Appendix – A list of possibly relevant equations

- Trigonometric identities:
  - Identity 1: $\sin^2\theta = \dfrac{1-\cos(2\theta)}{2}$
  - Identity 2: $\cos^2\theta = \dfrac{1+\cos(2\theta)}{2}$
  - Identity 3: $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
  - Identity 4: $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
  - Identity 5: $\sin\alpha\sin\beta = \dfrac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$
  - Identity 6: $\cos\alpha\cos\beta = \dfrac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$
  - Identity 7: $\sin\alpha\cos\beta = \dfrac{1}{2}[\sin(\alpha-\beta) + \sin(\alpha+\beta)]$
- Euler's formula: $e^{j\theta} = \cos\theta + j\sin\theta$
- Fundamental period of a periodic signal:
  - Continuous-time sinusoidal of the form $x(t) = A\cos(\omega t+\phi)$: $T_0 = 2\pi/\omega$
  - Discrete-time sinusoidal of the form $x[n] = A\cos(\Omega n+\phi)$: $N_0 = 2\pi m/\Omega$ if N_0 and m have no factors in common.
  - Continuous-time complex exponential of the form $x(t) = e^{j\omega t}$: $T_0 = 2\pi/|\omega|$
  - Discrete-time complex exponential of the form $x[n] = e^{j\Omega n}$: $N_0 = 2\pi m/|\Omega|$ if N_0 and m have no factors in common.
- Convolution sum: $x[n] * h[n] = \displaystyle\sum_{k=-\infty}^{+\infty} x[k]h[n-k]$
  - Commutative property: $x[n] * h[n] = h[n] * x[n]$
  - Distributive property: $x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$
  - Associative property: $x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n]$
- Convolution integral: $x(t) * h(t) = \displaystyle\int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$
  - Commutative property: $x(t) * h(t) = h(t) * x(t)$
  - Distributive property: $x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t)$
  - Associative property: $x(t) * [h_1(t) * h_2(t)] = [x(t) * h_1(t)] * h_2(t)$

- Properties of continuous-time LTI systems:
  - Memoryless: $h(t) = 0$ for $t \neq 0$.
  - Invertibility: $h(t) * h_1(t) = \delta(t)$ where $h_1(t)$ is the unit impulse response of the inverse system.
  - Causality: $h(t) = 0$ for $t < 0$.
  - Stability: $\int_{-\infty}^{+\infty} |h(t)|\, dt < \infty$

- Properties of discrete-time LTI systems:
  - Memoryless: $h[n] = 0$ for $n \neq 0$.
  - Invertibility: $h[n] * h_1[n] = \delta[n]$ where $h_1[n]$ is the unit impulse response of the inverse system.
  - Causality: $h[n] = 0$ for $n < 0$.
  - Stability: $\sum_{n=-\infty}^{+\infty} |h[n]| < \infty$

- Continuous-time Fourier series:
  - Formulas: Consider $x(t)$ periodic with fundamental period $T_0 = T$.
    * Synthesis: $x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$
    * Analysis: $a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt = \frac{1}{T}\int_T x(t) e^{-jk(2\pi/T)t} dt$
  - Properties: Consider $x(t)$ and $y(t)$ periodic with period T, $x(t) \leftrightarrow a_k$, $y(t) \leftrightarrow b_k$.
    * Linearity: $Ax(t) + By(t) \leftrightarrow Aa_k + Bb_k$
    * Time shift: $x(t - t_0) \leftrightarrow \left[e^{-jk(2\pi/T)t_0}\right] a_k$
    * Time reversal: $x(-t) \leftrightarrow a_{-k}$
    * Time scaling: $x(\alpha t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk(\alpha\omega_0)t}$
    * Multiplication: $x(t)y(t) \leftrightarrow \sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
    * Differentiation: $\frac{dx(t)}{dt} \leftrightarrow (jk\omega_0) a_k$
    * Parseval's relation: $\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$

- Discrete-time Fourier series:
  - Formulas: Consider $x[n]$ periodic with fundamental period $N_0 = N$.
    * Synthesis: $x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\Omega_0 n} = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}$
    * Analysis: $a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\Omega_0 n} = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk(2\pi/N)n}$
  - Properties: Consider $x[n]$ and $y[n]$ periodic with period N, $x[n] \leftrightarrow a_k$, $y[n] \leftrightarrow b_k$.
    * Linearity: $Ax[n] + By[n] \leftrightarrow Aa_k + Bb_k$
    * Time shift: $x[n - n_0] \leftrightarrow \left[e^{-jk(2\pi/N)n_0}\right] a_k$
    * Time reversal: $x[-n] \leftrightarrow a_{-k}$
    * Multiplication: $x[n]y[n] \leftrightarrow \sum_{l=\langle N \rangle} a_l b_{k-l}$
    * First difference: $x[n] - x[n-1] \leftrightarrow \left[1 - e^{-jk(2\pi/N)}\right] a_k$
    * Parseval's relation: $\frac{1}{N} \sum_{n=\langle N \rangle} |x[n]|^2 = \sum_{k=\langle N \rangle} |a_k|^2$

- Continuous-time Fourier transform:
  - Formulas:
    * Analysis: $X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$
    * Synthesis: $x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) e^{j\omega t} d\omega$
  - Properties: Consider $x(t) \leftrightarrow X(\omega)$, $y(t) \leftrightarrow Y(\omega)$.
    * Linearity: $ax(t) + by(t) \leftrightarrow aX(\omega) + bY(\omega)$
    * Time shift: $x(t - t_0) \leftrightarrow e^{-j\omega t_0} X(\omega)$
    * Time reversal: $x(-t) \leftrightarrow X(-\omega)$
    * Time scaling: $x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$
    * Differentiation in time: $\frac{dx(t)}{dt} \leftrightarrow (j\omega) X(\omega)$
    * Differentiation in frequency: $tx(t) \leftrightarrow j \frac{dX(\omega)}{d\omega}$
    * Multiplication: $x(t)y(t) \leftrightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\theta) Y(\omega - \theta) d\theta$
    * Convolution: $x(t) * y(t) \leftrightarrow X(\omega) Y(\omega)$
    * Parseval's relation: $\int_{-\infty}^{+\infty} |x(t)|^2 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(\omega)|^2 d\omega$

- Pairs:
  * Pair 1: $e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0)$
  * Pair 2: $x(t) = 1 \leftrightarrow 2\pi\delta(\omega)$
  * Pair 3: $x(t) = \begin{cases} 1, & |t| < T \\ 0, & |t| > T \end{cases} \leftrightarrow \dfrac{2\sin(\omega T)}{\omega}$
  * Pair 4: $\dfrac{\sin(Wt)}{\pi t} \leftrightarrow X(\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}$
  * Pair 5: $\delta(t) \leftrightarrow 1$
  * Pair 6: $e^{-at}u(t),\ \mathrm{Re}\{a\} > 0 \leftrightarrow \dfrac{1}{a + j\omega}$
  * Pair 7: $te^{-at}u(t),\ \mathrm{Re}\{a\} > 0 \leftrightarrow \dfrac{1}{(a + j\omega)^2}$

- Discrete-time Fourier transform:
  - Formulas:
    * Analysis: $X[\Omega] = \displaystyle\sum_{n=-\infty}^{+\infty} x[n]e^{-j\Omega n}$
    * Synthesis: $x[n] = \dfrac{1}{2\pi}\displaystyle\int_{2\pi} X[\Omega]e^{j\Omega n}d\Omega$
  - Properties: Consider $x[n] \leftrightarrow X[\Omega]$, $y[n] \leftrightarrow Y[\Omega]$.
    * Linearity: $ax[n] + by[n] \leftrightarrow aX[\Omega] + bY[\Omega]$
    * Time shift: $x[n - n_0] \leftrightarrow e^{-j\Omega n_0}X[\Omega]$
    * Time reversal: $x[-n] \leftrightarrow X[-\Omega]$
    * First difference in time: $x[n] - x[n-1] \leftrightarrow \left(1 - e^{-j\Omega}\right)X[\Omega]$
    * Differentiation in frequency: $nx[n] \leftrightarrow j\dfrac{dX[\Omega]}{d\Omega}$
    * Multiplication: $x[n]y[n] \leftrightarrow \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} X[\theta]Y[\Omega - \theta]d\theta$
    * Convolution: $x[n] * y[n] \leftrightarrow X[\Omega]Y[\Omega]$
    * Parseval's relation: $\displaystyle\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \dfrac{1}{2\pi}\int_{2\pi} |X[\Omega]|^2 d\Omega$
  - Pairs:
    * Pair 1: $\delta[n] \leftrightarrow 1$
    * Pair 2: $a^n u[n],\ |a| < 1 \leftrightarrow \dfrac{1}{1 - ae^{-j\Omega}}$
    * Pair 3: $(n+1)a^n u[n],\ |a| < 1 \leftrightarrow \dfrac{1}{(1 - ae^{-j\Omega})^2}$

- Laplace transform:
  - Formula: $X(s) = \int_{-\infty}^{+\infty} x(t)e^{-st}dt$
  - Properties: Consider $x(t) \leftrightarrow X(s)$ with ROC $= R$, $y(t) \leftrightarrow Y(s)$ with ROC $= R'$.
    - Linearity: $ax(t) + by(t) \leftrightarrow aX(s) + bY(s)$ with ROC $\supseteq (R \cap R')$
    - Time shift: $x(t - t_0) \leftrightarrow e^{-st_0}X(s)$ with ROC $= R$
    - Time reversal: $x(-t) \leftrightarrow X(-s)$ with ROC $= -R$
    - Time scaling: $x(at) \leftrightarrow \frac{1}{|a|}X\left(\frac{s}{a}\right)$ with ROC $= aR$
    - Differentiation in the time domain: $\frac{dx(t)}{dt} \leftrightarrow sX(s)$ with ROC $\supseteq R$
    - Differentiation in the s-domain: $-tx(t) \leftrightarrow \frac{dX(s)}{ds}$ with ROC $= R$
    - Convolution: $x(t) * y(t) \leftrightarrow X(s)Y(s)$ with ROC $\supseteq (R \cap R')$
  - Pairs:
    - Pair 1: $\delta(t) \leftrightarrow 1, \text{ for all } s$
    - Pair 2: $u(t) \leftrightarrow \frac{1}{s}, \ \mathrm{Re}\{s\} > 0$
    - Pair 3: $-u(-t) \leftrightarrow \frac{1}{s}, \ \mathrm{Re}\{s\} < 0$
    - Pair 4: $tu(t) \leftrightarrow \frac{1}{s^2}, \ \mathrm{Re}\{s\} > 0$
    - Pair 5: $-tu(-t) \leftrightarrow \frac{1}{s^2}, \ \mathrm{Re}\{s\} < 0$
    - Pair 6: $e^{-at}u(t) \leftrightarrow \frac{1}{s+a}, \ \mathrm{Re}\{s\} > -a$
    - Pair 7: $-e^{-at}u(-t) \leftrightarrow \frac{1}{s+a}, \ \mathrm{Re}\{s\} < -a$
    - Pair 8: $te^{-at}u(t) \leftrightarrow \frac{1}{(s+a)^2}, \ \mathrm{Re}\{s\} > -a$
    - Pair 9: $-te^{-at}u(-t) \leftrightarrow \frac{1}{(s+a)^2}, \ \mathrm{Re}\{s\} < -a$
- z-transform:
  - Formula: $X[z] = \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$
  - Properties: Consider $x[n] \leftrightarrow X[z]$ with ROC $= R$, $y[n] \leftrightarrow Y[z]$ with ROC $= R'$.
    - Linearity: $ax[n] + by[n] \leftrightarrow aX[z] + bY[z]$ with ROC $\supseteq (R \cap R')$
    - Time shift: $x[n - n_0] \leftrightarrow z^{-n_0}X[z]$ with ROC $= R$, except for the possible addition or deletion of $z = 0$ or $z = \infty$

    * Time reversal: $x[-n] \leftrightarrow X[1/z]$ with ROC $= 1/R$
    * Differentiation in the z-domain: $nx[n] \leftrightarrow -z\frac{dX[z]}{dz}$ with ROC $= R$
    * Convolution: $x[n] * y[n] \leftrightarrow X[z]Y[z]$ with ROC $\supseteq (R \cap R')$

- Pairs:
    * Pair 1: $\delta[n] \leftrightarrow 1,$ for all z
    * Pair 2: $\delta[n-n_0] \leftrightarrow z^{-n_0}$ for all z except 0 (if $n_0 > 0$) or ∞ (if $n_0 < 0$)
    * Pair 3: $u[n] \leftrightarrow \frac{1}{1-z^{-1}}$ for $|z| > 1$
    * Pair 4: $-u[-n-1] \leftrightarrow \frac{1}{1-z^{-1}}$ for $|z| < 1$
    * Pair 5: $a^n u[n] \leftrightarrow \frac{1}{1-az^{-1}}$ for $|z| > |a|$
    * Pair 6: $-a^n u[-n-1] \leftrightarrow \frac{1}{1-az^{-1}}$ for $|z| < |a|$
    * Pair 7: $na^n u[n] \leftrightarrow \frac{az^{-1}}{(1-az^{-1})^2}$ for $|z| > |a|$
    * Pair 8: $-na^n u[-n-1] \leftrightarrow \frac{az^{-1}}{(1-az^{-1})^2}$ for $|z| < |a|$

— End of Paper —