EE 3301 Solutions

Semester A 2022/23

Question 1 (25 marks)

(1.1) **(1 mark)** Write down your 10-digit number in the following format. **For example,** if your number is **9753689124,** write**:**

N1 = 9, N2 = 7, N3 = 5, N4 = 3, N5 = 6, N6 = 8, N7 = 9, N8 = 1, N9 = 2, N10= 4.

Consider four cities (nodes). The cost of a cable (link) (in units of 100,000 dollars) that connects any pair of these four cities is give in the following table.

	1	2	3	4
1		N1	N2	N3
2			N4	N5
3				N6
4				

For example, we can see from the table that the cost of the cable to connect City 1 and City 2 is N1.

Remark: The value Ni is the ith digit of your 10-digit number.

Solution

1.1 For the number **9753689124,** we write (as mentioned)**:**

N1 = 9, N2 = 7, N3 = 5, N4 = 3, N5 = 6, N6 = 8, N7 = 9, N8 = 1, N9 = 2, N10= 4.

Then the cost of a cable (link) (in units of 100,000 dollars) that connects any pair of these four cities is give in the following table.

	1	2	3	4
1		9	7	5
2			3	6
3				8
4				

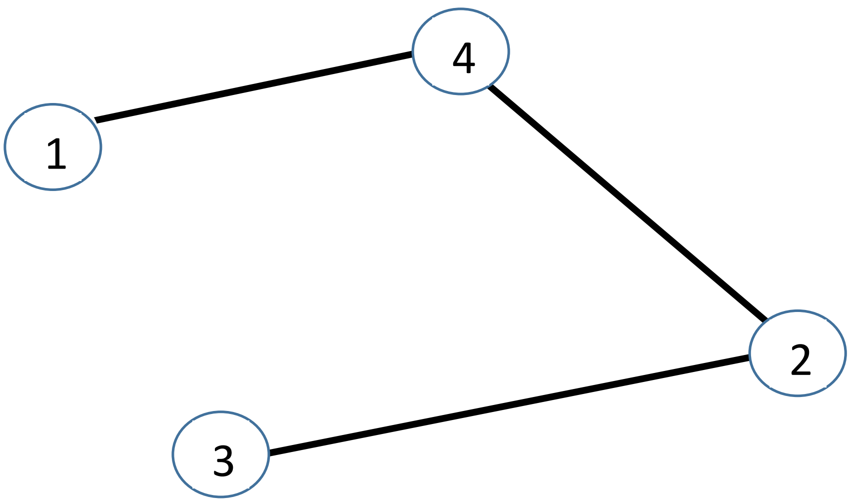
Two cities are said to be connected if there is a path between them.
(1.2) **(24 marks)** Answer the following questions

(a) You are required to connect the four cities by cables (links) at minimum cost such that all the four cities are connected to each other. What algorithm will you use to find the optimal solution? Provide the relevant optimal graph and the total cost. Show all steps. (5 marks)

Solution

We denote C(i,j) = C(j,i) to be the cost between nodes i and j.
The algorithm is Prim's Minimum Spanning Tree (MST) algorithm.
We start with Node 1. And grow the tree as follows. We add Node 4 because C(1,4)=5 and 5 = min(9,7,5). And we add the link (1,4) to the tree. Next, we add Node 2 because C(4,2) = 6 which is smaller than C(1,3) = 7 and C(2,1) = 9, so link (2,4) is added to the tree. Finally, we add Node 3 to the tree with the link (2,3) because C(2,3) = 3 = min(3,7,8).

Therefore, the links (1,4), (2,4) and (2,3) are in the spanning tree. The optimal graph is as follows.



C(1,4) = 5
C(2,4) = 6
C(2,3) = 3

And the total cost is equal to: 5 + 6 + 3 = 14.

(b) You are required to connect the four cities by cables (links) at minimum cost such that all the four cities remain connected to each other even if any one of the links break. Provide the relevant optimal graph and the total cost. Justify your answer. (5 marks)

Solution

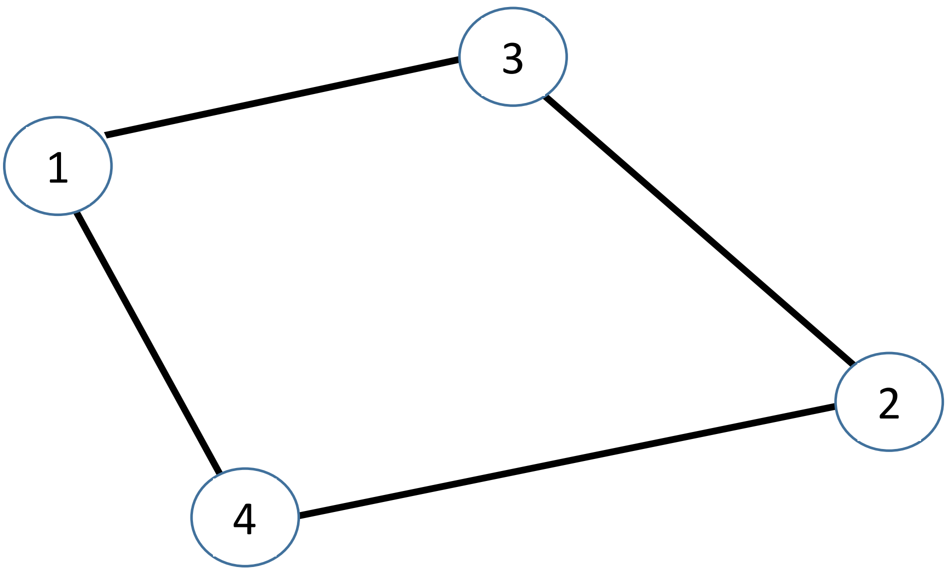
The ring topology has the least number of links such that the network remains connected subject to any link failure. Clearly in ring topology the network remains connected subject to any link failure and ring topology has n links what is only one less link than a tree topology.

All the possibilities of ring topologies are

1. 1-2-3-4-1 The cost is: C(1,2) + C(2,3) + C(3,4) + C(4,1) = 9+3+8+5 = 25
2. 1-2-4-3-1 The cost is: C(1,2) + C(2,4) + C(4,3) + C(3,1) = 9+6+8+7 = 30
3. 1-3-4-2-1 Same as 2
4. 1-3-2-4-1 The cost is: C(1,3) + C(3,2) + C(2,4) + C(4,1) = 7+3+6+5 = 21
5. 1-4-3-2-1 same as 1
6. 1-4-2-3-1 same as 4

Therefore, the least cost ring topology is 1-3-2-4-1 with the total cost of 21.

The optimal graph is as follows.



This is the least cost topology that is resilient to any link failure.

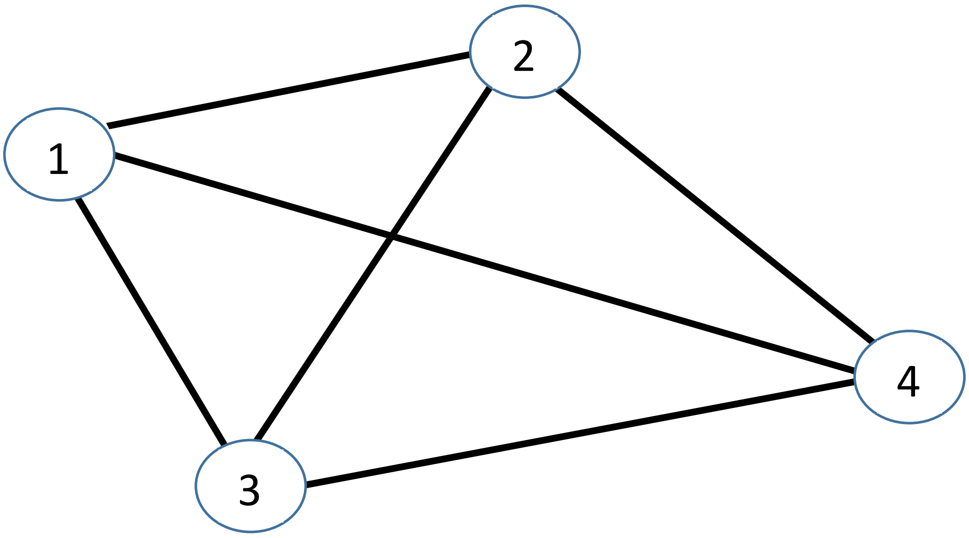
(c) Consider the fact that while a repair team is sent to repair a broken cable, another cable may break. Now you are required to connect the four cities by cables (links) at minimum cost such that all the four cities remain connected to each other even if any two of the links break. Provide the relevant optimal graph and the total cost. Justify your answer. (5 marks)

Solution

To connect the 4 nodes, the optimal graph now must be a fully connected graph (fully meshed graph) as shown below.

As we aim for resilience subject to 2-link failures, notice that each node must have more than two links adjacent to it. Otherwise, if they both fail, the node is disconnected. In a 4-node graph, only the fully connected graph topology satisfies this requirement. Notice also that if any two links are deleted the graph is still connected.

The optimal graph is as follows.



The total cost is

C(1,2) + C(1,3) + C(1,4) + C(2,3) + C(2,4) + C(3,4) = 9+7+5+3+6+8 = 38.

(d) Is the graph obtained in (a) above bipartite or non-bipartite? Justify your answer. (3 marks)

Answer: It is bipartite because its nodes can be divided into two disjoint sets U and V such that every edge connects a node in U to one in V. In particular, set U = {Node 1, Node 2} and V = Node 3. Node 4}. Then every edge in the graph connects a node in U to a Node in V.

Alternative answer: It is bipartite because it does not contain an odd circle.

(e) Is the graph obtained in (b) above bipartite or non-bipartite? Justify your answer. (3 marks)

Answer: It is bipartite because its nodes can be divided into two disjoint sets U and V, such that every edge connects a node in U to one in V. In particular, set U = {Node 1, Node 2} and V = Node 3. Node 4}. Then every edge in the graph connects a node in U to a Node in V.

Alternative answer: It is bipartite because it does not contain an odd circle.

(f) Does the graph obtained in (c) above have a Euler cycle? If the answer is yes, find it, and if the answer is no, justify your answer. (3 marks)

Answer: The graph does not have an Euler cycle because it has nodes with odd degree.

Question 2 (15 marks)

Consider again the four cities (nodes) from Problem 1. Problem 1 was about connectivity. This problem (Problem 2) is about capacity assignments. The aim here is to find the minimal capacity required on each link considering the resilience requirements discussed in Problem 1 and the end-to-end demands between the various pairs of nodes. The demands as well as the capacity assignments are bidirectional. The traffic demands $\{D_{ij}\}$ between the various pairs of nodes are given in the following table (in units of Tb/s).

	1	2	3	4
1		N9	N1	N6
2			N3	N7
3				N8
4				

Remark: The value Ni is the ith digit of your 10-digit number.

For example, we can see from the table that the total traffic demand (in both directions) between City 1 and City 2 is N9. The units of the traffic demands could be Tb/s, but as the units are well understood, so you do not need to include them in your solutions.

For the number **9753689124,** we have

N1 = 9, N2 = 7, N3 = 5, N4 = 3, N5 = 6, N6 = 8, N7 = 9, N8 = 1, N9 = 2, N10= 4.

Thus, the table is:

	1	2	3	4
1		2	9	8
2			5	9
3				1
4				

(a) **(6 Marks)** Consider a network based on the graph obtained in Problem 1.2 (a). Find the minimal value of the capacity assigned to each link in that network such that the traffic demands between all pair of nodes are satisfied. Show clearly how you obtain the optimal link capacities.

Solution

For Link (1,4), the required capacity is equal to: $\{D_{14}\} + \{D_{12}\} + \{D_{13}\} = 8 + 2 + 9 = 19$
For Link (2,4), the required capacity is equal to: $\{D_{12}\} + \{D_{34}\} + \{D_{24}\} + \{D_{13}\} = 2 + 1 + 9 + 9 = 21$.
For Link (2,3), the required capacity is equal to: $\{D_{13}\} + \{D_{23}\} + \{D_{34}\} = 9 + 5 + 1 = 15$.

(b) **(9 Marks)** Consider a network based on the graph obtained in Problem 1 (b). Explain in no more than 100 words how you will find the minimal value of the capacity assigned to each link in that network such that the traffic demands between all pair of nodes are satisfied subject to the resilient requirement that these traffic demands are satisfied even if any single link in the network fails. Choose one of the links in the graph obtained in Problem 1 (b) and find the optimal capacity on this link. Show all steps.

Solution

For each case of link failure, we calculate the required capacity for the remaining network (without the failed link the way we did in Question 2 (a)). This will give us the required capacity on each link for each case of link failure. Then for each link we take the maximum of the capacity requirements values for that link obtained for each case of link failure which is the required result.

Let's choose Link (1,4),

If Link (1,3) fails

For Link (1,4), the required capacity is equal to: $\{D_{14}\} + \{D_{12}\} + \{D_{13}\} = 8 + 2 + 9 = 19$

If Link (2,3) fails

For Link (1,4), the required capacity is equal to: $\{D_{14}\} + \{D_{12}\} + \{D_{34}\} = 8 + 2 + 1 = 11$.

If Link (2,4) fails

For Link (1,4), the required capacity is equal to: $\{D_{14}\} + \{D_{24}\} + \{D_{34}\} = 8 + 9 + 1 = 11$.

If Link (1,4) fails

This case is not relevant for calculation of the capacity for Link (1,4)

Therefore, the capacity requirement for Link (1,4) is Max (19, 11, 11) = 19.

<u>Question 3 (15 marks)</u>

A company aims to assess the benefit they obtain in terms of its revenue from sales as a function of its investment in advertising (advertising budget). The following table provides data for revenues in thousands of dollars from sales and advertising budget for six different markets.

Market	1	2	3	4	5	6
Revenue	32	50	498	249	297	95
Advertising budget	6 + N1	10 + N2	99 + N3	47 + N4	59 + N5	19 + N6

Remark: The value Ni is the ith digit of your 10-digit number.
For example, if your 10-digit number is 5697483715, the table should be updated to:

Market	1	2	3	4	5	6
Revenue	32	50	498	249	297	95
Advertising budget	11	16	108	54	63	27

3.1. (1 mark) Write down your 10-digit number.

<u>Solution</u>

5697483715

3.2. (1 mark) Update the table using your specific Ni values.

<u>Solution</u>

Market	1	2	3	4	5	6
Revenue	32	50	498	249	297	95
Advertising budget	11	16	108	54	63	27

3.3 (5 marks) The aim is to use linear regression and to find a linear function in the form of $y = ax + b$ to fit these data. In particular, provide a convex optimization formulation for this regression with an objective to find the optimal values for a and b using the least squares method. The objective function must include the specific numbers in the above table - not just a formula.

Solution

Problem formulation

Minimize

$$S = \sum_{i=1}^{n} (y_i - ax_i - b)^2$$
$$= (32 - 11a - b)^2 + (50 - 16a - b)^2 + (498 - 108a - b)^2 + (249 - 54a - b)^2 + (297 - 63a - b)^2 + (95 - 27a - b)^2.$$

3.4. (8 marks) Solve the convex optimization problem formulated in sub-question 3.3 in two ways: (1) using Excel Solver and (2) obtain the optimal values of *a* and *b* using the analytical solution by Excel and observe the consistency between (1) and (2).

Remark: *Name the Excel file "*Q3_Regression*" and upload it with your answers on Canvas.*

Solution

The solutions are provided in the Excel file "Q3_Regression".

Solving the above optimization problem using Excel Solver, we obtain the following optimal value for the decision variables *a* and *b*.

a = 4.921271
b = -25.339

And the optimal value of the objective function is 470.5907358

Solving the above optimization problem using the analytic solution by Excel, we obtain the following optimal value for the decision variables *a* and *b*.

a = 4.921275

b = -25.3393.

We observe that the results are consistent with insignificant discrepancies.

Question 4 (15 marks)

Consider a communications system that serves three users.

The first of the three users has the utility function $U_1(x_1) = 1 – a^{-x_1}$.

The second has the utility function of $U_2(x_2) = 1 – b^{-x_2}$.

The third has the utility function of $U_3(x_3) = 1 – c^{-x_3}$.

where the values of the parameters *a*, *b* and *c* are based on your first three digits N1, N2 and N3 in your 10-digit code. The value of parameter *a* is according to the following table.

N1	*a*
1	1.61
2	1.65
3	1.71
4	1.73
5	1.76
6	1.81
7	1.82
8	1.85
9	1.89

The values of parameters *b* and *c* are given by *b* = 3 + N2/9 (rounded to 2 decimal places) and c = 50 + N3.

4.1 (1 mark) Write down the values of N1, N2 and N3 according to your 10-digit number and provide the values of *a, b,* and *c* based on the values of N1, N2 and N3.

Remark: As in previous questions, the value Ni is the ith digit of your 10-digit number.

Solution

N1 = 5, N2 = 6, N3 = 9.

a = 1.76

b = 3 + 6/9 = 3.67

c = 50 + 9 = 59.

4.2 (5 marks) These utility functions represent a measure of satisfaction of the users as a function of the service rate measured in units of [10 Mb/s]. The total service rate provided by the system is 30 Mb/s. This means that the sum $x_1 + x_2 + x_3$ cannot be more than 3. The service provider aims to find the optimal values for x_1, x_2 and x_3 to maximize the total utility U = $U_1(x_1) + U_2(x_2) + U_3(x_3)$ of the three users.

Formulate this problem as a convex optimization problem and solve it using Excel Solver. Use the File Name: "utility_Q4". In your formulation, substitute the relevant values of *a, b* and *c* for of the symbols *a*, *b* and *c*. Write down the optimal values of x_1, x_2 and x_3 and of $U_1(x_1)$, $U_2(x_2)$ and $U_3(x_3)$ as well as the optimal sum of the utilities $U_1(x_1) + U_2(x_2) + U_3(x_3)$.

<u>Solution</u>

Problem formulation

The objective function is:

$U_1(x_1) + U_2(x_2) + U_3(x_3)$
$= 1 - a^{-x_1} + 1 - b^{-x_2} + 1 - c^{-x_3}$
$= 3 - a^{-x_1} - b^{-x_2} - c^{-x_3}$
$= 3 - (1.76)^{-x_1} - (3.67)^{-x_2} - (59)^{-x_3}$

The problem formulation is

Maximize $\quad 3 - (1.76)^{-x_1} - (3.67)^{-x_2} - (59)^{-x_3}$
Subject to:
$x_1 + x_2 + x_3 \leq 3.$

Solving the above optimization problem using Excel Solver, we obtain the following optimal value for the decision variables x_1, x_2 and x_3.

x_1 = 1.19156

x_2 = 1.158668

x_3 = 0.649772

We observe that the two users with the slower increasing

4.3 (9 marks) Now reduce the total capacity limitation from 3 to 2 **and solve the new problem in the same Excel file** "utility_Q4". Again, write down the new optimal values of x_1, x_2 and x_3 and of $U_1(x_1)$, $U_2(x_2)$ and $U_3(x_3)$ as well as the optimal sum of the utilities $U_1(x_1) + U_2(x_2) + U_3(x_3)$.
Provide interpretation to the differences in the optimal results – in particular, observe how the different optimal values of the variables vary as the capacity limitation become more stringent and provide explanation. Consider TCP (non-real-time) versus UDP (real-time) connections. And explain their relationships to utility functions of different services. When you upload the Excel file "utility_Q4", keep the last value of 2 for the total capacity limitation.

<u>Solution</u>

The solutions are provided in the Excel file "utility_Q4". We use different sheets "capacity3" and "Capacity 2" for the solutions of 4.2 and 4.3, but students can just use the same sheet by just changing the capacity constraint value from 3 to 2.

Solving the above optimization problem using Excel Solver, for the case with total capacity constraint equal 2, we obtain the following optimal value for the decision variables x_1, x_2 and x_3.

x_1 = 0.556007

x_2 = 0.882335

x_3 = 0.561659

We observe that users with the slower increasing utility lose more capacity than users with higher increasing utility as a function of the bitrate they receive. This is understandable as the latter can benefit more from an increase in bitrate. In our case the first user has the slowest increasing utility and the third one has the fastest increasing utility.

TCP (that was designed for non-real-time services) retransmits lost packets and reacts cooperatively to congestion control signals by reducing its rate during congestion. This is because non-real-time services can tolerate delays. They may be considered as having slow-increasing utility functions. On the other hand, UDP (used for non-real-time services) does not reduce its rate during congestion as it cannot tolerate delays. Such real-time delay-sensitive services can be considered as having slow-increasing utility functions up to a certain point and then fast-increasing up to the desired bit-rate and then the shape is flat or slowly increasing.

Question 5 (20 marks)

Consider a telecommunications provider that offers a connection service between two points through one link.
In this question, we will use units of 10 Gb/s, so the capacity of the link available to the provider is equal to C, where
$C = 7$ if N1 < 4,
$C = 8$ if 7 > N1 ≥ 4,
and $C = 9$ if N1 ≥ 7.

Various enterprises will bid for capacity on this link and the provider will choose the set of enterprises that will receive the service so that its revenue is maximized. That is, the objective of the provider is to maximize the total monthly income it receives from the enterprises. The following table provides a list of bids made by five enterprises that includes capacity requirements in units of 10 Gb/s and monthly payment offers in units of 1000 dollars. Note that the bids are confidential during the bidding process, so that one bidder does not know the bids made by other bidders. After the provide makes the final decision on the choice of the set of enterprises that will receive the service, all the chosen enterprises receive their bitrate capacity in full and those who are not chosen received nothing.

Enterprise	Bitrate capacity	Monthly payment
1	2	17
2	6	13
3	3	16
4	1	18

5.1 Formulate this problem as a 0,1 knapsack problem.

Solution

N1 = 9, so $C = 9$.

The formulation if the problem is:

Max $17x_1 + 13x_2 + 16x_3 + 16x_4$

Subject to:

$2x_1 + 6x_2 + 3x_3 + x_4 \leq 9$

x_i is binary, $i = 1, 2, 3, 4$.

5.2. Solve this problem using Excel solver. Call the Excel file "knapsack_Q_5".

Solution

See attached Excel file "knapsack_Q_5". The solution is:

$x_1 = 1$

$x_2 = 0$

$x_3 = 1$

$x_4 = 1$

The optimal solution of the objective function is 51.

5.3. Provide a greedy algorithm for this problem, prove that it gives an optimal solution, use it to solve the problem, and compare your result with the result of 5.2.

Solution

Notice that in this problem, the bids can be ordered in sequence:

Sequence order	Enterprise	Bitrate capacity	Monthly payment
1	4	1	18
2	1	2	17
3	3	3	16
4	2	6	13

The *i*th element in this sequence offers more payment for less capacity requirement than any other j element (enterprise) in this sequence for any $j > i$. Therefore, the greedy algorithm we will use this priority order to assign capacity as long as the capacity constraint is not violated and the relevant decision variables will obtain the value 1. The remaining enterprises will not be assigned capacity and their relevant decision variables will obtain the value 0.

We will use a proof by contradiction to prove that it gives an optimal solution.
Assume that we have an optimal solution such there are i and j enterprises where Enterprise i offers more than Enterprise j and requires less capacity, but Enterprise j wins the bid (its relevant optimal decision variable value is equal to 1) and Enterprise i loses the bid (its relevant optimal decision variable value is equal to 0).
In this case, we can make Enterprise i win the bid and Enterprise j lose the bid because Enterprise i requires less capacity than Enterprise j so the capacity constraint is not violated, and we will obtain a higher value of the objective function. This proves that the assumption that the solution is optimal is false which leads to contradiction.

QED

5.4 Provide a dynamic programming formulation **for this problem**. That is, provide: optimal value function, recurrence relation and boundary conditions. Notice the bolded words "for this problem" so do not use general terms like "weight" or the general parameter N.

Solution

Optimal value function

S(k,w) = the maximum value for the total revenue of all possible solutions of bitrate capacity requirements less than or equal to *w* consisting only of 0 or 1 items of types *k, k+1, …, 4*.

Recurrence relation

$$S(k,w) = \max_{x_k=0,1 \text{ (weight} \le w)} [x_k v_k + S(k+1, w - x_k w_k)].$$

Boundary conditions

$S(5,w) = 0, \; w \geq 0.$

$S(k,w) = -\infty, \; w < 0 \; for\ all \; k.$

Question 6 (10 marks)

In no more than 100 words, discuss limitations of Excel Solver in solving practical problems. In your discussion, consider the fact that Excel Solver is based on mathematical optimization models and algorithms and the limitations of these models and algorithms apply also to Excel Solver.

Solution

Excel Solver is based on mathematical optimization models and algorithms that apply to these models. It provides optimization solutions using these algorithms. For example, the Simplex algorithm can solve optimization problems that are formulated as Linear Programming problems. limitations of these models and algorithms apply also to Excel Solver. In solving practical problems, the models we use are often approximations of practical problems, so the solution introduces errors that we must understand and assess. Another limitation is that practical problems may be of large scalability and in such problems the running time and/or accuracy of the solutions may be unacceptable.