) \leq \sqrt{M}\sqrt{\sum_{m=1}^{M}(b_m - 1)^2}$. Then we square both sides and move M to the left-hand side, and we obtain $\sum_{m=1}^{M}(b_m - 1)^2 \geq \frac{1}{M}\left(\sum_{m=1}^{M}(b_m - 1)\right)^2$.

3. The span of $x$ and $y$ is given by
$$\{v \in \mathbb{R}^3 : \alpha x + \beta y, \text{where } \alpha, \beta \in \mathbb{R}\}.$$

Assume there are two vectors $v_1 = \alpha_1 x + \beta_1 y$ and $v_2 = \alpha_2 x + \beta_2 y$, and $a, b \in \mathbb{R}$. We have

$$av_1 + bv_2 = a(\alpha_1 x + \beta_1 y) + b(\alpha_2 x + \beta_2 y) = (a\alpha_1 + b\alpha_2)x + (a\beta_1 + b\beta_2)y,$$

which is also in the span of x and y. Hence, it is closed under addition and multiplication and thus it is a subspace of $\mathbb{R}^3$.

4.

a) $\mathcal{C}(A) = \{v \in \mathbb{R}^2 : v = \alpha\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \text{where } \alpha \in \mathbb{R}\}$ and $\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right\}$ is a basis.

$$Ax = 0 \Rightarrow \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow x_1 + 2x_2 = 0.$$

Hence, $\mathcal{N}(A) = \left\{v \in \mathbb{R}^2 : v = \alpha\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \text{where } \alpha \in \mathbb{R}\right\}$.

b) $Ax = b \Rightarrow \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \end{bmatrix} \Rightarrow x_1 + 2x_2 = 4.$

Direct observation gives a particular solution $x_p = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

General solution: $x = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \alpha\begin{bmatrix} 2 \\ -1 \end{bmatrix}$, where $\alpha \in \mathbb{R}$ .

5.

a) The generator matrix $G = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$.

The parity check matrix $H = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$.

b) No, it is not. One possible counter example: 00001000 has no inverse image.

c) All the codewords are:

00000000; 00010101; 00100110; 00110011;
01001001; 01011100; 01101111; 01111010;
10001010; 10011111; 10101100; 10111001;
11000011; 11010110; 11100101; 11110000.

The minimum weight of all non-zero codewords is 3, so it can correct at most 1 error.

d) The syndrome is $s = yH^T = [1, 0, 0, 0, 1, 1, 1, 1] \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = [0, 1,0,1]$

e) The second component and the last component are non-zero, which indicate that the second parity-check equation and the last are in error while the other two parity-check equations are without error. Since only $c_4$ occurs in the two erroneous equations but not in the two correct equations, according to the nearest-neighbor decoding, $c_4$ is wrong, so the decoder output is 1001.

6.

a) Yes, it is linear, since it satisfies

$$f_\theta(ax + by) = A_\theta(ax + by) = aA_\theta x + bA_\theta y = af_\theta(x) + bf_\theta(y).$$

b) $f_{\pi/4}^{-1}(x) = A_{-\pi/4}x$, where $A_{-\pi/4} = \begin{pmatrix} \cos\frac{\pi}{4} & \sin\frac{\pi}{4} \\ -\sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix}$.

c) $f_{\pi/4} \circ f_{\pi/2} = f_{3\pi/4}$.

d) It satisfies the four properties:

Closure: For any $f_{\theta_1} \circ f_{\theta_2}$, it means rotating the vector $x$ anti-clockwise by $\theta_2 + \theta_1$ degrees, which is multiples of $\pi/4$ and is still in the set.
Identity: $f_0$ is the identity.
Inverse: The inverse for $f_0$ is $f_0$, the inverse for other $f_\theta$ is $f_{2\pi-\theta}$
Associativity: $(f_{\theta_1} \circ f_{\theta_2}) \circ f_{\theta_3} = f_{\theta_1} \circ (f_{\theta_2} \circ f_{\theta_3})$.

e) $\langle \{f_0, f_{\pi/4}, f_{\pi/2}, f_{3\pi/4}, f_\pi, f_{5\pi/4}, f_{3\pi/2}, f_{7\pi/4}\},\ \circ\rangle$; $\langle \{f_0, f_{\pi/2}, f_\pi, f_{3\pi/2}\},\ \circ\rangle$;
$\langle \{f_0, f_\pi\},\circ\rangle$, ; $\langle \{f_0\},\circ\rangle$