

Test 3 (75 min.) Name: Ng Chung Wai Student ID: 5747463

Answer ALL questions.

1. (10 marks) Define a function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ by $f(x) = x_1 y_2 - x_2 y_1$, where y_1, y_2 are constants and $x = (x_1, x_2)$. Is f a linear function? Prove or disprove it.

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) \quad \text{where } \alpha = y_1, \beta = y_2$$

$$\alpha f(x) + \beta f(y) = \alpha(x_1 y_2 - x_2 y_1) + \beta(x_1 y_2 - x_2 y_1) = (\alpha + \beta)(x_1 y_1 - x_2 y_2)$$

Addition:

Multi:

2. (10 marks) Using Cauchy-Schwarz inequality to prove that

$$\sum_{m=1}^M (b_m - 1)^2 \geq \frac{1}{M} \left(\sum_{m=1}^M (b_m - 1) \right)^2$$

(Hint: Cauchy-Schwarz inequality: $|a^T b| \leq \|a\| \|b\|$.)

$$\sum (b_m - 1)^2 = \sum [(b_m - 1) + \dots + (b_m - 1)]^2$$

$$\frac{1}{M} \left[\sum (b_m - 1) \right]^2 = \frac{1}{M} \left[(b_m - 1) + \dots + (b_m - 1) \right]^2$$

$$\frac{1}{M} \left[\sum (b_m - 1) \right]^2 \leq \sum (b_m - 1)^2$$

$$\left[\sum (b_m - 1) \right]^2 \leq M \sum (b_m - 1)^2$$

$$\sum (b_m - 1)^2 \leq \sum (b_m - 1)^2 \leq M \sum (b_m - 1)^2$$

3. (10 marks) Let x and y be vectors in \mathbb{R}^3 , where $x = (1, 2, 3)$ and $y = (3, 2, 1)$. Is the span of x and y a subspace of \mathbb{R}^3 ? Prove or disprove it.

Yes, $\forall \alpha \in \mathbb{R}, y \in \mathbb{R}$, span of x and $y \in \mathbb{R}$

$$\text{Addition: } (1, 2, 3) + (3, 2, 1) = (4, 4, 4)$$

$$\text{Multiplication: } c(1, 2, 3) = (c, 2c, 3c)$$

4. Consider the equation $Ax = b$, where $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ and $b = \begin{bmatrix} 4 \\ 12 \end{bmatrix}$.

a) (7 marks) Determine the column space and the null space of A .

6 $C(A) = \{x \in \mathbb{R}^2 : \begin{bmatrix} 1 \\ 3 \end{bmatrix}\}$, basis is $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ since $2\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$

$Ax = 0 \quad \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_1 + 2x_2 = 0$

$N(A) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

b) (8 marks) Determine the general solution for x .

$Ax = b \quad \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \end{bmatrix}$

$x_1 + 2x_2 = 4$

GS: $x = 4 + \alpha \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, where $\alpha \in \mathbb{R}$

5. Consider the binary linear code C which appends four parity bits c_5, c_6, c_7, c_8 to four information bits u_1, u_2, u_3, u_4 in the following way. Put the four information bits in a 2×2 array and the four parity bits are obtained by two-dimensional even parity as shown below (i.e., each row and each column has an even number of 1's):

u_1	u_2	c_5
u_3	u_4	c_6
c_7	c_8	

a) (6 marks) The encoding function $f: \mathbb{B}^4 \rightarrow \mathbb{B}^8$ can be expressed in the form of $f(u) = uG$, where u is the row vector (u_1, u_2, u_3, u_4) . Determine the generator matrix G and the corresponding parity check matrix H .

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$G_{4 \times 8}$

$P_{4 \times 4}$

$\begin{matrix} & u & a & b & c \\ c & c & a & b & c \\ a & a & b & c & c \\ b & b & c & c & a \\ c & c & c & a & b \end{matrix}$

b) (6 marks) Is the function f surjective? Prove or disprove it.

Surjective

0

c) (6 marks) Determine the minimum distance of the code and state its error correction capability. Explain how you obtain the minimum distance.

$$\|x-y\|$$

d) (6 marks) Suppose the vector $(1,0,0,0,1,1,1,1)$ is received. Determine its syndrome.

Stallholses syndrome
Blzylers syndrome

e) (6 marks) Suppose nearest-neighbor decoding is used. Determine the decoder output for the received vector given in (d). Explain how you obtain the answer.

6. Consider the function $f_\theta: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $f_\theta(x) = A_\theta x$, where x is a 2-dim column vector, $A_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ is the rotation matrix with $0 \leq \theta < 2\pi$.

- a) (5 marks) Given any value of θ , is f_θ a linear function of x ? Prove or disprove it.

Suppose $B_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$

$$B_\theta A_\theta = \begin{pmatrix} 2\cos\theta & -2\sin\theta \\ 2\sin\theta & 2\cos\theta \end{pmatrix} = 2 \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

- b) (5 marks) Assume $\theta = \frac{\pi}{4}$. Write down the inverse function of $f_{\pi/4}$. Steps are not required.

as $\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$$f_{\frac{\pi}{4}} = A_{\frac{\pi}{4}} x$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} x$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} x$$

$$x = \sqrt{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} = f_{\frac{\pi}{4}}^{-1}$$

- c) (5 marks) Determine $f_{\pi/4} \circ f_{\pi/2}$, where the operator \circ denotes function composition. Steps are not required.

- d) (5 marks) Let $F = \{f_0, f_{\pi/4}, f_{\pi/2}, f_{3\pi/4}, f_\pi, f_{5\pi/4}, f_{3\pi/2}, f_{7\pi/4}\}$. Explain why $\langle F, \circ \rangle$ is a group. Your explanation can be based on the properties of the rotation matrix. Rigorous mathematical derivation is not required. CAN properties

closure
addition
identity matrix

- e) (5 marks) List all subgroups of $\langle F, \circ \rangle$.