In-Class Exercise 6

1. The continuous-time signal $x(t) = \sin(100\pi t + 1)$ is passed through an ideal continuous-time to discrete-time converter with the sampling period $T = 1/50$s to produce a discrete-time signal $x[n]$. Find $x[n]$. Can $x[n]$ uniquely represent $x(t)$?

2. Prove the multiplicative property of Fourier transform:

$$x_1(t) \cdot x_2(t) \leftrightarrow \frac{1}{2\pi} X_1(j\Omega) \otimes X_2(j\Omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(j\tau) X_2(j(\Omega - \tau)) d\tau$$

3. The continuous-time signal $x(t) = \sin(20\pi t) + \cos(40\pi t)$ is sampled at a sampling period T to obtain the discrete-time signal $x[n]$:

$$x[n] = \sin\left(\frac{\pi n}{5}\right) + \cos\left(\frac{2\pi n}{5}\right)$$

(a) Determine a choice for T consistent with this information.

(b) Is your choice for T in Part (a) unique? If so, explain why. If not, specify another choice of T consistent with the information given.

4. Consider sampling $x(t) = \cos(\Omega_0 t)$ with a sampling period T to produce $x[n]$. Determine the condition of Ω_0 and T if $x[n]$ is periodic.

5. The frequency which, under the sampling theorem, must be exceeded by the sampling frequency, is called the Nyquist rate. Determine the Nyquist rate of the following signal:

$$x(t) = \frac{\sin(8000t)}{\pi t}$$

Solution

1. Applying (7.1) we get:

$$x[n] = x(nT) = \sin(100\pi nT + 1) = \sin(2\pi n + 1) = \sin(1)$$

The $x(t)$ has a frequency of 100π and thus the sampling frequency should be larger than 200π in rad/s or 100 in Hz. However, the current sampling frequency is 50 Hz which is less than 100 Hz. As a result, $x[n]$ cannot uniquely represent $x(t)$.

We can also see that the constant $\sin(1)$ cannot represent the sinusoidal signal.

2.

Let $x(t) = x_1(t) \cdot x_2(t)$. Its Fourier transform is:

$$
\begin{aligned}
X(j\Omega) &= \int_{-\infty}^{\infty} x_1(t) \cdot x_2(t) e^{-j\Omega t} dt \\
&= \int_{-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(j\tau) e^{j\tau t} d\tau \right] \cdot x_2(t) e^{-j\Omega t} dt \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x_2(t) e^{-j(\Omega-\tau)t} dt \right] X_1(j\tau) d\tau \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_2(j(\Omega-\tau)) X_1(j\tau) d\tau
\end{aligned}
$$

3.(a)

$$x(nT) = \sin(20\pi nT) + \cos(40\pi nT)$$

Comparing it with $x[n] = \sin\left(\frac{\pi n}{5}\right) + \cos\left(\frac{2\pi n}{5}\right)$, we have:

$$20\pi nT = \frac{\pi n}{5} \quad \text{and} \quad 40\pi nT = \frac{2\pi n}{5}$$

Both give $T = 1/100$.

3.(b)

No. Another choice is $T = 11/100$:

$$\begin{aligned} x[n] &= x(nT) = \sin\left(20\pi n\frac{11}{100}\right) + \cos\left(40\pi n\frac{11}{100}\right) \\ &= \sin\left(\frac{11\pi n}{5}\right) + \cos\left(\frac{22\pi n}{5}\right) = \sin\left(\frac{\pi n}{5}\right) + \cos\left(\frac{2\pi n}{5}\right) \end{aligned}$$

Proof:

As in Example 7.3, we write more general equations:

$$20\pi nT = \frac{\pi n}{5} + 2k_1\pi \quad \text{and} \quad 40\pi nT = \frac{2\pi n}{5} + 2k_2\pi$$

where k_1 and k_2 are integers.

From the two equations, we get

$$100nT = n + 10k_1$$

and

$$200nT = 2n + 10k_2$$

As long as $2k_1 = k_2$, the equations are consistent. Choosing $k_1 = n$ yields $T = 11/100$.

4.

The discrete-time version of $x(t) = \cos(\Omega_0 t)$ is

$$x[n] = \cos(\Omega_0 nT), \quad \cdots, -1, 0, 1, \cdots$$

If $x[n]$ is periodic, we must have:

$$x[n] = x[n+N]$$

where $N > 0$ is the period in the sequence which should be an integer. We then have:

$$\cos(\Omega_0 nT) = \cos(\Omega_0 nT + 2K\pi) = \cos(\Omega_0(n+N)T) = \cos(\Omega_0 nT + N\Omega_0 T)$$

where $K > 0$ is another integer. We can deduce:

$$N = \frac{2K\pi}{\Omega_0 T}$$

If $x[n]$ is periodic, N must be an integer. That is, $\Omega_0 T$ is equal to a rational number times π.

5.

Using the result of Example 5.2, we know that $x(t) = \sin(W_0 t)/(\pi t)$ in the time domain corresponds to a rectangular pulse in the frequency domain in the frequency interval $[-W_0, W_0]$.

Hence the maximum signal frequency is 8000 rads^{-1} and the Nyquist rate is 16000 rads^{-1}.