Solutions to EE3210 Assignment 3

Problem 1: Given $x[n] = \alpha^n u[n]$ and $h[n] = \beta^n u[n]$, we have $x[k] = \alpha^k u[k]$ and $h[n-k] = \beta^{n-k} u[n-k]$. So we have

$$\begin{aligned} y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] &= \sum_{k=-\infty}^{+\infty} \alpha^k u[k] \beta^{n-k} u[n-k] \\ &= \beta^n \sum_{k=-\infty}^{+\infty} \left(\frac{\alpha}{\beta}\right)^k u[k]u[n-k]. \end{aligned}$$

We observe that

$$u[k]u[n-k] = \begin{cases} 1, & 0 \le k \le n \\ 0, & \text{otherwise.} \end{cases}$$

Then:

- For $n < 0$, since $u[k]u[n-k] = 0$ for all k, we have

$$y[n] = 0.$$

- For $n \ge 0$, since $u[k]u[n-k] = 1$ for $0 \le k \le n$, we have

$$y[n] = \beta^n \sum_{k=0}^{n} \left(\frac{\alpha}{\beta}\right)^k. \tag{1}$$

Thus:

(a) Solving (1) for $\alpha \ne \beta$, we have

$$y[n] = \beta^n \frac{1 - (\alpha/\beta)^{n+1}}{1 - \alpha/\beta} = \frac{\beta^{n+1} - \alpha^{n+1}}{\beta - \alpha}, \; n \ge 0.$$

In this case, for all n, we obtain

$$y[n] = \frac{\beta^{n+1} - \alpha^{n+1}}{\beta - \alpha} u[n].$$

(b) Solving (1) for $\alpha = \beta$, we have

$$y[n] = \beta^n \sum_{k=0}^{n} 1 = (n+1)\beta^n, \ n \geq 0.$$

In this case, for all n, we obtain

$$y[n] = (n+1)\beta^n u[n].$$

or, equivalently,

$$y[n] = (n+1)\alpha^n u[n].$$

Problem 2: Because of the commutative property, we have

$$\begin{aligned} x(t) * [h_1(t) * h_2(t)] &= [h_1(t) * h_2(t)] * x(t) \\ &= \int_{-\infty}^{+\infty} [h_1(\tau) * h_2(\tau)] x(t-\tau)\, d\tau \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h_1(m) h_2(\tau - m) x(t-\tau)\, dm\, d\tau. \end{aligned} \tag{2}$$

By changing the variable of integration in (2) from τ to $r = t - \tau$, we then have

$$\begin{aligned} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h_1(m) h_2(\tau - m) x(t-\tau)\, dm\, d\tau &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h_1(m) h_2(t - r - m) x(r)\, dm\, dr \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x(r) h_1(m) h_2(t - r - m)\, dr\, dm. \end{aligned}$$

On the other hand,

$$\begin{aligned} [x(t) * h_1(t)] * h_2(t) &= \int_{-\infty}^{+\infty} [x(\tau) * h_1(\tau)] h_2(t-\tau)\, d\tau \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x(m) h_1(\tau - m) h_2(t-\tau)\, dm\, d\tau \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x(m) h_1(\tau - m) h_2(t-\tau)\, d\tau\, dm. \end{aligned} \tag{3}$$

By changing the variable of integration in (3) from τ to $r = \tau - m$, we then have

$$\begin{aligned} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x(m) h_1(\tau - m) h_2(t-\tau)\, d\tau\, dm &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x(m) h_1(r) h_2(t - m - r)\, dr\, dm \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x(r) h_1(m) h_2(t - r - m)\, dr\, dm. \end{aligned}$$

Thus, the equality is proved.

Problem 3: In this cascade of two systems, the input to system B is exactly the output of system A.

(a) Since system A is an LTI system, if the output of system A is $ay_1(t) + by_2(t)$, by the property of linearity, this implies that the input to system A is $ax_1(t) + bx_2(t)$. Since system B is the inverse of system A, the response of system B to the input $ay_1(t) + by_2(t)$ must be $ax_1(t) + bx_2(t)$.

(b) Since system A is an LTI system, if the output of system A is $y_1(t - t_0)$, by the property of time invariance, this implies that the input to system A is $x_1(t - t_0)$. Since system B is the inverse of system A, the response of system B to the input $y_1(t - t_0)$ must be $x_1(t - t_0)$.

(c) By definition, an LTI system is one that possesses the properties of linearity and time invariance.

- From (a), we know that system B possesses the property of linearity, since given $y_1(t) \to x_1(t)$ and $y_2(t) \to x_2(t)$ it implies that $ay_1(t) + by_2(t) \to ax_1(t) + bx_2(t)$ for any complex constants a and b.
- From (b), we know that system B possesses the property of time invariance, since given $y_1(t) \to x_1(t)$ it implies that $y_1(t - t_0) \to x_1(t - t_0)$ for all t_0.

Therefore, system B is an LTI system.

Problem 4:

(a) The linear constant-coefficient difference equation that describes the relationship between the input $x[n]$ and the output $w[n]$ of system S_1 is

$$w[n] = aw[n-1] + x[n]. \tag{4}$$

(b) The linear constant-coefficient difference equation that describes the relationship between the input $w[n]$ and the output $y[n]$ of system S_2 is

$$y[n] = cy[n-1] + y[n-2] + bw[n]. \tag{5}$$

(c) From (5), we obtain

$$w[n] = \frac{1}{b}y[n] - \frac{c}{b}y[n-1] - \frac{1}{b}y[n-2] \tag{6}$$

and hence

$$w[n-1] = \frac{1}{b}y[n-1] - \frac{c}{b}y[n-2] - \frac{1}{b}y[n-3]. \tag{7}$$

Then, substituting $w[n]$ and $w[n-1]$ in (4) with (6) and (7), respectively, we have

$$\frac{1}{b}y[n] - \frac{c}{b}y[n-1] - \frac{1}{b}y[n-2] = \frac{a}{b}y[n-1] - \frac{ac}{b}y[n-2] - \frac{a}{b}y[n-3] + x[n]$$

and therefore

$$y[n] = (a+c)y[n-1] + (1-ac)y[n-2] - ay[n-3] + bx[n].$$