EE3210
Signals and Systems

Part 4: Linear Time-Invariant Systems

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Changes of Introduction_v1 Lecture Notes

- Page 11, change

Quiz 1	Week 6, 5-6pm, in class
Quiz 2	Week 10, 5-6pm, in class

to

Quiz 1	Week 7, 5-6pm, in class
Quiz 2	Week 11, 5-6pm, in class

Changes of Part3_v2 Lecture Notes

- Page 12, add:
  - Continuous-time systems: for all t, $0 < B < \infty$

$$|x(t)| \leq B \rightarrow |y(t)| \leq B$$

  - Discrete-time systems: for all n, $0 < B < \infty$

$$|x[n]| \leq B \rightarrow |y[n]| \leq B$$

Changes of Part3_v2 Lecture Notes (cont.)

- Page 13, change

$$|y[n]| = |\sin(n\pi)x[n]| \leq |\sin(n\pi)|\,|x[n]| \leq B$$

to

$$|y[n]| = |\sin(n\pi)x[n]| = |\sin(n\pi)|\,|x[n]| \leq |x[n]| \leq B$$

Changes of Part3_v2 Lecture Notes (cont.)

- Page 13, change

$$y[n] = \sum_{k=-\infty}^{n} u[k] = (n+1)u[n]$$

to

$$y[n] = \sum_{k=-\infty}^{n} u[k] = \begin{cases} 0, & n < 0 \\ n+1, & n \geq 0 \end{cases} = (n+1)u[n]$$

What is a Linear Time-Invariant (LTI) System?

- An LTI system is one that possesses the properties of linearity and time invariance.
  - As a consequence, if we can represent the input to an LTI system in terms of a linear combination of a set of basic signals, we can then use superposition to compute the output of the system in terms of its responses to these basic signals.
  - Such a representation provides considerable analytical convenience in dealing with LTI systems.

- Many systems encountered in nature can be successfully modeled as LTI systems.

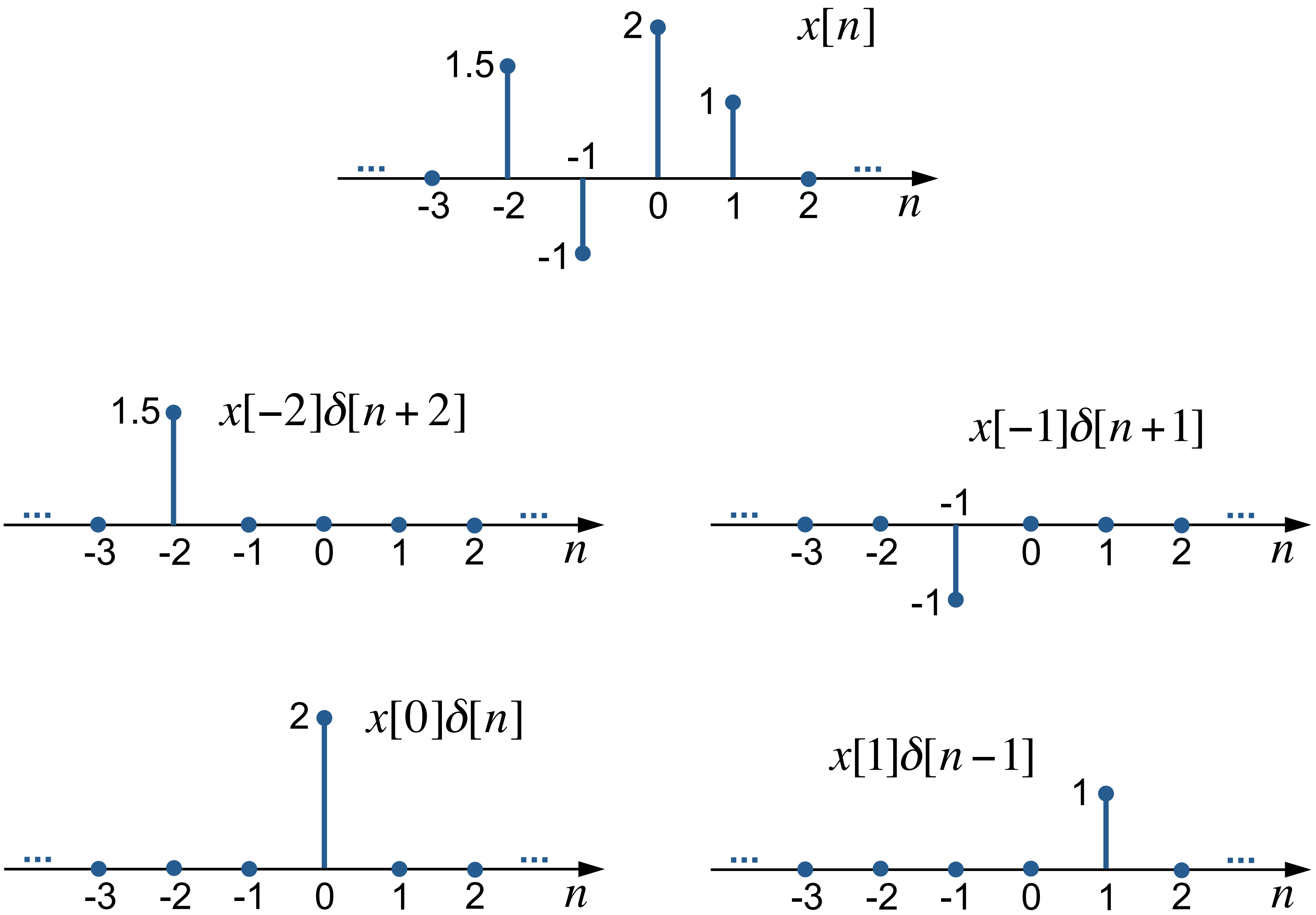
Representation of $x[n]$ in Terms of $\delta[n]$

- Any discrete-time signal $x[n]$ can be represented as

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k] \tag{1}$$

  which is a superposition of time-shifted unit impulses $\delta[n-k]$, each scaled by $x[k]$, and k is extended to $\pm\infty$.
  - Recall by definition $\delta[n-k] \neq 0$ only when $k = n$.
  - Therefore, for any value of n, only one of the terms on the right-hand side of (1) is nonzero, and the scaling associated with that term is precisely $x[n]$.

An Example



Discrete-Time Unit Impulse Response

- The response of a discrete-time LTI system to $\delta[n]$ is defined as the discrete-time unit impulse response, denoted by $h[n]$:

$$\delta[n] \rightarrow h[n]$$

- The property of time invariance implies that, for all k, we have

$$\delta[n-k] \rightarrow h[n-k]$$

- The property of superposition implies that

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k] \rightarrow y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

Convolution Sum

- The result

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] \tag{2}$$

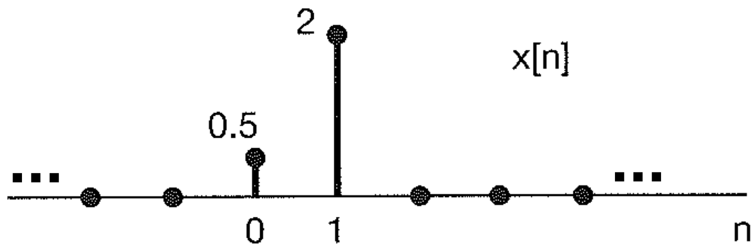
is known as the convolution sum of $x[n]$ and $h[n]$.

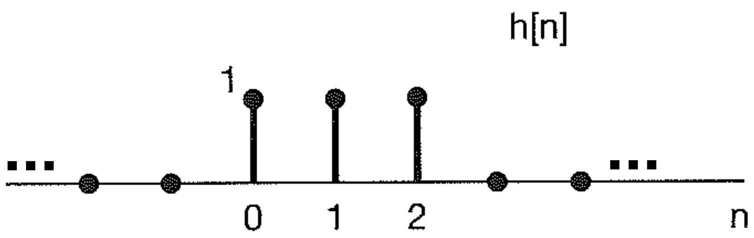
- From (2), we see that a discrete-time LTI system is completely characterized by its response to unit impulse signals.
- We will represent (2) symbolically as

$$y[n] = x[n] * h[n]$$

Example 1

- Consider an LTI system with input $x[n]$ and unit impulse response $h[n]$ given as

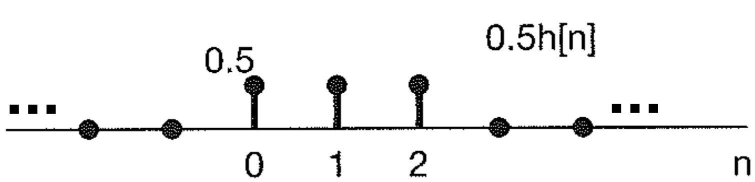


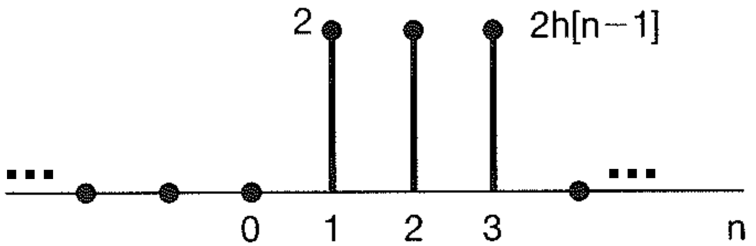


- For this case, $y[n]$ is simply

$$y[n] = x[0]h[n-0]+x[1]h[n-1] = 0.5h[n]+2h[n-1] \quad (3)$$

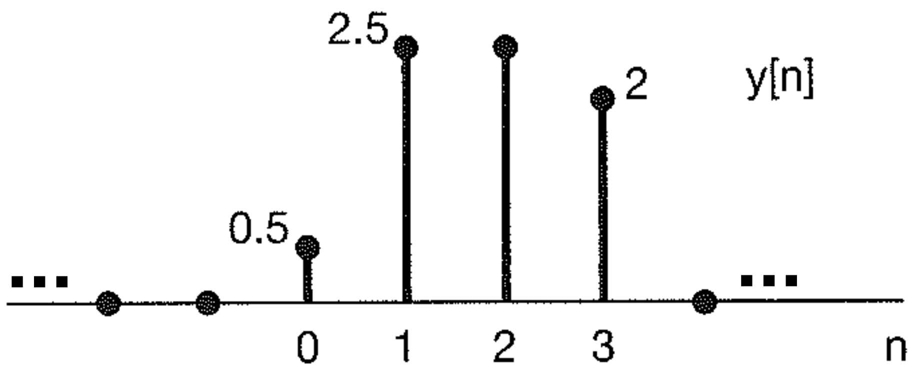
- We obtain $0.5h[n]$ and $2h[n-1]$ as





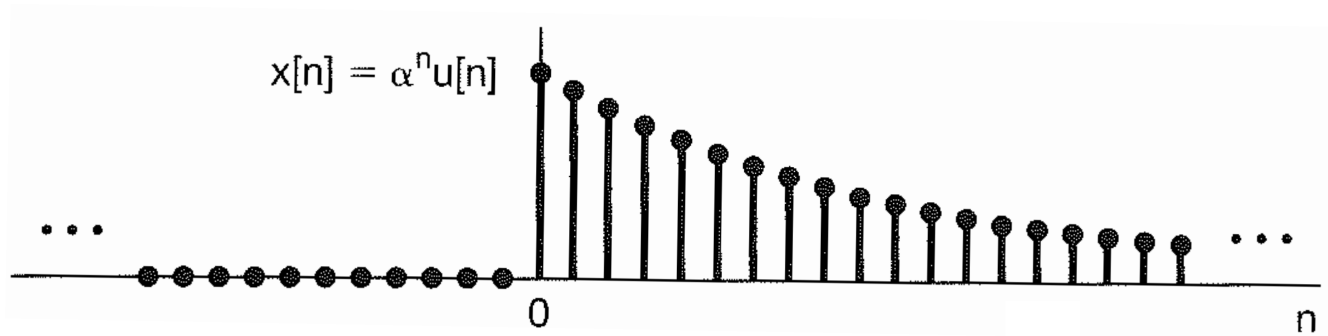
Example 1 (cont.)

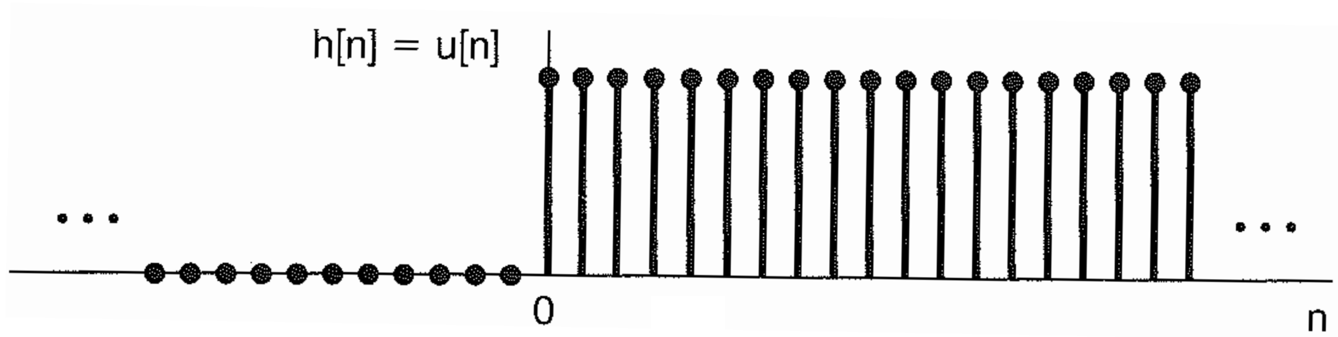
- The overall response $y[n]$ is obtained from (3) as



Example 2

- Consider an LTI system with input $x[n] = \alpha^n u[n]$, where $0 < \alpha < 1$, and unit impulse response $h[n] = u[n]$.





Example 2 (cont.)

- For $n < 0$, we have:

$$\left.\begin{array}{ll} h[n-k] = 0, & k \geq 0 \\ x[k] = 0, & k < 0 \end{array}\right\} \Rightarrow y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = 0$$

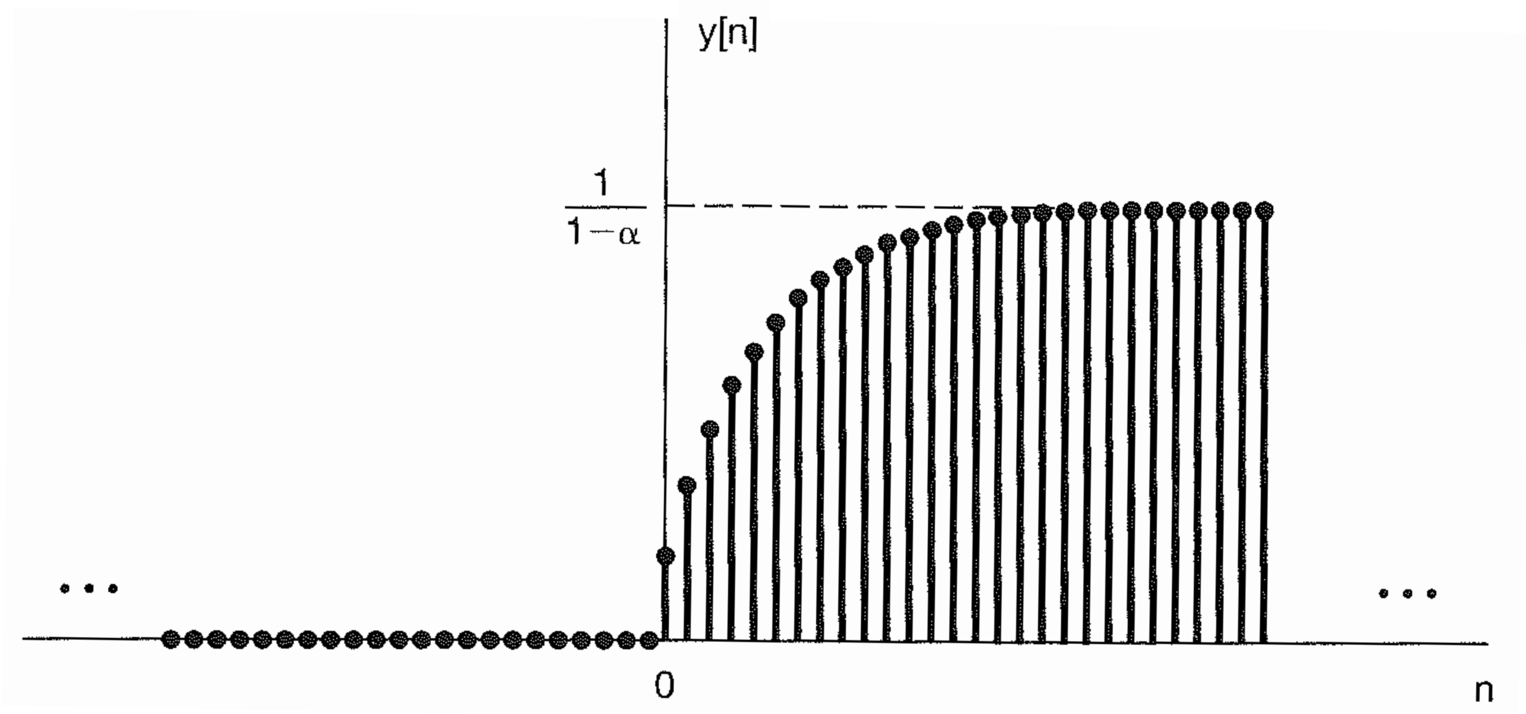
- For $n \geq 0$, we have:

$$x[k] = \begin{cases} \alpha^k, & k \geq 0 \\ 0, & k < 0 \end{cases} \text{ and } h[n-k] = \begin{cases} 1, & k \leq n \\ 0, & k > n \end{cases}$$

$$\Rightarrow y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = \sum_{k=0}^{n} \alpha^k = \frac{1-\alpha^{n+1}}{1-\alpha}$$

Example 2 (cont.)

- Thus, for all n, we obtain: $y[n] = \left(\dfrac{1-\alpha^{n+1}}{1-\alpha} \right) u[n]$



Properties of Convolution Sum

- The commutative property:

$$x[n] * h[n] = h[n] * x[n]$$

  - Recall by definition:

$$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] \tag{4}$$

  By changing the variable of summation in (4) from k to $m = n - k$, we have:

$$x[n] * h[n] = \sum_{m=-\infty}^{+\infty} x[n-m]h[m] = h[n] * x[n]$$

Properties of Convolution Sum (cont.)

- The distributive property:

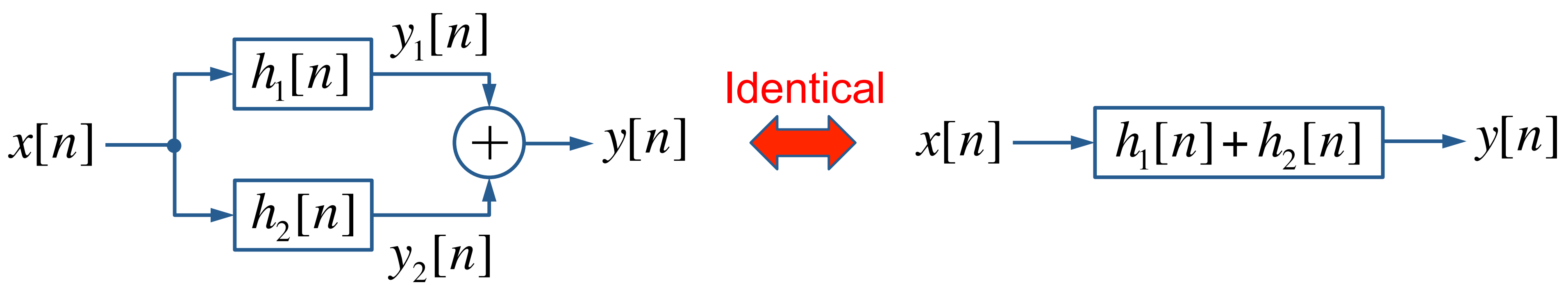
$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$$

- Using (4), we have:

$$\begin{aligned} x[n] * (h_1[n] + h_2[n]) \\ &= \sum_{k=-\infty}^{+\infty} x[k](h_1[n-k] + h_2[n-k]) \\ &= \sum_{k=-\infty}^{+\infty} x[k]h_1[n-k] + \sum_{k=-\infty}^{+\infty} x[k]h_2[n-k] \\ &= x[n] * h_1[n] + x[n] * h_2[n] \end{aligned}$$

Properties of Convolution Sum (cont.)

- Because of the distributive property, for a parallel interconnection of LTI systems, we have:



- Also, as a consequence of both the commutative and distributive properties, we have:

$$(x_1[n] + x_2[n]) * h[n] = x_1[n] * h[n] + x_2[n] * h[n]$$

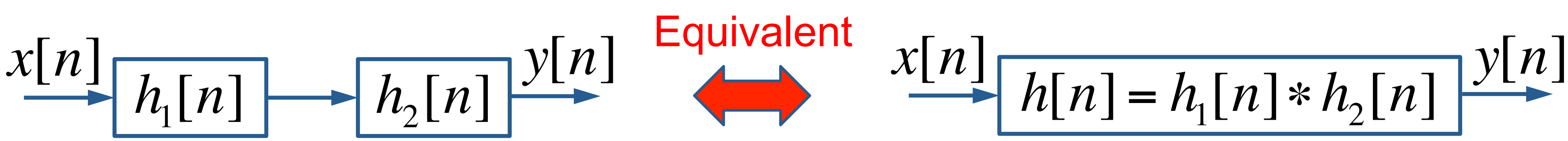
Properties of Convolution Sum (cont.)

- The associative property:

$$x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n]$$

  - We will prove this result in tutorial.

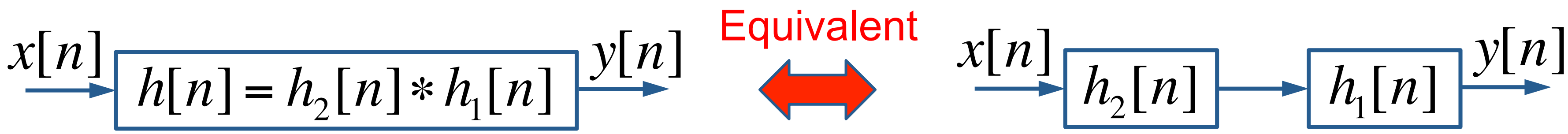
- From the associative property, a series interconnection of LTI systems is equivalent to a single system:



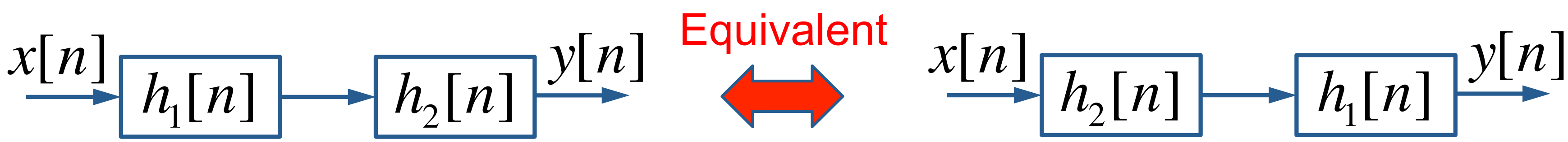
- From the commutative property, we have:

Properties of Convolution Sum (cont.)

- Using the associative property again, we have:



- Thus, as a consequence of both the commutative and associative properties, we have:

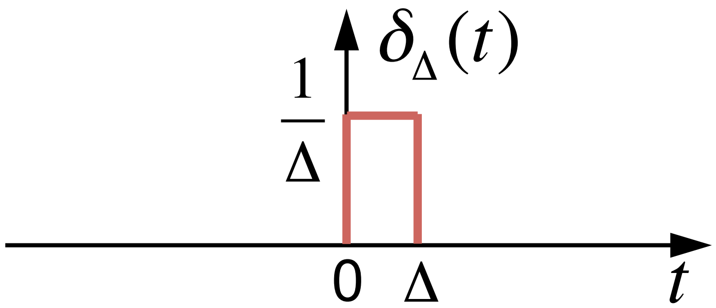


  - This means that the unit impulse response of a cascade of two LTI systems does not depend on the order in which they are cascaded.
  - This also holds for an arbitrary number of LTI systems.

Representation of $x(t)$ in Terms of $\delta(t)$

- Consider the signal $\delta_\Delta(t)$ defined by

$$\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta}, & 0 \le t < \Delta \\ 0, & \text{otherwise} \end{cases}$$



  - By definition, $\delta_\Delta(t)\Delta$ has unit amplitude for $0 \le t < \Delta$.
  - For any integer k, we can obtain $\delta_\Delta(t - k\Delta)$ as a time-shifted version of $\delta_\Delta(t)$, i.e.,

$$\delta_\Delta(t - k\Delta) = \begin{cases} \frac{1}{\Delta}, & k\Delta \le t < (k+1)\Delta \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

Representation of $x(t)$ in Terms of $\delta(t)$ (cont.)

- A continuous-time signal $x(t)$ can be approximated by

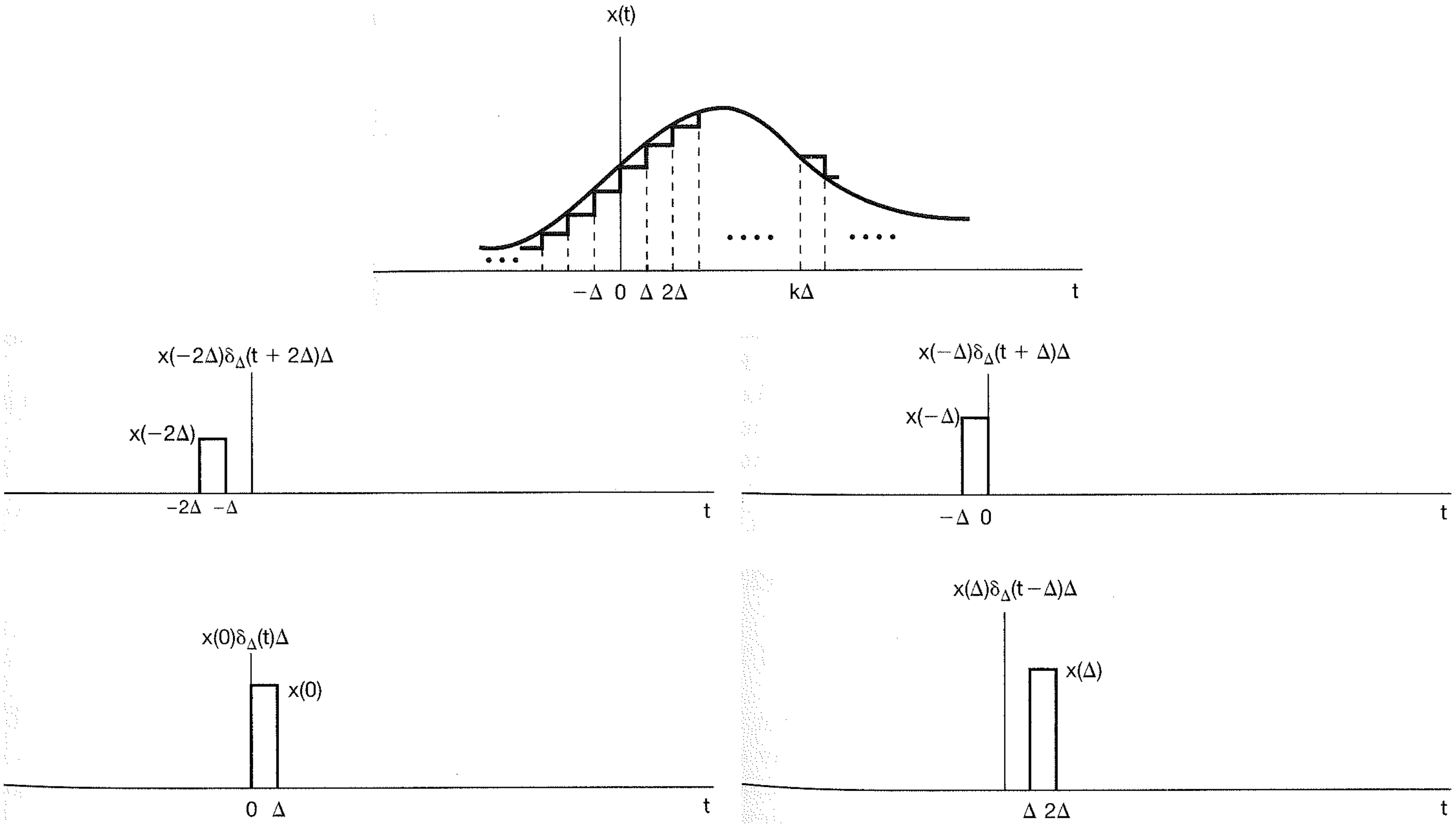
$$x_\Delta(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta)\delta_\Delta(t - k\Delta)\Delta \tag{6}$$

  i.e., a superposition of time-shifted signals $\delta_\Delta(t-k\Delta)\Delta$, each scaled by $x(k\Delta)$, and k is extended to $\pm\infty$.

  - Recall in (5) that, by definition, $\delta_\Delta(t - k\Delta)\Delta \neq 0$ only when $k\Delta \leq t < (k+1)\Delta$.
  - Therefore, for any value of t, only one term in the summation on the right-hand side of (6) is nonzero.

Representation of $x(t)$ in Terms of $\delta(t)$ (cont.)

- An example:



Representation of $x(t)$ in Terms of $\delta(t)$ (cont.)

- As we let Δ approach 0, $x_\Delta(t)$ approaches an integral of the form
$$\int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau$$
which is also equivalent to $x(t)$.

- This representation of $x(t)$ in terms of $\delta(t)$, i.e.,
$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau \tag{7}$$
can also be directly derived from the sampling property of the idealized signal $\delta(t)$.

Continuous-Time Unit Impulse Response

- The response of a continuous-time LTI system to $\delta(t)$ is defined as the continuous-time unit impulse response, denoted by $h(t)$, i.e., $\delta(t) \rightarrow h(t)$.
- The property of time invariance implies that, for all τ, we have $\delta(t-\tau) \rightarrow h(t-\tau)$.
- In (7), we can intuitively think of $x(t)$ as a sum of weighted shifted impulses, where the weight on the impulse $\delta(t-\tau)$ is $x(\tau)d\tau$.
- Then, the property of superposition implies that

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau \rightarrow y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

Convolution Integral

- The result

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau \tag{8}$$

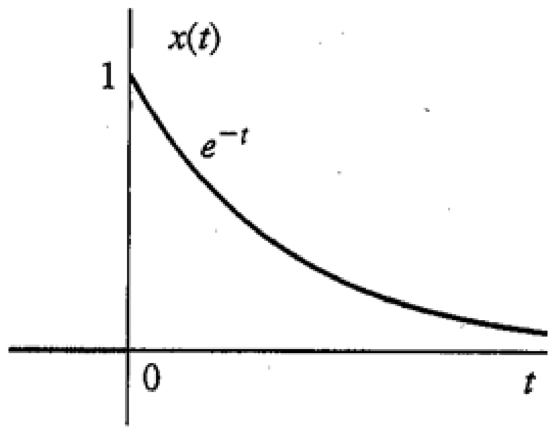
is referred to as the convolution integral of $x(t)$ and $h(t)$.

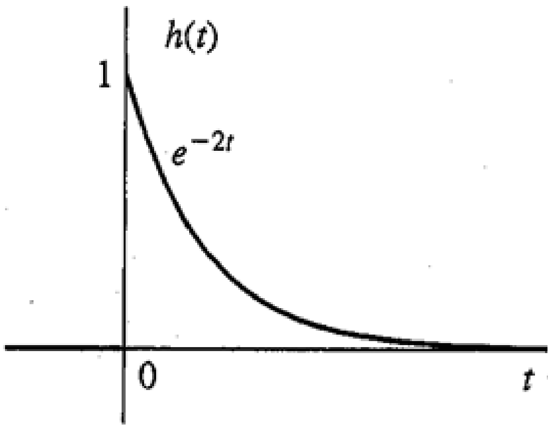
- From (8), we see that a continuous-time LTI system is completely characterized by its response to unit impulse signals.
- We will represent (8) symbolically as

$$y(t) = x(t) * h(t)$$

An Example

- Consider an LTI system with input $x(t) = e^{-t}u(t)$, and unit impulse response $h(t) = e^{-2t}u(t)$.





- By definition, we have

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau \\
&= \int_{-\infty}^{+\infty} e^{-\tau}u(\tau)e^{-2(t-\tau)}u(t-\tau)d\tau
\end{aligned}
$$

An Example (cont.)

- Note that:

$$u(\tau) = \begin{cases} 1, & \tau > 0 \\ 0, & \tau < 0 \end{cases} \quad \text{and} \quad u(t-\tau) = \begin{cases} 1, & \tau < t \\ 0, & \tau > t \end{cases}$$

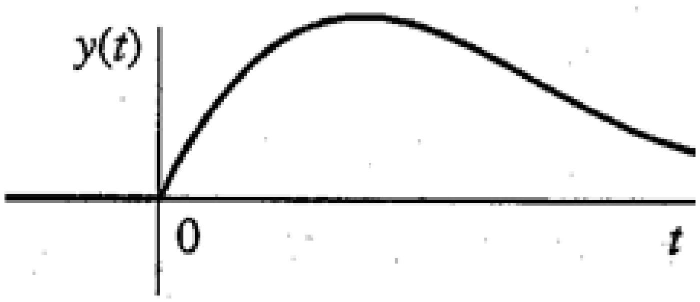
- Therefore, $u(\tau)u(t-\tau) = \begin{cases} 1, & 0 < \tau < t \\ 0, & \text{otherwise} \end{cases}$
- For $t < 0$, $y(t) = 0$.
- For $t > 0$, we have:

$$y(t) = e^{-2t} \int_0^t e^{\tau} d\tau = e^{-2t}(e^t - 1) = e^{-t} - e^{-2t}$$

An Example (cont.)

- Thus, for all t, we obtain:

$$y(t) = \left(e^{-t} - e^{-2t}\right) u(t)$$



Properties of Convolution Integral

- The commutative property:

$$x(t) * h(t) = h(t) * x(t)$$

- The distributive property:

$$x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t)$$

- The associative property:

$$x(t) * [h_1(t) * h_2(t)] = [x(t) * h_1(t)] * h_2(t)$$