In-Class Exercise 4

1. Compute the Fourier transform of

$$x(t) = e^{-\alpha|t|}, \quad \alpha > 0$$

   Then find the magnitude and phase of $X(j\Omega)$.

2. Compute the Fourier transform of $x(t) = \cos(100t)$.

3. Compute the Fourier transform of $x(t) = 1$.

4. Compute the Fourier transform of

$$x(t) = \sum_{k=0}^{\infty} \alpha^k \delta(t - kT), \quad |\alpha| < 1$$

5. Prove the conjugation property of Fourier transform:

$$x(t) \leftrightarrow X(j\Omega) \Rightarrow x^*(t) \leftrightarrow X^*(-j\Omega)$$

Then show that if $x(t)$ is real-valued, then the magnitude of Fourier transform is symmetric around $\Omega = 0$:

$$|X(j\Omega)| = |X(-j\Omega)|$$

6. Prove the frequency shifting property of Fourier transform:

$$x(t) \leftrightarrow X(j\Omega) \Rightarrow e^{j\Omega_0 t}x(t) \leftrightarrow X(j(\Omega - \Omega_0))$$

Then determine the Fourier transform of $x(t)\cos(\Omega_0 t)$ in terms of $X(j\Omega)$.

7. Given the inverse Fourier transform formula:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t}d\Omega$$

Show that

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt$$

which is the Fourier transform formula.

Solution

1.

The first step is to re-express $x(t)$ so that the absolute sign is removed:

$$x(t) = e^{-\alpha|t|} = \begin{cases} e^{-\alpha t}, & t > 0 \\ e^{\alpha t}, & t < 0 \end{cases}$$

We then apply (5.1) to obtain:

$$\begin{aligned} X(j\Omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt = \int_{-\infty}^{0} e^{\alpha t} e^{-j\Omega t} dt + \int_{0}^{\infty} e^{-\alpha t} e^{-j\Omega t} dt \\ &= \int_{-\infty}^{0} e^{-(j\Omega - \alpha)t} dt + \int_{0}^{\infty} e^{-(j\Omega + \alpha)t} dt \\ &= \frac{1}{-(j\Omega - \alpha)} e^{-(j\Omega - \alpha)t} \Big|_{-\infty}^{0} + \frac{1}{-(j\Omega + \alpha)} e^{-(j\Omega + \alpha)t} \Big|_{0}^{\infty} \\ &= -\frac{1}{j\Omega - \alpha} + \frac{1}{j\Omega + \alpha} \end{aligned}$$

Further simplification yields:

$$\begin{aligned} X(j\Omega) &= -\frac{1}{j\Omega-\alpha}\cdot\frac{-j\Omega-\alpha}{-j\Omega-\alpha}+\frac{1}{j\Omega+\alpha}\cdot\frac{-j\Omega+\alpha}{-j\Omega+\alpha} \\ &= \frac{2\alpha}{\alpha^2+\Omega^2} \end{aligned}$$

Hence $X(j\Omega)$ is real-valued because α is real-valued and positive.

Hence

$$|X(j\Omega)| = X(j\Omega) = \frac{2\alpha}{\alpha^2+\Omega^2}$$

and

$$\angle\left(X(j\Omega)\right) = 0$$

2.

According to (5.12):

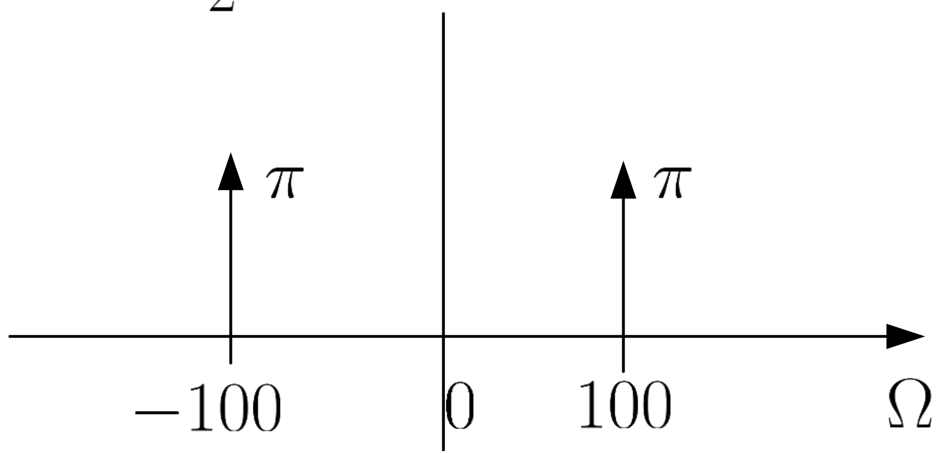
$$e^{j\Omega_0 t} \leftrightarrow 2\pi\delta(\Omega - \Omega_0)$$

Moreover,

$$\cos(\Omega_0 t) = \frac{e^{j\Omega_0 t} + e^{-j\Omega_0 t}}{2}$$

As a result,

$$\cos(100t) = \frac{e^{j100t} + e^{-j100t}}{2} \leftrightarrow \pi\delta(\Omega + 100) + \pi\delta(\Omega - 100)$$



3.

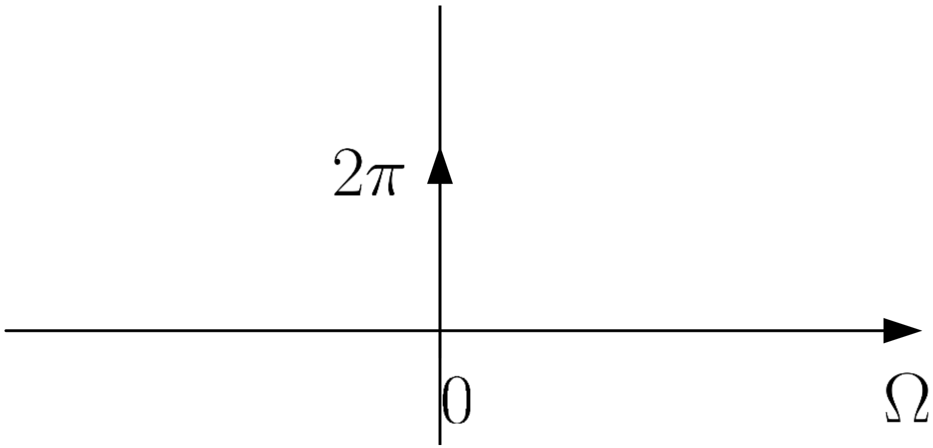
We can apply (5.12) again:

$$e^{j\Omega_0 t} \leftrightarrow 2\pi\delta(\Omega - \Omega_0)$$

A DC signal of $x(t) = 1$ corresponds to the frequency of $\Omega = 0$ as it can be expressed as $x(t) = e^{j\cdot 0\cdot t}$.

As a result,

$$1 = e^{j\cdot 0\cdot t} \leftrightarrow 2\pi\delta(\Omega)$$



4.

Applying (5.1) and using the properties of $\delta(t)$, we get:

$$
\begin{aligned}
X(j\Omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt \\
&= \int_{-\infty}^{\infty} \sum_{k=0}^{\infty} \alpha^k \delta(t-kT)e^{-j\Omega t}dt \\
&= \sum_{k=0}^{\infty} \alpha^k \int_{-\infty}^{\infty} \delta(t-kT)e^{-j\Omega t}dt \\
&= \sum_{k=0}^{\infty} \alpha^k e^{-jk\Omega T} \\
&= \sum_{k=0}^{\infty} \left(\alpha e^{-j\Omega T}\right)^k \\
&= \frac{1}{1-\alpha e^{-j\Omega T}}
\end{aligned}
$$

5.

Recall (5.1):

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$$

Let $x_1(t) = x^*(t)$. Its Fourier transform is:

$$X_1(j\Omega) = \int_{-\infty}^{\infty} x^*(t) e^{-j\Omega t} dt = \left(\int_{-\infty}^{\infty} x(t) e^{-j(-\Omega)t} dt \right)^* = X^*(-j\Omega)$$

If $x(t)$ is real-valued, then $x(t) = x^*(t)$. Their Fourier transforms should be identical. Hence we have:

$$X(j\Omega) = X^*(-j\Omega) \Rightarrow |X(j\Omega)| = |X(-j\Omega)|$$

6.

Recall (5.1):

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$$

Let $x_1(t) = e^{j\Omega_0 t} x(t)$. Its Fourier transform is:

$$X_1(j\Omega) = \int_{-\infty}^{\infty} e^{j\Omega_0 t} x(t) e^{-j\Omega t} dt = \int_{-\infty}^{\infty} x(t) e^{-j(\Omega - \Omega_0)t} dt = X(j(\Omega - \Omega_0))$$

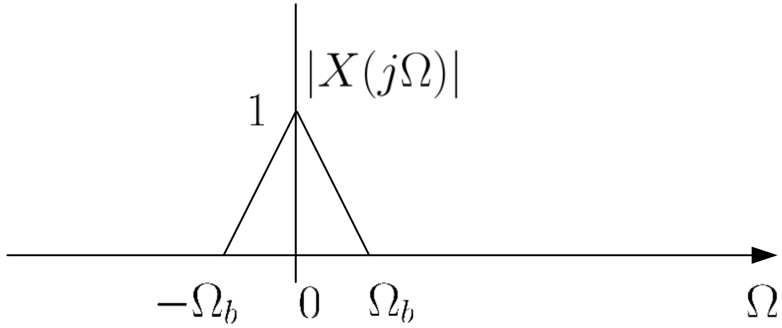
Let $y(t) = x(t)\cos(\Omega_0 t)$. Noting that

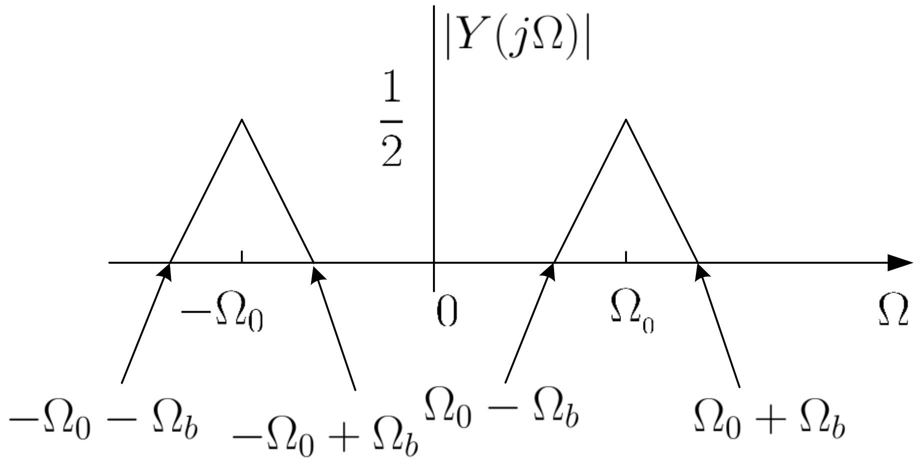
$$\cos(\Omega_0 t) = \frac{e^{j\Omega_0 t} + e^{-j\Omega_0 t}}{2}$$

and applying the frequency shifting properties yields

$$Y(j\Omega) = \frac{1}{2} X(j(\Omega + \Omega_0)) + \frac{1}{2} X(j(\Omega - \Omega_0))$$

A graphical illustration for a real-valued $x(t)$ is shown as follows:





Multiplying $x(t)$ by $\cos(\Omega_0 t)$ is also known as amplitude modulation in communications.

7.

With the use of inverse Fourier transform formula, we evaluate:

$$\begin{aligned}\int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt &= \int_{-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\lambda)e^{j\lambda t}d\lambda\right)e^{-j\Omega t}dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} e^{j\lambda t}\cdot e^{-j\Omega t}dt\right]X(j\lambda)d\lambda\end{aligned}$$

We see that the term inside the square bracket is the Fourier transform of $e^{j\lambda t}$. Applying (5.12), this term has the value of:

$$2\pi\delta(\Omega-\lambda)$$

As a result, we have:

$$
\begin{aligned}
\int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt &= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\Omega-\lambda)X(j\lambda)d\lambda \\
&= \int_{-\infty}^{\infty} \delta(\Omega-\lambda)X(j\lambda)d\lambda \\
&= X(j\Omega)\int_{-\infty}^{\infty} \delta(\Omega-\lambda)d\lambda \\
&= X(j\Omega)
\end{aligned}
$$