EE3210 Signals and Systems

Semester A 2023-2024

Assignment 1

Due Date: 11 October 2023

1. Let $x(t)$, $x_1(t)$ and $x_2(t)$ be three continuous-time signals.

   (a) Show that if $x(t)$ is an odd signal, then

   $$\int_{-\infty}^{\infty} x(t)dt = 0$$

   (b) Show that if $x_1(t)$ is an odd signal and $x_2(t)$ is an even signal, then $x_1(t)x_2(t)$ is an odd signal.

   (c) Decompose $x(t)$ into even and odd parts as $x(t) = x_e(t) + x_o(t)$ where

   $$x_e(t) = \frac{1}{2}[x(t) + x(-t)] \qquad \text{and} \qquad x_o(t) = \frac{1}{2}[x(t) - x(-t)]$$

   Show that

   $$\int_{-\infty}^{\infty} x^2(t)dt = \int_{-\infty}^{\infty} x_e^2(t)dt + \int_{-\infty}^{\infty} x_o^2(t)dt$$

2. A capacitor is discharged by connecting a resistor across its terminals at $t = 0$. The voltage across the terminals is:

   $$v(t) = e^{-3t}u(t)$$

   (a) Compute the energy of $v(t)$. Is $v(t)$ an energy signal?

   (b) Compute the power of $v(t)$. Is $v(t)$ a power signal?

3. Consider a continuous-time linear time-invariant system with input $x(t)$ and impulse response $h(t) = -2\delta(t-2) + \delta(t-10)$. Determine the system output $y(t)$ in terms of $x(t)$. Is the system stable? Is the system causal? Is the system memoryless? Briefly explain your answers.

4. Given a discrete-time system with input $x[n]$ and output $y[n]$:

   $$y[n] = T(x[n]) = x[n] + \frac{1}{x[n]}$$

   Determine whether the system is memoryless, invertible, stable, causal, linear, and/or time-invariant. Briefly explain your answers.

5. Consider a discrete-time linear time-invariant system with input $x[n] = u[-1-n]$ and impulse response $h[n] = (0.5)^n u[n]$. Compute the system output $y[n]$. Is the system stable? Why? Is the system causal? Why?

6. Determine the convolution of the following two discrete-time signals:

$$x[n] = \begin{cases} n^2 - 1, & -2 \leq n \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

and

$$h[n] = \begin{cases} n - 4, & 0 \leq n \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

7. Consider a continuous-time periodic signal $x(t)$ with fundamental period of $T = 2$. Within one period, $x(t)$ is expressed as:

$$x(t) = \begin{cases} 1, & 1 > t > 0 \\ 2, & 2 > t > 1 \end{cases}$$

(a) Compute the power of $x(t)$.

(b) Determine the Fourier series coefficients for $x(t)$.

(c) With the use of appropriate Fourier series properties or otherwise, find the Fourier series coefficients for $x(t)$ if it is modified as:

$$x(t) = \begin{cases} 10, & 1 > t > 0 \\ 20, & 2 > t > 1 \end{cases}$$

(d) With the use of appropriate Fourier series properties or otherwise, find the Fourier series coefficients for $x(t)$ if it is modified as:

$$x(t) = \begin{cases} 2, & 1 > t > 0 \\ 1, & 2 > t > 1 \end{cases}$$

(e) With the use of appropriate Fourier series properties or otherwise, find the Fourier series coefficients for $x(t)$ if it is modified as:

$$x(t) = \begin{cases} 2, & 0.5 > t > 0 \\ 1, & 1.5 > t > 0.5 \\ 2, & 2 > t > 1.5 \end{cases}$$

8. Choose a real-world system in daily life. The system can be a software or hardware. Identify the input, output and function of this system. Briefly explain if your chosen system is linear and time-invariant.

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Solution for Assignment 1

1(a)
If $x(t)$ is an odd, then $x(-t)+x(t)=0$. As a result, we have:

$$\begin{aligned}\int_{-\infty}^{\infty} x(t)dt &= \int_{-\infty}^{0} x(t)dt + \int_{0}^{\infty} x(t)dt \\ &= \int_{0}^{\infty} x(-u)du + \int_{0}^{\infty} x(t)dt, \quad t=-u \\ &= \int_{0}^{\infty} [x(-t)+x(t)]dt = 0\end{aligned}$$

1(b)
Let $y(t)=x_1(t)x_2(t)$. Then

$$y(-t)=x_1(-t)x_2(-t)=-x_1(t)x_2(t)=-y(t)$$

This implies that $y(t)$ is odd.

1(c)

$$\begin{aligned}\int_{-\infty}^{\infty} x^2(t)dt &= \int_{-\infty}^{\infty} [x_e(t)+x_o(t)]^2 dt \\ &= \int_{-\infty}^{\infty} x_e^2(t)dt + \int_{-\infty}^{\infty} x_o^2(t)dt + 2\int_{-\infty}^{\infty} x_e(t)x_o(t)dt\end{aligned}$$

Using the result of part (b), we know that $x_e(t)x_o(t)$ is an odd signal. Then, using the result of part (a), we have

$$\int_{-\infty}^{\infty} x_e(t)x_o(t)dt = 0$$

Therefore,

$$\int_{-\infty}^{\infty} x^2(t)dt = \int_{-\infty}^{\infty} x_e^2(t)dt + \int_{-\infty}^{\infty} x_o^2(t)dt$$

2(a)

$$E_v = \int_{-\infty}^{\infty} |v(t)|^2 dt = \int_{-\infty}^{\infty} (e^{-3t}u(t))^2 dt = \int_{0}^{\infty} e^{-6t}dt = -\frac{1}{6}e^{-6t}\Big|_0^{\infty} = \frac{1}{6}$$

It has finite energy and hence it is an energy signal.

2(b)
As energy is finite, its power is $P_v = 0$ and thus it is not a power signal.

3.

$$
\begin{aligned}
y(t) = x(t) \otimes h(t) &= \int_{-\infty}^{\infty} h(\lambda) x(t-\lambda) d\lambda \\
&= \int_{-\infty}^{\infty} \left[-2\delta(\lambda-2) + \delta(\lambda-10)\right] x(t-\lambda) d\lambda \\
&= \int_{-\infty}^{\infty} -2\delta(\lambda-2) x(t-\lambda) d\lambda + \int_{-\infty}^{\infty} \delta(\lambda-10) x(t-\lambda) d\lambda \\
&= -2x(t-2) + x(t-10)
\end{aligned}
$$

The system is **stable** because if $x(t)$ is bounded, $y(t)$ will also be bounded. (Or, the system is **stable** because $\int_{-\infty}^{\infty} |h(t)|\, dt = 3 < \infty$, that is, the impulse response is absolutely summable.)

The system is **causal** because the output $y(t)$ does not depend on any future input values. (Or, the system is **causal** because $h(t) = 0$ for $t < 0$.)

The system is **not memoryless** because the output at time t does not only depend on the input at time t.

4.
The system is **memoryless** because the output at time n only depends on the input at time n.

The system is **not invertible**. Reorganizing the input-output relationship as: $x^2[n] - y[n]x[n] + 1 = 0$, we can find that $x[n]$ is expressed in terms of $y[n]$ with two possibilities in solving the quadratic equation.

The system is **not stable**. It is because for a bounded input $x[n] = 0$, the output will be unbounded.

The system is **causal** because the output does not depend on the future input value.

The system is **not linear**. The proof is as follows:
Let $y_1[n] = T\{x_1[n]\}$, $y_2[n] = T\{x_2[n]\}$ and $y_3[n] = T\{x_3[n]\}$ with $x_3[n] = a \cdot x_1[n] + b \cdot x_2[n]$.

The system outputs for $x_1[n]$ and $x_2[n]$ are:

$$y_1[n] = x_1[n] + 1/x_1[n] \quad \text{and} \quad y_2[n] = x_2[n] + 1/x_2[n]$$

The system output for $x_3[n]$ is then:

$$
\begin{aligned}
y_3[n] &= x_3[n] + 1/x_3[n] = ax_1[n] + bx_2[n] + 1/\left(ax_1[n] + bx_2[n]\right) \\
&\neq ax_1[n] + bx_2[n] + 1/(ax_1[n]) + 1/(bx_2[n]) = ay_1[n] + by_2[n]
\end{aligned}
$$

The system is **time-invariant**. The proof is as follows:

First, we have $y[n-n_0] = x[n-n_0] + 1/x[n-n_0]$
Consider $x_1[n] = x[n-n_0]$, its system output is

$$y_1[n] = x_1[n] + 1/x_1[n] = x[n-n_0] + 1/x[n-n_0] = y[n-n_0]$$

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$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] = \sum_{m=-\infty}^{\infty} u[-m-1]\cdot(0.5)^{n-m}u[n-m]$$

$$= \sum_{m=-\infty}^{-1} (0.5)^{n-m}u[n-m]$$

$$= \sum_{l=1}^{\infty} (0.5)^{n+l}u[n+l]$$

For $n \geq -1$, all $\{u[n+l]\}$ correspond to 1 and we have:

$$y[n] = \sum_{l=1}^{\infty} (0.5)^{n+l} = (0.5)^n \sum_{l=1}^{\infty} (0.5)^l = (0.5)^n \cdot \frac{0.5}{1-0.5} = (0.5)^n$$

For $n < -1$, $u[n+l] = 1$ when $n + l \geq 0$ or $l \geq -n$

$$y[n] = \sum_{l=-n}^{\infty} (0.5)^{n+l} = (0.5)^n \sum_{l=-n}^{\infty} (0.5)^l = (0.5)^n \cdot \frac{(0.5)^{-n}}{1-0.5} = 2$$

Combining the results, we have:

$$y[n] = \begin{cases} (0.5)^n, & n \geq -1 \\ 2, & n < -1 \end{cases}$$

The system is stable because $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$.

The system is causal because $h[n] = 0$ for $n < 0$.

6.
Let $y[n]$ be the convolution output. Starting from $n = -2$, $y[n]$ = -12, -9, -2, 0, -10, -8, -6, -3. At other time instants, the output is 0.

7(a)

$$\begin{aligned} P_x &= \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 dt = \frac{1}{T}\int_0^T x^2(t)dt \\ &= \frac{1}{2}\left[\int_0^1 1^2 dt + \int_1^2 2^2 dt\right] = 2.5 \end{aligned}$$

7(b)
The signal has period $T = 2$ and fundamental frequency $\omega_0 = \pi$. Consider the period from $t = -1$ to $t = 1$ and use (4.4):

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt = \frac{1}{2}\left[\int_{-1}^{0} 2e^{-jk\pi t}dt + \int_0^1 e^{-jk\pi t}dt\right]$$

For $k = 0$,

$$a_0 = \frac{1}{2}\left[\int_{-1}^{0} 2dt + \int_{0}^{1} 1dt\right] = \frac{1}{2}[2+1] = 1.5$$

For $k \neq 0$,

$$\begin{aligned} a_k &= \frac{1}{2}\left[\int_{-1}^{0} 2e^{-jk\pi t}dt + \int_{0}^{1} e^{-jk\pi t}dt\right] \\ &= \frac{1}{2}\left[\frac{2}{-jk\pi} e^{-jk\pi t}\Big|_{-1}^{0} + \frac{1}{-jk\pi} e^{-jk\pi t}\Big|_{0}^{1}\right] \\ &= -\frac{1}{2jk\pi}\left[2 - 2e^{jk\pi} + e^{-jk\pi} - 1\right] \\ &= -\frac{1}{2jk\pi}\left[1 - 2e^{jk\pi} + e^{-jk\pi}\right] \end{aligned}$$

Combining the results, we have:

$$a_k = \begin{cases} 1.5, & k = 0 \\ -\dfrac{1}{2jk\pi}\left[1 - 2e^{jk\pi} + e^{-jk\pi}\right] & k \neq 0 \end{cases}$$

7(c)
Using the linearity property, now $x(t)$ is modified as $10x(t)$, so we have:

$$a_k = \begin{cases} 15, & k = 0 \\ -\dfrac{5}{jk\pi}\left[1 - 2e^{jk\pi} + e^{-jk\pi}\right] & k \neq 0 \end{cases}$$

7(d)
Using the time reversal property, we can just change a_k to a_{-k}

$$a_k = \begin{cases} 1.5, & k = 0 \\ \dfrac{1}{2jk\pi}\left[1 - 2e^{-jk\pi} + e^{jk\pi}\right] & k \neq 0 \end{cases}$$

(You can also use time-shift property with a shift of 1)

7(e)

Using the time shifting property, $x(t) \leftrightarrow a_k \Rightarrow x(t-0.5) \leftrightarrow e^{-j0.5k\pi}a_k$:

$$a_k = \begin{cases} 1.5, & k = 0 \\ -\dfrac{e^{-j0.5k\pi}}{2jk\pi}\left[1 - 2e^{jk\pi} + e^{-jk\pi}\right] & k \neq 0 \end{cases}$$

8.
There are many possible answers.

One example is a software that computes the moving average of stock prices. In this software system, the input is the close price of a stock to be investigated. The output is the moving average based on the previous close prices including the current one. It is used for analysis the stock, particularly to see if it is on the rising or falling trend.

Consider N-day moving average where $x[n]$ is the input and $y[n]$ is the output, the input-output relationship is then:

$$y[n] = \frac{1}{N}\left(x[n] + x[n-1] + \cdots + x[n-N+1]\right)$$

It is a difference equation, and hence the system is both linear and time-invariant.