1. (10 pts) Determine if the following statements or equations are true or false. It suffices to just provide an answer. You do not need to justify it.

(i) The series connection of two LTI systems is a LTI system.

(ii) The parallel connection of two LTI systems is a LTI system.

(iii) If the input to a LTI system is periodic, then the output is also periodic.

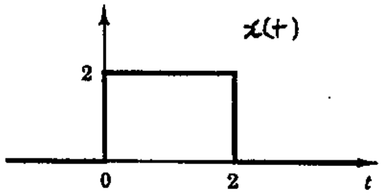
(iv) If $h(t)$ is the impulse response of a LTI system and $h(t)$ is periodic and nonzero, then the system is unstable.

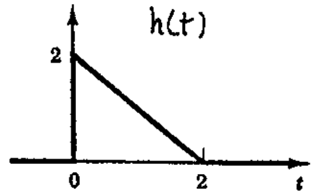
(v) If a LTI system is causal, then it is stable.

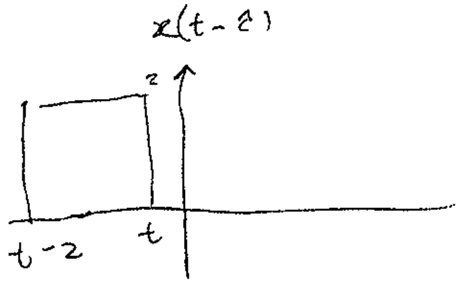
(vi) If $y(t) = x(t) * h(t)$, then $y(2t) = 2x(2t) * h(2t)$.

(vii) If $x(t)$ and $h(t)$ are odd, then $y(t) = x(t) * h(t)$ is even.

2. (5 pts) The impulse response of a LTI system and an input signal applied to the system are shown in the following figure. Determine the output response.







$$h(t) = \begin{cases} 2 - t & 0 \le t \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Integral limits: 1
Integrand: 1

1. $\underline{t < 0}$, $y(t) \equiv 0$.

2. $0 < t < 2$, $y(t) = \int_0^t (2-\tau)\, 2\, d\tau = 4t - t^2$

2. $2 < t < 4$, $y(t) = \int_{t-2}^{2} (2-\tau)\, 2\, d\tau = 4(2 - t + 2) - \left(2^2 - (t-2)^2\right)$

$$= 16 - 4t - 4 + t^2 - 4t + 4$$

$$= 16 - 8t + t^2$$

$t > 4$, $y(t) \equiv 0$

3. (6 pts) Let $x(t)$ be a periodic signal with period T. Suppose that its Fourier coefficients are given by

$$c_k = \begin{cases} 2, & k = 0 \\ (1/2)^{|k|}, & k \neq 0 \end{cases}$$

Answer and justify the following questions: (1) Is $x(t)$ real? (2) $\underbrace{\text{Is } x(t) \text{ even?}}_{3}$ (3) Calculate 3

$$\frac{1}{T}\int_T |x(t)|^2 dt.$$

1. Solution: (1) $x(t)$ is real, since

1. $$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\frac{2\pi}{T}t} = 2 + \sum_{k=1}^{\infty} 2\left(\tfrac{1}{2}\right)^k \cos k\frac{2\pi}{T}t$$

1. (2) $x(t)$ is even, $x(t) = x(-t)$

(3)

1. $$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |c_k|^2$$

1. $$= 2^2 + 2\sum_{k=1}^{\infty}\left(\tfrac{1}{2}\right)^{2k}$$

$$= 2^2 - 2 + 2\sum_{k=0}^{\infty}\left(\tfrac{1}{2}\right)^{2k}$$

$$= 2 + 2\sum_{k=0}^{\infty}\left(\tfrac{1}{4}\right)^{k}$$

1. $$= 2 + 2\cdot\frac{1}{1-\frac{1}{4}}$$

$$= 2 + \frac{8}{3} = \frac{16}{3}$$

4. (7 pts) (1) A LTI system has an impulse response $h(t) = e^{t}u(-t)$, where $u(t)$ is the unit step signal. (a) Give a differential equation description for the system. (b) Suppose that an input signal is given as $x(t) = e^{-t}u(t)$. Determine the output response.

Solution. Let $f(t) = e^{-t}u(t)$, $\Rightarrow h(t) = f(-t)$

1. $$H(\omega) = F(-\omega) = \frac{1}{1-j\omega} = \frac{Y(\omega)}{X(\omega)}$$

1. (a) $$-\frac{dy(t)}{dt} + y(t) = x(t), \text{ or } \frac{dy(t)}{dt} - y(t) = -x(t)$$

1. (b) $$X(\omega) = \frac{1}{1+j\omega}, \quad Y(\omega) = \frac{1}{(1-j\omega)(1+j\omega)}$$

$$Y(\omega) = \frac{A}{1-j\omega} + \frac{B}{1+j\omega}, \quad A = (1-s)Y(s)\Big|_{s=1} = \frac{1}{2}$$

$$B = (1+s)Y(s)\Big|_{s=-1} = \frac{1}{2}$$

1. $$= \frac{1}{2}\frac{1}{1-j\omega} + \frac{1}{2}\frac{1}{1+j\omega}$$

1. $$y(t) = \frac{1}{2}\left[e^{t}u(-t) + e^{-t}u(t)\right]$$

(2) Let $F(\omega)$ be the Fourier transform of $f(t)$. Show that

$$\int_{-\infty}^{\infty}\left|\frac{df(t)}{dt}\right|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\omega^2|F(\omega)|^2 d\omega.$$

Proof: $$\mathcal{F}\left\{\frac{df(t)}{dt}\right\} = j\omega F(\omega)$$

$$\Rightarrow \int_{-\infty}^{\infty}\left|\frac{df(t)}{dt}\right|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|j\omega F(\omega)|^2 d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\omega^2|F(\omega)|^2 d\omega$$

5. (9 pts) Consider the LTI system with frequency response

$$H(\omega) = \frac{j\omega + 1}{(j\omega)^2 + 7j\omega + 12}.$$

(1) Find the impulse response of this system.
(2) Determine a differential equation that describes the system.
(3) Find a block diagram realization (of any kind) consisting of adders, integrators, and coefficient multipliers for this system.
(4) Suppose that an input signal $x(t) = e^{-2t}u(t)$ is applied. Determine the output response.

Solution:

(1) $H(\omega) = \dfrac{j\omega+1}{(j\omega+3)(j\omega+4)}$

$H(s) = \dfrac{s+1}{(s+3)(s+4)} = \dfrac{A}{s+3} + \dfrac{B}{s+4} = -\dfrac{2}{s+3} + \dfrac{3}{s+4}$

2. $A = (s+3)\dfrac{s+1}{(s+3)(s+4)}\Big|_{s=-3} = -2$

$B = (s+4)\dfrac{s+1}{(s+3)(s+4)}\Big|_{s=-4} = 3$

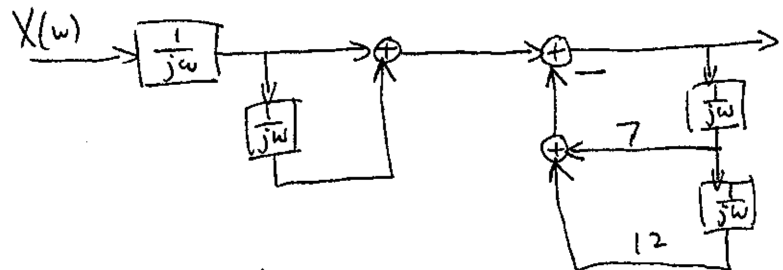
$h(t) = \left[-2e^{-3t} + 3e^{-4t}\right]u(t)$

(2) $\dfrac{Y(j\omega)}{X(\omega)} = H(\omega) = \dfrac{j\omega+1}{(j\omega)^2+7j\omega+12}$

2. $\dfrac{d^2y(t)}{dt^2} + 7\dfrac{dy(t)}{dt} + 12y(t) = \dfrac{dx(t)}{dt} + x(t)$

(3) $Y(\omega) = \dfrac{1}{j\omega}X(\omega) + \dfrac{1}{(j\omega)^2}X(\omega) - \dfrac{7}{j\omega}Y(\omega) - \dfrac{12}{(j\omega)^2}Y(\omega)$

2



(4) $X(\omega) = \dfrac{1}{j\omega+2}$

3 $Y(\omega) = H(\omega)X(\omega) = \dfrac{j\omega+1}{(j\omega+2)(j\omega+3)(j\omega+4)}$

$Y(s) = \dfrac{s+1}{(s+2)(s+3)(s+4)} = \dfrac{A}{s+2} + \dfrac{B}{s+3} + \dfrac{c}{s+4}$

$$A = \cancel{(s+2)} \frac{s+1}{\cancel{(s+2)}(s+3)(s+4)}\Bigg|_{s=-2} = -\frac{1}{2}$$

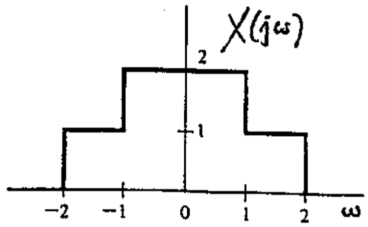
$$B = \cancel{(s+3)} \frac{s+1}{(s+2)\cancel{(s+3)}(s+4)}\Bigg|_{s=-3} = 2$$

$$C = \cancel{(s+4)} \frac{s+1}{(s+2)(s+3)\cancel{(s+4)}}\Bigg|_{s=-4} = -\frac{3}{2}$$

$$\Rightarrow Y(\omega) = -\frac{1}{2}\frac{1}{j\omega+2} + 2\frac{1}{j\omega+3} - \frac{3}{2}\frac{1}{j\omega+4}$$

$$y(t) = \left[-\frac{1}{2}e^{-2t} + 2e^{-3t} - \frac{3}{2}e^{-4t}\right]u(t)$$

6. (5 pts) The Fourier transform of a certain signal $x(t)$ is given in the following figure. Find $x(t)$ without performing any integration.



Solution :

$$X(j\omega) = Y_1(\omega) + Y_2(\omega)$$

2. $$Y_1(\omega) = \begin{cases} 1 & |\omega| < 2 \\ 0 & |\omega| > 2 \end{cases} \Rightarrow y_1(t) = \frac{\sin 2t}{\pi t}$$

2. $$Y_2(\omega) = \begin{cases} 1 & |\omega| < 1 \\ 0 & |\omega| > 1 \end{cases} \Rightarrow y_2(t) = \frac{\sin t}{\pi t}$$

1. $$\Rightarrow \quad x(t) = y_1(t) + y_2(t) = \frac{\sin 2t}{\pi t} + \frac{\sin t}{\pi t}$$

$$= \frac{\sin t}{\pi t}\left(1 + 2\cos t\right)$$