EE3210
Signals and Systems

Part 3: Basics of Systems

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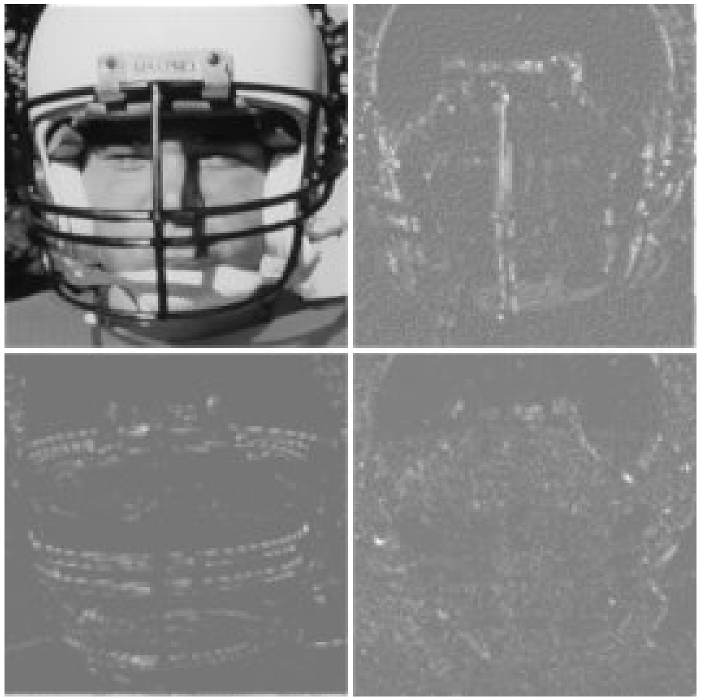
What is a System?

- A system responds to an input signal by producing an output signal, represented by a block diagram as:

- Examples:
  - Hardware realization:
    - Electrical circuit
    - Audio equipment
  - Software realization:
    - Stock market analysis
    - Image processing



Continuous-Time Systems

- A continuous-time system is a system where both input and output are continuous-time signals.

$$x(t) \longrightarrow \boxed{\text{Continuous-time system}} \longrightarrow y(t)$$

  - Represented symbolically as $x(t) \rightarrow y(t)$

- Example: For many physical systems, the input-output relationship can be represented as a first-order linear differential equation of the form

$$\frac{d\,y(t)}{d\,t} + ay(t) = bx(t)$$

where a and b are constants.

Discrete-Time Systems

- A discrete-time system is a system where both input and output are discrete-time signals.

  - Represented symbolically as $x[n] \rightarrow y[n]$

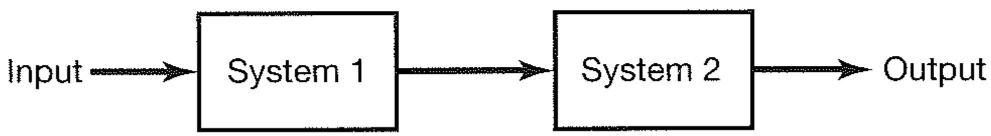
- Example: Many discrete-time systems can be modelled by a first-order linear difference equation of the form

$$y[n] + ay[n-1] = bx[n]$$

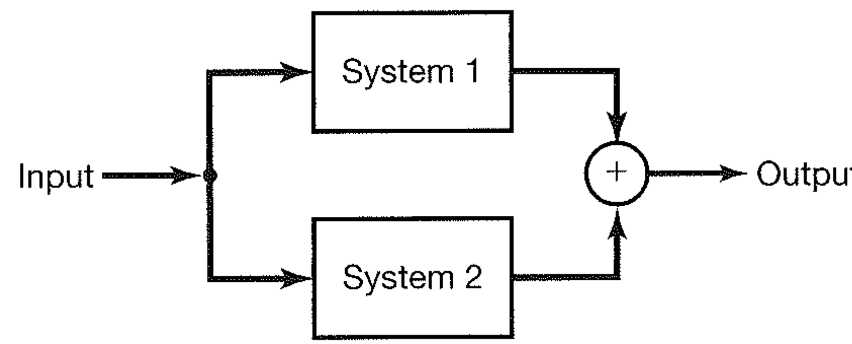
where a and b are constants.

Interconnections of Systems

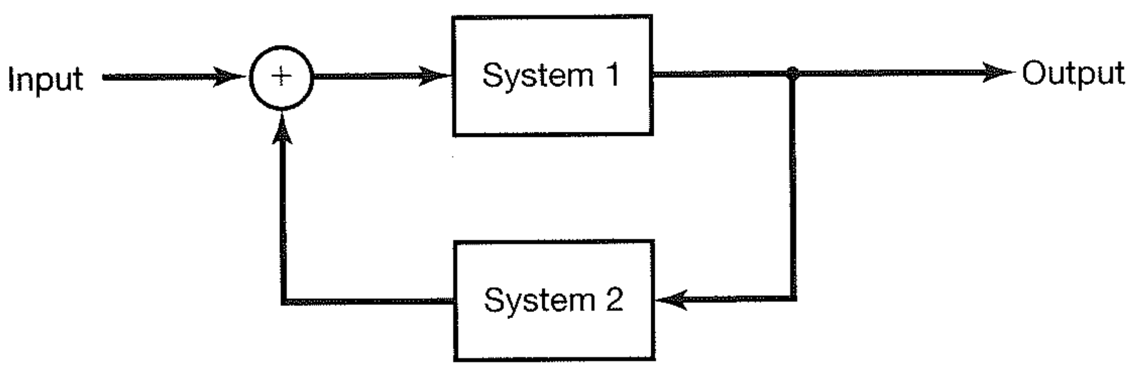
- Many real systems are built as interconnections of several subsystems.
- There are several ways one system can interact with one another:



Series (cascade) interconnection



Parallel interconnection



Feedback interconnection

Basic System Properties

- Systems with and without memory
- Invertibility and inverse systems
- Causality
- Stability
- Time invariance
- Linearity

Systems With and Without Memory

- A system is memoryless if its output for each value of the independent variable at a given time is dependent on the input at only that same time.

- Examples of a memoryless system:

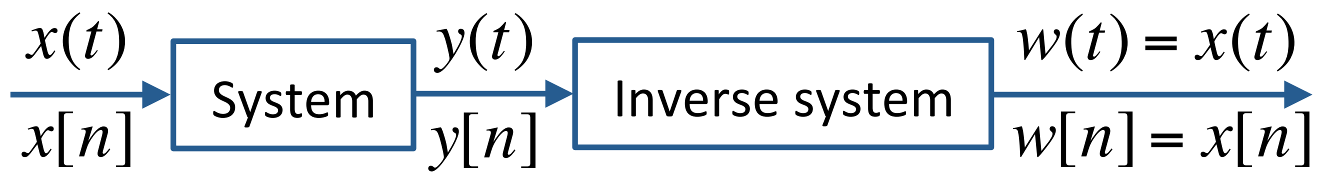
$$y(t) = Rx(t) \qquad\qquad y[n] = x^2[n]$$

- Examples of a system with memory:

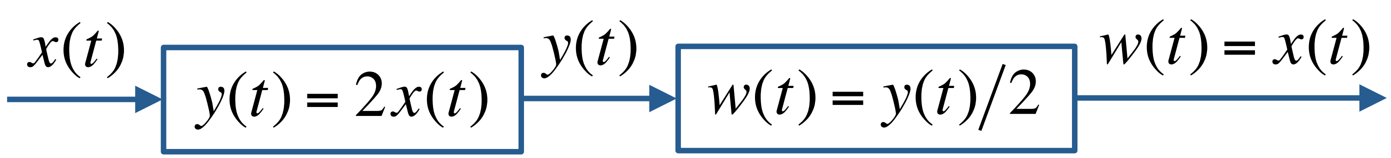
$$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)d\tau \qquad\qquad y[n] = \sum_{k=-\infty}^{n} x[k]$$

Invertibility and Inverse Systems

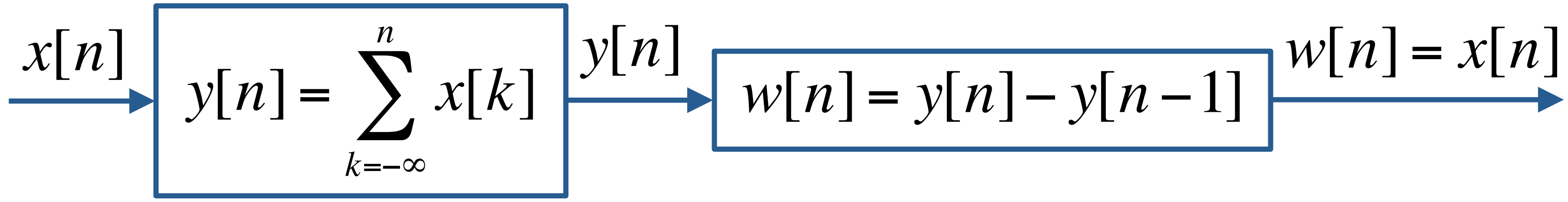
- A system is invertible if distinct inputs lead to distinct outputs.



- Examples of an invertible system:
  - Continuous time:



  - Discrete time:



Invertibility and Inverse Systems (cont.)

- Examples of a non-invertible system:
  - Continuous time:

$$y(t) = x^2(t)$$

    - For this system, we cannot determine the sign of the input from knowledge of the output.
  - Discrete time:

$$y[n] = 0$$

    - This system produces the zero output sequence for any input sequence.

Causality

- Causality refers to the fact that a physical system cannot predict the future values of the input.
- A system is causal if the output at any time depends on values of the input at only the present and past times.
- Examples of a causal system:

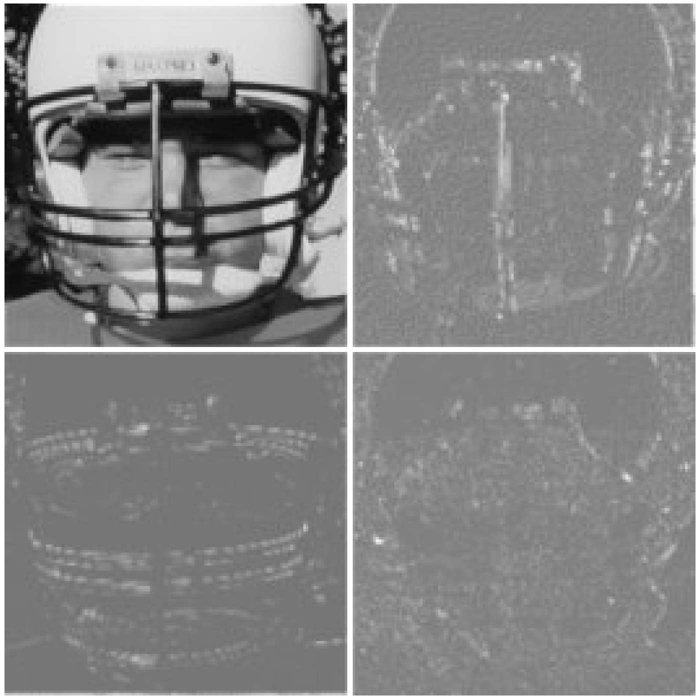
$$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)d\tau \qquad y[n] = \sum_{k=-\infty}^{n} x[k]$$

- Examples of a non-causal system:

$$y(t) = x(t+1) \qquad y[n] = x[n] - x[n+1]$$

Causality (cont.)

- Causality is a must for real-time applications.
- For other applications, this is not often an essential constraint:
  - Applications where the independent variable is not time, e.g., image processing.
  - Processing data that have been recorded previously, e.g., audio processing.

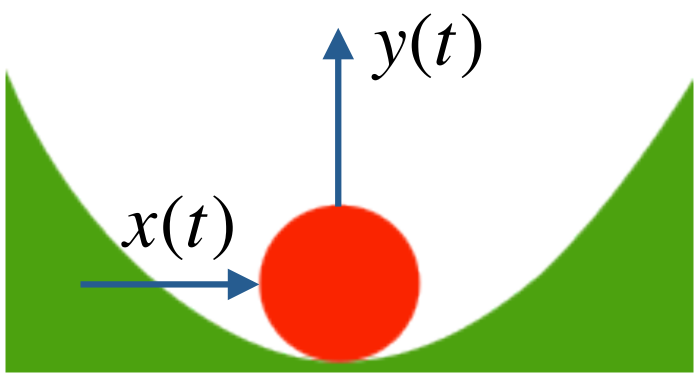


Stability

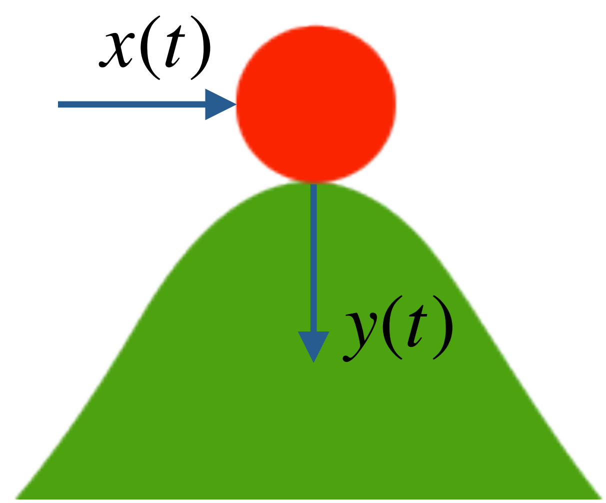
- Informally, a system is stable if small inputs lead to responses (i.e. outputs) that do not diverge.

$x(t)$: Horizontal force

$y(t)$: Vertical displacement



A stable system



An unstable system

Stability (cont.)

- Formal definition: If the input to a stable system is bounded in magnitude, the output must also be bounded in magnitude.
  - Continuous-time systems: for all t, $0 < B < \infty$

$$|x(t)| \leq B \to |y(t)| \leq B$$

  - Discrete-time systems: for all n, $0 < B < \infty$

$$|x[n]| \leq B \to |y[n]| \leq B$$

  - Called bounded-input bounded-output (BIBO) stable.

Stability (cont.)

- Examples of a stable system:
  - $y[n] = \sin(n\pi)x[n]$
    - Given $|x[n]| \leq B$ for all n, $0 < B < \infty$, then we have

$$|y[n]| = |\sin(n\pi)x[n]| = |\sin(n\pi)|\,|x[n]| \leq |x[n]| \leq B$$

- Examples of an unstable system:
  - $$y[n] = \sum_{k=-\infty}^{n} u[k] = \begin{cases} 0, & n < 0 \\ n+1, & n \geq 0 \end{cases} = (n+1)u[n]$$
    - Because $y[0] = 1$, $y[1] = 2$, $y[2] = 3$, . . ., $y[n]$ grows without bound, although the input to the system, which is a unit step $u[n]$, is bounded in magnitude.

Time Invariance

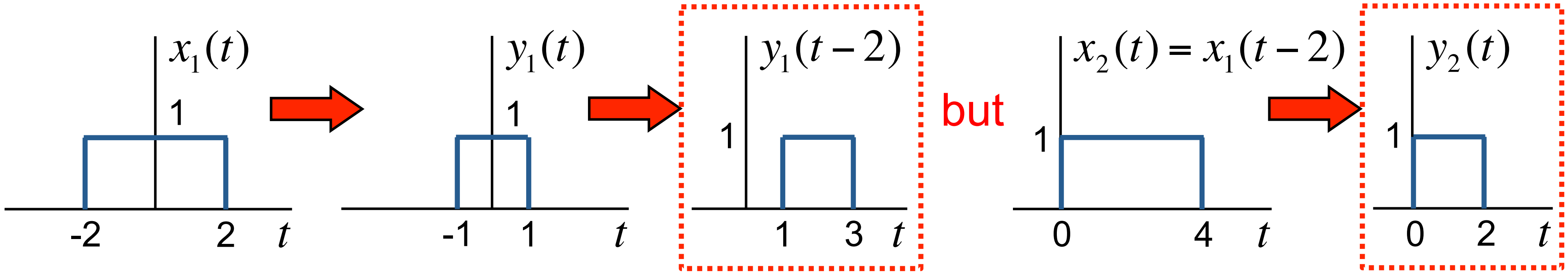
- A system is time invariant if a time shift in the input signal results in an identical time shift in the output signal. That is:
  - For continuous-time systems, if $x(t) \to y(t)$, then for all t_0 we have $x(t-t_0) \to y(t-t_0)$.
  - For discrete-time systems, if $x[n] \to y[n]$, then for all n_0 we have $x[n-n_0] \to y[n-n_0]$.

Time Invariance (cont.)

- Examples of a time-invariant system:
  - $y(t) = \sin[x(t)]$
    - Given $x_1(t)$ and letting $y_1(t) = \sin[x_1(t)]$, consider $x_2(t) = x_1(t - t_0)$. Then, we have $y_2(t) = \sin[x_2(t)] = \sin[x_1(t - t_0)]$. We also have $y_1(t - t_0) = \sin[x_1(t - t_0)]$. Thus, $y_2(t) = y_1(t - t_0)$.
  - $y[n] = x[n - 1]$
    - Given $x_1[n]$ and letting $y_1[n] = x_1[n - 1]$, consider $x_2[n] = x_1[n - n_0]$. Then, we have $y_2[n] = x_2[n - 1] = x_1[n - 1 - n_0]$. We also have $y_1[n - n_0] = x_1[n - n_0 - 1]$. Thus, $y_2[n] = y_1[n - n_0]$.

Time Invariance (cont.)

- Examples of a system that is not time invariant:
  - $y[n] = nx[n]$
    - Given $x_1[n]$ and letting $y_1[n] = nx_1[n]$, consider $x_2[n] = x_1[n - n_0]$. Then, we have $y_2[n] = nx_2[n] = nx_1[n-n_0]$, but we have $y_1[n-n_0] = (n-n_0)x_1[n-n_0]$. Thus, $y_2[n] \neq y_1[n - n_0]$.
  - $y(t) = x(2t)$
    - Demonstrate by counterexample:



Linearity

- A system is linear if it possesses two important properties known as the additivity property and the scaling property.

- For a continuous-time system, given $x_1(t) \to y_1(t)$ and $x_2(t) \to y_2(t)$, the system is linear if

(i) Additivity: $x_1(t) + x_2(t) \to y_1(t) + y_2(t)$

(ii) Scaling: $ax_1(t) \to ay_1(t)$ for any complex constant a.

  - (i) and (ii) can be combined into a single statement:

$$ax_1(t) + bx_2(t) \to ay_1(t) + by_2(t)$$

  where a and b are any complex constants.

Linearity (cont.)

- Similarly, for a discrete-time system, given $x_1[n] \rightarrow y_1[n]$ and $x_2[n] \rightarrow y_2[n]$, the system is linear if

(i) Additivity: $x_1[n] + x_2[n] \rightarrow y_1[n] + y_2[n]$

(ii) Scaling: $ax_1[n] \rightarrow ay_1[n]$ for any complex constant a.

  - (i) and (ii) can be combined into a single statement:

$$ax_1[n] + bx_2[n] \rightarrow ay_1[n] + by_2[n]$$

where a and b are any complex constants.

Linearity (cont.)

- It is straightforward to show from the definition of linearity that, if an input consists of the weighted sum of several signals, then the output is the weighted sum of the responses of the system to each of those signals.
  - This is known as the superposition property.

Linearity (cont.)

- For a linear continuous-time system, given $x_k(t)$, $k = 1, 2, 3, \ldots$, as a set of inputs to the system with corresponding outputs $y_k(t)$, $k = 1, 2, 3, \ldots$, following the superposition property, we have

$$x(t) = \sum_k a_k x_k(t) \rightarrow y(t) = \sum_k a_k y_k(t)$$

- Similarly, for a linear discrete-time system, given $x_k[n]$, $k = 1, 2, 3, \ldots$, as a set of inputs to the system with corresponding outputs $y_k[n]$, $k = 1, 2, 3, \ldots$, following the superposition property, we have

$$x[n] = \sum_k a_k x_k[n] \rightarrow y[n] = \sum_k a_k y_k[n]$$

Linearity (cont.)

- Examples of a linear system:
  - $y(t) = tx(t)$
    - Consider $x_1(t) \rightarrow y_1(t) = tx_1(t)$ and $x_2(t) \rightarrow y_2(t) = tx_2(t)$. Let $x_3(t) = ax_1(t) + bx_2(t)$. Then,
      $y_3(t) = tx_3(t) = atx_1(t) + btx_2(t) = ay_1(t) + by_2(t)$
  - $y[n] = x[n-1]$
    - Consider $x_1[n] \rightarrow y_1[n] = x_1[n-1]$ and $x_2[n] \rightarrow y_2[n] = x_2[n-1]$. Let $x_3[n] = ax_1[n] + bx_2[n]$. Then,
    $y_3[n] = x_3[n-1] = ax_1[n-1]+bx_2[n-1] = ay_1[n]+by_2[n]$

Linearity (cont.)

- Examples of a system that is not linear:
  - $y(t) = x^2(t)$
    - Consider $x_1(t) \rightarrow y_1(t) = x_1^2(t)$ and $x_2(t) \rightarrow y_2(t) = x_2^2(t)$. Let $x_3(t) = ax_1(t) + bx_2(t)$. Then,

$$\begin{aligned} y_3(t) = x_3^2(t) &= a^2x_1^2(t) + b^2x_2^2(t) + 2abx_1(t)x_2(t) \\ &= a^2y_1(t) + b^2y_2(t) + 2abx_1(t)x_2(t) \\ &\neq ay_1(t) + by_2(t) \end{aligned}$$

  - $y[n] = 2x[n] + 3$
    - Consider $x_1[n] \rightarrow y_1[n] = 2x_1[n] + 3$ and $x_2[n] \rightarrow y_2[n] = 2x_2[n] + 3$. Let $x_3[n] = ax_1[n] + bx_2[n]$. Then,

$$y_3[n] = 2x_3[n]+3 = 2ax_1[n]+2bx_2[n]+3 \neq ay_1[n]+by_2[n]$$