Solution

1.

1.(a)

The sample space is:

$S = \{PPP, PPF, PFP, PFF, FPP, FPF, FFP, FFF\}$

1.(b)

$U = \{PPF, PFF, FPF, FFF\}$

$V = \{PPP, PPF, PFP, PFF\}$

$X = \{PPP, PPF, PFP, FPP\}$

$Y = \{PFF, FPF, FFP, FFF\}$

1.(c)

Since $U \cap V = \{PPF, PFF\} \neq \emptyset$, hence U and V are not mutually exclusive.

1.(d)

Since $X \cap Y = \emptyset$, hence X and Y are mutually exclusive.

2.(a)

$$P(M \cup G) = P(M) + P(G) - P(M \cap G) = 0.6 + 0.4 - 0.2 = 0.8$$

2.(b)

Recall:

$$\begin{aligned} P(M \cup G \cup E) &= P(M) + P(G) + P(E) - P(M \cap G) - P(M \cap E) \\ &\quad -P(G \cap E) + P(M \cap G \cap E) \end{aligned}$$

Noting that $P(M \cup G \cup E) = 1$, and $P(G \cap E) = P(M \cap G \cap E) = 0$, we have:

$$1 = 0.6 + 0.4 + 0.3 - 0.2 - P(M \cap E) \Rightarrow P(M \cap E) = 0.1$$

Also, as $G \cap E = \emptyset$, hence we obtain:

$$P((M \cap G) \cup (M \cap E)) = P(M \cap G) + P(M \cap E) = 0.2 + 0.1 = 0.3$$

2.(c)

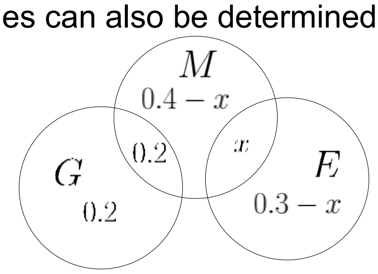
$$P(M) = P(M \cap G) + P(M \cap \overline{G}) \Rightarrow P(M \cap \overline{G}) = 0.6 - 0.2 = 0.4$$

As

$$P(M \cap \overline{G}) = 0.4 \neq P(M)P(\overline{G}) = 0.6 \times 0.6 = 0.36$$

$\overline{G}$ and M are not independent.

Alternatively, the probabilities can also be determined using the Venn diagram.



$$0.2 + 0.6 + 0.3 - x = 1 \Rightarrow x = 0.1$$

3.(a)
The sample space is {0, 20, 30, 50, 70, 80, 100, 120, 130,150}.

3.(b)
P(0)= 1/16; P(20)=1/16; P(30)= 1/16; P(50)= 3/16; P(70)= 2/16; P(80)= 2/16; P(100)= 3/16; P(120)= 1/16; P(130)= 1/16; P(150)= 1/16;

4.(a)
Denote F as event of choosing the fair coin, and denote H and T as the events of head and tail, respectively. We have:

$$P(F) = P(\overline{F}) = \frac{1}{2}$$

Hence:

$$P(H) = P(F)P(H|F) + P(\overline{F})P(H|\overline{F}) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot 0 = \frac{1}{4}$$

4.(b)

$$P(F|T) = \frac{P(F) \cdot P(T|F)}{P(T)} = \frac{P(F) \cdot P(T|F)}{1 - P(H)} = \frac{1/2(1/2)}{1 - 0.25} = \frac{1}{3}$$

5.
The probability of selecting box B conditioned to continuously draw k red balls is

$$P(B|kR) = \frac{P(kR|B)P(B)}{P(kR|A)P(A)+P(kR|B)P(B)},$$

where $P(A) = P(B) = 0.5$, $P(kR|A) = \left(\frac{1}{3}\right)^k$, $P(kR|B) = \left(\frac{3}{4}\right)^k$, then we have

$$P(B|kR) = \frac{P(kR|B)P(B)}{P(kR|A)P(A)+P(kR|B)P(B)} = \frac{9^k}{4^k+9^k}.$$

6.
First we may need to list out the possible numbers of H and S for the 8 lily pads:
HHHHHHHH $\Rightarrow C(8,0) = 1$ way
HHHHHHS $\Rightarrow C(7,1) = 7$ ways
HHHHSS $\Rightarrow C(6,2) = 15$ ways
HHSSS $\Rightarrow C(5,3) = 10$ ways
SSSS $\Rightarrow C(4,4) = 1$ way
Hence the total number of ways is 34.

7.
There are 6 people. If we assign the member(s) of one group, then the other group member(s) are automatically determined. Hence we only need to consider the first group with 1, 2 and 3 people only. The number of ways should be:

$C(6,1) + C(6,2)) + C(6,3)/2 = 6 + 15 + 10 = 31$

8.
Firstly, we may need to list out the possible numbers of dices:
1,2,5 $\Rightarrow P(3,3) = 3! = 6$ permutations.
1,3,4 $\Rightarrow P(3,3) = 3! = 6$ permutations.
1,1,6 $\Rightarrow P(3,3)/P(2,2) = 3!/2! = 3$ permutations. The denominator of 2! is to account for the overcounting due to the two equivalent "1".
2,2,4 $\Rightarrow P(3,3)/P(2,2) = 3!/2! = 3$ permutations.
2,3,3 $\Rightarrow P(3,3)/P(2,2) = 3!/2! = 3$ permutations.
Hence the total number of dice permutations is 21. The probability is $\frac{21}{6^3} = \frac{21}{216} = \frac{7}{72}$.

9.
The comics, novels and textbooks have $5!$, $4!$ and $2!$ different arrangements. The 3 groups have $3!$ different arrangements. Hence the total number of ways is:

$$5! \times 4! \times 2! \times 3! = 34560$$

10.
$P(AB) = P(AB\overline{C}) + P(ABC); P(AC) = P(AC\overline{B}) + P(ABC); P(BC) = P(BC\overline{A}) + P(ABC)$
Thus,
$P(AB) + P(AC) - P(BC) = P(AB\overline{C}) + P(AC\overline{B}) + P(ABC) - P(BC\overline{A})$
$= P(A(B \cup C)) - P(BC\overline{A})$
$\leq P(A(B \cup C))$
$\leq P(A)$

Note that it is possible to obtain the proof in different ways, e.g.,

Solution 2:

$$P(AB) + P(AC) - P(BC) \leq P(A) \tag{1}$$

Since

$$\begin{aligned} P(A) + P(B) &= P(AB) + P(A \cup B); \\ P(A) + P(C) &= P(AC) + P(A \cup C); \\ P(B) + P(C) &= P(BC) + P(B \cup C) \end{aligned}$$

(1) is equal to:

$$P(A) + P(B \cup C) \leq P(A \cup B) + P(A \cup C)$$

and we have:

$$P(A \cup B) + P(A \cup C) = P((A \cup B)(A \cup C)) + P(A \cup B \cup C) \geq P(A) + P(B \cup C)$$

Solution 3:

$$\begin{aligned} P(A) &\geq P(A(B \cup C)) = P(AB \cup AC) \\ &= P(AB) + P(AC) - P(ABC) \\ &\geq P(AB) + P(AC) - P(BC) \end{aligned}$$

Solution 4:

$$\begin{aligned} P(AB)+P(AC) &= P(A)P(B|A)+P(A)P(C|A) \\ &= P(A)\left(P(B|A)+P(C|A)\right) \\ &\le P(A) \end{aligned}$$

and thus:

$$P(AB)+P(AC)-P(BC)\le P(A)$$

11.
Assigning $A=\{\text{study hard}\}$ and $B=\{\text{pass}\}$, then we have $P(A)=0.9$, $P(\bar{A})=0.1$, $P(B\mid A)=0.95$, $P(\overline{B}\mid\overline{A})=0.9$, then we can find
(a)

$$\begin{aligned} P(\overline{A}|B) &= \frac{P(\overline{A}\cap B)}{P(B)} = \frac{P(\overline{A})P(B|\overline{A})}{P(A)P(B|A)+P(\overline{A})P(B|\overline{A})} \\ &= \frac{0.1\times 0.1}{0.9\times 0.95+0.1\times 0.1} = 0.0116 \end{aligned}$$

(b)

$$P(A|\overline{B}) = \frac{P(A\cap\overline{B})}{P(\overline{B})} = \frac{P(A)P(\overline{B}|A)}{P(A)P(\overline{B}|A)+P(\overline{A})P(\overline{B}|\overline{A})} = \frac{0.9\times 0.05}{0.9\times 0.05+0.1\times 0.9} = 0.3333$$

12.
$2N$ moves are needed. Among them, N must correspond to the up movement. Hence the total number of possible paths is $C(2N,N)$.

13.
There are four doors A, B, C, and D. Without loss of generality, assume that Peter chooses A door and E be the event that the host choose door D to show balloon. Let a, b, c, d be the event that the Rolex watch behind door A, B, C, D, respectively. There are 4 equally likely possibilities for what is behind the 4 doors, namely, P(a) = P(b) = P(c) = P(d) = 1/4.

Thus, we have:
P(E|a) = 1/3, P(E|b) = 1/2, P(E|c) = 1/2, P(E|d) = 0
and
P(E) = P(a) P(E|a) + P(b) P(E|b) + P(c) P(E|c) + P(d) P(E|d) = 1/3

If Peter does not switch his choice, his winning probability is:
P(a|E) = P(a) P(E|a) /P(E) = 1/4

But if he switches,
P(b|E) = P(b) P(E|b) /P(E) = 3/8
Or
P(c|E) = P(c) P(E|c) /P(E) = 3/8

Therefore, if he switches, the probability will be increased to 3/8.

We can also think another way as follows. If Peter does not switch, the probability of getting the watch is still 1/4. But if he switches, then the remaining probability is 3/4 for 2 doors as one door does not correspond to the watch has been opened, then the probability of choosing either one of them is 3/8.