In-Class Exercise 4

1. The cumulative distribution function (CDF) of a continuous random variable (RV) X is given as:

$$F(x) = \begin{cases} 0, & x < 0 \\ x^4, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$$

   Determine the probability that X has a value between 0.2 and 0.4.

2. Write down the formula to obtain a uniform RV $Y \sim \mathcal{U}(10, 20)$ in terms of $X \sim \mathcal{U}(0, 1)$.

3. Describe how to utilize uniform RVs to generate Bernoulli RVs with $p = 0.5$. Consider using the MATLAB command `rand`.

4. Use the definition in (2.19) to compute $\mathbb{E}\{R\}$ of the binomial RV R with parameters n and p, whose probability mass function (PMF) is:

$$p(r) = C(n, r)p^r(1 - p)^{n-r}, \quad 0 \le r \le n$$

5. Consider the experiment of rolling a dice where the outcome is the face number. Suppose the probability of obtaining a "1", "2", and "3" is the same, while the probability of obtaining a "4", "5", and "6" is the same. However, a "5" is twice as likely to be observed as a "1". What is the expected value of the face number?

6. Consider a data sequence only with letters "A", "B", "C", "D", and the current encoding scheme adopts a 2-bit code, 00, 01, 10, 11, respectively, i.e., each letter needs 2 bits for storage.

   Suppose we know that the probabilities of occurrence of "A", "B", "C" and "D" are 7/8, 1/16, 1/32, and 1/32, respectively. To utilize this information, we investigate encoding "A", "B", "C" and "D", by 0, 10, 110, and 111. Compute the expected number of bits per letter based on this strategy.

7. A RV K has symmetric probability mass function (PMF) such that $P_K(k) = P_K(-k)$, $k = \cdots, -1, 0, 1, \cdots$. Prove that all odd order moments are equal to zero, i.e., $\mathbb{E}\{K^n\} = 0$ for all odd numbers of n.

Solution

1.

Applying (2.10), the PDF is obtained as:

$$p(x) = \begin{cases} 4x^3, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Hence

$$P(0.2 \leq X \leq 0.4) = \int_{0.2}^{0.4} p(x)dx = x^4 \Big|_{0.2}^{0.4} = 0.024$$

Alternatively, the probability can also obtained directly by using CDF:

$$P(0.2 \leq X \leq 0.4) = F(0.4) - F(0.2) = 0.024$$

2. Substituting $a=10$ and $b=20$ in $Y=a+(b-a)X$, we obtain:

$$Y=10+10X$$

It is clear that the minimum and maximum values of Y are 10 and 20 when $X=0$ and $X=1$, respectively.

3.

The `rand` command produces a random number uniformly distributed between 0 and 1, while the Bernoulli RV is discrete and has 2 possible values, 0 with probability $1-p$, and 1 with probability p. Here, $p=0.5$ and we may simply assign RV=0 when the uniform number is between 0 and 0.5, and assign RV=1 when the uniform number is between 0.5 and 1, to produce a Bernoulli RV. For example,

```
>> rand
ans = 0.8147
```

We assign this as "1"

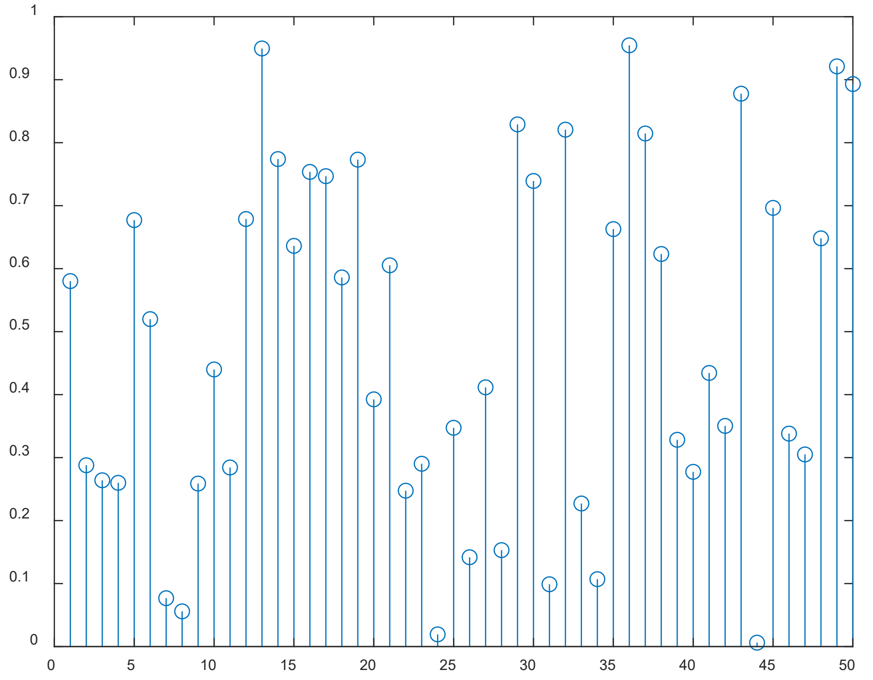
```
>> rand
ans = 0.9058
```

We assign this as "1"

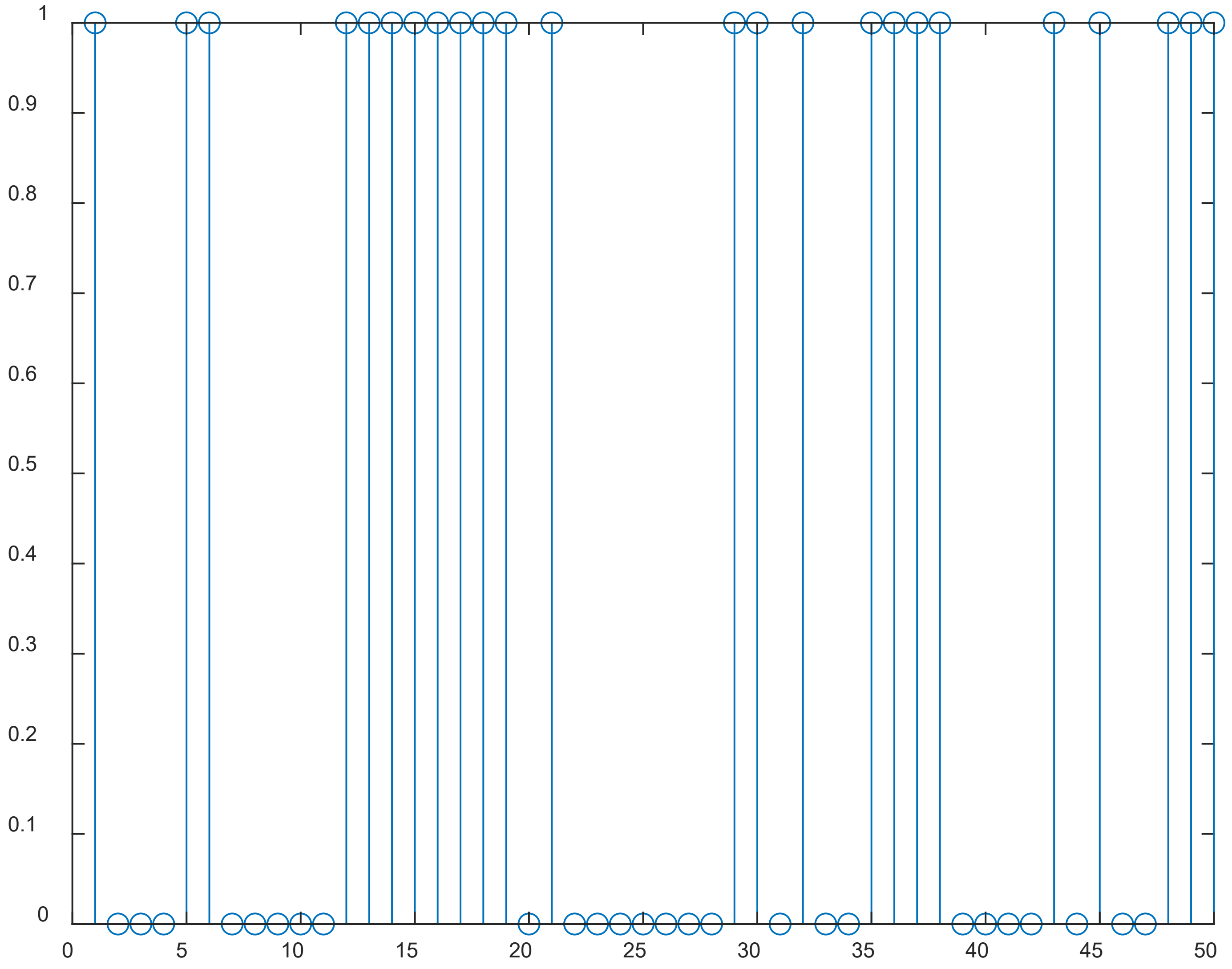
```
>> rand
ans = 0.1270
```

We assign this as "0"

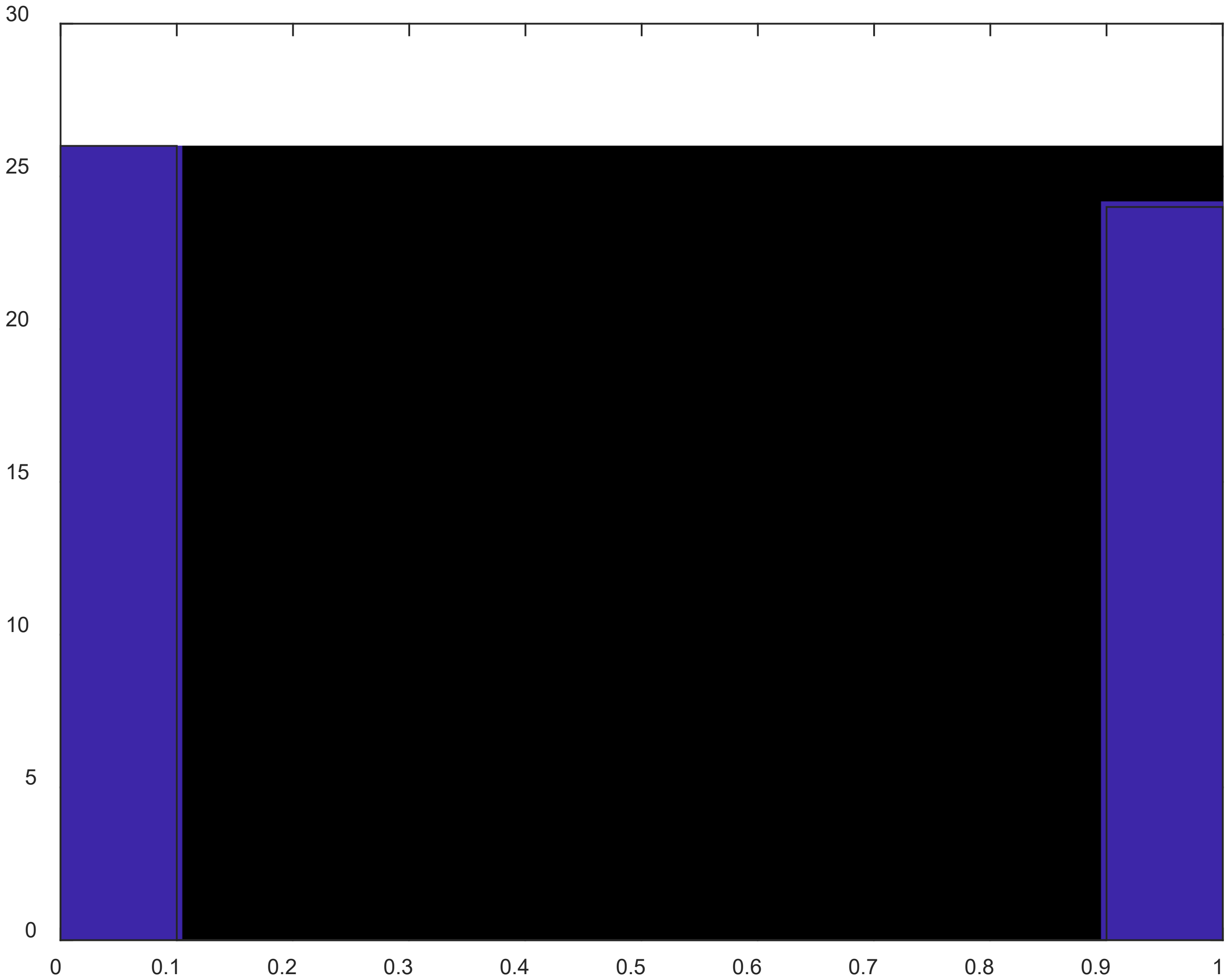
```
N=50;
u=rand(1,N);
for i=1:N
if u(i)<0.5
    b(i)=0;
else
    b(i)=1;
end
end
```



```
stem(1:N, b)
```



`hist(b)`



4.

$$
\begin{aligned}
\mathbb{E}\{R\} = \sum_{r=0}^{n} r p(r) &= \sum_{r=0}^{n} r \binom{n}{r} p^r (1-p)^{n-r} \\
&= \sum_{r=1}^{n} \frac{rn!}{(n-r)!r!} p^r (1-p)^{n-r} \\
&= \sum_{r=1}^{n} \frac{n!}{(n-r)!(r-1)!} p^r (1-p)^{n-r} \\
&= np \sum_{r=1}^{n} \frac{(n-1)!}{(n-r)!(r-1)!} p^{r-1} (1-p)^{n-r} \\
&= np \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{n-1-k} \\
&= np \left[p + (1-p) \right]^{n-1} = np
\end{aligned}
$$

5.

Assigning the random variable X as the face number, we have $1 \leq X \leq 6$. Let the probability of getting "1" be p. Then:

$$p(1) = p(2) = p(3) = p \quad \text{and} \quad p(4) = p(5) = p(6) = 2p$$

As the sum of all PMFs is 1, we easily obtain $p = 1/9$. The expected value of the face number is thus:

$$\mathbb{E}\{X\} = \frac{(1+2+3) + 2(4+5+6)}{9} = 4$$

6.

Assigning the random variable X as the codelength in terms of bit number, we have $1 \leq X \leq 3$. That is, for the sample space {A,B,C,D}, we have X(A)=1, X(B)=2, X(C)=X(D)=3.

Based on the given probability information, the PMF for X is $p(1) = 7/8$, $p(2) = 1/16$, and $p(3) = 1/32 + 1/32 = 1/16$. As a result,

$$\mathbb{E}\{X\} = 1 \cdot \frac{7}{8} + 2 \cdot \frac{1}{16} + 3 \cdot \frac{1}{16} = 1.1875$$

This means that the average number of bits for a letter is reduced from 2 to 1.1875.

Note that the idea of assigning shorter code words to the letters that occur more often is referred to as Huffman coding.

7.

Analogous to (2.21), we have:

$$
\begin{aligned}
\mathbb{E}\{K^n\} &= \sum_{k=-\infty}^{\infty} k^n P_K(k) \\
&= \sum_{k=-\infty}^{-1} k^n P_K(k) + \sum_{k=1}^{\infty} k^n P_K(k) \\
&= \sum_{l=1}^{\infty} (-l)^n P_K(-l) + \sum_{k=1}^{\infty} k^n P_K(k) \\
&= \sum_{l=1}^{\infty} (-l)^n P_K(l) + \sum_{k=1}^{\infty} k^n P_K(k) \\
&= \sum_{k=1}^{\infty} [(-k)^n + k^n] P_K(k)
\end{aligned}
$$

For odd n, $(-k)^n + k^n = 0$, and thus $\mathbb{E}\{K^n\} = 0$.