Euler Formula

$$e^{j\phi} = \cos(\phi) + j\sin(\phi)$$

$$\cos(\phi) = \frac{e^{j\phi} + e^{-j\phi}}{2} \qquad \sin(\phi) = \frac{e^{j\phi} - e^{-j\phi}}{2j}$$

Fourier Series

- The frequency domain representation of a continuous-time periodic signal

$$x(t) = x(t + T_p)$$

- The smallest T_p → fundamental period

$$\Omega_0 = \frac{2\pi}{T_p}$$

- Ω_0 → fundamental frequency

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t}$$

$$a_k = \frac{1}{T_p}\int_{-T_p/2}^{T_p/2} x(t)e^{-jk\Omega_0 t}\,dt, \qquad k = \ldots\text{-1,0,1,2,}\ldots$$

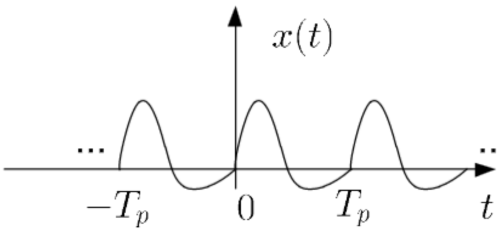
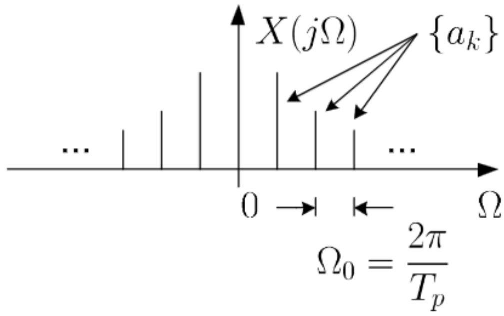
- a_k → Fourier series coefficients
  - It is generally complex

$$|a_k| = \sqrt{(\Re\{a_k\})^2 + (\Im\{a_k\}^2)}$$

$$\angle(a_k) = \tan^{-1}\left(\frac{\Im\{a_k\}}{\Re\{a_k\}}\right)$$

- Find $k = 0$ (base case) for a_0

- Then find $k \neq 0$ for a_k

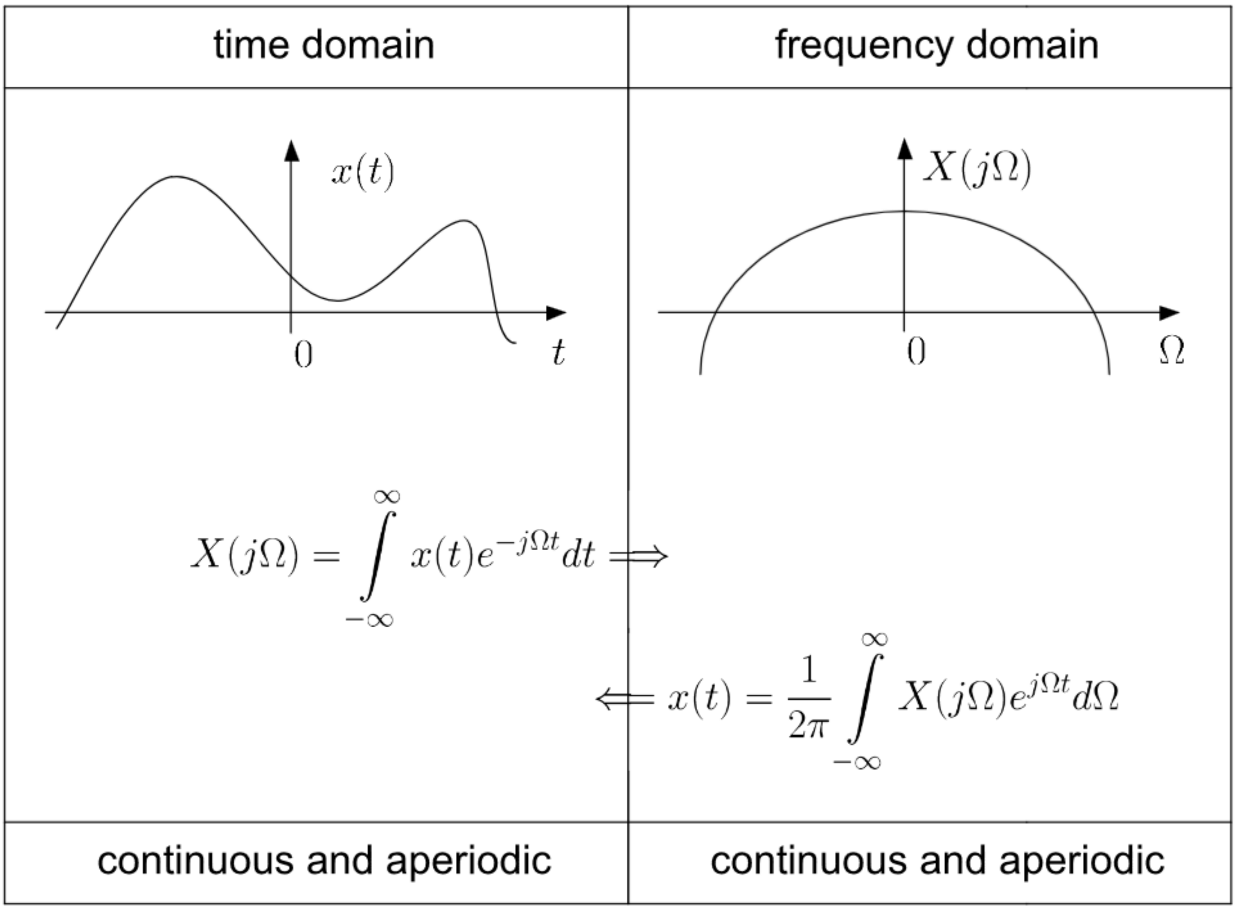
time domain	frequency domain
 $a_k = \frac{1}{T_p} \int_{-T_p/2}^{T_p/2} x(t) e^{-jk\Omega_0 t} dt \Longrightarrow$	 $\Longleftarrow x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t}$
continuous and periodic	discrete and aperiodic

Fourier Transform

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} \, dt$$

Inverse Fourier Transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} \, d\Omega$$



Periodic Signal Representation using Fourier Transform

- Ω_0 → tone of frequency
- With the use of $\delta(t)$

$$X(j\Omega) = 2\pi\delta(\Omega - \Omega_0)$$

Inverse Fourier Transform on Periodic Signal

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\Omega - \Omega_0)e^{j\Omega t}\, d\Omega = e^{j\Omega_0 t}$$

Fourier Transform and Linear Time-Invariant System

$$y(t) = x(t) \circledast h(t) \leftrightarrow Y(j\Omega) = X(j\Omega)H(j\Omega)$$

- It suggests converting the input and impulse response to the frequency domain, and then $y(t)$ is computed from the inverse Fourier transform of $X(j\Omega)H(j\Omega)$
- $H(j\Omega)$ → System frequency response

$$Y(j\Omega)\left[\sum_{k=0}^{N} a_k (j\Omega)^k\right] = X(j\Omega)\left[\sum_{k=0}^{M} b_k (j\Omega)^k\right]$$

$$H(j\Omega) = \frac{Y(j\Omega)}{X(j\Omega)} = \frac{\sum_{k=0}^{N} b_k (j\Omega)^k}{\sum_{k=0}^{M} a_k (j\Omega)^k}$$

Discrete-Time Fourier Transform (DTFT)

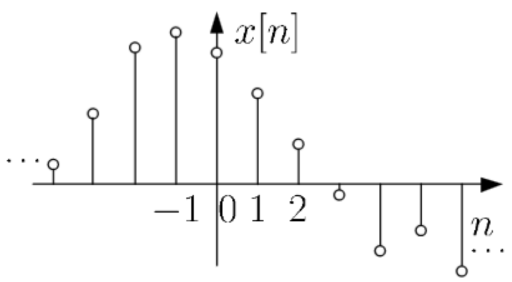
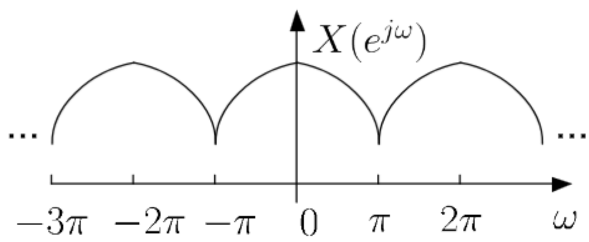
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$X(e^{j\omega}) \leftrightarrow X(e^{j(\omega+2k\pi)})$$

- It is periodic with a period 2π
- It is generally complex

Inverse Discrete-Time Fourier Transform

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n}\, d\omega$$

time domain	frequency domain
	
$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \Longrightarrow$	
	$\Longleftarrow x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$
discrete and aperiodic	continuous and periodic

DTFT and Linear Time-Invariant System

$$y[n] = x[n] \circledast h[n] \leftrightarrow Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

- This suggests converting the input and impulse response to the frequency domain, and then $y[n]$ is computed from the inverse DTFT of $X(e^{j\omega})H(e^{j\omega})$

$$Y(e^{j\omega})\sum_{k=0}^{N} a_k e^{-j\omega k} = X(e^{j\omega})\sum_{k=0}^{M} b_k e^{-j\omega k}$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{N} b_k e^{-j\omega k}}{\sum_{k=0}^{M} a_k e^{-j\omega k}}$$

Sampling

- It is converting a continuous-time signal $x(t)$ into a discrete-time signal $x[n]$

$$x[n] = x(t)\Big|_{t=nT} = x(nT), \qquad n = \ldots\text{-}1{,}0{,}1{,}2, \ldots$$

- T → sampling period
- $x[n]$ can reconstruct $x(t)$ if
  - $x(t)$ is bandlimited such that its Fourier transform $X(j\Omega) = 0$ for $|\Omega| \geq \Omega_b$ where Ω_b → bandwidth
  - Sampling period T is sufficiently small

Sampling Theorem

- Let $x(t)$ be a bandlimited continuous-time signal with

$$X(j\Omega) = 0, \qquad |\Omega| \geq \Omega_b$$

- Then $x(t)$ is uniquely determined by its samples $x[n] = x(nT)$ for $n = \ldots\text{-}1{,}0{,}1{,}2, \ldots$, if

$$\Omega_s = \frac{2\pi}{T} > 2\Omega_b$$

- Therefore, to avoid aliasing, the sampling frequency must >= $2\Omega_b$

Reconstruction

$$H(j\Omega) = \begin{cases} T, & -\Omega_c < \Omega < \Omega_c \\ 0, & \text{otherwise} \end{cases}$$

- Ω_c → a lowpass filter

$$\Omega_c = \frac{\Omega_s}{2} = \frac{\pi}{T}$$

Discrete-Time Signal Representation with z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

- z → continuous complex variable

Region of Convergence (ROC)

- It indicates when the z-transform of a sequence converges
- The set of values of z for which $X(z)$ converges

$$|X(z)| = \left| \sum_{n=-\infty}^{\infty} x[n] z^{-n} \right| \le \sum_{n=-\infty}^{\infty} \left| x[n] z^{-n} \right| < \infty$$

- If there is no value of z satisfies the converges, then z-transform does not exist
- That set of values of z → ROC

$$|z| > \lim_{n \to \infty} \left| \frac{x[n+1]}{x[n]} \right| = R_+$$

$$|z| < \lim_{m \to \infty} \left| \frac{x[-m]}{x[-m-1]} \right| = R_-$$

- The ROC for $X(z)$ is $R_+ < |z| < R_-$
  - ROC is a ring when $R_+ < R_-$
  - No ROC if $R_- < R_+$ and $X(z)$ does not exist

Existence of DTFT

- While the DTFT converges if

$$\left| X(e^{j\omega}) \right| = \left| \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \right| \le \sum_{n=-\infty}^{\infty} \left| x[n] e^{-j\omega n} \right| = \sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

- Then, it is possible that the DTFT of $x[n]$ does not exist

Poles and Zeros

- The set of values of z for which $X(z) = \pm\infty$ → the poles of $X(z)$
- The set of values of z for which $X(z) = 0$ → the zeros of $X(z)$

$$X(z) = \frac{z^N \sum_{k=0}^{M} b_k z^{M-k}}{z^M \sum_{k=0}^{N} a_k z^{N-k}}$$

- If $M > N$ → there are $M - N$ poles at zero location
- If $M < N$ → there are $N - M$ zeros at zero location

Finite-Duration and Infinite-Duration Sequences

- Finite-duration sequence → values of $x[n]$ are nonzero only for a finite time interval
  - Otherwise, it is an infinite-duration sequence
    - Right-sided → if $x[n] = 0$ for $n < N_+ < \infty$, where N_+ is an integer
    - Left-sided → if $x[n] = 0$ for $n > N_- > -\infty$, where N_- is an integer
    - Two-sided → neither right-sided nor left-sided

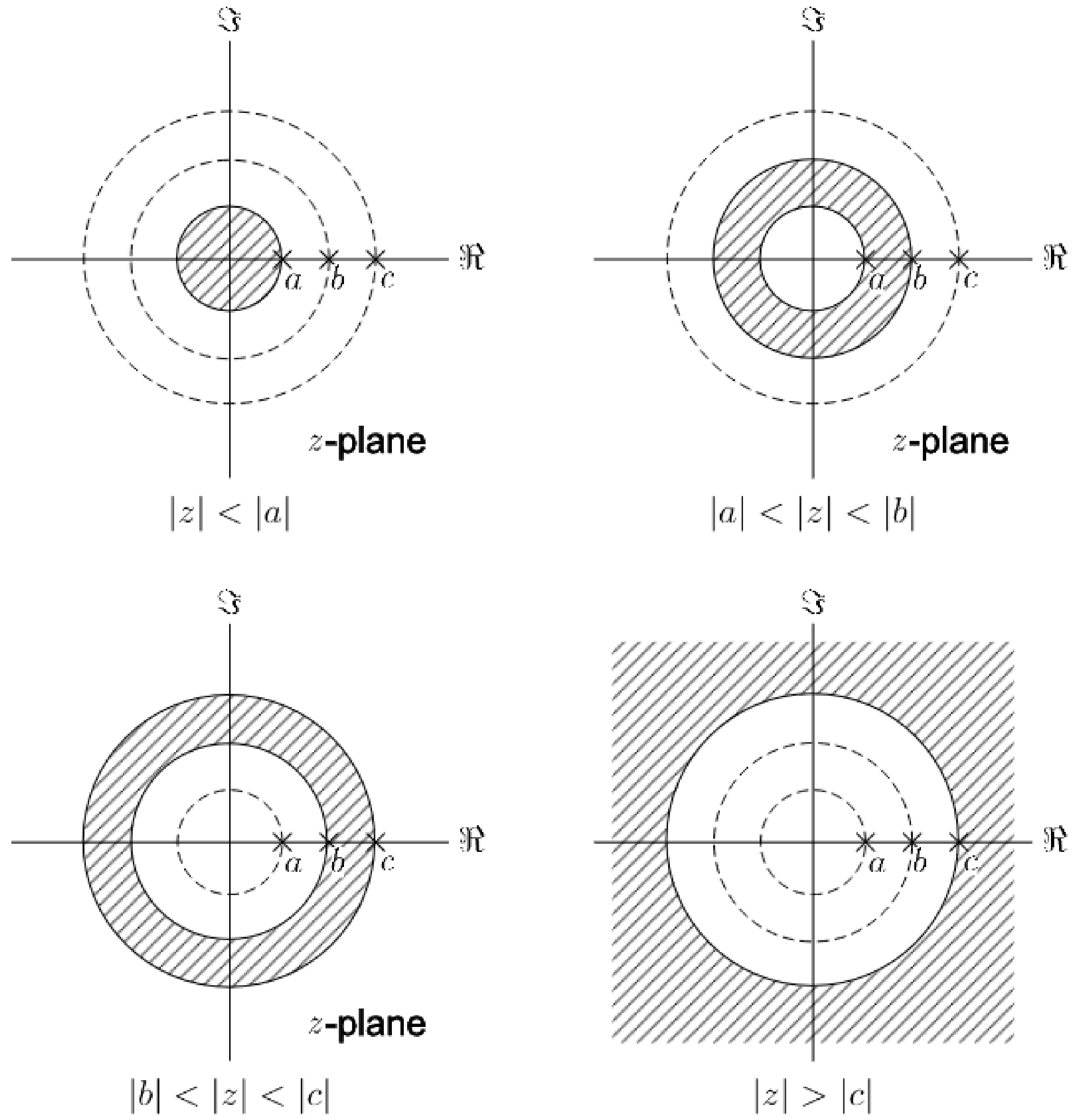
Table of z-transforms for Common Sequences

Sequence	Transform	ROC
$\delta[n]$	1	All z
$\delta[n-m]$	z^{-m}	$\lvert z\rvert > 0$, $m > 0$; $\lvert z\rvert < \infty$, $m < 0$
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
$a^n \cos(bn) u[n]$	$\frac{1-a\cos(b)z^{-1}}{1-2a\cos(b)z^{-1}+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
$a^n \sin(bn) u[n]$	$\frac{a\sin(b)z^{-1}}{1-2a\cos(b)z^{-1}+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$

Summary of ROC Properties

1. There are four possible shapes for ROC
   1. Entire region except $z = 0$ and/or $z = \infty$
   2. A ring
   3. Inside a circle in the z-plane centred at the origin
   4. Outside a circle in the z-plane centred at the origin
2. The DTFT of a sequence $x[n]$ exists iff the ROC of the z-transform of $x[n]$ includes the unit circle
3. The ROC cannot contain any poles
4. When $x[n]$ is a finite-duration sequence, the ROC is the entire z-plane expect $z = 0$ and/or $z = \infty$
5. When $x[n]$ is a right-sided sequence, the ROC is of the form $|z| > |p_{\max}|$ where $p_{\max}$ is the pole with the largest magnitude in $X(z)$
6. When $x[n]$ is a left-sided sequence, the ROC is of the form $|z| < |p_{\min}|$ where $p_{\min}$ is the pole with the smallest magnitude in $X(z)$
7. When $x[n]$ is a two-sided sequence, the ROC is of the form $|p_a| < |z| < |p_b|$ where p_a and p_b are two poles with the successive magnitudes in $X(z)$ such that $|p_a| < |p_b|$

8. The ROC must be a connected region



Causality and Stability Investigation with ROC

- The causality condition is when $h[n] = 0$ for $n < 0$
  - If the system is causal and $h[n]$ is of finite duration, the ROC should include ∞
  - If the system is causal and $h[n]$ is of infinite duration, the ROC is of the form $|z| > |p_{\max}|$ and should include ∞
- The stability condition is when $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$
  - This also means that the DTFT of $h[n]$ exists

Inverse z-transform

- Using partial fraction expansion

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

1. Determine the N nonzero poles $c_1, c_2, \ldots, c_N$
2. Case 1 → $M < N$ and all poles are of the first order
    1. $X(z) = \sum_{k=1}^{N} \frac{A_k}{1-c_k z^{-1}}$
    2. Find $A_k = (1-c_k z^{-1})X(z)\Big|_{z=c_k}$
    3. Perform inverse z-transform by inspection
3. Case 2 → $M \geq N$ and all poles are of the first order
    1. $X(z) = \sum_{l=0}^{M-N} B_l z^{-1} + \sum_{k=1}^{N} \frac{A_k}{1-c_k z^{-1}}$
    2. Find B_l by using the long division of the numerator by the denominator
    3. Find $A_k = (1-c_k z^{-1})X(z)\Big|_{z=c_k}$
    4. Perform inverse z-transform by inspection
4. Case 3 → $M < N$ with multiple-order poles
    1. $X(z) = \sum_{k=1,k\neq i}^{N} \frac{A_k}{1-c_k z^{-1}} + \sum_{m=1}^{s} \frac{C_m}{(1-c_i z^{-1})^m}$
    2. Find $A_k = (1-c_k z^{-1})X(z)\Big|_{z=c_k}$
    3. Find $C_m = \frac{1}{(s-m)!(-c_i)^{s-m}} \cdot \frac{d^{s-m}}{dw^{s-m}}[(1-c_i w)^s X(w^{-1})]\Big|_{w=c_i^{-1}}$
    4. Perform inverse z-transform by inspection
5. Case 4 → $M \geq N$ with multiple-order poles
    1. $X(z) = \sum_{l=0}^{M-N} B_l z^{-l} + \sum_{k=1,k\neq i}^{N} \frac{A_k}{1-c_k z^{-1}} + \sum_{m=1}^{s} \frac{C_m}{(1-c_i z^{-1})^m}$
    2. Find $A_k = (1-c_k z^{-1})X(z)\Big|_{z=c_k}$
    3. Find B_l by using the long division of the numerator by the denominator
    4. Find $C_m = \frac{1}{(s-m)!(-c_i)^{s-m}} \cdot \frac{d^{s-m}}{dw^{s-m}}[(1-c_i w)^s X(w^{-1})]\Big|_{w=c_i^{-1}}$
    5. Perform inverse z-transform by inspection

- Using power series expansion

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \cdots + x[-1]z^1 + x[0] + x[1]z^{-1} + x[2]z^2 + \ldots$$

Transfer Function of Linear Time-Invariant System

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$Y(z)\sum_{k=0}^{N} a_k z^{-k} = X(z)\sum_{k=0}^{M} b_k z^{-k}$$

- The transfer function $H(z)$ defined as:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

- This suggests taking the z-transform for $x[n]$ and $h[n]$ → **$X(z)H(z)$
  - Perform inverse z-transform of $X(z)H(z)$ → $y[n]$