EE3331 Assignment 2

1. N number of shots are required to reach no less than 90% probability of hitting the target at least once. If the probability of hitting the target is exactly 90%. Then, the probability of missing the target will be $1 - 0.9 = 0.1 = 10\%$. Probability of hitting the target in a shot is $P(H) = 0.2$. Probability of missing the target in a shot is $P(M) = 1 - P(H) = 1 - 0.2 = 0.8$. Therefore, the probability of missing the target in n number of shots is $(P(M))^n = 0.8^n$. Since, $(0.8^n) \leq 0.1 \implies 0.8^{10} = 0.1073741824$ and n must be an integer. Therefore, $n \geq 11$.
2. Now, we have mean $\mu = 72$ and when $P(x \geq 96) = 2.5\%$. Therefore, $p = 1 - 0.025 = 0.975$, by the Z-score table, we can know when $p = 0.975 \implies z = 1.96$. By $z = \frac{x-\mu}{\sigma}$, $\sigma = 12.2$. To find $P(60 \leq x \leq 84)$, we can find their corresponding z-score and do subtraction on their p-value. First, z-score of 60 is $z = \frac{60-72}{12.2} = -0.984$, then $p(60) = 0.16354$. Secondly, z-score of 84 is $z = \frac{84-72}{12.2} = 0.984$, then $p(84) = 0.83646$. Therefore, $P(60 \leq x \leq 84) = p(84) - p(60) = 0.83646 - 0.16354 = 0.67292$.
3. If $y^2 + Xy + 1 = 0$ has real roots, which means its discriminant Δ is equal or greater than 0. $X^2 - 4(1)(1) \geq 0 \implies X \geq \pm 2$. Since the range of $X \in (1, 6)$, therefore we only need to find the probability of $P(X \geq 2)$. Therefore, $P(X \geq 2) = \frac{\int_2^6 dx}{5} = \frac{4}{5}$.
4. Letting X denote the random variables that is defined as the absolute difference of two fair dice,

$$P\{X=0\} = P\{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\} = \frac{6}{36}$$
$$P\{X=1\} = P\{(1,2),(2,3),(3,4),(4,5),(5,6),(2,1),(3,2),(4,3),(5,4),(6,5)\} = \frac{10}{36}$$
$$P\{X=2\} = P\{(1,3),(2,4),(3,5),(4,6),(3,1),(4,2),(5,3),(6,4)\} = \frac{8}{36}$$
$$P\{X=3\} = P\{(1,4),(2,5),(3,6),(4,1),(5,2),(6,3)\} = \frac{6}{36}$$
$$P\{X=4\} = P\{(1,5),(2,6),(5,1),(6,2)\} = \frac{4}{36}$$
$$P\{X=5\} = P\{(1,6),(6,1)\} = \frac{2}{36}$$

   Therefore, the probability that the absolute difference is odd number is $P(X=1) + P(X=3) + P(X=5) = \frac{10}{36} + \frac{6}{36} + \frac{2}{36} = 0.5$
5. Letting X equal to the number of head ("successes") that appear, then X is a binomial random variable with parameters ($n = 100$, $p = 0.1$). Hence, $P\{X = 10\} = \binom{100}{10} 0.1^{10}(1-0.1)^{100-10} = 0.132$. Let $\lambda = np = 100 \times 0.1 = 10$, $P\{X = 10\} = e^{-10}\frac{10^{10}}{10!} = 0.12511$. Poisson distribution is more suitable because when n is large and p is small, it can be derived as a limiting case to the binomial distribution by the law of rare events, so it is a good approximation of the binomial distribution.
6. (a)

$$\text{By} \sum_{n=1}^{\infty} p(x_i) = 1$$

$$\begin{aligned} \frac{1}{2} + \sum_{x=1}^{\infty} \frac{1}{\alpha^x} &= 1 \\ \frac{1}{\alpha} + \frac{1}{\alpha^2} + \frac{1}{\alpha^3} + \cdots + \frac{1}{\alpha^\infty} &= \frac{1}{2} \\ \frac{1}{\alpha} \times \frac{1}{1 - \frac{1}{\alpha}} &= \frac{1}{2} \\ \frac{1}{\alpha} \times \frac{\alpha}{\alpha - 1} &= \frac{1}{2} \\ a &= 3 \end{aligned}$$

(b)

$$F_X(x) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{1}{2}, & \text{if } 0 \leq x < 1 \\ \frac{5}{6}, & \text{if } 1 \leq x < 2 \\ 1, & \text{if } 2 \leq x \end{cases}$$

7. (a)

$$\mathbb{E}\{X\} = -2 \times \frac{1}{4} + -1 \times \frac{1}{4} + 0 \times \frac{1}{4} + 1 \times \frac{1}{4} = -\frac{1}{2}$$

$$\mathbb{E}\{X^2\} = (-2)^2 \times \frac{1}{4} + (-1)^2 \times \frac{1}{4} + 0^2 \times \frac{1}{4} + 1^2 \times \frac{1}{4} = \frac{3}{2}$$

$$\text{var}(X) = \frac{3}{2} - \left(-\frac{1}{2}\right)^2 = \frac{5}{4}$$

(b)

$$\text{By var}(X) = \mathbb{E}\{X^2\} - (\mathbb{E}\{X\})^2 \text{ and } \mathbb{E}\{aX + b\} = a\mathbb{E}\{X\} + b$$

$$\mathbb{E}\{Y\} = \mathbb{E}\{\mid 2X + (-4) \mid\} = 2\mathbb{E}\{X\} - 4 = 2 \times -\frac{1}{2} - 4 = \mid -5 \mid = 5$$

$$\begin{aligned} \text{var}(Y) &= \mathbb{E}\{Y^2\} - (\mathbb{E}\{Y\})^2 \\ \text{var}(aX + b) &= \mathbb{E}\{(aX + b)^2\} - (\mathbb{E}\{aX + b\})^2 \\ &= \mathbb{E}\{a^2X^2 + 2abX + b^2\} - (a\mathbb{E}\{X\} + b)^2 \\ &= a^2\mathbb{E}\{X^2\} + 2ab\mathbb{E}\{X\} + b^2 - a^2\mathbb{E}\{X\}^2 - 2ab\mathbb{E}\{X\} - b^2 \\ &= a^2(\mathbb{E}\{X^2\} - (E\{X\})^2) \\ &= a^2\text{var}(X) \end{aligned}$$

$$\text{var}\{(2X + (-4))\} = 2^2 \times \frac{5}{4} = 5$$

8. Letting X denote the random variable that is defined as the sum of two fair dice,

$$P\{X = 10\} = P\{(4, 6), (5, 5), (6, 4)\} = \frac{3}{36}$$

Therefore, apply the probability mass function (pmf) of binomial random variable, with parameters ($n = 5$, $p = \frac{3}{36}$). $P\{0\}$ is the event did not occurs within 5 trials.

$$1 - P\{0\} = 1 - \binom{5}{0}\left(\frac{3}{36}\right)^0\left(1 - \frac{3}{36}\right)^{5-0} = 0.353$$

9. (a)

$$\text{By } \frac{d}{da}F(a) = f(a)$$

$$\text{Therefore, } f_X(x) = \begin{cases} \frac{1}{4}, & \text{for } -2 \le x \le 2 \\ 0, & \text{for otherwise} \end{cases}$$

(b)

$$\text{By } \mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)\,dx$$

$$\begin{aligned} \mathbb{E}\{X^4\} &= \int_{-2}^{2} \frac{1}{4} \times x^4\,dx \\ &= \frac{1}{4}\frac{x^5}{5}\ |_{-2}^{2} \\ &= \frac{16}{5} \end{aligned}$$

10. (a)

It can be considered as the probability mass function of a binomial random variable, which X is the number of wins, m is the total trials of games and p is 0.6 for winning probability. Therefore, its pmf is,

$$p(X) = \binom{m}{X}0.6^X(1 - 0.6)^{m-X}$$

(b)

$$\begin{aligned} \mathbb{E}\{X\} &= \sum_{i=0}^{n} ip(i) \\ &= \sum_{i=0}^{n} i\binom{n}{i}p^i(1-p)^{n-i} \\ &= \sum_{i=1}^{n} \frac{in!}{(n-i)!i!}p^i(1-p)^{n-i} \\ &= \sum_{i=1}^{n} \frac{n!}{(n-i)!(i-1)!}p^i(1-p)^{n-i} \\ &= np\sum_{i=1}^{n} \frac{(n-1)!}{(n-i)!(i-1)!}p^{i-1}(1-p)^{n-i} \\ &= np\sum_{k=0}^{n-1} \binom{n-1}{k}p^k(1-p)^{n-1-k} \\ &= np[p + (1-p)]^{n-1} \\ &= np \end{aligned}$$

$$\mathbb{E}\{X\} = 0.6m$$

11. (a)

It is a binomial random variable κ with parameters of $(n = 3,\ p)$.

$$\begin{aligned} P(\kappa = -6000) &= (1-p)^3 \\ P(\kappa = -4000 + 1000) = P(\kappa = -3000) &= 3(1-p)^2 p \\ P(\kappa = -2000 + 2000) = P(\kappa = 0) &= 3p^2(1-p) \\ P(\kappa = 3000) &= p^3 \end{aligned}$$

(b)

The $\mathbb{E}\{k\} = \mathbb{E}(1000X - 2000(3-X))$ for X heads, then we can profit $1000X$ and lose $2000(3-X)$

$\mathbb{E}\{k\} = 3000\mathbb{E}\{X\} - 6000$ and by $\mathbb{E}\{X\} = np = 3p \implies \mathbb{E}\{k\} = 9000p - 6000$

$$\text{To find the minimum value of } p \text{ that the gambler will lose any money}$$

$$\begin{aligned} 9000p - 6000 &\geq 0 \\ p &\geq \frac{2}{3} \end{aligned}$$

12. (a)

It is a binomial distribution with parameters of $n = 5$ and $p = 0.5$ for winning and losing both.

Since at least won three games to end the series, therefore the minimum n will be 3 and for both team can end the series, to multiple the probability by 2, because they are disjoint event.

$$\begin{aligned} P(N=3) &= \binom{3}{3} 0.5^3 \times 0.5^0 \times 2 = \frac{1}{4} \\ P(N=4) &= \binom{4}{3} 0.5^3 \times 0.5 \times 2 = \frac{1}{2} \\ P(N=5) &= \binom{5}{3} 0.5^3 \times 0.5^2 \times 2 = \frac{5}{8} \end{aligned}$$

(b)

$$\begin{aligned} P(W=0) &= \binom{3}{0} 0.5^0 \times 0.5^3 = \frac{1}{8} \\ P(W=1) &= \binom{4}{1} 0.5^1 \times 0.5^3 = \frac{1}{4} \\ P(W=2) &= \binom{5}{2} 0.5^2 \times 0.5^3 = \frac{5}{16} \\ P(W=3) &= \binom{5}{3} 0.5^3 \times 0.5^2 = \frac{5}{16} \end{aligned}$$

(c)

$$
\begin{aligned}
P(L=0) &= \binom{3}{3}0.5^3 \times 0.5^0 = \frac{1}{8} \\
P(L=1) &= \binom{4}{3}0.5^3 \times 0.5^1 = \frac{1}{4} \\
P(L=2) &= \binom{5}{3}0.5^3 \times 0.5^2 = \frac{5}{16} \\
P(L=3) &= \binom{5}{2}0.5^2 \times 0.5^3 = \frac{5}{16}
\end{aligned}
$$

13. We have $\mathbb{E}\{X\} = 7$ and $\text{var}(X) = 3$. By,

$$
\begin{aligned}
E[X] &= \int_{\alpha}^{\beta} \frac{x}{\beta - \alpha} dx \\
&= \frac{\beta^2 - \alpha^2}{2(\beta - \alpha)} \\
&= \frac{\beta + \alpha}{2}
\end{aligned}
$$

$$
\mathbb{E}\{X\} = \beta + \alpha = 14
$$

$$
\begin{aligned}
\text{By var}(X) &= \frac{(\beta - \alpha)^2}{12} \\
\frac{(\beta - \alpha)^2}{12} &= 3 \\
\beta - \alpha &= \pm 6
\end{aligned}
$$

$$
\text{When } \beta - \alpha = -6 \implies \alpha = \beta + 6
$$

$$
\begin{aligned}
\text{Then } \beta + \beta + 6 &= 14 \\
\beta &= 4 \\
\alpha &= 4 + 6 = 10
\end{aligned}
$$

$$
\text{When } \beta - \alpha = +6 \implies \beta = \alpha + 6
$$

$$
\begin{aligned}
\text{Then } \alpha + \alpha + 6 &= 14 \\
\alpha &= 4 \\
\beta &= 10
\end{aligned}
$$

We only consider the case that $\alpha \leq \beta$, therefore we uses the case of $\alpha = 4,\ \beta = 10$.

$$
f(x) = \begin{cases} \frac{1}{6}, & 4 < x < 10 \\ 0, & \text{otherwise} \end{cases}
$$