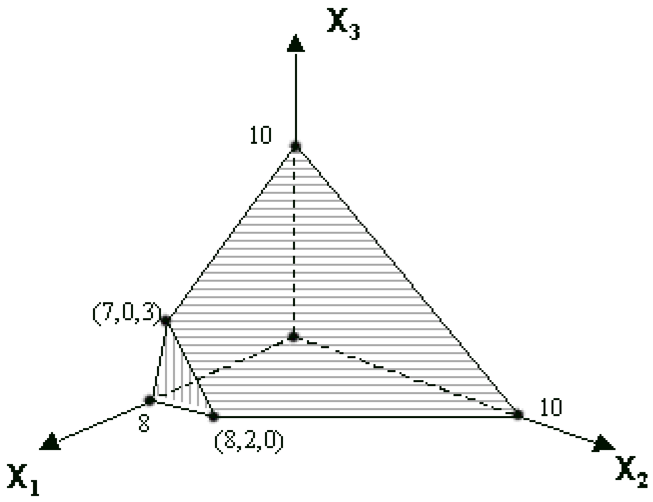
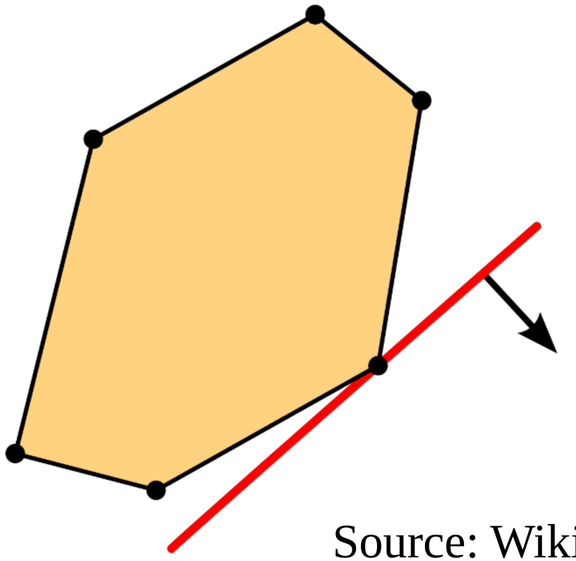
Linear Programming



Source: http://home.ubalt.edu/ntsbarsh/opre640a/nonlinear.htm



Source: Wikipedia

Linear Programming (LP)

Maximize $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$

Subject to $\mathbf{A}\boldsymbol{x} \leq \boldsymbol{b}$

And $\boldsymbol{x} \geq \mathbf{0}$

Integer Linear Programming (ILP)

Maximize $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$

Subject to $\mathbf{A}\boldsymbol{x} \leq \boldsymbol{b}$

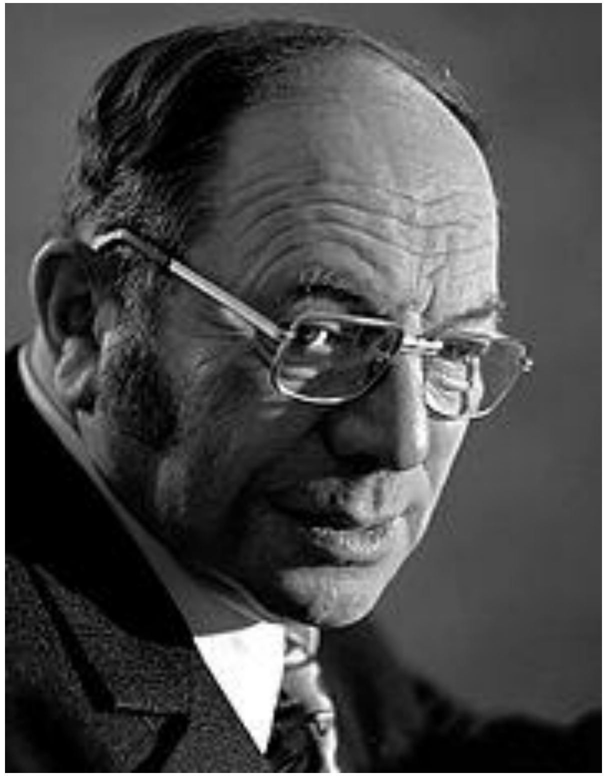
And $\boldsymbol{x} \geq \mathbf{0}$

And $\boldsymbol{x}$ integer

Leonid Kantorovich (1912 – 1986)

In 1939, he provided formulation and solution of linear programming problems.

- Nobel Prize in Economics (1975): “for their contributions to the theory of optimum allocation of resources” together with Tjalling Charles Koopmans.
- Stalin Prize (1960)
- Order of the Patriotic War
- Medal For Defense of Leningrad
- Stalin Prize (1949).



Source: Wikiperdia

“In 1947, Dantzig devised the simplex method, an important tool for solving linear programming problems in diverse applications, such as allocating resources, scheduling production and workers, planning investment portfolios and formulating marketing and military strategies.”

Source: https://news.stanford.edu/news/2005/may25/dantzigobit-052505.html

Based on his work tools are developed "that shipping companies use to determine how many planes they need and where their delivery trucks should be deployed. The oil industry long has used linear programming in refinery planning, as it determines how much of its raw product should become different grades of gasoline and how much should be used for petroleum-based byproducts. It is used in manufacturing, revenue management, telecommunications, advertising, architecture, circuit design and countless other areas".[

Source: Joe Holley (2005). "Obituaries of George Dantzig". In: *Washington Post*, May 19, 2005; B06

"through his research in mathematical theory, computation, economic analysis, and applications to industrial problems, Dantzig contributed more than any other researcher to the remarkable development of linear programming"

Source: Robert Freund (1994). "Professor George Dantzig: Linear Programming Founder Turns 80". In: *SIAM News*, November 1994.

Credit: Wikipedia

LP – Transformation

We introduced the format

Maximize $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$
Subject to $\mathbf{A}\boldsymbol{x} \leq \boldsymbol{b}$
And $\boldsymbol{x} \geq \mathbf{0}$

$\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$ is the objective function
$\boldsymbol{x}$ is a vector of decision variables
$\mathbf{A}\boldsymbol{x} \leq \boldsymbol{b}$ and $\boldsymbol{x} \geq \mathbf{0}$ are constraints

But what if we want to minimize instead of maximize?
What if some constraints are "$\leq$" some are "$\geq$" some are "$=$" ??
What if some of the variables may be unconstrained (can be also negative) ??
No Problem – all these cases can be handled by LP.

Maximize $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$ is equivalent to Minimize $-\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$

$5X_1+7X_2 \leq 500$ is equivalent to $-5X_1 - 7X_2 \geq -500$

or to $5X_1+7X_2 + S_1 = 500$

$S_1 \geq 0$ is called a slack variable – it has a zero coefficient in the objective function, but its optimal value can be positive.

If the practical problem requires the constraint $5X_1+7X_2 = 500$
How do you transform it to $\mathbf{A}\boldsymbol{x} \leq \boldsymbol{b}$?

Answer

Instead of $5X_1+7X_2 = 500$ you write:

$5X_1+7X_2 \leq 500$ **AND** $5X_1+7X_2 \geq 500$

or

$5X_1+7X_2 \leq 500$ **AND** $-5X_1 - 7X_2 \leq -500$

Then, it is transformed to the format $\mathbf{A}\boldsymbol{x} \leq \boldsymbol{b}$

What if X_1 is unconstrained but you require $\boldsymbol{x} \geq \mathbf{0}$??

Then, you set $X_1 = U_1 - V_1$ and $U_1 \geq 0$, and $V_1 \geq 0$.

For more information on LP Transformation see:

https://econweb.ucsd.edu/~jsobel/172aw02/notes5.pdf

Linear Programming – A Geometric Approach

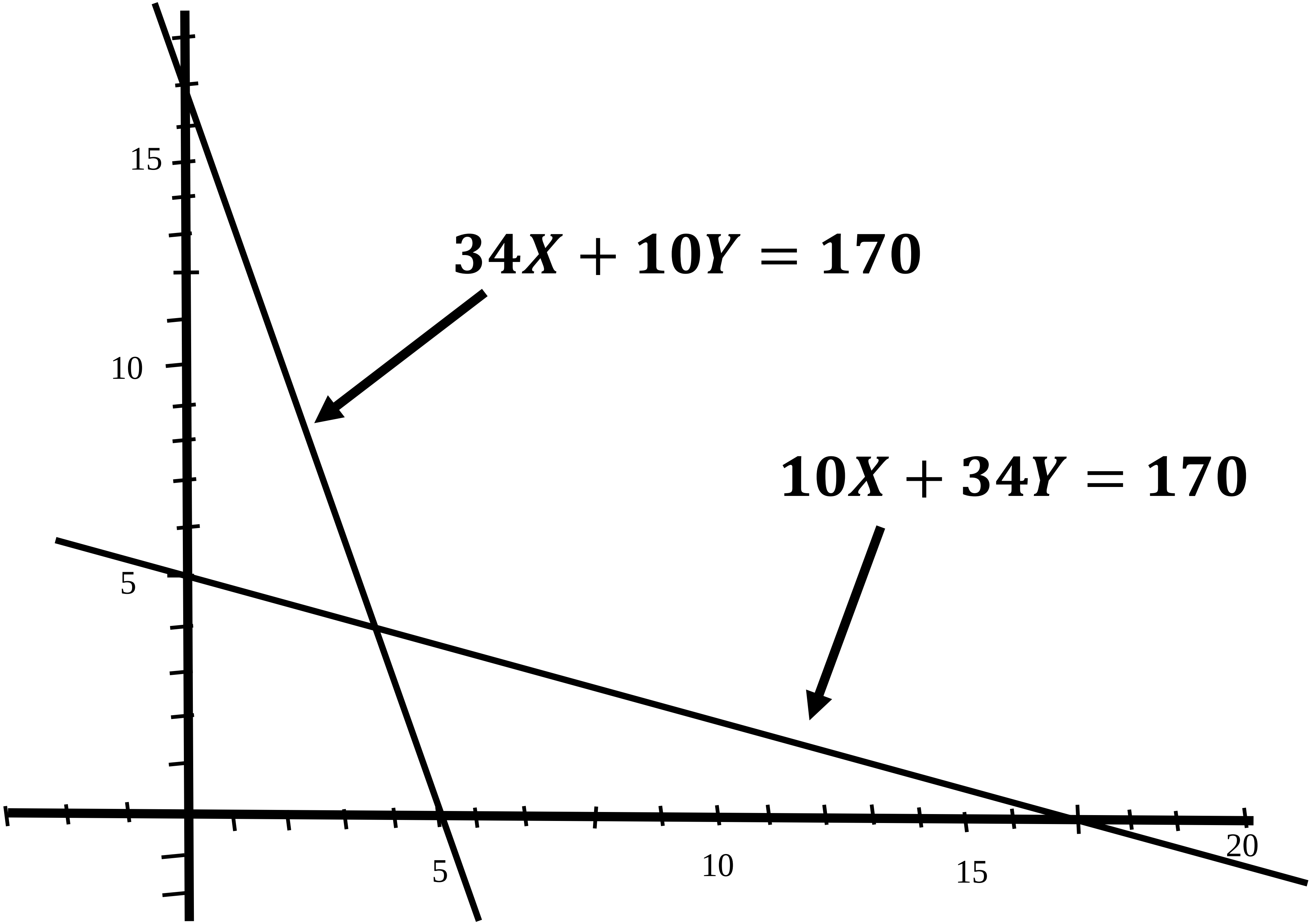
Consider the Following Simple Example

Maximize:

$$\boldsymbol{P = 12X + 8Y}$$

Subject to:

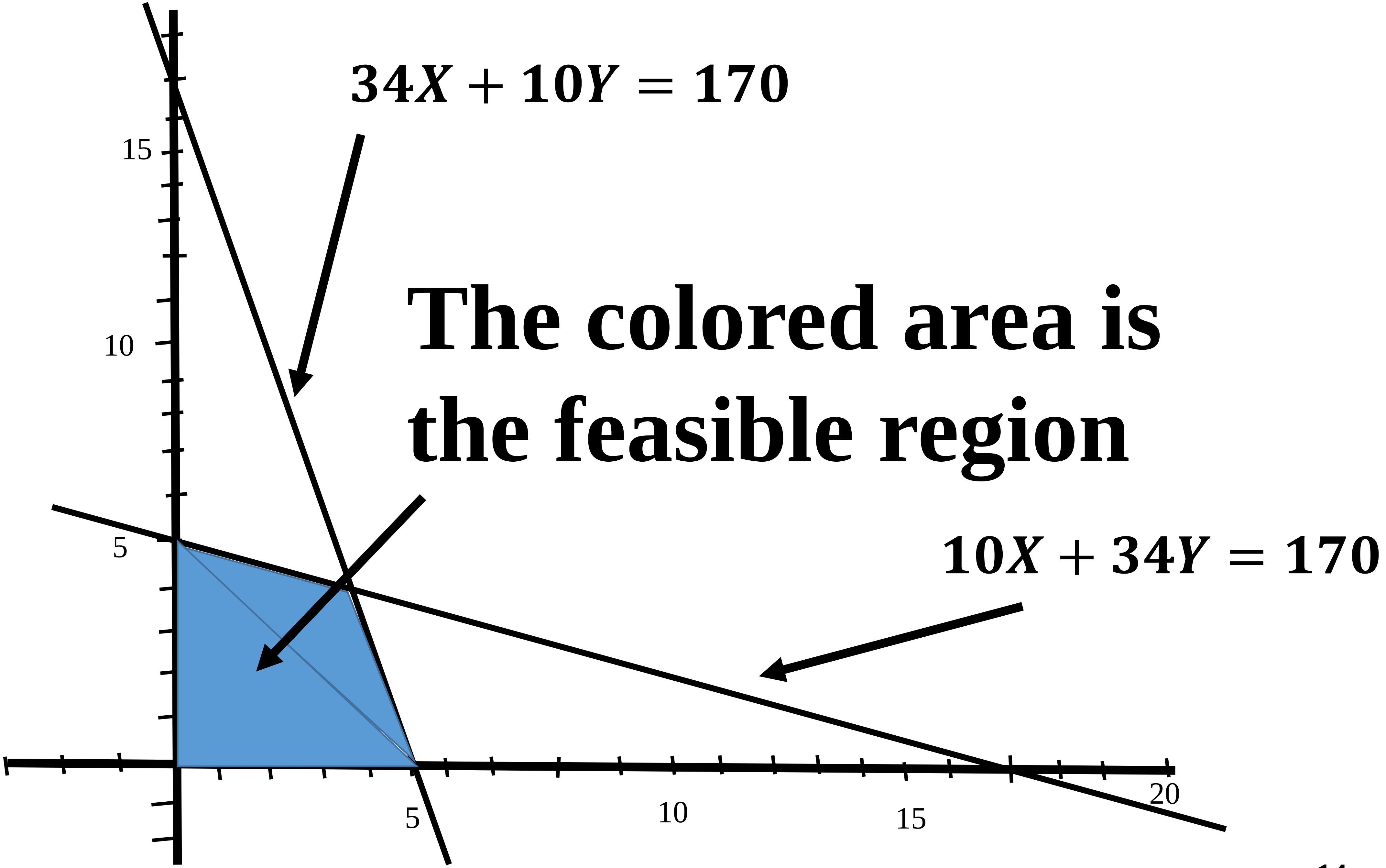
$$\boldsymbol{10X + 34Y \leq 170}$$
$$\boldsymbol{34X + 10Y \leq 170}$$
$$\boldsymbol{X \geq 0}$$
$$\boldsymbol{Y \geq 0}$$


$34X + 10Y = 170$
$10X + 34Y = 170$
15
10
5
5
10
15
20

The Feasible Region

The feasible region is the set of points {X,Y} that satisfies the constraints.

If the feasible region is the empty set, the problem does not have any feasible solution. No feasible solutions implies no optimal solution. If this is the case, go to the next problem.


$34X + 10Y = 170$
The colored area is the feasible region
$10X + 34Y = 170$
15
10
5
5
10
15
20

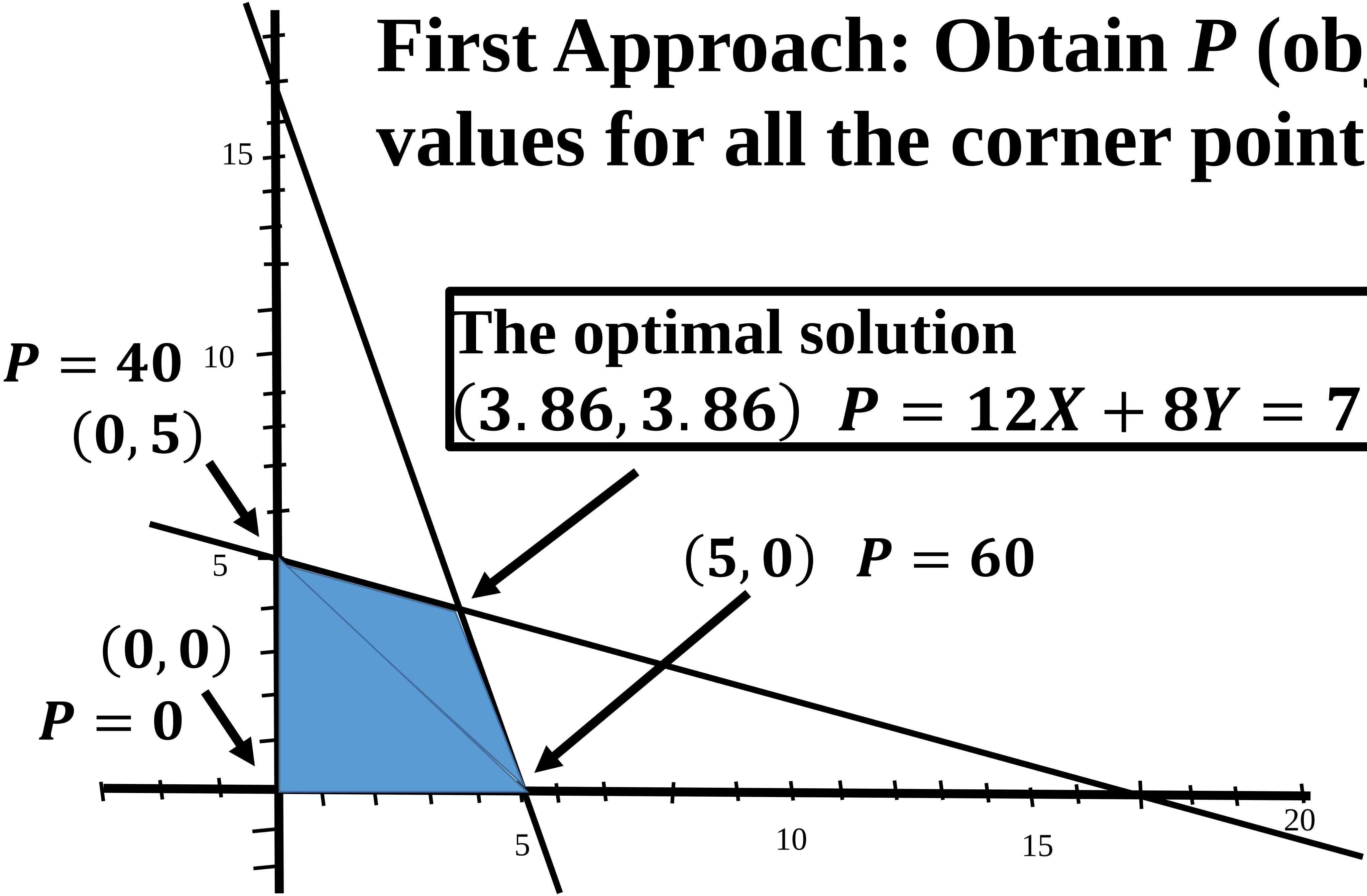
Fundamental Theorem of Linear Programming (LP)

The fundamental theorem of LP says that if there is a solution to an LP problem, then it will occur at one or more corner points or the boundary between two corner points.

In other words, the solution is going to be on the edge of the region, not in the middle. A corner point is a vertex of the feasible region, so we need to figure out where those are.

Source: https://people.richland.edu/james/ictcm/2006/geometric.html

First Approach: Obtain *P* (objective) values for all the corner points



Second Approach: Considering the slope of between the variables in the objective function

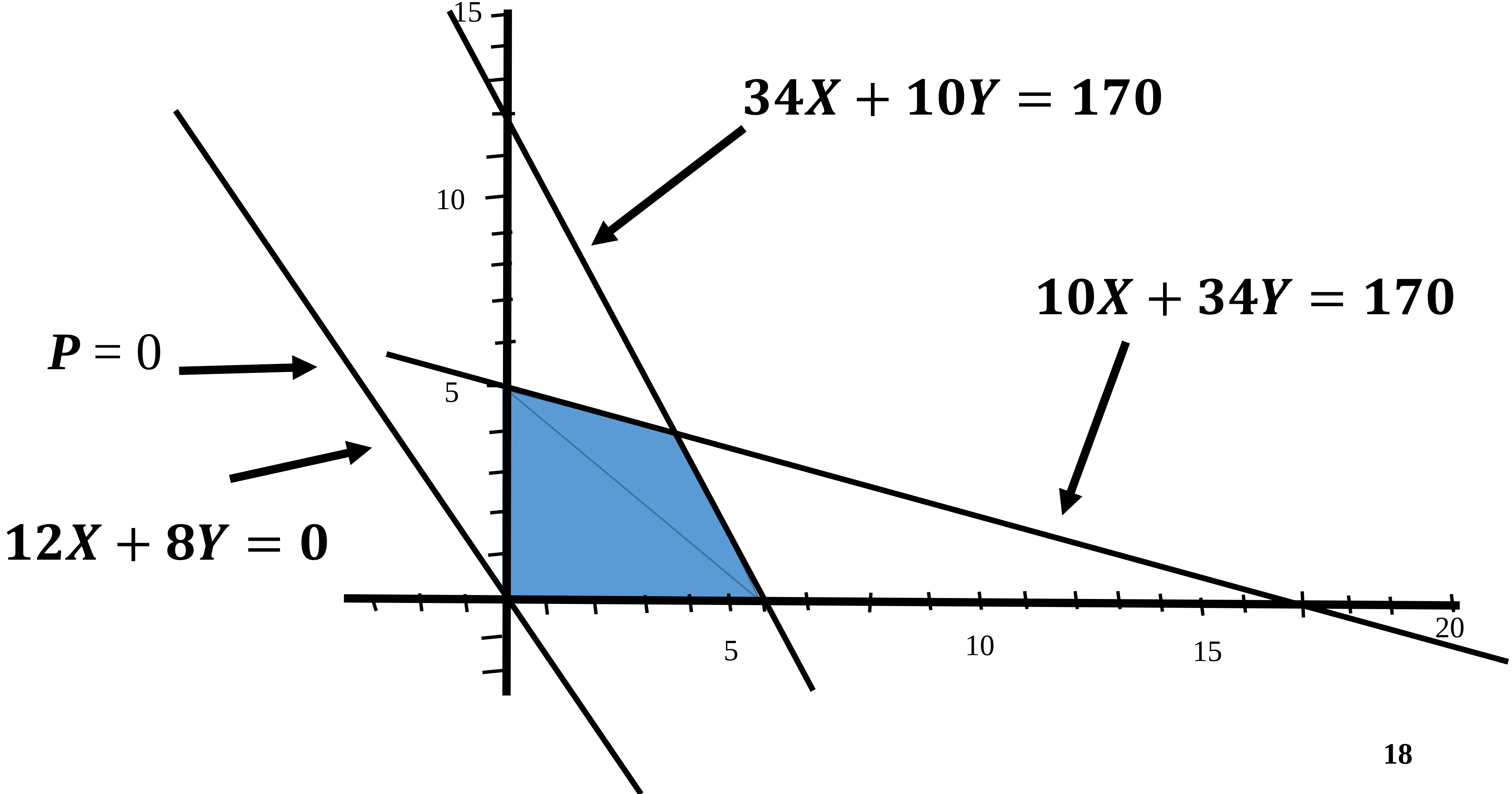
Objective Function: $\boldsymbol{P = 12X + 8Y}$

Let us consider all the set of points $\{X, Y\}$ that $P = 0$

$$12X + 8Y = 0$$

or

$$Y = -\frac{3}{2}X$$


$34X + 10Y = 170$
$10X + 34Y = 170$
$P = 0$
$12X + 8Y = 0$
15
10
5
5
10
15
20

Objective Function

$$\boldsymbol{P = 12X + 8Y}$$

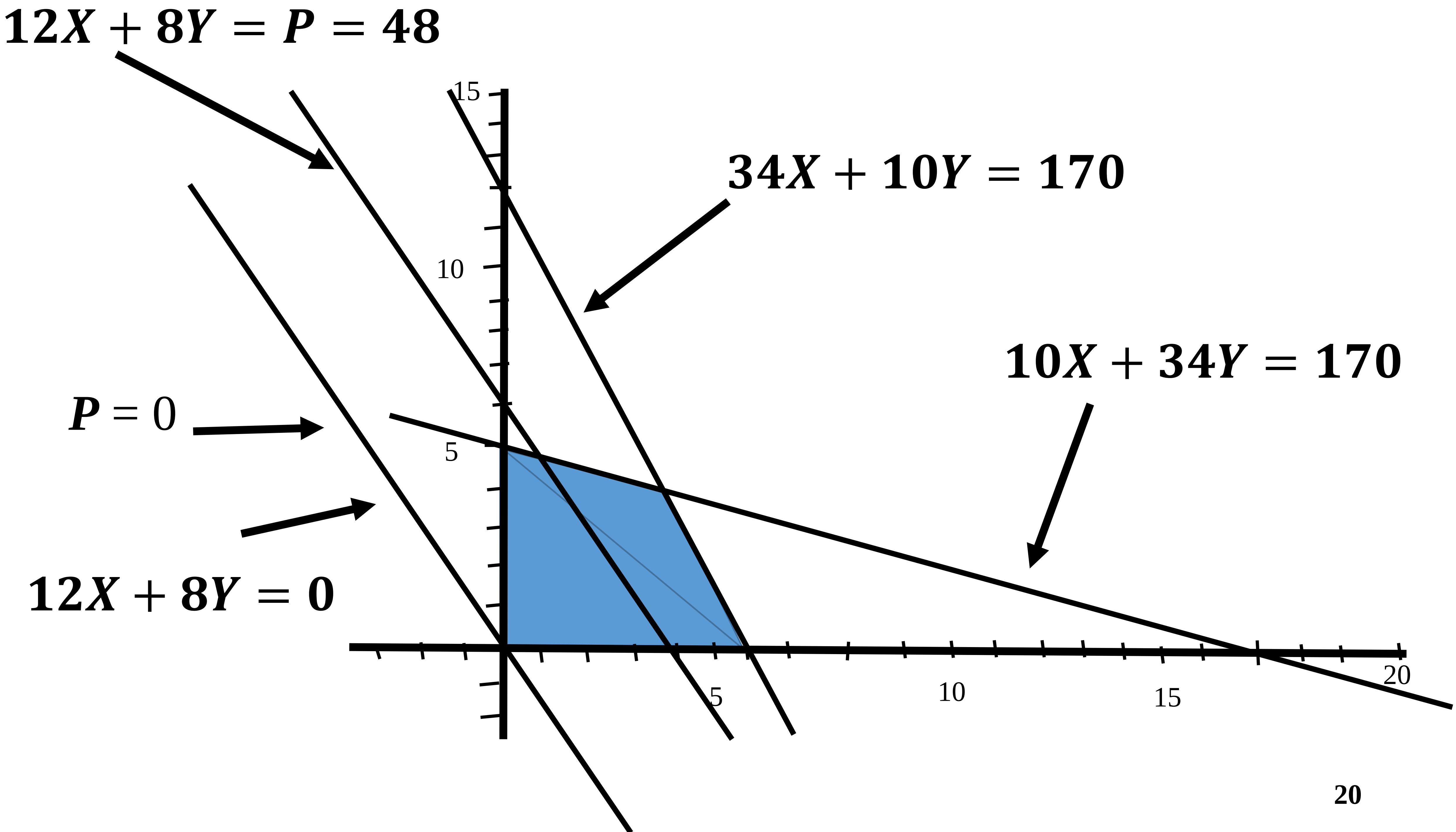
Let us consider all the set of points $\{X, Y\}$ that $P = C$

$$12X + 8Y = C$$

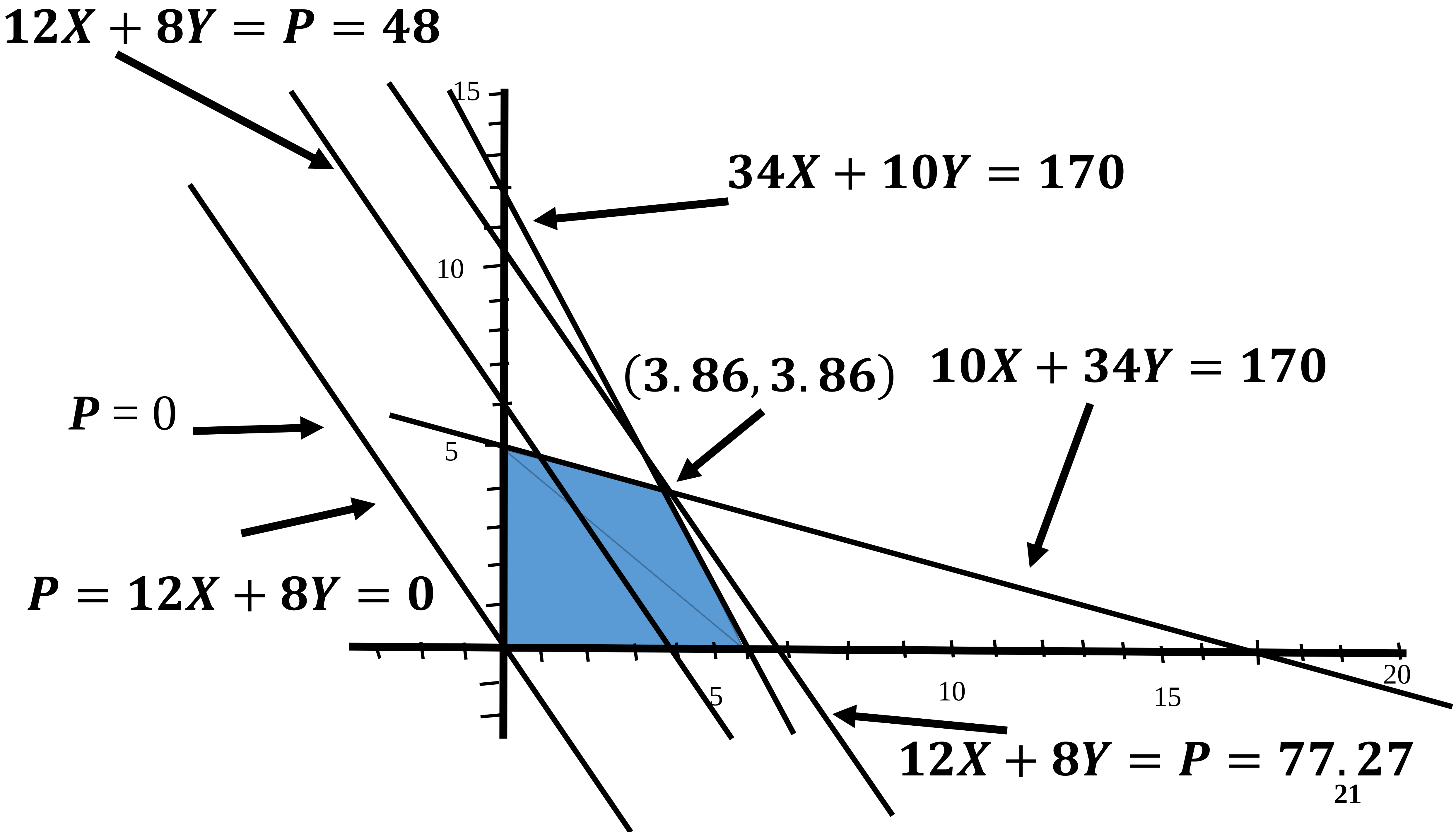
or

$$Y = -\frac{3}{2}X + \frac{C}{8}$$

Conclusion: all the points on a straight line that has slope of (-3/2) give the same values of ***P.***



$12X + 8Y = P = 48$
$34X + 10Y = 170$
$10X + 34Y = 170$
$P = 0$
$12X + 8Y = 0$
15
10
5
5
10
15
20


$12X + 8Y = P = 48$
15
$34X + 10Y = 170$
10
$(3.86, 3.86)$
$10X + 34Y = 170$
$P = 0$
5
$P = 12X + 8Y = 0$
5
10
15
20
$12X + 8Y = P = 77.27$

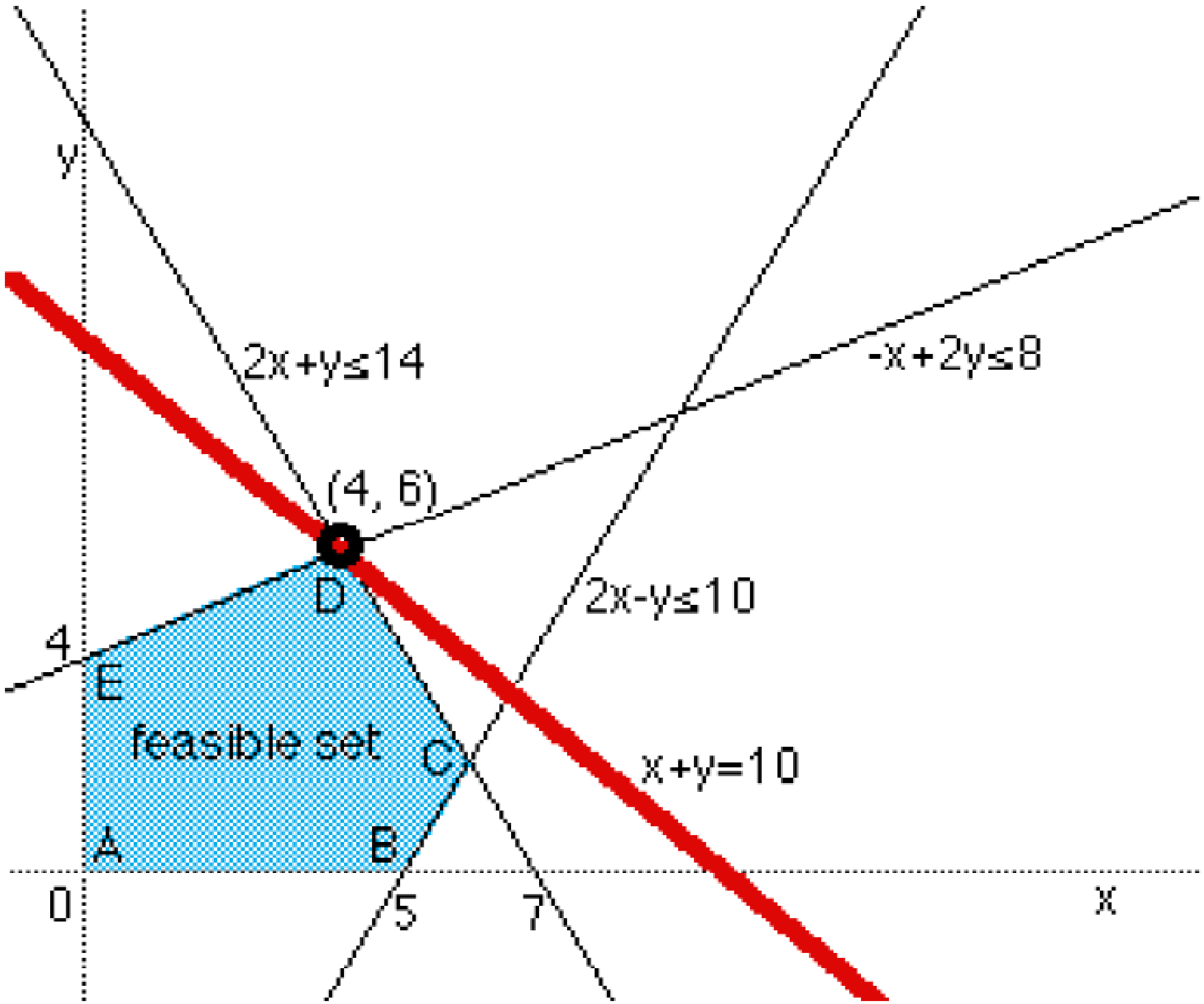
Example 2

Consider the linear programming problem:

$$
\begin{array}{lrcrcr}
\text{Maximize} & x & + & y & & \\
\text{subject to} & 2x & + & y & \leq & 14 \\
 & -x & + & 2y & \leq & 8 \\
 & 2x & - & y & \leq & 10 \\
 & \multicolumn{5}{l}{x \geq 0,\ y \geq 0}
\end{array}
$$

Source: M. A. Schulze, "**Linear Programming for Optimization",** Perceptive Scientific Instruments, Inc.
https://www.markschulze.net/LinearProgramming.pdf

Geometric solution of Example 2



Source: M. A. Schulze, "**Linear Programming for Optimization",** Perceptive Scientific Instruments, Inc.
https://www.markschulze.net/LinearProgramming.pdf

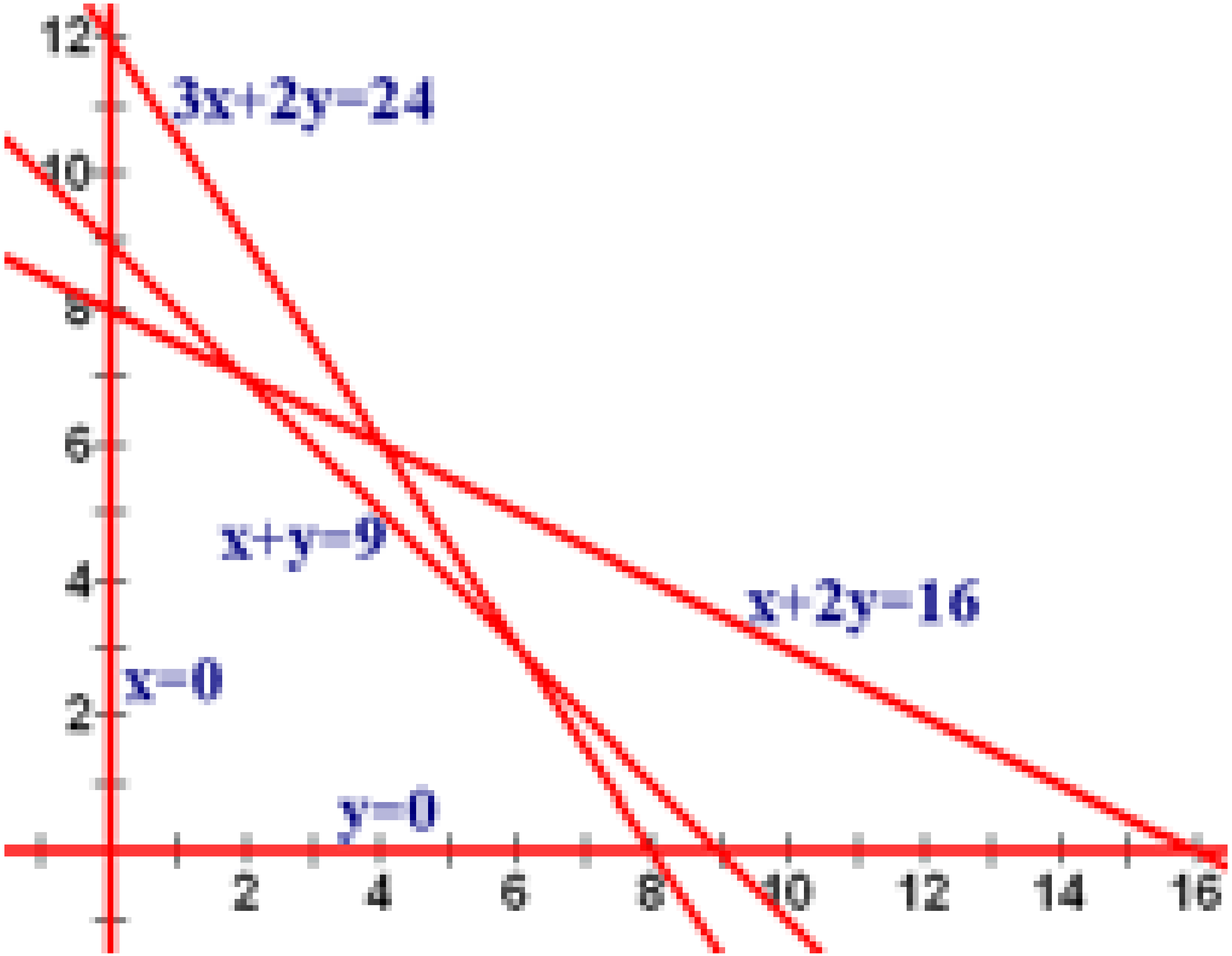
Example 3

Maximize: $P = 40X + 30Y$

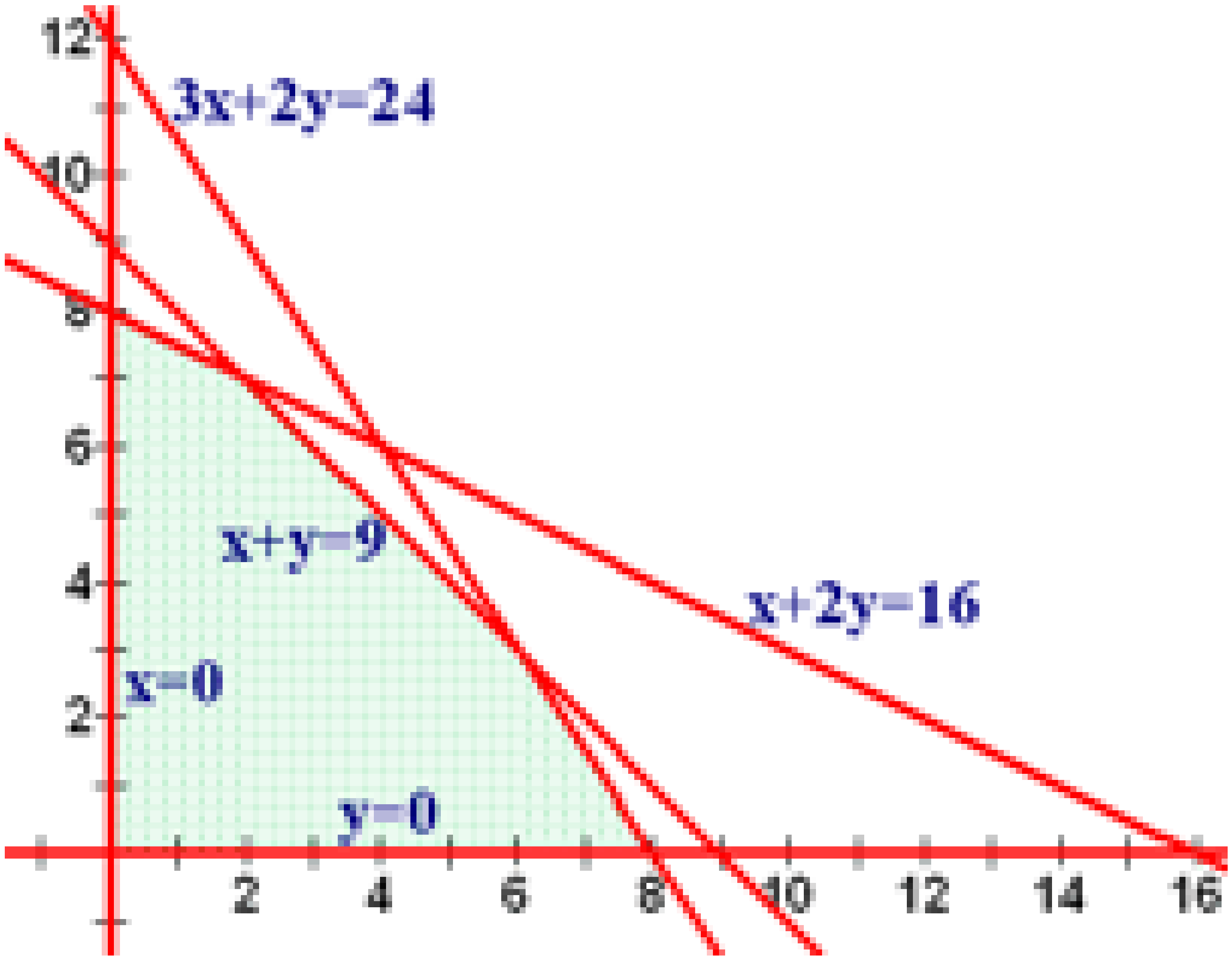
Subject to:

$$X + 2Y \leq 16$$
$$X + Y \leq 9$$
$$3X + 2Y \leq 24$$
$$X \geq 0$$
$$Y \geq 0$$

Source: J. Jones https://people.richland.edu/james/ictcm/2006/geometric.html



Source: James Jones <https://people.richland.edu/james/ictcm/2006/geometric.html>
https://people.richland.edu/james/ictcm/2006/ ; <https://people.richland.edu/james/>



Source: James Jones <https://people.richland.edu/james/ictcm/2006/geometric.html>
https://people.richland.edu/james/ictcm/2006/ ; <https://people.richland.edu/james/>

x	y	P = 40x + 30y
0	0	0 + 0 = 0
0	8	0 + 240 = 240
2	7	80+210 = 290
6	3	240 + 90 = 330
8	0	320 + 0 = 320

Maximum

Source: https://people.richland.edu/james/ictcm/2006/cornerpoints.html

Other Objective Functions with the Same Constraints

Let's say that you had the same feasible region, but different objective functions.

- Maximize P = 40x + 30y (the original objective function)
- Maximize P = 20x + 30y
- Maximize P = 10x + 30y
- Maximize P = 50x + 30y
- Maximize P = 30x + 30y

Source: https://people.richland.edu/james/ictcm/2006/cornerpoints.html

x	y	40x+30y	20x+30y	10x+30y	50x+30y	30x+30y
0	0	0	0	0	0	0
0	8	240	240	**240**	240	240
2	7	290	**250**	230	310	**270**
6	3	**330**	210	150	390	**270**
8	0	320	160	80	**400**	240

The bolded numbers are the maximal values for P.

Source: https://people.richland.edu/james/ictcm/2006/cornerpoints.html

Multiple Solutions

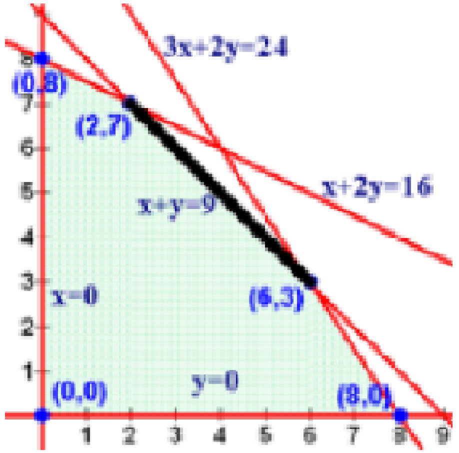
Notice that last one gives us a slight problem. The maximum value of 270 occurs not only when $x = 2$ and $y = 7$ but also when $x = 6$ and $y = 3$. In fact, it also occurs when $x = 3$ and $y = 6$, when $x = 4$ and $y = 5$, when $x = 5$ and $y = 4$, when $x = 4.6$ and $y = 4.4$, and any other point on the line segment between (2,7) and (6,3).

This is the case where the fundamental theorem of linear programming mentioned that the solution was the boundary between two corner points. Every point between (2,7) and (6,3) is on the line $x + y = 9$.

But not every point on the line $x + y = 9$ is a solution. For example, $x = 1$ and $y = 8$ is not even in the feasible region. So, we need to restrict our domain to just the portion of the line segment we need.

So now we can answer that last problem.

The maximize value of $P = 30x + 30y$ is 270 when $x + y = 9$ and $2 \le x \le 6$.



Source: https://people.richland.edu/james/ictcm/2006/cornerpoints.html

Homework

Solve the previous LP examples using Excel Solver. Compare the results with the above solutions and correct if necessary. Upload your work to Canvas Discussions (even if you could not match the above results). Active discussions and mutual help are encouraged.

So far, we considered cases with only two decision variables, but LP applies to an arbitrary number of decision variables, denoted in various ways, e.g., $\{X_1, X_2, X_3, \ldots, X_n\}$ or $\{X_{ij}\}$, etc.

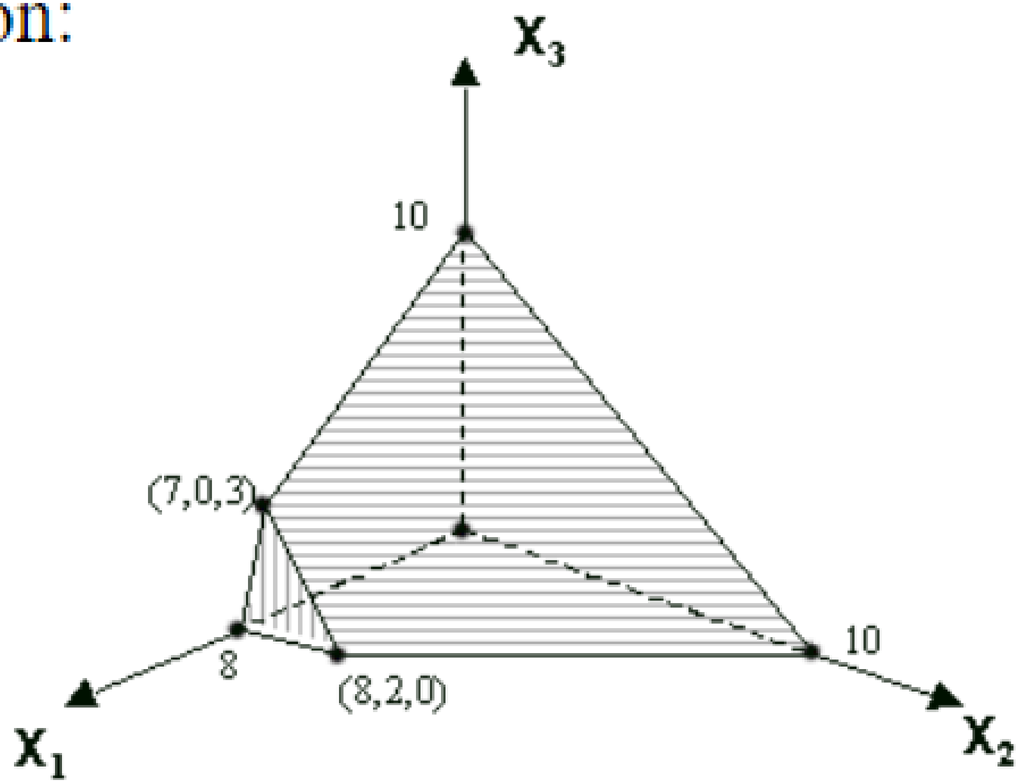
We now show the feasible region for an example of three decision variables: X_1, X_2, X_3.

Consider the following feasible region:

$$X_1 + X_2 + X_3 \leq 10$$

$$3X_1 + X_3 \leq 24$$

X_1, X_2, and $X_3 \geq 0$.



Source: http://home.ubalt.edu/ntsbarsh/opre640a/nonlinear.htm

LP Transformation

Decision variables are not restricted to be nonnegative.

Maximize:

$$\boldsymbol{P = 5X + 10Y}$$

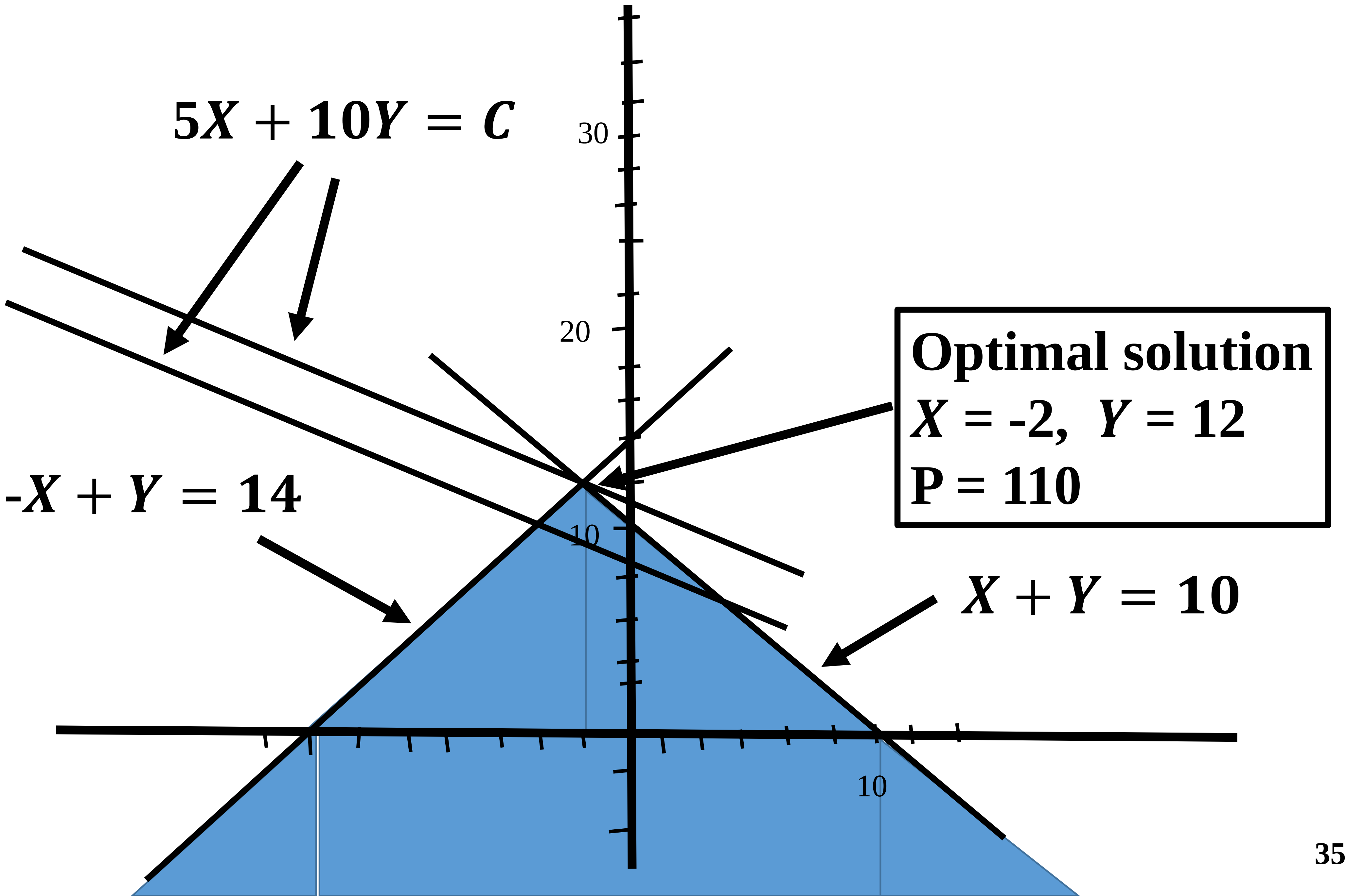
Subject to:

$$\boldsymbol{X + Y \leq 10}$$

$$\boldsymbol{-X + Y \leq 14}$$

$\boldsymbol{X}$ unconstrained

$\boldsymbol{Y}$ unconstrained



$5X + 10Y = C$
30
20
10
10
Optimal solution
X = -2, Y = 12
P = 110
$-X + Y = 14$
$X + Y = 10$

Equivalent LP where decision variables are restricted to be nonnegative

Set $X = U_1 - V_1$, and $Y = U_2 - V_2$

Then, $P = 5X + 10Y = 5(U_1 - V_1) + 10(U_2 - V_2)$

$$P = 5U_1 - 5V_1 + 10U_2 - 10V_2$$

Subject to:

$$X + Y = U_1 - V_1 + U_2 - V_2 \leq 10$$

$$-U_1 + V_1 + U_2 - V_2 = \leq 14$$

$$U_1 \geq 0, \quad U_2 \geq 0$$

$$V_1 \geq 0, \quad V_2 \geq 0$$

Homework

Consider the following optimization problem

Maximize:

$$\boldsymbol{P = AX + BY}$$

Subject to:

$$\boldsymbol{X + Y \leq C}$$

$$\boldsymbol{-X + Y \leq D}$$

$\boldsymbol{X}$ unconstrained

$\boldsymbol{Y}$ unconstrained

Homework (Cont'd)

1. **Choose the parameters: *A*, *B*, *C*, and *D*.**
2. **Solve the LP problem using the geometric method.**
3. **Solve the LP problem using Excel Solver.**
4. **Transform the problem to an equivalent LP problem where all the variables are nonnegative and solve it using Excel Solver. Make sure that the results are consistent.**

Prove that relaxing the constraints will not worsen the optimal solution and may improve it.

Consider the optimization problem
Minimize $f(x)$
Subject to: $x \in A$.
Assume that x^* solves this problem, i.e., $f(x^*) \leq f(x)$, for all $x \in A$.
Let $A \subset B$ (A is a subset of B), and assume that $\hat{x}$ solves the problem
Minimize $f(x)$
Subject to: $x \in B$.
Prove that $f(\hat{x}) \leq f(x^*)$.

Linear Programming Applications

Bob's Bakery Problem

Bob's bakery sells **bagels** and **muffins**.

To bake a dozen bagels Bob needs 5 cups of flour, 2 eggs, and one cup of sugar.

To bake a dozen muffins Bob needs 4 cups of flour, 4 eggs and two cups of sugar.

Bob can sell bagels in \$10/dozen and muffins in \$12/dozen.

Bob has 50 cups of flour, 30 eggs and 20 cups of sugar.

How many bagels and muffins should Bob bake in order to maximize his revenue?

Source: https://courses.cs.washington.edu/courses/cse421/10au/lectures/lp.pdf

LP formulation: Bob's bakery

	Bagels	Muffins	Avail.
Flour	5	4	50
Eggs	2	4	30
Sugar	1	2	20
Revenue	10	12	

$$A = \begin{pmatrix} 5 & 4 \\ 2 & 4 \\ 1 & 2 \end{pmatrix}$$

$$b = \begin{bmatrix} 10 & 12 \end{bmatrix} \qquad c = \begin{bmatrix} 50 \\ 30 \\ 20 \end{bmatrix}$$

$$\begin{aligned} &\text{Maximize } 10x_1+12x_2 \\ &\text{s.t.} \quad 5x_1+4x_2 \le 50 \\ &\qquad\ \ 2x_1+4x_2 \le 30 \\ &\qquad\ \ x_1+2x_2 \le 20 \\ &\qquad\ \ x_1 \ge 0,\ x_2 \ge 0 \end{aligned}$$

$$\begin{aligned} &\text{Maximize } b \cdot x \\ &\text{s.t.} \quad Ax \le c \\ &\qquad\ \ x \ge 0. \end{aligned}$$

Source: https://courses.cs.washington.edu/courses/cse421/10au/lectures/lp.pdf

Homework

Solve the numerical example of the Bob's Bakery problem using Excel Solver and upload to Canvas Discussions. Compare your results with the results of your classmates and correct if necessary. Again, active discussions and mutual help are encouraged.

Washing Machine Production Problem

Source: https://www.fm-magazine.com/issues/2019/feb/linear-programming-microsoft-excel.html

An example from management accounting

Beacon Co. is a manufacturer of washing machines. It currently sells two models of washing machines: the Arkel and the Kallex. At the start of every production cycle, Beacon must decide how many units of each washing machine to produce, given its available resources. In the coming production cycle, Beacon faces key resource constraints. In particular, it has only 3,132 hours of labour, 1,440 feet of rubber hosing, and 200 drums available.

Selling each Arkel unit earns the company a profit of $350 while selling each Kallex unit earns the company a profit of $300. At the same time, manufacturing each Arkel unit requires 18 hours of labour, 6 feet of rubber hosing, and 1 drum, while manufacturing each Kallex unit requires 12 hours of labour, 8 feet of rubber hosing, and 1 drum. Details of the relevant facts are summarised in the table "Summary of Production of Washing Machines".

Based on these facts and the assumption that 100% of production will be sold, Beacon must decide how many units of each washing machine to produce in the coming production run to maximise profits.

Source: https://www.fm-magazine.com/issues/2019/feb/linear-programming-microsoft-excel.html

Summary of production of washing machines

	Arkel	Kallex	Total available
Labour hours	18 hours per unit	12 hours per unit	3,132 hours
Rubber hosing	6 feet per unit	8 feet per unit	1,440 feet
Drums	1 per unit	1 per unit	200 drums
Profits	$350 per unit	$300 per unit	

Source: https://www.fm-magazine.com/issues/2019/feb/linear-programming-microsoft-excel.html

LP Formulation

Maximise profits:	$350X_1$	+	$300X_2$		} Objective function
Subject to:	X_1	+	X_2	≤ 200	} Drum constraint
	$18X_1$	+	$12X_2$	$\leq 3{,}132$	} Labour constraint
	$6X_1$	+	$8X_2$	$\leq 1{,}440$	} Hosing constraint
	X_1			≥ 0	} Variable lower bound
			X_2	≥ 0	} Variable lower bound

Source: https://www.fm-magazine.com/issues/2019/feb/linear-programming-microsoft-excel.html

Homework

Solve the numerical example of the Washing Machine Production problem using Excel Solver and upload to Canvas Discussions. Compare your results with the results of your classmates and correct if necessary.
Share your thoughts on if the linearity of the objective function and constraints are practical.
Again, active discussions and mutual help are encouraged.

Car Production Problem

Suppose a particular Ford plant can build Escorts at the rate of one per minute, Explorer at the rate of one every 2 minutes, and Lincoln Navigators at the rate of one every 3 minutes. The vehicles get 25, 15, and 10 miles per gallon, respectively, and Congress mandates that the average fuel economy of vehicles produced be at least 18 miles per gallon. Ford loses $1000 on each Escort, but makes a profit of $5000 on each Explorer and $15,000 on each Navigator. What is the maximum profit this Ford plant can make in one 8-hour day?

Source: M. A. Schulze, "**Linear Programming for Optimization",** Perceptive Scientific Instruments, Inc.
https://www.markschulze.net/LinearProgramming.pdf

Adapted from: G. Strang, *Linear Algebra and its Applications*, 3rd ed. San Diego: Harcourt Brace Jovanovich, 1988.

LP Formulation

The *cost function* is the profit Ford can make by building x Escorts, y Explorers, and z Navigators, and we want to maximize it:

$$-1000x + 5000y + 15000z \qquad (1.3.1)$$

The *constraints* arise from the production times and Congressional mandate on fuel economy. There are 480 minutes in an 8-hour day, and so the production times for the vehicles lead to the following limit:

$$x + 2y + 3z \leq 480 \qquad (1.3.2)$$

The average fuel economy restriction can be written:

$$25x + 15y + 10z \geq 18(x + y + z) \qquad (1.3.3)$$

which simplifies to:

$$7x - 3y - 8z \geq 0 \qquad (1.3.4)$$

There is an additional implicit constraint that the variables are all non-negative: $x, y, z \geq 0$.

Source: M. A. Schulze, "**Linear Programming for Optimization",** Perceptive Scientific Instruments, Inc.
https://www.markschulze.net/LinearProgramming.pdf

LP Formulation (cont'd)

This production planning problem can now be written succinctly as:

$$\begin{aligned} \text{Maximize} \quad & -1000x + 5000y + 15000z \\ \text{subject to} \quad & x + 2y + 3z \leq 480 \\ & 7x - 3y - 8z \geq 0 \\ & x, y, z \geq 0 \end{aligned} \tag{1.3.5}$$

The solution to this problem is x=132.41, y=0, and z=115.86, yielding a cost function value of 1,605,517.24. Note that for some problems, non-integer values of the variables may not be desired. Solving a linear programming problem for integer values of the variables only is called *integer programming* and is a significantly more difficult problem. The solution to an integer programming problem is not necessarily close to the solution of the same problem solved without the integer constraint. In this example, the optimal solution if x, y, and z are constrained to be integers is x=132, y=1, and z=115 with a resulting cost function value (profit) of $1,598,000.

Source: M. A. Schulze, "**Linear Programming for Optimization",** Perceptive Scientific Instruments, Inc.
https://www.markschulze.net/LinearProgramming.pdf

Homework

1. For the above Car Production problem, try to round (up and down) the LP results to consider other integer solutions. For each case, check feasibility (i.e., check that the constraints are satisfied). For the feasible integer solutions, calculate the value of the objective function and compare with the provided optimal integer solution. Demonstrate that such rounding approach cannot guarantee optimal integer solution.
2. Comment on the practicability of linear cost functions in the case of the Car Production problem.
3. Solve the numerical example of the Car Production problem using Excel Solver and upload to Canvas Discussions. Compare your results with the results of your classmates and correct if necessary. Again, active discussions and mutual help are encouraged.

Mini-project

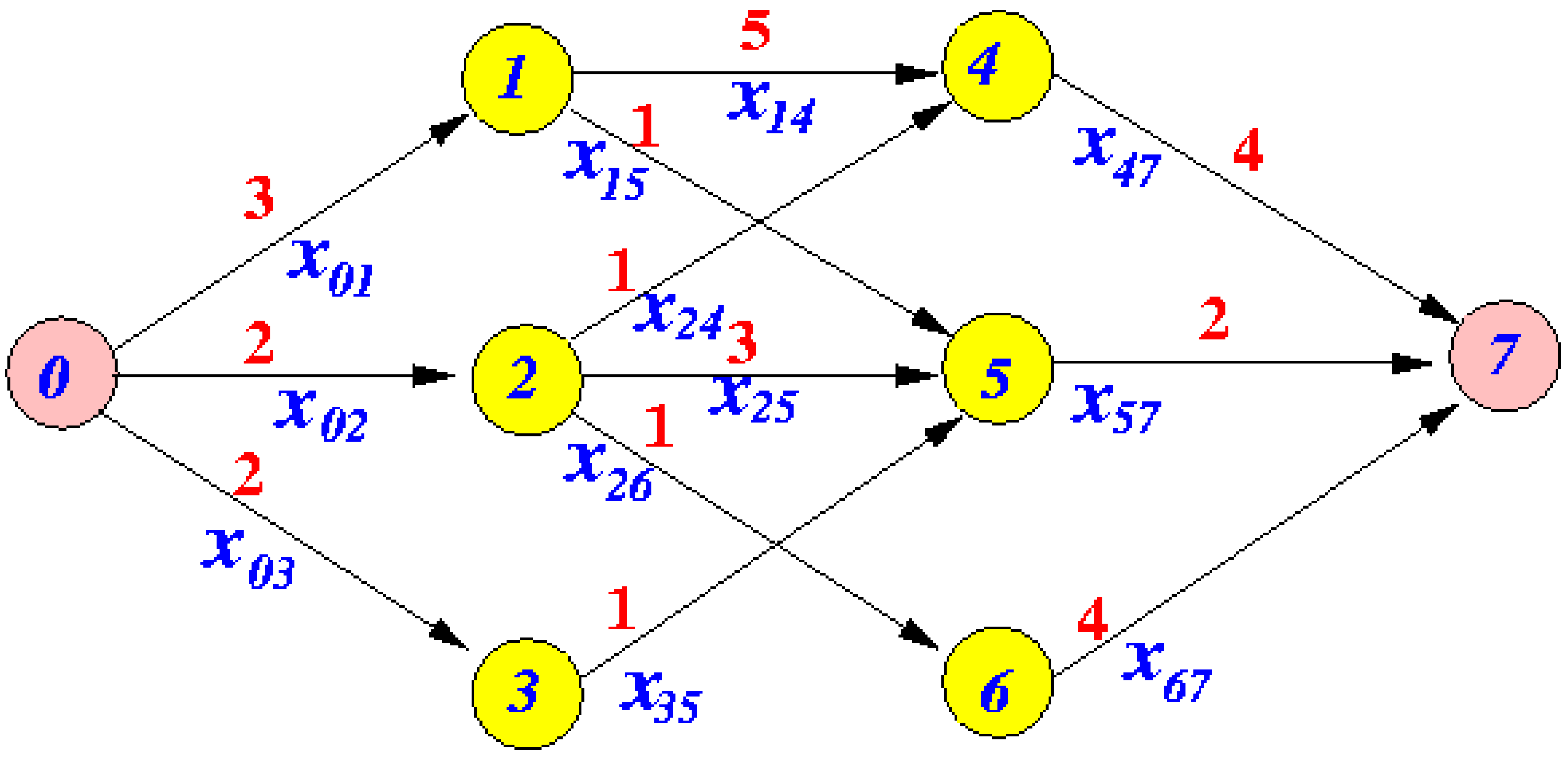
1. Consider an LP with a large number of variables (over 20) and solve it using excel.
2. Change the variables to be constrained to be integer.
3. Compare the running time of the two cases LP and ILP.
4. Demonstrate that the running time of ILP is significantly higher than that of LP, especially, as the number of variables increases.
5. Compare and discuss the values of the optimal solutions of the LP versus the ILP.

Max Flow Problem

Maximize the traffic flow from Node 0 to Node 7, subject to capacity and flow conservation constraints.

X_{ij} =
Traffic on link *ij*.

The numbers on the links are capacity limitations.



source: http://www.mathcs.emory.edu/~cheung/Courses/323/Syllabus/NetFlow/max-flow-lp.html

Constraints

Capacity constraints: Traffic on a link cannot be more than the link capacity limitation.

Flow conservation constraints: Total traffic enters an internal node must be equal to total traffic that exits that internal node. All nodes excluding Nodes 0 and 7 are internal nodes.

Capacity Constraints

$$X_{01} \leq 3 \qquad X_{25} \leq 3$$
$$X_{02} \leq 2 \qquad X_{26} \leq 1$$
$$X_{03} \leq 2 \qquad X_{35} \leq 1$$
$$X_{14} \leq 5 \qquad X_{47} \leq 4$$
$$X_{15} \leq 1 \qquad X_{57} \leq 2$$
$$X_{24} \leq 1 \qquad X_{67} \leq 4$$

Credit: http://www.mathcs.emory.edu/~cheung/Courses/323/Syllabus/NetFlow/max-flow-lp.html

Flow Conservation Constraints

Node *1)* $X_{01} = X_{14} + X_{15}$
Node *2)* $X_{02} = X_{24} + X_{25} + X_{26}$
Node *3)* $X_{03} = X_{35}$
Node *4)* $X_{14} + X_{24} = X_{47}$
Node *5)* $X_{15} + X_{25} + X_{35} = X_{57}$
Node *6)* $X_{26} = X_{67}$

Non-negativity constraints: $X_{ij} \geq 0$, for all ij

Credit: http://www.mathcs.emory.edu/~cheung/Courses/323/Syllabus/NetFlow/max-flow-lp.html

The Objective Function

Maximize: $\quad \text{Flow} = X_{01} + X_{02} + X_{03}$

Optimal Solution

$X_{01} = 3$
$X_{02} = 2$
$X_{03} = 1$
$X_{14} = 3$
$X_{15} = 0$
$X_{24} = 1$

$X_{25} = 1$
$X_{26} = 0$
$X_{35} = 1$
$X_{47} = 4$
$X_{57} = 2$
$X_{67} = 0$

Max Flow = 6

Credit: http://www.mathcs.emory.edu/~cheung/Courses/323/Syllabus/NetFlow/max-flow-lp.html

Homework

Solve the previous Max Flow problem using Excel Solver. Compare your results with the above solution and correct if necessary. Upload your work to Canvas Discussions (even if you could not match the above results). Again, active discussions and mutual help are encouraged.

Max Flow Min Cut Theorem

"In a flow network, the maximum amount of flow passing from the source (start node) to the sink (end node) is equal to the total weight (capacity) of the edges in the minimum cut, i.e. the smallest total weight (capacity) of the edges which if removed would disconnect the source from the sink."

Source: https://en.wikipedia.org/wiki/Max-flow_min-cut_theorem

Examples of Cuts

Cut 1: (0,1), (0,2), (0,3)

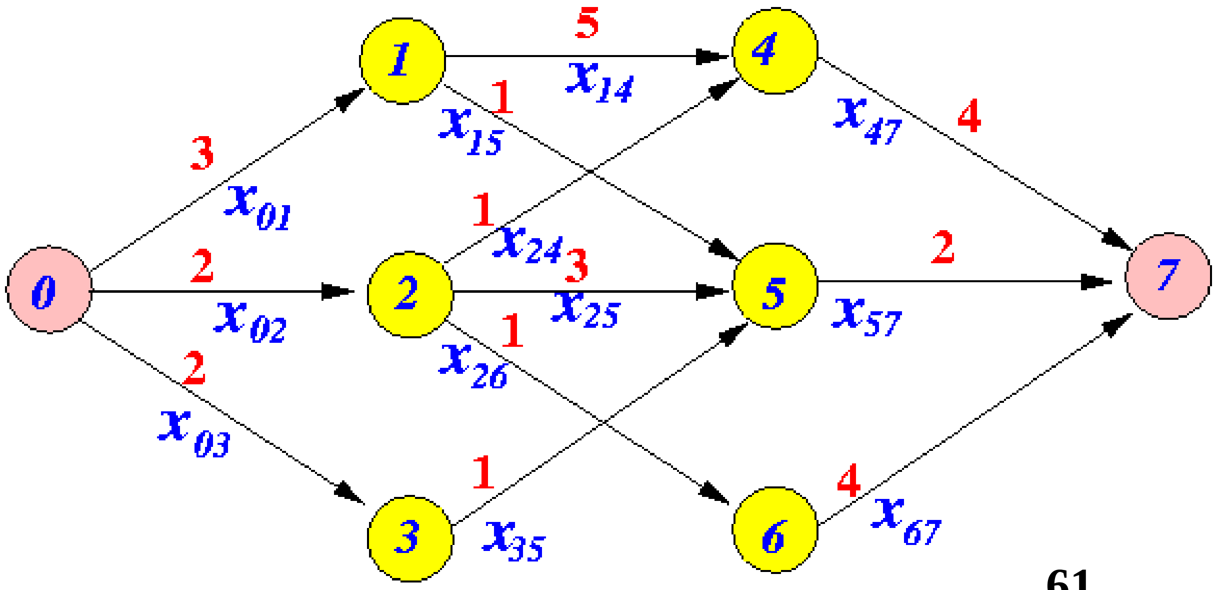
Cut 2: (1,4), (2,4), (1,5), (2,5), (2,6), (3,5)

Cut 3: (1,4), (2,4), (1,5), (2,5), (2,6), (0,3)

Cut 4: (4,7), (5,7), (6,7)

Cut 5: (47), (57), (26)

Other Cuts???



Cuts

$G = (V, E)$ is a directed graph.

V is the set of vertices and E is the set of edges.

$s \in V$ is the source (start node)

$t \in V$ the sink (end node)

c_{uv} is the capacity of edge $(u,v) \in E$

An **s-t cut** $C = (S, T)$ is a partition of V such that $s \in S$ and $t \in T$. That is, s-t cut is a division of the vertices of the network into two parts, with the source in one part and the sink in the other.

Source: https://en.wikipedia.org/wiki/Max-flow_min-cut_theorem

Cuts (cont'd)

The **cut-set** X_C of a cut C is the set of edges that connect the source part of the cut to the sink part (edges directed from a node in S to a node in T).

$$X_C := \{(u, v) \in E : u \in S, v \in T\}$$

Source: https://en.wikipedia.org/wiki/Max-flow_min-cut_theorem

Cuts (cont'd)

The **capacity** of an s-t cut, denoted *c(S,T)*, is the total capacity of its edges,

$$c(S,T) \coloneqq \sum_{(u,v)\in X_C} c_{uv}$$

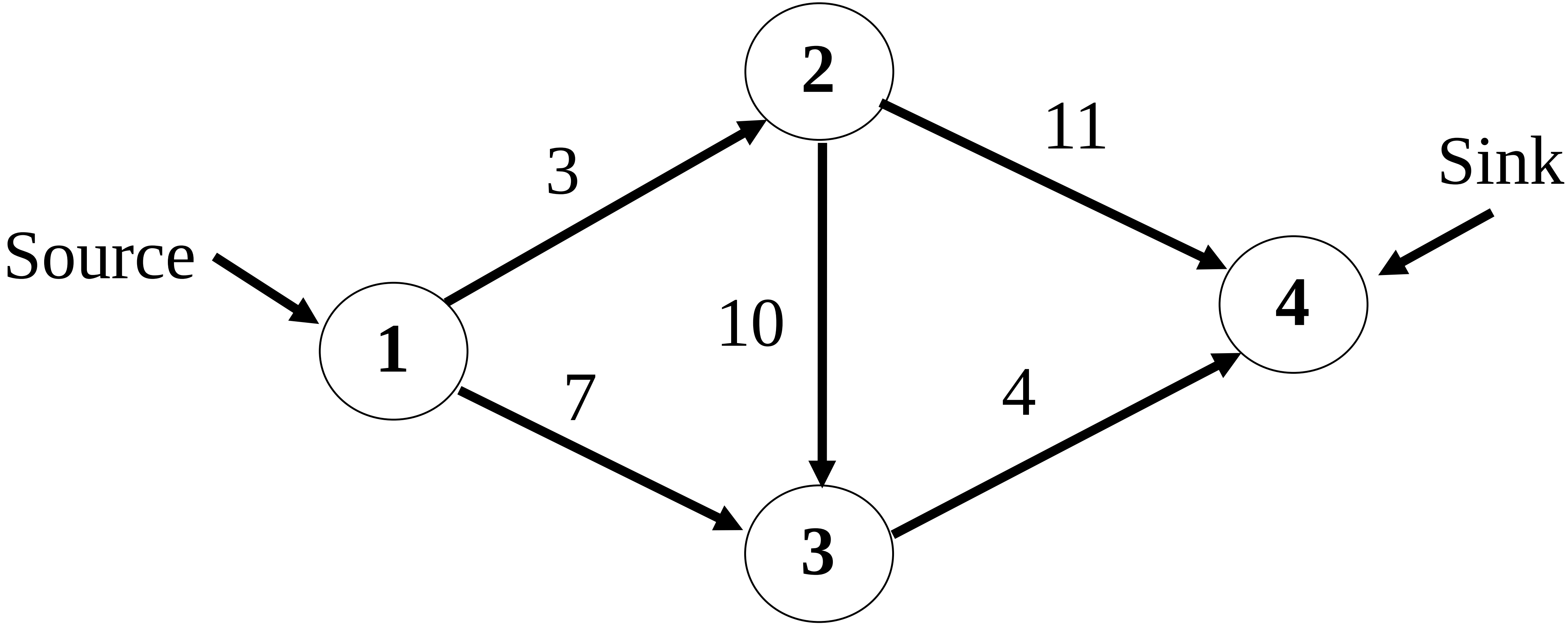
Source: https://en.wikipedia.org/wiki/Max-flow_min-cut_theorem

Minimum s-t Cut Problem

Minimize c(S, T), that is, determine S and T such that the capacity of the S-T cut is minimal.

Source: https://en.wikipedia.org/wiki/Max-flow_min-cut_theorem

Find the Minimum Cut



The Transportation Problem

Suppose a company has m warehouses and n retail outlets. A single product is to be shipped from the warehouses to the outlets. Each warehouse has a given level of supply, and each outlet has a given level of demand. We are also given the transportation costs between every pair of warehouse and outlet, and these costs are assumed to be linear. More explicitly, the assumptions are:

- The total supply of the product from warehouse i is a_i, where $i = 1, 2, \ldots, m$.
- The total demand for the product at outlet j is b_j, where $j = 1, 2, \ldots, n$.
- The cost of sending one unit of the product from warehouse i to outlet j is equal to c_{ij}, where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. The total cost of a shipment is linear in the size of the shipment.

Source: https://personal.utdallas.edu/~scniu/OPRE-6201/documents/TP1-Formulation.pdf

The Transportation Problem (cont'd)

The problem of interest is to determine an optimal transportation scheme between the warehouses and the outlets, subject to the specified supply and demand constraints.

Graphically, a transportation problem is often visualized as a network with m source nodes, n sink nodes, and a set of $m \times n$ "directed arcs." This is depicted in Figure TP-1.

We now proceed with a linear-programming formulation of this problem.

The Decision Variables

A transportation scheme is a complete specification of how many units of the product should be shipped from each warehouse to each outlet. Therefore, the decision variables are:

x_{ij} = the size of the shipment from warehouse i to outlet j, where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Source: https://personal.utdallas.edu/~scniu/OPRE-6201/documents/TP1-Formulation.pdf

The Transportation Problem (cont’d)

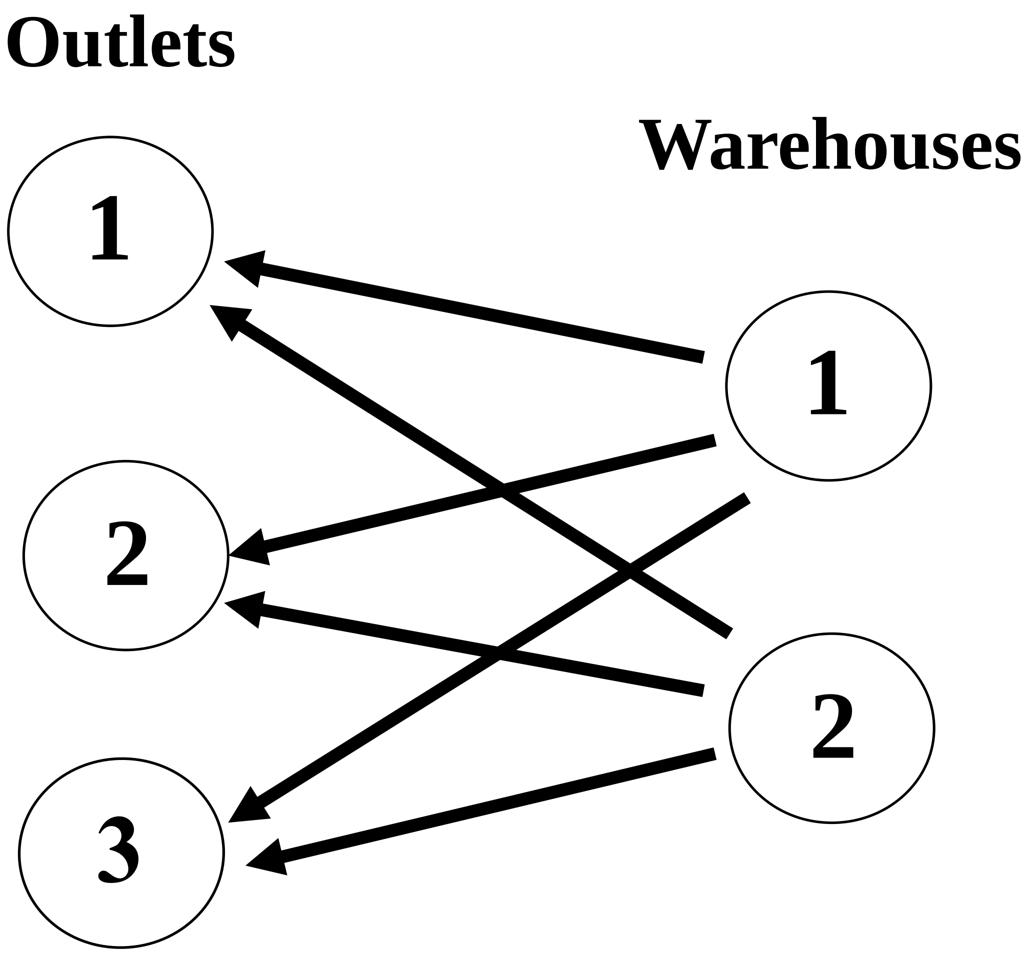


Figure TP-1: The Transportation Problem

The Transportation Problem (cont'd) – Objective function

Consider the shipment from warehouse i to outlet j. For any i and any j, the transportation cost per unit is c_{ij}; and the size of the shipment is x_{ij}. Since we assume that the cost function is linear, the total cost of this shipment is given by $c_{ij}x_{ij}$. Summing over all i and all j now yields the overall transportation cost for all warehouse-outlet combinations. That is, our objective function is:

$$\text{Minimize} \quad \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} .$$

Source: https://personal.utdallas.edu/~scniu/OPRE-6201/documents/TP1-Formulation.pdf

The Transportation Problem (cont'd) - The Constraints

Consider warehouse i. The total outgoing shipment from this warehouse is the sum $x_{i1} + x_{i2} + \cdots + x_{in}$. In summation notation, this is written as $\sum_{j=1}^{n} x_{ij}$. Since the total supply from warehouse i is a_i, the total outgoing shipment cannot exceed a_i. That is, we must require

$$\sum_{j=1}^{n} x_{ij} \leq a_i\,, \quad \text{for } i = 1, 2, \ldots, m.$$

Consider outlet j. The total incoming shipment at this outlet is the sum $x_{1j}+x_{2j}+\cdots+x_{mj}$. In summation notation, this is written as $\sum_{i=1}^{m} x_{ij}$. Since the demand at outlet j is b_j, the total incoming shipment should not be less than b_j. That is, we must require

$$\sum_{i=1}^{m} x_{ij} \geq b_j\,, \quad \text{for } j = 1, 2, \ldots, n.$$

This results in a set of $m + n$ functional constraints. Of course, as physical shipments, the x_{ij}'s should be nonnegative.

Source: https://personal.utdallas.edu/~scniu/OPRE-6201/documents/TP1-Formulation.pdf

The Transportation Problem (cont'd)

LP Formulation

In summary, we have arrived at the following formulation:

$$\text{Minimize} \quad \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$$

Subject to:

$$\sum_{j=1}^{n} x_{ij} \leq a_i \qquad \text{for } i = 1, 2, \ldots, m$$

$$\sum_{i=1}^{m} x_{ij} \geq b_j \qquad \text{for } j = 1, 2, \ldots, n$$

$$x_{ij} \geq 0 \quad \text{for } i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n.$$

This is a linear program with $m \times n$ decision variables, $m + n$ functional constraints, and $m \times n$ nonnegativity constraints.

Source: https://personal.utdallas.edu/~scniu/OPRE-6201/documents/TP1-Formulation.pdf

The Transportation Problem (cont'd) - A Numerical Example

As a simple example, suppose we are given: $m = 3$ and $n = 2$; $a_1 = 45$, $a_2 = 60$, and $a_3 = 35$; $b_1 = 50$ and $b_2 = 60$; and finally, $c_{11} = 3$, $c_{12} = 2$, $c_{21} = 1$, $c_{22} = 5$, $c_{31} = 5$, and $c_{32} = 4$. Then, substitution of these values into the above formulation leads to the following explicit problem:

$$
\begin{array}{llllllllll}
\text{Minimize} & 3x_{11} & +2x_{12} & +x_{21} & +5x_{22} & +5x_{31} & +4x_{32} & & & \\
\text{Subject to:} & & & & & & & & & \\
& x_{11} & +x_{12} & & & & & \leq & 45 & (1) \\
& & & x_{21} & +x_{22} & & & \leq & 60 & (2) \\
& & & & & x_{31} & +x_{32} & \leq & 35 & (3) \\
& x_{11} & & +x_{21} & & +x_{31} & & \geq & 50 & (4) \\
& & x_{12} & & +x_{22} & & +x_{32} & \geq & 60 & (5)
\end{array}
$$

$$x_{ij} \geq 0 \quad \text{for } i = 1, 2, 3 \text{ and } j = 1, 2.$$

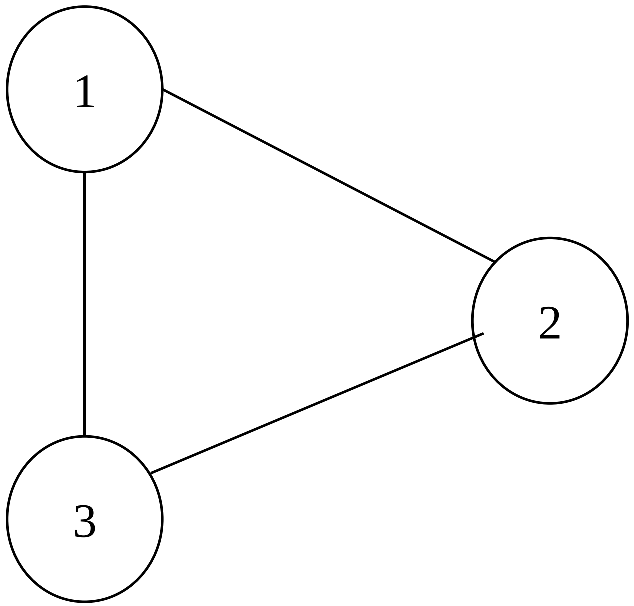
Source: https://personal.utdallas.edu/~scniu/OPRE-6201/documents/TP1-Formulation.pdf

Homework

Solve the numerical example of the transportation problem using Excel Solver. Compare the results with your classmates and correct if necessary. Again and again, active discussions and mutual help are encouraged.

Traffic Routing Problem

We consider the three node network shown in the figure below. All the links in this network are assumed to be bidirectional, and the traffic demand (total in both directions) between node i and j is denoted D_{ij} . The bandwidth capacity of link {ij} is denoted B_{ij} . Traffic between nodes 1 and 2 is the sum of the direct traffic between 1 and 2 denoted X_{12} and the traffic between 1 and 2 that is routed through Node 3, denoted X_{132}. Accordingly, we must satisfy the constraint $X_{12} + X_{132} = D_{12}$.



Credit: M. Pióro and D. Medhi, Routing, Flow, and Capacity Design in Communication and Computer Networks. San Mateo, CA, USA: Morgan Kaufmann, Jul. 2004.

Traffic Routing Problem (cont'd)

Similarly, traffic between nodes 1 and 3 is the sum of the direct traffic between 1 and 3 denoted X_{13} and the traffic between 1 and 3 that is routed through Node 2, denoted X_{123}. Accordingly, we must satisfy the constraint $X_{13} + X_{123} = D_{13}$.

Also, denoting the direct traffic between 2 and 3 by X_{23} and the traffic between 2 and 3 that is routed through Node 1 by X_{213}, we have the constraint $X_{23} + X_{213} = D_{23}$.

Credit: M. Pióro and D. Medhi, Routing, Flow, and Capacity Design in Communication and Computer Networks. San Mateo, CA, USA: Morgan Kaufmann, Jul. 2004.

Traffic Routing Problem (cont'd)

To satisfy bandwidth capacity constraints, we must have

$$X_{13}+X_{132}+X_{213} \leq B_{13}$$
$$X_{12}+X_{123}+X_{213} \leq B_{12}$$
$$X_{23}+X_{132}+X_{123} \leq B_{23}$$

Notes: All parameters and variables used above are defined in the same units (e.g., [Gb/s]) and their values are non-negative.

Credit: M. Pióro and D. Medhi, Routing, Flow, and Capacity Design in Communication and Computer Networks. San Mateo, CA, USA: Morgan Kaufmann, Jul. 2004.

Traffic Routing Problem (cont’d)

Assume that the total cost of transmission and switching that may include amortized Capital Expenditures (CAPEX) and Operating Expenses OPEX (including energy cost) for a two hop connection is twice as that of one hop connection. The objective is to minimize the total cost. Accordingly, the objective is:

Minimize: $P = X_{12}+X_{13}+X_{23}+2X_{132}+2X_{213}+2X_{123}$

Credit: M. Pióro and D. Medhi, Routing, Flow, and Capacity Design in Communication and Computer Networks. San Mateo, CA, USA: Morgan Kaufmann, Jul. 2004.

Traffic Routing Problem (cont'd)

Accordingly, the LP Formulation is as follows.

Minimize: $P = X_{12}+X_{13}+X_{23}+2X_{132}+2X_{213}+2X_{123}$

Subject to:

$X_{12} + X_{132} = D_{12}$

$X_{13} + X_{123} = D_{13}$

$X_{23} + X_{213} = D_{23}$

$X_{13}+X_{132}+X_{213} \leq B_{13}$

$X_{12}+X_{123}+X_{213} \leq B_{12}$

$X_{23}+X_{132}+X_{123} \leq B_{23}$

$X_{12} \geq 0$

$X_{13} \geq 0$

$X_{23} \geq 0$

$X_{123} \geq 0$

$X_{132} \geq 0$

$X_{213} \geq 0$

Credit: M. Pióro and D. Medhi, Routing, Flow, and Capacity Design in Communication and Computer Networks. San Mateo, CA, USA: Morgan Kaufmann, Jul. 2004.

Homework

Consider the case of
$D_{12} = 4$, $D_{23} = 3$, $D_{13} = 6$, and $B_{12} = B_{13} = B_{23} = 5$.

Write the LP Formulation for this case and solve the LP using the excel solver to obtain the optimal solution.

Credit: M. Pióro and D. Medhi, Routing, Flow, and Capacity Design in Communication and Computer Networks. San Mateo, CA, USA: Morgan Kaufmann, Jul. 2004.

Capacity Assignment Problem

Consider an undirected network and a set of demands $\{D_{ij}\}$.
Assume that all the demands are met through shortest path Routing.
Your task is to set the values of the link capacity C_{ij} for each link (i,j)$\in E$ such that the sum $\sum_{(\mathrm{i,j})\in V} C_{\mathrm{ij}}$ is minimized.
How will you do it? Write clearly all the steps.
Write your answer in Chat.

Note that roman (i,j) is used for links and italic *ij* is used for origin destination pair.

Capacity Assignment Solution

1. Run Dijkstra algorithm separately for each origin-destination pair $\{ij\}$.
2. Consider each link {ij} $\in E$ in the network separately, and for each link add up the demands D_{ij} of each origin-destination pair $\{ij\}$ that passes through that link based on their shortest path algorithm.
3. This sum of the demands D_{ij} of the paths that pass through that link based on their shortest path algorithm is equal to the optimal capacity C_{ij} for link {ij} $\in E$.

Why? On each link, the sum of the demands must be satisfies and any additional capacity assignment will not be optimal.

Challenge: Capacity Assignment Optimization for Network Survivability

Now consider again that all the end-to-end connections use their shortest path for routing, and each of them has a demand requirement D_{ij} of each origin-destination (OD) pairs *{ij}*. Develop an algorithm that minimizes the total capacities as before but will guarantee that for any link failure there is still sufficient capacity to route all the required demands of the OD pairs assuming they use the shortest paths after each link failure.

Duality in Linear Programming

Primal LP

Maximize $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$

Subject to $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$

And $\boldsymbol{x} \geq \boldsymbol{0}$

Dual LP **(And vice versa)**

Minimize $\boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}$

Subject to $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} \geq \boldsymbol{c}$

And $\boldsymbol{y} \geq \boldsymbol{0}$

Various other primal-dual kinds.

Primal	Dual	Note
Maximize $\mathbf{c}^{\mathrm{T}}\mathbf{x}$ subject to $A\mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \geq 0$	Minimize $\mathbf{b}^{\mathrm{T}}\mathbf{y}$ subject to $A^{\mathrm{T}}\mathbf{y} \geq \mathbf{c}$, $\mathbf{y} \geq 0$	This is called a "symmetric" dual problem
Maximize $\mathbf{c}^{\mathrm{T}}\mathbf{x}$ subject to $A\mathbf{x} \leq \mathbf{b}$	Minimize $\mathbf{b}^{\mathrm{T}}\mathbf{y}$ subject to $A^{\mathrm{T}}\mathbf{y} = \mathbf{c}$, $\mathbf{y} \geq 0$	This is called an "asymmetric" dual problem
Maximize $\mathbf{c}^{\mathrm{T}}\mathbf{x}$ subject to $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq 0$	Minimize $\mathbf{b}^{\mathrm{T}}\mathbf{y}$ subject to $A^{\mathrm{T}}\mathbf{y} \geq \mathbf{c}$	

Source: https://en.wikipedia.org/wiki/Dual_linear_program

The **dual** of a given linear program (LP) is another LP that is derived from the original (the **primal**) LP in the following schematic way:

- Each variable in the primal LP becomes a constraint in the dual LP;
- Each constraint in the primal LP becomes a variable in the dual LP;
- The objective direction is inversed – maximum in the primal becomes minimum in the dual and vice-versa.

Source: https://en.wikipedia.org/wiki/Dual_linear_program

Economic interpretation of LP Duality

Optimal ***y*** values of the dual LP (aka shadow prices aka dual prices) is the change in the primal optimal objective value per small relaxation of each constraint.
It represents the maximal amount of money the company management is willing to pay for additional unit of resource.

Proof of the weak duality theorem

$$\begin{array}{lcl} \max \mathbf{c} \cdot \mathbf{x} : & & \min \mathbf{b} \cdot \mathbf{y} : \\ A\mathbf{x} \leq \mathbf{b} & \leq & A^T\mathbf{y} \geq \mathbf{c} \\ \mathbf{x} \geq 0 & & \mathbf{y} \geq 0 \end{array}$$

(P)

$$\max c_1x_1 + c_2x_2 + \cdots + c_nx_n :$$
$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2$$
$$\cdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m$$
$$x_1, x_2, \ldots, x_n \geq 0$$

(D)

$$\min b_1y_1 + b_2y_2 + \cdots + b_my_m :$$
$$a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m \geq c_1$$
$$a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m \geq c_2$$
$$\cdots$$
$$a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m \geq c_n$$
$$y_1, y_2, \ldots, y_m \geq 0$$

[Source: https://www.coursera.org/lecture/approximation-algorithms-part-2/proof-of-weak-duality-theorem-eAkFN](https://www.coursera.org/lecture/approximation-algorithms-part-2/proof-of-weak-duality-theorem-eAkFN)

Homework: Watch the video in the link for the full proof and publish it (with full credit) in Canvas Discussions. Write it in your own handwriting.

Strong Duality Theorem

Maximize $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$
Subject to $\mathbf{A}\boldsymbol{x} \le \boldsymbol{b}$
And $\boldsymbol{x} \ge \mathbf{0}$

$=$

Minimize $\boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}$
Subject to $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} \ge \boldsymbol{c}$
And $\boldsymbol{y} \ge \mathbf{0}$

The max-flow min-cut theorem follows from the above strong duality theorem. See the link below for details.

https://en.wikipedia.org/wiki/Max-flow_min-cut_theorem#Linear_program_formulation

Simple Example

Maximize $3x_1 + 4x_2$

Subject to: $5x_1 + 6x_2 \leq 7$

$x_1 \geq 0, \quad x_2 \geq 0.$

Find the dual of this LP problem.

This example was modified from: https://en.wikipedia.org/wiki/Dual_linear_program

Simple Example (cont'd)

Primal

Maximize $3x_1 + 4x_2$

Subject to: $5x_1 + 6x_2 \leq 7$

$x_1 \geq 0, \quad x_2 \geq 0.$

Dual

Minimize $7y_1$

Subject to: $5y_1 \geq 3$

$6y_1 \geq 4$

$y_1 \geq 0.$

Homework: Use Excel to solve the two problems and verify that the optimal solution for the objective function of the primal is equal to that of the dual. See Excel file dual (Sheet1 and Sheet2).

This example was modified from: https://en.wikipedia.org/wiki/Dual_linear_program

Farmer Example

Consider a farmer who may grow wheat and barley with the set provision of some L land, F fertilizer and P pesticide. To grow one unit of wheat, one unit of land, F_1 units of fertilizer and P_1 units of pesticide must be used. Similarly, to grow one unit of barley, one unit of land, F_2 units of fertilizer and P_2 units of pesticide must be used.

The primal problem would be the farmer deciding how much wheat (x_1) and barley (x_2) to grow if their sell prices are S_1 and S_2 per unit.

Maximize:	$S_1 \cdot x_1 + S_2 \cdot x_2$	(maximize the revenue from producing wheat and barley)
subject to:	$x_1 + x_2 \leq L$	(cannot use more land than available)
	$F_1 \cdot x_1 + F_2 \cdot x_2 \leq F$	(cannot use more fertilizer than available)
	$P_1 \cdot x_1 + P_2 \cdot x_2 \leq P$	(cannot use more pesticide than available)
	$x_1, x_2 \geq 0$	(cannot grow negative amounts).

Source: https://en.wikipedia.org/wiki/Dual_linear_program

Farmer Example (cont'd)

Primal LP

Maximize: $S_1 \cdot x_1 + S_2 \cdot x_2$ (maximize the revenue from producing wheat and barley)

subject to: $x_1 + x_2 \leq L$ (cannot use more land than available)

$F_1 \cdot x_1 + F_2 \cdot x_2 \leq F$ (cannot use more fertilizer than available)

$P_1 \cdot x_1 + P_2 \cdot x_2 \leq P$ (cannot use more pesticide than available)

$x_1, x_2 \geq 0$ (cannot grow negative amounts).

Dual LP

Minimize: $L \cdot y_L + F \cdot y_F + P \cdot y_P$ (minimize the total cost of the means of production as the "objective function")

subject to: $y_L + F_1 \cdot y_F + P_1 \cdot y_P \geq S_1$ (the farmer must receive no less than S_1 for his wheat)

$y_L + F_2 \cdot y_F + P_2 \cdot y_P \geq S_2$ (the farmer must receive no less than S_2 for his barley)

$y_L, y_F, y_P \geq 0$ (prices cannot be negative).

Source: https://en.wikipedia.org/wiki/Dual_linear_program