EE2302 Foundations of Information Engineering Semester A, 2022/23

Solutions to Test 1

1)

a) The power set of B is $\{\phi, \{2\}, \{3\}, \{6\}, \{2,3\}, \{2,6\}, \{3,6\}, \{2,3,6\}\}$.

b) $B \times (A \cap C) = \{2, 3, 6\} \times \{2, 5\} = \{(2, 2), (2, 5), (3, 2), (3, 5), (6, 2), (6, 5)\}$.

c) $|C \cup D| = |C| + |D| - |C \cap D| = 5 + 5 - 2 = 8$.

2)

a) Yes. Assume $x \in A$, then let $x = 8p + 7$ for $\forall p \in \mathbb{Z}$. Rewrite it as $x = 4(2p + 1) + 3$ so x mod $4 = 3$ always holds and $x \in B$. Since $x \in A$ implies $x \in B$, it follows that $A \subseteq B$.

b) No. To disprove this statement, we need to show $\exists x \in B$ and this $x \notin A$. It is easy to see that $3 \in B$ but $3 \notin A$, which is a counterexample.

3)

a) $A \cup B = \{x \in \mathbb{R}: 0 \leq x \leq 1 \text{ or } 3 < x \leq 4\}$.

b) They are equal. We have $|A| \leq |A \cup B|$ because there exists an injection $f(x) = x$ from A to $A \cup B$, and $|A \cup B| \leq |A|$ because there exists an injection $g(x) = x/4$ from $A \cup B$ to A. That is, $|A| = |A \cup B|$.

Alternative solution: $|A| = |A \cup B|$ because there exists a bijection f from A to $A \cup B$ defined by $f(x) = 2x$ when $0 \leq x \leq 0.5$ and $f(x) = 2x + 2$ when $0.5 < x \leq 1$.

4)

a) $p = 1, q = 7$.

b) Such values cannot be found since both $(1,5)$ and $(3,5)$ are in f, which cannot be injective.

c) $p = 4, q = 7$.

5)

a) f is injective: assume $x_1, x_2 \in \mathbb{Z}$ and $f(x_1) = f(x_2) = 2x_1 = 2x_2 \to x_1 = x_2$.
f is not surjective: for example, $y = 3 \in \mathbb{Z}$ but there does not exist $x \in \mathbb{Z}$ such that $f(x) = 3$.

b) f is not injective: we can find $f(4) = 4/2 = 2 = f(5) = (5 - 1)/2$, but $4 \neq 5$.
f is surjective: for any $y \in \mathbb{Z}$, there exists $x = 2y \in \mathbb{Z}$ such that $f(x) = 2y/2 = y$.

6)

a) $f(x) = x^2 + 4x - 3 = (x+2)^2 - 7 \in (-3, 2]$, so the co-domain Y of the function f is $Y = \{y \in \mathbb{R}: -3 < y \leq 2\}$.

b) g is injective: $g(z_1) = 1/z_1 = g(z_2) = 1/z_2 \rightarrow z_1 = z_2$.
g is surjective: For any $x \in \mathbb{R}$ and $0 < x \leq 1$, there exists $z \in \mathbb{R}$ and $z \geq 1$ such that $g(z) = 1/z = x$. Hence, g is a bijection.
Let $g(z) = 1/z = x \rightarrow z = 1/x$, so its inverse function $g^{-1}(x) = 1/x$.

c) $f \circ g = f\big(g(z)\big) = f(1/z) = (1 + 4z - 3z^2)/z^2$.
$f \circ g$ is an injection since $g: \mathrm{Z} \rightarrow X$ and $f: X \rightarrow Y$ are both injections.

7) Let these two integers be p and q, respectively. Suppose both p and q are not less than 50, then their sum $p + q \geq 50 + 50 = 100$. Hence, the statement is proved.

8)

a) T is not a partial order relation since it is not antisymmetric.
Consider $sTt \leftrightarrow l(s) \leq l(t)$ and $tTs \leftrightarrow l(t) \leq l(s)$, we have $l(s) = l(t)$. Let $t = 0$ and $s = 1$, it is easy to see $l(s) = l(t) = 1$ but $0 \neq 1$.

b) R is reflexive: $\forall x \in \mathbb{R}_+, xRx$ is true since $x^2 \leq x^2$.
R is antisymmetric: $\forall x, y \in \mathbb{R}_+,\ xRy \wedge yRx \rightarrow x^2 = y^2 \rightarrow x = y$.
R is transitive: $\forall x, y. z \in \mathbb{R}_+, xRy \wedge yRz \leftrightarrow x^2 \leq y^2 \leq z^2 \rightarrow xRz$.

9)

a) R is reflexive: $\forall x \in \mathbb{B}^\infty, xRx$ is true since $g(x) = g(x)$.
R is symmetric: $\forall x, y \in \mathbb{B}^\infty, xRy \rightarrow g(x) = g(y) \rightarrow yRx$.
R is transitive: $\forall x, y, z \in \mathbb{B}^\infty, xRy \wedge\ yRz \rightarrow g(x) = g(y) = g(z) \rightarrow xRz$.
So R is an equivalence relation and the number of its distinct equivalence classes is 8.

List them: [000…], [001…], [010…], [011…], [100…], [101…], [110…], [111…].

b) S is reflexive: $\forall x \in \mathbb{B}^\infty, xSx$ is true since $g(x) \leq g(x)$.
S is not symmetric: This can be proved by giving a counterexample. Let x be an infinite string starts with 000 while y starts with 001. Then $0 = g(x) \leq g(y) = 1$ and xSy is true. On the other hand, we do not have ySx since $g(y) \leq g(x)$ is false.
S is transitive: $\forall x, y, z \in \mathbb{B}^\infty, xSy \wedge\ ySz \rightarrow g(x) \leq g(y) \leq g(z) \rightarrow xSz$.
S is not antisymmetric: $\forall x, y \in \mathbb{B}^\infty$, $xSy \wedge\ ySx \rightarrow g(x) = g(y) \nrightarrow x = y$ by giving a counterexample $g(x) = g(y) = 0$ but $x = 0000 \ldots \neq y = 0001 \ldots$.

So S is neither an equivalence relation nor a partial order.

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