<u>**Solution**</u>

1.(a)

Taking the DTFT on both sides yields

$$Y(e^{j\omega})\left(1-ae^{-j\omega}\right)=X(e^{j\omega})\left(1-\frac{1}{a}e^{-j\omega}\right)\Rightarrow H(e^{j\omega})=\frac{1-\frac{1}{a}e^{-j\omega}}{1-ae^{-j\omega}}$$

1.(b)

$$\begin{aligned}|H(e^{j\omega})|^2=H(e^{j\omega})H^*(e^{j\omega}) &= \frac{1-\frac{1}{a}e^{-j\omega}}{1-ae^{-j\omega}}\cdot\frac{1-\frac{1}{a}e^{j\omega}}{1-ae^{j\omega}}=\frac{1+\frac{1}{a^2}-\frac{2}{a}\cos(\omega)}{1+a^2-2a\cos(\omega)}\\ &= \frac{1}{a^2}\end{aligned}$$

Hence we have:

$$|H(e^{j\omega})|=\frac{1}{|a|}$$

2.(a)
The impulse response is not a finite-duration sequence.

2.(b)
The system is not causal. It is because $h[n] \neq 0$ for $n < 0$.

2.(c)
The system is not stable. It is because $\sum_{n=-\infty}^{\infty} |h[n]| \to \infty$.

2.(d)
By making use of

$$a^n u[-n-1] \leftrightarrow -\frac{1}{1-az^{-1}}, \quad |z| < |a|$$

and the time-shifting property, we get

$$(0.5)(0.5)^{n-1}u[-n] \leftrightarrow -\frac{0.5z^{-1}}{1-0.5z^{-1}}, \quad |z| < 0.5$$

2.(e)

$$H(z) = \frac{1}{1-2z}$$

Hence there is no zero but only one pole at $z = 0.5$.

2.(f)

$$H(z) = \frac{Y(z)}{X(z)} = -\frac{0.5z^{-1}}{1-0.5z^{-1}}$$
$$\Rightarrow Y(z)\left(1-0.5z^{-1}\right) = -0.5z^{-1}X(z)$$
$$\Rightarrow y[n] - 0.5y[n-1] = -0.5x[n-1]$$

2.(g)
The z transform of $x[n] = \delta[n] - 0.5\delta[n-1]$ is:

$$X(z) = 1 - 0.5z^{-1}, \quad |z| > 0$$

Hence $Y(z)$ is:

$$Y(z) = H(z)X(z) = -0.5z^{-1}, \quad |z| > 0$$

$\Rightarrow y[n] = -0.5\delta[n-1]$

3.

$$H(z) = \frac{3+3z^{-1}}{1+2.5z^{-1}+z^{-2}} = \frac{3z(z+1)}{(z+0.5)(z+2)} = \frac{1}{1+0.5z^{-1}} + \frac{2}{1+2z^{-1}}$$

Since the system is stable, its ROC must include the unit circle and hence it is $0.5 < |z| < 2$. Hence $h[n]$ is:

$$h[n] = (-0.5)^n u[n] - 2(-2)^n u[-n-1]$$

4.(a)

There are only two frequency components in $x(t)$: 100 and 200. Apparently, the fundamental frequency is 100.

4.(b)

According to sampling theorem, the sampling frequency must exceed 400 (or $200/\pi$ in Hz).

4.(c)

$$x[n] = x(t)|_{t=nT} = \sin(5n+1) + 2\cos(10n+2)$$

4.(d)

$x[n]$ is not periodic as there is no integer N which satisfies $x[n] = x[n+N]$.

5.

Since the system is causal, the ROC should be $|z| > |a_0|$. As a result, the impulse response of the system is:

$$h[n] = b_0(-a_0)^n u[n] + b_1 \delta[n]$$

When the input is $x[n] = \delta[n]$, $y[n] = h[n]$. Hence we have the following equations:

$$y[0] = b_0 + b_1 = 4$$

$$y[1] = b_0(-a_0) = -a_0 b_0 = 3$$

$$y[2] = b_0(-a_0)^2 = a_0^2 b_0 = 2$$

From the last two equations, we get

$$a_0 = -\frac{2}{3}$$

$$b_0 = 4.5$$

From the first equation, we then get

$$b_1 = -0.5$$

6.
Investigating

$$x[n] = (0.5)^{3n} u[3n-3]$$

for different n:

$$x[0] = 0$$
$$x[1] = (0.5)^3 = 0.125$$
$$x[2] = (0.5)^6 = (0.125)^2$$

Hence we can re-express $x[n]$ as:

$$x[n] = \begin{cases} 0, & n \leq 0 \\ (0.125)^n, & n \geq 1 \end{cases}$$

Taking z transform on $x[n]$, we have:

$$
\begin{aligned}
X(z) &= \sum_{n=1}^{\infty} \left(0.125z^{-1}\right)^n, \quad \left|0.125z^{-1}\right| < 1 \Rightarrow |z| > 0.125 \\
&= \frac{0.125z^{-1}}{1-0.125z^{-1}}
\end{aligned}
$$

The ROC is $|z| > 0.125$.