Solutions to EE3210 Tutorial 5 Problems

Problem 1: Because of the commutative property, we have

$$\begin{aligned} x[n] * (h_1[n] * h_2[n]) &= (h_1[n] * h_2[n]) * x[n] \\ &= \sum_{k=-\infty}^{+\infty} (h_1[k] * h_2[k]) x[n-k] \\ &= \sum_{k=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} h_1[m] h_2[k-m] x[n-k]. \end{aligned} \tag{1}$$

By changing the variable of summation in (1) from k to $r = n - k$, we then have

$$\begin{aligned} \sum_{k=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} h_1[m] h_2[k-m] x[n-k] &= \sum_{r=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} h_1[m] h_2[n-r-m] x[r] \\ &= \sum_{r=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} x[r] h_1[m] h_2[n-r-m]. \end{aligned}$$

On the other hand,

$$\begin{aligned} (x[n] * h_1[n]) * h_2[n] &= \sum_{k=-\infty}^{+\infty} (x[k] * h_1[k]) h_2[n-k] \\ &= \sum_{k=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} x[m] h_1[k-m] h_2[n-k] \\ &= \sum_{m=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} x[m] h_1[k-m] h_2[n-k]. \end{aligned} \tag{2}$$

By changing the variable of summation in (2) from k to $r = k - m$, we then have

$$\begin{aligned} \sum_{m=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} x[m] h_1[k-m] h_2[n-k] &= \sum_{m=-\infty}^{+\infty} \sum_{r=-\infty}^{+\infty} x[m] h_1[r] h_2[n-m-r] \\ &= \sum_{r=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} x[r] h_1[m] h_2[n-r-m]. \end{aligned}$$

Thus, the equality is proved.

Problem 2: Given $x[n] = (-\frac{1}{2})^n u[n-4]$ and $h[n] = 4^n u[2-n]$, we have $x[k] = (-\frac{1}{2})^k u[k-4]$ and $h[n-k] = 4^{n-k} u[k-(n-2)]$. So we have

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = \sum_{k=-\infty}^{+\infty} \left(-\frac{1}{2}\right)^k 4^{n-k} u[k-4]u[k-(n-2)]$$
$$= 4^n \sum_{k=-\infty}^{+\infty} \left(-\frac{1}{8}\right)^k u[k-4]u[k-(n-2)].$$

We observe that

$$u[k-4]u[k-(n-2)] = \begin{cases} u[k-4], & n-2 \leq 4 \\ u[k-(n-2)], & n-2 > 4 \end{cases}$$

Then:

- For $n-2 \leq 4$, i.e., $n \leq 6$, we have

$$y[n] = 4^n \sum_{k=-\infty}^{+\infty} \left(-\frac{1}{8}\right)^k u[k-4] = 4^n \sum_{k=4}^{+\infty} \left(-\frac{1}{8}\right)^k = 4^n \sum_{k=0}^{+\infty} \left(-\frac{1}{8}\right)^{k+4} = \left(\frac{8}{9}\right)\left(-\frac{1}{8}\right)^4 4^n.$$

- For $n-2 > 4$, i.e., $n > 6$, we have

$$y[n] = 4^n \sum_{k=-\infty}^{+\infty} \left(-\frac{1}{8}\right)^k u[k-(n-2)] = 4^n \sum_{k=n-2}^{+\infty} \left(-\frac{1}{8}\right)^k = 4^n \sum_{k=0}^{+\infty} \left(-\frac{1}{8}\right)^{k+n-2} = \left(\frac{8^3}{9}\right)\left(-\frac{1}{2}\right)^n.$$

Thus, for all n, we obtain

$$y[n] = \left(\frac{8^3}{9}\right)\left(-\frac{1}{2}\right)^n u[n-7] + \left(\frac{8}{9}\right)\left(-\frac{1}{8}\right)^4 4^n u[6-n].$$

Alternatively, we can write

$$y[n] = \left(\frac{8^3}{9}\right)\left(-\frac{1}{2}\right)^n u[n-6] + \left(\frac{8}{9}\right)\left(-\frac{1}{8}\right)^4 4^n u[5-n]$$

because we observe that, when $n = 6$,

$$\left(\frac{8^3}{9}\right)\left(-\frac{1}{2}\right)^n = \left(\frac{8}{9}\right)\left(-\frac{1}{8}\right)^4 4^n = \frac{8}{9}.$$

Problem 3: Given $x(t) = u(t) - 2u(t-2) + u(t-5)$ and $h(t) = e^{2t}u(1-t)$, we have

$$x(\tau) = u(\tau) - 2u(\tau - 2) + u(\tau - 5) \tag{3}$$

and

$$h(t-\tau) = e^{2(t-\tau)}u(\tau - [t-1]).$$

We observe in (3) that

$$x(\tau) = \begin{cases} 1, & 0 < \tau < 2 \\ -1, & 2 < \tau < 5 \\ 0, & \text{otherwise.} \end{cases}$$

Thus,

$$\begin{aligned} y(t) &= \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau \\ &= \int_0^2 e^{2(t-\tau)}u(\tau - [t-1])d\tau - \int_2^5 e^{2(t-\tau)}u(\tau - [t-1])d\tau. \end{aligned}$$

This can be written as

$$y(t) = \begin{cases} \displaystyle\int_0^2 e^{2(t-\tau)}d\tau - \int_2^5 e^{2(t-\tau)}d\tau, & t < 1 \\ \displaystyle\int_{t-1}^2 e^{2(t-\tau)}d\tau - \int_2^5 e^{2(t-\tau)}d\tau, & 1 < t < 3 \\ \displaystyle -\int_{t-1}^5 e^{2(t-\tau)}d\tau, & 3 < t < 6 \\ 0, & 6 < t. \end{cases}$$

Therefore, we obtain $y(t)$ as

$$y(t) = \begin{cases} \dfrac{1}{2}\left[e^{2t} - 2e^{2(t-2)} + e^{2(t-5)}\right], & t < 1 \\ \dfrac{1}{2}\left[e^{2} - 2e^{2(t-2)} + e^{2(t-5)}\right], & 1 < t < 3 \\ \dfrac{1}{2}\left[e^{2(t-5)} - e^{2}\right], & 3 < t < 6 \\ 0, & 6 < t. \end{cases}$$