Solution

1.(a)

As the joint PDF should integrate to 1, we first obtain:

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y)dxdy &= \int_{-0.5}^{0.5}\int_{-0.5}^{0.5}(k+3y^2)dxdy \\ &= \left(\int_{-0.5}^{0.5} dx\right)\left(\int_{-0.5}^{0.5}(k+3y^2)dy\right) \\ &= ky+y^3\Big|_{-0.5}^{0.5} = k+0.25\end{aligned}$$

Equating $k+0.25=1$ yields:

$$k=0.75$$

1.(b)

For $-0.5 < y < 0.5$, the marginal PDF of Y is:

$$p(y) = \int_{-\infty}^{\infty} p(x,y)dx = \int_{-0.5}^{0.5} (0.75 + 3y^2)dx$$
$$= 0.75 + 3y^2$$

The complete marginal PDF of Y is then:

$$p(y) = \begin{cases} 0.75 + 3y^2, & -0.5 < y < 0.5 \\ 0, & \text{otherwise} \end{cases}$$

2.(a)

From the CDF, the PMF of K is determined as:

$$P_K(k) = \begin{cases} 0.1, & k = -2 \\ 0.2, & k = 4 \\ 0.3, & k = 6 \\ 0.4, & k = 7 \end{cases}$$

Clearly, $P(K > 0)$ is $P_K(4) + P_K(6) + P_K(7) = 0.9$.

Hence $P_{X|X>0}(x)$ is:

$$P_{K|K>0}(k) = \begin{cases} 2/9, & k = 4 \\ 3/9, & k = 6 \\ 4/9, & k = 7 \end{cases}$$

2.(b)

The conditional CDF is the:

$$F_{K|K>0}(k) = \begin{cases} 0, & k < 4 \\ 2/9, & 4 \le k < 6 \\ 5/9, & 6 \le k < 7 \\ 1, & k \ge 7 \end{cases}$$

2.(c)

$$\mathbb{E}\{K|K>0\} = 4 \cdot \frac{2}{9} + 6 \cdot \frac{3}{9} + 7 \cdot \frac{4}{9} = 6$$

$$\mathbb{E}\{K^2|K>0\} = 4^2 \cdot \frac{2}{9} + 6^2 \cdot \frac{3}{9} + 7^2 \cdot \frac{4}{9} = \frac{112}{3}$$

$$\mathrm{var}(K^2|K>0) = \mathbb{E}\{K^2|K>0\} - (\mathbb{E}\{K|K>0\})^2 = \frac{4}{3}$$

3.(a)

As the face number is between 1 and 4, the possible values of X and Y are 1, 2, 3 and 4. For X, we have:

$$P(X=1)=P(\{(1,1),(1,2),(1,3),(1,4),(2,1),(3,1),(4,1)\})$$
$$P(X=2)=P(\{(2,2),(2,3),(2,4),(3,2),(4,2)\})$$
$$P(X=3)=P(\{(3,3),(3,4),(4,3)\})$$
$$P(X=4)=P(\{(4,4)\})$$

For fair dice, each corresponds to probability of $(1/4)^2$. Hence, the PMF of X is

$$p(x)=\begin{cases} 7/16, & x=1 \\ 5/16, & x=2 \\ 3/16, & x=3 \\ 1/16, & x=4 \\ 0, & \text{otherwise} \end{cases}$$

Similarly, for Y, we have:

$$
\begin{aligned}
P(Y=1) &= P(\{(1,1)\}) \\
P(Y=2) &= P(\{(1,2),(2,2),(2,1)\}) \\
P(Y=3) &= P(\{(1,3),(2,3),(3,3),(3,2),(3,1)\}) \\
P(Y=4) &= P(\{(1,4),(2,4),(3,4),(4,4),(4,3),(4,2),(4,1)\})
\end{aligned}
$$

Hence, the PMF of Y is

$$
p(y) = \begin{cases} 1/16, & y=1 \\ 3/16, & y=2 \\ 5/16, & y=3 \\ 7/16, & y=4 \\ 0, & \text{otherwise} \end{cases}
$$

3.(b)

We can first observe that only $Y \geq X$ is possible. For example, for $(O_1, O_2) = (1, 2)$ or $(O_1, O_2) = (2, 1)$, $X = 1$ and $Y = 2$ while $(O_1, O_2) = (2, 2)$, $X = Y = 2$. That is, when $O_1 = O_2$, there is one combination while when $O_1 \neq O_2$, there are two combinations.

Furthermore $P(Y < X) = 0$. As a result, the joint PMF of X and Y is:

4	0	0	0	1/16
3	0	0	1/16	1/8
2	0	1/16	1/8	1/8
1	1/16	1/8	1/8	1/8
X/Y	1	2	3	4

3.(c)
Using the table in 3.(b), we easily obtain:

$$P(Y \geq X+1) = P(Y > X) = \frac{3}{4}$$

3.(d)
From the table in 3.(b), $P_{XY}(1,1) = 1/16$.

From 3.(a), we have $P_X(1) = 7/16$ and $P_Y(1) = 1/16$. Hence we have:

$$P_{X|Y}(1|1) = \frac{P_{XY}(1,1)}{P_Y(1)} = 1$$

$$P_{Y|X}(1|1) = \frac{P_{XY}(1,1)}{P_X(1)} = \frac{1}{7}$$

4.(a)

Let $\mu = \mathbb{E}\{X\}$. Let the events of having a head and tail be H and T, respectively. Clearly, $P(H) = p$ and $P(T) = 1 - p$.

We first condition on the result of the first coin toss:

$$\begin{aligned}\mathbb{E}\{X\} &= \mathbb{E}\{X|H\}P(H) + \mathbb{E}\{X|T\}P(T) \\ &= (\mu + 1) \cdot p + (1 - p)E\{X|T\}\end{aligned}$$

To find $\mathbb{E}\{X|T\}$, we need to condition on the result of the second coin toss:

$$\begin{aligned}\mathbb{E}\{X|T\} &= \mathbb{E}\{X|TH\}P(H) + \mathbb{E}\{X|TT\}P(T) \\ &= 2p + (1 + \mathbb{E}\{X|T\})(1 - p) \\ \Rightarrow \mathbb{E}\{X|T\} &= \frac{p+1}{p}\end{aligned}$$

Note that for $\mathbb{E}\{X|TT\}$, because the first two tosses are TT, we have wasted the first coin toss and will start from the second toss, resulting in the sum of 1 and $\mathbb{E}\{X|T\}$.

As a result, we get:

$$\mu = p(1+\mu) + (1-p)\frac{p+1}{p} \Rightarrow \mu = \mathbb{E}\{X\} = \frac{1}{p(1-p)}$$

4.(b)

For a fair coin, $p = 0.5$. Using the result in 4.(a), we get $\mathbb{E}\{X\} = 4$.

That is, on average, we need to have 4 tosses to obtain a tail, followed by a head, in two successive trials.

5.(a)

As $X \sim (X_L, X_U)$ and $Y \sim (Y_L, Y_U)$, their PDFs can be written as:

$$p(x) = \begin{cases} \dfrac{1}{X_U - X_L}, & X_L < x < X_U \\ 0, & \text{otherwise} \end{cases}$$

and

$$p(y) = \begin{cases} \dfrac{1}{Y_U - Y_L}, & Y_L < y < Y_U \\ 0, & \text{otherwise} \end{cases}$$

As X and Y are independent, their joint PDF is thus equal to

$$p(x, y) = p(x)p(y)$$

According to

$$\mathbb{E}\{g(X,Y)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)p(x,y)dxdy$$

Now $g(X,Y) = P = (X+Y)^2 R$. We have:

$$\begin{aligned}
\mathbb{E}\{P\} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x+y)^2 Rp(x,y)dxdy \\
&= R\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x+y)^2 p(x)p(y)dxdy \\
&= R\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 p(x)p(y)dxdy + R\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y^2 p(x)p(y)dxdy \\
&\quad +2R\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyp(x)p(y)dxdy
\end{aligned}$$

Ignoring R, the first term is

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 p(x)p(y)dxdy &= \left(\int_{-\infty}^{\infty} p(y)dy\right)\left(\int_{-\infty}^{\infty} x^2 p(x)dx\right) \\ &= 1\cdot\int_{X_L}^{X_U}\frac{x^2}{X_U-X_L}dx \\ &= \frac{1}{3}(X_U^2+X_UX_L+X_L^2)\end{aligned}$$

Similarly, the second term is:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y^2 p(x)p(y)dxdy = \frac{1}{3}(Y_U^2+Y_UY_L+Y_L^2)$$

For the last term, we have:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyp(x)p(y)dxdy = \frac{1}{4}(X_U+X_L)(Y_U+Y_L)$$

Finally, we obtain:

$$\mathbb{E}\{P\} = \frac{R}{3}(X_U^2 + X_U X_L + X_L^2 + Y_U^2 + Y_U Y_L + Y_L^2) + \frac{R}{2}(X_U + X_L)(Y_U + Y_L)$$

5.(b)
Substituting $X_L = 9$, $X_U = 11$, $Y_L = 0.5$, $Y_U = 1.5$ and $R = 1$ into the result of 5.(a), we obtain:

$$\mathbb{E}\{P\} = 121.3367$$

5.(c)
As X, $Y$and R are independent, their joint PDF is thus:

$$p(x, y, r) = p(x)p(y)p(r)$$

Extending the concept of expected value to 3 variable, the mean power is:

$$
\begin{aligned}
\mathbb{E}\{P\} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x+y)^2 r p(x,y,r)\,dx\,dy\,dr \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x+y)^2 r p(x)p(y)p(r)\,dx\,dy\,dr \\
&= \left(\int_{-\infty}^{\infty} r p(r)\,dr\right)\cdot\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x+y)^2 p(x)p(y)\,dx\,dy\right)
\end{aligned}
$$

Using the result of 5.(a), we obtain:

$$
\begin{aligned}
\mathbb{E}\{P\} = &\frac{R_U+R_L}{6}(X_U^2+X_UX_L+X_L^2+Y_U^2+Y_UY_L+Y_L^2)+ \\
&\frac{R_U+R_L}{4}(X_U+X_L)(Y_U+Y_L)
\end{aligned}
$$