What is the magnitude $|x|$?

$$|x| = \sqrt{(\mathbb{R}(x)^2 + \mathbb{I}(x)^2}$$

What is the phase $\angle(x)$?

$$\angle(x) = \tan^{-1}\left(\frac{\mathbb{I}(x)}{\mathbb{R}(x)}\right)$$

What is the complex conjugate of x?

$$x^* = \mathbb{R}(x) - j\mathbb{I}(x)$$

What is the even function of signal?

$$x_e(t) = x_e(-t) \text{ or } x_e[n] = x_e[-n]$$
$$x_e(t) = \frac{1}{2}[x(t) + x(-t)] \text{ or } x_e[n] = \frac{1}{2}[x[n] + x[-n]]$$

What is the odd function of signal

$$x_o(t) = -x_o(-t) \text{ or } x_o[n] = -x_o[-n]$$
$$x_o(t) = \frac{1}{2}[x(t) - x(-t)] \text{ or } x_o[n] = \frac{1}{2}[x[n] - x[-n]]$$

What is the energy of $x(t)$ or $x[n]$?

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2\, dt \text{ or } \sum_{n=-\infty}^{\infty} |x[n]|^2$$

What is the power of $x(t)$ or $x[n]$?

- If the signal energy is ∞, then use power formula
- Signal power is the time average of signal energy

$$P_x = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2\, dt \text{ or } \lim_{N\to\infty} \frac{1}{2N+1}\sum_{n=-N}^{N} |x[n]|^2$$

How to identify a signal is energy or power?

- Energy signal if $0 < E_x < \infty$, indicating its $P_x = 0$
- Power signal if $0 < P_x < \infty$, indicating its $E_x = \infty$

What is the unit impulse in a continuous-time signal?

- Only to determine the value of $x(t)$ at the impulse location $t = t_0$

$$\delta(t) = 0,\ \text{t} \neq 0$$
$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$$

What is the sifting property?

- Consider $\delta(t)$ as the building block of any continuous-time signal
- $x(t)$ is the sum of infinite impulse functions and each with amplitude $x(\tau)$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\, d\tau$$

What is the unit step in a continuous-time signal?

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$

How does $u(t)$ can be expressed in terms of $\delta(t)$?

$$u(t) = \int_{-\infty}^{\infty} u(\tau)\delta(t-\tau)\, d\tau = \int_{0}^{\infty} \delta(t-\tau)\, d\tau$$

What is the sine (or cosine) wave form of the signal?

- Amplitude $A > 0$
- Radian frequency ω
- Phase $\phi \in [0, 2\pi)$

$$x(t) = A\cos(\omega t + \phi)$$

What is the fundamental period T_0?

$$T_0 = \frac{2\pi}{\omega} = \frac{1}{f}$$

What is the sine (or cosine) wave form of the complex signal?

$$x(t) = Ae^{j(\omega t + \phi)}$$
$$e^{j\phi} = \cos(\phi) + j\sin(\phi)$$

What are the $\cos(\phi)$ and $\sin(\phi)$ in Euler's formula

$$\cos(\phi) = \frac{e^{j\phi} + e^{-j\phi}}{2}$$
$$\sin(\phi) = \frac{e^{j\phi} - e^{-j\phi}}{2j}$$

What is the unit impulse in a discrete-time signal?

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

What is the unit step?

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n \leq 0 \end{cases}$$

What is the sifting property formula in discrete time?

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

What are the six basic system properties?

- Memoryless
  - If its output at a given time is dependent only on the input at the same time
  - The system does not have memory to store any input values because it just operates on the current input
- Invertibility
  - If distinct inputs lead to distinct outputs
- Linearity
  - If it obeys the principle of superposition
- Time-invariance
  - If a time-shift of input causes a corresponding shift in output
- Causality
  - If the output does not depend on future input
- Stability
  - Every bounded input produces a bounded output

What is the principle of superposition?

$$x_3[n] = ax_1[n] + bx_2[n]$$
$$y_3[n] = ay_1[n] + by_2[n]$$

What is the linear time-invariant (LTI) system?

- Impulse response
- Apply convolution to describe the relationship between input, output, and impulse response

What is the impulse response in LTI system?

- The output of the LTI system when the input is the unit impulse $\delta(t)$ or $\delta[n]$

What is the convolution formula?

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] = x[n] \circledast h[n]$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau = x(t) \circledast h(t)$$

What are the three properties of convolution?

- Commutative

$$x[m]h[n-m] = h[m]x[n-m]$$

- Associative
- Distributive

How to use the impulse response to check the system's causality?

$$\begin{aligned} h(t) &= 0, \quad t < 0 \\ h[n] &= 0, \quad n < 0 \end{aligned}$$

How to use the impulse response to check the system's stability?

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$
$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

What is the formula of the sum of a geometric series?

$$\begin{aligned} s_n &= ar^0 + ar^1 + \cdots + ar^{n-1} \\ &= \sum_{k=1}^{n} ar^{k-1} \\ &= \frac{a(1-r^n)}{1-r} \end{aligned}$$

What is the linear constant coefficient difference equation?

- It is useful to check the linearity and time-invariance of a system

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

What is the linear constant coefficient differential equation?

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

What is the relationship between the difference equation and system input and output?

$$y[n] = \frac{1}{a_0}\left(-\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]\right)$$
$$x[n] = \frac{1}{b_0}\left(\sum_{k=0}^{N} a_k y[n-k] - \sum_{k=1}^{M} b_k x[n-k]\right)$$

What is the Fourier Series?

- The frequency domain representation of a continuous-time periodic signal

What is the fundamental frequency?

$$\Omega_0 = \frac{2\pi}{T_p}$$

How to implement Fourier series into $x(t)$?

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t} \qquad\qquad t \in (-\infty, \infty)$$

$$a_k = \frac{1}{T_p} \int_{-T_p/2}^{T_p/2} x(t) e^{-jk\Omega_0 t}\, dt \quad k = \cdots -1, 0, 1, 2 \cdots$$

What are the general steps to find the Fourier series coefficients a_k?

- Find Ω_0
- Find $T_p = \frac{2\pi}{\Omega_0}$
- Find the coefficient a_0

What are the six properties of the Fourier series?

- Linearity
- Time shifting
- Time reversal
- Time scaling
- Multiplication
- Conjugation
- Parseval's relation

What is the Parseval's relation?

- To compute the power in either the time domain or frequency domain

$$\frac{1}{T_p} \int_{-T_p/2}^{T_p/2} |x(t)|^2\, dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

What is the Fourier transform?

- To solve the aperiodic continuous-time signal into frequency domain
- $X(j\Omega)$ is a function of frequency Ω and known as spectrum

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t}\, dt$$

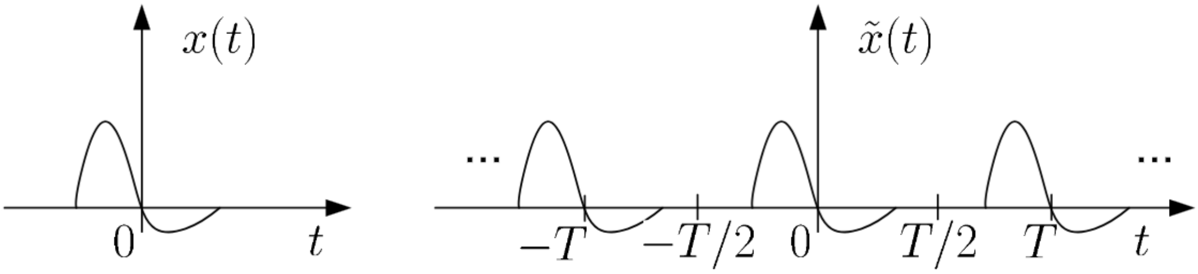
What is the inverse Fourier transform?

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t}\, d\Omega$$

How to derive Fourier transform from Fourier series?

- Start with an aperiodic $x(t)$

- Construct its periodic version $\tilde{x}(t)$ with period T



- By using the formula of the Fourier series coefficients

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t)e^{-jk\Omega_0 t}\,dt$$

$$x(t) = \begin{cases} \tilde{x}(t) & \text{For } |t| < T/2 \\ 0 & \text{For } |t| > T/2 \end{cases}$$

- Substituting $\Omega = k\Omega_0$

$$X(jk\Omega_0) = \int_{-\infty}^{\infty} x(t)e^{-jk\Omega_0 t}\,dt$$

$$a_k = \frac{1}{T}X(jk\Omega_0)$$

- By $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t}$

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T}X(jk\Omega_0)e^{jk\Omega_0 t} = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \Omega_0 X(jk\Omega_0)e^{jk\Omega_0 t}$$

- As $T \to \infty$ or $\Omega_0 \to_0$, $\tilde{x}(t) \to x(t)$

$$\begin{aligned} x(t) &= \lim_{\Omega_0 \to 0} \tilde{x}(t) \\ &= \lim_{\Omega_0 \to 0} \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \Omega_0 X(jk\Omega_0)e^{jk\Omega_0 t} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t}\,d\Omega \end{aligned}$$

How to implement Fourier transform in continuous-time periodic signal with the use of $\delta(t)$?

- Consider an impulse in the frequency domain

$$X(j\Omega) = 2\pi\delta(\Omega - \Omega_0)$$

- Take the inverse Fourier transform of $X(j\Omega)$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\Omega - \Omega_0)e^{j\Omega t}\,d\Omega = e^{j\Omega_0 t}$$

- As result, the Fourier transform pair is:

$$e^{j\Omega_0 t} \leftrightarrow 2\pi\delta(\Omega - \Omega_0)$$

- The Fourier transform pair for a continuous-time periodic signal is:

$$\sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t} \leftrightarrow \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\Omega - k\Omega_0)$$

What are the ten properties of Fourier Transform?

- Linearity

$$ax(t) + by(t) \leftrightarrow aX(j\Omega) + bY(j\Omega)$$

- Time shifting

$$x(t) \leftrightarrow X(j\Omega) \implies x(t - t_0) \leftrightarrow e^{-j\Omega t_0} X(j\Omega)$$

- Time reversal

$$x(t) \leftrightarrow X(j\Omega) \implies x(-t) \leftrightarrow X(-j\Omega)$$

- Time scaling

$$x(t) \leftrightarrow X(j\Omega) \implies x(\alpha t) \leftrightarrow \frac{1}{|\alpha|} X\left(\frac{j\Omega}{\alpha}\right)$$

- Multiplication

$$x(t) \times y(t) \leftrightarrow \frac{1}{2\pi} X(j\Omega) \circledast Y(j\Omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\tau) Y(j(\Omega - \tau))\, d\tau$$

- Conjugation

$$x(t) \leftrightarrow X(j\Omega) \impliedby x^*(t) \leftrightarrow X^*(-j\Omega)$$

- Parseval's relation
  - Address the energy of $x(t)$

$$\int_{-\infty}^{\infty} |x(t)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\Omega)|^2\, dt$$

- Convolution

$$x(t) \circledast y(t) \leftrightarrow X(j\Omega) Y(j\Omega)$$

- Differentiation

$$\frac{dx(t)}{dt} \leftrightarrow j\Omega X(j\Omega) \implies \frac{d^k x(t)}{dt^k} \leftrightarrow (j\Omega)^k X(j\Omega)$$

- Integration

$$\int_{-\infty}^{t} x(\tau)\, d\tau \leftrightarrow \frac{1}{j\Omega} X(j\Omega) + \pi X(0)\delta(\Omega)$$

How to determine a signal is periodic or not?

- $x(t)$ or $x[n]$ is said to be periodic if there exists $T > 0$ or a positive integer N such that

$$\begin{aligned} x(t) &= x(t+T) \\ x[n] &= x[n+N] \end{aligned}$$

- The smallest T and N are called the fundamental period
- Cosine and sine functions are both periodic with period 2π

Examples:

$$\begin{aligned} x(t) &= \cos(10\pi t) \\ &= \cos(10\pi t + 2\pi) \\ &= x\left(t + \frac{2\pi}{10\pi}\right) \\ &= x\left(t + \frac{1}{5}\right) \\ T &\implies \frac{1}{5} \end{aligned}$$

$$\begin{aligned} x[n] &= \cos\left(\frac{2\pi n}{3}\right) \\ &= \cos\left(\frac{2\pi n}{3} + 2\pi\right) \\ &= \cos\left(\frac{2\pi(n+3)}{3}\right) \\ &= x[n+3] \\ N &\implies 3 \end{aligned}$$

How to determine a signal is even or odd?

- $x_e(t) = x_e(-t)$ or $x_e[n] = x_e[-n]$
- $x_o = -x_o(-t)$ or $x_o[n] = -x_o[-n]$
- $x_e(t) = \frac{1}{2}[x(t) + x(-t)] \text{ or } x_e[n] = \frac{1}{2}[x[n] + x[-n]]$
- $x_o(t) = \frac{1}{2}[x(t) - x(-t)] \text{ or } x_o[n] = \frac{1}{2}[x[n] - x[-n]]$

Examples:

$$\begin{aligned} x[n] &= 1 + 2n - 3n^2 \\ x_e[n] &= \frac{1}{2}[(1 + 2n - 3n^2) + (1 + 2(-n) - 3(-n)^2)] = \frac{1}{2}[2 - 6n^2] = 1 - 3n^2 \\ x_o[n] &= \frac{1}{2}[(1 + 2n - 3n^2) - (1 + 2(-n) - 3(-n)^2)] = \frac{1}{2}[4n] = 2n \end{aligned}$$

- As results, $x[n] = 1 + 2n - 3n^2$ satisfying both even and odd properties, so it is neither even nor odd

How to determine a signal is energy or power?

- Energy signal if $0 < E_x < \infty$, indicating its $P_x = 0$
- Power signal if $0 < P_x < \infty$, indicating its $E_x = \infty$

$$
\begin{aligned}
E_x &= \int_{-\infty}^{\infty} |x(t)|^2 \, dt \\
&= \sum_{n=-\infty}^{\infty} |x[n]|^2
\end{aligned}
$$

$$
\begin{aligned}
P_x &= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt \\
&= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2
\end{aligned}
$$

Examples:

- Given a signal $x(t) = e^{-3t}u(t)$,

$$
\begin{aligned}
E &= \int_{-\infty}^{\infty} \left|e^{-3t}u(t)\right|^2 dt \\
&= \int_{0}^{\infty} e^{-6t} \, dt \qquad\qquad u(t) \text{ is a unit step function} \\
&= -\frac{1}{6}e^{-6t}\Big|_0^\infty \\
&= \frac{1}{6}
\end{aligned}
$$

$$
\begin{aligned}
P &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left|e^{-3t}u(t)\right|^2 dt \qquad\qquad T_0 = 2T \\
&= \lim_{T\to\infty} \frac{1}{2T} \int_{0}^{T} e^{-6t} \, dt \\
&= \lim_{T\to\infty} \frac{1}{2T} \left(-\frac{1}{6}e^{-6t}\right)\Big|_0^T \\
&= 0
\end{aligned}
$$

- As result, $x(t) = e^{-3t}u(t)$ is an energy signal

How to compute the convolution of a system and determine its properties?

- $y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] = x[n] \circledast h[n]$
- $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau) \, d\tau = x(t) \circledast h(t)$
- Stability:

$$
\int_{-\infty}^{\infty} |h(t)| \, dt < \infty
$$

$$
\sum_{n=-\infty}^{\infty} |h[n]| < \infty
$$

- Causality:

$$\begin{aligned} h(t) &= 0, \quad t < 0 \\ h[n] &= 0, \quad n < 0 \end{aligned}$$

Examples:

- Given $x[n] = u[-1-n]$ and $h[n] = (0.5)^n u[n]$, compute the system output $y[n]$

$$\begin{aligned} y[n] &= \sum_{m=-\infty}^{\infty} u[-m-1](0.5)^{n-m}u[n-m] \\ &= \sum_{m=-\infty}^{-1} 0.5^{n-m}u[n-m] \qquad\qquad u[-m-1] \text{ is a unit step function} \\ &= \sum_{l=1}^{\infty} 0.5^{n+l}u[n+l] \qquad\qquad \text{changing variable: } l = -m \end{aligned}$$

- Compute the different condition of $u[n+l]$ for $n \geq 1$ and $n < -1$
- For $n \geq -1$, $u[n+l] = 1$, because $u[-1+1] = u[0]$

$$y[n] = \sum_{l=1}^{\infty} 0.5^{n+l} = 0.5^n \sum_{l=1}^{\infty} 0.5^l = 0.5^n \times \frac{0.5}{1-0.5} = 0.5^n$$

- For $n < -1$, $u[n+l] = 1$ when $n+l \geq 0$ or $l \geq -n$, because $n+l \geq 0$

$$y[n] = \sum_{l=-n}^{\infty} 0.5^{n+l} = 0.5^n \sum_{l=-n}^{\infty} (0.5)^l = 0.5^n \times \frac{0.5^{-n}}{1-0.5} = 2$$

- As result,

$$y[n] = \begin{cases} 0.5^n, & n \geq -1 \\ 2, & n < -1 \end{cases}$$

- Compute the stability and causality

$$\sum_{-\infty}^{\infty} |0.5^n u[n]| = \sum_{0}^{\infty} 0.5^n \leq \infty$$

$$h[-1] = 0.5^{-1}u[-1] = 0 \qquad\qquad \text{take arbitrary number that is} < 0$$

How to compute the Fourier transform of a signal?

Examples:

- Given $x(t) = e^{-\alpha|t|}$, $\alpha > 0$
- Re-express $x(t)$ to remove the absolute sign

$$x(t) = e^{\alpha|t|} = \begin{cases} e^{-\alpha t}, & t > 0 \\ e^{\alpha t}, & t < 0 \end{cases}$$

- Apply $X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t}\, dt$

$$
\begin{aligned}
X(j\Omega) &= \int_{-\infty}^{0} e^{\alpha t} e^{-j\Omega t}\, dt + \int_{0}^{\infty} e^{-\alpha t} e^{-j\Omega t}\, dt \\
&= \int_{-\infty}^{0} e^{-(j\Omega - \alpha)t}\, dt + \int_{0}^{\infty} e^{-(j\Omega + \alpha)t}\, dt \\
&= \frac{1}{-(j\Omega - \alpha)} e^{-(j\Omega - \alpha)t} \Big|_{\infty}^{0} + \frac{1}{-(j\Omega + \alpha)} e^{-(j\Omega + \alpha)t} \Big|_{0}^{\infty} \\
&= -\frac{1}{j\Omega - \alpha} + \frac{1}{j\Omega + \alpha}
\end{aligned}
$$

- Simplify by multiplying its conjunction $x^* = \mathbb{R}(x) - j\mathbb{I}(x)$

$$
\begin{aligned}
X(j\Omega) &= -\frac{1}{j\Omega - \alpha} \times \frac{-j\Omega - \alpha}{-j\Omega - \alpha} + \frac{1}{j\Omega + \alpha} \times \frac{-j\Omega + \alpha}{-j\Omega + a} \\
&= \frac{2\alpha}{\alpha^2 + \Omega^2}
\end{aligned}
$$

- Compute its magnitude and phase, by using $|x| = \sqrt{(\mathbb{R}(x)^2 + \mathbb{I}(x)^2}$ and $\angle(x) = \tan^{-1}\left(\frac{\mathbb{I}(x)}{\mathbb{R}(x)}\right)$

$$
|X(j\Omega)| = \sqrt{(\frac{2\alpha}{\alpha^2 + \Omega^2})^2} = \frac{2\alpha}{\alpha^2 + \Omega^2}
$$

$$
\angle(X(j\Omega)) = \tan^{-1}\left(\frac{0}{\frac{2\alpha}{\alpha^2 + \Omega^2}}\right) = 0
$$