In-Class Exercise 5

1. Determine the discrete-time Fourier transform (DTFT) of $x[n] = (0.5)^n u[n]$.

2. Consider a discrete-time linear time-invariant system with input $x[n]$, output $y[n]$ and impulse response $h[n]$. Let the DTFT of $h[n]$ be $H(e^{j\omega})$.

   (a) If $x[n] = e^{j\omega_1 n}$, determine $y[n]$ in terms of $H(e^{j\omega})$.

   (b) Extend the result of (a) when

$$x[n] = \sum_{k=1}^{K} \alpha_k e^{j\omega_k n}$$

3. Let $X(e^{j\omega})$ denote the DTFT of $x[n]$. Prove:

(a) The DTFT of $x^*[n]$ is $X^*(e^{-j\omega})$.

(b) If $x[n]$ is real-valued, then:

$$|X(e^{j\omega})| = |X(e^{-j\omega})|$$

(c) The DTFT of $x^*[-n]$ is $X^*(e^{j\omega})$.

4. Prove that

$$x[n] = e^{j\omega_0 n} \leftrightarrow X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l)$$

is a DTFT pair where $\omega_0 \in (-\pi, \pi)$. Then determine the DTFTs of $x[n] = 1$ and $x[n] = \cos(0.5n)$ in $(-\pi, \pi)$.

5. Consider a discrete-time linear time-invariant system with input:

$$x[n] = \alpha^n u[n], \quad |\alpha| < 1$$

and impulse response:

$$h[n] = \beta^n u[n], \quad |\beta| < 1$$

Determine the DTFT of the system output $y[n]$.

Solution

1.

According to (6.4), the DTFT is:

$$
\begin{aligned}
X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty} (0.5)^n u[n]e^{-j\omega n} \\
&= \sum_{n=0}^{\infty}(0.5)^n e^{-j\omega n} \\
&= \sum_{n=0}^{\infty} \left(0.5e^{-j\omega}\right)^n \\
&= \frac{1-\left(0.5e^{-j\omega}\right)^{\infty}}{1-0.5e^{-j\omega}} = \frac{1}{1-0.5e^{-j\omega}}
\end{aligned}
$$

2.(a)
According to convolution, we have:

$$
\begin{aligned}
y[n] &= h[n] \otimes x[n] \\
&= \sum_{m=-\infty}^{\infty} h[m]x[n-m] \\
&= \sum_{m=-\infty}^{\infty} h[m]e^{j\omega_1(n-m)} \\
&= e^{j\omega_1 n} \sum_{m=-\infty}^{\infty} h[m]e^{-j\omega_1 m} = e^{j\omega_1 n} H(e^{j\omega_1})
\end{aligned}
$$

2.(b)

Extending the results to multiple complex tones, we have:

$$
\begin{aligned}
y[n] &= \sum_{m=-\infty}^{\infty} h[m]x[n-m] \\
&= \sum_{m=-\infty}^{\infty} h[m] \sum_{k=1}^{K} \alpha_k e^{j\omega_k(n-m)} \\
&= \sum_{k=1}^{K} \alpha_k e^{j\omega_k n} \sum_{m=-\infty}^{\infty} h[m] e^{-j\omega_k m} \\
&= \sum_{k=1}^{K} \alpha_k e^{j\omega_k n} H(e^{j\omega_k})
\end{aligned}
$$

3.(a)

Recall (6.4):

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

The DTFT of $x^*[n]$ is:

$$\sum_{n=-\infty}^{\infty} x^*[n]e^{-j\omega n} = \left(\sum_{n=-\infty}^{\infty} x[n]e^{-j(-\omega)n}\right)^* = X^*(e^{-j\omega})$$

3.(b)

If $x(t)$ is real-valued, then $x[n] = x^*[n]$. Their DTFTs should be identical. Hence we have:

$$X(e^{j\omega}) = X^*(e^{-j\omega}) \Rightarrow |X(e^{j\omega})| = |X(e^{-j\omega})|$$

3.(c)

The DTFT of $x^*[-n]$ is:

$$\sum_{n=-\infty}^{\infty} x^*[-n]e^{-j\omega n} = \left(\sum_{n=-\infty}^{\infty} x[-n]e^{j\omega n}\right)^* = \left(\sum_{m=\infty}^{-\infty} x[m]e^{-j\omega m}\right)^*, m=-n$$

$$= \left(\sum_{m=-\infty}^{\infty} x[m]e^{-j\omega m}\right)^* = X^*(e^{j\omega})$$

4.

We know that DTFT is periodic with period 2π. When using inverse DTFT, we consider the interval $(-\pi,\pi)$ which corresponds to the impulse $2\pi\delta(\omega-\omega_0)$. Applying (6.2) yields:

$$
\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\sum_{l=-\infty}^{\infty} 2\pi\delta(\omega-\omega_0-2\pi l)\right]e^{j\omega n}d\omega \\
&= \int_{-\pi}^{\pi}\delta(\omega-\omega_0)e^{j\omega n}d\omega \\
&= \int_{-\pi}^{\pi}\delta(\omega-\omega_0)e^{j\omega_0 n}d\omega \\
&= e^{j\omega_0 n}\int_{-\pi}^{\pi}\delta(\omega-\omega_0)d\omega \\
&= e^{j\omega_0 n}
\end{aligned}
$$

Hence

$$x[n] = e^{j\omega_0 n} \leftrightarrow X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l)$$

is proved.

For a constant $x[n] = 1$, it corresponds to $\omega_0 = 0$. As a result, its DTFT is:

$$X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - 2\pi l) \Rightarrow X(e^{j\omega}) = 2\pi\delta(\omega), \quad \omega \in (-\pi, \pi)$$

As $\cos(0.5n) = (e^{j0.5n} + e^{-j0.5n})/2$, its DTFT is then:

$$X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} \pi \left[\delta(\omega + 0.5 - 2\pi l) + \delta(\omega - 0.5 - 2\pi l)\right]$$

$$\Rightarrow X(e^{j\omega}) = \pi\delta(\omega + 0.5) + \pi\delta(\omega - 0.5), \quad \omega \in (-\pi, \pi)$$

5.
Generalizing the results of Question 1 yields:

$$X(e^{j\omega}) = \frac{1}{1-\alpha e^{-j\omega}}$$

$$H(e^{j\omega}) = \frac{1}{1-\beta e^{-j\omega}}$$

According to (6.22), the DTFT of $y[n]$ is then:

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) = \frac{1}{1-\alpha e^{-j\omega}} \cdot \frac{1}{1-\beta e^{-j\omega}}$$