EE2331 Data Structures and Algorithms:

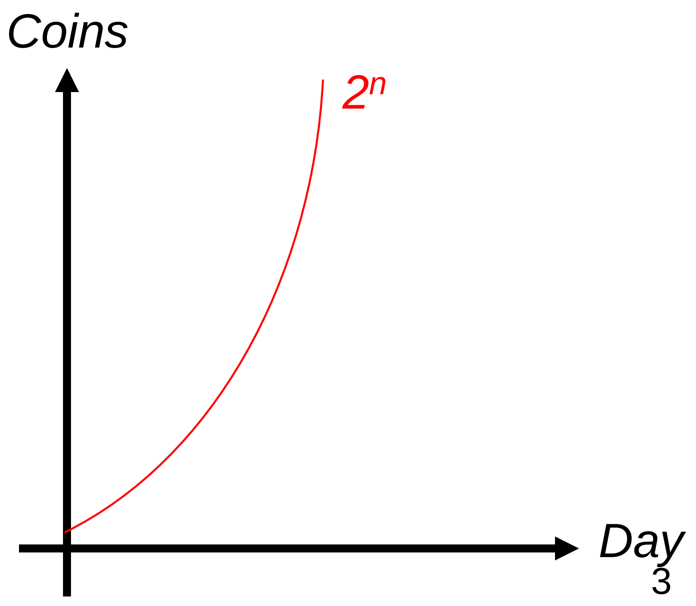
Analysis of Algorithms

The Story

- Once upon a time in a country, a warrior won a crucial battle and saved the country
- The King was very happy and gave a trunk of golden coins to the hero
- The warrior said: the trunk of coins would be too heavy for me to transport. That will be grateful if I can take a coin today and double the coins on each following subsequent day
- The King thought for a while and accepted his request
- But the country went bankrupt after a month!

Growth Rate

- The coins progressed as follows: 1, 2, 4, 8, 16, 32, 64, 128, …
- The number sequence looks insignificant at the very beginning
- But grows extremely fast (exponentially)
  - $Day_0 = 1$
  - $Day_{10} = 1{,}024$
  - $Day_{20} = 1{,}048{,}576$
  - $Day_{30} = 1{,}073{,}741{,}824$
  - $Day_{40} = 1{,}099{,}511{,}627{,}776$
  - Coins taken on $day_n = 2^n$



Analysis of Algorithms

- Often several different algorithms are available to solve the same problem. These algorithms can work but may not be efficiency.
  - May be impractical for large input size
  - May run extremely slow for particular inputs

- We want to know the efficiency and complexity of algorithms so as to compare them and make a wise choice.

Algorithms Analysis Example

- Find the sum of 1 + 2 + 3 + 4 + … + 998 + 999

- Method 1:
  - 1 + 2 = 3
  - 3 + 3 = 6
  - 6 + 4 = 10
  - …
  - 498,501 + 999 = 499,500

998 addition

- Method 2:
  - (1 + 999) x 999 / 2
  - = 499,500

1 addition, 1 multiplication, 1 division

Algorithms Analysis Example

- Find the sum of 1 + 2 + 3 + 4 + … + 999,999

- Method 1:
  - 1 + 2 = 3
  - 3 + 3 = 6
  - 6 + 4 = 10
  - …
  - 498,998,500,001 + 999,999 = 499,999,500,000

999,998 addition!

- Method 2:
  - (1 + 999,999) x 999,999 / 2
  - = 499,999,500,000

Still 1 addition, 1 multiplication, 1 division! (independent of the input size)

Algorithms Analysis Example

Method 1:

```
int sumOfSeries(int n) {
    int i, sum = 0;
    for (i = 1; i < n; i++)
        sum += i;
    return sum;
}
```

n - 1 addition

Which one is better?

Method 2:

```
int sumOfSeries(int n) {
    return (1 + n) * n / 2;
}
```

1 addition, 1 multiplication, 1 division

Complexity of Algorithms

Two Types of Complexity

- Complexity
  - Time Complexity (time requirement)
  - Space Complexity (memory space requirement)
- Usually the problems have a natural “size” called *n*
  - i.e. the amount of data to be processed
- Describe the resources used as a function of *n*
  - i.e. the amount of time taken, the amount of memory required
- Usually interested in **average case** and **worse case**
  - Average case: the amount of time/memory expected to take on typical input data
  - Worse case: the amount of time/memory would take on the worst possible input

Time Complexity

- Fixed part (ignored)
  - Time for declaring variables, assigning values
- Variable part
  - Vary according to input value (e.g. for-loop)
  - Conditional instructions (if-then-else)
- The time requirement $T(P)$ of program P is
  - $T(P) = C + Tp(n)$
  - C is the fixed part
  - $Tp(n)$ is the variable part depends on problem size n
- To analyze time complexity, concentrate solely on the estimation of $Tp(n)$

Space Complexity

- Fixed part (ignored)
  - Space for instruction, constant, variables
- Variable part
  - Vary according to input value (e.g. linked list)
  - Recursion stack space
- The space requirement $S(P)$ of program P is
  - $S(P) = C + Sp(n)$
  - C is the fixed part
  - $Sp(n)$ is the variable part depends on problem size n
- To analyze space complexity, concentrate solely on the estimation of $Sp(n)$

An Example Program

```
#include <iostream>

int main(int argc, char *argv[]) {

    int i, n, sum = 0;

    cin >> n;

    for(i = 0; i < n; i++)
        sum += i;

    return 0;
}
```

Constant time, C_1

Constant time, C_2

Variable time, depends on n
$= C_3 \times n$

Constant time, C_4

Total execution time
$= C_1 + C_2 + C_3 \times n + C_4$
$\approx C_3 \times n$ (if n is very large)

Analysis

- The exact value of C_i is not important, but the order of magnitude is important
- Usually C_i is a very small number
- $n * C_i$ would be a very significant number if n is a very large number
- e.g. suppose C_i is 1ms
  - If n is 1, execution time is 1ms
  - If n is 10, execution time is 10ms
  - If n is 1 million, execution time is 1,000s
- C_i is machine dependent
- To simplify our analysis, simply count how many times each instruction is executed in the algorithm.

Count the No. of Operations

```
int i, n, sum = 0;
```

This instruction being executed once

```
cin >> n;
```

This instruction being executed once

```
for(i = 0; i < n; i++)
    sum += i;
```

This block being executed *n* times

Total execution
$= 1 + 1 + n$
$= n + 2$
$\approx n$ (if n is very large)

Count the No. of Operations

```
//Code A
sum += i;
```

This instruction being executed once

```
//Code B
for(i = 0; i < n; i++)
        sum += i;
```

This code being executed *n* times

```
//Code C
for(i = 0; i < n; i++)
    for(j = 0; j < n; j++)
        sum += i * j;
```

This code being executed n^2 times!

Asymptotic Complexity

- Asymptotic complexity is a way of expressing the main component of the cost of an algorithm.
- For example, when analyzing some algorithm, one might find that the time (or the number of steps) it takes to complete a problem of size n is given by

  $$T(n) = 4n^2 - 2n + 2$$

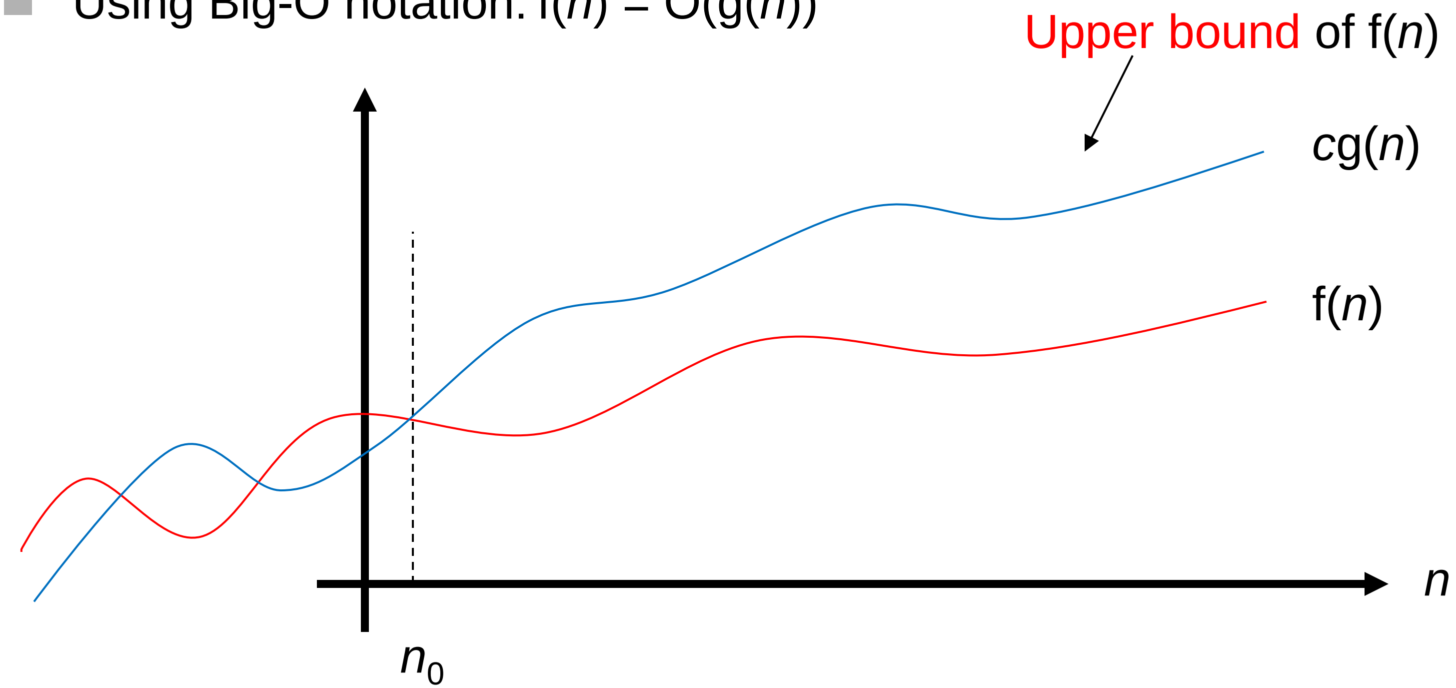
- If we ignore constants (which makes sense because those depend on the particular hardware the program is run on) and slower growing terms (i.e. 2n), we could say T(n) grows at the order of n^2 and write:

  $$T(n) = O(n^2)$$

- The letter O is used because the rate of growth of a function is also called its *Order*. Basically, it tells you how fast a function grows or declines.

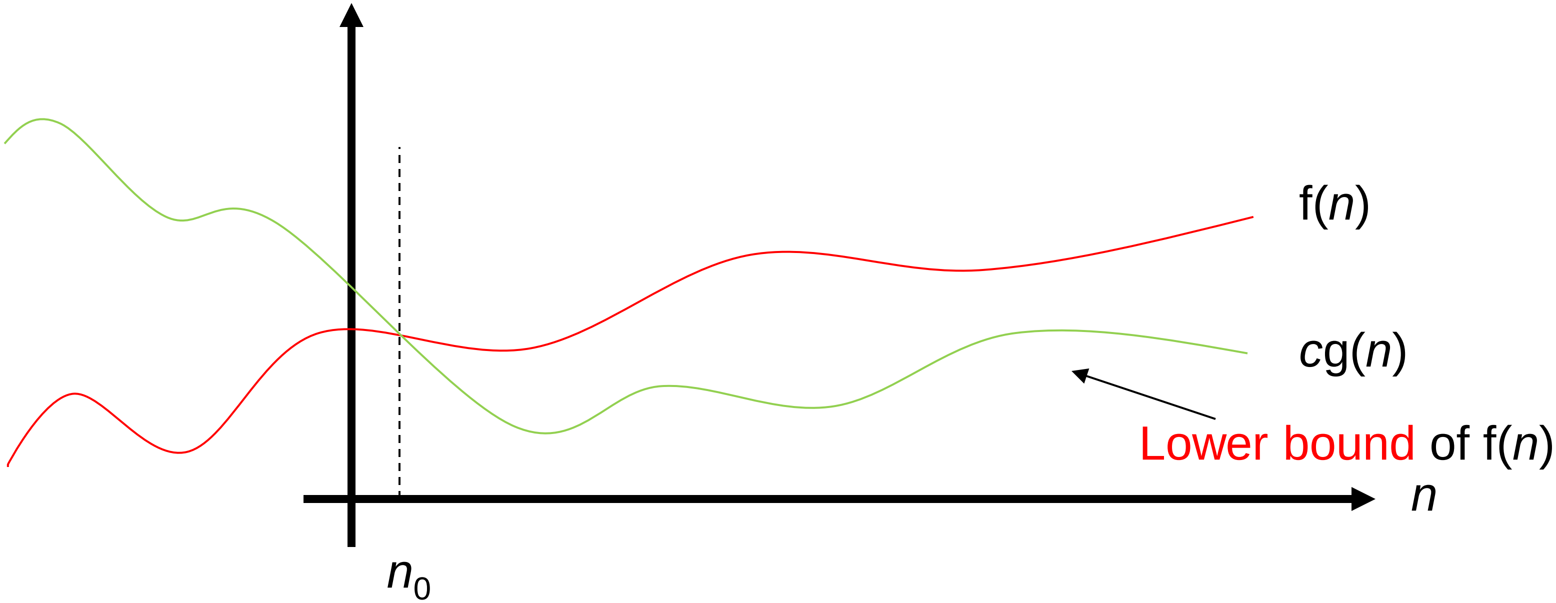
Asymptotic Notation O

- Big-O notation defines an upper bound of an algorithm's running time.
- We say that a function f(n) is of the order of g(n), iff there exists constant $c > 0$ and n_0 such that $f(n) \le cg(n)$ for all $n \ge n_0$
- In other words, f(n) is at most a constant times of g(n) for sufficiently large of values of n
- Using Big-O notation: $f(n) = O(g(n))$



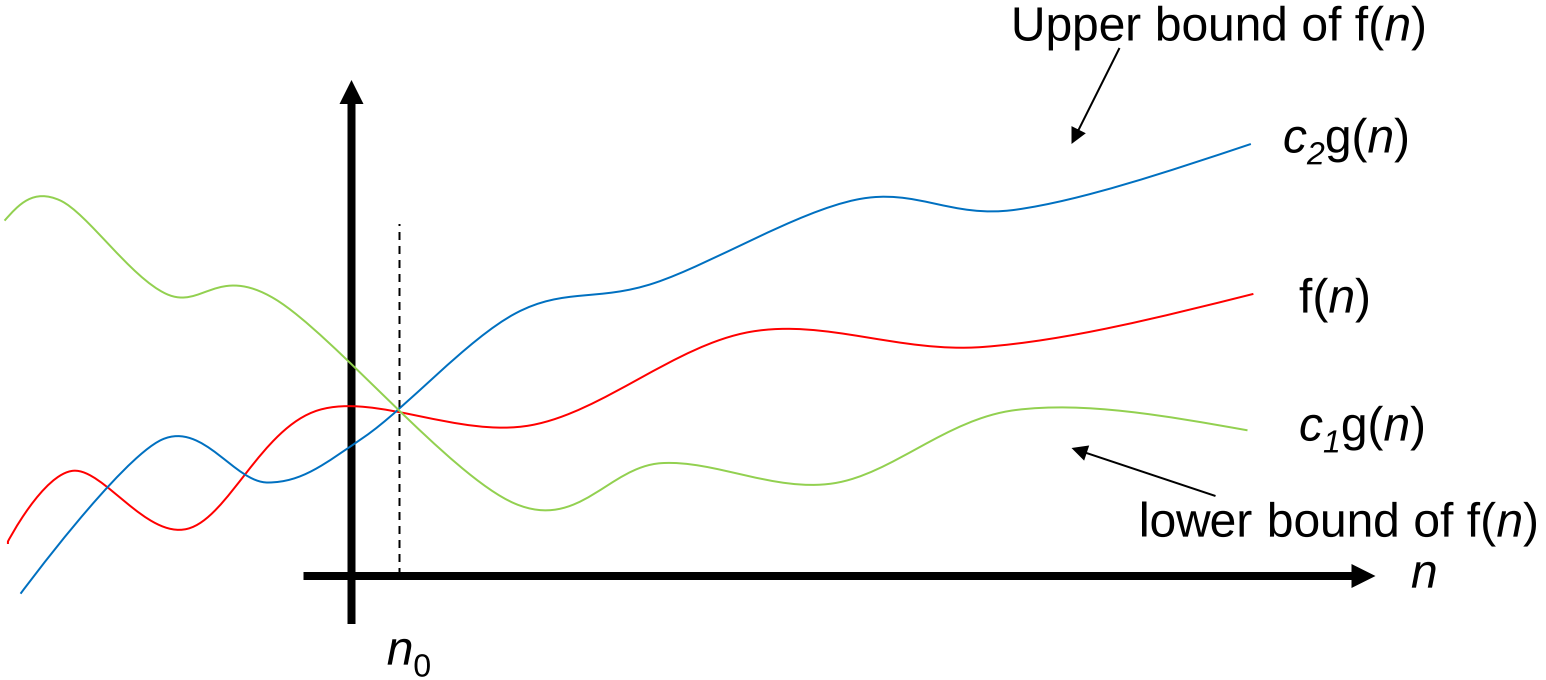
Asymptotic Notation Ω

- Big-Omega notation defines a lower bound of an algorithm's running time.
- $f(n) = \Omega(g(n))$ iff there exists constant $c > 0$ and n_0 such that $f(n) \geq cg(n)$ for all $n \geq n_0$



Asymptotic Notation Θ

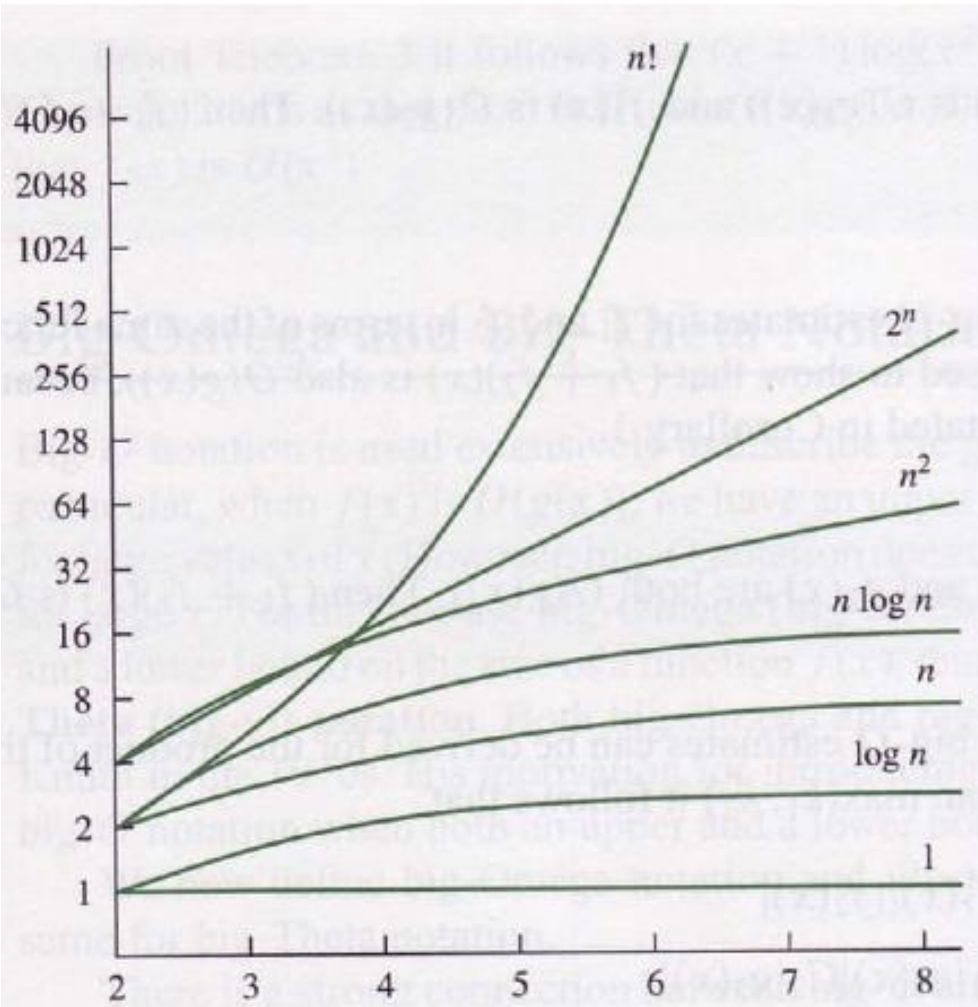
- Big-Theta notation defines an exact bound of an algorithm's running time.
- $f(n) = \Theta(g(n))$ iff there exists constant $c_1 > 0$, $c_2 > 0$ and n_0 such that $c_1 g(n) \le f(n) \le c_2 g(n)$ for all n, $n \ge n_0$



Important Complexity Classes

- $O(1)$: Constant time
- $O(\log_2 n)$: Logarithmic time
- $O(n)$: Linear time
- $O(n \log_2 n)$: Log-linear time
- $O(n^2)$: Quadratic time
- $O(n^3)$: Cubic time
- $O(n^k)$: Polynomial time
- $O(2^n)$: Exponential time

Important Complexity Classes



Factorial time

Exponential time

Quadratic time

Log-linear time

Linear time

Logarithmic time

Constant time

Increasing complexity

Becomes not feasible when *n* grows larger

Practical Problem Sizes

- Practical problem sizes that can be handled by algorithms of different complexity classes.

Complexity class		Maximum n	Example applications
constant time	O(1)	Unlimited	random number generation, hashing
logarithmic	O(log n)	Effectively unlimited	binary search
linear	O(n)	$n < 10^{10}$	sum of a list, sequential search, vector product
log-linear	O(n log n)	$n < 10^{8}$	fast sorting algorithms
quadratic	$O(n^2)$	$n < 10^{5}$	2D matrix addition, insertion sort
cubic	$O(n^3)$	$n < 10^{3}$	2D matrix multiplication
exponential	$O(a^n)$ for $a > 1$	$n < 36$ for 2^n	traveling salesman, placement and routing in VLSI, many other optimization problems
Factorial	O(n!)	$n < 15$	enumerate the permutations of n objects

Count the No. of Execution

```
int i, n, sum = 0, product = 1;
cin >> n;
for(i = 0; i < n; i++)
    sum += i;
for(i = 1; i < n; i++)
    product *= i;
printf("%d %d", sum, product);
```

1 time

1 time

n times

(*n* – 1) times

1 time

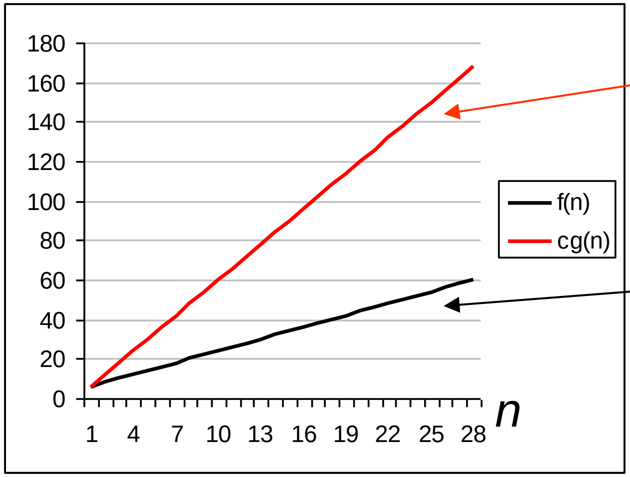
Total execution

$= 1 + 1 + n + (n\text{-}1) + 1$

$= 2n + 2$

Big O Notation

- If total execution is $2n + 2$
- $f(n) = 2n + 2$
- $f(n) = 2n + 2 \le 2n + 2n$ (for $n \ge 1$)
- $f(n) = 2n + 2 \le 4n$ (for $n \ge 1$)
- $f(n) = 2n + 2 \le cg(n)$ ($c = 4$, $g(n) = n$, $n_0 = 1$)
- $f(n) = O(g(n))$
- So, time complexity of the above code = $O(n)$



$cg(n)$: upper bound of $f(n)$

$f(n)$: always smaller than $cg(n)$ for $n \ge n_0$

Why Using Big-O

- We concern about the bounds on the performance of algorithms, rather than giving exact speeds.
- The actual running time depends upon our processor speed, the condition of our processor cache, etc. It's all very complicated in the concrete details, and moreover not relevant to the essence of the algorithm.
- Big-O provides a simple bounded approximation of the complexity order.
  - **Prove** the running time is always less then some **upper bound** *for sufficiently large input size*
  - **Derive** the **average** running time for a *random* input
  - Count the no. of iterations (loops) and function calls
  - Don’t Count variable declaration and other constant time items

Big O Notation Examples

- If total execution is $2n^2 + 6n + 4$
- $= O(n^2)$

- If total execution is $7n^3 + 2n^2 + 6n + 4$
- $= O(n^3)$

- If total execution is $(n + 1)/2$
- $= O(n)$

- If total execution is $2n^2 + \log n$
- $= O(n^2)$
- The fastest growing one dominates the order of $f(n)$

Example: Matrix Addition

- The problem size is given by N, the dimension of the matrix.

```
#define N 10
typedef int Matrix[N][N];

1   void MatrixAdd(Matrix A, Matrix B, Matrix C) {
2      for (int i = 0; i < N; i++)
3         for (int j = 0; j < N; j++)
4            C[i][j] = A[i][j] + B[i][j];
5      return;
6   }
```

line	step count	frequency
1	0	0
2	1	N+1
3	1	N(N+1)
4	1	N^2
5	1	1
6	0	0

Total = $2N^2 + 2N + 2$
= $O(N^2)$

Exercises

```
//Code A
int findMax(int a[], int n) {
    int i, max = a[0];
    for(i = 1; i < n; i++)
        if (a[i] > max)
            max = a[i];
    return max;
}
```

```
//Code C
for(i = 0; i < n; i++)
    for(j = 0; j < i; j++)
        sum++;
```

```
//Code B
for(i = 0; i < n; i++)
    for(j = 0; j < n * n; j++)
        sum++;
```

What are the Big-O of these 3 pieces of code?

Other Considerations

- Small no. of input
  - The growth rate of the running time may be less important than the constant factor when the input size is small
- Difficult to understand
  - An efficient but complicated algorithm may not be desirable because it is difficult to be maintained
  - e.g. recursive vs. non-recursive algorithm
- Space requirement
  - If an efficient algorithm uses too much space and is implemented with the slow secondary storage, it may negate the efficiency