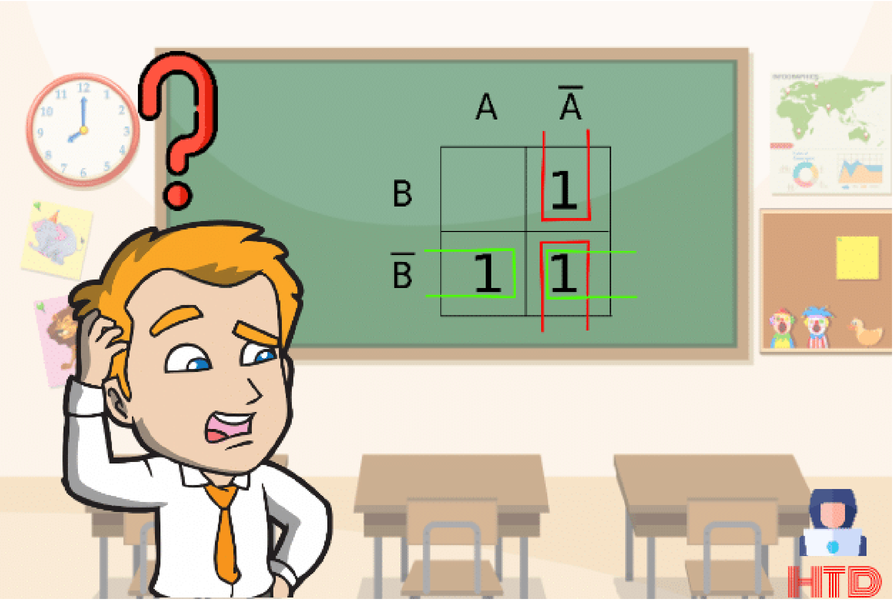
EE2000 Logic Circuit Design

Recap Lecture 2 – Karnaugh Map and Quine-McCluskey (QM) Method



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2.1 Karnaugh Map

- In 1953, Maurice Karnaugh introduced a map method known as **Karnaugh map (K-map)**
- A straightforward procedure for minimizing Boolean functions in a tabular form
- Graphical representation of a truth table
- Minterm is used in the cell of the K-map
- *n*-variable function has 2^n cells:
  - Two-variable K-map has 4 cells
  - Three-variable K-map has 8 cells
  - Four-variable K-map has 16 cells

Four-variable K-map

$b = 1$

cd \ ab	00	01	11	10
00	a'b'c'd'	a'bc'd'	abc'd'	ab'c'd'
01	a'b'c'd	a'bc'd	abc'd	ab'c'd
11	a'b'cd	a'bcd	abcd	ab'cd
10	a'b'cd'	a'bcd'	abcd'	ab'cd'

$d = 1$

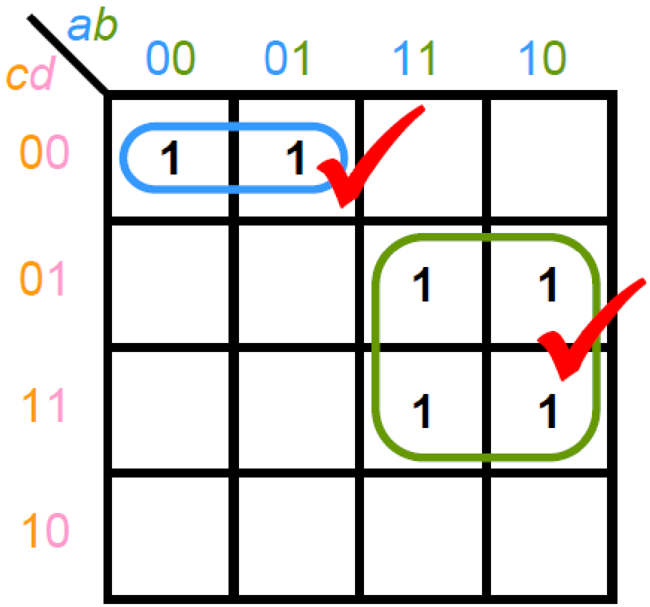
$c = 1$

$a = 1$

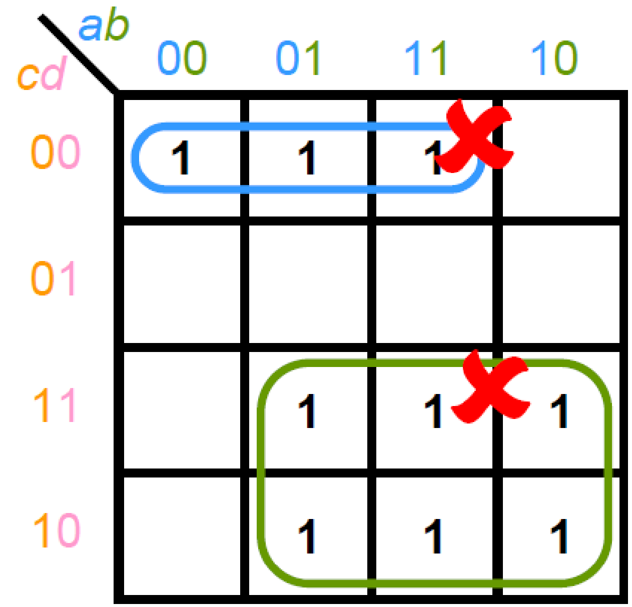
cd \ ab	00	01	11	10
00	m_0	m_4	m_{12}	m_8
01	m_1	m_5	m_{13}	m_9
11	m_3	m_7	m_{15}	m_{11}
10	m_2	m_6	m_{14}	m_{10}

Note the Gray code order of the rows and columns

Summary



Group size is power of 2 (e.g. 2, 4, 8)



Other group size is illegal

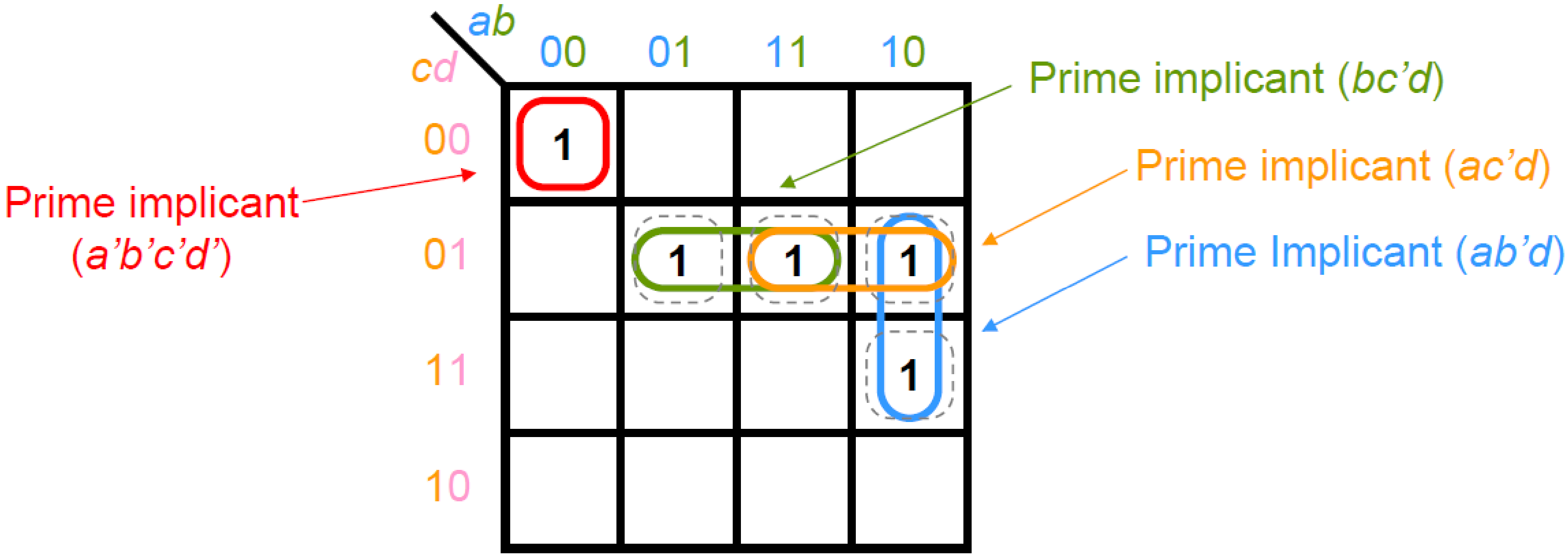
- Booleans function to be minimized by K-map are always in Canonical SOP or POS (will discuss later) form
- Arrange cells in 1-bit difference
- Group adjacent cells in group size of 2^n, e.g. 2, 4, 8
- Apply adjacency law

2.2 Minimization using Karnaugh Map

- Group adjacent cells in group size of 2^n, e.g. 2, 4, 8
- Rules:
  1. Find the fewest groups that can cover all cells marked with 1s.
  2. The groups should be as large as possible.
- Goal:
  1. Reduce the number of product terms to minimum
  2. Save the cost

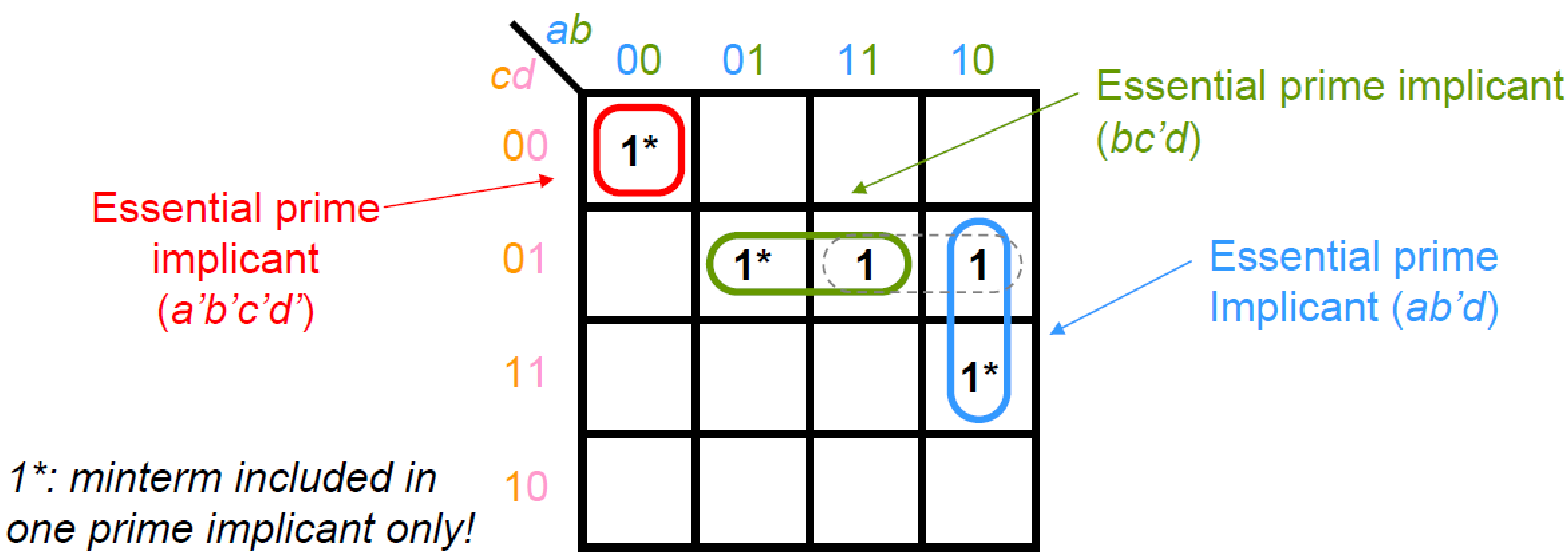
Terminology

Prime Implicant: An implicant that is not fully contained in any one other implicant.



Terminology

Essential Prime Implicant: If a minterm is included in only one prime implicant, that prime implicant is essential prime implicant.

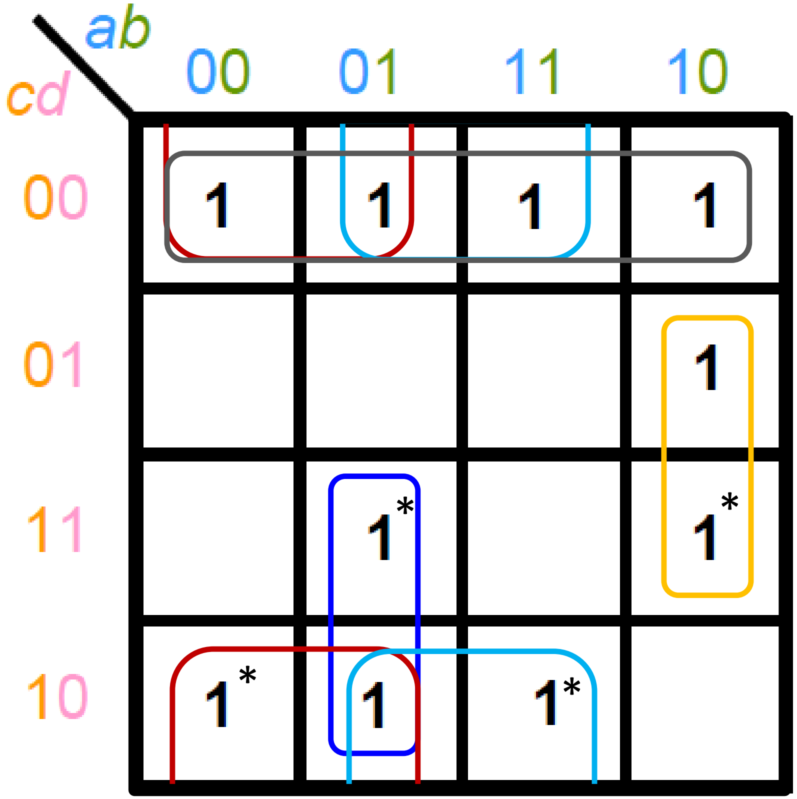


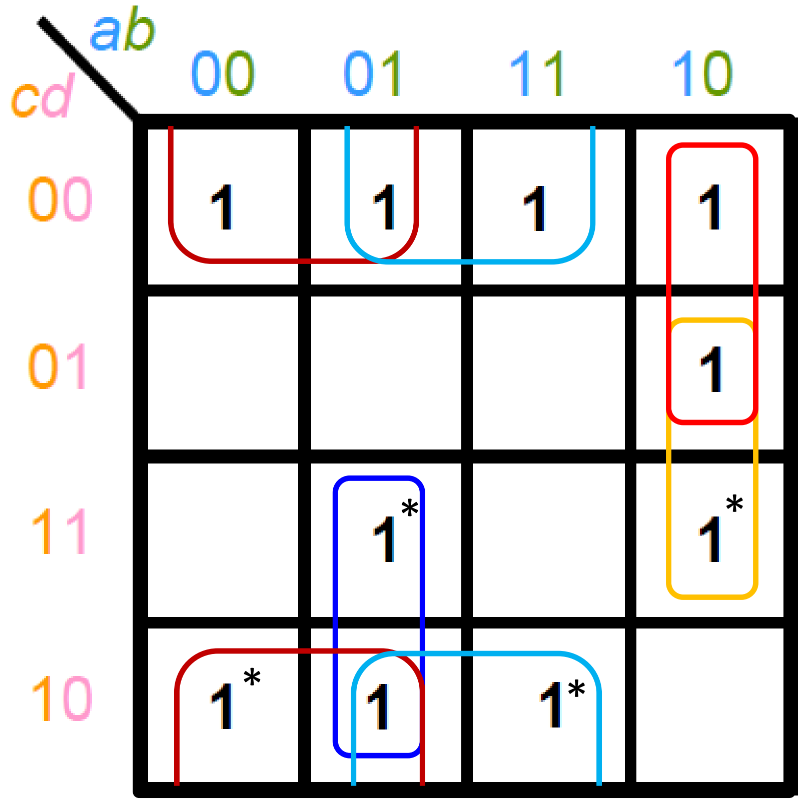
Systematic Approach

- Rules:
  1. Find the fewest groups that can cover all cells marked with 1s.
  2. The groups should be as large as possible.
- Approach:
  1. Determine all PIs.
  2. Select EPIs.
  3. Add PI to include the remaining minterm.

Exercise

1. Identify all PIs.
2. Select all EPIs.
3. Add PIs of remaining minterms.





POS

Simplify $f(a, b, c, d) = \Sigma m(0, 1, 2, 5, 8, 9, 10)$ in POS form

cd \ ab	00	01	11	10
00	1	0	0	1
01	1	1	0	1
11	0	0	0	0
10	1	0	0	1

Fill the 1s and 0s into the map

cd \ ab	00	01	11	10
00	1	0	0	1
01	1	1	0	1
11	0	0	0	0
10	1	0	0	1

Group the 0s using the same procedure as grouping the 1s

$f'(a, b, c, d) = ab + cd + bd'$

$f(a, b, c, d) = (a'+b')(c'+d')(b'+d)$

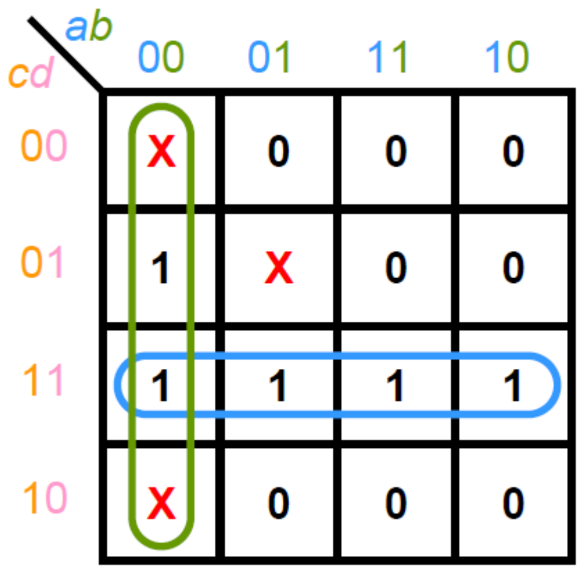
2.3 Boolean Functions with Don't Care Cases

The output of Boolean functions are **incompletely specified functions**,

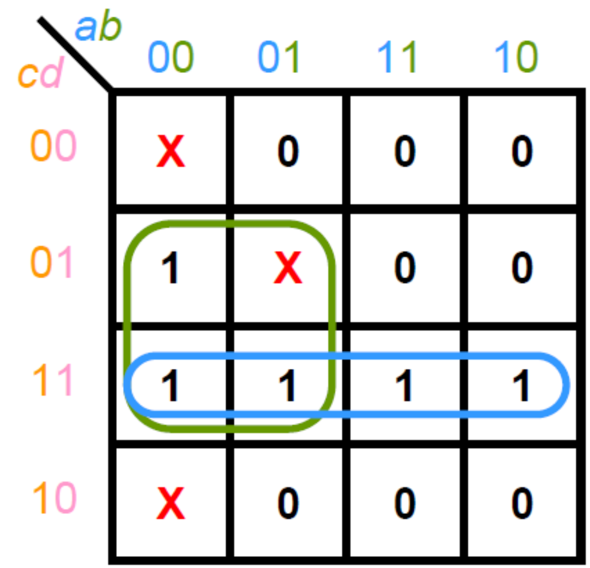
- For some input conditions, the outputs are unspecified
- Input condition has no effects to the function
- Output values are defined as **don't Care**
- Don't Care term can be minterm / maxterms
- Don't Care term indicates by an ×, *d*, ϕ or φ

Solutions

1. Identify PIs that must include all 1s but don't care term × is optional.
2. Use × when possible to create larger group size.
3. Select the EPIs first , then remaining PIs



$f(a, b, c, d) = a'b' + cd$



$f(a, b, c, d) = a'd + cd$

Choose to include those Xs that give largest PIs

2.4 Quine-McCluskey (QM) Method

- Developed by W. V. Quine and E. J. McCluskey in 1956
- Functionally identical to Karnaugh map
- More efficient in computer algorithms
- Ease to handle large number of variables

For number of variables that is less than or equal to 4, we use K-map; otherwise, QM method will be more efficient.

Procedure of QM-method (4-variable)

Step 1: Partition (Group minterms by the number of 1's)

w	x	y	z	Minterms
0	0	0	0	m_0
0	0	0	1	m_1
0	0	1	0	m_2
0	0	1	1	m_3
0	1	0	0	m_4
0	1	0	1	m_5
0	1	1	0	m_6
0	1	1	1	m_7
1	0	0	0	m_8
1	0	0	1	m_9
1	0	1	0	m_{10}
1	0	1	1	m_{11}
1	1	0	0	m_{12}
1	1	0	1	m_{13}
1	1	1	0	m_{14}
1	1	1	1	m_{15}

Minterms	$wxyz$
m_0	0000
m_1	0001
m_2	0010
m_4	0100
m_8	1000
m_3	0011
m_5	0101
m_6	0110
m_9	1001
m_{10}	1010
m_{12}	1100
m_7	0111
m_{11}	1011
m_{13}	1101
m_{14}	1110
m_{15}	1111

Procedure of QM-method (4-variable)

Step 1: Partition (Group minterms by the number of 1's)

Simplify $f(a, b, c, d) = \Sigma m(1, 4, 5, 6, 8, 9, 10, 12, 14)$

Minterms	$abcd$
m_1	0001
m_4	0100
m_8	1000
m_5	0101
m_6	0110
m_9	1001
m_{10}	1010
m_{12}	1100
m_{14}	1110

Procedure of QM-method (4-variable)

Step 2: Combine (Apply adjacency property to each pair of terms in consecutive groups)

- Combine adjacent group implicants into (n-1) variable implicants
- Mark the changed bit with "-" and tick the combined impicants

Minterms	$abcd$
m_1	0001 ✓
m_4	0100
m_8	1000
m_5	0101 ✓
m_6	0110
m_9	1001 ✓
m_{10}	1010
m_{12}	1100
m_{14}	1110

Minterms	$abcd$
m_1, m_5	0-01
m_1, m_9	-001

Procedure of QM-method (4-variable)

Step 2: Combine (Apply adjacency property to each pair of terms in consecutive groups)

- Combine adjacent group implicants into (n-1) variable implicants
- Mark the changed bit with "-" and tick the combined impicants

Minterms	$abcd$
m_1	0001 ✓
m_4	0100 ✓
m_8	1000 ✓
m_5	0101 ✓
m_6	0110 ✓
m_9	1001 ✓
m_{10}	1010 ✓
m_{12}	1100 ✓
m_{14}	1110 ✓

Minterms	$abcd$
m_1, m_5	0-01
m_1, m_9	-001
m_4, m_5	010-
m_4, m_6	01-0 ✓
m_4, m_{12}	-100 ✓
m_8, m_9	100-
m_8, m_{10}	10-0
m_8, m_{12}	1-00
m_6, m_{14}	-110 ✓
m_{10}, m_{14}	1-10
m_{12}, m_{14}	11-0 ✓

Minterms	$abcd$
m_4, m_6, m_{12}, m_{14}	-1-0

Procedure of QM-method (4-variable)

Step 3: Identify Prime Implicants (PIs)

-All unmarked terms

Minterms	$abcd$
m_1	0001 ✓
m_4	0100 ✓
m_8	1000 ✓
m_5	0101 ✓
m_6	0110 ✓
m_9	1001 ✓
m_{10}	1010 ✓
m_{12}	1100 ✓
m_{14}	1110 ✓

Minterms	$abcd$
m_1, m_5	0-01 PI_3
m_1, m_9	-001 PI_4
m_4, m_5	010- PI_5
m_4, m_6	01-0 ✓
m_4, m_{12}	-100 ✓
m_8, m_9	100- PI_6
m_8, m_{10}	10-0 ✓
m_8, m_{12}	1-00 ✓
m_6, m_{14}	-110 ✓
m_{10}, m_{14}	1-10 ✓
m_{12}, m_{14}	11-0 ✓

Minterms	$abcd$
m_4, m_6, m_{12}, m_{14}	-1-0 PI_1
$m_8, m_{10}, m_{12}, m_{14}$	1--0 PI_2

Procedure of QM-method (4-variable)

Step 4: Generate PI chart

- Identify minterms that are covered by only 1 PI
- Identify essential PIs

$f(a, b, c, d) = \Sigma m(1, 4, 5, 6, 8, 9, 10, 12, 14)$

PI	Minterms	$abcd$	1	4	5	6	8	9	10	12	14
PI_1	m_4, m_6, m_{12}, m_{14}	-1-0		x		x				x	x
PI_2	$m_8, m_{10}, m_{12}, m_{14}$	1--0					x		x	x	x
PI_3	m_1, m_5	0-01	x		x						
PI_4	m_1, m_9	-001	x					x			
PI_5	m_4, m_5	010-		x	x						
PI_6	m_8, m_9	100-					x	x			

Procedure of QM-method (4-variable)

Step 4: Generate PI chart

- Identify minterms that are covered by only 1 PI

- Identify essential PIs

PI	Minterms	$abcd$	1	4	5	6	8	9	10	12	14
PI_1	m_4, m_6, m_{12}, m_{14}	-1-0		x		x				x	x
PI_2	$m_8, m_{10}, m_{12}, m_{14}$	1--0					x		x	x	x
PI_3	m_1, m_5	0-01	x		x						
PI_4	m_1, m_9	-001	x					x			
PI_5	m_4, m_5	010-		x	x						
PI_6	m_8, m_9	100-					x	x			

$\therefore$ PI_1 and PI_2 are essential PIs.

Procedure of QM-method (4-variable)

Step 5: Reduce PI chart

- Remove the rows of EPIs and the columns that covered by them

PI	Minterms	$abcd$	1	5	9
PI_3	m_1, m_5	0-01	x	x	
PI_4	m_1, m_9	-001	x		x
PI_5	m_4, m_5	010-		x	
PI_6	m_8, m_9	100-			x

Solution 1

PI	Minterms	$abcd$	1	5	9
PI_3	m_1, m_5	0-01	x	x	
PI_4	m_1, m_9	-001	x		x

Solution 2

PI	Minterms	$abcd$	1	5	9
PI_3	m_1, m_5	0-01	x	x	
PI_6	m_8, m_9	100-			x

Solution 3

PI	Minterms	$abcd$	1	5	9
PI_4	m_1, m_9	-001	x		x
PI_5	m_4, m_5	010-		x	

Procedure of QM-method (4-variable)

Step 6: Express the Boolean Function

PI	Minterms	$abcd$
PI_1	m_4, m_6, m_{12}, m_{14}	-1-0
PI_2	$m_8, m_{10}, m_{12}, m_{14}$	1--0
PI_3	m_1, m_5	0-01
PI_4	m_1, m_9	-001

$$f(a,b,c,d) = \mathrm{PI}_1 + \mathrm{PI}_2 + \mathrm{PI}_3 + \mathrm{PI}_4$$

$$= bd' + ad' + a'c'd + b'c'd$$

PI	Minterms	$abcd$
PI_1	m_4, m_6, m_{12}, m_{14}	-1-0
PI_2	$m_8, m_{10}, m_{12}, m_{14}$	1--0
PI_3	m_1, m_5	0-01
PI_6	m_8, m_9	100-

$$f(a,b,c,d) = \mathrm{PI}_1 + \mathrm{PI}_2 + \mathrm{PI}_3 + \mathrm{PI}_6$$

$$= bd' + ad' + a'c'd + ab'c'$$

PI	Minterms	$abcd$
PI_1	m_4, m_6, m_{12}, m_{14}	-1-0
PI_2	$m_8, m_{10}, m_{12}, m_{14}$	1--0
PI_4	m_1, m_9	-001
PI_5	m_4, m_5	010-

$$f(a,b,c,d) = \mathrm{PI}_1 + \mathrm{PI}_2 + \mathrm{PI}_4 + \mathrm{PI}_5$$

$$= bd' + ad' + b'c'd + a'bc'$$

Exercise (Don't Care Case)

Step 1-3 (Partition, Combine, List PIs): Include Don't Care minterms

Simplify $f(a, b, c, d) = \Sigma m(4, 8, 9, 10, 12, 15) + \Sigma d(2, 6, 13)$

Minterms	$abcd$
m_2	0010 ✔
m_4	0100 ✔
m_8	1000 ✔
m_6	0110 ✔
m_9	1001 ✔
m_{10}	1010 ✔
m_{12}	1100 ✔
m_{13}	1101 ✔
m_{15}	1111 ✔

Minterms	$abcd$
m_2, m_6	0-10 PI_2
m_2, m_{10}	-010 PI_3
m_4, m_6	01-0 PI_4
m_4, m_{12}	-100 PI_5
m_8, m_9	100- ✔
m_8, m_{10}	10-0 PI_6
m_8, m_{12}	1-00 ✔
m_9, m_{13}	1-01 ✔
m_{12}, m_{13}	110- ✔
m_{13}, m_{15}	11-1 PI_7

Minterms	$abcd$
m_8, m_9, m_{12}, m_{13}	1-0- PI_1

Exercise (Don't Care Case)

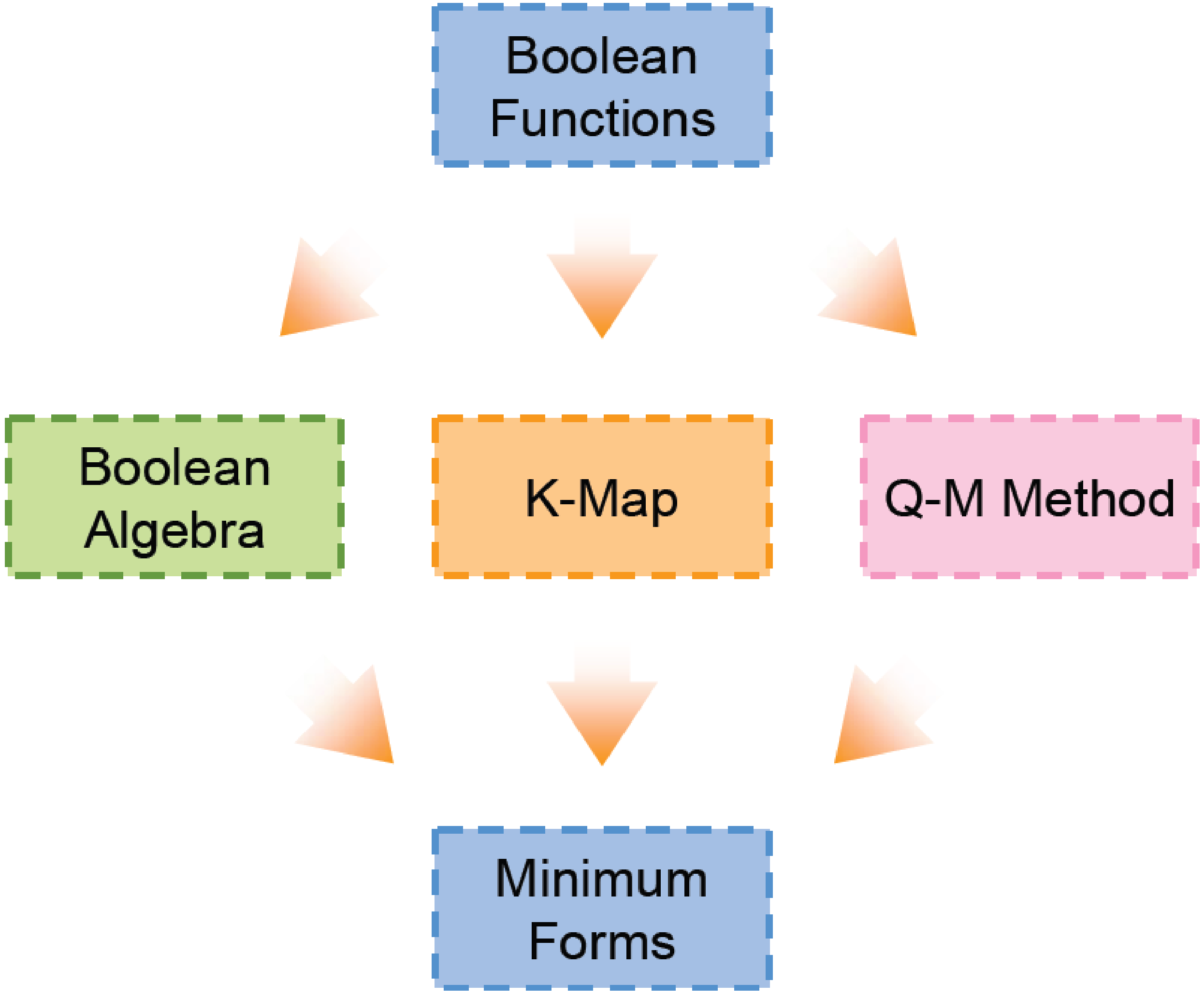
Step 4: Generate PI chart

- Exclude Don't Care Minterms

Simplify $f(a, b, c, d) = \Sigma m(4, 8, 9, 10, 12, 15) + \Sigma d(2, 6, 13)$

PI	Minterms	$abcd$	4	8	9	10	12	15
PI_1	m_8, m_9, m_{12}, m_{13}	1-0-						
PI_2	m_2, m_6	0-10						
PI_3	m_2, m_{10}	-010						
PI_4	m_4, m_6	01-0						
PI_5	m_4, m_{12}	-100						
PI_6	m_8, m_{10}	10-0						
PI_7	m_{13}, m_{15}	11-1						

Summary



Boolean Functions
Boolean Algebra
K-Map
Q-M Method
Minimum Forms

Multiple Output Problems

$c \backslash ab$	00	01	11	10
0	m_0	m_2	m_6	m_4
1	m_1	m_3	m_7	m_5

$$f(a,b,c) = \sum m(2,3,7)$$

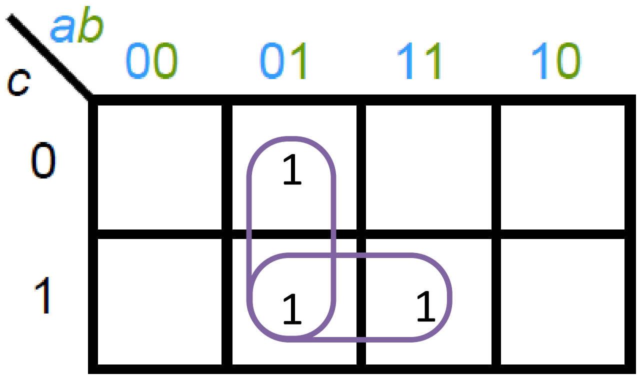
$c \backslash ab$	00	01	11	10
0		1		
1		1	1	

$$g(a,b,c) = \sum m(4,5,7)$$

$c \backslash ab$	00	01	11	10
0				1
1			1	1

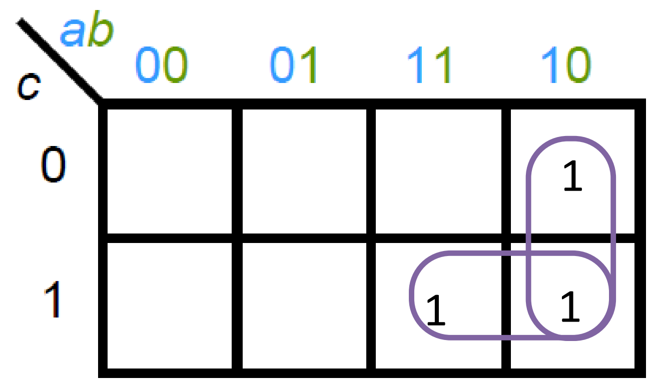
Multiple Output Problems

$f(a,b,c) = \sum m(2,3,7)$



$f(a,b,c) = a'b + bc$

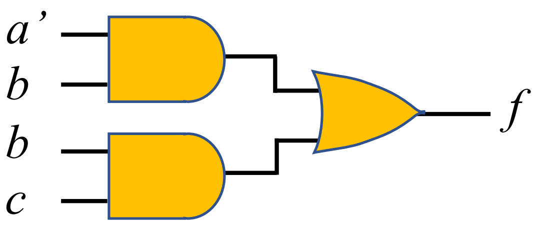
$g(a,b,c) = \sum m(4,5,7)$



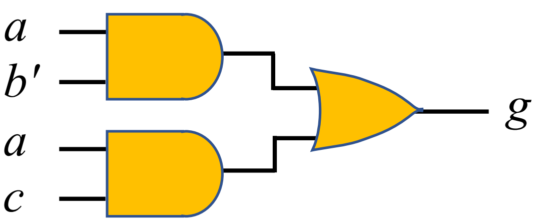
$g(a,b,c) = ab' + ac$

Multiple Output Problems

$f(a, b, c) = a'b + bc$



$g(a, b, c) = ab' + ac$



12 gate inputs and 6 gates (how to reduce the cost?)

Multiple Output Problems

c \ ab	00	01	11	10
0	m_0	m_2	m_6	m_4
1	m_1	m_3	m_7	m_5

$$f(a,b,c) = \sum m(2,3,7)$$

c \ ab	00	01	11	10
0		1		
1		1	1	

$$f(a,b,c) = a'b + bc$$

$$g(a,b,c) = \sum m(4,5,7)$$

c \ ab	00	01	11	10
0				1
1			1	1

$$g(a,b,c) = ab' + ac$$

Multiple Output Problems

$c \backslash ab$	00	01	11	10
0	m_0	m_2	m_6	m_4
1	m_1	m_3	m_7	m_5

$f(a,b,c) = \sum m(2,3,7)$

$c \backslash ab$	00	01	11	10
0		1		
1		1	1	

$f(a,b,c) = a'b + abc$

$g(a,b,c) = \sum m(4,5,7)$

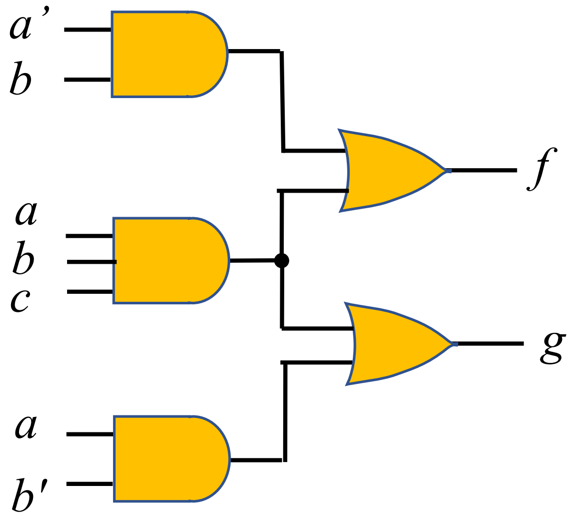
$c \backslash ab$	00	01	11	10
0				1
1			1	1

$g(a,b,c) = ab' + abc$

Multiple Output Problems

$f(a,b,c) = a'b + abc$

$g(a,b,c) = ab' + abc$



11 gate inputs and 5 gates!!!

Multiple Output Problems

Minterms	abc
m_0	000
m_1	001
m_2	010
m_4	100
m_3	011
m_5	101
m_6	110
m_7	111

$$f(a,b,c) = \sum m(2,3,7)$$

$$g(a,b,c) = \sum m(4,5,7)$$

- *fg*: mark as '-' when exist in respective function, else '0'.
- Terms can be combined when they have a common '-'
- After combination, any tag with a '0' will be '0', else '-'.
- Terms are marked if they are covered by the combined terms.

Minterms	abc fg
m_2	010 -0 ✓
m_4	100 0- ✓
m_3	011 -0 ✓
m_5	101 0- ✓
m_7	111 --

Minterms	abc fg
m_2,m_3	01- -0
m_4,m_5	10- 0-
m_3,m_7	-11 -0
m_5,m_7	1-1 0-

Multiple Output Problems

Minterms	abc fg	
m_2	010 -0 ✓	
m_4	100 0- ✓	
m_3	011 -0 ✓	
m_5	101 0- ✓	
m_7	111 --	PI$_1$

Minterms	abc fg	
m_2,m_3	01- -0	PI$_2$
m_4,m_5	10- 0-	PI$_3$
m_3,m_7	-11 -0	PI$_4$
m_5,m_7	1-1 0-	PI$_5$

PI	Minterms	abc	fg	f			g		
				2	3	7	4	5	7
PI$_1$	m_7	111	--			x			x
PI$_2$	m_2, m_3	01-	-0	x	x				
PI$_3$	m_4, m_5	10-	0-				x	x	
PI$_4$	m_3, m_7	-11	-0		x	x			
PI$_5$	m_5, m_7	1-1	0-					x	x

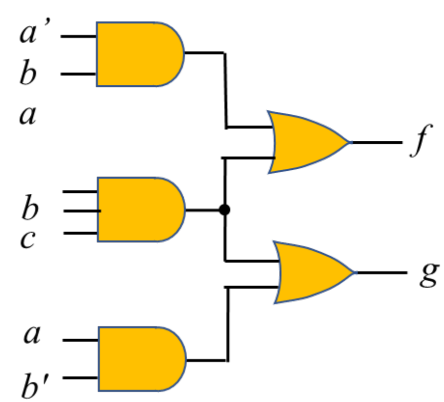
Multiple Output Problems

PI	Minterms	abc	fg	f			g		
				2	3	7	4	5	7
PI_1	m_7	111	--			x			x
PI_2	m_2, m_3	01-	-0	x	x				
PI_3	m_4, m_5	10-	0-				x	x	
PI_4	m_3, m_7	-11	-0		x	x			
PI_5	m_5, m_7	1-1	0-					x	x

PI	Minterms	abc	fg	f	g
				7	7
PI_1	m_7	111	--	x	x
PI_4	m_3, m_7	-11	-0	x	
PI_5	m_5, m_7	1-1	0-		x

$$f(a,b,c) = \mathrm{PI}_1 + \mathrm{PI}_2 = abc + a'b$$

$$g(a,b,c) = \mathrm{PI}_1 + \mathrm{PI}_3 = abc + ab'$$



Exercise

Reduce the following functions that will use the least number of gates and gate inputs.

$f(a,b,c,d)$

cd \ ab	00	01	11	10
00	1	1		1
01	1	1		1
11				
10	1	1		

$g(a,b,c,d)$

cd \ ab	00	01	11	10
00				1
01				1
11			1	1
10	1	1	1	1