# Solutions to EE3210 Tutorial 10 Problems

**Problem 1:** From Tutorial 9 Problem 3, we know as a Fourier transform pair that

$$h(t) = \text{sinc}(t) \leftrightarrow H(\omega) = \begin{cases} 1, & |\omega| < \pi \\ 0, & |\omega| > \pi. \end{cases}$$

We also know as a Fourier transform pair that

$$x(t) = 1 \leftrightarrow X(\omega) = 2\pi\delta(\omega).$$

Therefore, we have

$$Y(\omega) = X(\omega)H(\omega) = 2\pi\delta(\omega).$$

Taking the inverse Fourier transform by inspection, we obtain

$$y(t) = 1.$$

**Problem 2:** Given $x(t) = e^{-t}u(t)$ and $h(t) = e^{-2t}u(t)$, we derive the Fourier transform of $x(t)$ and $h(t)$ as

$$X(\omega) = \frac{1}{1 + j\omega} \quad \text{and} \quad H(\omega) = \frac{1}{2 + j\omega}.$$

Now, using the convolution property of the Fourier transform, we obtain the Fourier transform of $y(t)$ as

$$Y(\omega) = X(\omega)H(\omega) = \frac{1}{(1 + j\omega)(2 + j\omega)}. \tag{1}$$

Making the substitution of $v$ for $j\omega$ in (1), we obtain the rational function

$$G(v) = \frac{1}{(v + 1)(v + 2)}. \tag{2}$$

The partial-fraction expansion for (2) is

$$G(v) = \frac{A_1}{v + 1} + \frac{A_2}{v + 2}$$

where

$$A_1 = [(v+1)G(v)]|_{v=-1} = \frac{1}{v+2}\bigg|_{v=-1} = 1$$

and

$$A_2 = [(v+2)G(v)]|_{v=-2} = \frac{1}{v+1}\bigg|_{v=-2} = -1.$$

Therefore,

$$Y(\omega) = \frac{1}{1+j\omega} - \frac{1}{2+j\omega}.$$

Taking inverse Fourier transforms by inspection and using the linearity property, we get

$$y(t) = e^{-t}u(t) - e^{2t}u(t) = \left(e^{-t} - e^{-2t}\right)u(t).$$

**Problem 3:** Applying the Fourier transform to both sides of the difference equation, and using the properties of time shift and linearity, we have

$$Y[\Omega] - \frac{3}{4}e^{-j\Omega}Y[\Omega] + \frac{1}{8}e^{-j\Omega 2}Y[\Omega] = 2X[\Omega]. \tag{3}$$

Rearranging (4) and using the convolution property, we obtain the frequency response of the system as

$$H[\Omega] = \frac{Y[\Omega]}{X[\Omega]} = \frac{2}{1 - \frac{3}{4}e^{-j\Omega} + \frac{1}{8}e^{-j\Omega 2}} = \frac{2}{\left(1 - \frac{1}{2}e^{-j\Omega}\right)\left(1 - \frac{1}{4}e^{-j\Omega}\right)}.$$

Given $x[n] = \left(\frac{1}{4}\right)^n u[n]$, we derive the Fourier transform of $x[n]$ as

$$X[\Omega] = \frac{1}{1 - \frac{1}{4}e^{-j\Omega}}.$$

Then, we have

$$Y[\Omega] = X[\Omega]H[\Omega] = \frac{2}{\left(1 - \frac{1}{2}e^{-j\Omega}\right)\left(1 - \frac{1}{4}e^{-j\Omega}\right)^2}. \tag{4}$$

Making the substitution of $v$ for $e^{-j\Omega}$ in (4), we obtain the rational function

$$G(v) = \frac{2}{\left(1 - \frac{1}{2}v\right)\left(1 - \frac{1}{4}v\right)^2} = \frac{-64}{(v-2)(v-4)^2}. \tag{5}$$

The partial-fraction expansion for (5) is

$$G(v) = \frac{A_1}{v-4} + \frac{A_2}{(v-4)^2} + \frac{A_3}{v-2}$$

where

$$A_3 = [(v-2)\,G(v)]\Big|_{v=2} = \frac{-64}{(v-4)^2}\Bigg|_{v=2} = -16$$

$$A_2 = \left[(v-4)^2\,G(v)\right]\Big|_{v=4} = \frac{-64}{v-2}\Bigg|_{v=4} = -32$$

$$A_1 = \left\{\frac{d}{dv}\left[(v-4)^2\,G(v)\right]\right\}\Bigg|_{v=4} = \left\{\frac{d}{dv}\left[\frac{-64}{v-2}\right]\right\}\Bigg|_{v=4} = \frac{64}{(v-2)^2}\Bigg|_{v=4} = 16.$$

Therefore,

$$G(v) = \frac{16}{v-4} + \frac{-32}{(v-4)^2} + \frac{-16}{v-2} = \frac{-4}{1-\frac{1}{4}v} + \frac{-2}{\left(1-\frac{1}{4}v\right)^2} + \frac{8}{1-\frac{1}{2}v}$$

and

$$Y[\Omega] = \frac{-4}{1-\frac{1}{4}e^{-j\Omega}} + \frac{-2}{\left(1-\frac{1}{4}e^{-j\Omega}\right)^2} + \frac{8}{1-\frac{1}{2}e^{-j\Omega}}.$$

Taking inverse Fourier transforms by inspection and using the linearity property, we get

$$y[n] = \left[-4\left(\frac{1}{4}\right)^n - 2(n+1)\left(\frac{1}{4}\right)^n + 8\left(\frac{1}{2}\right)^n\right]u[n].$$