EE3210
Signals and Systems

Part 2: Basics of Signals

香港城市大學
City University
of Hong Kong

Instructor: Dr. Jun Guo

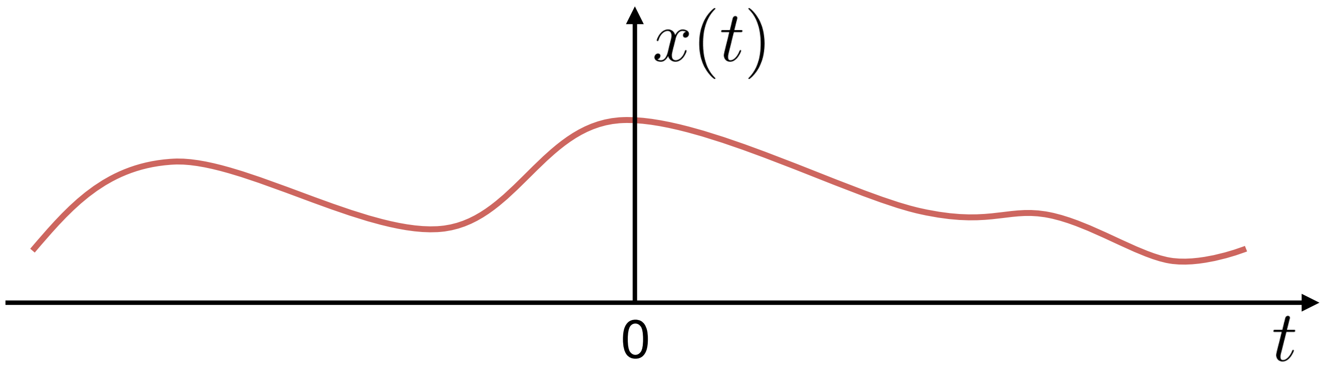
DEPARTMENT OF ELECTRONIC ENGINEERING

What is a Signal?

- A signal, which is represented mathematically as a function of one or more independent variables (e.g., time, space, distance, etc.), contains information about the behavior or nature of some phenomenon.
- Examples:
  - Voltage/current: A function of time, continuous
  - Stock market index: A function of time, discrete
  - Audio: A function of time, continuous/discrete
  - Image: A function of space, continuous/discrete

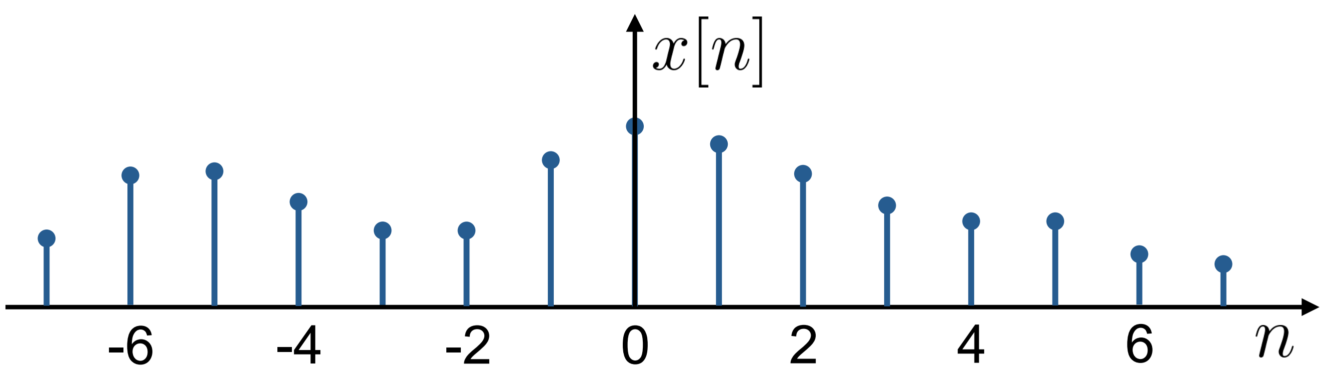
Continuous-Time Signals

- Continuous-time signals are defined for a continuum of values $x(t)$ as a function of the continuous-time independent variable t.



Discrete-Time Signals

- Discrete-time signals are defined only at discrete times n, i.e., for integer values of the independent variable, for a discrete set of values $x[n]$.
  - $x[n]$ is also called a discrete-time sequence.
  - In the case of a very important class of discrete-time signals arising from the sampling of continuous-time signals, n is also called the sample number.



Energy and Power: Continuous-Time Signals

- The total energy in a continuous-time signal $x(t)$ over the time interval $t_1 \leq t \leq t_2$ is defined as

$$\int_{t_1}^{t_2} |x(t)|^2 dt$$

where $|x|$ denotes the magnitude of the (possibly complex) number x.

- The time-averaged power of $x(t)$ is obtained as

$$\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |x(t)|^2 dt$$

Energy and Power: Discrete-Time Signals

- Similarly, the total energy in a discrete-time signal $x[n]$ over the time interval $n_1 \leq n \leq n_2$ is defined as

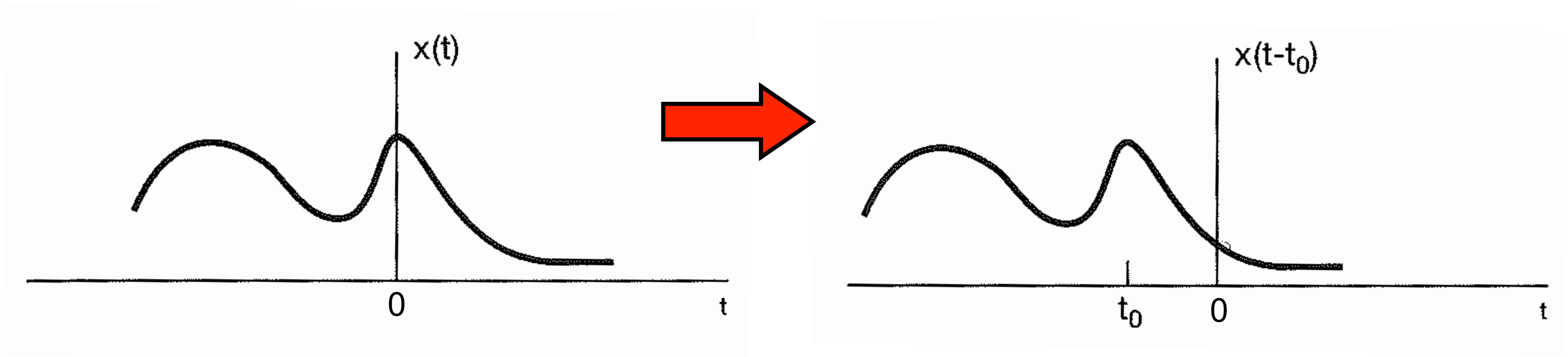
$$\sum_{n=n_1}^{n_2} |x[n]|^2$$

- The time-averaged power of $x[n]$ is obtained as

$$\frac{1}{n_2 - n_1 + 1} \sum_{n=n_1}^{n_2} |x[n]|^2$$

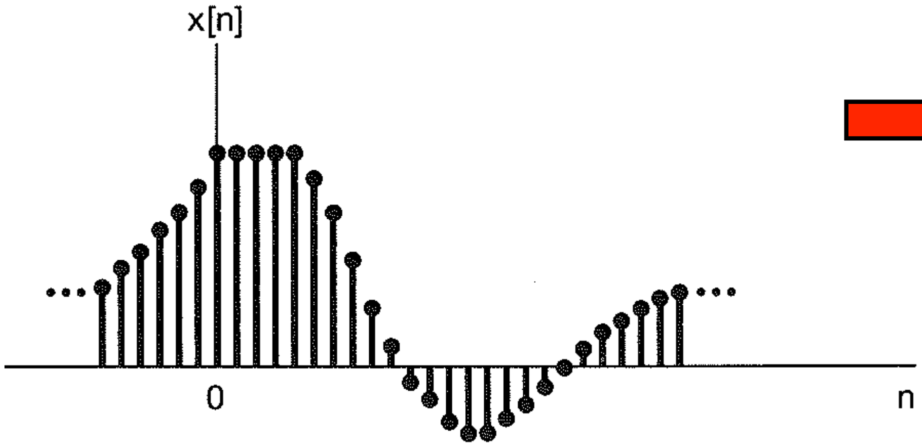
Signal Transformations: Time Shift

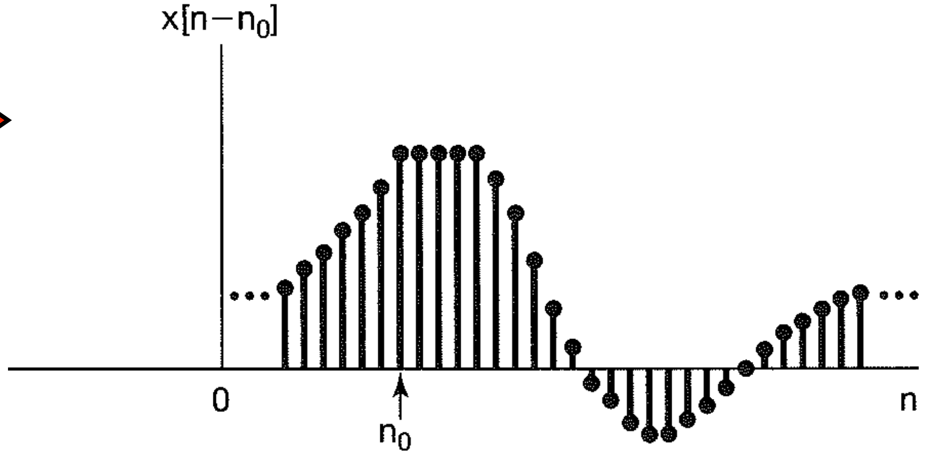
- For a continuous-time signal $x(t)$, $x(t-t_0)$ represents a delayed (if t_0 is positive) or advanced (if t_0 is negative) version of $x(t)$.



Signal Transformations: Time Shift (cont.)

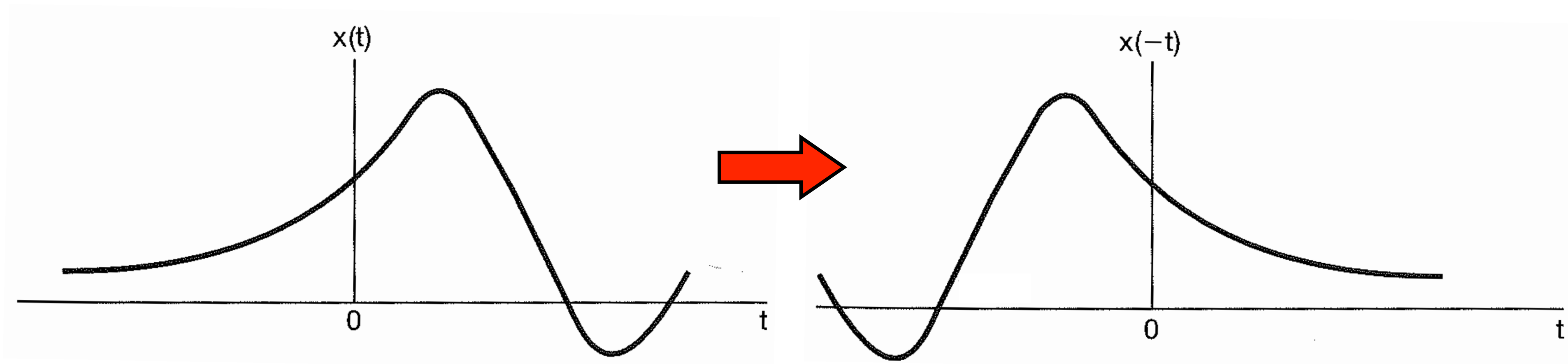
- Similar transformation can be defined for a discrete-time signal $x[n]$ to obtain its time shifted version $x[n - n_0]$.





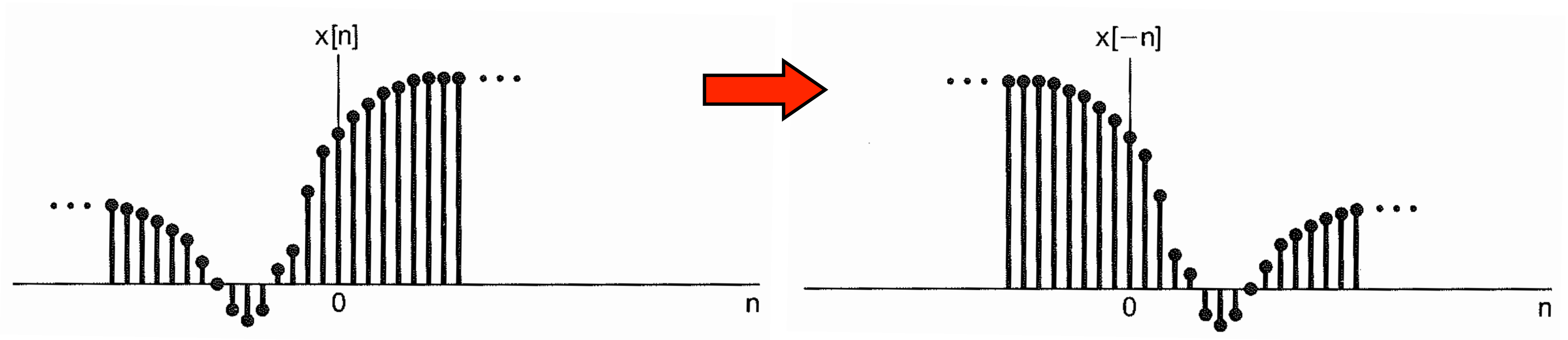
Signal Transformations: Time Reversal

- For a continuous-time signal $x(t)$, $x(-t)$ is obtained from $x(t)$ by a reflection about $t=0$.



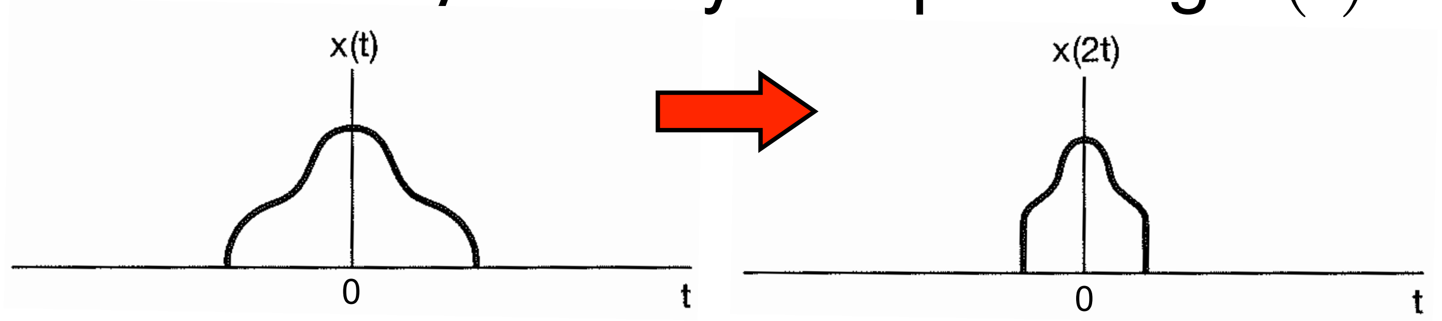
Signal Transformations: Time Reversal (cont.)

- Similar transformation can be defined for a discrete-time signal $x[n]$ to obtain $x[-n]$ by reversing $x[n]$.

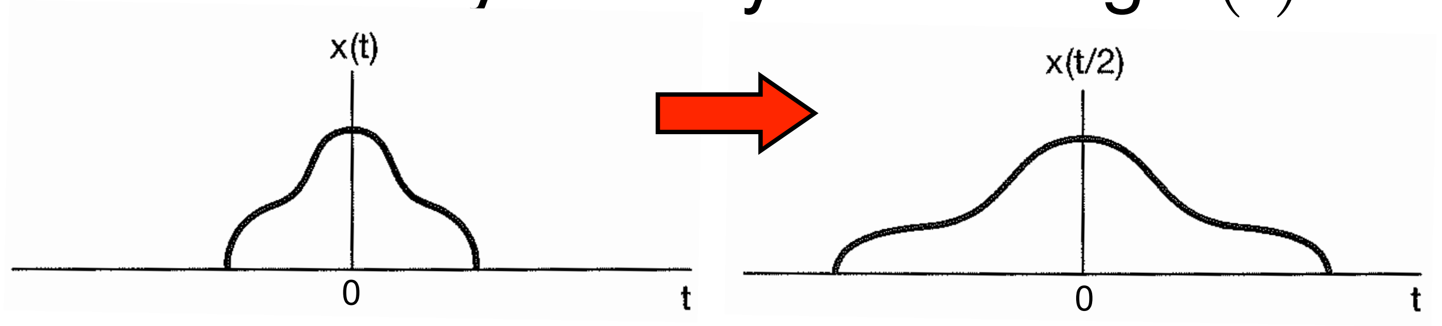


Signal Transformations: Time Scaling

- For a continuous-time signal $x(t)$:
  - $x(2t)$ is obtained by linearly compressing $x(t)$.



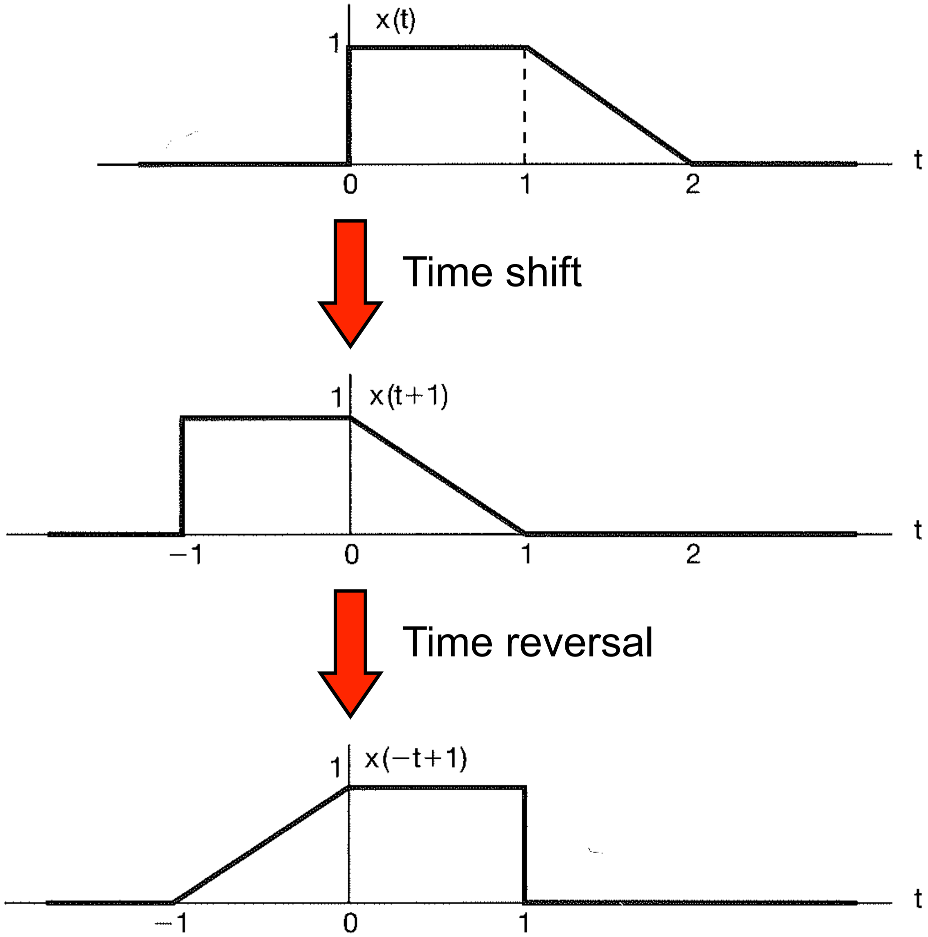
  - $x(t/2)$ is obtained by linearly stretching $x(t)$.



- Can we obtain $x[2n]$, $x[n/2]$ from a discrete-time signal $x[n]$?

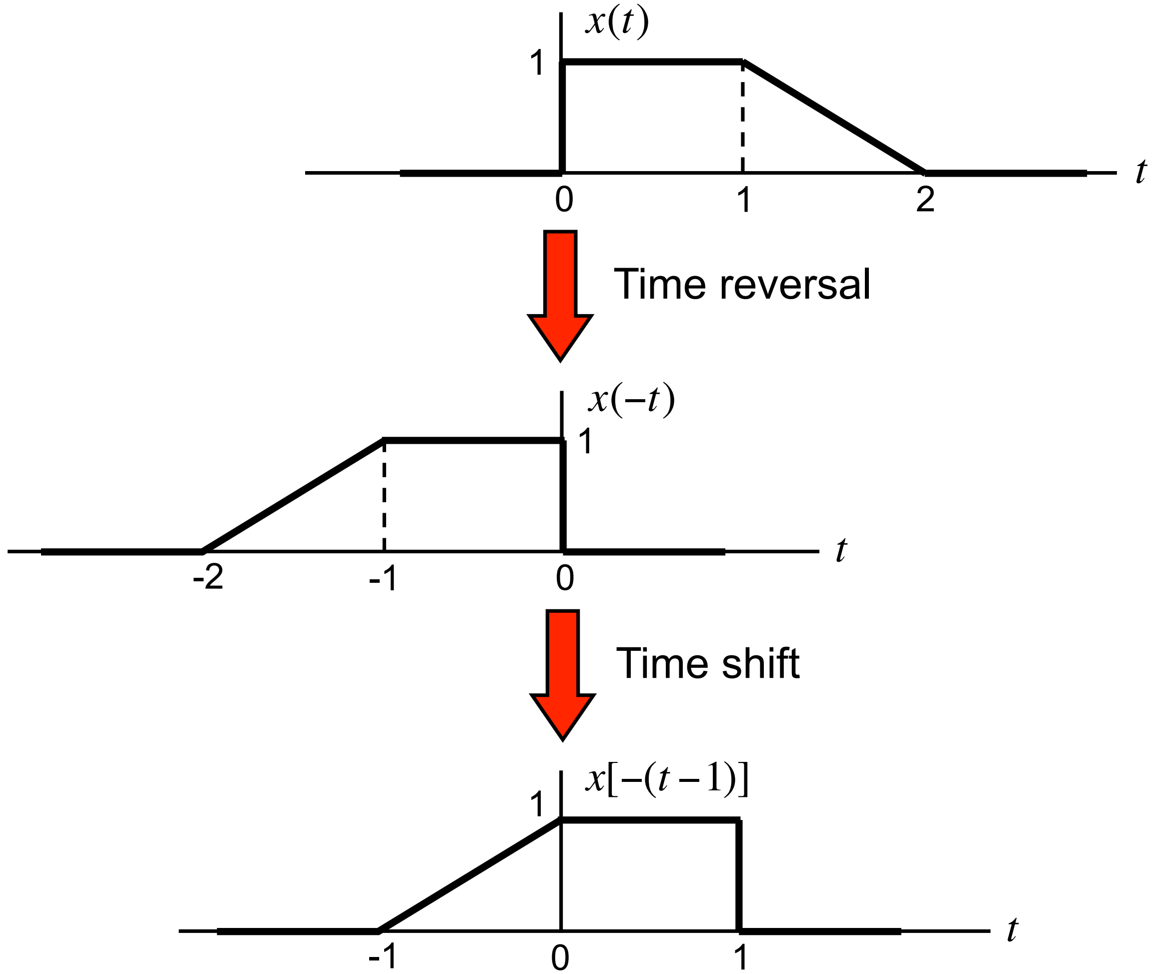
Combined Transformations: Example 1

- Obtain $x(-t+1)$ from $x(t)$:
  - We can do time shift first followed by time reversal.
  - Can we do time reversal first followed by time shift?



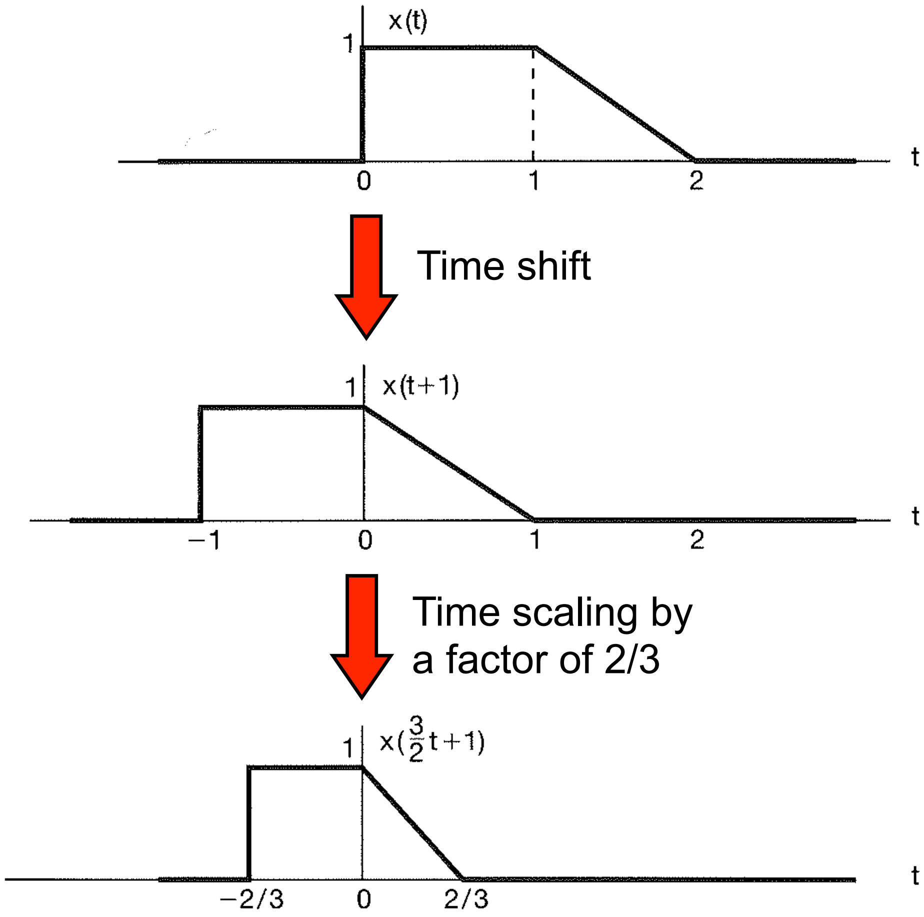
Example 1 (cont.)

- Rewrite $x(-t+1)$ as $x[-(t-1)]$



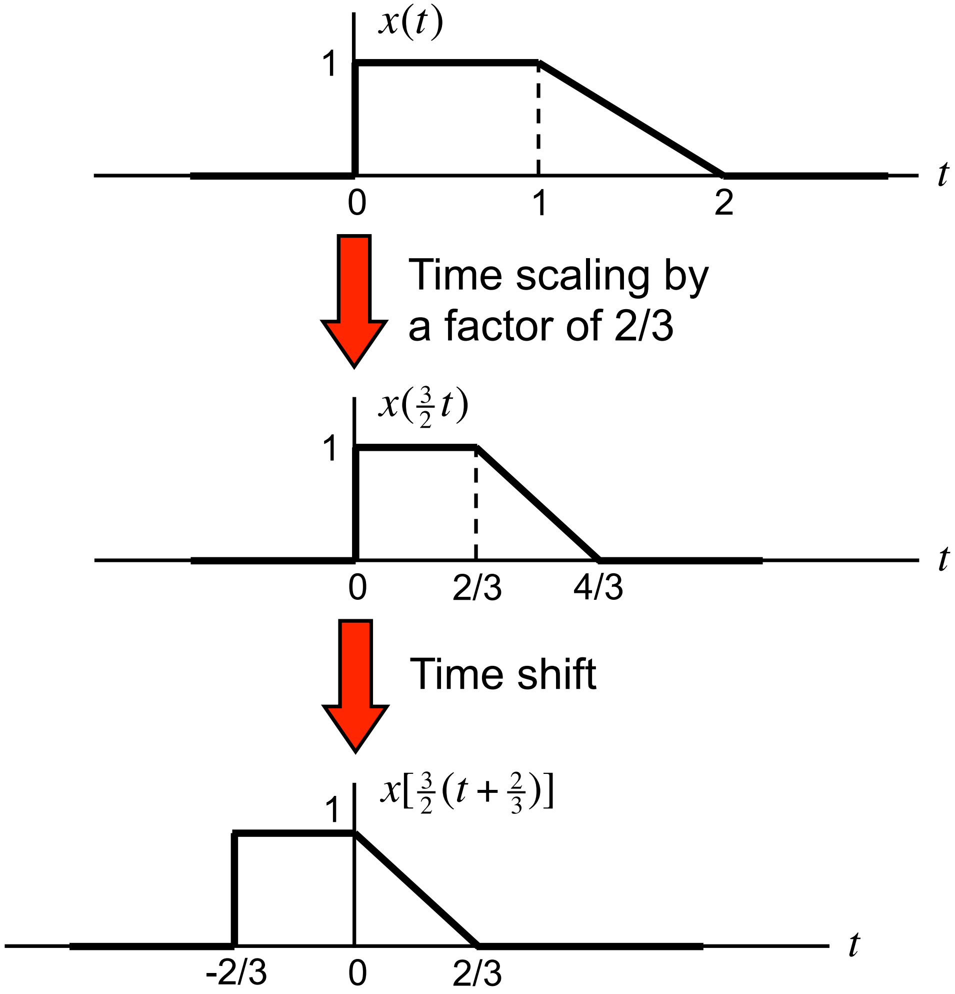
Combined Transformations: Example 2

- Obtain $x(\frac{3}{2}t+1)$ from $x(t)$:
  - We can do time shift first followed by time scaling.
  - Can we do time scaling first followed by time shift?



Example 2 (cont.)

- Rewrite $x(\frac{3}{2}t+1)$ as $x[\frac{3}{2}(t+\frac{2}{3})]$

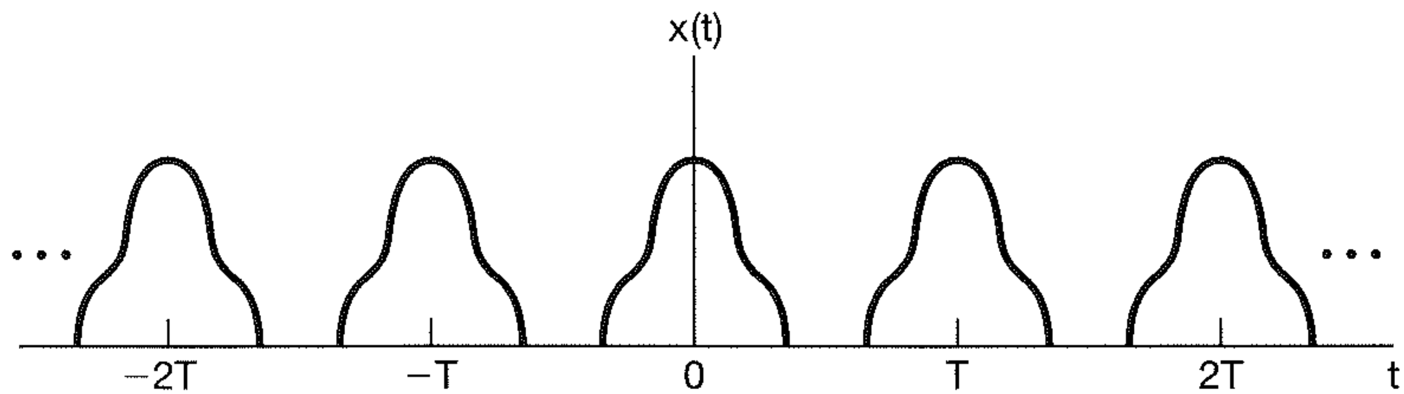


Continuous-Time Periodic Signals

- A continuous-time signal $x(t)$ is periodic if there is a positive value of T for which

$$x(t) = x(t+T) \tag{1}$$

for all values of t.



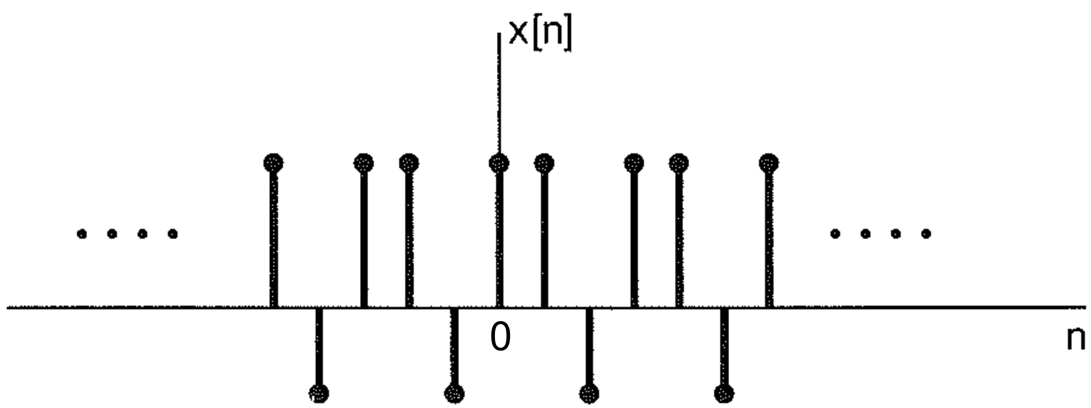
- Fundamental period: The smallest positive value of T for which (1) holds.

Discrete-Time Periodic Signals

- A discrete-time signal $x[n]$ is periodic with period N, where N is a positive integer, if

$$x[n] = x[n+N] \tag{2}$$

for all values of n.



- Fundamental period: The smallest positive value of N for which (2) holds.

Analog Frequency versus Digital Frequency

- For a continuous-time periodic signal, its period T can be any real value in the range $(0, \infty)$. Therefore, its frequency $f = 1/T$ can be arbitrarily large, i.e., when $T \to 0$, $f \to \infty$.

- For a discrete-time periodic signal, since the smallest possible value of its period N is 1, its frequency $f = 1/N$ is bounded by 1, i.e., $f \in (0, 1]$.

Even and Odd Signals

- A signal $x(t)$ or $x[n]$ is referred to as even if

$$x(-t) = x(t) \text{ or } x[-n] = x[n]$$

- A signal $x(t)$ or $x[n]$ is referred to as odd if

$$x(-t) = -x(t) \text{ or } x[-n] = -x[n]$$

- Any signal can be broken into a sum of two signals, one is even, and one is odd:
  - Even part: $\mathcal{E}\{x(t)\} = \frac{1}{2}[x(t) + x(-t)]$
  - Odd part: $\mathcal{O}\{x(t)\} = \frac{1}{2}[x(t) - x(-t)]$
  - Same results apply in the discrete-time case.

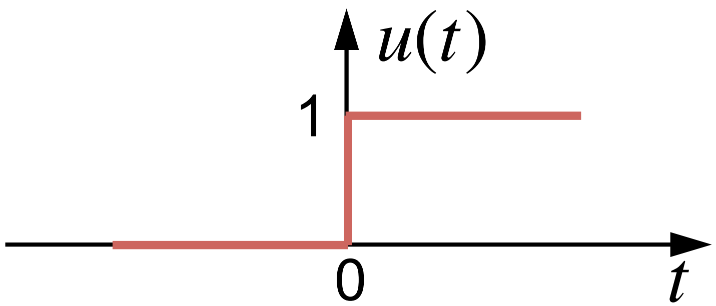
Basic Signals

- Four basic continuous-time and discrete-time signals:
  - Unit step
  - Unit impulse
  - Sinusoidal
  - Complex exponential
- These signals can be used as basic building blocks for construction and representation of other signals.

Continuous-Time Unit Step

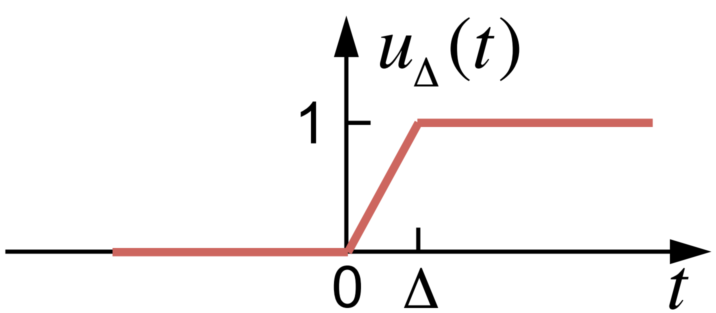
- The continuous-time unit step signal, denoted by $u(t)$, is defined as:

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$



  - Note: $u(t)$ is discontinuous at $t = 0$.

- The idealized signal $u(t)$ can be approximated by $u_\Delta(t)$.



$$u(t) = \lim_{\Delta \to 0} u_\Delta(t)$$

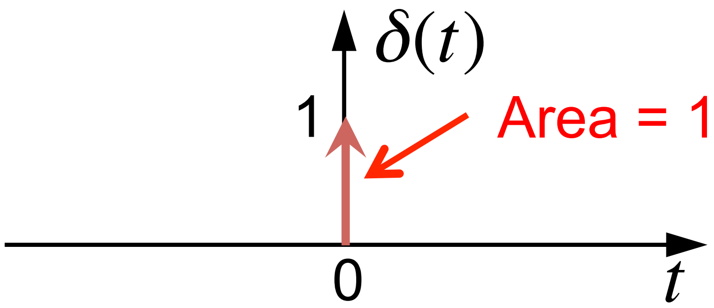
Continuous-Time Unit Impulse

- The continuous-time unit impulse signal, denoted by $\delta(t)$, is defined as:

$$\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases}$$

which satisfies the identity:

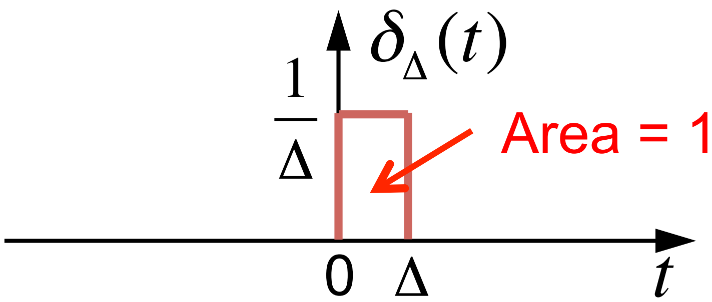
$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$



- Note: $\delta(t)$ is not a function in the traditional sense.

Continuous-Time Unit Impulse (cont.)

- The idealized signal $\delta(t)$ can be approximated by $\delta_\Delta(t)$.



$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$$

- Sampling property: Consider a function $x(t)$ that is continuous at an arbitrary point t_0. Then,
  - $x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$
  - $\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0)\int_{-\infty}^{\infty} \delta(t-t_0)dt = x(t_0)$

Relationship between $u(t)$ and $\delta(t)$

- $u(t)$ is the running integral of $\delta(t)$:

$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau \tag{3}$$

  - By changing the variable of integration in (3) from τ to $\sigma = t - \tau$, we have:

$$u(t) = -\int_{\infty}^{0} \delta(t-\sigma) d\sigma = \int_{0}^{\infty} \delta(t-\sigma) d\sigma$$

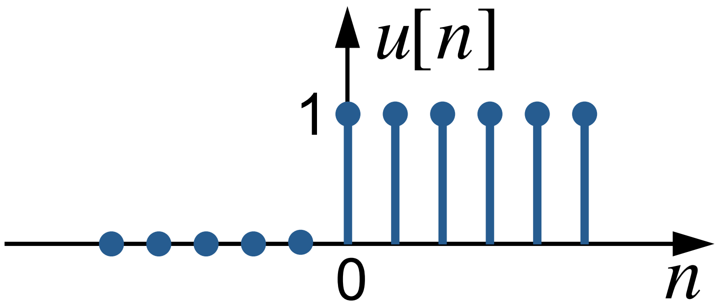
- $\delta(t)$ is the first derivative of $u(t)$:

$$\delta(t) = \frac{d\,u(t)}{d\,t}$$

Discrete-Time Unit Step

- The discrete-time unit step signal, denoted by $u[n]$, is defined as:

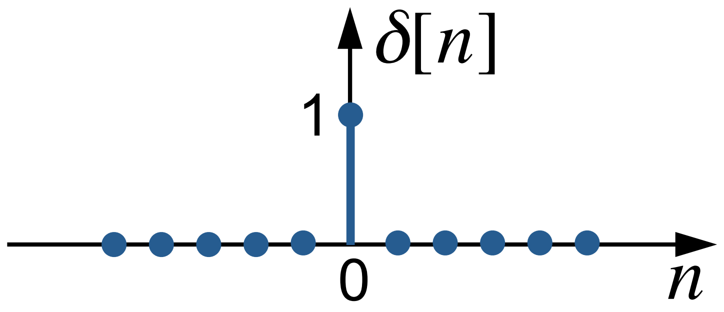
$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$



Discrete-Time Unit Impulse

- The discrete-time unit impulse signal, denoted by $\delta[n]$, is defined as:

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$



  - $\delta[n]$ is also referred to as the unit sample sequence.

- Sampling property:

$$x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]$$

Relationship between $u[n]$ and $\delta[n]$

- $u[n]$ is the running sum of $\delta[n]$:

$$u[n] = \sum_{m=-\infty}^{n} \delta[m] \qquad (4)$$

  - By changing the variable of summation in (4) from m to $k = n - m$, we have:

$$u[n] = \sum_{k=\infty}^{0} \delta[n-k] = \sum_{k=0}^{\infty} \delta[n-k]$$

- $\delta[n]$ is the first difference of $u[n]$:

$$\delta[n] = u[n] - u[n-1]$$

Continuous-Time Sinusoidal

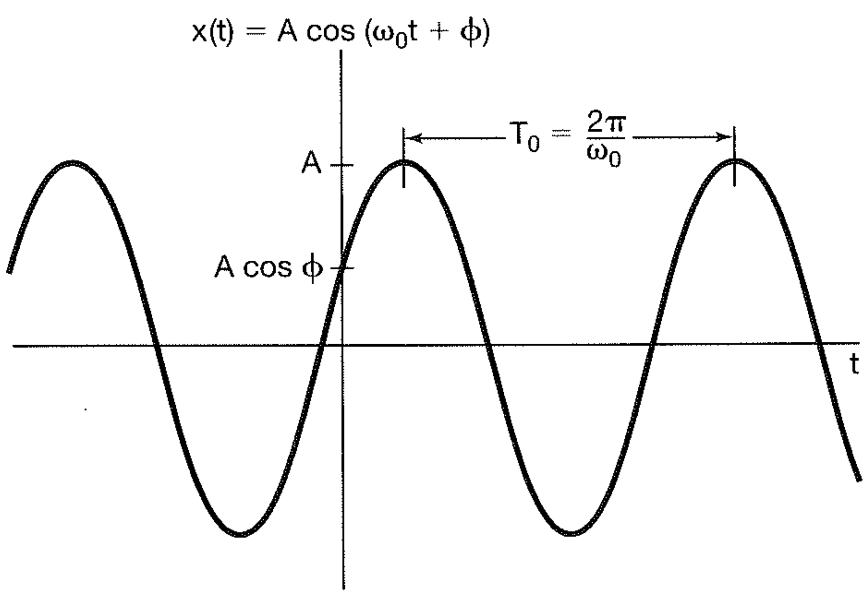
- The continuous-time sinusoidal signal has the general form

$$x(t) = A\cos(\omega_0 t + \phi)$$

  - Note: In this course, we use the notation ω for analog frequency.
  - Note: With seconds as the units of t, the units of ϕ and ω_0 are radians and radians per second, respectively.

Properties

- Distinct signals $x(t)$ for distinct values of ω_0.
- The larger ω_0, the higher is the rate of oscillation in $x(t)$.
- $x(t)$ is periodic for any value of ω_0 with the fundamental period T_0 given by $T_0 = 2\pi/\omega_0$.



Continuous-Time Complex Exponential

- The continuous-time complex exponential signal is of the form

$$x(t) = Ce^{at}$$

where C and a are, in general, complex numbers.

  - Let C be expressed in polar form: $C = |C|e^{j\theta}$
  - Let a be expressed in Cartesian form: $a = r + j\omega_0$
  - Then

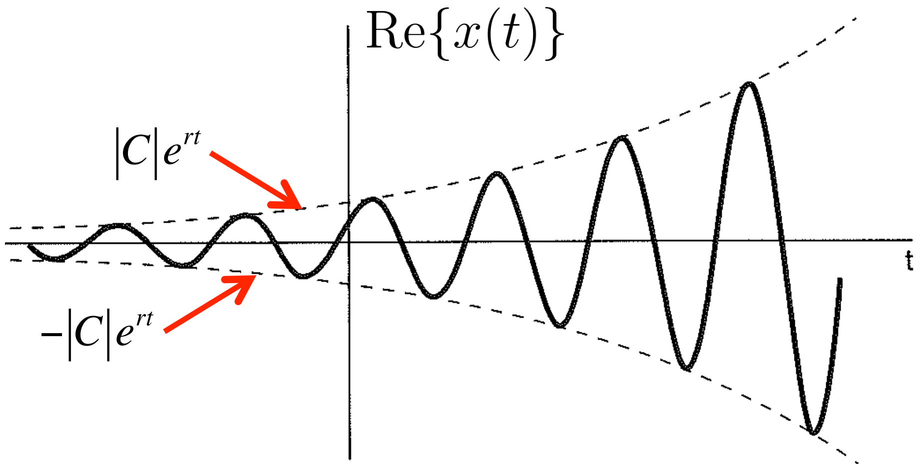
$$x(t) = |C|e^{j\theta}e^{(r+j\omega_0)t} = |C|e^{rt}e^{j(\omega_0 t+\theta)} \quad (5)$$

  - Using Euler's formula, we can expand (5) further as

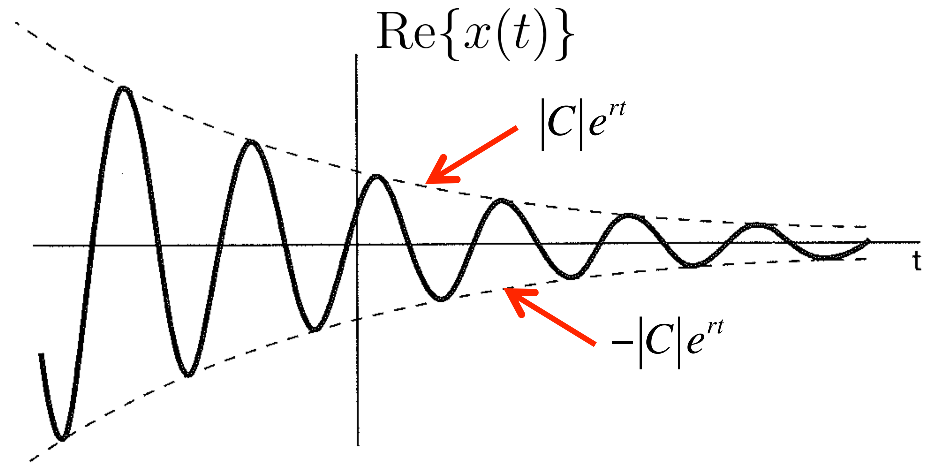
$$x(t) = |C|e^{rt}\cos(\omega_0 t + \theta) + j|C|e^{rt}\sin(\omega_0 t + \theta) \quad (6)$$

Observations

- We observe in (6) that:
  - For $r = 0$, $\mathrm{Re}\{x(t)\}$ and $\mathrm{Im}\{x(t)\}$ are sinusoidal.
  - For $r > 0$, they are sinusoidal signals multiplied by a growing exponential.
  - For $r < 0$, they are sinusoidal signals multiplied by a decaying exponential.



$r > 0$



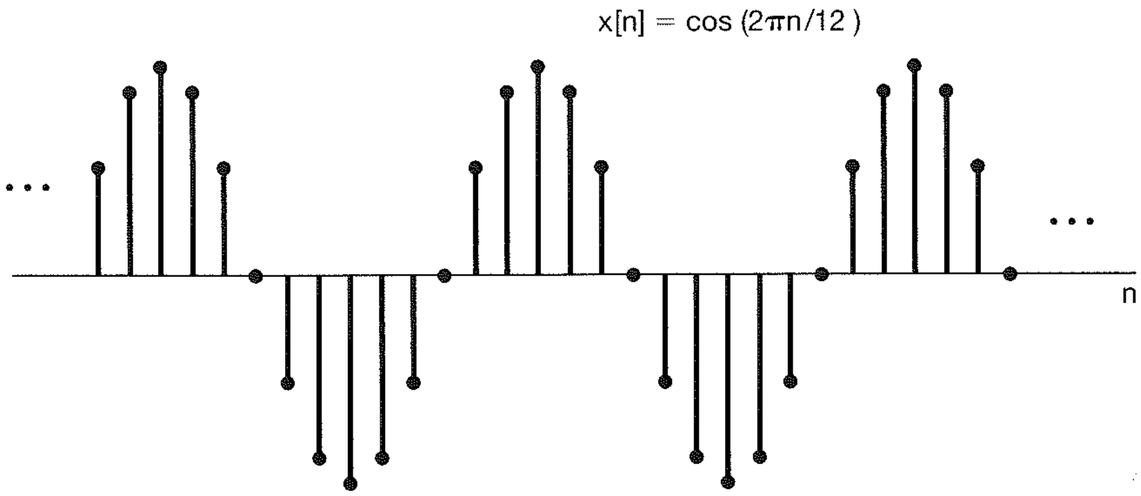
$r < 0$

Discrete-Time Sinusoidal

- The discrete-time sinusoidal signal has the general form

$$x[n] = A\cos(\Omega_0 n + \phi)$$

  - Note: In this course, we use the notation Ω instead of ω for digital frequency.
  - Note: If we take n to be dimensionless, then both ϕ and Ω_0 have units of radians.



Properties

- $x[n]$ at frequency $\Omega_0 + 2\pi k$ is the same as that at frequency Ω_0 for any integer k, since

$$A\cos[(\Omega_0 + 2\pi k)n + \phi] = A\cos(\Omega_0 n + \phi) \tag{7}$$

  - We need only consider a frequency interval of length 2π in which to choose Ω_0.
  - We will use the interval $0 \leq \Omega_0 \leq 2\pi$ or the interval $-\pi \leq \Omega_0 \leq \pi$ on most occasions.

Properties (cont.)

- Implied by (7), the discrete-time sinusoidal signal does not have a continually increasing rate of oscillation with an increasing Ω_0.
  - Consider Ω_0 in the interval $0 \leq \Omega_0 \leq 2\pi$. As illustrated in the figure on the next page:
    - For $0 \leq \Omega_0 \leq \pi$, the rate of oscillation $\uparrow$ as $\Omega_0 \uparrow$.
    - For $\pi \leq \Omega_0 \leq 2\pi$, the rate of oscillation $\downarrow$ as $\Omega_0 \uparrow$.
    - In particular, for $\Omega_0 = \pi$,

$$\cos(\pi n) = (-1)^n$$

so that this signal oscillates rapidly, changing sign at each point in time.

Properties (cont.)

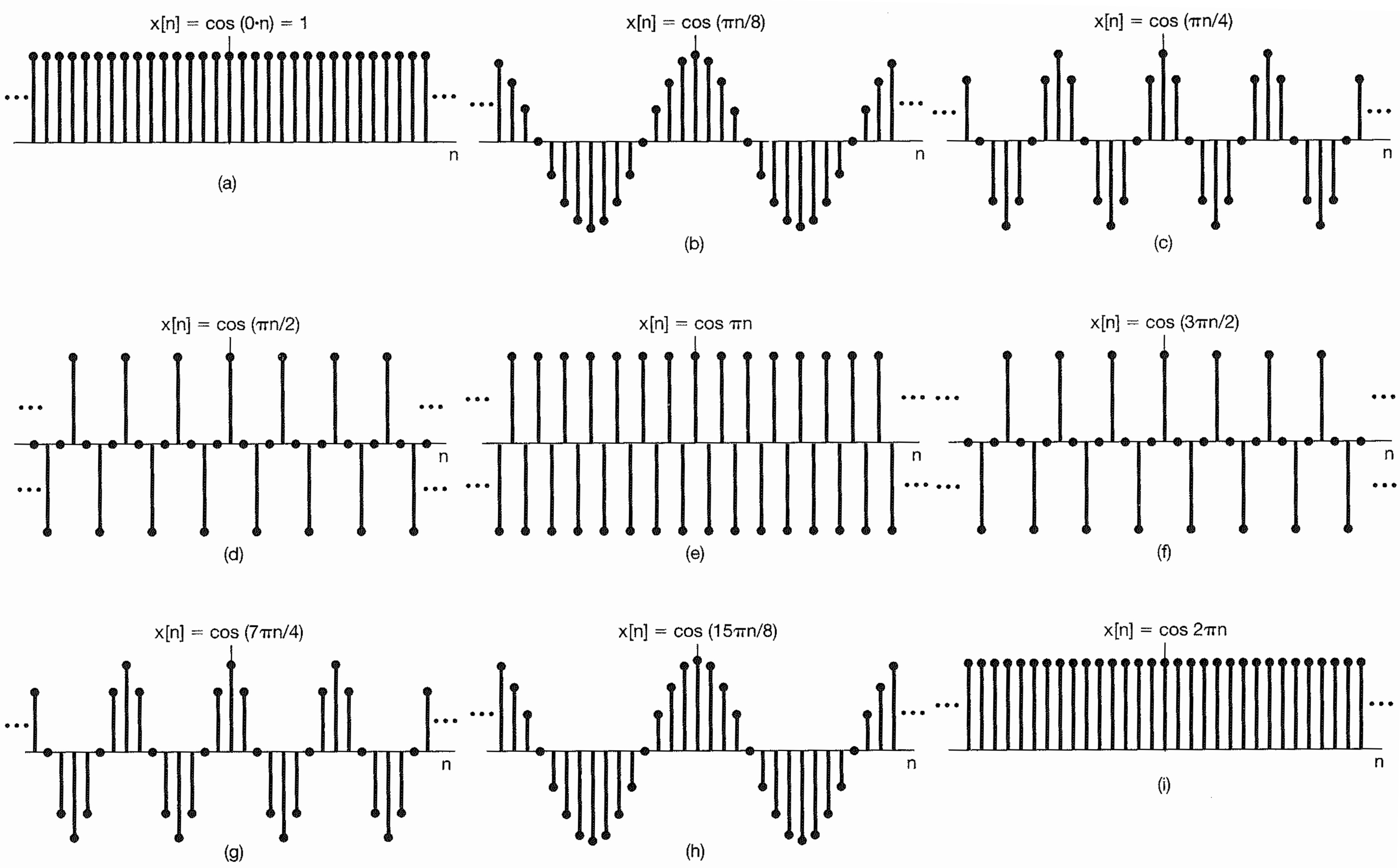


Figure 1.27 Discrete-time sinusoidal sequences for several different frequencies.

Properties (cont.)

- For the discrete-time sinusoidal signal to be periodic, we must have

$$A\cos(\Omega_0 n + \phi) = A\cos(\Omega_0 n + \Omega_0 N + \phi)$$

where the period N is necessarily a positive integer.

  - This requires that $\Omega_0 N = 2\pi k$, or equivalently, $\Omega_0/(2\pi) = k/N$, where k is an integer.
  - Thus, the signal is periodic only if $\Omega_0/(2\pi)$ is a rational number.
  - The fundamental period N_0 is given by $N_0 = 2\pi k/\Omega_0$ if N_0 and k have no factors in common.

Examples

- Is $x[n] = \cos(n/6)$ periodic?
  - Answer: $\Omega_0 = 1/6 \Rightarrow \Omega_0/(2\pi)$ is irrational $\Rightarrow$ aperiodic.
- Is $y[n] = \cos(8\pi n/31)$ periodic?
  - Answer: $\Omega_0 = 8\pi/31 \Rightarrow \Omega_0/(2\pi)$ is rational $\Rightarrow$ periodic.
- What is the fundamental period of $y[n]$?
  - Answer: $N_0 = 31$.

Discrete-Time Complex Exponential

- The discrete-time complex exponential signal is defined by
$$x[n] = C\alpha^n$$
where C and α are, in general, complex numbers.
  - Let C be expressed in polar form: $C = |C|e^{j\theta}$
  - Let α be expressed in polar form: $\alpha = |\alpha|e^{j\Omega_0}$
  - Then
$$x[n] = |C||\alpha|^n e^{j(\Omega_0 n+\theta)} \tag{8}$$
  - Using Euler's formula, we can expand (8) further as

$$x[n] = |C||\alpha|^n \cos(\Omega_0 n + \theta) + j|C||\alpha|^n \sin(\Omega_0 n + \theta) \tag{9}$$

Observations

- We observe in (9) that:
  - For $|\alpha| = 1$, $\mathrm{Re}\{x[n]\}$ and $\mathrm{Im}\{x[n]\}$ are sinusoidal.
  - For $|\alpha| > 1$, they are sinusoidal sequences multiplied by a growing exponential.
  - For $|\alpha| < 1$, they are sinusoidal sequences multiplied by a decaying exponential.