3210 ASS 2

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1) $x(t) = e^{-2|t-1|}$ $\begin{cases} e^{-2(t-1)}, & t<1 \\ e^{2(t-1)}, & t\geq 1 \end{cases}$

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt = \int_{-\infty}^{\infty} e^{-2|t-1|} e^{-j\Omega t} dt$$

$$= \int_{-\infty}^{1} e^{2(t-1)} e^{-j\Omega t} dt + \int_{1}^{\infty} e^{-2(t-1)} e^{-j\Omega t} dt$$

$$= \int_{-\infty}^{1} e^{2t-2-j\Omega t} dt + \int_{1}^{\infty} e^{-2t+2-j\Omega t} dt$$

$$= e^{-2}\int_{-\infty}^{1} e^{t(2-j\Omega)} dt + e^{2}\int_{1}^{\infty} e^{t(-2-j\Omega)} dt$$

$$= \frac{e^{-2}}{2-j\Omega}\left[e^{t(2-j\Omega)}\right]_{-\infty}^{1} + \frac{e^{2}}{-2-j\Omega}\left[e^{t(-2-j\Omega)}\right]_{1}^{\infty}$$

$$= \frac{e^{-2}}{2-j\Omega}\left(e^{2-j\Omega}\right) - \frac{e^{2}}{2+j\Omega}\left(-e^{-2-j\Omega}\right)$$

$$= \frac{e^{-j\Omega}}{2-j\Omega} + \frac{e^{-j\Omega}}{2+j\Omega}$$

$$= \frac{e^{-j\Omega}(2+j\Omega+2-j\Omega)}{4+\Omega^2}$$

$$= \frac{4e^{-j\Omega}}{4+\Omega^2}$$

2) $y[n] - \frac{1}{2}y[n-1] = x[n] + 2x[n-1] + x[n-2]$

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n] e^{-j\omega n}$$

DTFT $y[n-1]$: $\sum_{m=-\infty}^{\infty} y[m] e^{-j\omega(m+1)}$, where $m = n-1$, $n = m+1$

$$= \sum y[m] e^{-j\omega m} \cdot e^{-j\omega}$$

$$= e^{-j\omega} Y(e^{j\omega})$$

DTFT $x[n-2]$ $= \sum_{k=-\infty}^{\infty} x[k] e^{-j\omega(k+2)}$, where $k = n-2$, $n = k+2$

$$= e^{-2j\omega} X(e^{j\omega})$$

i.e. $Y(e^{j\omega})(1 - 0.5e^{-j\omega}) = X(e^{j\omega})(1 + 2e^{-j\omega} + e^{-2j\omega})$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1+2e^{-j\omega}+e^{-2j\omega}}{1-0.5e^{-j\omega}}$$

9) $x[n] = u[-n-1]$ $\quad$ $h[n] = 0.5^n u[n]$

$X_z = \dfrac{-1}{1-z^{-1}}, |z|<1$ $\quad$ $H_z = \dfrac{1}{1-0.5z^{-1}}, |z|>0.5$

$Y_z = X_z \cdot H_z = \dfrac{-1}{1-z^{-1}} \cdot \dfrac{1}{1-0.5z^{-1}} = \dfrac{-1}{(1-z^{-1})(1-0.5z^{-1})} = \dfrac{A}{1-z^{-1}} + \dfrac{B}{1-0.5z^{-1}}$

$A(1-0.5z^{-1}) + B(1-z^{-1}) = \dfrac{-1}{(1-z^{-1})(1-0.5z^{-1})}$

Compare coefficients

$A + B = -1$ ①

$-0.5A - B = 0$ ②

①: $B = -1-A$

put ① into ②

②: $-0.5A + 1 + A = 0$

$0.5A = -1$

$A = -2$

Then $B = 1$

$A = (1-z^{-1})Y(z)\Big|_{z=1} = \dfrac{-1}{1-0.5z^{-1}}\Big|_{z=1} = -2$

$B = (1-0.5z^{-1})Y(z)\Big|_{z=0.5} = \dfrac{-1}{1-z^{-1}}\Big|_{z=0.5} = 1$

i.e. $Y_z = \dfrac{-2}{1-z^{-1}} + \dfrac{1}{1-0.5z^{-1}}$

Inverse z transform

$\therefore y[n] = 2u[-n-1] + 0.5^n u[n]$

3) $H(e^{jw}) = \dfrac{1-\frac{1}{2}e^{-jw}+e^{-3jw}}{1+\frac{1}{2}e^{-jw}+\frac{3}{4}e^{-2jw}} = \dfrac{Y(e^{jw})}{X(e^{jw})}$

$X(e^{jw})\left(1-\frac{1}{2}e^{-jw}+e^{-3jw}\right) = Y(e^{jw})\left(1+\frac{1}{2}e^{-jw}+\frac{3}{4}e^{-2jw}\right)$

$X(e^{jw}) - \frac{1}{2}e^{-jw}X(e^{jw}) + e^{-3jw}X(e^{jw}) = Y(e^{jw}) + \frac{1}{2}e^{-jw}Y(e^{jw}) + \frac{3}{4}e^{-2jw}Y(e^{jw})$

$x[n] - \frac{1}{2}x[n-1] + x[n-3] = y[n] + \frac{1}{2}y[n-1] + \frac{3}{4}y[n-2]$

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4) (a) $p[n] = x[n] + x[n] \otimes h_1[n]$

$y[n] = p[n] \otimes h_2[n] = (x[n] + x[n] \otimes h_1[n]) \otimes h_2[n]$

$= x[n] \otimes h_2[n] + x[n] \otimes h_1[n] \otimes h_2[n]$

$= x[n] \otimes (h_2[n] + h_1[n] \otimes h_2[n])$

Hence $h[n] = h_2[n] + h_1[n] \otimes h_2[n]$

$= \sum_{k=-\infty}^{\infty} a^n u[n] + \beta \delta[k-1] a^{n-k} u[n-k]$

$= \beta a^{n-1} u[n-1] + a^n u[n]$

$\therefore h[n] = \begin{cases} \beta a^{n-1} + a^n, & \text{for } n \geq 1 \\ 0 \end{cases}$ //

(b) $H(z) = \frac{Y(z)}{X(z)} = \beta a^{n-1} u[n-1] + a^n u[n]$

$= \frac{\beta}{1-az^{-1}} + \frac{1}{1-az^{-1}}$

$= \frac{\beta+1}{1-az^{-1}}$ //

(c) $\frac{Y(z)}{X(z)} = \frac{\beta+1}{1-az^{-1}}$

$Y(z)(1-az^{-1}) = (\beta+1)X(z)$

$Y(z) - az^{-1}Y(z) = \beta X(z) + X(z)$

$y[n] - ay[n-1] = \beta x[n] + x[n]$ //

(d) $y[n] = \beta x[n] + x[n] + ay[n-1]$

$\because$ Depend on past and current input

$\therefore$ Not Causal //

(e) Let denominator $1-az^{-1} = 0$

$\frac{z}{z-a} = 0$

$z = a$

$\therefore$ When $z = |a|$, would be stable //

5) $x[n] = \begin{cases} ae^{j(w_0 n + \phi)}, & 0 \le n \le N-1 \\ 0 \end{cases}$

(a) $X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jwn} = \sum_{n=0}^{N-1} ae^{j(w_0 n+\phi)} e^{-jwn} = \sum_{n=0}^{N-1} ae^{jw_0 n + j\phi - jwn}$

$= ae^{j\phi} \sum_{n=0}^{N-1} e^{(w_0 - w)n}$

$= ae^{j\phi}\left[\frac{1 - e^{(w_0-w)N}}{1 - e^{w_0 - w}}\right]$

(b) $|X(e^{jw})| = |a| \left| \frac{1 - e^{N(w_0 - w)}}{1 - e^{w_0 - w}} \right|$

To maximize, minimize denominator

$\ell\ (1 - e^{w_0 - w}) \to 0$

$-e^{w_0 - w} \to -1$

$e^{w_0 - w} \to 1$

$(w_0 - w)\ln e \to \ln 1$

$w_0 - w \to 0$

$w_0 \to w$

$\therefore X(e^{jw})$ maximized when w_0 tends to w, max $|X(e^{jw})| = |a|$

6) $x(t) = \sin\left(\frac{\pi}{2} t\right)$, $T = 1s$

$x[nT] = \sin\left[\frac{\pi}{2}(nT)\right]$

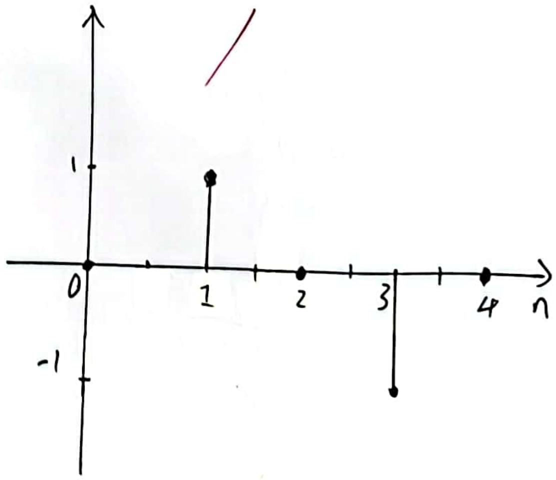
when $n=0$, $x[0] = \sin[0] = 0$

$x[1] = \sin\left[\frac{\pi}{2} \cdot T\right] = \sin\left(\frac{\pi}{2}\right) = 1$

$x[2] = \sin(\pi) = 0$

$x[3] = \sin \frac{3\pi}{2} = -1$

$x[4] = \sin(2\pi) = 0$



$\therefore x[n]$ is periodic

9) $H(z) = \dfrac{z^{-2}}{(1-0.5z^{-1})(1-3z^{-1})}$, $0.5 < z < 3$

(a) Given stable, $x[n] = u[n]$

$X(z) = \dfrac{1}{1-z^{-1}}$

$Y(z) = H(z)X(z) = \dfrac{z^{-2}}{(1-z^{-1})(1-0.5z^{-1})(1-3z^{-1})}$

Case 1 Partial

$Y(z) = \dfrac{A}{1-z^{-1}} + \dfrac{B}{1-0.5z^{-1}} + \dfrac{C}{1-3z^{-1}}$

$A: (1-z^{-1})Y(z)\Big|_1 = \dfrac{z^{-2}}{(1-0.5z^{-1})(1-3z^{-1})}\Big|_1 = -1$

$B: (1-0.5z^{-1})Y(z)\Big|_{0.5} = \dfrac{0.5^{-2}}{(1-0.5^{-1})(1-3\times0.5^{-1})} = 0.8$

$C = (1-3z^{-1})Y(z)\Big|_3 = \dfrac{3^{-2}}{(1-3^{-1})(1-0.5\cdot3^{-1})} = 0.2$

$= \dfrac{-1}{1-z^{-1}} + \dfrac{0.8}{1-0.5z^{-1}} + \dfrac{0.2}{1-3z^{-1}}$, $1<z<3$

$\therefore y[n] = -u[n] + 0.8(0.5)^n u[n] - 0.2(3)^n u[-n-1]$

(b) Given causal, $x[n] = \delta[n]$

$X(z) = 1$, $\in$ all z

$Y(z) = H(z)X(z) = \dfrac{z^{-2}}{(1-0.5z^{-1})(1-3z^{-1})}$

$$\begin{array}{r} \frac{1}{1.5} \\ 1.5z^{-2} - 3.5z^{-1} + 1 \,\overline{)\, z^{-2}} \\ -\frac{7}{3}z + \frac{2}{3} \end{array}$$

Case 2 partial

$Y(z) = \dfrac{2}{3} + \dfrac{A}{1-0.5z^{-1}} + \dfrac{B}{1-3z^{-1}}$

$A: (1-0.5z^{-1})Y(z)\Big|_{0.5} = \dfrac{0.5^{-2}}{(1-3\cdot0.5^{-1})} = -0.8$

$B: (1-3z^{-1})Y(z)\Big|_3 = \dfrac{3^{-2}}{(1-0.5\cdot3^{-1})} = \dfrac{2}{15}$

$Y(z) = \dfrac{2}{3} + \dfrac{-0.8}{1-0.5z^{-1}} + \dfrac{2}{15}\left(\dfrac{1}{1-3z^{-1}}\right)$, $z>3$

$\therefore y[n] = \frac{2}{3}\delta[n] - 0.8(0.5)^n u[n] + \frac{2}{15}(3)^n u[n]$

7) $x[n] = \begin{cases} na^n, & 1 \le n \le N \\ 0 \end{cases}$

$$x(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=1}^{N} na^n z^{-n}$$

$$= az^{-1} + 2a^2 z^{-2} + \dots + Na^N z^{-N}$$

$$az^{-1} x(z) = a^2 z^{-2} + 2a^3 z^{-3} + \dots + Na^{N+1} z^{-N-1}$$

$$x(z)[1 - az^{-1}] = az^{-1} + a^2 z^{-2} + \dots + a^N z^{-N} + Na^{N+1} z^{-N+1}$$

$$= \frac{(az^{-1})[1-(az^{-1})^{N+1}]}{1-az^{-1}} + N(az^{-1})^{N+1}$$

$$x(z) = \frac{az^{-1}}{(1-az^{-1})^2} \cdot [1-(az^{-1})^{N+1}] + \frac{N(az^{-1})^{N+1}}{1-az^{-1}}$$

$\therefore$ ROC is all values of z, except when $z = a$

10) (a) A denoising system takes in a noisy signal and outputs a clean signal with useful information.

Input : Noisy digital signals or images, e.g. MRI scans

Output : Clean, denoised signals or images

Functions : Using mathematical concepts, sparsity, to distinguish meaningful signals from unwanted noise.

Principle : Exploiting sparsity phenomenon, meaningful information is surrounded by relative quiet. Mathematical transform are used to compress, remove noises and emphasizing signal.

(b) Use fourier transform to extract $\cos(100\pi t)$.

Use Inverse Fourier Transform to compress signal.