In-Class Exercise 2

1. Consider a discrete-time system S with input $x[n]$ and output $y[n]$. This system is obtained through a series interconnection of a system S_1 followed by another system S_2. The input and output relationships for S_1 and S_2 are:

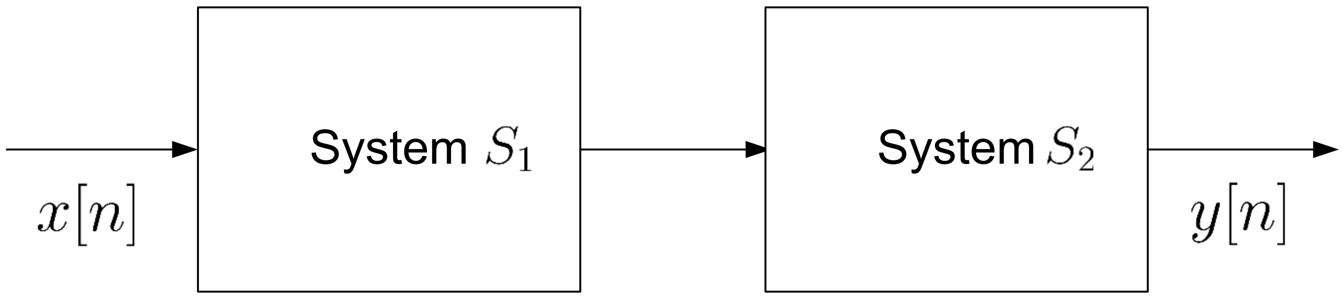
$$S_1 : y_1[n] = 2x_1[n] + 4x_1[n-1]$$

$$S_2 : y_2[n] = x_2[n-2] + \frac{1}{2}x_2[n-3]$$

where $x_1[n]$ and $x_2[n]$ denote inputs and $y_1[n]$ and $y_2[n]$ denote outputs.

(a) Determine the input-output relationship for system S, i.e., find the equation that relates $x[n]$ and $y[n]$.

(b) Does the input-output relationship of system S change if we first pass $x[n]$ through S_2 and then S_1?



2. Determine whether the following discrete-time system, with input signal $x[n]$ and output signal $y[n]$ is memoryless, invertible, stable, causal, linear, and/or time-invariant:

$$y[n] = ax[n+1] + b, \quad 0 < |a| < \infty, \ 0 < |b| < \infty$$

3. Determine whether the following continuous-time system, with input signal $x(t)$ and output signal $y(t)$ is memoryless, invertible, stable, causal, linear, and/or time-invariant:

$$y(t) = \cos\left[x(t)\right]$$

4. Consider the following discrete-time system, with input signal $x[n]$ and output signal $y[n]$:

$$y[n] = x[n-1] - y[n-1]$$

where $y[n] = 0$ for $n < 0$.

(a) Determine $y[n]$ when $x[n] = \delta[n]$.
(b) Determine $y[n]$ when $x[n] = u[n]$.

5. Compute the output $y[n]$ if the input is $x[n] = a^n u[n]$ and the linear time-invariant system impulse response is $h[n] = b^n u[n]$ with $|b| \geq 1$. Is the system stable? Why? Is the system causal? Why?

6. Determine $y[n] = x[n] \otimes h[n]$ where $x[n]$ and $h[n]$ are

$$x[n] = \begin{cases} 3, & n = -1 \\ 2, & n = 1 \\ 6, & n = 2 \\ 0, & \text{otherwise} \end{cases}$$

and

$$h[n] = \begin{cases} 2, & n = -1 \\ 4, & n = 0 \\ 7, & n = 1 \\ 0, & \text{otherwise} \end{cases}$$

7. Compute the output $y(t)$ if the input is $x(t) = u(t-3) - u(t-5)$ and the linear time-invariant system impulse response is $h(t) = e^{-3t}u(t)$. Is the system stable? Why? Is the system causal? Why?

8. Compute the impulse response $h[n]$ for a LTI system which is characterized by the following difference equation:

$$y[n] = x[n-1] + 2x[n-2] + 3x[n-3]$$

9. Define the area under a continuous-time signal $v(t)$ as:

$$A_v = \int_{-\infty}^{\infty} v(t)dt$$

Show that if $y(t) = x(t) \otimes h(t)$, then

$$A_y = A_x \cdot A_h$$

10. Denote $h[n]$ as the impulse response of a discrete-time linear time-invariant system. If the system is also memoryless, then determine the form of $h[n]$.

Solution

1(a)

Let $w[n]$ be the after output after passing through S_1:

$$w[n] = 2x[n] + 4x[n-1]$$

Passing $w[n]$ through S_2 yields:

$$\begin{aligned} y[n] &= w[n-2] + \frac{1}{2}w[n-3] \\ &= (2x[n-2] + 4x[n-3]) + \frac{1}{2}(2x[n-3] + 4x[n-4]) \\ &= 2x[n-2] + 5x[n-3] + 2x[n-4] \end{aligned}$$

1(b)

Let $v[n]$ be the after output after passing through S_2:

$$v[n] = x[n-2] + \frac{1}{2}x[n-3]$$

Passing $w[n]$ through S_1 yields:

$$\begin{aligned} y[n] &= 2v[n] + 4v[n-1] \\ &= 2\left(x[n-2] + \frac{1}{2}x[n-3]\right) + 4\left(x[n-3] + \frac{1}{2}x[n-4]\right) \\ &= 2x[n-2] + 5x[n-3] + 2x[n-4] \end{aligned}$$

Realizing that S_1 and S_2 are LTI systems and using the commutative property, we can also understand that there will be no change in the output.

2.

$$y[n] = ax[n+1] + b, \quad 0 < |a| < \infty,\ 0 < |b| < \infty$$

<u>Memoryless</u>

The system is not memoryless because output $y[n]$ at time n does not only depend on $x[n]$ at time n. In fact, $y[n]$ depends on input at time $n+1$.

<u>Invertibility</u>

The system is invertible. By reorganizing the equation and we see that $x[n]$ can be computed from $y[n]$ using:

$$y[n] = ax[n+1] + b \Rightarrow x[n] = \frac{y[n-1] - b}{a}$$

Stability

If $x[n]$ is bounded, then $y[n] = ax[n+1] + b$ is bounded for bounded a and b. As a result, $y[n]$ is bounded and the system is stable. In a more rigorous manner, we have:

$$|y[n]| = |ax[n+1] + b| \leq |a| \cdot |x[n+1]| + |b|$$

where $|x[n+1]| < \infty$ or $|x[n]| < \infty$ must give $|y[n]| < \infty$.

Causality

It is not causal because $y[n]$ depends on future input, namely, $x[n+1]$.

Linearity

The system outputs for $x_1[n]$ and $x_2[n]$ are:

$$y_1[n] = ax_1[n+1] + b \quad \text{and} \quad y_2[n] = ax_2[n+1] + b$$

Consider $x_3[n] = cx_1[n] + dx_2[n]$, its system output is then:

$$\begin{aligned}
y_3[n] &= ax_3[n+1] + b \\
&= a\left(cx_1[n+1] + dx_2[n+1]\right) + b \\
&= acx_1[n+1] + adx_2[n+1] + b \\
&= c\left(ax_1[n+1] + b\right) + d\left(ax_2[n+1] + b\right) + b - bc - bd \\
&\neq cy_1[n] + dy_2[n]
\end{aligned}$$

As a result, this system is not linear.

Time-invariance

First, we have:

$$y[n-n_0] = ax[n-n_0+1] + b$$

Consider $x_1[n] = x[n-n_0]$, its system output is

$$\begin{aligned} y_1[n] &= ax_1[n+1] + b \\ &= ax[n-n_0+1] + b \\ &= y[n-n_0] \end{aligned}$$

Hence the system is time-invariant.

3.

$$y(t) = \cos\left[x(t)\right]$$

Memoryless

The system is memoryless because the output $y(t)$ at time t only depends on $x(t)$ at time t.

Invertibility

The system is not invertible. By reorganizing the equation and we see that $x(t)$ cannot be computed from $y(t)$ because there are infinite possibilities of $x(t)$

$$x(t) = \cos^{-1}(y(t)) + 2n\pi, \quad n = \cdots, -1, 0, 1, \cdots$$

Stability

If $x(t)$ is bounded, then $y(t) = \cos\left[x(t)\right]$ must be bounded because $1 \geq |\cos\left[x(t)\right]|$.

Causality

It is causal because $y(t)$ at time t depends on $x(t)$ up to time t.

Linearity

The system outputs for $x_1(t)$ and $x_2(t)$ are:

$$y_1(t) = \cos\left[x_1(t)\right] \qquad \text{and} \qquad y_2(t) = \cos\left[x_2(t)\right]$$

Consider $x_3(t) = ax_1(t) + bx_2(t)$, its system output is then:

$$\begin{aligned} y_3(t) &= \cos\left[x_3(t)\right] \\ &= \cos\left[ax_1(t) + bx_2(t)\right] \\ &\neq a\cos\left[x_1(t)\right] + b\cos\left[x_2(t)\right] \\ &= ay_1(t) + by_2(t) \end{aligned}$$

As a result, this system is not linear.

Time-invariance

First, we have:

$$y(t-t_0) = \cos\left[x(t-t_0)\right]$$

Consider $x_1(t) = x(t-t_0)$, its system output is

$$\begin{aligned} y_1(t) &= \cos\left[x_1(t)\right] \\ &= \cos\left[x(t-t_0)\right] \\ &= y(t-t_0) \end{aligned}$$

Hence the system is time-invariant.

4(a)

When $x[n]=\delta[n]$ and using $y[-1]=0$, we start with $n=0$:

$$
\begin{aligned}
y[0] &= \delta[-1]-y[-1]=0 \\
y[1] &= \delta[0]-y[0]=1 \\
y[2] &= \delta[1]-y[1]=-1 \\
y[3] &= \delta[2]-y[2]=1
\end{aligned}
$$

We may deduce the general form of $y[n]$ as:

$$
y[n]=(-1)^{n-1}u[n-1]
$$

4(b)

When $x[n]=u[n]$ and using $y[-1]=0$, we start with $n=0$:

$$
\begin{aligned}
y[0] &= u[-1]-y[-1]=0 \\
y[1] &= u[0]-y[0]=1 \\
y[2] &= u[1]-y[1]=0 \\
y[3] &= u[2]-y[2]=1
\end{aligned}
$$

We may deduce the general form of $y[n]$ as:

$$
y[n]=\begin{cases} 1, & n=1,3,5,\cdots \\ 0, & \text{otherwise} \end{cases}
$$

5.

Using (3.11), we have:

$$
\begin{aligned}
y[n] &= \sum_{m=-\infty}^{\infty} x[m]h[n-m] \\
&= \sum_{m=-\infty}^{\infty} a^m u[m] b^{m-n} u[n-m] \\
&= \sum_{m=0}^{\infty} a^m b^{m-n} u[n-m] \\
&= \sum_{k=n}^{-\infty} a^{n-k} b^k u[k], \quad k = n-m \\
&= a^n \sum_{k=-\infty}^{n} (a^{-1}b)^k u[k]
\end{aligned}
$$

Since $u[k]=0$ for $k<0$, $y[n]=0$ for $n<0$.

For $n \geq 0$, we then have:

$$\begin{aligned} y[n] &= a^n \sum_{k=0}^{n} (a^{-1}b)^k \\ &= a^n \frac{1-(a^{-1}b)^{n+1}}{1-a^{-1}b} \\ &= \frac{a^{n+1}-b^{n+1}}{a-b} \end{aligned}$$

Combining the results, we have:

$$y[n] = \frac{a^{n+1}-b^{n+1}}{a-b} u[n]$$

Since $\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^{\infty} |b| = \infty$, the system is not stable. Moreover, the system is causal because $h[n]=0$ for $n<0$.

6.
Using (3.11) again, we have:

$$\begin{aligned} y[n] &= \sum_{m=-\infty}^{\infty} x[m]h[n-m] \\ &= x[-1]h[n+1] + x[1]h[n-1] + x[2]h[n-2] \end{aligned}$$

Try $n=-2$:

$$y[-2] = x[-1]h[-1] + x[1]h[-3] + x[2]h[-4] = 3 \cdot 2 = 6$$

Try $n=-1$:

$$y[-1] = x[-1]h[0] + x[1]h[-2] + x[2]h[-3] = 3 \cdot 4 = 12$$

Try $n=0$:

$$y[-1]=x[-1]h[1]+x[1]h[-1]+x[2]h[-2]=3\cdot 7+2\cdot 2=25$$

Compute $y[n]$ for other values of n and combine the results, we get:

$$y[n]=\begin{cases}6, & n=-2\\ 12, & n=-1\\ 25, & n=0\\ 20, & n=1\\ 38, & n=2\\ 42, & n=3\\ 0, & \text{otherwise}\end{cases}$$

7.

Using the convolution for continuous-time case, we have:

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{-\infty}^{\infty} e^{-3\tau}u(\tau)\left[u(t-3-\tau)-u(t-5-\tau)\right]d\tau$$
$$= \int_{0}^{\infty} e^{-3\tau}u(t-3-\tau)d\tau - \int_{0}^{\infty} e^{-3\tau}u(t-5-\tau)d\tau$$

Let the first and second components be $y_1(t)$ and $y_2(t)$ such that $y(t) = y_1(t) - y_2(t)$.

$$\begin{aligned} y_1(t) &= \int_{0}^{\infty} e^{-3\tau}u(t-3-\tau)d\tau, \quad \lambda = t-3-\tau \\ &= \int_{t-3}^{-\infty} e^{-3(t-3-\lambda)}u(\lambda)d(-\lambda) \\ &= \int_{-\infty}^{t-3} e^{-3(t-3-\lambda)}u(\lambda)d\lambda \end{aligned}$$

When $t-3<0$ or $t<3$, the integral will only involve the zero part of $u(\lambda)$ because $u(\lambda)=0$ for $\lambda<0$. Hence $y_1(t)=0$ for $t<3$. For $t>3$, we have:

$$\begin{aligned} y_1(t) &= \int_0^{t-3} e^{-3(t-3-\lambda)} d\lambda = e^{-3(t-3)} \int_0^{t-3} e^{3\lambda} d\lambda \\ &= e^{-3(t-3)} \cdot \frac{1}{3} e^{3\lambda}\Big|_0^{t-3} = e^{-3(t-3)} \cdot \frac{1}{3}\left(e^{3(t-3)}-1\right) = \frac{1}{3}\left(1-e^{-3(t-3)}\right) \end{aligned}$$

That is,

$$y_1(t) = \frac{1}{3}\left(1-e^{-3(t-3)}\right) u(t-3)$$

Similarly, $y_2(t)$ is:

$$y_2(t) = \frac{1}{3}\left(1-e^{-3(t-5)}\right) u(t-5)$$

Combining the results yields:

$$y(t) = \frac{1}{3}\left(1 - e^{-3(t-3)}\right) u(t-3) - \frac{1}{3}\left(1 - e^{-3(t-5)}\right) u(t-5)$$

or

$$y(t) = \begin{cases} 0, & t < 3 \\ \frac{1}{3}\left(1 - e^{-3(t-3)}\right) u(t-3), & 3 < t < 5 \\ \frac{1}{3}\left(e^{-3(t-5)} - e^{-3(t-3)}\right), & t > 5 \end{cases}$$

Since $\int_{-\infty}^{\infty} |h[t]| dt = \int_{0}^{\infty} e^{-3t} dt < \infty$, the system is stable. Moreover, the system is causal because $h(t) = 0$ for $t < 0$.

8.
Using (3.12), we have:

$$y[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m]$$
$$= \cdots h[0]x[n] + h[1]x[n-1] + h[2]x[n-2] + h[3]x[n-3] + \cdots$$

It is seen that only $h[1]$, $h[2]$ and $h[3]$ are nonzero. That is, the impulse response is:

$$h[n] = \delta[n-1] + 2\delta[n-2] + 3\delta[n-3]$$

9.

$$
\begin{aligned}
A_y = \int_{-\infty}^{\infty} y(t)dt &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau dt \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)h(\lambda)d\tau d(\lambda+\tau), \quad \lambda = t-\tau \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)h(\lambda)d\tau d\lambda \\
&= \left(\int_{-\infty}^{\infty} x(\tau)d\tau\right)\left(\int_{-\infty}^{\infty} h(\lambda)d\lambda\right) \\
&= A_x \cdot A_h
\end{aligned}
$$

10.

Let $x[n]$ and $y[n]$ be the system input and output, respectively. Expanding the convolution formula yields:

$$\begin{aligned} y[n]=x[n]\otimes h[n] &= \sum_{m=-\infty}^{\infty} h[m]x[n-m] \\ &= \cdots h[-2]x[n+2]+h[-1]x[n+1]+ \\ &\quad h[0]x[n]+h[1]x[n-1]+h[2]x[n-2]+\cdots \end{aligned}$$

If the system is memoryless, $y[n]$ at time n only depends on $x[n]$ at the same time, implying that

$$y[n]=x[n]\otimes h[n]=h[0]x[n]$$

That is, $h[n]=K\delta[n]$ where $K=h[0]$ is a constant.