Unit 1

Sets

Albert Sung

The Barber Paradox

- The barber is a man in town who shaves those and only those men who do not shave themselves.
- *Q:* Who shaves the barber?

The Barber Paradox

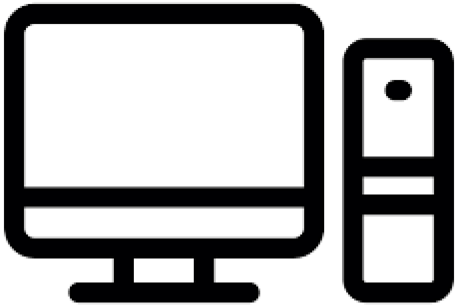
- (1-min video) https://www.youtube.com/watch?v=qQs2ZHV_WBk

The Halting Problem

```
i = 1
while i != 10:
      i += 2
print('Hello world!')
```

```
i = 2
while i != 10:
      i += 2
print('Hello world!')
```

Can you write a program to check whether any given program will halt or not?



Outline of Unit 1

- 1.1 Basic Concepts
- 1.2 Proofs Involving Sets
- 1.3 Functions
- 1.4 Russell's Paradox
- 1.5 The Halting Problem

Unit 1.1

Basic Concepts

Sets

- A set is a collection of objects.
- A is a subset of B, written as $A \subseteq B$, if every member of A is also a member of B.
  - It is a proper subset of B if B contains some elements that are not in A.
    - i.e., A is not the same as B.
- B is then said to be a superset of A.
- The cardinality of a set A is defined as the number of elements in the set.
- It is denoted by $|A|$.
  - If $|A|$ is finite, A is called a finite set.
  - Otherwise, A is called an infinite set.

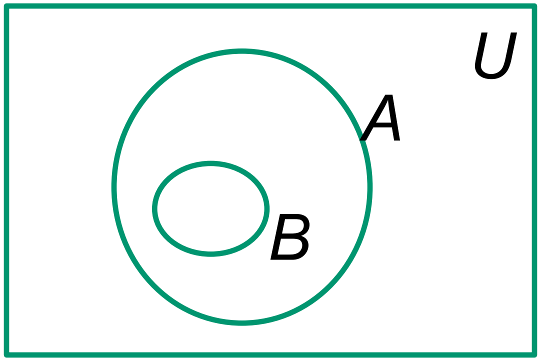
Some Common Sets in Math

Set	Symbols
Natural Numbers*	$\mathbb{N} = \{1, 2, 3, \dots\}$
Whole Numbers	$\mathbb{N} \cup \{0\}$ or $\mathbb{Z}_{\geq 0}$
Integers	$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
Binary Numbers	$\mathbb{B} = \{0, 1\}$
Rational Numbers	$\mathbb{Q}$
Real Numbers	$\mathbb{R}$
Complex Numbers	$\mathbb{C}$

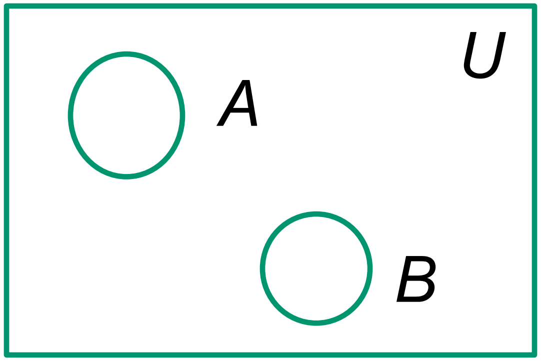
*In some convention, 0 is included in the set of natural numbers.

Relationship between Sets

- A universal set U is a set containing everything that we are considering.
- Venn diagram
  - U is represented by a rectangular box.
  - Subsets of U (e. g. A and B) are represented by circles (more precisely, regions inside closed curves).
- A and B are disjoint if they have no elements in common.

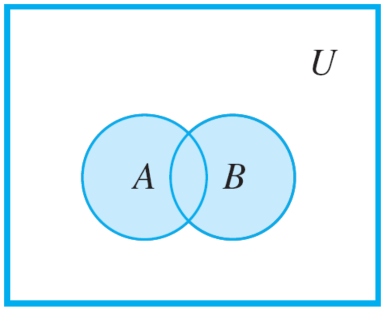


B is a subset of A.

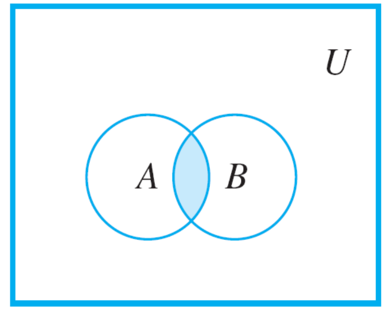


A and B are disjoint.

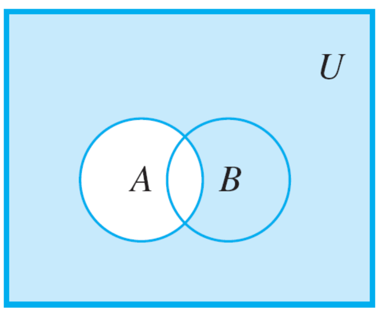
Fundamental Operations



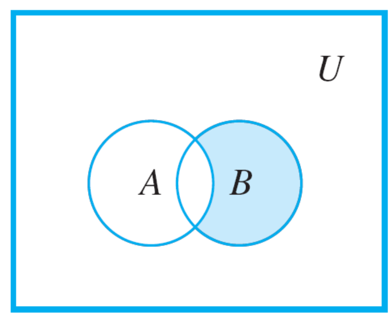
Union: $A \cup B$



Intersection: $A \cap B$



Complement: A^c or $\bar{A}$



Difference: $B \setminus A$

Mutually Disjoint Sets

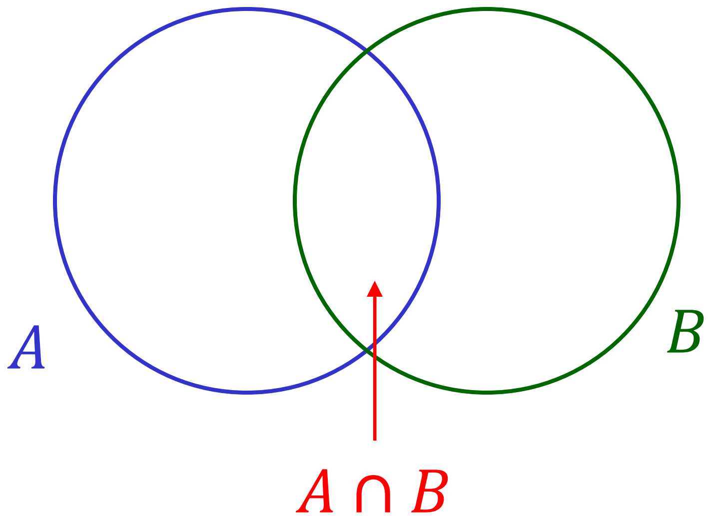
- The n sets, $A_1, A_2, \ldots, A_n$ are mutually disjoint if any two of them have no elements in common, i.e., $A_i \cap A_j = \emptyset$ for all $i, j = 1, 2, \ldots, n$ and $i \neq j$.

- Example: If $A \cap B \cap C = \emptyset$, are A, B and C mutually disjoint?

Inclusion-Exclusion Principle

- For finite sets A and B,

$$|A \cup B| = |A| + |B| - |A \cap B|$$



When adding $|A|$ and $|B|$, $A \cap B$ has been counted twice. That's why we need to subtract it.

This result can be generalized to more than two sets (to be discussed in the tutorial).

Classwork

Consider the numbers 1, 2, …, 100.

How many of them are divisible by 2 or by 3?

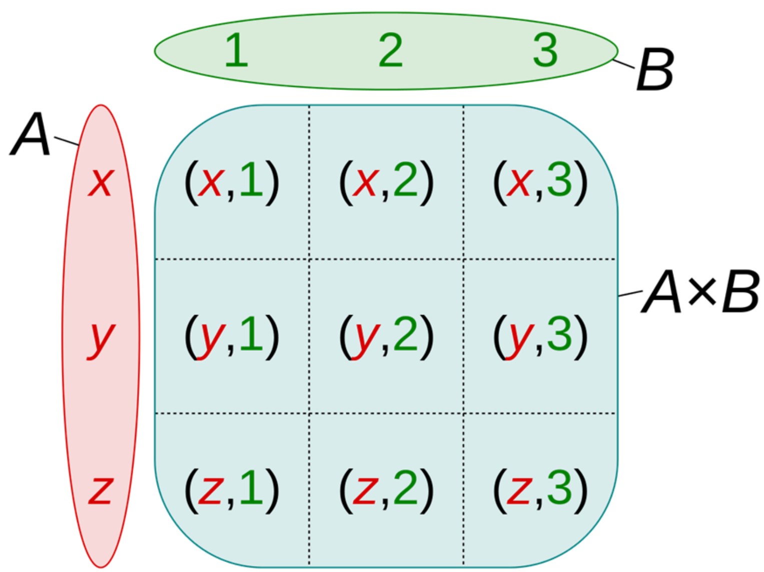
- Solution:

Cartesian Product

- The Cartesian product $A \times B$ of the sets A and B is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$.

$$A \times B \triangleq \{(a, b) | a \in A \ \wedge b \in B\}.$$

- Example:



Ordered pair:
- The order is important: $(a, b) \neq (b, a)$

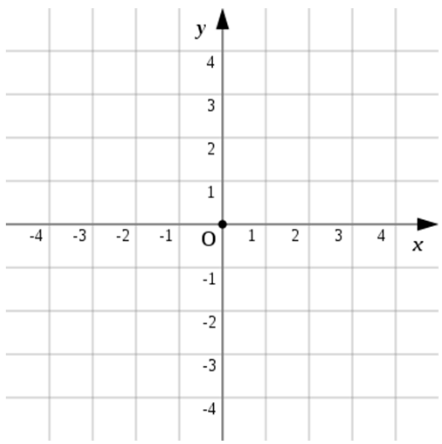
What is $|A \times B|$?

Cartesian Product

- The Cartesian product can be generalized to more than two sets, e.g., $A \times B \times C$.
- If the same set is involved, we write

$$\underbrace{A \times A \times \cdots \times A}_{n} = A^n$$

- For example, the x-y plane is $\mathbb{R}^2$.



Power Set

- Given a set A, the set of all its subsets, denoted by $\mathcal{P}(A)$, is called the power set of A.

- Example:
  - Suppose $A = \{1, 2, 3\}$.
  - List all subsets of A:

    $\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$ and $\{1, 2, 3\}$.
  - Hence,

    $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$.

Cardinality of Power Set

- Suppose $|A| = n$.
- What is $|\mathcal{P}(A)|$?

$A = \{1, \ 2, \ 3, \ldots, \ n\}$.

Subset: $\{ \qquad \qquad \qquad \}$

For each element, there are two possibilities:

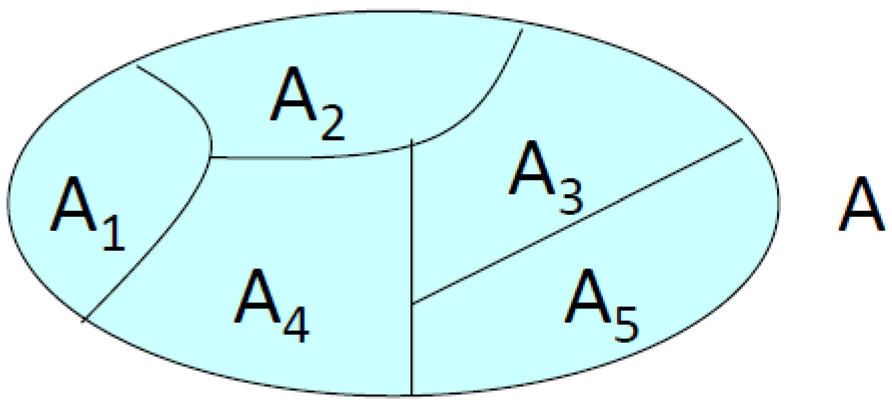
- Put it into the subset, or not.

Partition

- A collection of non-empty sets $\{A_1, A_2, \ldots, A_n\}$ is a partition of a set A iff

i. $A = A_1 \cup A_2 \cup \cdots \cup A_n$, and

ii. $A_1, A_2, \ldots, A_n$ are mutually disjoint.



Note: Partition itself is a set.

Unit 1.2

Proofs Involving Sets

To prove $A \subseteq B$ using element argument

Proposition: $A \subseteq B$.

Proof:

Assume $x \in A$.

< Explain what $x \in A$ means >

$\vdots$

$\vdots$ *apply algebra, logic, …*

$\vdots$

< that's what $x \in B$ means >

Therefore, $x \in B$.

Since $x \in A$ implies $x \in B$, it follows that $A \subseteq B$.

Q.E.D.

Example

- Prove that $A \subseteq B$, where

$$A = \{m \in \mathbb{Z}: m = 6r + 12 \text{ for some } r \in \mathbb{Z}\},$$
$$B = \{n \in \mathbb{Z}: n = 3s \text{ for some } s \in \mathbb{Z}\}.$$

- *Proof:*
  - Assume $x \in A$.
  - Then $x = 6r + 12 = 3(2r + 4)$ for some integer r.
  - Rewrite it as $x = 3s$, where $s = 2r + 4$ is an integer.
  - Therefore, $x \in B$.
  - Since $x \in A$ implies $x \in B$, it follows that $A \subseteq B$.

Q.E.D.

Another Example

- Prove $A \cap B \subseteq A$.

- *Proof*:
  - Assume $x \in A \cap B$.
  - By the definition of intersection, $x \in A$ and $x \in B$.
  - Therefore, $x \in A$ (by simplification rule in propositional logic).

Q.E.D.

Set Equality

- Two sets are the same (or equal) if and only if
  - they contain the same elements, or equivalently,
  - each is a subset of the other.

$$A = B \quad \Leftrightarrow \quad A \subseteq B \text{ and } B \subseteq A.$$

- You need to prove both directions: (i) $A \subseteq B$, and (ii) $B \subseteq A$.

Example

- Prove $A = B \cap C$, where $A = \{n \in \mathbb{Z}: 6|n\}$, $B = \{n \in \mathbb{Z}: 2|n\}$, and $C = \{n \in \mathbb{Z}: 3|n\}$.

  - Note: $a|b$ means a divides b.
  - To prove the above result, we need the following lemma (to be proved in Unit 4).

Euclid's Lemma: If p is prime and $p|ab$, then $p|a$ or $p|b$, for all integers a and b.

  - A lemma is a true statement often used to prove a theorem.

Proof (first part)

Proof:

- (First part) We need to prove that $A \subseteq B \cap C$.
  - Let $x \in A$, which means $6|x$.
  - Then $x = 6r = (2)(3)r$, where r is an integer.
  - Therefore, $2|x$, implying $x \in B$.
  - Similarly, $3|x$, implying $x \in C$.
  - Hence, $x \in B$ and $x \in C$, meaning $x \in B \cap C$.

Proof (second part)

- (Second part) We need to prove that $B \cap C \subseteq A$.
  - Let $x \in B \cap C$, which means $x \in B$ and $x \in C$.
  - Then $x = 2s$ and $x = 3t$ for some integers s and t.
  - Since $2s = 3t$, we have $2|3t$.
  - Since 2 is a prime, by Euclid's Lemma, $2|3$ or $2|t$.
  - Since $2 \nmid 3$, it follows that $2|t$.
  - So, $x = 3t = (3)(2u) = 6u$, where u is an integer.
  - Hence, $x \in A$.

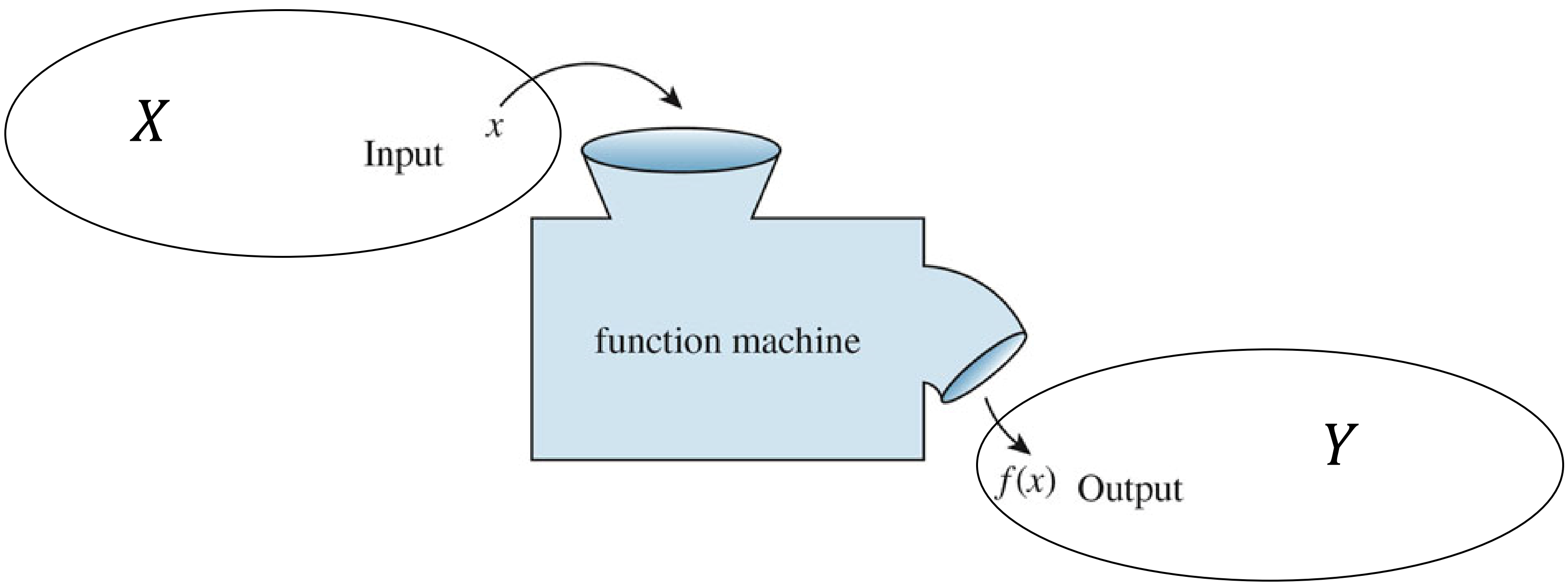
Q.E.D.

Unit 1.3

Functions

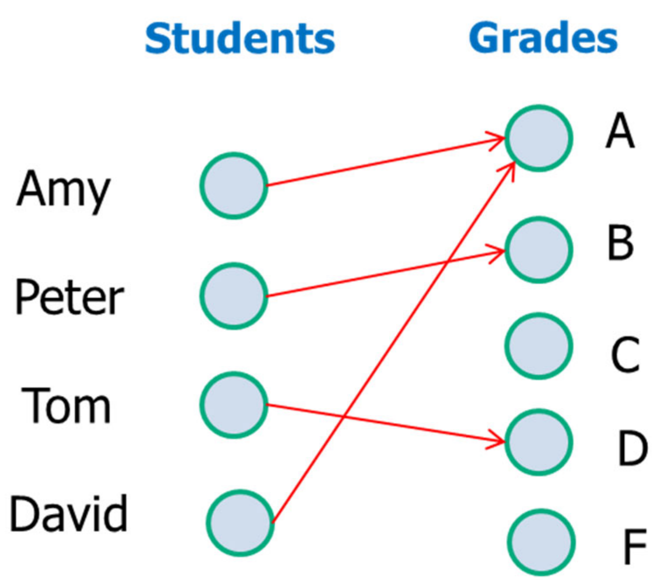
Functions

- A function f from X to Y, denoted by $f : X \rightarrow Y$, (or f maps X to Y) is an assignment of each element of X to exactly one element of Y.
  - X and Y are nonempty sets.



Example

- Consider the Grade Assignment Function f which maps a set of students to a set of grades.
  - f assigns each student exactly one grade.



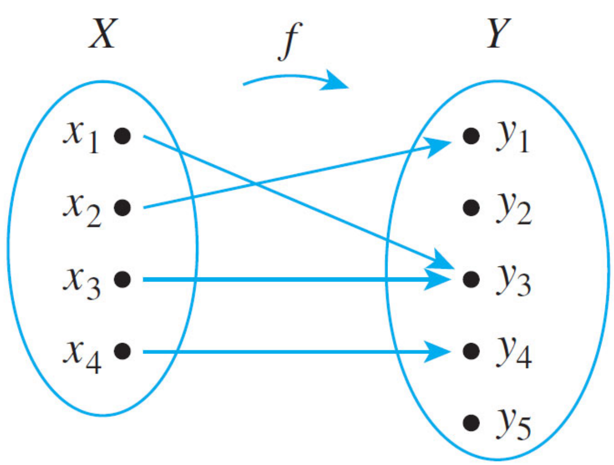
No student is assigned more than one grade.

No student has no grade assigned.

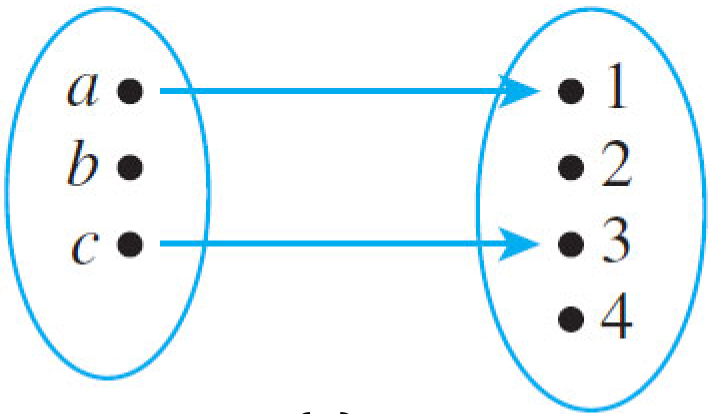
Arrow Diagrams

- A function $f : X \to Y$ can be represented by an arrow diagram.
- An arrow is drawn from each element in X to its corresponding unique element in Y under f.

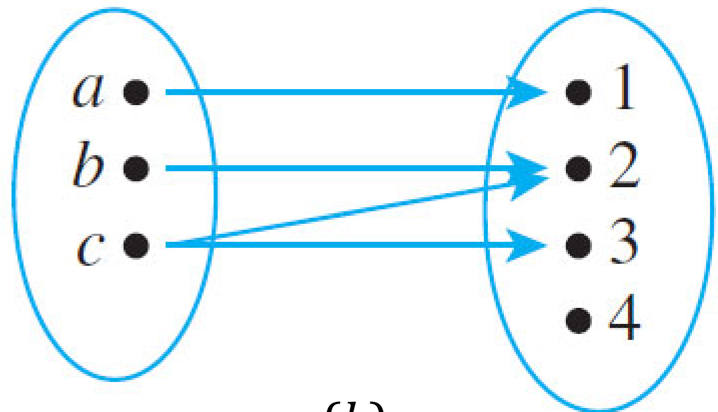
  - Every element in X points to a unique element in Y.
  - No element of X has two arrows coming out of it.



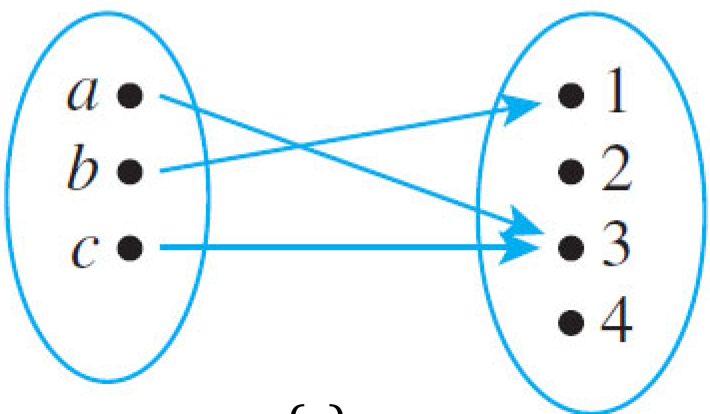
Are They Functions?



(a)



(b)



(c)

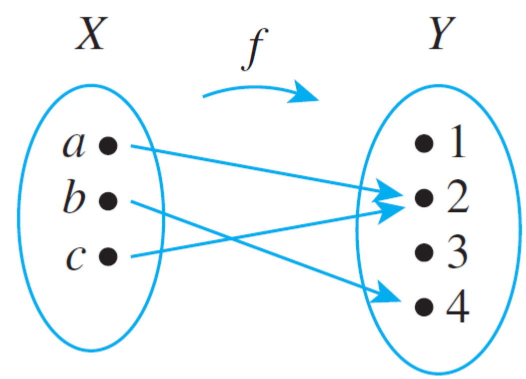
Terminologies

Consider a function $f: X \to Y$.

- X is called the domain of f while Y is called the co-domain of f.
- Given $x \in X$ and $f(x) = y \in Y$, y is called the image of x under f.
- The range of f is the set of images of all elements in X.
  - Note: range $\subseteq$ co-domain.
- Given $y \in Y$, the inverse image of y is the set of all elements $x \in X$ such that $f(x) = y$.

Classwork

a) What are the domain, co-domain and range of f?



b) What is the image of a under f?

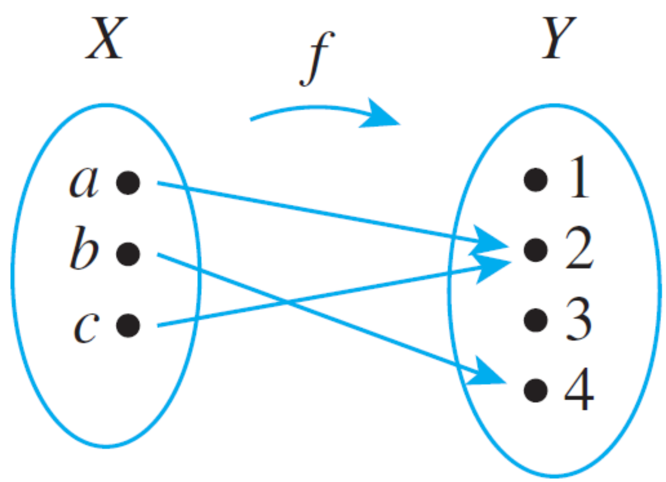
c) What is the inverse image of 2 under f?

d) What is the inverse image of 3 under f?

A Function is a Set

- A function $f: X \rightarrow Y$ is a subset of the Cartesian product between X and Y.

- $X \times Y =$
  $\{(a,1),(a,2),(a,3),(a,4),$
  $(b,1),(b,2),(b,3),(b,4),$
  $(c,1),(c,2),(c,3),(c,4)\}$



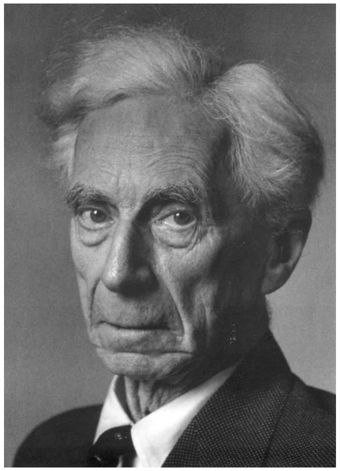
- $f = \{(a,2),(b,4),(c,2)\}$

Unit 1.4

Russell's Paradox

Naïve Set Theory

- In naïve set theory, a set is simply a *collection of objects*.
- Given any *property*, there is a set which contains all objects that have the property.
  - For example, students enrolled in EE2302 this semester form a set.
  - The set is empty if no object has the property.
- Russell found that a paradox arises!



Bertrand Russell (1872-1970), a British philosopher, logician, and writer.

Can $X \in X$?

- Russell's paradox is based on this construction:

$$\boxed{S = \{X \mid X \notin X\}}$$

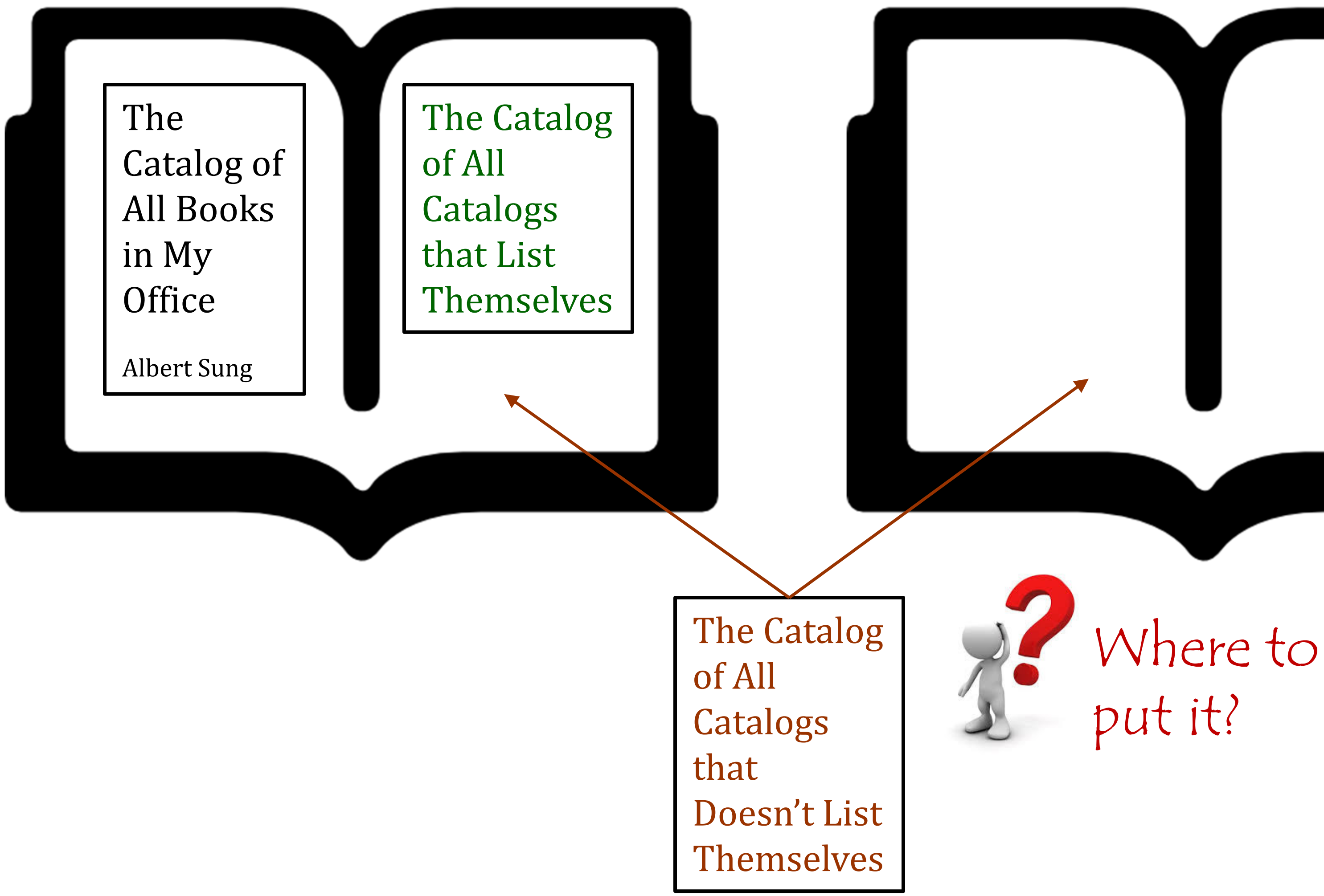
  - Can a set be a member of itself?
  - Or equivalently, can $X \in X$?

Suppose I create a catalog of all books in my office.

This is the catalog.

The Catalog of All Catalogs that List Themselves

The Catalog of All Catalogs that Doesn't List Themselves



Russell's Paradox

- Let S be the set of all sets that are not members of themselves:

$$S = \{X \mid X \notin X\}.$$

Note: X is a set.

- *Q:* Is S an element of itself?
  - i.e., is $S \in S$?
- *A:* Neither yes nor no, because either way leads to a contradiction:
  - Suppose $S \in S$. By the defining property of S, $S \notin S$.
  - Suppose $S \notin S$. By the defining property of S, $S \in S$.

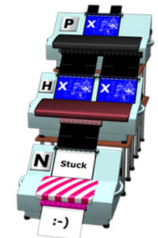
Remarks

- The barber's paradox is a popular version of Russell's paradox.
  - Russell uses it to explain the paradox to layman.
- To resolve Russell's paradox, a set has to be *properly* defined.
- Russell's paradox facilitates the development of axiomatic set theory.
  - There are different ways to do it... (details omitted)

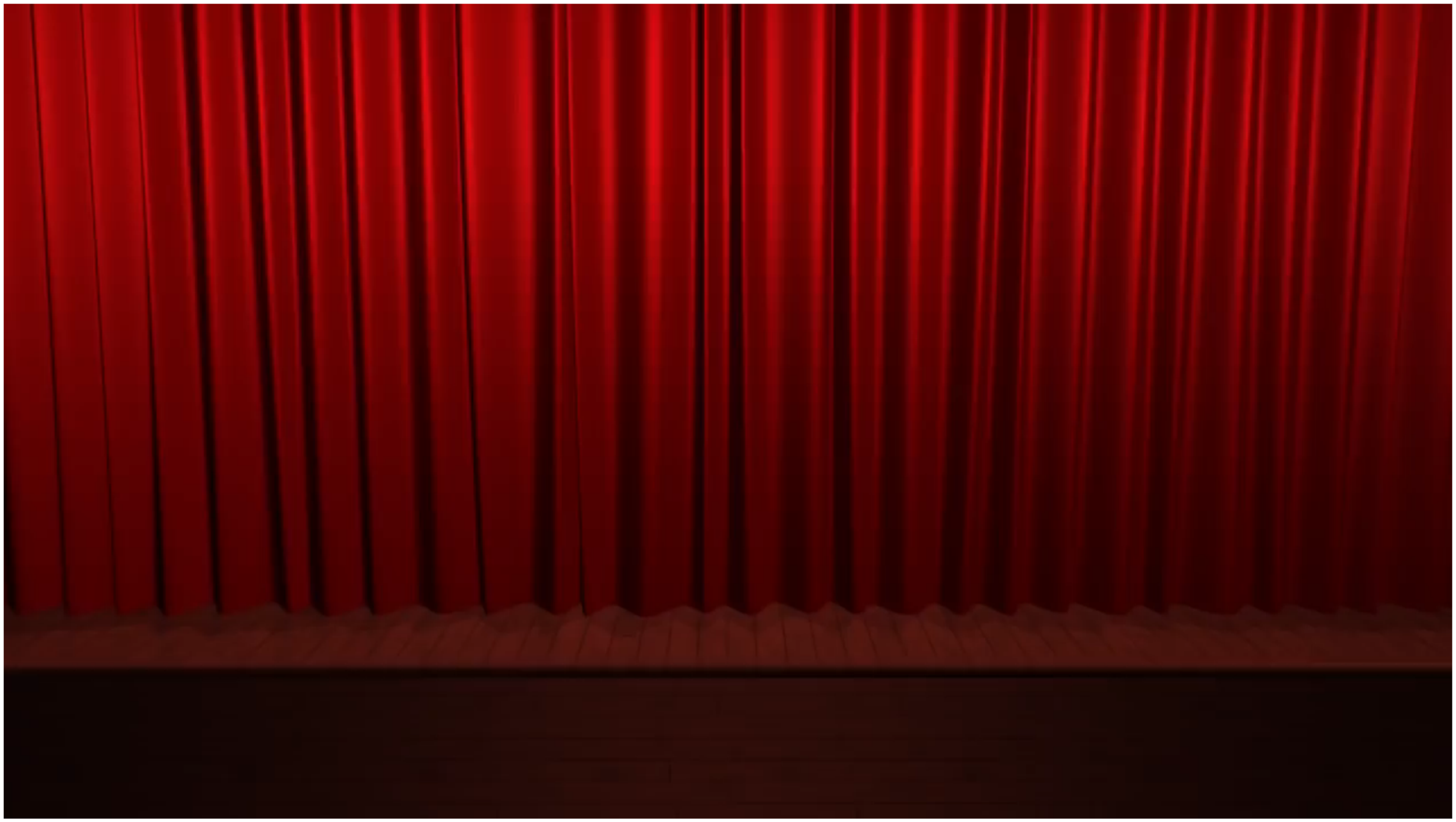
Unit 1.5

Halting Problem

The Halting Problem



(8 min): https://www.youtube.com/watch?v=92WHN-pAFCs



The Halting Theorem

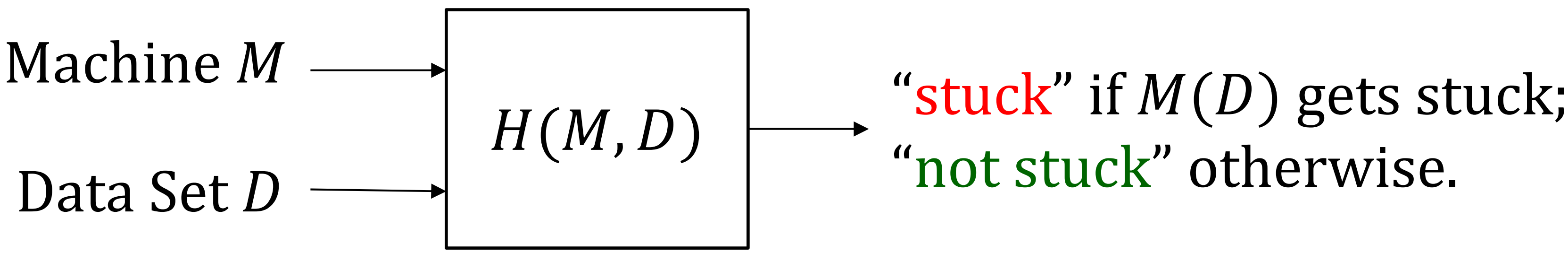
Theorem: There is no computer algorithm that

i. accepts any algorithm M and data set D as input, and then

ii. correctly outputs “stuck” or “not stuck” to indicate whether (or not) M terminates in a finite number of steps when M is run with data set D.

Proof by Contradiction

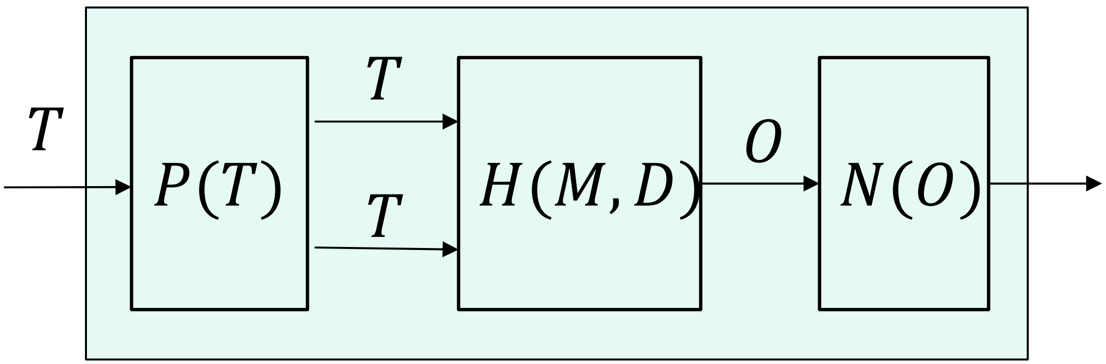
- Assume there is a halting machine:



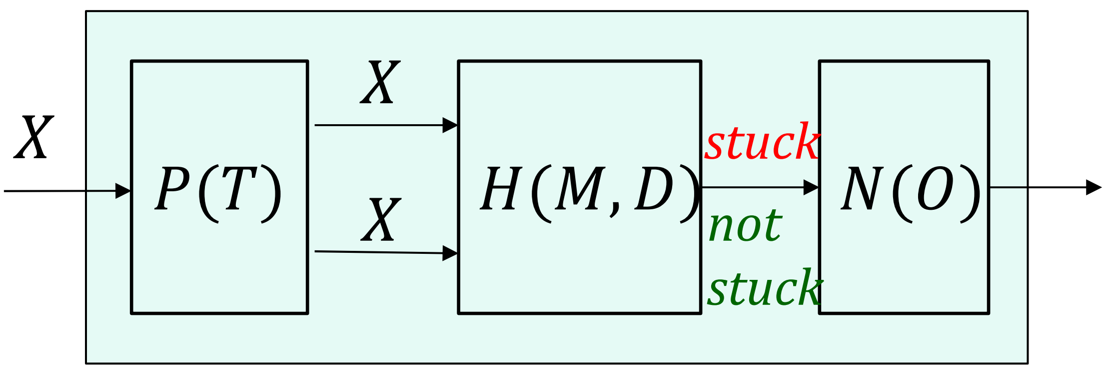
- Try to show that the existence of H leads to a contradiction.

Contradiction Arises…

- Construct machine X as follows:



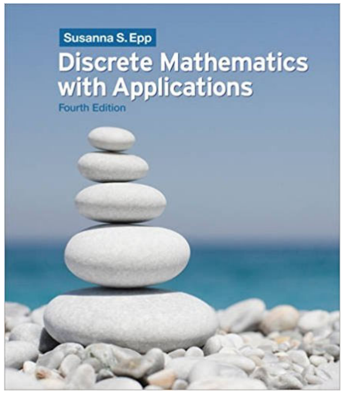
"smile" if O = "stuck";
loop forever otherwise.



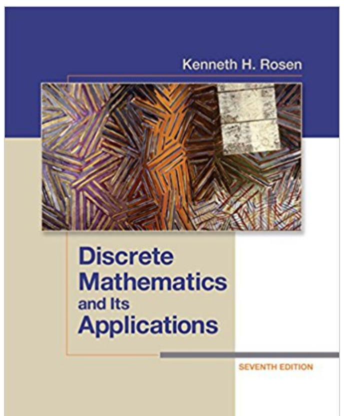
"smile"
loop forever

?!

Recommended Reading



- Chapter 6 and Section 7.1, S. S. Epp, *Discrete Mathematics with Applications*, 4th ed., Brooks Cole, 2010.



- Sections 2.1 and 2.2, K. H. Rosen, *Discrete Mathematics and its Applications*, 7th ed., McGraw-Hill Education, 2011.