EE3210 Cheatsheet Final Exam

Formula regarding to Complex Number

Magnitude $|x|$

$$|x| = \sqrt{(\Re\{x\})^2 + (\Im\{x\})^2} = \sqrt{x \cdot x^*} \tag{1}$$

Phase $\angle(x)$

$$\angle(x) = \tan^{-1}\left(\frac{\Im\{x\}}{\Re\{x\}}\right) \tag{2}$$

Complex Conjugate of x

$$x^* = \Re\{x\} - j\Im\{x\} \tag{3}$$

Periodicity

If there exists $T > 0$ such that

$$x(t) = x(t+T) \tag{4}$$

for all t, the smallest T is called the fundamental period.

If there exists a positive integer N such that

$$x[n] = x[n+N] \tag{5}$$

for all n, the smallest N is called the fundamental period.

If a signal is not periodic, then it is aperiodic.

Even and Odd Function

The signal is an even function if

$$x_e(t) = x_e(-t) \text{ or } x_e[n] = x_e[-n] \tag{6}$$

The signal is an odd function if

$$x_o(t) = -x_o(-t) \text{ or } x_o[n] = -x_o[-n] \tag{7}$$

Any signal can be represented by a sum of even and odd signals

$$x(t) = x_e(t) + x_o(t) \text{ or } x[n] = x_e[n] + x_o[n] \tag{8}$$

where

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)] \text{ and } x_o(t) = \frac{1}{2}[x(t) - x(-t)] \tag{9}$$

$$x_e[n] = \frac{1}{2}[x[n] + x[-n]] \text{ and } x_o[n] = \frac{1}{2}[x[n] - x[-n]] \tag{10}$$

Energy and Power

Energy of a signal is defined as

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt \qquad \text{or} \qquad \sum_{n=-\infty}^{\infty} |x[n]|^2 \tag{11}$$

If the signal energy is ∞, then to use power of $x(t)$ or $x[n]$ as the measure, which is defined as

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt \qquad \text{or} \qquad \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 \tag{12}$$

Signal power is the time average of the signal energy.

A signal is energy signal if $0 < E_x < \infty$, indicating its $P_x = 0$.

A signal is power signal if $0 < P_x < \infty$, indicating its $E_x = \infty$.

Unit Impulse

The unit impulse $\delta(t)$ has the following characteristics,

$$\delta(t) = 0, \qquad t \neq 0 \tag{13}$$

and

$$\int_{-\infty}^{\infty} \delta(t) \, dt = 1 \tag{14}$$

Sifting Property

$\delta(t)$ can be the building block of any continuous-time signal,

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\, d\tau \tag{15}$$

That is, imagining $x(t)$ as a sum of infinite impulse functions and each with amplitude $x(\tau)$.

Unit step

The unit step function $u(t)$ has the following form,

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases} \tag{16}$$

$u(t)$ can be expressed in terms of $\delta(t)$ as

$$u(t) = \int_{-\infty}^{\infty} u(\tau)\delta(t-\tau)\, d\tau = \int_{0}^{\infty} \delta(t-\tau)\, d\tau \tag{17}$$

Conversely, to express $\delta(t)$ in terms of $u(t)$ as

$$\delta(t) = \frac{du(t)}{dt} \tag{18}$$

Sinusoid

It is a sine or cosine wave of the following form,

$$x(t) = A\cos(\omega t + \phi) \tag{19}$$

which is characterised by three parameters, amplitude $A > 0$, radian frequency ω and phase $\phi \in [0, 2\pi)$

Fundamental period T_0 is determined as

$$\begin{aligned} x(t) = x(t+T_0) &= A\cos(\omega(t+T_0)+\phi) = A\cos(\omega t + 2\pi + \phi) \\ \implies \omega T_0 &= 2\pi \\ \implies T_0 &= \frac{2\pi}{\omega} = \frac{1}{f} \end{aligned} \tag{20}$$

For the complex-valued case, it has the following form,

$$x(t) = Ae^{j(\omega t+\phi)} \tag{21}$$

Using the Euler formula

$$e^{j\phi} = \cos(\phi) + j\sin(\phi) \tag{22}$$

According to (22), to obtain that

$$\cos(\phi) = \frac{e^{j\phi} + e^{-j\phi}}{2} \tag{23}$$

and

$$\sin(\phi) = \frac{e^{j\phi} - e^{-j\phi}}{2} \tag{24}$$

Unit Impulse in Discrete-Time

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases} \tag{25}$$

Unit Step in Discrete-Time

$$u[n] = \begin{cases} 1, n \geq 0 \\ 0, n < 0 \end{cases} \tag{26}$$

Sifting Property in Discrete-Time

$\delta[n]$ can be served as the building block of any discrete-time signal $x[n]$ as

$$\begin{aligned} x[n] &= \cdots + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + \dots \\ &= \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \end{aligned} \tag{27}$$

$u[n]$ can be expressed in terms of $\delta[n]$ as

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k] \tag{28}$$

Conversely, $\delta[n]$ can be expressed in terms of $u[n]$ as

$$\delta[n] = u[n] - u[n-1] \tag{29}$$

Basic System Properties

Memoryless

A system is memoryless if its output at a given time is dependent only on the input at that same time, i.e., $y(t)$ at time t depends only on $x(t)$ at time t; $y[n]$ at time n depends only on $x[n]$ at time n.

Invertibility

A system is invertible if distinct inputs lead to distinct outputs, as known as if an inverse system exists

Linearity

A system is linear if it obeys principle of superposition.

$$\mathcal{T}\{ax_1(t) + bx_2(t)\} = a\mathcal{T}\{x_1(t)\} + b\mathcal{T}\{x_2(t)\} = ay_1(t) + by_2(t) \tag{30}$$

and

$$\mathcal{T}\{ax_1[n] + bx_2[n]\} = a\mathcal{T}\{x_1[n]\} + b\mathcal{T}\{x_2[n]\} = ay_1[n] + by_2[n] \tag{31}$$

A standard approach to determine the linearity of a system is given as follows. Let

$$y_i[n] = \mathcal{T}\{x_i[n]\}, \qquad i = 1, 2, 3 \tag{32}$$

with

$$x_3[n] = ax_1[n] + bx_2[n] \tag{33}$$

If $y_3[n] = ay_1[n] + by_2[n]$, then the system is linear.

Time-Invariance

A system is time-invariant if a time-shift of input causes a corresponding shift in output as followings,

$$y(t) = \mathcal{T}\{x(t)\} \rightarrow y(t-t_0) = \mathcal{T}\{x(t-t_0)\} \tag{34}$$

and

$$y[n] = \mathcal{T}\{x[n]\} \rightarrow y[n-n_0] = \mathcal{T}\{x[n-n_0]\} \tag{35}$$

That is, the system response is independent of time.

Causality

A system is causal if the output $y(t)$ or $y[n]$ at time t or n depends on input $x(t)$ or $x[n]$ up to time t or n.

That is, in casual system, output does not depends on the future input.

Stability

A system is stable if every bounded input $x(t)$ or $x[n]$ produces a bounded output $y(t)$ or $y[n]$ for all t or n, or if the bounded-input bounded-output criterion is satisfied.

That is

$$|y(t)| < B \qquad \text{if } |x(t)| < A, \qquad |A| < \infty,\ \ |B| < \infty \tag{36}$$

and

$$|y[n]| < B \qquad \text{if } |x[n]| < A, \qquad |A| < \infty,\ \ |B| < \infty \tag{37}$$

Linear Time-Invariant System (LTI)

Impulse Response

The impulse response $h(t)$ or $h[n]$ is the output of a LTI system when the input is the unit impulse $\delta(t)$ or $\delta[n]$

Convolution

The convolution of $x[n]$ and $h[n]$ is defined as

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] = x[n] \circledast h[n] \tag{38}$$

Similarly, the convolution of $x(t)$ and $h(t)$ is defined as

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau = x(t) \circledast h(t) \tag{39}$$

There are three properties in convolution:

Commutative,

$$\begin{aligned} x[n] \circledast h[n] &= h[n] \circledast x[n] \\ \sum_{m=-\infty}^{\infty} x[m]h[n-m] &= \sum_{m=-\infty}^{\infty} h[m]x[n-m] \end{aligned} \tag{40}$$

Associative,

$$x[n] \circledast (h_1[n] \circledast h_2[n]) = (x[n] \circledast h_1[n]) \circledast h_2[n] \tag{41}$$

Distributive,

$$\begin{aligned} y[n] &= x[n] \circledast (h_1[n] + h_2[n]) \\ &= x[n] \circledast h_1[n] + x[n] \circledast h_2[n] \end{aligned} \tag{42}$$

Causality and Stability in LTI

A LTI system is causal if its impulse response satisfies

$$\begin{aligned} h(t) &= 0, \qquad t < 0 \\ h[n] &= 0, \qquad n < 0 \end{aligned} \tag{43}$$

A LTI system is stable if its impulse response satisfies

$$\begin{aligned} \int_{-\infty}^{\infty} |h(t)|\,dt < \infty \\ \sum_{n=-\infty}^{\infty} |h[n]| < \infty \end{aligned} \tag{44}$$

Fourier Series

Fourier series is the frequency domain representation of a continuous-time periodic signal.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t}, \qquad t \in (-\infty, \infty) \tag{45}$$

where

$$a_k = \frac{1}{T_p} \int_{-T_p/2}^{T_p/2} x(t) e^{-jk\Omega_0 t} \, dt, \qquad k = \cdots -1, 0, 1, 2 \cdots \tag{46}$$

That is, every periodic signal can be expressed as a sum of harmonically related complex sinusoids with frequencies $\cdots - \Omega_0, 0, \Omega_0, 2\Omega_0, 3\Omega_0, \cdots$, where the fundamental frequency Ω_0 is called the first harmonic.

a_k is generally complex, so to use (1) and (2) for its representation,

$$|a_k| = \sqrt{(\Re\{a_k\})^2 + (\Im\{a_k\}^2)} \tag{47}$$

and

$$\angle(a_k) = \tan^{-1}\left(\frac{\Im\{a_k\}}{\Re\{a_k\}}\right) \tag{48}$$

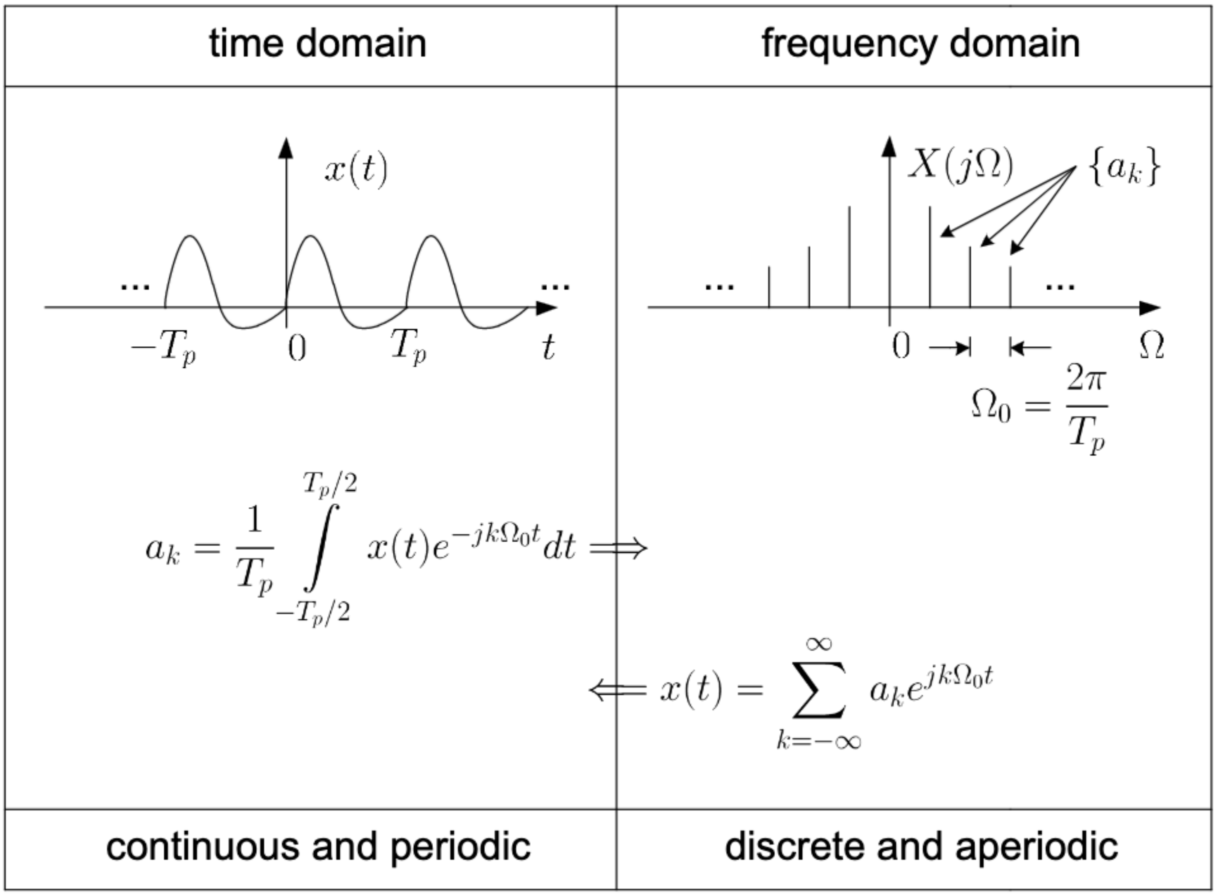
According to (4), that $x(t)$ is periodic if there exists $T_p > 0$ such that

$$x(t) = x(t + T_p), \qquad t \in (-\infty, \infty) \tag{49}$$

The smallest T_p is called fundamental period.

The fundamental frequency Ω_0 can be computed as

$$\Omega_0 = \frac{2\pi}{T_p} \tag{50}$$



Properties of Fourier Series

Linearity

Let $x(t) \leftrightarrow a_k$ and $y(t) \leftrightarrow b_k$ be two Fourier series pairs with the same period of T_p,

$$Ax(t) + By(t) \leftrightarrow Aa_k + Bb_k \tag{51}$$

Time Shifting

A shift of t_0 in $x(t)$ causes a multiplication of $e^{-jk\Omega_0 t_0}$ in a_k as

$$x(t) \leftrightarrow a_k \implies x(t - t_0) \leftrightarrow e^{-jk\Omega_0 t_0} a_k = e^{-jk(2\pi)/T_p t_0} a_k \tag{52}$$

Time Reversal

$$x(t) \leftrightarrow a_k \implies x(-t) \leftrightarrow a_{-k} \tag{53}$$

Time Scaling

For a time-scaled version of $x(t)$, $x(\alpha t)$ where $\alpha \neq 0$ is a real number, is defined as

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t} \implies x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(\alpha\Omega_0)t} \tag{54}$$

Multiplication

Let $x(t) \leftrightarrow a_k$ and $y(t) \leftrightarrow b_k$ be two Fourier series pairs with the same period of T_p, is defined as

$$x(t)y(t) \leftrightarrow \sum_{l=-\infty}^{\infty} a_l b_{k-l} \tag{55}$$

Conjugation

$$x(t) \leftrightarrow a_k \implies x^*(t) \leftrightarrow a^*_{-k} \tag{56}$$

Parseval's Relation

The Parseval's relation addresses the power of $x(t)$ as

$$\frac{1}{T_p} \int_{-T_p/2}^{T_p/2} |x(t)|^2 \, dt = \sum_{k=-\infty}^{\infty} |a_k|^2 \tag{57}$$

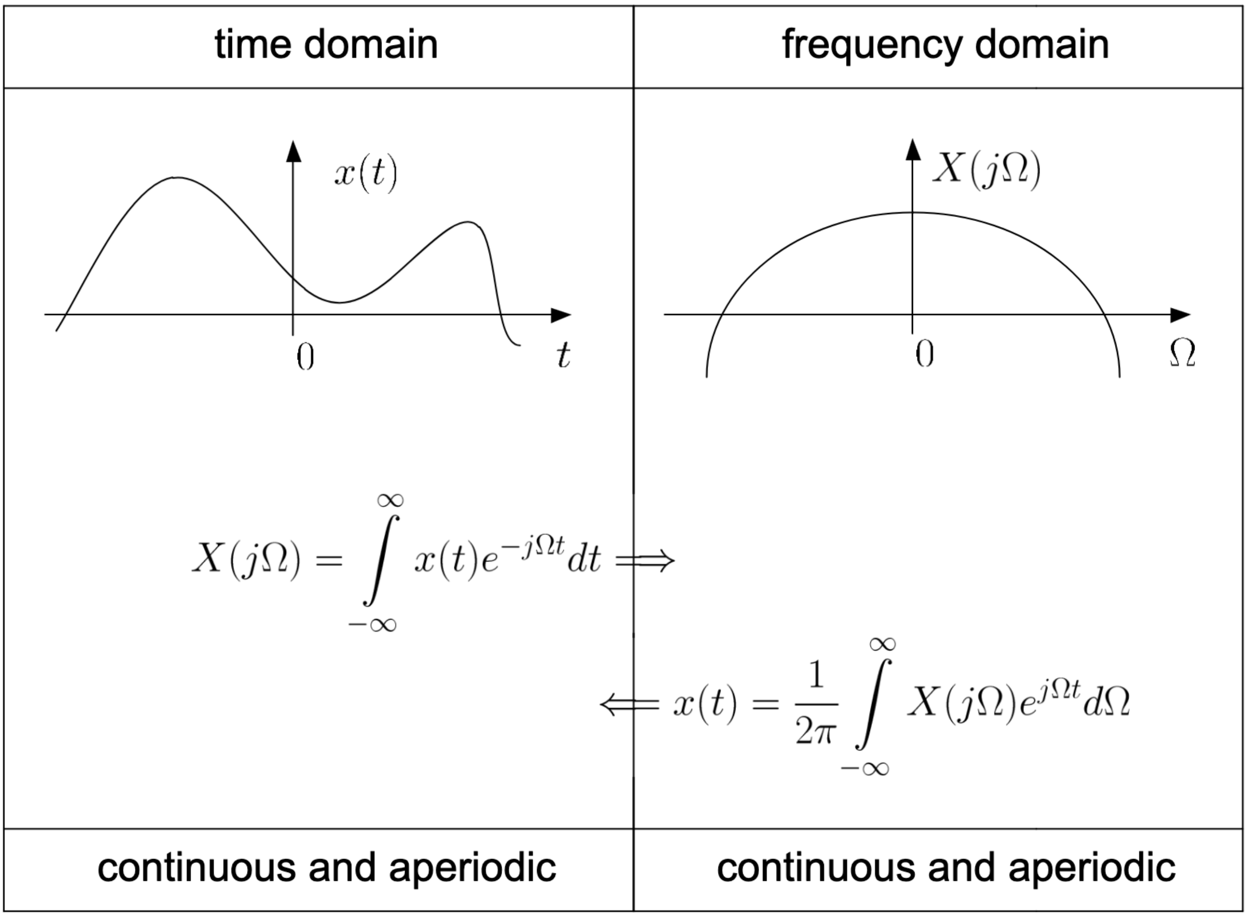
Fourier Transform

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} \, dt \tag{58}$$

where $X(j\Omega)$ is a function of frequency Ω, also know as spectrum.

The inverse Fourier transform is defined as

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} \, d\Omega \tag{59}$$



Periodic Signal Representation using Fourier Transform

Fourier transform can be used to represent continuous-time periodic signals with the use of $\delta(t)$.

Instead of time domain, to consider impulse in the frequency domain as

$$X(j\Omega) = 2\pi\delta(\Omega - \Omega_0) \tag{60}$$

The inverse Fourier transform is defined as

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\Omega - \Omega_0)e^{j\Omega t}\, d\Omega = e^{j\Omega_0 t} \tag{61}$$

As a result, the Fourier transform pair is

$$e^{j\Omega_0 t} \leftrightarrow 2\pi\delta(\Omega - \Omega_0) \tag{62}$$

Properties of Fourier Transform

Linearity

Let $x(t) \leftrightarrow X(j\Omega)$ and $y(t) \leftrightarrow Y(j\Omega)$ be two Fourier transform pairs, and it is defined as

$$ax(t) + by(t) \leftrightarrow aX(j\Omega) + bY(j\Omega) \tag{63}$$

Time Shifting

A shift of t_0 in $x(t)$ causes a multiplication of $e^{-j\Omega t_0}$ in $X(j\Omega)$, as

$$x(t) \leftrightarrow X(j\Omega) \implies x(t - t_0) \leftrightarrow e^{-j\Omega t_0} X(j\Omega) \tag{64}$$

Time Reversal

$$x(t) \leftrightarrow X(j\Omega) \implies x(-t) \leftrightarrow X(-j\Omega) \tag{65}$$

Time Scaling

For a time-scaled version of $x(t)$, $x(\alpha t)$ where $\alpha \neq 0$ is a real number, it is defined as

$$x(t) \leftrightarrow X(j\Omega) \implies x(\alpha t) \leftrightarrow \frac{1}{|\alpha|} X\left(\frac{j\Omega}{\alpha}\right) \tag{66}$$

Multiplication

Let $x(t) \leftrightarrow X(j\Omega)$ and $y(t) \leftrightarrow Y(j\Omega)$ be two Fourier transform pairs, it is defined as

$$x(t) \cdot y(t) \leftrightarrow \frac{1}{2\pi} X(j\Omega) \circledast Y(j\Omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\tau) Y(j(\Omega - \tau))\, d\tau \tag{67}$$

Conjugation

$$x(t) \leftrightarrow X(j\Omega) \implies x^*(t) \leftrightarrow X^*(-j\Omega) \tag{68}$$

Parseval's Relation

The Parseval's relation address the energy of $x(t)$ defined as

$$\int_{-\infty}^{\infty} |x(t)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\Omega)|^2\, dt \tag{69}$$

Convolution

Let $x(t) \leftrightarrow X(j\Omega)$ and $y(t) \leftrightarrow Y(j\Omega)$ be two Fourier transform pairs, it is defined as

$$x(t) \circledast y(t) \leftrightarrow X(j\Omega)Y(j\Omega) \tag{70}$$

Differentiation

Differentiating $x(t)$ w.r.t. t corresponds to multiply $X(j\Omega)$ by $j\Omega$ in the frequency domain is defined as

$$\frac{dx(t)}{dt} \leftrightarrow j\Omega X(j\Omega) \implies \frac{d^k x(t)}{dt^k} \leftrightarrow (j\Omega)^k X(j\Omega) \tag{71}$$

Integration

$$\int_{-\infty}^{t} x(\tau)\, d\tau \leftrightarrow \frac{1}{j\Omega} X(j\Omega) + \pi X(0)\delta(\Omega) \tag{72}$$

Fourier Transform and LTI System

$$y(t) = x(t) \circledast h(t) \leftrightarrow Y(j\Omega) = X(j\Omega)H(j\Omega) \tag{73}$$

This suggests to convert the input and impulse response to frequency domain, then $y(t)$ can be computed from inverse Fourier transform of $X(j\Omega)H(j\Omega)$.

$H(j\Omega)$ represents the LTI system in the frequency domain, is called the system frequency response.

As the input and output of a LTI system satisfy the differential equation

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \tag{74}$$

Therefore, the system frequency response $H(j\Omega)$ can also be computed as

$$H(j\Omega) = \frac{Y(j\Omega)}{X(j\Omega)} = \frac{\sum_{k=0}^{M} b_k (j\Omega)^k}{\sum_{k=0}^{M} a_k (j\Omega)^k} \tag{75}$$

Discrete-Time Fourier Transform (DTFT)

With the use of sampled version of a continuous-time signal $x(t)$, to obtain the discrete-time Fourier transform (DTFT) as follows,

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \tag{76}$$

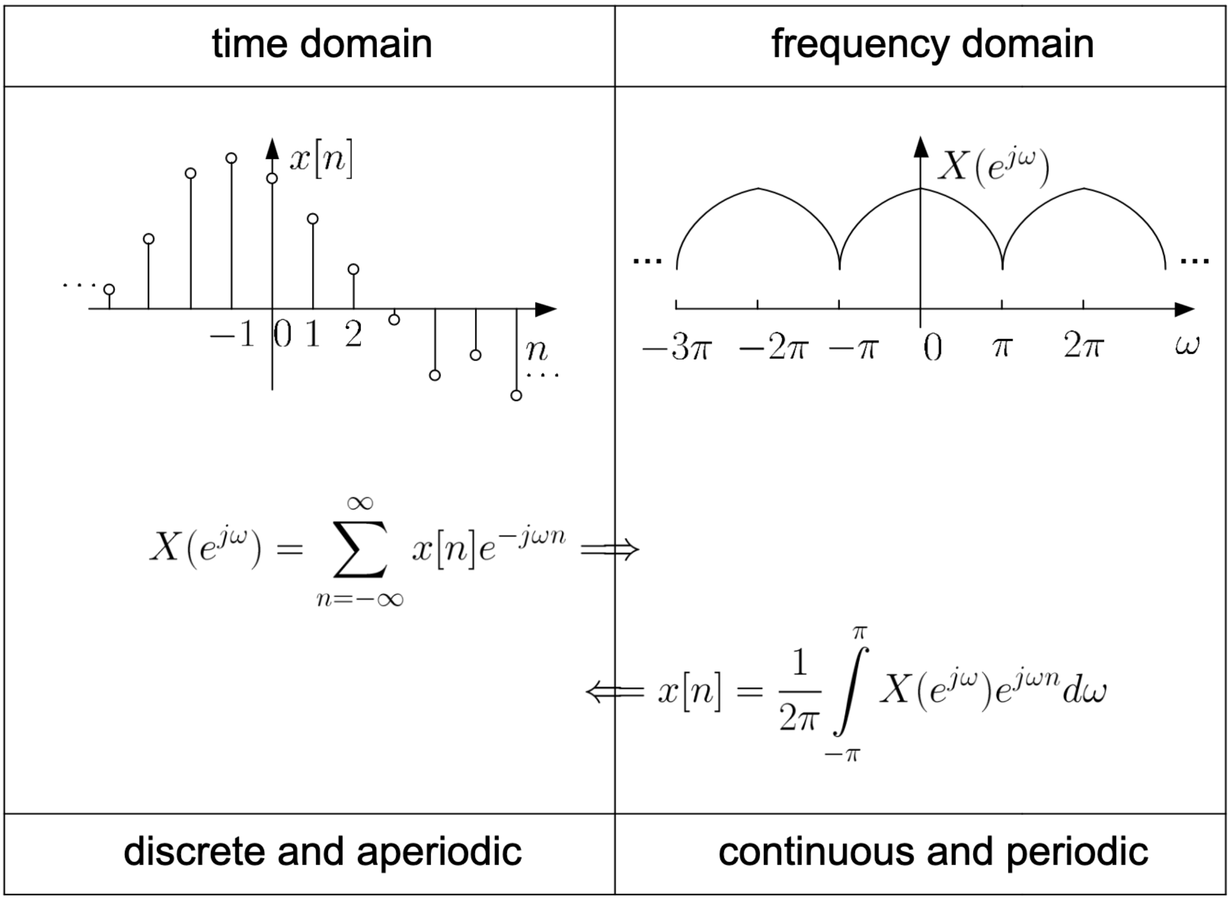
where $\omega = \Omega T$ as the discrete-time frequency.

It is also periodic with period 2π as follows,

$$X(e^{j(\omega+2k\pi)}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega+2k\pi)n} \tag{77}$$

The inverse DTFT is defined as

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\, d\omega \tag{78}$$



Properties of DTFT

Linearity

If $x_1[n] \leftrightarrow X_1(e^{j\omega})$ and $x_2[n] \leftrightarrow X_2(e^{j\omega})$ are two DTFT pairs, then

$$ax_1[n] + bx_2[n] \leftrightarrow aX_1(e^{j\omega}) + bX_2(e^{j\omega}) \tag{79}$$

Time Shifting

A shift of n_0 in $x[n]$ causes a multiplication of $e^{-j\omega n_0}$ in $X(e^{j\omega})$ as follows

$$x[n] \leftrightarrow X(e^{j\omega}) \implies x[n-n_0] = e^{-j\omega n_0} X(e^{j\omega}) \tag{80}$$

Time Reversal

The DTFT pairs of $x[-n]$ is defined as

$$x[n] \leftrightarrow X(e^{j\omega}) \implies x[-n] \leftrightarrow X(e^{-j\omega}) \tag{81}$$

Multiplication

Multiplication in the time domain corresponds to convolution in the frequency domain is defined as

$$x_1[n] \cdot x_2[n] \leftrightarrow X_1(e^{j\omega}) \tilde{\circledast} X_2(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(e^{j\tau})\, X_2(e^{j(\omega-\tau)}) d\tau \tag{82}$$

where $\tilde{\circledast}$ denotes convolution within one period.

Conjugation

The DTFT pair for $x^*[n]$ is defined as

$$x[n] \leftrightarrow X(e^{j\omega}) \implies x^*[n] \leftrightarrow X^*(e^{-j\omega}) \tag{83}$$

Multiplication by an Exponential Sequence

Multiplying $x[n]$ by $e^{j\omega_0 n}$ in time domain corresponds to a shift of w_0 in the frequency domain, it is defined as

$$x[n] \leftrightarrow X(e^{j\omega}) \implies e^{j\omega_0 n} x[n] \leftrightarrow X(e^{j(\omega-\omega_0)}) \tag{84}$$

Differentiation

Differentiating $X(e^{j\omega})$ w.r.t. w corresponds to multiply $x[n]$ by n, it is defined as

$$x[n] \leftrightarrow X(e^{j\omega}) \implies nx[n] \leftrightarrow j \frac{dX(e^{j\omega})}{d\omega} \tag{85}$$

Parseval's Relation

The Parseval's relation addresses the energy of $x[n]$

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|X(e^{j\omega})\right|^2 d\omega \tag{86}$$

Convolution

If $x_1[n] \leftrightarrow X_1(e^{j\omega})$ and $x_2[n] \leftrightarrow X_2(e^{j\omega})$ are two DTFT pairs, then

$$x_1[n] \circledast x_2[n] \leftrightarrow X_1(e^{j\omega})X_2(e^{j\omega}) \tag{87}$$

DTFT and LTI

$$y[n] = x[n] \circledast h[n] \leftrightarrow Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) \tag{88}$$

This suggests to convert the input and impulse response to frequency domain, then $y[n]$ is computed from inverse DTFT of $X(e^{j\omega})H(e^{j\omega})$.

$H(e^{j\omega})$ represents the LTI system in the frequency domain, is called the system frequency response.

Since the input and output of a discrete-time LTI system satisfy the difference equation as follows

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \tag{89}$$

Therefore, the system frequency response can be computed as

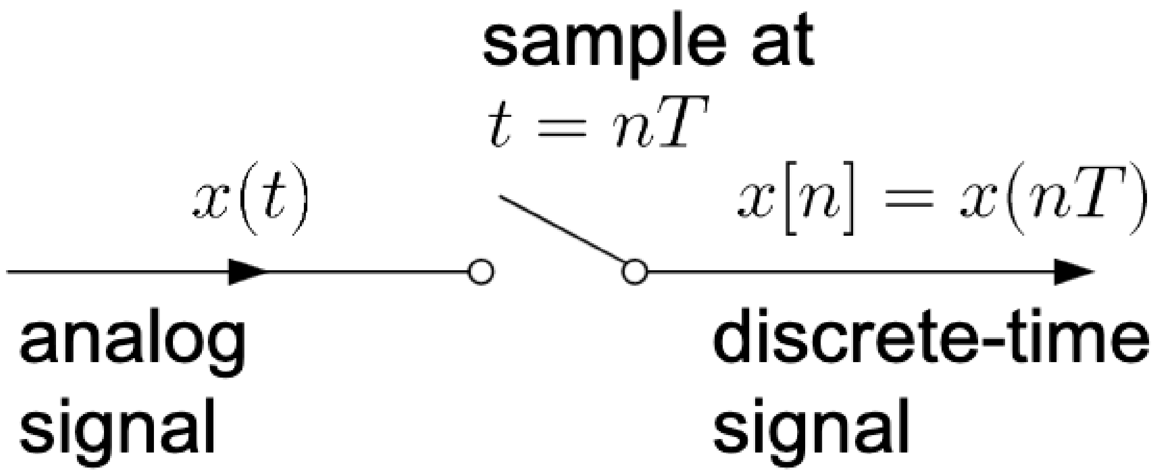
$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{M} a_k e^{-j\omega k}} \tag{90}$$

Sampling and Reconstruction

Sampling

It is a process of converting a continuous-time signal $x(t)$ into a discrete-time signal $x[n]$.

$x[n]$ is obtained by extracting $x(t)$ every T seconds where T is known as the sampling period or interval.



Therefore, its relationship between $x(t)$ and $x[n]$ is

$$x[n] = x(t)|_{t=nT} = x(nT), \qquad n = \cdots -1, 0, 1, 2, \cdots \tag{91}$$

$x[n]$ can uniquely represent $x(t)$ or use $x[n]$ to reconstruct $x(t)$ if $x(t)$ is bandlimited such that its Fourier transform $X(j\Omega) = 0$ for $|\Omega| \geq \Omega_b$ where Ω_b is called the bandwidth and the sampling period T is sufficiently small.

In the time domain, $x_s t$ is obtained by multiplying $x(t)$ by the impulse train $i(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$, as follows

$$x_s(t) = x(t) \sum_{k=-\infty}^{\infty} \delta(t - kT) = \sum_{k=-\infty}^{\infty} x[k]\delta(t - kT) \tag{92}$$

Let the sampling frequency in radian be $\Omega_s = 2\pi/T$, it is defined as

$$I(j\Omega) = \Omega_s \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s) \tag{93}$$

Therefore, $X_s j\Omega$ is determined as

$$X_{s(j\Omega)} = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j(\Omega - k\Omega_0)) \tag{94}$$

which is the sum of infinite copies of $X(j\Omega)$ scaled by $1/T$.

Sampling Theorem

Let $x(t)$ be a bandlimited continuous-time signal with

$$X(j\Omega) = 0, \qquad |\Omega| \geq \Omega_b \tag{95}$$

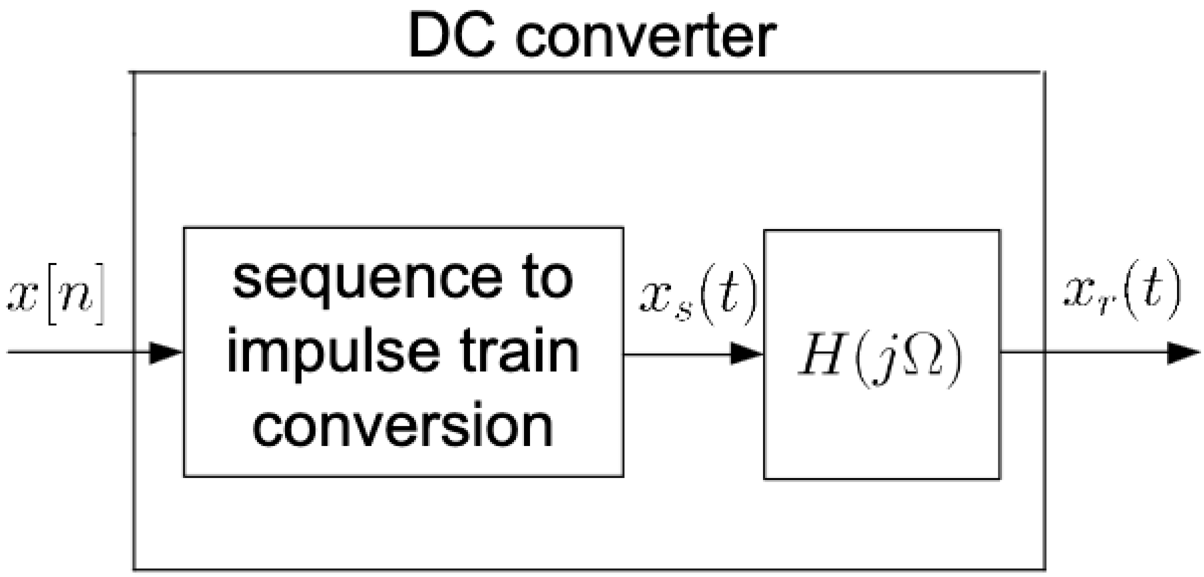
Then $x(t)$ is uniquely determined by its samples $x[n] = x(nT)$, $n = \cdots - 1, 0, 1, 2 \cdots$, if

$$\Omega_s = \frac{2\pi}{T} > 2\Omega_b \tag{96}$$

Therefore, to avoid aliasing, the sampling frequency must exceed $2\Omega_b$.

Reconstruction

It is a process of transforming $x[n]$ back to $x(t)$ via a discrete-time to continuous-time (DC) converter.



Therefore, the requirements of $H(j\Omega)$ are

$$H(j\Omega) = \begin{cases} T, & -\Omega_c < \Omega < \Omega_c \\ 0, & \text{otherwise} \end{cases} \tag{97}$$

where $\Omega_b < \Omega_c < \Omega_s - \Omega_b$, which is a lowpass filter.

Set Ω_c as the average of Ω_b and $(\Omega_s - \Omega_b)$, as follows

$$\Omega_c = \frac{\Omega_s}{2} = \frac{\pi}{T} \tag{98}$$

z Transform

The z transform of $x[n]$, denoted by $X(z)$, is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \tag{99}$$

where z is a continuous complex variable.

z can be expressed as

$$z = re^{j\omega} \tag{100}$$

where $r = |z| > 0$ is magnitude and $\omega = \angle(z)$ is angle of z.

From (100), z transform can be written as

$$X(z)|_{z=re^{j\omega}} = X(rej\omega) = \sum_{n=-\infty}^{\infty} (x[n]r^{-n})e^{-j\omega n} \tag{101}$$

Region of Convergence (ROC)

ROC indicates when z transform of a sequence converges.

Generally there exists some z such that

$$|X(z)| = \left| \sum_{n=-\infty}^{\infty} x[n]z^{-n} \right| \to \infty \tag{102}$$

where the z transform does not converge

The set of values of z for which $X(z)$ converges, as

$$|X(z)| = \left| \sum_{n=-\infty}^{\infty} x[n]z^{-n} \right| \leq \sum_{n=-\infty}^{\infty} |x[n]z^{-n}| < \infty \tag{103}$$

It is called the ROC, which must be specified along with $X(z)$ in order for the z transform to be complete. If there is no value of z satisfies (103), then z transform does not exists.

Poles and Zeros

Values of z for which $X(z) = 0$ are the zeros of $X(z)$.

Values of z for which $X(z) = \pm\infty$ are the poles of $X(z)$.

Finite-Duration and Infinite-Duration Sequences

Finite-duration sequence is that the values of $x[n]$ are nonzero only for a finite time interval.

Otherwise, $x[n]$ is called an infinite-duration sequence, with three variants,

1. If $x[n] = 0$ for $n < N_+ < \infty$ where N_+ is an integer, then it is right sided.
2. If $x[n] = 0$ for $n > N_- > -\infty$ where N_- is an integer, then it is left-sided.
3. If it is neither right-sided nor left-sided, then it is two-sided.

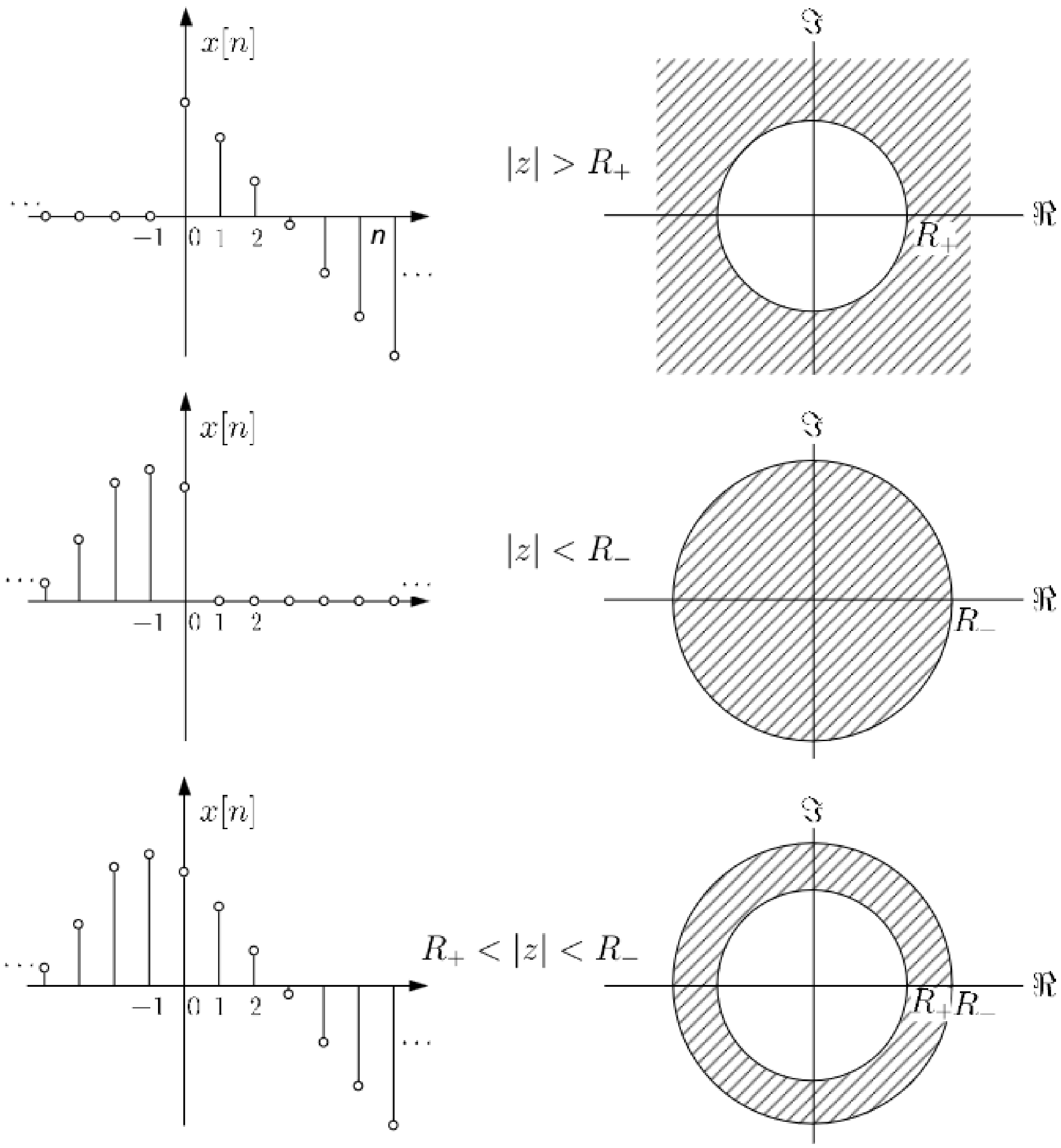
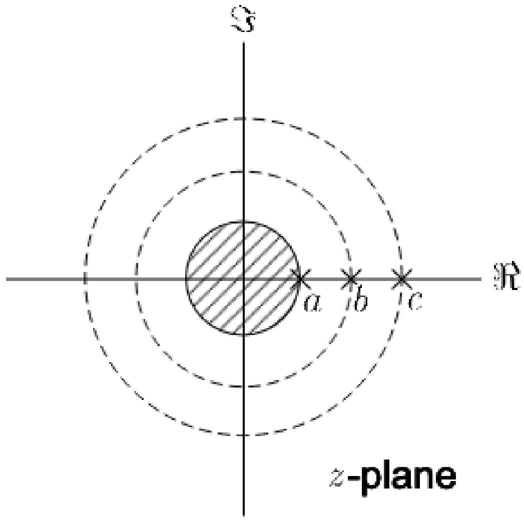


Table of z Transform

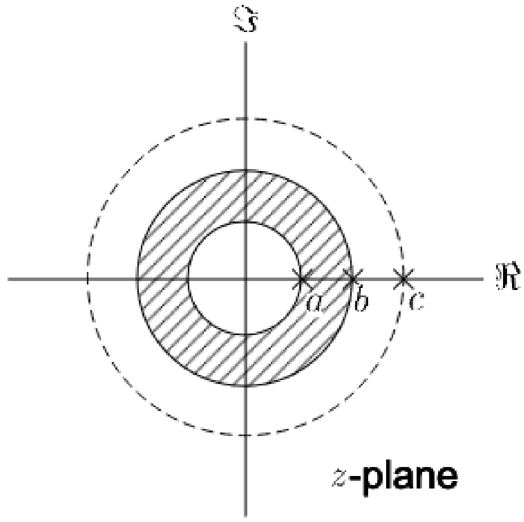
Sequence	Transform	ROC
$\delta[n]$	1	All z
$\delta[n-m]$	z^{-m}	$\lvert z\rvert > 0$, $m > 0$; $\lvert z\rvert < \infty$, $m < 0$
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
$a^n \cos(bn)u[n]$	$\frac{1-a\cos(b)z^{-1}}{1-2a\cos(b)z^{-1}+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
$a^n \sin(bn)u[n]$	$\frac{a\sin(b)z^{-1}}{1-2a\cos(b)z^{-1}+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$

Summary of ROC Properties

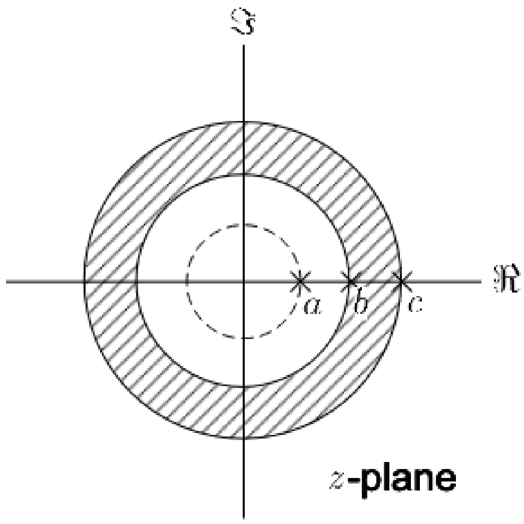
1. There are four possible shapes for ROC, namely the entire region except possibly $z=0$ and/or $z=\infty$, a ring, or inside or outside a circle in the z plane centred at origin.
2. The DTFT of a sequence $x[n]$ exists iff the ROC of z transform of $x[n]$ includes the unit circle.
3. The ROC cannot contain any poles.
4. When $x[n]$ is a finite-duration sequence, the ROC is the entire z plane except possibly $z=0$ and/or $z=\infty$.
5. When $x[n]$ is a right-sided sequence, the ROC is of the form $|z| > |p_{\max}|$ where $p_{\max}$ is the pole with the largest magnitude in $X(z)$.
6. When $x[n]$ is a left-sided sequence, the ROC is of the form $|z| < |p_{\min}|$ where $p_{\min}$ is the pole with the smallest magnitude in $X(z)$.
7. When $x[n]$ is a a two-sided sequence, the ROC is of the form $|p_a| < z < |p_b|$.
8. The ROC must be a connected region.



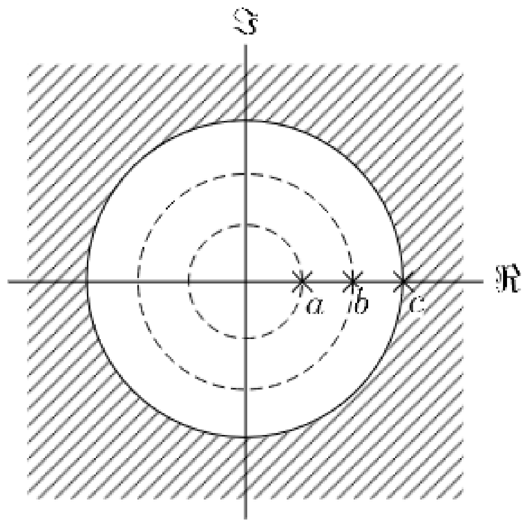
$|z| < |a|$



$|a| < |z| < |b|$



$|b| < |z| < |c|$



$|z| > |c|$

Properties of z Transform

Linearity

Let $x_1[n] \leftrightarrow X_1(z)$ and $x_2[n] \leftrightarrow X_2(z)$ be two z transform pairs with ROCs $\mathcal{R}_{x_1}$ and $\mathcal{R}_{x_2}$, respectively, it is defined as

$$ax_1[n] + bx_2[n] \leftrightarrow aX_1(z) + bX_2(z) \tag{104}$$

Its ROC is denoted by $\mathcal{R}$, which includes $\mathcal{R}_{x_1} \cap \mathcal{R}_{x_2}$, that is $\mathcal{R}$ contains at least the intersection of $\mathcal{R}_{x_1}$ and $\mathcal{R}_{x_2}$.

Time Shifting

A time-shift of n_0 in $x[n]$ causes a multiplication of z^{-n_0} in $X(z)$, it is defined as

$$x[n-n_0] \leftrightarrow z^{-n_0}X(z) \tag{105}$$

Multiplication by an Exponential Sequence

If multiply $x[n]$ by z_0^n in the time domain, the variable z will be changed to z/z_0 in the z transform domain, it is defined as

$$z_0^n x[n] \leftrightarrow X(z/z_0) \tag{106}$$

If the ROC for $x[n]$ is $R_+ < |z| < R_-$, then the ROC for $z_0^n x[n]$ is $|z_0|R_+ < |z| < |z_0|R_-$.

Differentiation

Differentiating $X(z)$ w.r.t. z corresponds to multiply $x[n]$ by n in the time domain, it is defined as

$$nx[n] \leftrightarrow -z\frac{dX(z)}{dz} \tag{107}$$

Conjugation

The z transform pair for $x^*[n]$ is defined as

$$x^*[n] \leftrightarrow X^*(z^*) \tag{108}$$

Time Reversal

The z transform pair for $x[-n]$ is defined as

$$x[-n] \leftrightarrow X(z^{-1}) \tag{109}$$

If the ROC for $x[n]$ is $R_+ < |z| < R_-$, then the ROC for $x[-n]$ is $1/R_- < |z| < 1/R_+$.

Convolution

Let $x_1[n] \leftrightarrow X_1(z)$ and $x_2[n] \leftrightarrow X_2(z)$ be two z transform pairs with ROCs $\mathcal{R}_{x_1}$ and $\mathcal{R}_{x_2}$, respectively, it is defined as

$$x_1[n] \circledast x_2[n] \leftrightarrow X_1(z)X_2(z) \tag{110}$$

and its ROC includes $\mathcal{R}_{x_1} \cap \mathcal{R}_{x_1}$.

Causality and Stability Investigation with ROC

The causality condition is same with (43) as

$$h[n] = 0, \qquad n < 0 \tag{111}$$

If the system is causal and $h[n]$ is of finite duration, the ROC should include ∞.

If the system is causal and $h[n]$ is of infinite duration, the ROC is of the form $|z| > |p_{\max}|$ and should include ∞.

The stability condition is same with (44) as

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty \tag{112}$$

Inverse z Transform

The z transform and inverse z transform are one-to-one mapping as

$$x[n] \leftrightarrow X(z) \tag{113}$$

Partial Fraction Expansion

Consider $X(z)$ is a rational function in z^{-1}, as

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \tag{114}$$

Determine the N nonzero poles, $c_1, c_2, \ldots, c_N$.

If $M < N$ and all poles are of first order,

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - c_k z^{-1}} \tag{115}$$

$$A_k = (1 - c_k z^{-1}) X(z)|_{z=c_k} \tag{116}$$

Perform inverse z transform for the fraction by inspection.

If $M \geq N$ and all poles are of first order,

$$X(z) = \sum_{l=0}^{M-N} B_l z^{-l} + \sum_{k=1}^{N} \frac{A_k}{1 - c_k z^{-1}} \tag{117}$$

B_l are obtained by long division of the numerator by the denominator, such as

$$X(z) = \frac{z^{-2} - 2z^{-1} + 4}{0.5z^{-2} - 1.5z^{-1} + 1}, \qquad |z| > 1 \tag{118}$$

Then, B_l can be found as

$$\begin{array}{r} 2 \\ 0.5z^{-2} - 1.5z^{-1} + 1 \overline{\smash{)}\, z^{-2} - 2z^{-1} + 4} \\ \underline{z^{-2} - 3z^{-1} + 2} \\ z^{-1} + 2 \end{array}$$

After finding B_l, using (116) to find A_k.

If $M < N$ with multiple-order poles

If $X(z)$ has a s order pole at $z = c_i$ with $s \geq 2$, then it is defined as

$$X(z) = \sum_{k=1, k \neq i}^{N} \frac{A_k}{1 - c_k z^{-1}} + \sum_{m=1}^{s} \frac{C_m}{(1 - c_i z^{-1})^m} \tag{119}$$

C_m can be computed as

$$C_m = \frac{1}{(s-m)!(-c_i)^{s-m}} \cdot \frac{d^{s-m}}{dw^{s-m}} [(1 - c_i w)^s X(w^{-1})] \Big|_{w = c_i^{-1}} \tag{120}$$

A_k can be found by using (116).

If $M \geq N$ with multiple order poles

$$X(z) = \sum_{l=0}^{M-N} B_l z^{-1} + \sum_{k=1, k \neq i}^{N} \frac{A_k}{1 - c_k z^{-1}} + \sum_{m=1}^{s} \frac{C_m}{(1 - c_i z^{-1})^m} \tag{121}$$

Transfer Function $H(z)$ of Linear Time-Invariant System

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \tag{122}$$

Applying z transform on (122) with the use of the linearity and time shifting properties, it is defined as

$$Y(z)\sum_{k=0}^{N} a_k z^{-k} = X(z)\sum_{k=0}^{M} b_k z^{-k} \tag{123}$$

Then, the transfer function, denoted by $H(z)$, is defined as

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \tag{124}$$

Laplace Transform

The Laplace transform of $x(t)$, denoted by $X(s)$, is defined as

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt \tag{125}$$

where s is a continuous complex variable.

To express s as

$$s = \sigma + j\Omega \tag{126}$$

where σ and Ω are the real and imaginary parts of s respectively.

According to (126), the Laplace transform can be written as

$$X(\sigma + j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\Omega)t}\,dt = \int_{-\infty}^{\infty} (x(t)e^{-\sigma t})e^{-j\Omega t}\,dt \tag{127}$$

Region of Convergence (ROC)

ROC indicates when Laplace transform of $x(t)$ converges, that is if

$$|X(s)| = \left|\int_{-\infty}^{\infty} x(t)e^{-st}\,dt\right| \to \infty \tag{128}$$

Then, the Laplace transform does not converge at point s.

Therefore, the Laplace transform exists if

$$|X(\sigma + j\Omega)| \leq \int_{-\infty}^{\infty} \left| x(t)e^{-(\sigma+j\Omega)t} \right| dt = \int_{-\infty}^{\infty} \left| x(t)e^{-\sigma t} \right| dt < \infty \tag{129}$$

The set of values of σ which satisfies (129) is called the ROC.

Poles and Zeros

Values of s for which $X(s) = 0$ are the zeros of $X(s)$.

Values of s for which $X(s) = \pm\infty$ are the poles of $X(s)$.

Finite-Duration and Infinite-Duration Signals

Finite-Duration Singal

If the values of $x(t)$ are nonzero only for a finite time interval, then it is a finite-duration signal. If $x(t)$ is absolutely integrable, then the ROC of $X(s)$ is the entire s plane.

That is,

$$x(t) = \begin{cases} \text{nonzero}, & T_1 < t < T_2 \\ 0, & \text{otherwise} \end{cases} \tag{130}$$

It is also absolutely integrable,

$$\int_{-\infty}^{\infty} |x(t)| \, dt = \int_{T_1}^{T_2} |x(t)| \, dt < \infty \tag{131}$$

Infinite-Duration Signal

It $x(t)$ is not finite-duration, then it is an infinite-duration signal.

1. If $x(t) = 0$ for $t < T_1 < \infty$, then it is right-sided.
2. If $x(t) = 0$ for $t > T_2 > -\infty$, then it is left-sided.
3. If it is neither right-sided nor left sided, then it is two-sided.

Table of Laplace Transforms

Signal	Transform	ROC
$\delta(t)$	1	All s
$\delta(t-T)$	e^{-sT}	All s
$e^{-at}u(t)$	$\frac{1}{s+a}$	$\Re\{s\} > -a$
$-e^{-at}u(-t)$	$\frac{1}{s+a}$	$\Re\{s\} < -a$
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$	$\frac{1}{(s+a)^n}$	$\Re\{s\} > -a$
$-\frac{t^{n-1}}{(n-1)!}e^{-at}u(-t)$	$\frac{1}{(s+a)^n}$	$\Re\{s\} < -a$
$e^{-at}\cos(bt)u(t)$	$\frac{s+a}{(s+a)^2+b^2}$	$\Re\{s\} > -a$
$e^{-at}\sin(bt)u(t)$	$\frac{b}{(s+a)^2+b^2}$	$\Re\{s\} > -a$

Summary of ROC Properties

1. The ROC of $X(s)$ consists of a region parallel to the $j\Omega$ axis in the s plane. There are four possible cases, namely, the entire region, right-half plane (region includes ∞), left-half plane (region includes $-\infty$) and single strip (region bounded by two poles).
2. The Fourier transform of a signal $x(t)$ exists iff the ROC of the Laplace transform of $x(t)$ includes the $j\Omega$ axis.
3. For a rational $X(s)$, its ROC cannot contain any poles.
4. When $x(t)$ is finite-duration and absolutely integrable, the ROC is the entire s plane.
5. When $x(t)$ is right-sided, the ROC is the right-half plane to the right of the rightmost pole.
6. When $x(t)$ is left-sided, the ROC is left-half plane to the left of the leftmost pole.
7. When $x(t)$ is two-sided, the ROC is the form $\Re\{p_a\} > \Re\{s\} > \Re\{p_b\}$ where p_a and p_b are two poles of $X(s)$ with the successive values in real part.
8. The ROC must be a connected region.

Properties of Laplace Transform

Linearity

Let $x_1(t) \leftrightarrow X_1(s)$ and $x_2(t) \leftrightarrow X_2(s)$ be two Laplace transform pairs with ROCs $\mathcal{R}_{x_1}$ and $\mathcal{R}_{x_2}$ respectively, it is defined as

$$ax_1(t) + bx_2(t) \leftrightarrow aX_1(s) + bX_2(s) \tag{132}$$

Its ROC is denoted by $\mathcal{R}$, which contains at least the intersection of $\mathcal{R}_{x_1}$ and $\mathcal{R}_{x_2}$.

Time Shifting

A time-shift of t_0 in $x(t)$ causes a multiplication of e^{-st_0} in $X(s)$, that is

$$x(t) \leftrightarrow X(s) \implies x(t - t_0) \leftrightarrow e^{st_0}X(s) \tag{133}$$

The ROC for $x(t - t_0)$ is identical to $X(s)$.

Multiplication by an Exponential Signal

$$x(t) \leftrightarrow X(s) \implies e^{s_0 t}x(t) \leftrightarrow X(s - s_0) \tag{134}$$

If the ROC for $x(t)$ is $\mathcal{R}$, then the ROC for $e^{s_0 t}x(t)$ is $\mathcal{R} + \Re\{s_0\}$, that is shifted by $\Re\{s_0\}$. If $X(s)$ has a pole (zero) at $s = a$, then $X(s - s_0)$ has a pole (zero) at $s = a + s_0$.

Differentiation in s Domain

Differentiating $X(s)$ w.r.t. s corresponds to multiply $x(t)$ by $-t$ in the time domain, that is,

$$x(t) \leftrightarrow X(s) \implies -tx(t) \leftrightarrow \frac{dX(s)}{ds} \tag{135}$$

The ROC for $tx(t)$ is identical to that of $X(s)$.

Conjugation

The Laplace transform pair for $x^*(t)$ is defined as

$$x(t) \leftrightarrow X(s) \implies x^*(t) \leftrightarrow X^*(s^*) \tag{136}$$

The ROC for $x^*(t)$ is identical to $X(s)$.

Time Reversal

The Laplace transform pair for $x(-t)$ is defined as

$$x(t) \leftrightarrow X(s) \implies x(-t) \leftrightarrow X(-s) \tag{137}$$

The ROC will be reversed too.

Convolution

Let $x_1(t) \leftrightarrow X_1(s)$ and $x_2(t) \leftrightarrow X_2(s)$ be two Laplace transform pairs with ROCs $\mathcal{R}_{x_1}$ and $\mathcal{R}_{x_2}$ respectively, it is defined as

$$x_1(t) \circledast x_2(t) \leftrightarrow X_1(s)X_2(s) \tag{138}$$

and its ROC includes $\mathcal{R}_{x_1} \cap \mathcal{R}_{x_2}$.

Differentiation in Time Domain

Differentiating $x(t)$ w.r.t. t corresponds to multiply $X(s)$ by s in the s domain, that is defined as

$$x(t) \leftrightarrow X(s) \implies \frac{dx(t)}{dt} \leftrightarrow sX(s) \tag{139}$$

Its ROC includes the ROC for $x(t)$.

Integration

$$x(t) \leftrightarrow X(s) \implies \int_{-\infty}^{t} x(\tau)\, d\tau \leftrightarrow \frac{1}{s}X(s) \tag{140}$$

If the ROC for $x(t)$ is $\mathcal{R}$, then the ROC for $\int_{-\infty}^{t} x(\tau)\, d\tau$ includes $\mathcal{R} \cap \{\Re\{s\} > 0\}$.

Causality and Stability Investigation with ROC

The causality condition is same with (43), which is

$$h(t) = 0, \qquad t < 0 \tag{141}$$

If the system is causal and $h(t)$ is of infinite duration, the ROC must be the right-half plane. If $H(s)$ is rational and its ROC is the right-half plane, then the system must be causal.

Inverse Laplace Transform

Partial Fraction Expansion

Consider $X(s)$ is a rational function in s, that is

$$X(s) = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k} \tag{142}$$

To obtain the partial fraction expansion, first to determine N nonzero poles, $c_1, c_2, \ldots, c_N$.

If $M < N$ and all poles are first order,

$$X(s) = \sum_{k=1}^{N} \frac{A_k}{s - c_k} \tag{143}$$

and A_k can be computed as

$$A_k = (s - c_k)X(s)|_{s=c_k} \tag{144}$$

If $M \geq N$ and all poles are first order,

$$X(s) = \sum_{l=0}^{M-N} B_l s^l + \sum_{k=1}^{N} \frac{A_k}{s - c_k} \tag{145}$$

and B_l are obtained by long division of the numerator by the denominator, A_k can be obtained using (144).

If $M < N$ with multiple-order poles,

Assuming that $X(s)$ has a r order pole at $s = c_i$, with $r \geq 2$.

$$X(s) = \sum_{k=1, k \neq i}^{N} \frac{A_k}{s - c_k} + \sum_{m=1}^{r} \frac{C_m}{(s - c_i)^m} \tag{146}$$

and C_m can be computed as

$$C_m = \frac{1}{(r-m)!} \cdot \frac{d^{r-m}}{ds^{r-m}} [(s - c_i)^r X(s)] \bigg|_{s=c_i} \tag{147}$$

If $M \geq N$ with multiple-order poles,

$$X(s) = \sum_{l=0}^{M-N} B_l s^l + \sum_{k=1, k\neq i}^{N} \frac{A_k}{s - c_k} + \sum_{m=1}^{r} \frac{C_m}{(s - c_i)^m} \tag{148}$$

Transfer Function of LTI System

The differential equation is defined as

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \tag{149}$$

Applying Laplace transform with the use of linearity property, it can be defined as

$$Y(s) \sum_{k=0}^{N} a_k s^k = X(s) \sum_{k=0}^{M} b_k s^k \tag{150}$$

Therefore, the transfer function, denoted by $H(s)$ is defined as

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k} \tag{151}$$

Miscellaneous

Geometric Series Formulas

$\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}$	$\sum_{k=0}^{N} a^k = \frac{1-a^{N+1}}{1-a}$
$\sum_{k=1}^{\infty} a^k = \frac{a}{1-a}$	$\sum_{k=1}^{N} a^k = \frac{a(1-a^{N+1})}{1-a}$
$\sum_{k=N_1}^{N_2} a^k = \frac{a^{N_1}-a^{N_2+1}}{1-a}$	$\sum_{k=1}^{N} k = \frac{N(N+1)}{2}$

Changing Subject on Summation

$$X(z) = \sum_{n=-\infty}^{-1} -a^n z^{-n} = -\sum_{m=1}^{\infty} a^{-m} z^m = -\sum_{m=1}^{\infty} (a^{-1} z)^m$$

LTI System

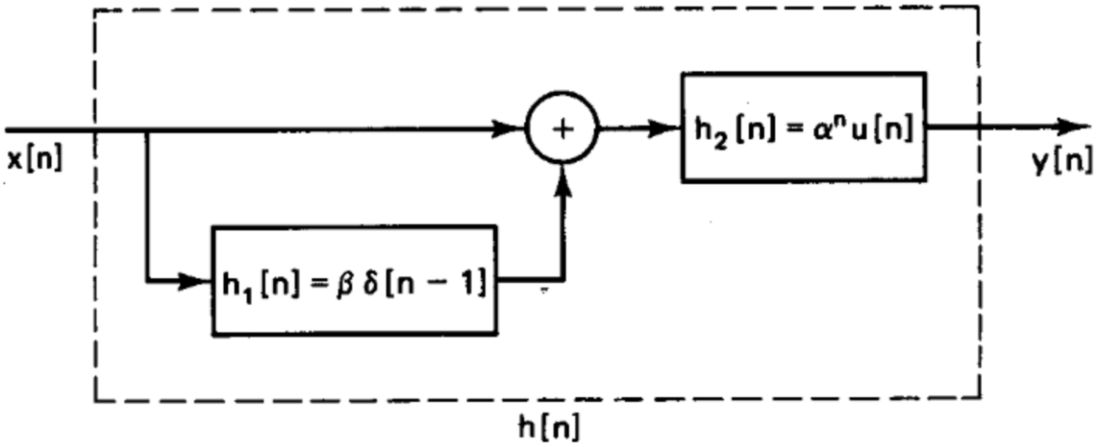


Figure 1

$$
\begin{aligned}
y[n] &= (x[n] + x[n] \otimes h_1[n]) \otimes h_2[n] \\
&= (x[n] \otimes \delta[n] + x[n] \otimes h_1[n]) \otimes h_2[n] \\
&= x[n] \otimes ([\delta[n] + h_1[n]]) \otimes h_2[n]
\end{aligned}
$$