EE3210
Signals and Systems

Part 9: Fourier Transform

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Fourier Transform vs. Fourier Series

- Recall that Fourier series representations of periodic signals are all in the form of a linear combination of harmonically related complex exponentials.
  - In the case of periodic signals, we consider only harmonically related complex exponentials because we want to make sure that the Fourier series representation of a periodic signal is again a periodic signal with the same fundamental period.

Fourier Transform vs. Fourier Series (cont.)

- In contrast, we will see that, with Fourier transform, aperiodic signals can also be represented through a linear combination of complex exponentials but are infinitesimally close in frequency.
  - Intuitively, this makes sense because, in the case of aperiodic signals, we do not have the constraint of periodicity.
  - Consequently, the Fourier transform representation of an aperiodic signal in terms of a linear combination takes the form of an integral rather than a sum.

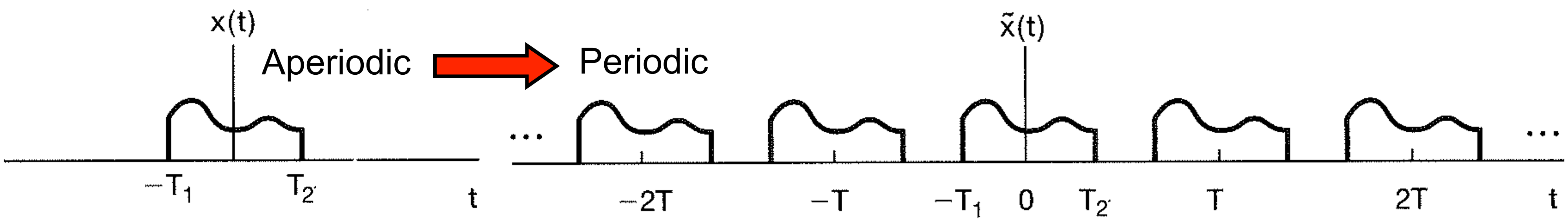
Basic Idea of Fourier Transform

- The basic idea of Fourier transform is to view an aperiodic signal as a periodic signal with an infinite period.
- This allows us to examine the limiting behaviour of the Fourier series representation of the periodic signal, which then leads to the Fourier transform representation of the aperiodic signal.

Basic Idea of Fourier Transform (cont.)

- Specifically, in the Fourier series representation of a periodic signal, as the period T or N increases, the fundamental frequency ω_0 or Ω_0 decreases, and hence the harmonically related components become closer in frequency.
- As the period approaches ∞, the frequency components form a continuum, and the Fourier series sum becomes an integral.

Fourier Transform Representation of Continuous-Time Aperiodic Signals



- Consider an aperiodic signal $x(t)$ with finite duration.
  - That is, for some numbers T_1 and T_2, $x(t) = 0$ outside the interval $-T_1 < t < T_2$.
- From $x(t)$, we can construct a periodic signal $\tilde{x}(t)$ with fundamental period $T_0 = T$ for which $x(t)$ is one period.
  - This requires that $T_1 + T_2 < T$.
  - As $T \to \infty$, $\tilde{x}(t)$ approaches $x(t)$.

Fourier Transform Representation of Continuous-Time Aperiodic Signals (cont.)

- Recall the Fourier series representation of the periodic signal $\tilde{x}(t)$ that is of the form:

$$\boxed{\text{Synthesis: } \tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}} \quad (1)$$

where

$$\boxed{\text{Analysis: } a_k = \frac{1}{T}\int_T \tilde{x}(t) e^{-jk\omega_0 t} dt} \quad (2)$$

- The fundamental frequency of the Fourier series is $\omega_0 = 2\pi/T$.

Fourier Transform Representation of Continuous-Time Aperiodic Signals (cont.)

- Since $\tilde{x}(t) = x(t)$ over a period that includes the interval $-T_1 < t < T_2$, and also since $x(t) = 0$ outside this interval, we can rewrite (2) as:

$$a_k = \frac{1}{T}\int_{-T_1}^{T_2} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-\infty}^{+\infty} x(t)e^{-jk\omega_0 t}dt$$

- Then, defining $X(\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$, we have

$$a_k = \frac{1}{T}X(k\omega_0) \qquad (3)$$

Fourier Transform Representation of Continuous-Time Aperiodic Signals (cont.)

- Now, combining (1) and (3), we can express $\tilde{x}(t)$ in terms of $X(\omega)$ as

$$\tilde{x}(t) = \frac{1}{T} \sum_{k=-\infty}^{+\infty} X(k\omega_0) e^{jk\omega_0 t}$$

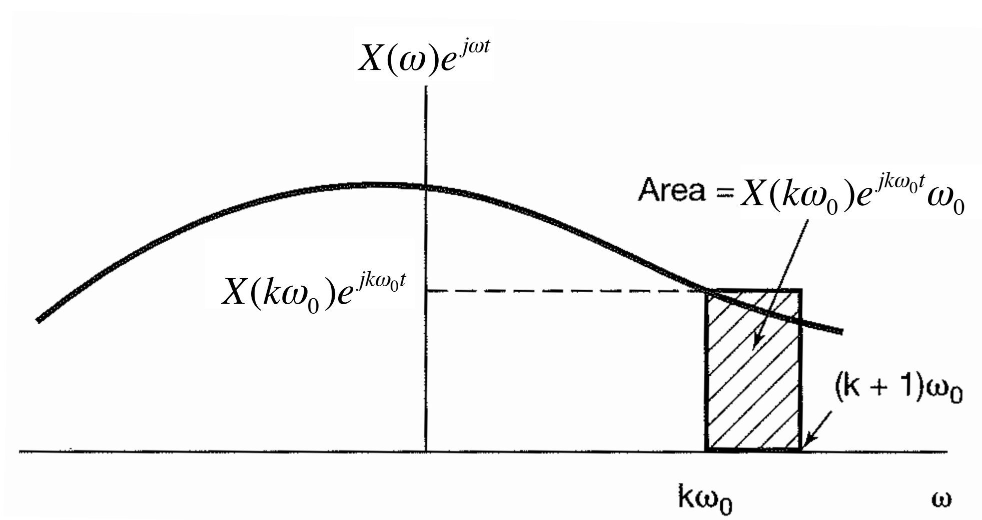
which is equivalent to

$$\tilde{x}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} X(k\omega_0) e^{jk\omega_0 t} \omega_0 \qquad (4)$$

since $2\pi/T = \omega_0 \Rightarrow 1/T = \omega_0/(2\pi)$.

Fourier Transform Representation of Continuous-Time Aperiodic Signals (cont.)

- As $T \to \infty$, $\tilde{x}(t)$ approaches $x(t)$, and consequently, in the limit, (4) becomes a representation of $x(t)$.
- Furthermore, as $T \to \infty$, ω_0 approaches 0, and the right-hand side of (4) approaches an integral of the form



$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)e^{j\omega t}\,d\omega$$

Fourier Transform Representation of Continuous-Time Aperiodic Signals (cont.)

- Therefore, using the fact that $\tilde{x}(t) \to x(t)$ as $T \to \infty$, we have:

$$\boxed{\text{Synthesis: } x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)e^{j\omega t}d\omega} \tag{5}$$

where

$$\boxed{\text{Analysis: } X(\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt} \tag{6}$$

- (5) and (6) are referred to as the continuous-time Fourier transform pair, with $X(\omega)$ referred to as the Fourier transform of $x(t)$.

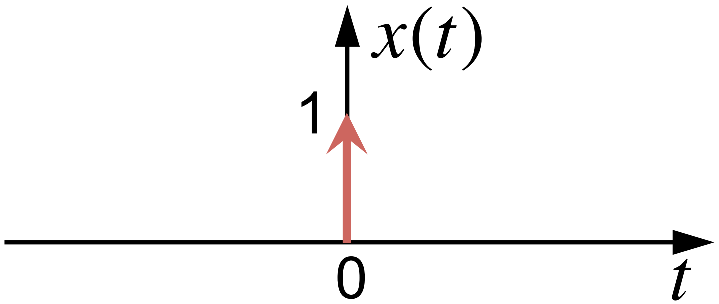
Observations

- We observe in (5) that:
  - An aperiodic signal $x(t)$ can be represented as a linear combination of complex exponentials, although the complex exponentials occur at a continuum of frequencies.
    - The weight on the complex exponential $e^{j\omega t}$ is $X(\omega)d\omega/(2\pi)$.
  - The Fourier transform $X(\omega)$ of $x(t)$ provides us with the information on how $x(t)$ is composed of complex exponentials at different frequencies.
    - For this reason, $X(\omega)$ is commonly referred to as the spectrum of $x(t)$.

Example 1

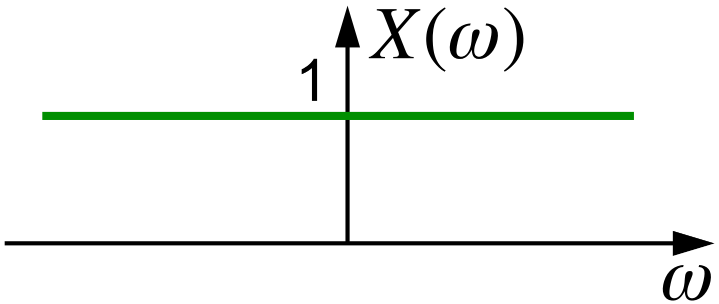
- Consider the continuous-time unit impulse, i.e.,

$$x(t) = \delta(t)$$



- Applying the analysis formula (6), we obtain

$$X(\omega) = \int_{-\infty}^{+\infty} \delta(t) e^{-j\omega t} dt = 1$$

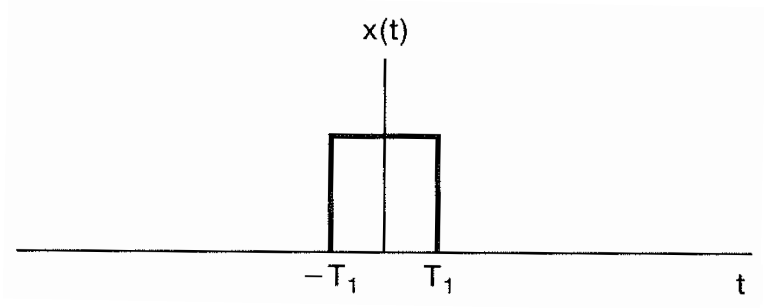


  - That is, $\delta(t)$ has a Fourier transform consisting of equal contributions at all frequencies.

Example 2

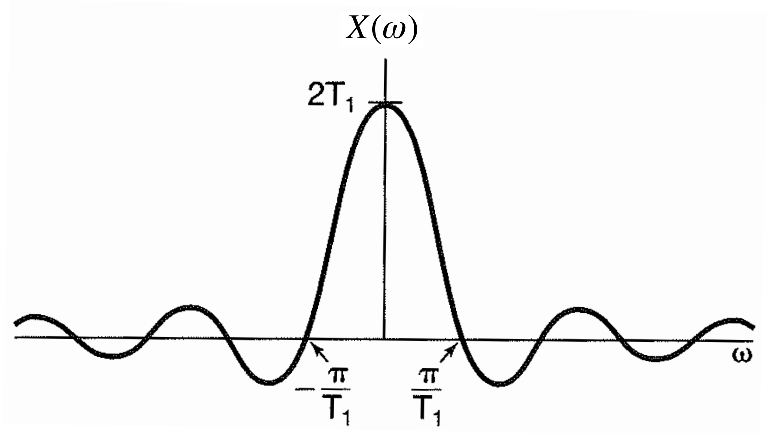
- Consider the rectangular pulse signal

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}$$



- Applying the analysis formula (6), we obtain

$$X(\omega) = \int_{-T_1}^{T_1} e^{-j\omega t} dt = \frac{2\sin(\omega T_1)}{\omega}$$
$$= 2T_1 \operatorname{sinc}\left(\frac{\omega T_1}{\pi}\right)$$

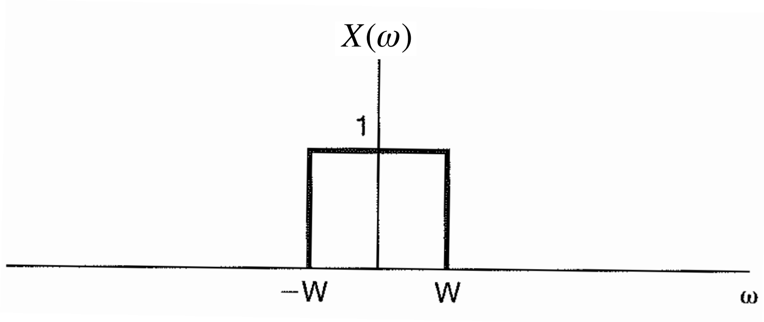


- Note: $\operatorname{sinc}(\cdot)$ is known as the sinc function, defined by $\operatorname{sinc}(\theta) = \sin(\pi\theta)/(\pi\theta)$.

Example 3

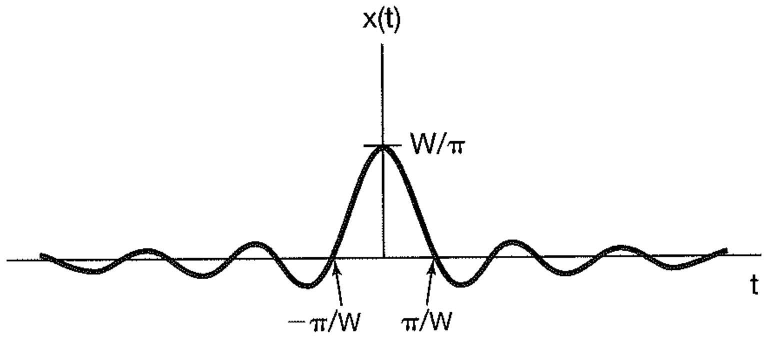
- Consider the signal $x(t)$ whose Fourier transform is

$$X(\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}$$



- Applying the synthesis formula (5), we obtain

$$x(t) = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t} d\omega = \frac{\sin(Wt)}{\pi t}$$
$$= \frac{W}{\pi}\,\mathrm{sinc}\left(\frac{Wt}{\pi}\right)$$

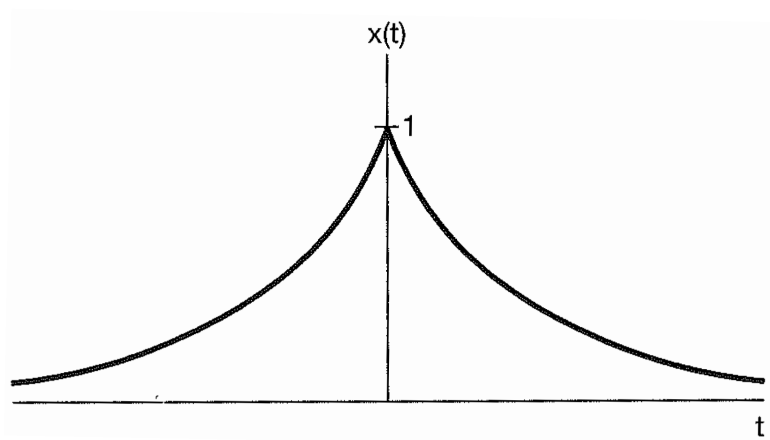


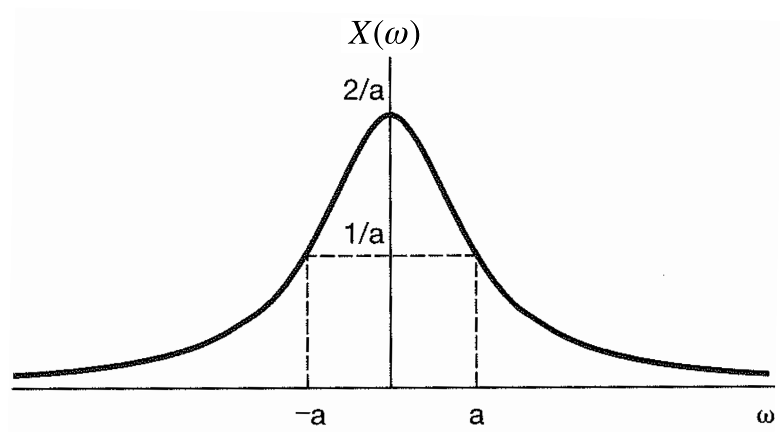
Example 4

- Consider the signal $x(t) = e^{-a|t|}$, $a > 0$.
- Applying the analysis formula (6), we obtain

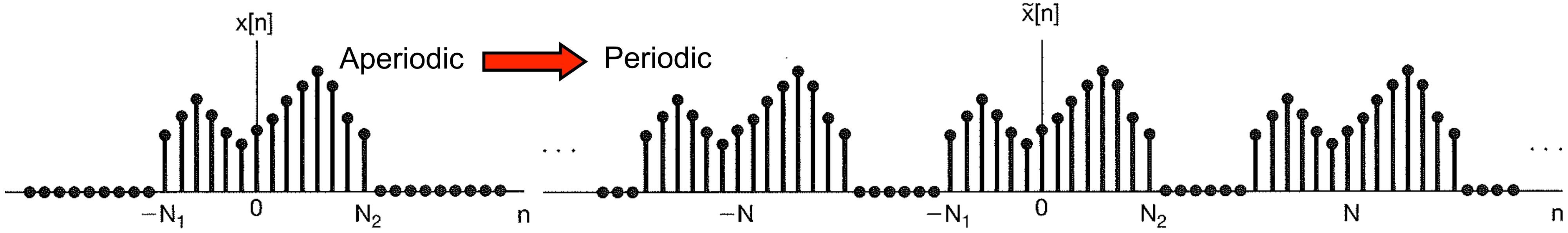
$$X(\omega) = \int_{-\infty}^{0} e^{at} e^{-j\omega t} dt + \int_{0}^{+\infty} e^{-at} e^{-j\omega t} dt$$

$$= \frac{1}{a - j\omega} + \frac{1}{a + j\omega} = \frac{2a}{a^2 + \omega^2}$$





Fourier Transform Representation of Discrete-Time Aperiodic Signals



- Consider an aperiodic signal $x[n]$ with finite duration.
  - That is, for some integers N_1 and N_2, $x[n] = 0$ outside the range $-N_1 \leq n \leq N_2$.
- From $x[n]$, we can construct a periodic signal $\tilde{x}[n]$ with fundamental period $N_0 = N$ for which $x[n]$ is one period.
  - This requires that $N_1 + N_2 < N$.
  - As $N \to \infty$, $\tilde{x}[n]$ approaches $x[n]$.

Fourier Transform Representation of Discrete-Time Aperiodic Signals (cont.)

- Recall the Fourier series representation of the periodic signal $\tilde{x}[n]$ that is of the form:

$$\boxed{\text{Synthesis: } \tilde{x}[n] = \sum_{k=\langle N \rangle} a_k e^{jk\Omega_0 n}} \tag{7}$$

where

$$\boxed{\text{Analysis: } a_k = \frac{1}{N} \sum_{n=\langle N \rangle} \tilde{x}[n] e^{-jk\Omega_0 n}} \tag{8}$$

- The fundamental frequency of the Fourier series is $\Omega_0 = 2\pi/N$.

Fourier Transform Representation of Discrete-Time Aperiodic Signals (cont.)

- Since $\tilde{x}[n] = x[n]$ over a period that includes the range $-N_1 \leq n \leq N_2$, and also since $x[n] = 0$ outside this range, we can rewrite (8) as:

$$a_k = \frac{1}{N}\sum_{n=-N_1}^{N_2} x[n]e^{-jk\Omega_0 n} = \frac{1}{N}\sum_{n=-\infty}^{+\infty} x[n]e^{-jk\Omega_0 n}$$

- Then, defining $X[\Omega] = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\Omega n}$, which is periodic in Ω with period 2π, we have

$$a_k = \frac{1}{N}X[k\Omega_0] \qquad (9)$$

Fourier Transform Representation of Discrete-Time Aperiodic Signals (cont.)

- Now, combining (7) and (9), we can express $\tilde{x}[n]$ in terms of $X[\Omega]$ as

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=\langle N \rangle} X[k\Omega_0] e^{jk\Omega_0 n} = \frac{1}{2\pi} \sum_{k=q}^{q+N-1} X[k\Omega_0] e^{jk\Omega_0 n} \Omega_0 \tag{10}$$

since $\sum_{k=\langle N \rangle}$ has the same effect as $\sum_{k=q}^{q+N-1}$ for any integer number q, and $2\pi/N = \Omega_0 \Rightarrow 1/N = \Omega_0/(2\pi)$.

- As $N \to \infty$, $\tilde{x}[n]$ approaches $x[n]$, and consequently, in the limit, (10) becomes a representation of $x[n]$.

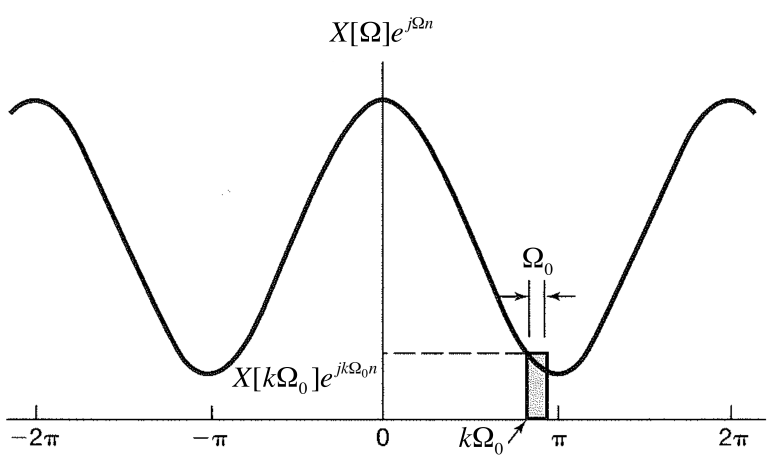
Fourier Transform Representation of Discrete-Time Aperiodic Signals (cont.)

- Furthermore, as $N \to \infty$, $\Omega_0 \to 0$, and we have

$$(q\Omega_0) < (k\Omega_0) < (q\Omega_0 + N\Omega_0 - \Omega_0) \to (q\Omega_0 + 2\pi)$$

  since $N\Omega_0 \equiv 2\pi$.

- Therefore, the right-hand side of (10) approaches an integral of the form $\frac{1}{2\pi}\int_{2\pi} X[\Omega]e^{j\Omega n}d\Omega$.
  - That is, the integration can be performed over any interval of length 2π, since $X[\Omega]e^{j\Omega n}$ is periodic in Ω with period 2π.



Fourier Transform Representation of Discrete-Time Aperiodic Signals (cont.)

- Thus, using the fact that $\tilde{x}[n] \to x[n]$ as $N \to \infty$, we have:

$$\boxed{\text{Synthesis: } x[n] = \frac{1}{2\pi}\int_{2\pi} X[\Omega] e^{j\Omega n} d\Omega} \quad (11)$$

where

$$\boxed{\text{Analysis: } X[\Omega] = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\Omega n}} \quad (12)$$

- (11) and (12) are referred to as the discrete-time Fourier transform pair, with $X[\Omega]$ referred to as the Fourier transform of $x[n]$.

Observations

- We observe in (11) that:
  - Similar to its continuous-time counterpart, an aperiodic signal $x[n]$ can be represented as a linear combination of complex exponentials that are infinitesimally close in frequency.
    - The weight on the complex exponential $e^{j\Omega n}$ is $X[\Omega]d\Omega/(2\pi)$.
  - The Fourier transform $X[\Omega]$ of $x[n]$ provides us with the information on how $x[n]$ is composed of complex exponentials at different frequencies.
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Example 1

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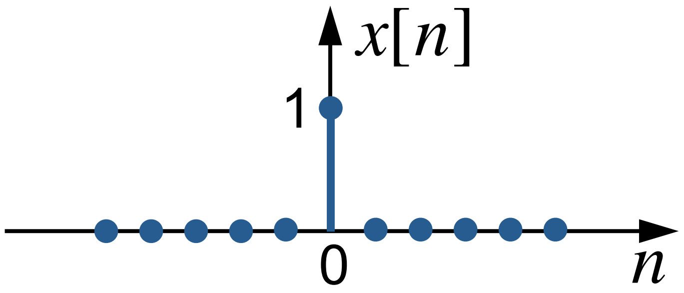
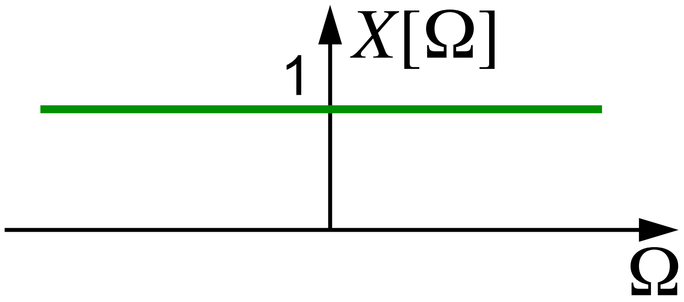


- Applying the analysis formula (12), we obtain

$$X[\Omega] = \sum_{n=-\infty}^{+\infty} \delta[n] e^{-j\Omega n} = 1$$

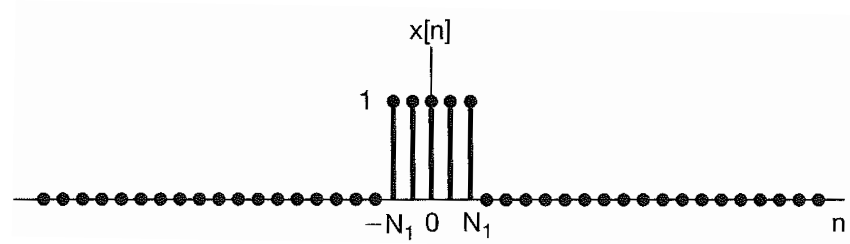


  - That is, $\delta[n]$ has a Fourier transform consisting of equal contributions at all frequencies.

Example 2

- Consider the rectangular pulse signal

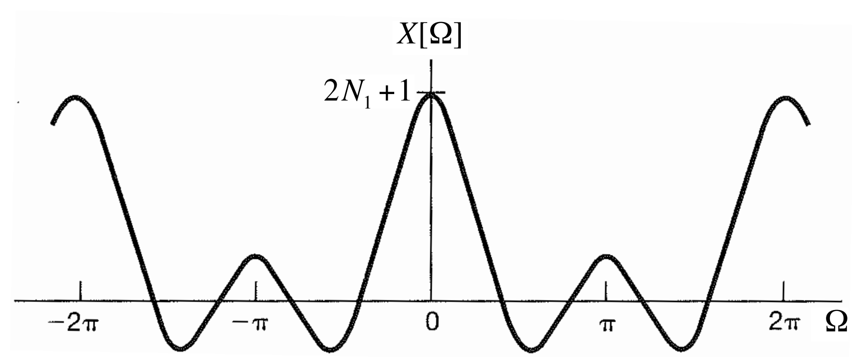
$$x[n] = \begin{cases} 1, & |n| \le N_1 \\ 0, & |n| > N_1 \end{cases}$$



- Applying the analysis formula (12), we obtain

$$X[\Omega] = \sum_{n=-N_1}^{+N_1} e^{-j\Omega n}$$

$$= \frac{\sin[\Omega(N_1 + 1/2)]}{\sin(\Omega/2)}$$



Example 3

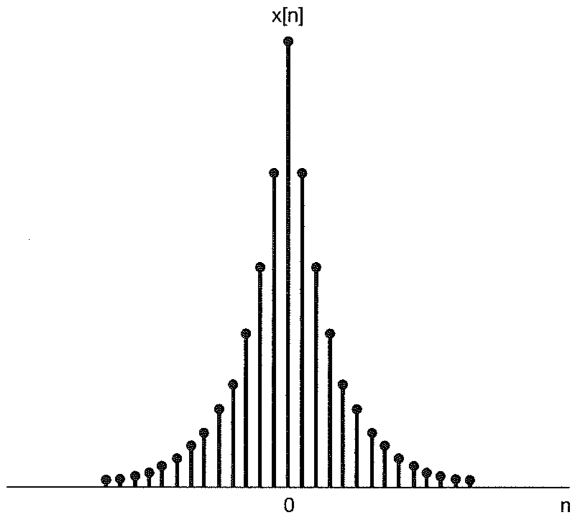
- Consider $x[n] = a^{|n|}$, $|a| < 1$.
- Applying the analysis formula (12), we obtain

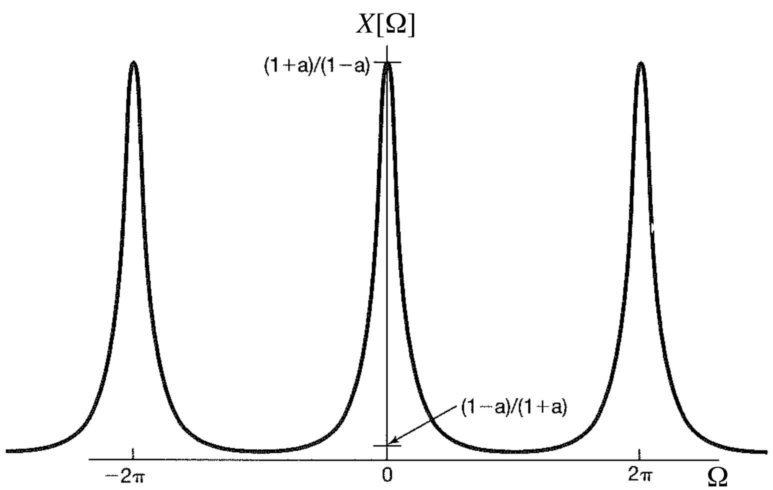
$$X[\Omega] = \sum_{n=-\infty}^{-1} a^{-n} e^{-j\Omega n} + \sum_{n=0}^{+\infty} a^{n} e^{-j\Omega n}$$

$$= \sum_{m=1}^{+\infty} a^{m} e^{j\Omega m} + \sum_{n=0}^{+\infty} a^{n} e^{-j\Omega n}$$

$$= \frac{ae^{j\Omega}}{1 - ae^{j\Omega}} + \frac{1}{1 - ae^{-j\Omega}}$$

$$= \frac{1 - a^2}{1 - 2a\cos\Omega + a^2}$$





Convergence of Fourier Transform

- Fourier transform works not only for signals with finite duration but also for an extremely broad class of signals of infinite duration.

- Convergence issues of continuous-time Fourier transform are similar to those of continuous-time Fourier series.
  - In particular, any signal $x(t)$ that is continuous or that has a finite number of discontinuities has a Fourier transform if it is absolutely integrable, i.e.,

$$\int_{-\infty}^{+\infty} |x(t)|\, dt < \infty$$

Convergence of Fourier Transform (cont.)

- The discrete-time Fourier transform of a signal $x[n]$ exists if $x[n]$ is absolutely summable, i.e.,

$$\sum_{n=-\infty}^{+\infty} |x[n]| < \infty$$

Properties of Fourier Transform

- Here, we will describe several important properties of continuous-time and/or discrete-time Fourier transform, including the convolution property that forms the basis for frequency-domain analysis of LTI systems.
  - A summary of these and other important properties of continuous-time Fourier transform can be found in Table 4.1 on Page 331 of the textbook.
  - A summary of these and other important properties of discrete-time Fourier transform can be found in Table 5.1 on Page 394 of the textbook.

Properties of Fourier Transform (cont.)

- For notational convenience, we will use:
  - $x(t) \leftrightarrow X(\omega)$ to indicate the paring of a continuous-time signal $x(t)$ and its Fourier transform.
  - $x[n] \leftrightarrow X[\Omega]$ to indicate the paring of a discrete-time signal $x[n]$ and its Fourier transform.
- We will also use the notation $\mathcal{F}\{\cdot\}$ to indicate the Fourier transform and the notation $\mathcal{F}^{-1}\{\cdot\}$ to indicate the inverse Fourier transform, i.e.,

$$X(\omega) = \mathcal{F}\{x(t)\} \;\leftrightarrow\; x(t) = \mathcal{F}^{-1}\{X(\omega)\}$$

$$X[\Omega] = \mathcal{F}\{x[n]\} \;\leftrightarrow\; x[n] = \mathcal{F}^{-1}\{X[\Omega]\}$$

Linearity

- Continuous-time Fourier transform:
  - If $x(t) \leftrightarrow X(\omega)$ and $y(t) \leftrightarrow Y(\omega)$, then

$$ax(t) + by(t) \leftrightarrow aX(\omega) + bY(\omega)$$

where a and b are arbitrary constants.

- Discrete-time Fourier transform:
  - If $x[n] \leftrightarrow X[\Omega]$ and $y[n] \leftrightarrow Y[\Omega]$, then

$$ax[n] + by[n] \leftrightarrow aX[\Omega] + bY[\Omega]$$

where a and b are arbitrary constants.

Time Shift

- Continuous-time Fourier transform:
  - If $x(t) \leftrightarrow X(\omega)$, then

$$x(t - t_0) \leftrightarrow e^{-j\omega t_0} X(\omega)$$

- Discrete-time Fourier transform:
  - If $x[n] \leftrightarrow X[\Omega]$, then

$$x[n - n_0] \leftrightarrow e^{-j\Omega n_0} X[\Omega]$$

Time Reversal

- Continuous-time Fourier transform:
  - If $x(t) \leftrightarrow X(\omega)$, then

$$x(-t) \leftrightarrow X(-\omega)$$

- Discrete-time Fourier transform:
  - If $x[n] \leftrightarrow X[\Omega]$, then

$$x[-n] \leftrightarrow X[-\Omega]$$

Time Scaling

- Continuous-time Fourier transform:
  - If $x(t) \leftrightarrow X(\omega)$, then

$$x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

Differentiation in Time

- Continuous-time Fourier transform:
  - If $x(t) \leftrightarrow X(\omega)$, then

$$\frac{dx(t)}{dt} \leftrightarrow (j\omega)X(\omega)$$

First Difference in Time

- Discrete-time Fourier transform:
  - If $x[n] \leftrightarrow X[\Omega]$, then

$$x[n] - x[n-1] \leftrightarrow \left(1 - e^{-j\Omega}\right) X[\Omega]$$

Differentiation in Frequency

- Continuous-time Fourier transform:
  - If $x(t) \leftrightarrow X(\omega)$, then

$$tx(t) \leftrightarrow j\frac{dX(\omega)}{d\omega}$$

- Discrete-time Fourier transform:
  - If $x[n] \leftrightarrow X[\Omega]$, then

$$nx[n] \leftrightarrow j\frac{dX[\Omega]}{d\Omega}$$

Parseval's Relation

- Continuous-time Fourier transform:
  - If $x(t) \leftrightarrow X(\omega)$, then

$$\int_{-\infty}^{+\infty} |x(t)|^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(\omega)|^2 d\omega$$

- Discrete-time Fourier transform:
  - If $x[n] \leftrightarrow X[\Omega]$, then

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X[\Omega]|^2 d\Omega$$

- $|X(\omega)|^2$ and $|X[\Omega]|^2$ are called the energy density spectrum of $x(t)$ and $x[n]$, respectively.

Multiplication

- Continuous-time Fourier transform:
  - If $x(t) \leftrightarrow X(\omega)$ and $y(t) \leftrightarrow Y(\omega)$, then

$$x(t)y(t) \leftrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\theta)Y(\omega - \theta)d\theta$$

- Discrete-time Fourier transform:
  - If $x[n] \leftrightarrow X[\Omega]$ and $y[n] \leftrightarrow Y[\Omega]$, then

$$x[n]y[n] \leftrightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} X[\theta]Y[\Omega - \theta]d\theta$$

where a and b are arbitrary constants.

Convolution

- Continuous-time Fourier transform:
  - If $x(t) \leftrightarrow X(\omega)$ and $y(t) \leftrightarrow Y(\omega)$, then

$$x(t) * y(t) \leftrightarrow X(\omega)Y(\omega)$$

- Discrete-time Fourier transform:
  - If $x[n] \leftrightarrow X[\Omega]$ and $y[n] \leftrightarrow Y[\Omega]$, then

$$x[n] * y[n] \leftrightarrow X[\Omega]Y[\Omega]$$

Convolution (cont.)

- Proof for the continuous-time case:

$$
\begin{aligned}
&\mathcal{F}\{x(t) * y(t)\} \\
&= \int_{-\infty}^{+\infty} [x(t) * y(t)] e^{-j\omega t} dt = \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} x(\tau) y(t-\tau) d\tau \right] e^{-j\omega t} dt \\
&= \int_{-\infty}^{+\infty} x(\tau) \left[\int_{-\infty}^{+\infty} y(t-\tau) e^{-j\omega t} dt \right] d\tau \\
&= \int_{-\infty}^{+\infty} x(\tau) \left[\int_{-\infty}^{+\infty} y(m) e^{-j\omega (m+\tau)} dm \right] d\tau \\
&= \int_{-\infty}^{+\infty} x(\tau) e^{-j\omega \tau} d\tau \int_{-\infty}^{+\infty} y(m) e^{-j\omega m} dm \\
&= X(\omega) Y(\omega)
\end{aligned}
$$

Convolution (cont.)

- Proof for the discrete-time case:

$$
\begin{aligned}
&\mathcal{F}\{x[n] * y[n]\} \\
&= \sum_{n=-\infty}^{+\infty} (x[n] * y[n]) e^{-j\Omega n} = \sum_{n=-\infty}^{+\infty} \left(\sum_{k=-\infty}^{+\infty} x[k] y[n-k] \right) e^{-j\Omega n} \\
&= \sum_{k=-\infty}^{+\infty} x[k] \sum_{n=-\infty}^{+\infty} y[n-k] e^{-j\Omega n} = \sum_{k=-\infty}^{+\infty} x[k] \sum_{m=-\infty}^{+\infty} y[m] e^{-j\Omega(m+k)} \\
&= \sum_{k=-\infty}^{+\infty} x[k] e^{-j\Omega k} \sum_{m=-\infty}^{+\infty} y[m] e^{-j\Omega m} \\
&= X[\Omega] Y[\Omega]
\end{aligned}
$$