Solutions to EE3210 Tutorial 10 Problems

Problem 1:

(a) Using the analysis formula of the continuous-time Fourier transform, we have

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt = \int_{-\infty}^{+\infty} \sum_{k=0}^{+\infty} \alpha^k \delta(t-kT)e^{-j\omega t}dt$$

$$= \sum_{k=0}^{+\infty} \alpha^k \int_{-\infty}^{+\infty} \delta(t-kT)e^{-j\omega t}dt = \sum_{k=0}^{+\infty} \alpha^k e^{-jk\omega T} = \sum_{k=0}^{+\infty} \left(\alpha e^{-j\omega T}\right)^k$$

$$= \frac{1}{1-\alpha e^{-j\omega T}}.$$

(b) Using the analysis formula of the discrete-time Fourier transform, we have

$$X[\Omega] = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\Omega n} = \sum_{n=-\infty}^{+\infty} 4^n u[-n]e^{-j\Omega n} = \sum_{l=-\infty}^{+\infty} 4^{-l}u[l]e^{j\Omega l} = \sum_{l=0}^{+\infty} \left(\frac{1}{4}e^{j\Omega}\right)^l$$

$$= \frac{1}{1-\frac{1}{4}e^{j\Omega}}.$$

Problem 2:

(a) Using the time reversal property, we have

$$x(-t) \leftrightarrow X(-\omega).$$

Then, using the time shift property, we have

$$x[-(t-1)] \leftrightarrow e^{-j\omega}X(-\omega)$$

and

$$x[-(t+1)] \leftrightarrow e^{j\omega}X(-\omega).$$

Therefore, using the linearity property, we obtain

$$x(1-t) + x(-1-t) \leftrightarrow e^{-j\omega}X(-\omega) + e^{j\omega}X(-\omega) = 2\cos\omega X(-\omega).$$

(b) Using the time shift property, we have

$$x(t-6) \leftrightarrow e^{-j6\omega}X(\omega).$$

Then, using the time scaling property, we have

$$x(3t-6) \leftrightarrow \frac{1}{3}e^{-j2\omega}X\left(\frac{\omega}{3}\right).$$

(c) Using the differentiation in time property, we have

$$\frac{dx(t)}{dt} \leftrightarrow (j\omega)X(\omega).$$

Applying this property again, we have

$$\frac{d^2x(t)}{dt^2} \leftrightarrow (j\omega)^2X(\omega) = -\omega^2X(\omega).$$

Finally, using the time shift property, we have

$$\frac{d^2x(t-1)}{dt^2} \leftrightarrow -\omega^2e^{-j\omega}X(\omega).$$

Problem 3:

(a) Using the synthesis formula of the continuous-time Fourier transform, we have

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)e^{j\omega t}d\omega = \int_{-\infty}^{+\infty} \delta(\omega-\omega_0)e^{j\omega t}d\omega = e^{j\omega_0 t}.$$

Remark. In this case, $x(t) = e^{j\omega_0 t}$ is in fact a periodic signal. Although continuous-time periodic signals in general do not satisfy the absolutely integrable condition, i.e.,

$$\int_{-\infty}^{+\infty} |x(t)|dt < \infty, \tag{1}$$

they have Fourier transforms if impulse functions such as $\delta(\omega-\omega_0)$ are permitted in the transform. This implies that (1) is a sufficient but not necessary condition for the existence of Fourier transforms. ■

(b) Using the synthesis formula of the continuous-time Fourier transform, we have

$$\begin{aligned}
x(t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)e^{j\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} 2[\delta(\omega-1)-\delta(\omega+1)]e^{j\omega t}d\omega + \frac{1}{2\pi}\int_{-\infty}^{+\infty} 3[\delta(\omega-2\pi)+\delta(\omega+2\pi)]e^{j\omega t}d\omega \\
&= \frac{1}{\pi}\left(e^{jt}-e^{-jt}\right) + \frac{3}{2\pi}\left(e^{j2\pi t}+e^{-j2\pi t}\right) \\
&= \frac{3}{\pi}\cos(2\pi t) + j\frac{2}{\pi}\sin t.
\end{aligned}$$