Lecture 2. Deterministic Signal Analysis

- Time-Domain Analysis
- Frequency-Domain Analysis

Time-Domain Analysis

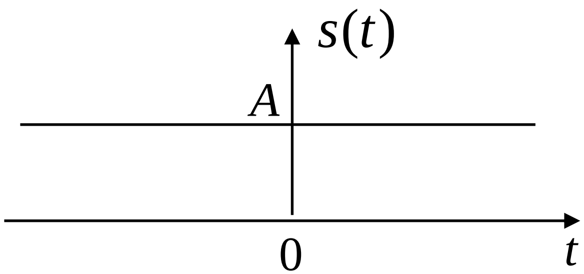
- Signal Representation in Time Domain
- Signal Energy and Signal Power
- Signal Transmission Through an LTI System

Signals in Time Domain

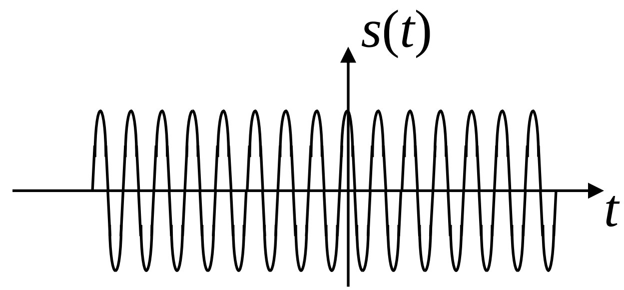
- A signal is a set of data or information, which can be represented as a function of time t: $s(t)$
- Deterministic Signal vs. Random Signal
  - ✓ Deterministic signal is a signal whose physical description is known completely, either in a mathematical form or a graphical form: Expression of $s(t)$ is known. Lecture 2
  - ✓ Random signal is a signal that cannot be predicted precisely, but known in terms of probabilistic description: $s(t)$ is a random process. Lecture 5
- Other Classification of Signals:
  - ✓ Periodic signal vs. Aperiodic signal
  - ✓ Continuous-time vs. Discrete-time signal
  - ✓ Analog signal vs. Digital signal

Examples of Deterministic Signals

- Constant signal: $s(t) = A$

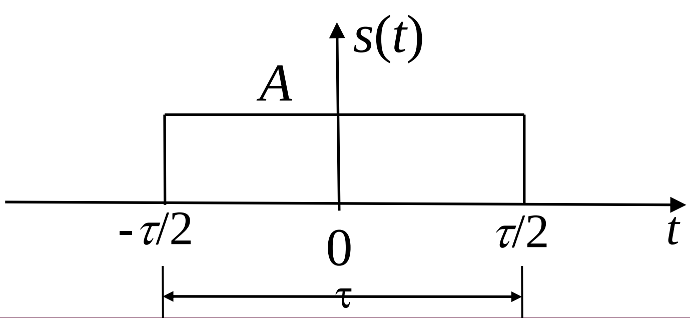


- Sinusoidal signal: $s(t) = \cos(2\pi f_0 t)$



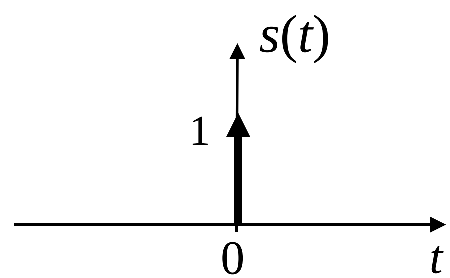
- Rectangular pulse:

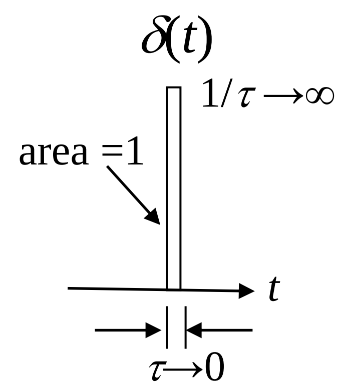
$$s(t) = \begin{cases} A & -\tau/2 \le t \le \tau/2 \\ 0 & \text{otherwise} \end{cases}$$



- Unit impulse: $s(t) = \delta(t)$

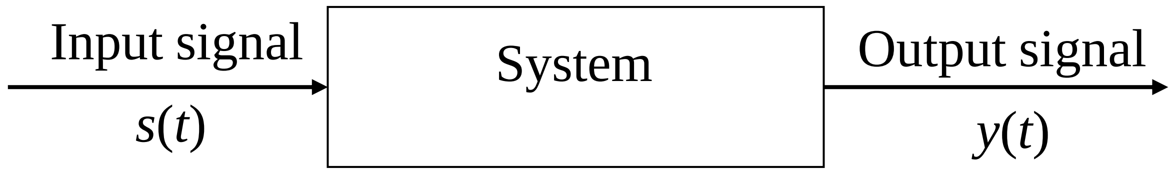
$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t)\,dt = 1$$





Linear Time Invariant (LTI) System

- System: A system is an entity that processes a set of signals (inputs) to yield another set of signals (outputs).



- Linear and Time Invariant:
  - ✓ Linear: If $s_1(t) \rightarrow y_1(t)$, and $s_2(t) \rightarrow y_2(t)$, then $a_1 s_1(t) + a_2 s_2(t) \rightarrow a_1 y_1(t) + a_2 y_2(t)$ for coefficients a_1 and a_2.
  - ✓ Time Invariant : If $s(t) \rightarrow y(t)$, then $s(t-T) \rightarrow y(t-T)$ for delay T.
- Impulse response $h(t)$: If the input signal is a unit impulse $\delta(t)$, then the output signal is called the impulse response of the system: $\delta(t) \rightarrow h(t)$

Linear Time Invariant (LTI) System

- For an LTI system: the output signal $y(t)$ is the convolution of the input signal $s(t)$ and the impulse response of the system $h(t)$.

$$\text{Input signal } s(t) \longrightarrow \boxed{\text{System}} \longrightarrow \text{Output signal } y(t)$$

$$\delta(t) \rightarrow h(t)$$

$$\delta(t-n\Delta\tau) \rightarrow h(t-n\Delta\tau)$$

$$s(n\Delta\tau)\delta(t-n\Delta\tau)\,\Delta\tau \rightarrow s(n\Delta\tau)h(t-n\Delta\tau)\,\Delta\tau$$

$$s(t)=\lim_{\Delta\tau\to 0}\sum_{n=-\infty}^{\infty} s(n\Delta\tau)\delta(t-n\Delta\tau)\,\Delta\tau \rightarrow \lim_{\Delta\tau\to 0}\sum_{n=-\infty}^{\infty} s(n\Delta\tau)h(t-n\Delta\tau)\,\Delta\tau = y(t)$$

$$=\int_{-\infty}^{\infty} s(\tau)h(t-\tau)d\tau = s(t) * h(t)$$

$$s(t) \longrightarrow \boxed{\begin{array}{c}\text{Impulse Response}\\ h(t)\end{array}} \longrightarrow y(t) = s(t) * h(t) = \int_{-\infty}^{\infty} s(\tau)h(t-\tau)d\tau$$

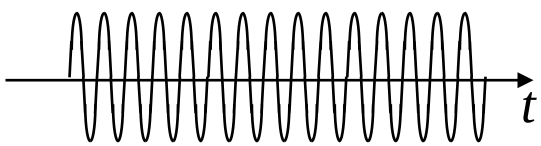
Frequency-Domain Analysis

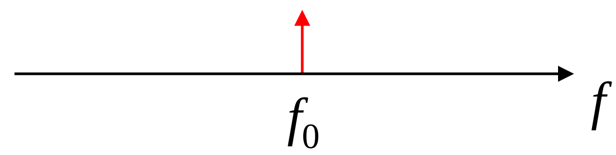
- Fourier Transform
- Energy Spectrum, Power Spectrum and Signal Bandwidth
- Signal Transmission Through an LTI System

Signals in Frequency Domain

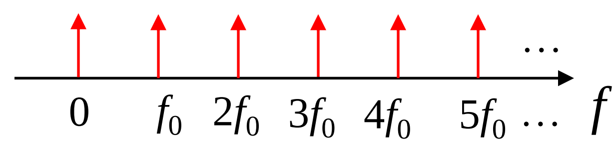
Time domain | **Frequency domain**

- $\cos(2\pi f_0 t)$





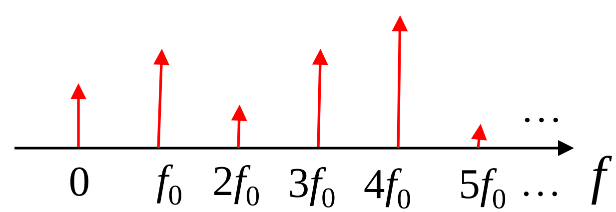
- $\sum_{n=0}^{\infty} \cos(2\pi n f_0 t)$ $\sum_{n=0}^{\infty} \cos(2\pi n f_0 t + \theta_n)$



- $a_0 + \sum_{n=1}^{\infty} a_n \cos(2\pi n f_0 t) + \sum_{n=1}^{\infty} b_n \sin(2\pi n f_0 t)$

$$= \sum_{n=-\infty}^{\infty} s_n e^{j2\pi n f_0 t}$$

$s_0 = a_0$, $s_n = (a_n - j\, b_n)/2$, $s_{-n} = (a_n + j\, b_n)/2$.



$$\boxed{\begin{aligned} \cos(2\pi f_0 t) &= \tfrac{1}{2}\left(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right) \\ \sin(2\pi f_0 t) &= \tfrac{1}{2j}\left(e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}\right) \end{aligned}}$$

Fourier Series

- $s(t)=\sum_{n=-\infty}^{\infty} s_n e^{j2\pi n f_0 t}$. Let $T_0=1/f_0$.

  $s(t+T_0)=\sum_{n=-\infty}^{\infty} s_n e^{j2\pi n f_0 (t+T_0)}=\sum_{n=-\infty}^{\infty} s_n e^{j2\pi n f_0 t+j2\pi n}=s(t)$

  s(t) is a periodic signal with period T_0!

- Fourier Series: $$\boldsymbol{s(t)=\sum_{n=-\infty}^{\infty} s_n e^{j2\pi n f_0 t}} \qquad \boldsymbol{s_n=\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} s(t)e^{-j2\pi n f_0 t}dt}$$

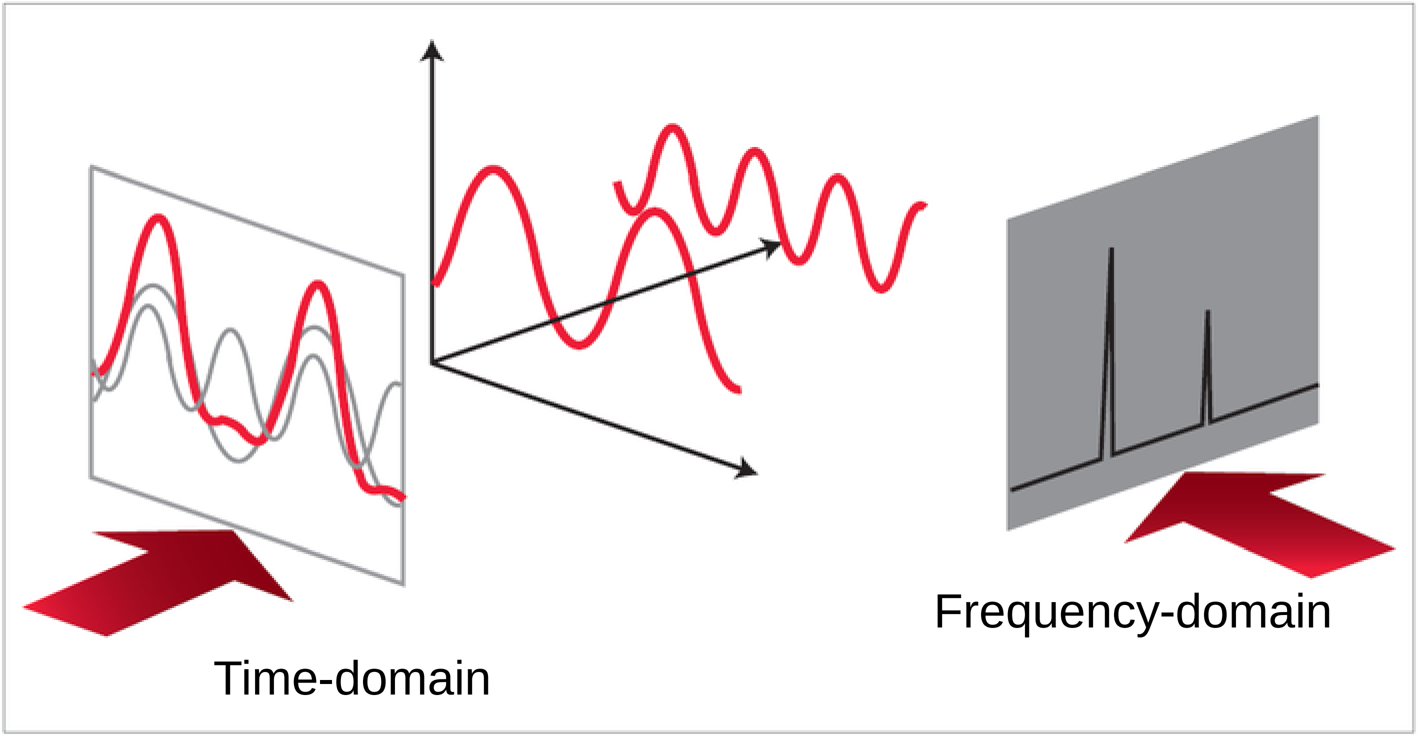
✓ Any periodic signal with period T_0 can be expressed as a sum of sinusoidal signals, each with the frequency an integer number of $1/T_0$.

✓ To obtain s_n: $\int_{-T_0/2}^{T_0/2} s(t)e^{-j2\pi m f_0 t}dt=\sum_{n=-\infty}^{\infty} s_n\int_{-T_0/2}^{T_0/2} e^{j2\pi (n-m) f_0 t}dt$

$$=s_m T_0$$

$$\boxed{\int_{-T_0/2}^{T_0/2} e^{j2\pi m f_0 t}dt=\begin{cases}T_0 & m=0\\ 0 & m\neq 0\end{cases}}$$

$$\Rightarrow \quad \boldsymbol{s_n=\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} s(t)e^{-j2\pi n f_0 t}dt}$$


Frequency-domain
Time-domain

From Fourier Series to Fourier Transform

For an aperiodic signal $s(t)$, construct a periodic signal $s'(t)$ by repeating the signal $s(t)$ every T_0 seconds: $s(t) = \lim_{T_0\to\infty} s'(t)$.

- ✓ Fourier Series: $s'(t) = \sum_{n=-\infty}^{\infty} s'_n e^{j2\pi n f_0 t}$, $s'_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} s'(t) e^{-j2\pi n f_0 t} dt$
- ✓ Let $S'(f) = \int_{-T_0/2}^{T_0/2} s'(t) e^{-j2\pi f t} dt$. Then $s'_n = \frac{S'(nf_0)}{T_0}$

$$s'(t) = \sum_{n=-\infty}^{\infty} \frac{S'(nf_0)}{T_0} e^{j2\pi n f_0 t} = \sum_{n=-\infty}^{\infty} S'(nf_0) f_0 e^{j2\pi n f_0 t}$$

- ✓ With $T_0 \to \infty$, $f_0 \to 0$.

$$s(t) = \lim_{T_0\to\infty} s'(t) = \lim_{f_0\to 0} \sum_{n=-\infty}^{\infty} S'(nf_0) f_0 e^{j2\pi n f_0 t} = \int_{-\infty}^{\infty} S(f) e^{j2\pi f t} df$$

$$S(f) = \lim_{T_0\to\infty} S'(f) = \lim_{T_0\to\infty} \int_{-T_0/2}^{T_0/2} s'(t) e^{-j2\pi f t} dt = \int_{-\infty}^{\infty} s(t) e^{-j2\pi f t} dt$$

Fourier Transform

Fourier Transform

Given a time-domain signal $s(t)$, its Fourier transform is defined as follows.

Fourier transform: $$S(f) = \int_{-\infty}^{\infty} s(t) e^{-j2\pi ft} dt$$

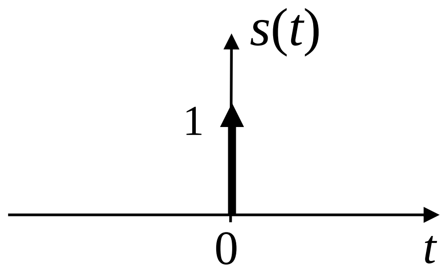
The time-domain signal $s(t)$ can be expressed by $S(f)$ using an inverse transform.

Inverse Fourier transform: $$s(t) = \int_{-\infty}^{\infty} S(f) e^{j2\pi ft} df$$

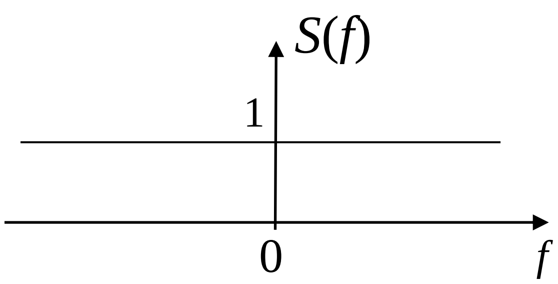
- (Fourier) spectrum of $s(t)$: $S(f)$
- Magnitude spectrum of $s(t)$: $|S(f)|$

$$s(t) \Leftrightarrow S(f)$$

Example 1: Spectrum of Unit Impulse $\delta(t)$

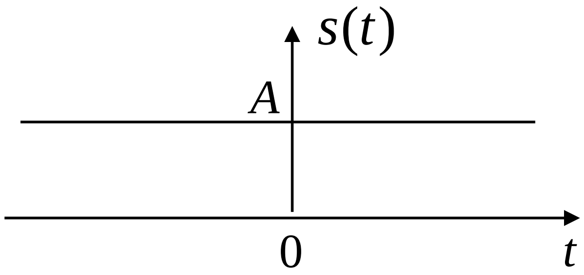


$$s(t) = \delta(t): \quad \delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t)dt = 1$$

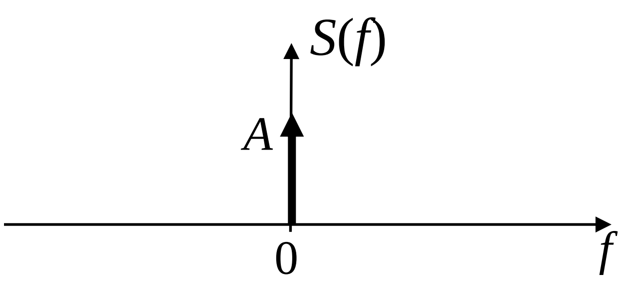


$$S(f) = \int_{-\infty}^{\infty} \delta(t)e^{-j2\pi ft}dt = 1$$

Example 2: Spectrum of Constant Signal



$$s(t) = A$$



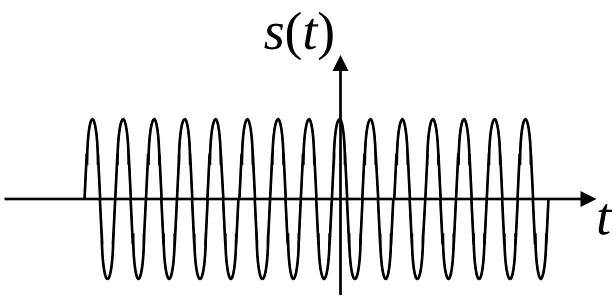
$$S(f) = A\int_{-\infty}^{\infty} e^{-j2\pi ft}dt$$

$$S(0) = A\int_{-\infty}^{\infty} e^{-j2\pi 0t}dt = A\int_{-\infty}^{\infty} 1dt = \infty$$

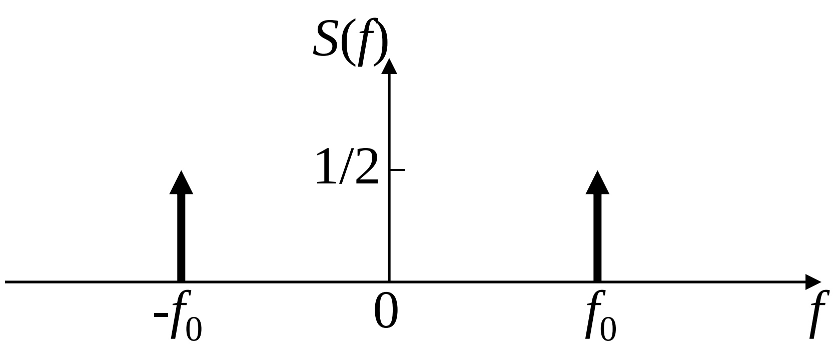
$$S(f) = A\int_{-\infty}^{\infty} e^{-j2\pi ft}dt = 0 \quad \text{for} \quad f \neq 0$$

$$\boxed{S(f) = A\delta(f)}$$

Example 3: Spectrum of Sinusoidal Signal



$$s(t) = \cos(2\pi f_0 t)$$



$$S(f) = \int_{-\infty}^{\infty} \cos 2\pi f_0 t \cdot e^{-j2\pi ft} dt$$

$$= \int_{-\infty}^{\infty} \tfrac{1}{2}(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}) \cdot e^{-j2\pi ft} dt$$

$$= \tfrac{1}{2} \int_{-\infty}^{\infty} e^{-j2\pi(f-f_0)t} dt + \tfrac{1}{2} \int_{-\infty}^{\infty} e^{-j2\pi(f+f_0)t} dt$$

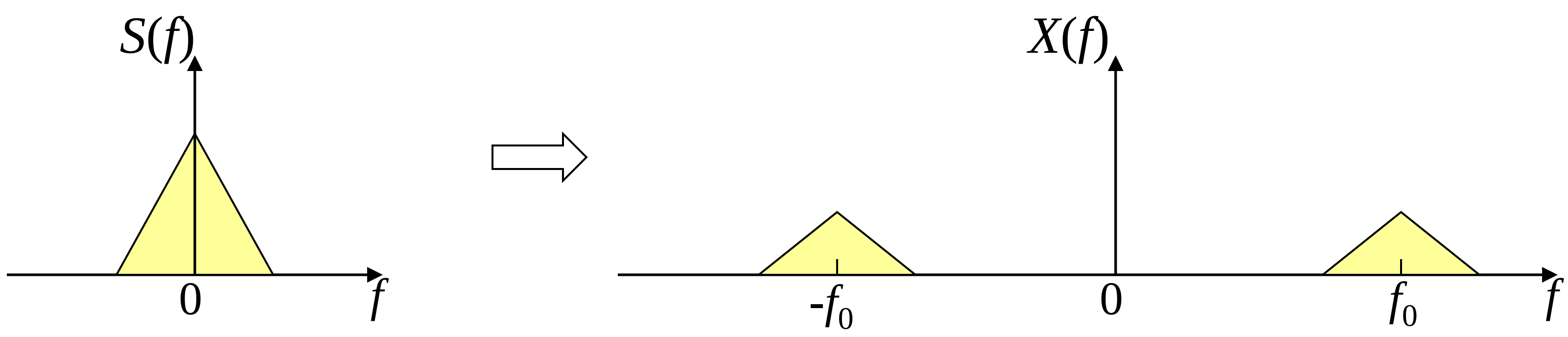
$$= \tfrac{1}{2}(\delta(f-f_0) + \delta(f+f_0))$$

Example 4: Spectrum of $s(t)\cos(2\pi f_0 t)$

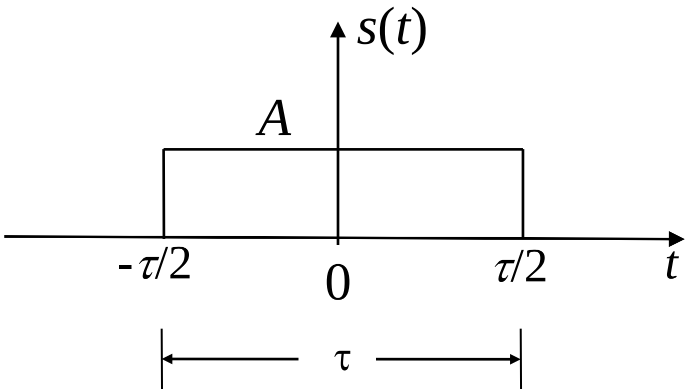
$$x(t) = s(t)\cos(2\pi f_0 t)$$

$$X(f) = \int_{-\infty}^{\infty} s(t)\cos(2\pi f_0 t)\cdot e^{-j2\pi ft}dt = \int_{-\infty}^{\infty} s(t)\cdot \tfrac{1}{2}(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t})\cdot e^{-j2\pi ft}dt$$

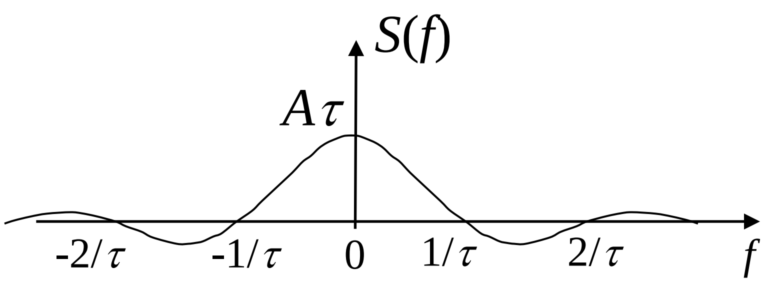
$$= \tfrac{1}{2}\int_{-\infty}^{\infty} s(t)\cdot e^{-j2\pi(f-f_0)t}dt + \tfrac{1}{2}\int_{-\infty}^{\infty} s(t)\cdot e^{-j2\pi(f+f_0)t}dt = \tfrac{1}{2}[S(f-f_0)+S(f+f_0)]$$



Example 5: Spectrum of Single Rectangular Pulse



$$s(t) = \begin{cases} A & -\tau/2 \le t \le \tau/2 \\ 0 & \text{otherwise} \end{cases}$$



$$S(f) = A\int_{-\tau/2}^{\tau/2} e^{-j2\pi ft}dt \; = A\cdot\frac{e^{-j\pi f\tau} - e^{+j\pi f\tau}}{-j2\pi f}$$

$$= A\tau\cdot\frac{\sin(\pi f\tau)}{\pi f\tau} = A\tau\,\mathrm{sinc}(f\tau)$$

$$\mathrm{sinc}(x) \triangleq \frac{\sin(\pi x)}{\pi x}$$

- sinc function is an even, oscillating function with a decreasing magnitude.
- It has unit peak at x=0, and zero crossing points at x= non-zero integers.

Properties of Fourier Transform

$\alpha s_1(t)+\beta s_2(t)$	$\Leftrightarrow$	$\alpha S_1(f)+\beta S_2(f)$	Linearity
$s_1(t)s_2(t)$	$\Leftrightarrow$	$S_1(f)*S_2(f)$	Convolution
$s_1(t) * s_2(t)$	$\Leftrightarrow$	$S_1(f)\cdot S_2(f)$	
$S(t)$	$\Leftrightarrow$	$s(-f)$	Duality
$s(t-\tau)$	$\Leftrightarrow$	$S(f)e^{-j2\pi f\tau}$	Time shift
$s(t)e^{-j2\pi f_0 t}$	$\Leftrightarrow$	$S(f+f_0)$	Frequency shift
$s(t)\cos(2\pi f_0 t)$	$\Leftrightarrow$	$\frac{1}{2}[S(f-f_0)+S(f+f_0)]$	Modulation
$s(at)$	$\Leftrightarrow$	$\frac{1}{\lvert a\rvert}S\left(\frac{f}{a}\right)$	Time scale

(for any real $a \neq 0$)

Review Examples 2 & 4

$$s(t)=\delta(t) \quad \Leftrightarrow \quad S(f)=1$$

$$s(t)=1 \quad \Leftrightarrow \quad S(f)=\delta(f)$$

Duality: $S(t) \Leftrightarrow s(-f)$

$$x(t)=s(t)\cos(2\pi f_0 t)$$

$$\Leftrightarrow$$

$$X(f)=S(f)*\left[\tfrac{1}{2}(\delta(f-f_0)+\delta(f+f_0))\right]$$

$$=\tfrac{1}{2}[S(f-f_0)+S(f+f_0)]$$

Modulation:

$$s_1(t)\cos(2\pi f_0 t)$$

$$\Leftrightarrow$$

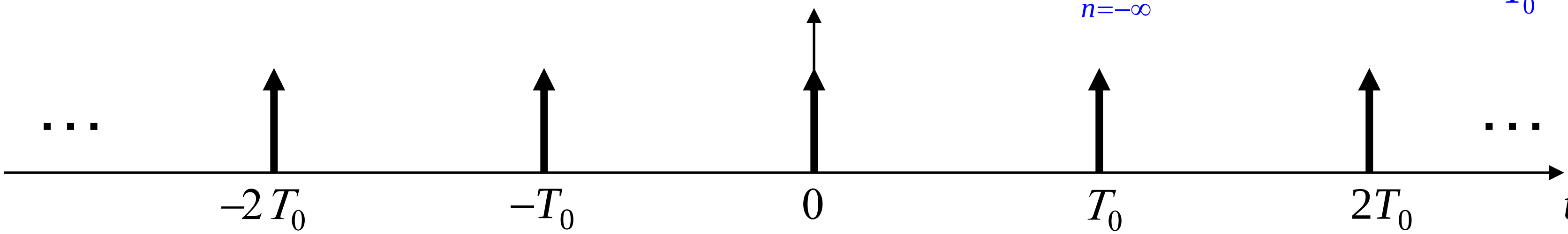
$$\tfrac{1}{2}[S(f-f_0)+S(f+f_0)]$$

Convolution:

$$s_1(t)s_2(t) \Leftrightarrow S_1(f)*S_2(f)$$

Example 6: Spectrum of Impulse Train

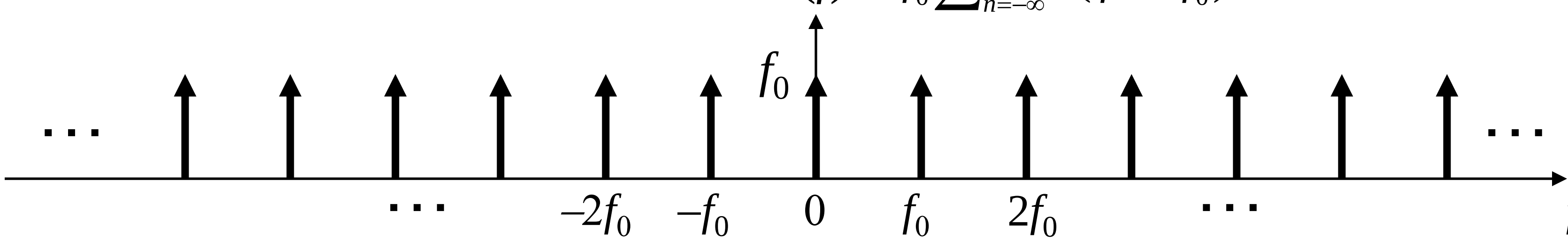
$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0) = \sum_{n=-\infty}^{\infty} s_n e^{j2\pi n f_0 t} \qquad f_0 = \frac{1}{T_0}$$



$$S(f) = \sum_{n=-\infty}^{\infty} s_n \delta(f - nf_0)$$

$$s_n = \frac{1}{T_0}\int_0^{T_0} s(t) e^{-j2\pi n f_0 t} dt = f_0 \int_0^{1/f_0} \delta(t) e^{-j2\pi n f_0 t} dt = f_0$$

$$S(f) = f_0 \sum_{n=-\infty}^{\infty} \delta(f - nf_0)$$



Example 7: Spectrum of Periodic Signal

- For periodic signal $s(t)$ with period T_0, define $s_{T_0}(t)$ as

$$s_{T_0}(t) = \begin{cases} s(t) & -T_0/2 < t < T_0/2 \\ 0 & otherwise \end{cases}$$

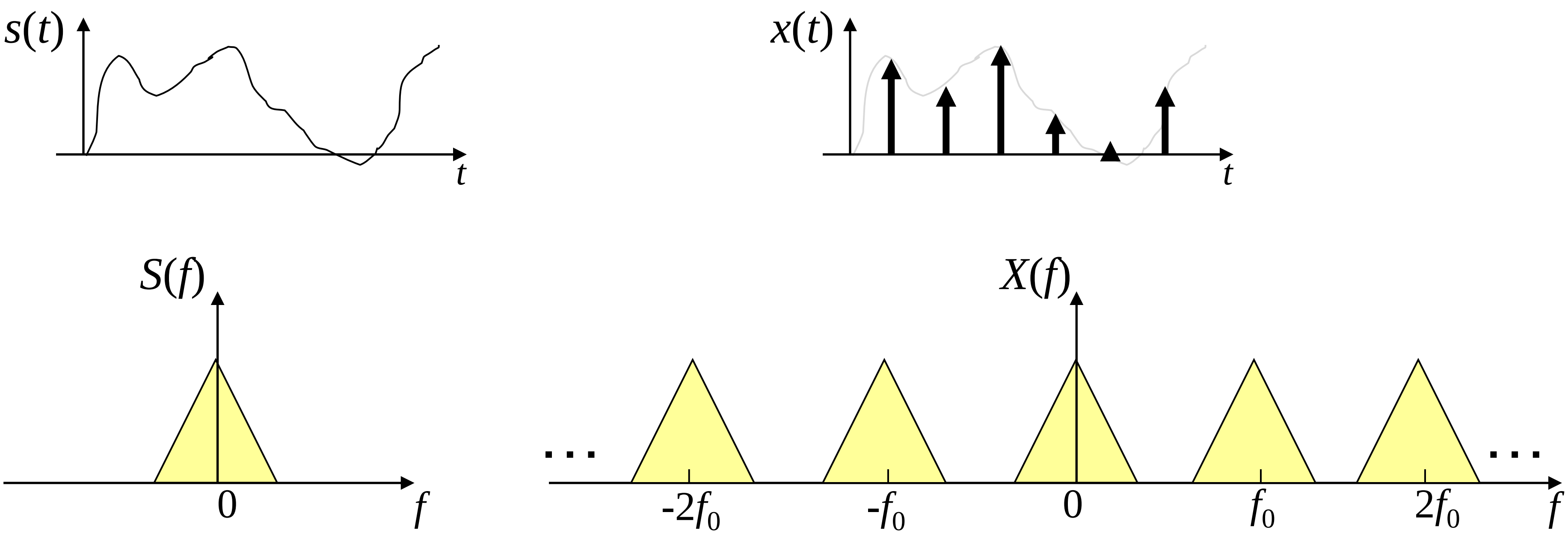
- $$s(t) = \sum_{n=-\infty}^{\infty} s_{T_0}(t - nT_0) = s_{T_0}(t) * \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$$

- $$S(f) = S_{T_0}(f) \cdot f_0 \sum_{n=-\infty}^{\infty} \delta(f - nf_0) = f_0 \sum_{n=-\infty}^{\infty} S_{T_0}(nf_0) \delta(f - nf_0)$$

Example 8: Spectrum of Sampled Signal

$$x(t) = s(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$$

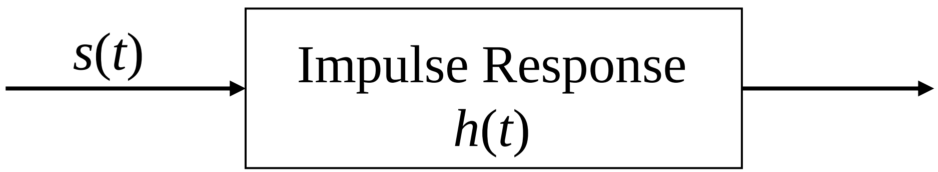
$$X(f) = S(f) * f_0 \sum_{n=-\infty}^{\infty} \delta(f - nf_0) = f_0 \sum_{n=-\infty}^{\infty} S(f - nf_0)$$



Signal Transmission through an LTI System

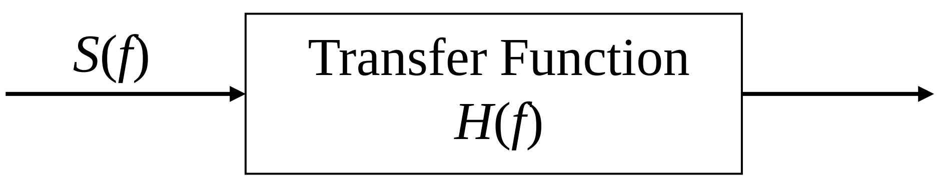
Signal Transmission over an LTI System

- Time Domain:



$$y(t) = s(t) * h(t) = \int_{-\infty}^{\infty} s(\tau)h(t-\tau)d\tau$$

- Frequency Domain:

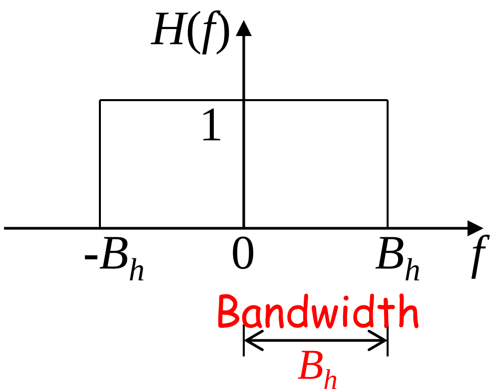


$$Y(f) = S(f) \cdot H(f)$$

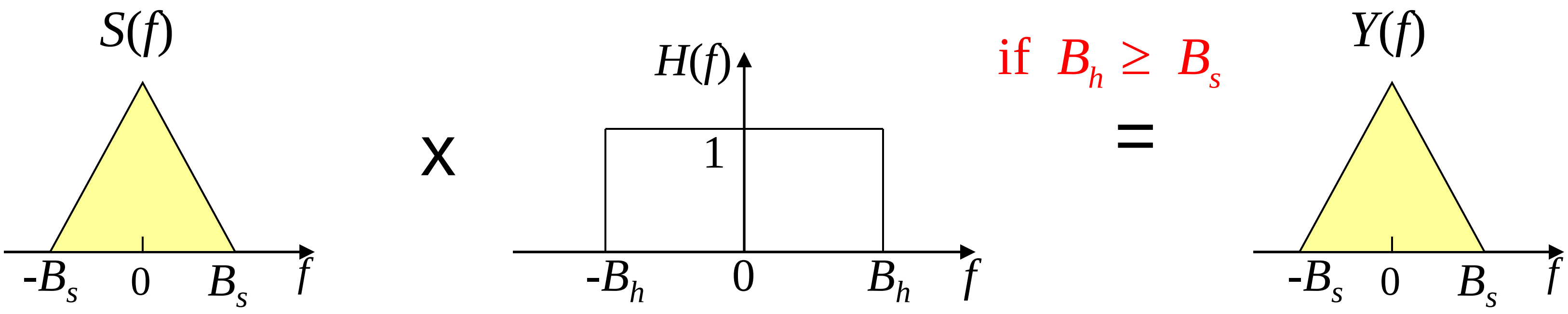
$$s(t) \Leftrightarrow S(f)$$
$$h(t) \Leftrightarrow H(f)$$
$$y(t) \Leftrightarrow Y(f)$$

Ideal Lowpass System

- Transfer Function $H(f)$ of an ideal lowpass system:

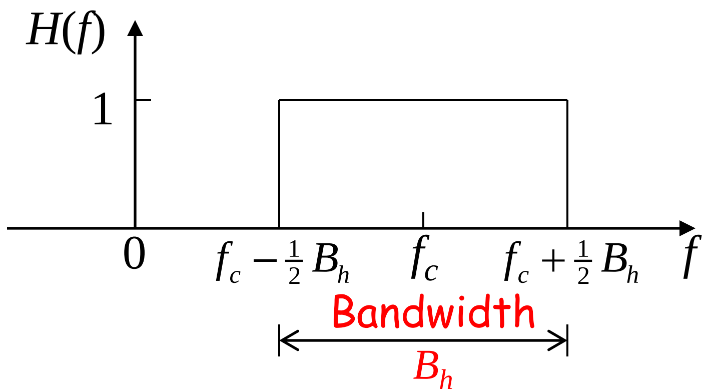


- For a baseband input signal with bandwidth B_s:



Ideal Bandpass System

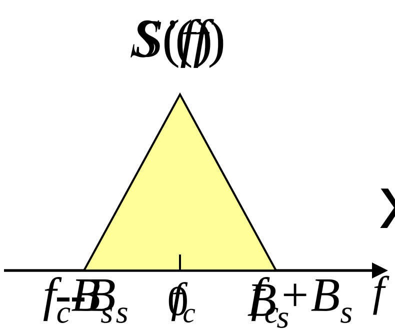
- Transfer Function $H(f)$ of an ideal bandpass system:



- For a baseband input signal with bandwidth B_s:

If: 1) the center frequency of $S(f)$ is shifted to f_c

2) $B_h \geq 2B_s$



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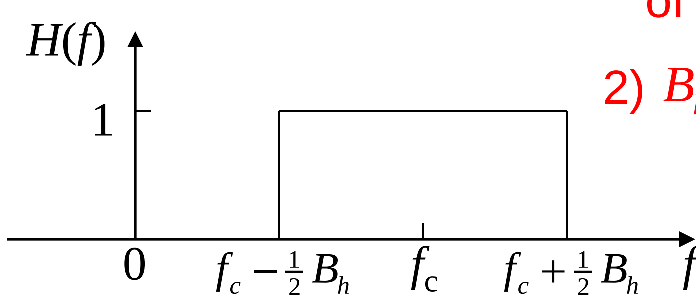

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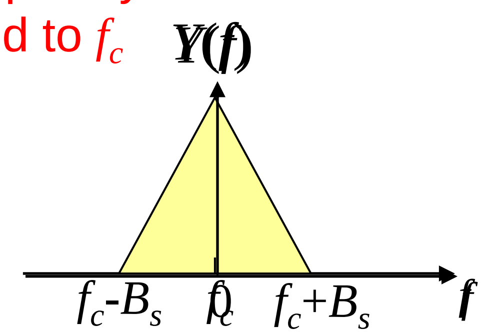
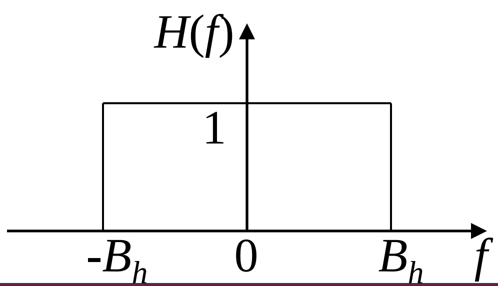


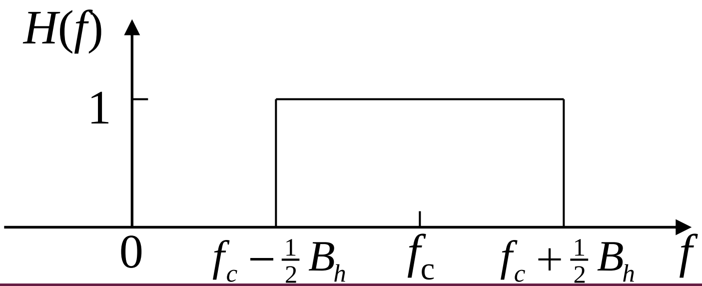
Baseband Channel and Bandpass Channel

- **Baseband channel**
  - A baseband channel efficiently passes frequency components from dc (zero) to the cutoff frequency B_h Hz.
  - Examples: coaxial cable

- **Bandpass channel**
  - A bandpass channel efficiently passes frequency components within a certain band, say, between $f_c - \frac{1}{2}B_h$ and $f_c + \frac{1}{2}B_h$ Hz.
  - Examples: EM wave, fibre

In this course, a baseband channel and a bandpass channel are modeled as an ideal low-pass LTI system and an ideal bandpass LTI system, respectively.





Energy Spectrum, Power Spectrum

Energy-type Signal and Power-type Signal

- Energy-type Signal: A signal is an energy-type signal if and only if its energy is positive and finite.
  - ✓ s(t) is an energy-type signal if and only if $0 < E_s = \int_{-\infty}^{\infty} |s(t)|^2 \, dt < \infty.$

- Power-type Signal: A signal is a power-type signal if and only if its power is positive and finite.
  - ✓ s(t) is a power-type signal if and only if $0 < P_s = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |s(t)|^2 \, dt < \infty.$

How to determine if a signal is an energy-type signal or a power-type signal from the frequency domain?

Energy and Energy Spectrum

- Energy of energy-type signal s(t):

$$E_s = \int_{-\infty}^{\infty} |s(t)|^2\, dt = \int_{-\infty}^{\infty} s(t)s^*(t)dt = \int_{-\infty}^{\infty} s(t)\left[\int_{-\infty}^{\infty} S^*(f)e^{-j2\pi ft}df\right]dt$$

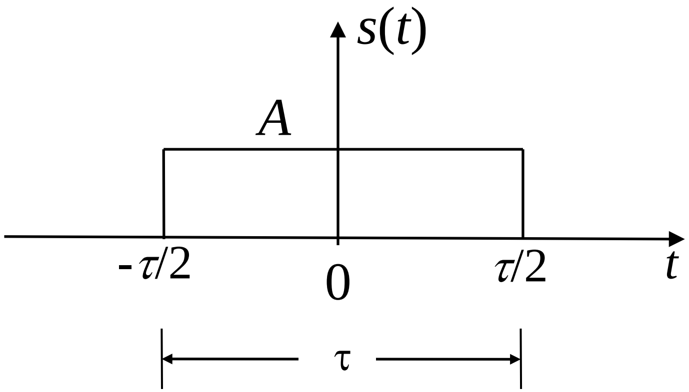
$$= \int_{-\infty}^{\infty} S^*(f)\left[\int_{-\infty}^{\infty} s(t)e^{-j2\pi ft}dt\right]df = \int_{-\infty}^{\infty} S^*(f)S(f)df = \int_{-\infty}^{\infty} |S(f)|^2\, df$$

$$= \int_{-\infty}^{\infty} U_s(f)df$$

Parseval's Theorem: $E_s = \int_{-\infty}^{\infty} |s(t)|^2\, dt = \int_{-\infty}^{\infty} |S(f)|^2\, df$

- Energy spectrum: $U_s(f) \triangleq |S(f)|^2$

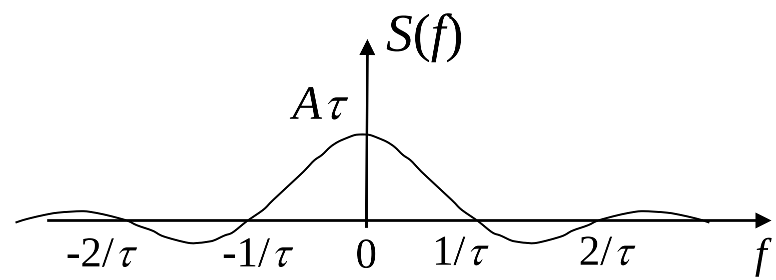
Example 9: Energy Spectrum of Single Rectangular Pulse



$$s(t) = \begin{cases} A & -\tau/2 \le t \le \tau/2 \\ 0 & \text{otherwise} \end{cases}$$

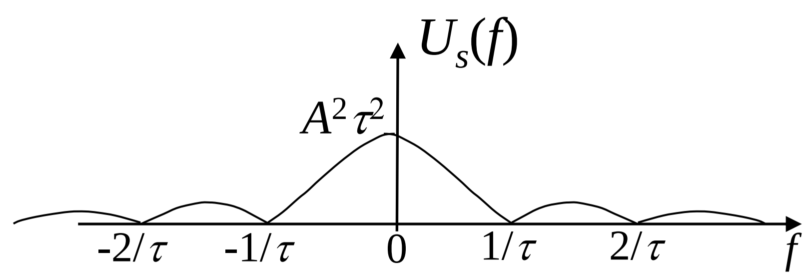
- Fourier spectrum:

$$S(f) = A\tau \operatorname{sinc}(f\tau)$$



- Energy spectrum:

$$U_s(f) = |S(f)|^2 = A^2\tau^2 \operatorname{sinc}^2(f\tau)$$



Power and Power Spectrum

- Power of power-type signal s(t):

$$s_T(t) \triangleq \begin{cases} s(t) & -T/2 < t < T/2 \\ 0 & otherwise \end{cases}$$

$$P_s = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |s(t)|^2\, dt = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |s_T(t)|^2\, dt$$

$$= \lim_{T\to\infty} \frac{1}{T}\int_{-\infty}^{\infty} |S_T(f)|^2\, df = \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{1}{T} |S_T(f)|^2\, df = \int_{-\infty}^{\infty} G_s(f) df$$

- Power spectrum:

$$G_s(f) \triangleq \lim_{T\to\infty} \frac{1}{T} |S_T(f)|^2$$

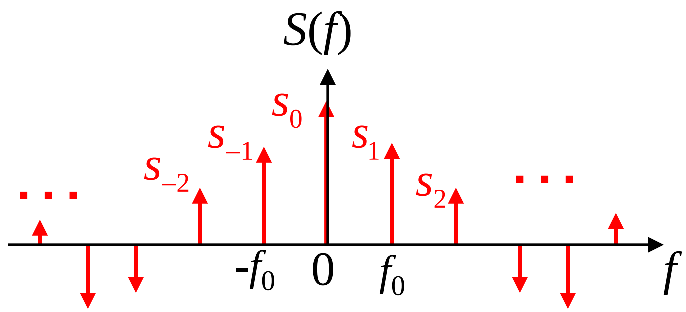
$$G_s(f) \Leftrightarrow \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} s(t+\tau)s^*(t)dt$$

Example 10: Power Spectrum of Periodic Signal

For periodic signal $s(t)$ with period T_0: $s(t)=\sum_{n=-\infty}^{\infty} s_n e^{j2\pi n f_0 t}$ $\qquad f_0 = 1/T_0$

- Fourier spectrum:

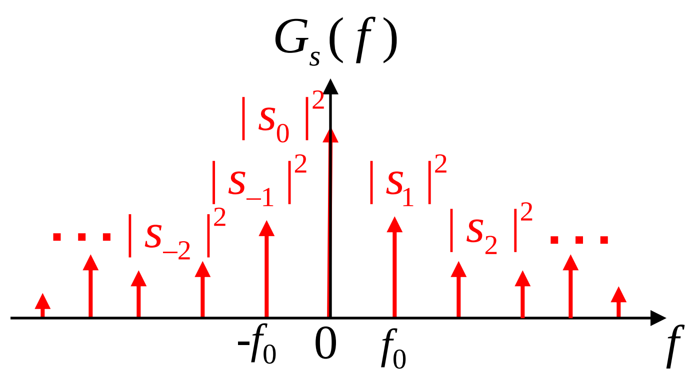
$$S(f)=\sum_{n=-\infty}^{\infty} s_n \delta(f-nf_0)$$



- Power spectrum:

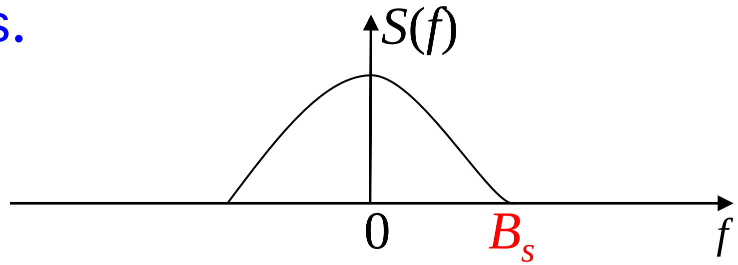
$$G_s(f) \Leftrightarrow \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} s(t+\tau)s^*(t)dt = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} s(t+\tau)s^*(t)dt = \sum_{n=-\infty}^{\infty}|s_n|^2 e^{j2\pi n f_0 \tau}$$

$$G_s(f)=\sum_{n=-\infty}^{\infty}|s_n|^2 \delta(f-nf_0)$$



Signal Bandwidth

- Bandwidth of signal $s(t)$: the amount of **positive** frequency spectrum that signal $s(t)$ occupies.



- Effective Bandwidth: x% of the signal's power (energy) are included.