In-Class Exercise 1

1. Consider a discrete-time signal of the form:

$$x[n] = \cos(\omega_0 n)$$

Determine the condition of ω_0 if $x[n]$ is periodic.

2. A continuous-time signal $x(t)$ can be written as the sum of an even function $x_e(t)$ and odd function $x_o(t)$:

$$x(t) = x_e(t) + x_o(t)$$

Show that

$$x_e(t) = \frac{1}{2}\left[x(t) + x(-t)\right] \quad \text{and} \quad x_o(t) = \frac{1}{2}\left[x(t) - x(-t)\right]$$

3. Determine the real and imaginary parts as well as magnitude and phase of

$$\frac{a + jb}{b - jc}$$

4. Determine the imaginary part of $x(t)$:

$$x(t) = \sqrt{2}e^{j\pi/4}\cos(3t + 2\pi)$$

5. Determine if the following signal is energy or power signal and then compute its energy or power:

$$x(t) = \begin{cases} e^{-at}, \ 0 < t < \infty, & a > 0 \\ 0, \quad\quad \text{otherwise} \end{cases}$$

6. Consider a periodic continuous-time signal $x(t)$. Can it be an energy or power signal? If it can be an energy or power signal, will its energy or power be changed with time shift, time scaling and time reversal?

7. Given

$$r[n] = nu[n] = \begin{cases} n, & n \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Find $y[n] = 2r[1-n]$.

8. Evaluate the following expression:

$$\int_{-\infty}^{\infty} \sin(3t)\delta\left(t - \frac{\pi}{2}\right) dt$$

9. Determine the fundamental period of the continuous-time signal $x(t)$:

$$x(t) = 2\cos(10t + 1) - \sin(4t - 1)$$

10. Compute

$$\int_{-\infty}^{\infty} \delta(2t) dt$$

Solution

1. If $x[n]$ is periodic, we can write

$$x[n] = x[n+N]$$

where N is a positive integer. Since the cosine wave has a period of 2π, we can set up the condition as follows:

$$x[n] = \cos(\omega_0 n) = \cos(\omega_0 n + 2\pi K) = x[n+N] = \cos(\omega_0 (n+N))$$

where K is another positive integer. Hence we have:

$$2\pi K = \omega_0 N \Rightarrow N = \frac{2\pi K}{\omega_0}$$

That is, for periodic $x[n]$, $2\pi K/\omega_0$ must be an integer for an integer K, or ω_0 is equal to a rational number times π.

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>> n=0:100;
x=cos(0.2.*n);
stem(n,x)
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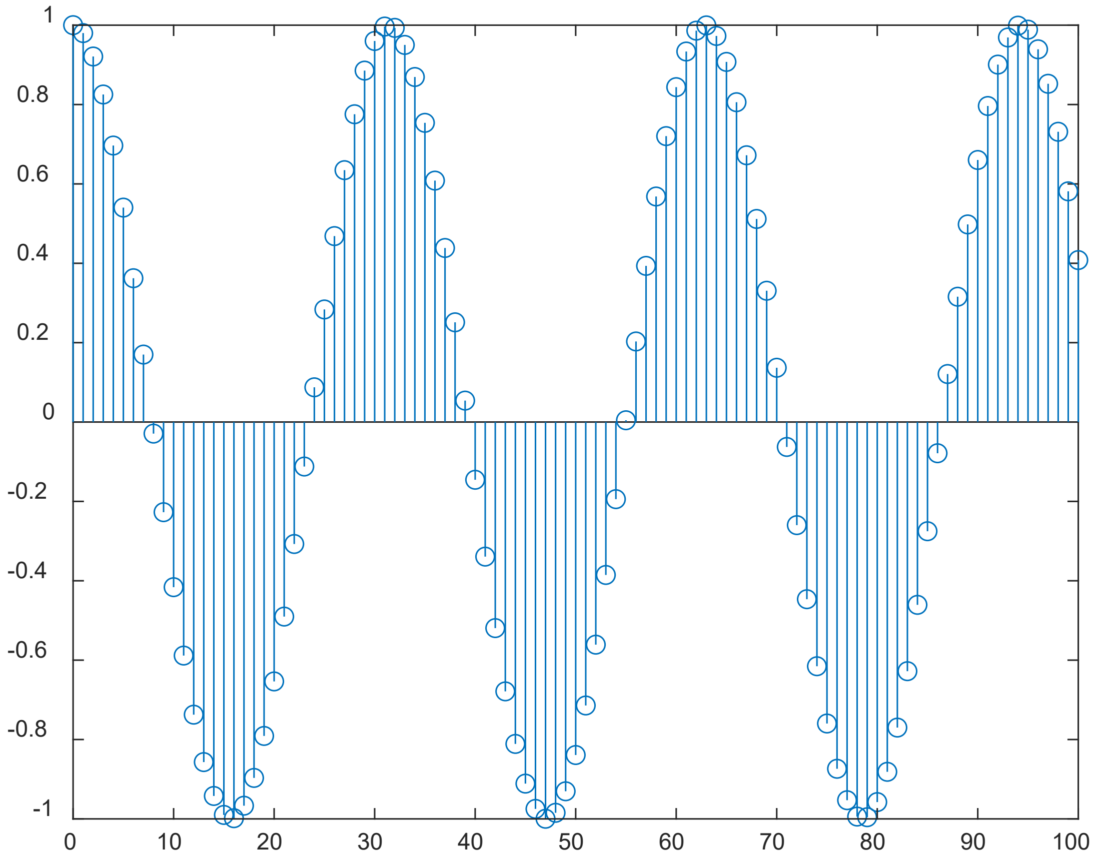

$x[n] \neq x[n+N]$ for all positive integers N.

```
>> n=0:100;
x=cos(0.06*pi.*n);
stem(n,x)
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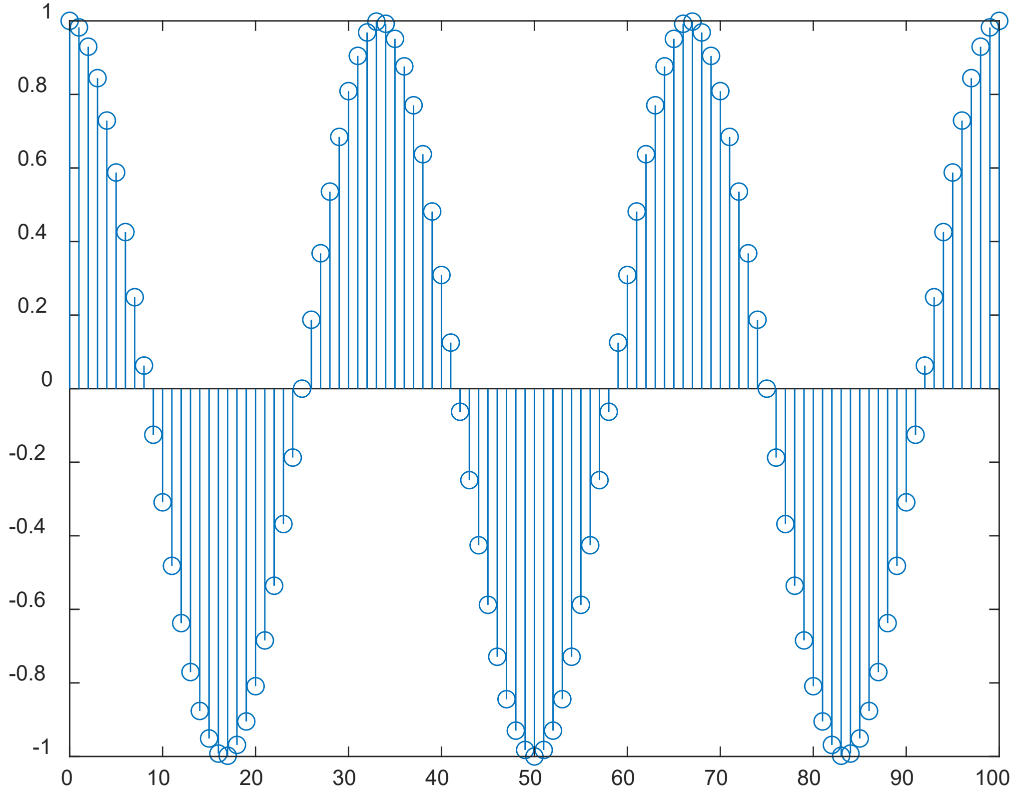

$$x[n] = x[n+100]$$

Why N=100?

According to previous analysis with $\omega_0 = 0.06\pi$, we have:

$$N = \frac{2\pi K}{0.06\pi} = \frac{100K}{3} \Rightarrow K = 3 \text{ to give } N = 100$$

It is also easy to check that:

$$x[n+100] = \cos(0.06\pi(n+100) = \cos(0.06\pi n + 6\pi) = \cos(0.06\pi n) = x[n]$$

Can we have 2π instead of 6π?

For discrete-time sinusoid, it is stated that $\omega_0 \in (0, \pi)$. For example, it is easy to check that

$$\cos(2.06\pi n) = \cos(2\pi n + 0.06\pi n) = \cos(0.06\pi n)$$
$$\cos(1.94\pi n) = \cos(2\pi n - 0.06\pi n) = \cos(0.06\pi n)$$

You can explain why $\omega_0 \in (0, \pi)$ after studying Ch. 6.

2. Given

$$x(t) = x_e(t) + x_o(t)$$

Since $x_e(t)$ is even and $x_o(t)$ is odd, we can write $x(-t)$ as:

$$x(-t) = x_e(-t) + x_o(-t) = x_e(t) - x_o(t)$$

Adding $x(t)$ and $x(-t)$ and then dividing by 2 gives $x_e(t)$.

Similarly, subtracting $x(-t)$ from $x(t)$, and then dividing by 2 gives $x_o(t)$.

3. The first step is to make the denominator real:

$$\frac{a+jb}{b-jc} = \frac{a+jb}{b-jc} \cdot \frac{b+jc}{b+jc} = \frac{ab-bc+j(b^2+ac)}{b^2+c^2}$$

Hence the real and imaginary parts are, respectively:

$$\frac{ab-bc}{b^2+c^2} \qquad \text{and} \qquad \frac{b^2+ac}{b^2+c^2}$$

The phase is then:

$$\tan^{-1}\left(\frac{b^2+ac}{ab-bc}\right)$$

Finally, the magnitude can be computed as:

$$\sqrt{\frac{a+jb}{b-jc} \cdot \frac{a-jb}{b+jc}} = \sqrt{\frac{a^2+b^2}{b^2+c^2}}$$

4. Using Euler's formula:

$$\begin{aligned} x(t) &= \sqrt{2}e^{j\pi/4}\cos(3t+2\pi) = \sqrt{2}(\cos(\pi/4)+j\sin(\pi/4))\cos(3t+2\pi) \\ &= \sqrt{2}\cos(\pi/4)\cos(3t+2\pi) + j\sqrt{2}\sin(\pi/4)\cos(3t+2\pi) \\ &= \cos(3t+2\pi) + j\cos(3t+2\pi) = \cos(3t) + j\cos(3t) \end{aligned}$$

Hence $\Im\{x(t)\} = \cos(3t)$.

5. We see that $x(t)$ is decaying from $t=0$ to $t=\infty$, and is zero otherwise. Thus it is an energy signal. Using (2.13), we get:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_0^{\infty} e^{-2at} dt = \left.\frac{e^{-2at}}{-2a}\right|_0^{\infty} = \frac{1}{2a} < \infty$$

6. It cannot be an energy signal because the signal is of infinite duration and does not decay to zero because of the repeated structure. Performing integration will lead to infinity.

   It can be a power signal because performing integration in one period will yield a finite value.

   The powers of $x(t-t_0)$, $x(-t)$ and $x(ct)$ are identical to that of $x(t)$.

7.

$$y[n] = 2r[-n+1] = 2(-n+1)u[-n+1] = \begin{cases} -2n+2, & -n+1 \geq 0 \\ 0, & -n+1 < 0 \end{cases}$$
$$= \begin{cases} -2n+2, & n \leq 1 \\ 0, & n > 1 \end{cases}$$

8.

$$\int_{-\infty}^{\infty} \sin(3t)\delta\left(t-\frac{\pi}{2}\right)dt = \int_{-\infty}^{\infty} \sin\left(3\cdot\frac{\pi}{2}\right)\delta\left(t-\frac{\pi}{2}\right)dt$$
$$= \sin\left(\frac{3\pi}{2}\right)\int_{-\infty}^{\infty} \delta\left(t-\frac{\pi}{2}\right)dt$$
$$= -1\cdot 1 = -1$$

9.

There are two sine waves: one with radian frequency 10 and the other with 4. For $2\cos(10t+1)$, according to (2.26), the fundamental period is:

$$T_1 = \frac{2\pi}{10} = \frac{\pi}{5}$$

While for $\sin(4t-1)$, the fundamental period is:

$$T_2 = \frac{2\pi}{4} = \frac{\pi}{2}$$

Then the fundamental frequency of $x(t)$ should satisfy both, i.e., it should be the LCM of $\pi/5$ or $\pi/2$, which is π.

10.
With the use of (2.19), we have:

$$\int_{-\infty}^{\infty} \delta(2t)dt = \int_{-\infty}^{\infty} \delta(\lambda) d\left(\frac{\lambda}{2}\right), \quad \lambda = 2t$$
$$= \frac{1}{2}\int_{-\infty}^{\infty} \delta(\lambda) d\lambda = \frac{1}{2}$$

Note that it can be deduced that $\delta(2t) = \frac{1}{2}\delta(t)$.

You may also imagine that when applying the scaling of $\delta(2t)$, the signal is compressed, i.e., the width is reduced by half while the amplitude (height) remains unchanged, the total area is reduced by half, resulting in $0.5\delta(t)$.