EE3210
Signals and Systems

Part 6: Introduction to Fourier Analysis

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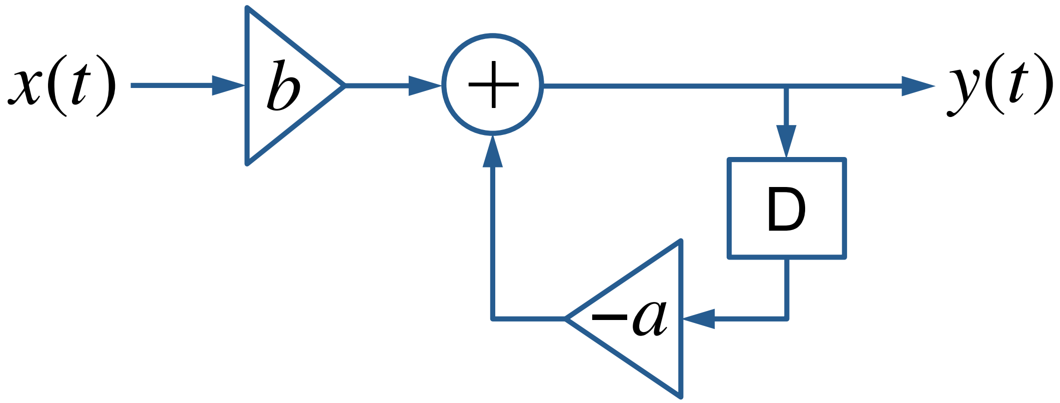
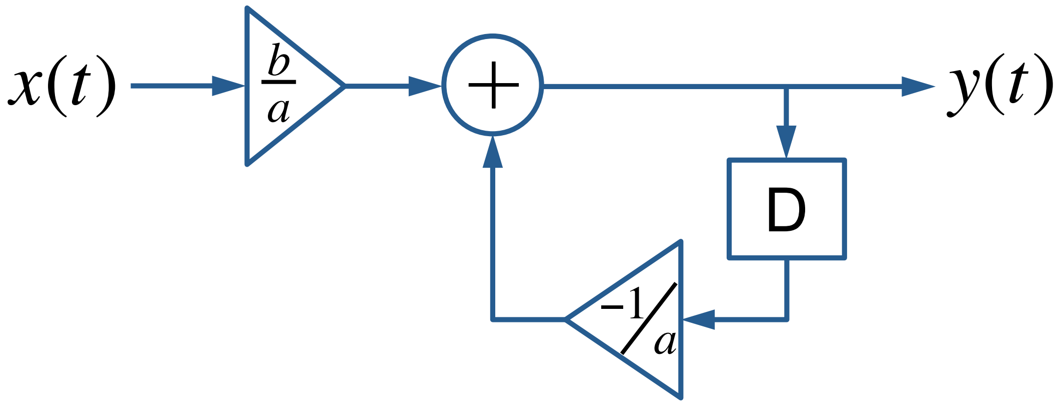
Changes of Part5_v1 Lecture Notes

- Page 41, change the figure



to



Changes of Part5_v1 Lecture Notes (cont.)

- Page 42, change the equation

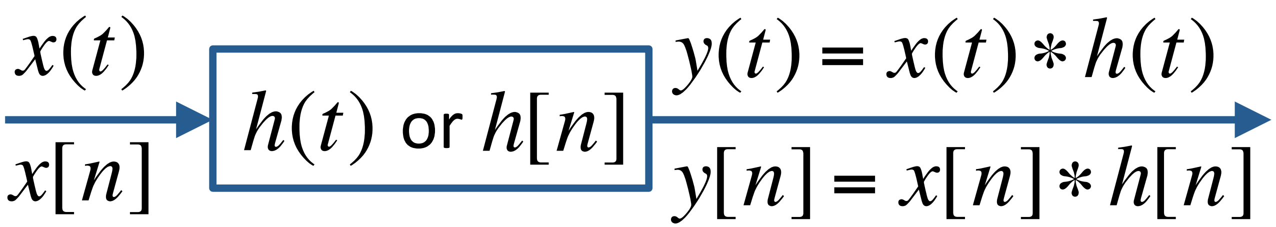
$$\int_{-\infty}^{t} \frac{dy(t)}{dt} = y(t) = \int_{-\infty}^{t} [bx(\tau) - ay(\tau)] d\tau$$

to

$$\int_{-\infty}^{t} \frac{dy(\tau)}{d\tau} d\tau = y(t) = \int_{-\infty}^{t} [bx(\tau) - ay(\tau)] d\tau$$

Recall: Unit Impulses as Basic Signals

- So far, with the representation of the input to an LTI system in terms of unit impulse signals, we have been able to develop convolution sum or convolution integral for analyzing LTI systems based on the notion of unit impulse response.
  - This analysis is, however, done purely in time domain, as all the relevant signals are represented as functions of time.



Complex Exponentials as Basic Signals

- Now, let us explore the representation of the input to an LTI system in terms of complex exponential signals.
- We will see that, by utilizing complex exponentials as basic signals, we will be able to develop more powerful frequency-domain methods for analyzing LTI systems.

Example 1

- Consider an LTI system for which the input $x(t)$ and the output $y(t)$ are related by

$$y(t) = x(t-3) \tag{1}$$

- If $x(t) = e^{j2t}$, i.e., the input is a complex exponential, then we have:

$$y(t) = e^{j2(t-3)} = e^{-j6}e^{j2t}$$

  - We observe that the output is the same complex exponential e^{j2t} scaled by a complex constant e^{-j6}.

Example 2

- Consider again the LTI system described by (1) with the input signal $x(t) = \cos(4t) + \cos(7t)$.
  - Using Euler's formula, we can rewrite $x(t)$ as

$$x(t) = \frac{1}{2}e^{j4t} + \frac{1}{2}e^{-j4t} + \frac{1}{2}e^{j7t} + \frac{1}{2}e^{-j7t}$$

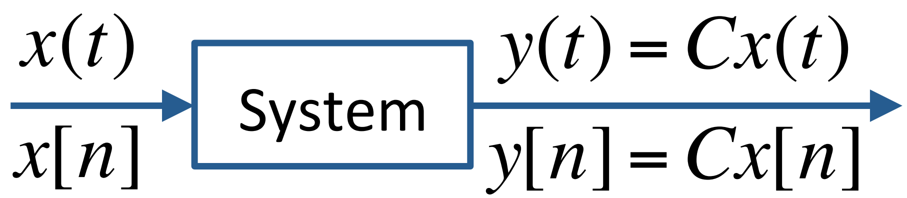
which is a linear combination of complex exponentials.

- From (1), we have:

$$y(t) = \frac{1}{2}e^{-j12}e^{j4t} + \frac{1}{2}e^{j12}e^{-j4t} + \frac{1}{2}e^{-j21}e^{j7t} + \frac{1}{2}e^{j21}e^{-j7t}$$

which is also a linear combination of the same complex exponential signals.

Eigenfunction of a System



- An input signal $x(t)$ or $x[n]$ to which the system responds by producing an output signal

$$y(t) = Cx(t) \text{ or } y[n] = Cx[n]$$

where C is a (possibly complex) constant, is referred to as an eigenfunction of the system.

  - C is referred to as the system's eigenvalue.

Complex Exponentials as Eigenfunctions of Continuous-Time LTI Systems

- Consider the set of complex exponential signals of the form e^{st} in continuous time, where s is a complex number.

- Consider a continuous-time LTI system with impulse response $h(t)$. For an input $x(t) = e^{st}$, we have:

$$y(t) = h(t) * x(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau$$

$$= \int_{-\infty}^{+\infty} h(\tau)e^{s(t-\tau)}d\tau = e^{st}\int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau$$

Complex Exponentials as Eigenfunctions of Continuous-Time LTI Systems (cont.)

- Let
$$H(s) = \int_{-\infty}^{+\infty} h(\tau) e^{-s\tau} d\tau \qquad (2)$$
and assume that the integral on the right-hand side of (2) converges.

- Then, the response of the system to e^{st} is of the form:
$$y(t) = H(s)e^{st}$$

- Hence, complex exponentials of the form e^{st} are eigenfunctions of continuous-time LTI systems.
  - $H(s)$ is the eigenvalue associated with e^{st}.

An Example

- Consider again the LTI system described by (1) on Page 5 with the input signal $x(t) = e^{j2t}$.

- From (1), we obtain the unit impulse response of the system as

$$h(t) = \delta(t-3)$$

- Thus, with $s = j2$ in this case, the eigenvalue $H(s)$ associated with the eigenfunction e^{j2t} is obtained from (2) as

$$H(s) = \int_{-\infty}^{+\infty} \delta(\tau - 3) e^{-s\tau} d\tau = e^{-3s} = e^{-j6}$$

Complex Exponentials as Eigenfunctions of Discrete-Time LTI Systems

- Consider the set of complex exponential signals of the form z^n in discrete time, where z is a complex number.
- Consider a discrete-time LTI system with impulse response $h[n]$. For an input $x[n] = z^n$, we have:

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k]$$

$$= \sum_{k=-\infty}^{+\infty} h[k]z^{n-k} = z^n \sum_{k=-\infty}^{+\infty} h[k]z^{-k}$$

Complex Exponentials as Eigenfunctions of Discrete-Time LTI Systems (cont.)

- Let $H[z] = \sum_{k=-\infty}^{+\infty} h[k]z^{-k}$ and assume that the summation on the right-hand side converges.
- Then, the response of the system to z^n is of the form:

$$y[n] = H[z]z^n$$

- Hence, complex exponentials of the form z^n are eigenfunctions of discrete-time LTI systems.
  - $H[z]$ is the eigenvalue associated with z^n.

Observations

- For both continuous time and discrete time, if the input to an LTI system could be represented as a linear combination of complex exponentials, then the output can be easily found, which is also a linear combination of the same complex exponential signals. That is:
  - Continuous-time LTI systems:

$$x(t) = \sum_k a_k e^{s_k t} \to y(t) = \sum_k a_k H(s_k) e^{s_k t}$$

  - Discrete-time LTI systems:

$$x[n] = \sum_k a_k z_k^n \to y[n] = \sum_k a_k H[z_k] z_k^n$$

Observations (cont.)

- This motivates us to consider the question of how broad a class of signals could be represented as a linear combination of complex exponentials.
  - We will first examine this question and develop Fourier series representation of periodic signals.
  - Then, we will study the problem and develop Fourier transform and its generalizations, i.e., Laplace transform and Z-transform, for aperiodic signals.
- This will not only allow us to calculate the response of more complex LTI systems, but also provide the basis for frequency-domain analysis of LTI systems.