Solution

1.(a)
The total probability should be 1, so we have:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y)dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \alpha e^{-(4x+5y)}dxdy = \frac{\alpha}{20} = 1 \Rightarrow \alpha = 20$$

1.(b)

$$F(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} p(u,v)dudv = \begin{cases} \int_{-\infty}^{y}\int_{-\infty}^{x} 20e^{-(4u+5v)}dudv, \\ \qquad\qquad\qquad\qquad 0, \end{cases}$$
$$= \begin{cases} (1-e^{-4x})(1-e^{-5y}), & x>0,\ y>0, \\ 0, & \text{otherwise.} \end{cases}$$

1.(c)

$$P(X<1,\ Y<2) = \int_0^2\int_0^1 20e^{-(4x+5y)}dxdy = (1-e^{-4})(1-e^{-10}) = 0.9816$$

2.
Let $Z = X - Y$. As $\mathbb{E}\{X\} = \mathbb{E}\{Y\} = 0$, we have $\mathbb{E}\{Z\} = 0$.

Because $\mathbb{E}\{Z\} = 0$, and X and Y are independent of each other, we have:

$$\mathbb{E}\{Z^2\} = \text{var}(Z) = \mathbb{E}\{(X-Y)^2\} = \mathbb{E}\{X^2\} + \mathbb{E}\{Y^2\} = 2$$

Then we have

$$\text{var}(|X-Y|) = \text{var}(|Z|) = \mathbb{E}\{|Z|^2\} - [\mathbb{E}\{|Z|\}]^2 = \mathbb{E}\{Z^2\} - [\mathbb{E}\{|Z|\}]^2,$$

Using

$$\mathbb{E}\{|Z|\} = \int_{-\infty}^{+\infty} |z|\frac{1}{2\sqrt{\pi}}e^{-z^2/4}dz = \frac{1}{\sqrt{\pi}}\int_0^{+\infty} ze^{-z^2/4}dz = 2\sqrt{\frac{1}{\pi}}$$

Therefore,

$$\text{var}(|X-Y|) = 2 - \frac{4}{\pi}$$

Note that the answer can be validated using MATLAB:

```
>> X= randn(1,1000000);
Y= randn(1,1000000);
Z=abs(X-Y);
var(Z)
ans = 0.7277
```

3.
Denote head and tail as H and T, respectively. The sample space is S={HH, HT, TH, TT}, and each outcome has the probability of 1/4. We can then construct:

outcome	X	Y
HH	0	1
HT	1	0
TH	1	1
TT	2	0

Finally, we obtain:

$P_{XY}(x,y)$	$Y=0$	$Y=1$
$X=0$	0	$\frac{1}{4}$
$X=1$	$\frac{1}{4}$	$\frac{1}{4}$
$X=2$	$\frac{1}{4}$	0

4.
By observing that $P_{NK}(n,k)$ can be factorized as:

$$P_{NK}(n,k) = \frac{100^n e^{-100}}{n!} \times \binom{100}{k} p^k (1-p)^{100-k}$$

where one corresponds to the Poisson distribution and another corresponds to the binomial distribution, we easily get:

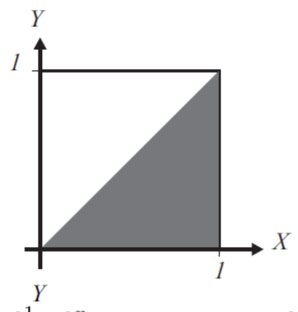
$$P_N(n) = \begin{cases} \frac{100^n e^{-100}}{n!}, & n = 0, 1, \cdots, \\ 0, & \text{otherwise} \end{cases}$$

$$P_K(k) = \begin{cases} \binom{100}{k} p^k (1-p)^{100-k}, & k = 0, 1, \cdots, 100 \\ 0, & \text{otherwise} \end{cases}$$

5.(a)
The total probability should be 1, so we have:

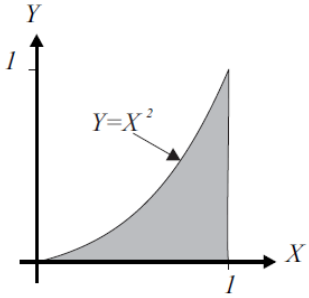
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y)dxdy = \int_0^1\int_0^1 cxy^2 dxdy = \frac{c}{6} = 1 = 6$$

5.(b)
For $P(X>Y)$, the region is:



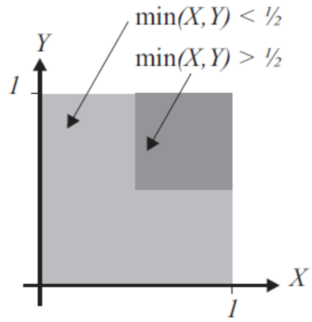
$$P(X>Y) = \int_0^1 \int_0^x 6xy^2 dydx = \int_0^1 2x^4 dx = 0.4$$

5.(c)
For $P(X^2 > Y)$, the region is:



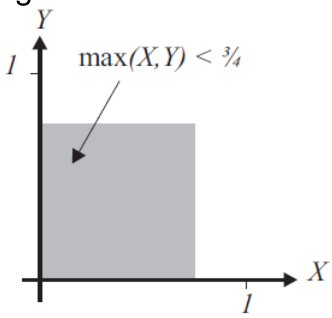
$$P(X > Y) = \int_0^1 \int_0^{x^2} 6xy^2 dy dx = 0.25$$

5.(d)
For $P(\min(X,Y) \leq 0.5)$, it is the L-shape region:



$$P(\min(X,Y) \leq 0.5) = 1 - P(\min(X,Y) \geq 0.5) = 1 - \int_{0.5}^1 \int_{0.5}^1 6xy^2 dx dy = 11/32$$

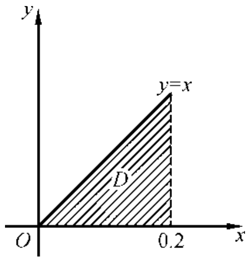
5.(e)
For $P(\max(X,Y) \leq 0.75)$, the region is:



$$P(\max(X,Y) \leq 0.75) = \int_0^{0.75} \int_0^{0.75} 6xy^2 dx dy = (3/4)^5 = 243/1024$$

6.
Since random variables X, Y are independent of each other, the joint PDF is:

$$p(x,y) = P_X(x)\cdot P_Y(y) = \begin{cases} \dfrac{1}{0.2}\cdot 5e^{-5y}, \\ 0, \end{cases} = \begin{cases} 25e^{-5y}, & 0 < x < 0.2 \text{ and } y > 0, \\ 0, & \text{otherwise.} \end{cases}$$



According to the above figure, we have:

$$P(Y \le X) = \int_0^{0.2}\int_0^x 25e^{-5y}dydx = \int_0^{0.2}(-5e^{5x}+5)dx = e^{-1} = 0.3679.$$

7.

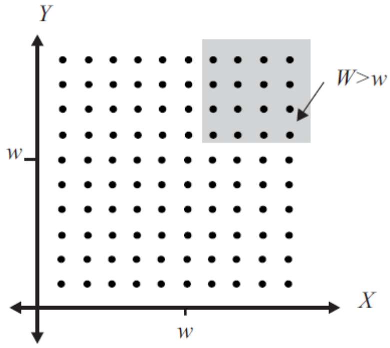
$$\begin{aligned} \mathbb{E}\{\hat{A}\} &= \mathbb{E}\left\{\frac{1}{N-1}\sum_{n=1}^{N} r_n\right\} = \frac{1}{N-1}\sum_{n=1}^{N}\mathbb{E}\{r_n\} = \frac{1}{N-1}\sum_{n=1}^{N}\mathbb{E}\{A+w_n\} \\ &= \frac{1}{N-1}\sum_{n=1}^{N}(A+\mathbb{E}\{w_n\}) = \frac{1}{N-1}\sum_{n=1}^{N}(A+0) = \frac{N}{N-1}A \end{aligned}$$

$$\begin{aligned} \mathrm{var}(\hat{A}) &= \mathbb{E}\left\{\left(\frac{1}{N-1}\sum_{n=1}^{N} r_n - \mathbb{E}\{A\}\right)^2\right\} \\ &= \mathbb{E}\left\{\left(\frac{1}{N-1}\sum_{n=1}^{N} r_n - \frac{N}{N-1}A\right)^2\right\} \\ &= \frac{1}{(N-1)^2}\mathbb{E}\left\{\left(\sum_{n=1}^{N} w_n + nA - nA\right)^2\right\} = \frac{1}{(N-1)^2}\mathbb{E}\left\{\left(\sum_{n=1}^{N} w_n\right)^2\right\} \\ &= \frac{1}{(N-1)^2}\sum_{n=1}^{N}\sum_{m=1}^{N}\mathbb{E}\left\{w_n w_m\right\} = \frac{1}{(N-1)^2}\sum_{n=1}^{N}\mathbb{E}\left\{w_n^2\right\} = \frac{N\sigma_w^2}{(N-1)^2} \end{aligned}$$

We then apply (3.29) to obtain:

$$\mathrm{MSE}(\hat{A}) = \mathrm{var}(\hat{A}) + (A - \mathbb{E}\{A\})^2 = \frac{N\sigma_w^2}{(N-1)^2} + \frac{A^2}{(N-1)^2}$$

8.(a)
We first consider $W=\min(X,Y)>w$, the region is:



$$P(W>w)=P(\min(X,Y)>w)=P(X>w,Y>w)=0.01(10-w^2)$$

Then noting that for $w=1,2,\cdots,10$:

$$P(W=w)=P(W>w-1)-P(W>w)=0.01(10-w+1)^2-0.01(10-w)^2=0.01(21-2w)$$

Finally, we obtain:

$$P_W(w)=\begin{cases}0.01(21-2w), & w=1,2,\cdots,10\\ 0, & \text{otherwise}\end{cases}$$

8.(b)
Given the event $A=\{\min(X,Y)>5\}$, we first compute $P(A)$:

$$P(A)=P(X>5,Y>5)=\sum_{x=6}^{10}\sum_{y=6}^{10}0.01=0.25$$

Hence $P_{XY|A}(x,y)$ is:

$$P_{XY|A}(x,y)=\begin{cases}0.04, & x=6,7,\cdots 10,\ y=6,7,\cdots,10\\ 0, & \text{otherwise}\end{cases}$$

9.
Given the event $A=\{X+Y\le 1\}$, we first compute $P(A)$:

$$P(A)=\int_0^1\int_0^{1-x}6e^{-(2x+3y)}dydx=1-3e^{-2}+2e^{-3}$$

Hence $P_{XY|A}(x,y)$ is:

$$P_{XY|A}(x,y)=\begin{cases}\dfrac{6e^{-(2x+3y)}}{1-3e^{-2}+2e^{-3}}, & x+y\le 1,\ x\ge 0,\ y\ge 0\\ 0, & \text{otherwise}\end{cases}$$

10.
We compute the marginal PDF $P_Y(y)$ first. For $0 \le y \le 1$, we have:

$$P_Y(y) = \int_0^1 (x+y)dx = \frac{2y+1}{2}$$

Hence:

$$P_{X|Y}(x|y) = \frac{P_{XY}(x,y)}{P_Y(y)} = \begin{cases} \frac{2(x+y)}{2y+1}, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Due to the symmetry between X and Y, we have:

$$P_{Y|X}(y|x) = \frac{P_{XY}(x,y)}{P_X(x)} = \begin{cases} \frac{2(x+y)}{2x+1}, & 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$