

## Euler Formula

$$e^{j\phi} = \cos(\phi) + j \sin(\phi)$$

$$\cos(\phi) = \frac{e^{j\phi} + e^{-j\phi}}{2} \quad \sin(\phi) = \frac{e^{j\phi} - e^{-j\phi}}{2j}$$

## Fourier Series

- The frequency domain representation of a continuous-time periodic signal

$$x(t) = x(t + T_p)$$

- The smallest  $T_p \rightarrow$  fundamental period

$$\Omega_0 = \frac{2\pi}{T_p}$$

- $\Omega_0 \rightarrow$  fundamental frequency

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t}$$

$$a_k = \frac{1}{T_p} \int_{-T_p/2}^{T_p/2} x(t) e^{-jk\Omega_0 t} dt, \quad k = \dots -1, 0, 1, 2, \dots$$

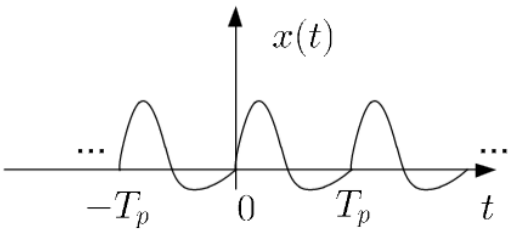
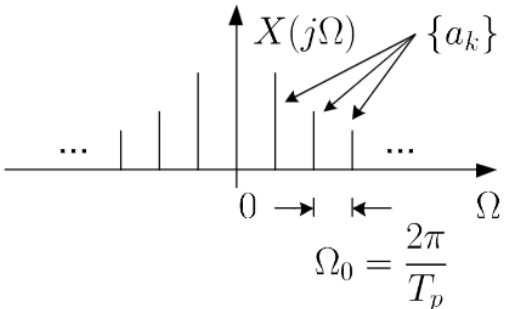
- $a_k \rightarrow$  Fourier series coefficients
  - It is generally complex

$$|a_k| = \sqrt{(\Re\{a_k\})^2 + (\Im\{a_k\})^2}$$

$$\angle(a_k) = \tan^{-1} \left( \frac{\Im\{a_k\}}{\Re\{a_k\}} \right)$$

## Fourier Series Coefficients

- Find  $k = 0$  (base case) for  $a_0$
- Then find  $k \neq 0$  for  $a_k$

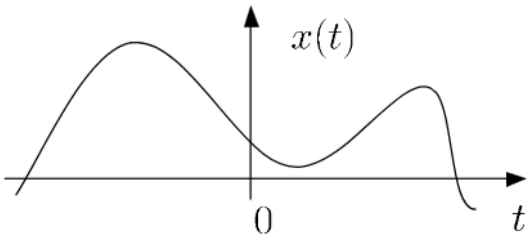
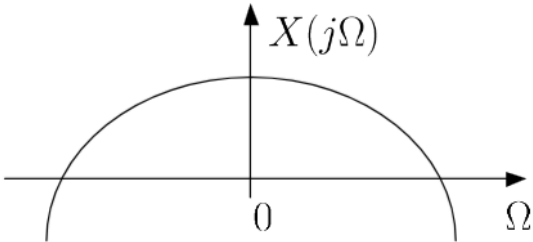
time domain	frequency domain
 $a_k = \frac{1}{T_p} \int_{-T_p/2}^{T_p/2} x(t) e^{-jk\Omega_0 t} dt \Rightarrow$	 $\Omega_0 = \frac{2\pi}{T_p}$
$\Leftarrow x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t}$	
continuous and periodic	discrete and aperiodic

## Fourier Transform

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$$

## Inverse Fourier Transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega$$

time domain	frequency domain
 $X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t} dt \Rightarrow$ $\Leftarrow x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t} d\Omega$	
continuous and aperiodic	continuous and aperiodic

## Periodic Signal Representation using Fourier Transform

- $\Omega_0 \rightarrow$  tone of frequency
- With the use of  $\delta(t)$

$$X(j\Omega) = 2\pi\delta(\Omega - \Omega_0)$$

## Inverse Fourier Transform on Periodic Signal

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi\delta(\Omega - \Omega_0)e^{j\Omega t} d\Omega = e^{j\Omega_0 t}$$

## Fourier Transform and Linear Time-Invariant System

$$y(t) = x(t) \circledast h(t) \leftrightarrow Y(j\Omega) = X(j\Omega)H(j\Omega)$$

- It suggests converting the input and impulse response to the frequency domain, and then  $y(t)$  is computed from the inverse Fourier transform of  $X(j\Omega)H(j\Omega)$
- $H(j\Omega) \rightarrow$  System frequency response

$$Y(j\Omega) \left[ \sum_{k=0}^N a_k(j\Omega)^k \right] = X(j\Omega) \left[ \sum_{k=0}^M b_k(j\Omega)^k \right]$$

$$H(j\Omega) = \frac{Y(j\Omega)}{X(j\Omega)} = \frac{\sum_{k=0}^N b_k(j\Omega)^k}{\sum_{k=0}^M a_k(j\Omega)^k}$$

## Discrete-Time Fourier Transform (DTFT)

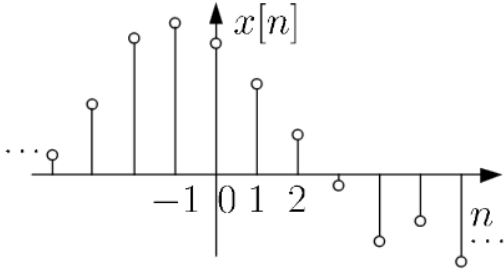
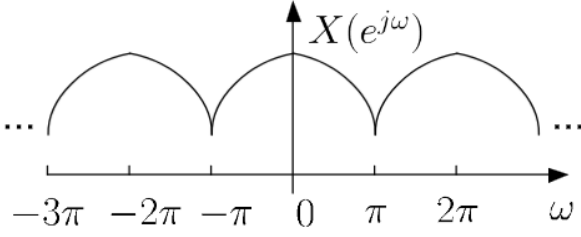
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$X(e^{j\omega}) \leftrightarrow X(e^{j(\omega+2k\pi)})$$

- It is periodic with a period  $2\pi$
- It is generally complex

## Inverse Discrete-Time Fourier Transform

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n} d\omega$$

time domain	frequency domain
 <p>A discrete-time signal <math>x[n]</math> is plotted on a stem plot. The horizontal axis is labeled <math>n</math> and has tick marks at -1, 0, 1, and 2. The vertical axis is labeled <math>x[n]</math>. The signal is periodic with period 4, with peaks at <math>n = 0, 4, 8, \dots</math> and troughs at <math>n = -4, -8, \dots</math>. Ellipses at both ends indicate the signal continues infinitely.</p> $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \Rightarrow$ $\Leftarrow x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n} d\omega$	 <p>The Discrete-time Fourier Transform <math>X(e^{j\omega})</math> is plotted on a continuous curve. The horizontal axis is labeled <math>\omega</math> and has tick marks at <math>-3\pi, -2\pi, -\pi, 0, \pi, 2\pi</math>. The vertical axis is labeled <math>X(e^{j\omega})</math>. The plot shows a periodic series of overlapping bell-shaped curves centered at <math>\omega = 0, \pm 2\pi, \dots</math> and troughs at <math>\omega = \pm\pi, \dots</math>. Ellipses at both ends indicate the periodic nature of the spectrum.</p>
discrete and aperiodic	continuous and periodic

## DTFT and Linear Time-Invariant System

$$y[n] = x[n] \otimes h[n] \leftrightarrow Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

- This suggests converting the input and impulse response to the frequency domain, and then  $y[n]$  is computed from the inverse DTFT of  $X(e^{j\omega})H(e^{j\omega})$

$$Y(e^{j\omega}) \sum_{k=0}^N a_k e^{-j\omega k} = X(e^{j\omega}) \sum_{k=0}^M b_k e^{-j\omega k}$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^N b_k e^{-j\omega k}}{\sum_{k=0}^M a_k e^{-j\omega k}}$$

## Sampling

- It is converting a continuous-time signal  $x(t)$  into a discrete-time signal  $x[n]$

$$x[n] = x(t) \Big|_{t=nT} = x(nT), \quad n = \dots -1, 0, 1, 2, \dots$$

- $T \rightarrow$  sampling period
- $x[n]$  can reconstruct  $x(t)$  if
  - $x(t)$  is bandlimited such that its Fourier transform  $X(j\Omega) = 0$  for  $|\Omega| \geq \Omega_b$  where  $\Omega_b \rightarrow$  bandwidth
  - Sampling period  $T$  is sufficiently small

## Sampling Theorem

- Let  $x(t)$  be a bandlimited continuous-time signal with

$$X(j\Omega) = 0, \quad |\Omega| \geq \Omega_b$$

- Then  $x(t)$  is uniquely determined by its samples  $x[n] = x(nT)$  for  $n = \dots -1, 0, 1, 2, \dots$ , if

$$\Omega_s = \frac{2\pi}{T} > 2\Omega_b$$

- Therefore, to avoid aliasing, the sampling frequency must  $\geq 2\Omega_b$

## Reconstruction

$$H(j\Omega) = \begin{cases} T, & -\Omega_c < \Omega < \Omega_c \\ 0, & \text{otherwise} \end{cases}$$

- $\Omega_c \rightarrow$  a lowpass filter

$$\Omega_c = \frac{\Omega_s}{2} = \frac{\pi}{T}$$

## Discrete-Time Signal Representation with z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

- $z \rightarrow$  continuous complex variable

### Region of Convergence (ROC)

- It indicates when the z-transform of a sequence converges
- The set of values of  $z$  for which  $X(z)$  converges

$$|X(z)| = \left| \sum_{n=-\infty}^{\infty} x[n]z^{-n} \right| \leq \sum_{n=-\infty}^{\infty} |x[n]z^{-n}| < \infty$$

- If there is no value of  $z$  satisfies the converges, then z-transform does not exist
- That set of values of  $z \rightarrow$  ROC

$$|z| > \lim_{n \rightarrow \infty} \left| \frac{x[n+1]}{x[n]} \right| = R_+$$

$$|z| < \lim_{m \rightarrow \infty} \left| \frac{x[-m]}{x[-m-1]} \right| = R_-$$

- The ROC for  $X(z)$  is  $R_+ < |z| < R_-$ 
  - ROC is a ring when  $R_+ < R_-$
  - No ROC if  $R_- < R_+$  and  $X(z)$  does not exist

### Existence of DTFT

- While the DTFT converges if

$$|X(e^{j\omega})| = \left| \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \right| \leq \sum_{n=-\infty}^{\infty} |x[n]e^{-j\omega n}| = \sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

- Then, it is possible that the DTFT of  $x[n]$  does not exist

## Poles and Zeros

- The set of values of  $z$  for which  $X(z) = \pm\infty \rightarrow$  the poles of  $X(z)$
- The set of values of  $z$  for which  $X(z) = 0 \rightarrow$  the zeros of  $X(z)$

$$X(z) = \frac{z^N \sum_{k=0}^M b_k z^{M-k}}{z^M \sum_{k=0}^N a_k z^{N-k}}$$

- If  $M > N \rightarrow$  there are  $M - N$  poles at zero location
- If  $M < N \rightarrow$  there are  $N - M$  zeros at zero location

## Finite-Duration and Infinite-Duration Sequences

- Finite-duration sequence  $\rightarrow$  values of  $x[n]$  are nonzero only for a finite time interval
  - Otherwise, it is an infinite-duration sequence
    - Right-sided  $\rightarrow$  if  $x[n] = 0$  for  $n < N_+ < \infty$ , where  $N_+$  is an integer
    - Left-sided  $\rightarrow$  if  $x[n] = 0$  for  $n > N_- > -\infty$ , where  $N_-$  is an integer
    - Two-sided  $\rightarrow$  neither right-sided nor left-sided

## Table of z-transforms for Common Sequences

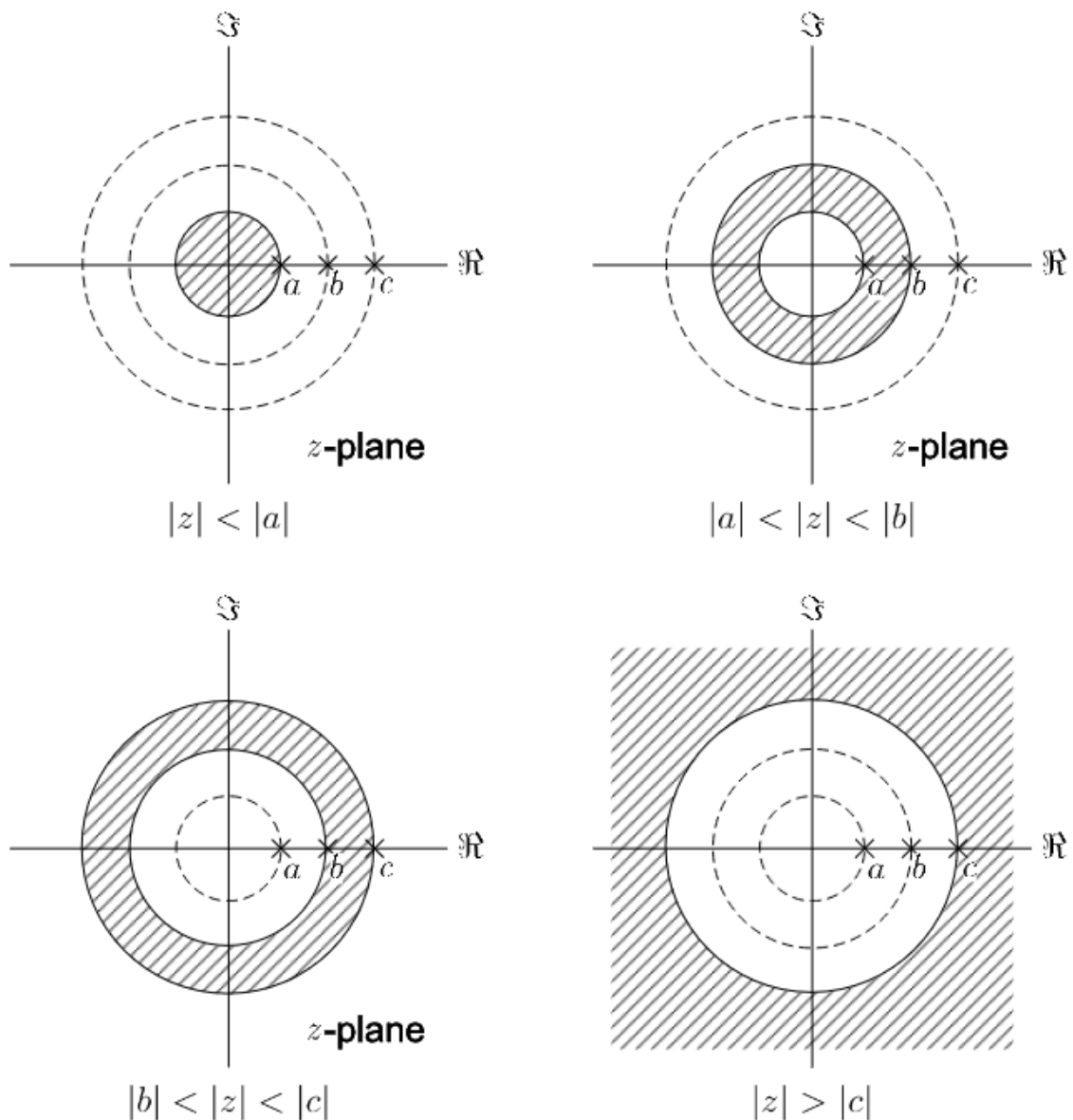
Sequence	Transform	ROC
$\delta[n]$	<b>1</b>	<b>All <math>z</math></b>
$\delta[n - m]$	$z^{-m}$	$ z  > 0, m > 0;  z  < \infty, m < 0$
$a^n u[n]$	$\frac{1}{1 - az^{-1}}$	$ z  >  a $
$-a^n u[-n - 1]$	$\frac{1}{1 - az^{-1}}$	$ z  <  a $
$na^n u[n]$	$\frac{az^{-1}}{(1 - az^{-1})^2}$	$ z  >  a $
$-na^n u[-n - 1]$	$\frac{az^{-1}}{(1 - az^{-1})^2}$	$ z  <  a $
$a^n \cos(bn) u[n]$	$\frac{1 - a \cos(b)z^{-1}}{1 - 2a \cos(b)z^{-1} + a^2 z^{-2}}$	$ z  >  a $
$a^n \sin(bn) u[n]$	$\frac{a \sin(b)z^{-1}}{1 - 2a \cos(b)z^{-1} + a^2 z^{-2}}$	$ z  >  a $



## Summary of ROC Properties

1. There are four possible shapes for ROC
  1. Entire region except  $z = 0$  and/or  $z = \infty$
  2. A ring
  3. Inside a circle in the  $z$ -plane centred at the origin
  4. Outside a circle in the  $z$ -plane centred at the origin
2. The DTFT of a sequence  $x[n]$  exists iff the ROC of the  $z$ -transform of  $x[n]$  includes the unit circle
3. The ROC cannot contain any poles
4. When  $x[n]$  is a finite-duration sequence, the ROC is the entire  $z$ -plane except  $z = 0$  and/or  $z = \infty$
5. When  $x[n]$  is a right-sided sequence, the ROC is of the form  $|z| > |p_{\max}|$  where  $p_{\max}$  is the pole with the largest magnitude in  $X(z)$
6. When  $x[n]$  is a left-sided sequence, the ROC is of the form  $|z| < |p_{\min}|$  where  $p_{\min}$  is the pole with the smallest magnitude in  $X(z)$
7. When  $x[n]$  is a two-sided sequence, the ROC is of the form  $|p_a| < |z| < |p_b|$  where  $p_a$  and  $p_b$  are two poles with the successive magnitudes in  $X(z)$  such that  $|p_a| < |p_b|$

8. The ROC must be a connected region



## Causality and Stability Investigation with ROC

- The causality condition is when  $h[n] = 0$  for  $n < 0$ 
  - If the system is causal and  $h[n]$  is of finite duration, the ROC should include  $\infty$
  - If the system is causal and  $h[n]$  is of infinite duration, the ROC is of the form  $|z| > |p_{\max}|$  and should include  $\infty$
- The stability condition is when  $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$ 
  - This also means that the DTFT of  $h[n]$  exists

## Inverse z-transform

- Using partial fraction expansion

$$X(z) = \frac{\sum_{k=0}^M b_k z^{-k}}{\sum_{k=0}^N a_k z^{-k}}$$

- Determine the  $N$  nonzero poles  $c_1, c_2, \dots, c_N$
- Case 1  $\rightarrow M < N$  and all poles are of the first order

1.  $X(z) = \sum_{k=1}^N \frac{A_k}{1 - c_k z^{-1}}$

2. Find  $A_k = (1 - c_k z^{-1})X(z) \Big|_{z=c_k}$

3. Perform inverse z-transform by inspection

- Case 2  $\rightarrow M \geq N$  and all poles are of the first order

1.  $X(z) = \sum_{l=0}^{M-N} B_l z^{-l} + \sum_{k=1}^N \frac{A_k}{1 - c_k z^{-1}}$

2. Find  $B_l$  by using the long division of the numerator by the denominator

3. Find  $A_k = (1 - c_k z^{-1})X(z) \Big|_{z=c_k}$

4. Perform inverse z-transform by inspection

- Case 3  $\rightarrow M < N$  with multiple-order poles

1.  $X(z) = \sum_{k=1, k \neq i}^N \frac{A_k}{1 - c_k z^{-1}} + \sum_{m=1}^s \frac{C_m}{(1 - c_i z^{-1})^m}$

2. Find  $A_k = (1 - c_k z^{-1})X(z) \Big|_{z=c_k}$

3. Find  $C_m = \frac{1}{(s-m)!(-c_i)^{s-m}} \cdot \frac{d^{s-m}}{dw^{s-m}} [(1 - c_i w)^s X(w^{-1})] \Big|_{w=c_i^{-1}}$

4. Perform inverse z-transform by inspection

- Case 4  $\rightarrow M \geq N$  with multiple-order poles

1.  $X(z) = \sum_{l=0}^{M-N} B_l z^{-l} + \sum_{k=1, k \neq i}^N \frac{A_k}{1 - c_k z^{-1}} + \sum_{m=1}^s \frac{C_m}{(1 - c_i z^{-1})^m}$

2. Find  $A_k = (1 - c_k z^{-1})X(z) \Big|_{z=c_k}$

3. Find  $B_l$  by using the long division of the numerator by the denominator

4. Find  $C_m = \frac{1}{(s-m)!(-c_i)^{s-m}} \cdot \frac{d^{s-m}}{dw^{s-m}} [(1 - c_i w)^s X(w^{-1})] \Big|_{w=c_i^{-1}}$

5. Perform inverse z-transform by inspection

- Using power series expansion

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \dots + x[-1]z^1 + x[0] + x[1]z^{-1} + x[2]z^{-2} + \dots$$

## Transfer Function of Linear Time-Invariant System

$$\sum_{k=0}^N a_k y[n-k] = \sum_{k=0}^M b_k x[n-k]$$

$$Y(z) \sum_{k=0}^N a_k z^{-k} = X(z) \sum_{k=0}^M b_k z^{-k}$$

- The transfer function  $H(z)$  defined as:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^M b_k z^{-k}}{\sum_{k=0}^N a_k z^{-k}}$$

- This suggests taking the z-transform for  $x[n]$  and  $h[n] \rightarrow **X(z)H(z)$ 
  - Perform inverse z-transform of  $X(z)H(z) \rightarrow y[n]$