In-Class Exercise 7

1. Determine the z transform of

$$x[n] = \left(\frac{1}{5}\right)^n u[n-3]$$

   Specify its region of convergence (ROC).

2. Determine the discrete-time Fourier transforms (DTFTs) of $x[n] = (0.5)^n u[n]$ and $y[n] = 2^n u[n]$.

3. Determine the z transform of

$$x[n] = (0.5)^n (u[n+5] - u[n-5])$$

   Specify its ROC.

4. Consider the discrete-time signal $x[n]$:

$$x[n] = (-1)^n u[n] + \alpha^n u[-n - n_0]$$

where α is a complex number and n_0 is an integer.

Given that the ROC of $X(z)$ is $1 < |z| < 2$, determine the constraints/requirements on α and n_0, if any.

5. Consider the z transform of a discrete-time signal $h[n]$:

$$H(z) = \frac{1 - 2z^{-1}}{(1 + 0.3z^{-1})(1 - 0.5z^{-1})(1 - 0.7z^{-1})(1 + 0.9z^{-1})}$$

Determine the zero(s) and pole(s) of $H(z)$. Determine the all the possible ROCs for $H(z)$.

6. Determine the z transform of

$$x[n] = \begin{cases} 0, & n < 0 \\ n, & 0 \leq n \leq N-1 \\ N, & n \geq N \end{cases}$$

with $N \geq 1$. Specify its ROC. (Hint: express $x[n]$ as a difference between $Nu[n]$ and its time-shift version)

7. Determine the z transform of

$$x[n] = \begin{cases} \left(\dfrac{1}{3}\right)^n \cos\left(\dfrac{\pi}{4}n\right), & n \leq 0 \\ 0, & n > 0 \end{cases}$$

Specify its ROC.

Solution

1.

$$
\begin{aligned}
X(z) &= \sum_{n=-\infty}^{\infty} \left(\frac{1}{5}\right)^n u[n-3] z^{-n} \\
&= \sum_{n=3}^{\infty} \left(\frac{1}{5}z^{-1}\right)^n, \quad \left|\frac{1}{5}z^{-1}\right| < 1 \Rightarrow |z| > \frac{1}{5} \\
&= \left(\frac{1}{5}z^{-1}\right)^3 \frac{1}{1-\frac{1}{5}z^{-1}} \\
&= \frac{1}{125}\frac{z^{-3}}{1-\frac{1}{5}z^{-1}}
\end{aligned}
$$

The ROC is $|z| > 0.2$. Note that another approach is to use the time-shift property as in Example 8.10.

2.
The z transform of $x[n] = a^n u[n]$ is (See Example 8.2 or Table 8.1):

$$X(z) = \frac{z}{z-a}, \quad |z| > |a|$$

Now $a = 0.5$ so we have:

$$X(z) = \frac{z}{z-0.5}, \quad |z| > 0.5$$

Since the ROC includes unit circle, the DTFT exists and its expression is obtained easily by putting $z = e^{j\omega}$:

$$X(e^{j\omega}) = \frac{e^{j\omega}}{e^{j\omega} - 0.5}$$

Alternatively, we can compute the DTFT from its definition:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} (0.5)^n u[n] e^{-j\omega n} = \sum_{n=0}^{\infty} (0.5)^n e^{-j\omega n} = \sum_{n=0}^{\infty} (0.5e^{-j\omega})^n$$

As in z transform, the DTFT exists if $|X(e^{j\omega})| < \infty$. We see that

$$|X(e^{j\omega})| \leq \sum_{n=0}^{\infty} |0.5e^{-j\omega}|^n = \sum_{n=0}^{\infty} (0.5)^n = \frac{1}{1-0.5} = 2 < \infty$$

As a result, the DTFT exists and has the form of

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} (0.5e^{-j\omega})^n = \frac{1}{1-0.5e^{-j\omega}}$$

which also agrees with Example 6.1.

On the other hand, the z transform of $y[n] = 2^n u[n]$ is:

$$Y(z) = \frac{z}{z-2}, \quad |z| > 2$$

Since the ROC does not include unit circle, the DTFT does not exist. Alternatively, we can compute the DTFT from its definition:

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} 2^n u[n] e^{-j\omega n} = \sum_{n=0}^{\infty} 2^n e^{-j\omega n} = \sum_{n=0}^{\infty} (2e^{-j\omega})^n$$

As in z transform, the DTFT exists if $|Y(e^{j\omega})| < \infty$. We see that

$$|Y(e^{j\omega})| \leq \sum_{n=0}^{\infty} |2e^{-j\omega}|^n = \sum_{n=0}^{\infty} 2^n = 1 + 2 + 2^2 + \cdots \to \infty$$

This means that the DTFT does not exist.

3.

$$
\begin{aligned}
X(z) &= \sum_{n=-5}^{4} \left(0.5z^{-1}\right)^{n} \\
&= \left(\frac{0.5}{z}\right)^{-5} + \left(\frac{0.5}{z}\right)^{-4} + \cdots + \left(\frac{0.5}{z}\right)^{4} \\
&= \left(0.5z^{-1}\right)^{-5} \frac{1-\left(0.5z^{-1}\right)^{10}}{1-0.5z^{-1}} \\
&= \frac{\left(0.5z^{-1}\right)^{-5} - \left(0.5z^{-1}\right)^{5}}{1-0.5z^{-1}}
\end{aligned}
$$

The ROC is $0 < |z| < \infty$.

Note that it is a finite-duration sequence, which is similar to Examples 8.5 and 8.6.

4.

From Examples 8.2 to 8.4, we can see that $1 < |z|$ is due to $(-1)^n u[n]$ while $|z| < 2$ is due to $\alpha^n u[-n - n_0]$.

As a result, we can deduce that $|\alpha| = 2$.

On the other hand, the time-shift n_0 does not affect the ROC of $1 < |z| < 2$, and thus its value can be any integer.

5.

Expressing $H(z)$ in terms of z instead of z^{-1} yields:

$$H(z) = \frac{z^3(z-2)}{(z+0.3)(z-0.5)(z-0.7)(z+0.9)}$$

Hence we see that there are four zeros, three at zero and one is 2.

There are also four poles at -0.3, 0.5, 0.7 and -0.9.

Four poles correspond to 5 possible ROCs:

$$\begin{gathered} |z| < 0.3 \\ 0.3 < |z| < 0.5 \\ 0.5 < |z| < 0.7 \\ 0.7 < |z| < 0.9 \\ |z| > 0.9 \end{gathered}$$

6.

Based on the given hint, $x[n]$ can be written as:

$$x[n] = nu[n] - (n - N)u[n - N]$$

Let $x_1[n] = nu[n]$ and $x_2[n] = (n - N)u[n - N]$ such that

$$x[n] = x_1[n] - x_2[n]$$

Using Table 8.1, we get:

$$x_1[n] = nu[n] \leftrightarrow \frac{z^{-1}}{(1 - z^{-1})^2}, \quad |z| > 1$$

Applying the time shifting property:

$$x[n-n_0] \leftrightarrow z^{-n_0}X(z)$$

we obtain

$$x_2[n]=(n-N)u[n-N]=x_1[n-N] \leftrightarrow z^{-N}\frac{z^{-1}}{(1-z^{-1})^2}, \quad |z|>1$$

As a result,

$$X(z)=\frac{z^{-1}(1-z^{-N})}{(1-z^{-1})^2}, \quad |z|>1$$

7.

Using the Euler formula, we have:

$$\left(\frac{1}{3}\right)^n \cos\left(\frac{\pi}{4}n\right) = \frac{1}{2}\left[\left(\frac{1}{3}e^{j\frac{\pi}{4}}\right)^n + \left(\frac{1}{3}e^{-j\frac{\pi}{4}}\right)^n\right]$$

For the first component, we have:

$$\begin{aligned}
&\sum_{n=-\infty}^{0} \left(\frac{1}{3}e^{j\frac{\pi}{4}}\right)^n z^{-n}, \quad m=-n \\
&= \sum_{m=0}^{\infty} \left(3e^{-j\frac{\pi}{4}}z\right)^m, \quad \left|3e^{-j\frac{\pi}{4}}z\right| < 1 \Rightarrow |z| < \frac{1}{3} \\
&= \frac{1}{1-3e^{-j\frac{\pi}{4}}z}
\end{aligned}$$

Similarly, the second component is:

$$\sum_{n=-\infty}^{0} \left(\frac{1}{3}e^{-j\frac{\pi}{4}}\right)^n z^{-n} = \frac{1}{1-3e^{j\frac{\pi}{4}}z}$$

where the ROC is also $|z| < 1/3$. Combining the results yields:

$$\begin{aligned} X(z) &= \frac{1}{2}\left[\frac{1}{1-3e^{j\frac{\pi}{4}}z} + \frac{1}{1-3e^{-j\frac{\pi}{4}}z}\right] \\ &= \frac{1-3\cos(\frac{\pi}{4})z}{1-6\cos(\frac{\pi}{4})z+9z^2} \end{aligned}$$

and the ROC is $|z| < \dfrac{1}{3}$.