Unit 8

Linearity (with solution)

Question 1: Vector Space

Consider the set of all binary n-vectors, $\{0, 1\}^n$

- Addition of two vectors is defined by

$$(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n),$$

where the addition of two bits is defined by modulo-2 addition (i.e., logical XOR).

- Scalar multiplication is defined by

$$c(x_1, \dots, x_n) = (cx_1, \dots, cx_n), \quad \text{for } c \in \{0, 1\},$$

where multiplication of two bits is defined by usual multiplication (i.e., $0 \cdot 0 = 0 \cdot 1 = 0$ and $1 \cdot 1 = 1$).

Is it a vector space?

Q.1 (solution)

- Commutative and associative conditions are satisfied because of the property of XOR.
- Zero condition is satisfied since $x + \mathbf{0} = x$.
- Inverse condition is satisfied since $x + x = \mathbf{0}$.
- Associative and Unitarity conditions for scale multiplication are satisfied due to the property of usual multiplication.
- It is straightforward to check that the two distributive conditions are also satisfied.

Question 2: Subspace

The set of all real polynomials (with usual addition and scalar multiplication) is a vector space.

- The degree of a polynomial is the highest of the degrees of its monomials (individual terms) with ***non-zero*** coefficients.

Is each of the following sets its subspaces? Why?

a) The set of all real polynomials with degree less than n;

b) The set of all real polynomials with degree equal to n.

Q.2 (solution)

a) The set consists of all real polynomials in the form of $p = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}$.
(Note: some coefficients a_i may be zero)

- Closed under addition:

  $p + q = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_{n-1} + b_{n-1})x^{n-1}$
  is still in the set.

- Closed under scalar multiplication:

  $cp = (ca_0 + ca_1 x + \cdots + ca_{n-1} x^{n-1})$ is still in the set.

- Therefore, it is a subspace.

Q.2 (solution)

b) The set consists of all real polynomials in the form of $p = a_0 + a_1 x + \cdots + a_n x^n, a_n \neq 0$.

- Not closed under addition if $a_n = -b_n$:

  $p + q = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n$

  $= (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_{n-1} - b_{n-1})x^{n-1}$,
  whose degree is at most $n - 1$.

- Not closed under scalar multiplication if $c = 0$:

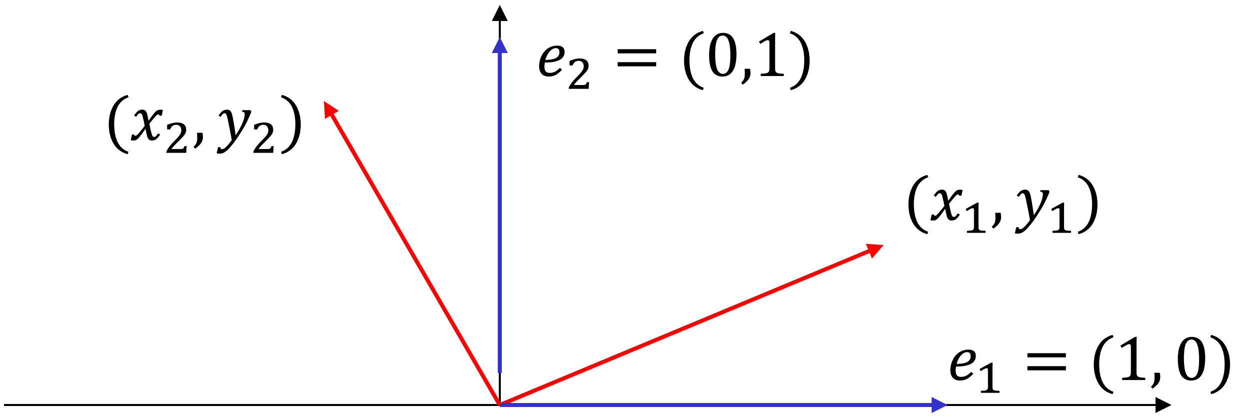
  $0p = 0$, whose degree is 0.

- Therefore, it is **not** a subspace.

Additional Explanation on Q.2(b)

- To better understand the solution (which applies to any value of n), you may consider a concrete example, say, $n = 2$.
- The set of all polynomials with degree equal to 2 is
$$\{ax^2 + bx + c : a > 0\}.$$
  - Note that a must be positive, for otherwise the degree of the polynomial is less than 2.
- Consider x^2 and $-x^2$, which are both in the above set. The set is not closed under addition because $x^2 + (-x^2) = 0$, which has degree 0.
- It can also be proved that the set is not closed under scalar multiplication by choosing the scalar to be 0.

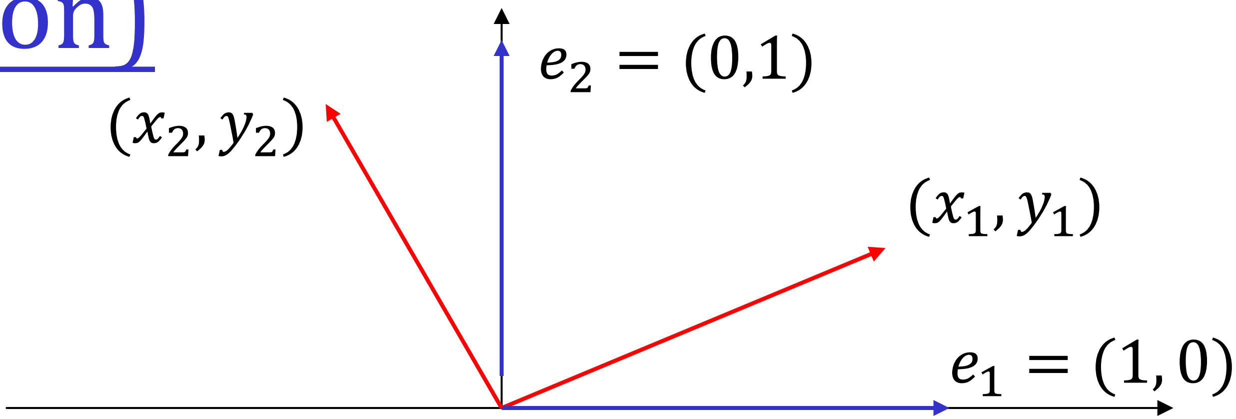
Question 3: Rotation



Consider anti-clockwise rotations of e_1 and e_2 by 30^o.

a) Find (x_1, y_1) and (x_2, y_2).
b) Consider an arbitrary vector $v = (x, y)$. Express v as a linear combination of e_1 and e_2.
c) What is the resultant vector after rotating v by 30^o?
d) What is the corresponding rotation matrix?

Q.3 (solution)



a) $(x_1, y_1) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right), (x_2, y_2) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

b) $v = xe_1 + ye_2$

c) $v' = x\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) + y\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ (by linearity)

$= \left(\frac{\sqrt{3}}{2}x - \frac{1}{2}y,\ \frac{1}{2}x + \frac{\sqrt{3}}{2}y\right)$

d) $A = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

Question 4: Projection

Consider the straight line $y = \frac{x}{2}$ in the 2-dimensional space.

a) Find the matrix that projects any vector to the above line.
b) Hence, find the projection of (3, 2) onto the above line.

Q.4 (solution)

a) The slope is ½. Consider a right-angled triangle with base 2, height 1, and hypotenuse $\sqrt{5}$. Therefore, $\cos\theta = \frac{2}{\sqrt{5}}$ and $\sin\theta = \frac{1}{\sqrt{5}}$.

The projection matrix is

$$P = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}.$$

b) The vector after projection is

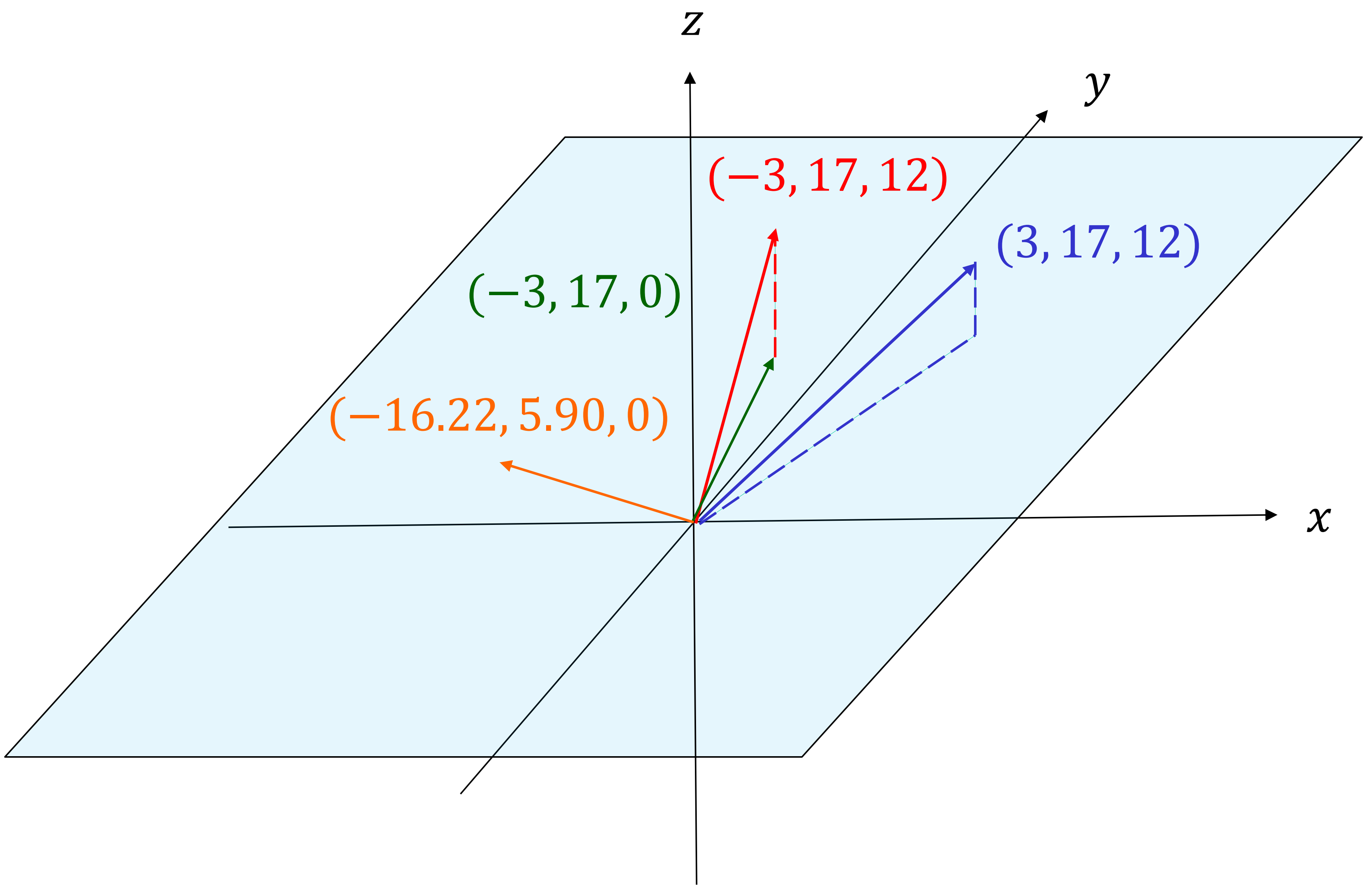
$$\frac{1}{5}\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 16 \\ 8 \end{bmatrix}.$$

Question 5: Geometric Transformations

- Consider the vector (3, 17, 12). First, it is reflected across the *y*-*z* plane. Next, it is projected onto the *x*-*y* plane. Lastly, it is rotated anti-clockwise by 60^{o} on the *x*-*y* plane. What is the *x*-component of the resultant vector? Round your answer to 2 decimal places.

Q5. Solution

- Solution:
  - Reflection across the y-z plane: $(-3, 17, 12)$.
  - Projection onto the x-y plane: $(-3, 17, 0)$.
  - Rotation anti-clockwise by 60º on the x-y plane:
    $$\begin{bmatrix} \cos 60^o & -\sin 60^o \\ \sin 60^o & \cos 60^o \end{bmatrix} \begin{bmatrix} -3 \\ 17 \end{bmatrix} = \begin{bmatrix} -16.22 \\ 5.90 \end{bmatrix}.$$
  - Hence, the x-component is -16.22.


z
y
$(-3, 17, 12)$
$(3, 17, 12)$
$(-3, 17, 0)$
$(-16.22, 5.90, 0)$
x