Discrete-Time Fourier Transform

Chapter Intended Learning Outcomes:

(i)   Represent discrete-time signals using discrete-time Fourier transform

(ii)   Understand the properties of discrete-time Fourier transform

(iii)   Understand the relationship between discrete-time Fourier transform and linear time-invariant system

Discrete-Time Signals in Frequency Domain

For continuous-time signals, we can use Fourier series and Fourier transform to study them in frequency domain.

With the use of sampled version of a continuous-time signal $x(t)$, we can obtain the discrete-time Fourier transform (DTFT) or Fourier transform of discrete-time signals as follows.

We start with studying the sampled signal $x_s(t)$ produced by multiplying $x(t)$ by the impulse train $i(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$:

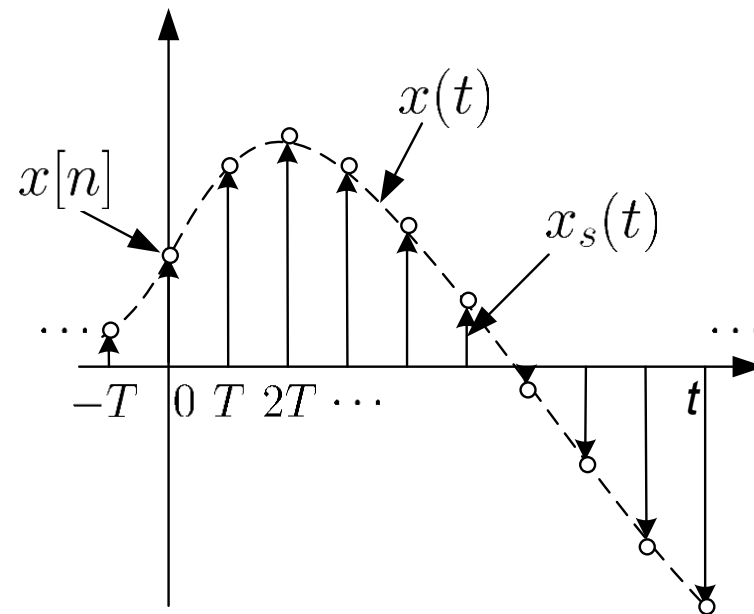$$x_s(t) = x(t) \cdot i(t) = \sum_{k=-\infty}^{\infty} x(t)\delta(t - kT) \qquad (6.1)$$

Fig. 6.1: Continuous-time signal multiplied by impulse train

Using (2.20) and assigning $x[k] = x(kT)$, (6.1) becomes:

$$x_s(t) = \sum_{k=-\infty}^{\infty} x(kT)\delta(t - kT) = \sum_{k=-\infty}^{\infty} x[k]\delta(t - kT) \qquad (6.2)$$

where $x_s(t)$ is still a continuous-time signal, although $x[n]$ is discrete-time.

Taking Fourier transform of $x_s(t)$ with the use of the properties of $\delta(t)$, we obtain:

$$X_s(j\Omega) = \int_{-\infty}^{\infty} x_s(t)e^{-j\Omega t}dt = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x[n]\delta(t-nT)e^{-j\Omega t}dt$$

$$= \sum_{n=-\infty}^{\infty} x[n] \int_{-\infty}^{\infty} \delta(t-nT)e^{-j\Omega t}dt = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega nT} \qquad (6.3)$$

Defining $\omega = \Omega T$ as the discrete-time frequency parameter and writing $X_s(j\Omega)$ as $X(e^{j\omega})$, (6.3) becomes

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \qquad (6.4)$$

which is the DTFT of the discrete-time signal $x[n]$.

As in Fourier transform, $X(e^{j\omega})$ is also called spectrum and is a continuous function of the frequency parameter ω.

Nevertheless, $X(e^{j\omega})$ is periodic with period 2π:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega+2k\pi)n} = X(e^{j(\omega+2k\pi)}) \quad (6.5)$$

for any integer k.

To convert $X(e^{j\omega})$ to $x[n]$, we use inverse DTFT:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega \qquad (6.6)$$

which is obtained by putting (6.4) into (6.6):

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}X(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\sum_{m=-\infty}^{\infty}x[m]e^{-j\omega m}\right]e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}x[m]\int_{-\pi}^{\pi}e^{-j\omega m}e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}x[m]\frac{2\sin((n-m)\pi)}{n-m}$$

$$= x[n] \tag{6.7}$$

Note that $\sin((n-m)\pi)/(n-m) = 0$ if $m \neq n$ while when $m = n$, we have $\int_{-\pi}^{\pi}e^{-j\omega m}e^{j\omega n}d\omega = \int_{-\pi}^{\pi}d\omega = 2\pi$.
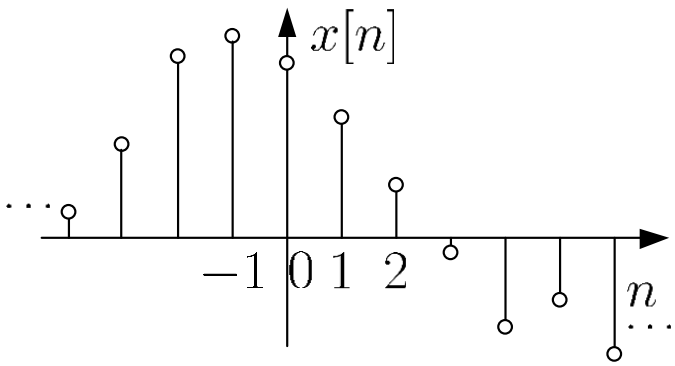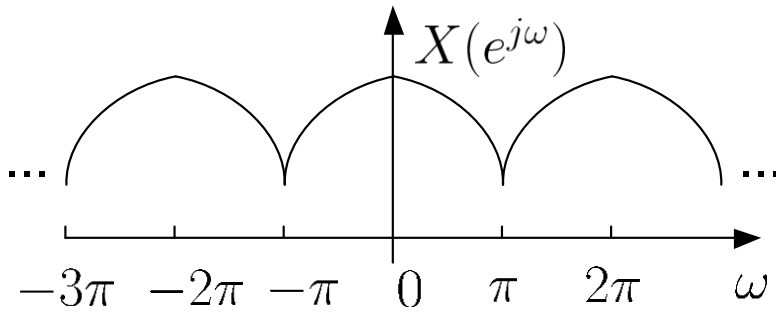
time domain	frequency domain

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \implies$$

$$\impliedby x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n} d\omega$$

discrete and aperiodic	continuous and periodic

Fig.6.2: Illustration of DTFT

As $X(e^{j\omega})$ is generally complex, we can illustrate $X(e^{j\omega})$ using the magnitude and phase spectra, i.e., $|X(e^{j\omega})|$ and $\angle(X(e^{j\omega}))$:

$$|X(e^{j\omega})| = \sqrt{(\Re\{X(e^{j\omega})\})^2 + (\Im\{X(e^{j\omega})\})^2} \qquad (6.8)$$

and

$$\angle(X(e^{j\omega})) = \tan^{-1}\left(\frac{\Im\{X(e^{j\omega})\}}{\Re\{X(e^{j\omega})\}}\right) \qquad (6.9)$$

where both are continuous in frequency and periodic with period 2π.

<u>Example 6.1</u>
Find the DTFT of $x[n]$ which has the form of:

$$x[n] = a^n u[n], \quad |a| < 1$$

Using (6.4), the DTFT is:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^n u[n] e^{-j\omega n} = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n$$

As $|ae^{-j\omega}| = |a| < 1 \Rightarrow |ae^{-j\omega}|^{\infty} = 0$ and applying the geometric sum formula, we have

$$X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

where we see that $X(e^{j\omega})$ is complex.

Example 6.2
Find the DTFT of $x[n] = \delta[n]$.

Using (6.4), we have

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n]e^{-j\omega n} = e^{-j\omega \cdot 0} = 1$$

Analogous to Example 5.4 that the spectrum of the continuous-time $\delta(t)$ has unit amplitude at all frequencies, the spectrum of $\delta[n]$ also has unit amplitude at all frequencies in $(-\pi, \pi)$.

Example 6.3
Find the DTFT of $x[n] = u[n] - u[n-N]$. Plot the magnitude and phase spectra for $N = 10$.

Using (6.4), we have

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=0}^{N-1} e^{-j\omega n} = \frac{e^{-j\omega \cdot 0}\left(1 - e^{-j\omega N}\right)}{1 - e^{-j\omega}} = \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}}$$

We can also further express $X(e^{j\omega})$ as:

$$X(e^{j\omega}) = \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}} = \frac{e^{-j\omega N/2}}{e^{-j\omega/2}} \cdot \frac{e^{j\omega N/2} - e^{-j\omega N/2}}{e^{j\omega/2} - e^{-j\omega/2}} = e^{-j\omega(N-1)/2} \cdot \frac{\sin(\omega N/2)}{\sin(\omega/2)}$$

In doing so, $|X(e^{j\omega})|$ and $\angle(X(e^{j\omega}))$ can be written in closed-forms as:

$$|X(e^{j\omega})| = \left| \frac{\sin(\omega N/2)}{\sin(\omega/2)} \right|$$

and

$$\angle(X(e^{j\omega})) = -\frac{\omega(N-1)}{2} + \angle\left(\frac{\sin(\omega N/2)}{\sin(\omega/2)} \right)$$

Although $(\sin(\omega N/2)/\sin(\omega/2))$ is real, its phase is π if it is negative while the phase is 0 if it is positive.

Note that we generally employ (6.8) and (6.9) for magnitude and phase computation.

In using MATLAB to plot $|X(e^{j\omega})|$ and $\angle(X(e^{j\omega}))$, we utilize the command `sinc` so that there is no need to separately handle the "0/0" cases due to the sine functions. Recall:

$$\text{sinc}(u) = \frac{\sin(\pi u)}{\pi u}$$

As a result, we have:

$$\frac{\sin(\omega N/2)}{\sin(\omega/2)} = \frac{\sin(\omega \cdot N\pi/(2\pi))}{\omega \cdot N\pi/(2\pi)} \cdot \frac{\omega \cdot N\pi}{2\pi} \cdot \frac{\omega\pi/(2\pi)}{\sin(\omega\pi/(2\pi))} \cdot \frac{2\pi}{\omega\pi}$$

$$= N \cdot \frac{\text{sinc}(\omega N/(2\pi))}{\text{sinc}(\omega/(2\pi))}$$

The key MATLAB code for $N = 10$ is

```
N=10;                      %N=10
w=0:0.01*pi:2*pi;    %successive frequency point
                                 %separation is 0.01pi
dtft=N.*sinc(w.*N./2./pi)./(sinc(w./2./pi)).*exp(-
j.*w.*(N-1)./2);       %define DTFT function
subplot(2,1,1)
Mag=abs(dtft);        %compute magnitude
plot(w./pi,Mag);      %plot magnitude
subplot(2,1,2)
Pha=angle(dtft);      %compute phase
plot(w./pi,Pha);      %plot phase
```

There are 201 uniformly-spaced points for plotting the continuous functions $|X(e^{j\omega})|$ and $\angle(X(e^{j\omega}))$.

Magnitude spectrum
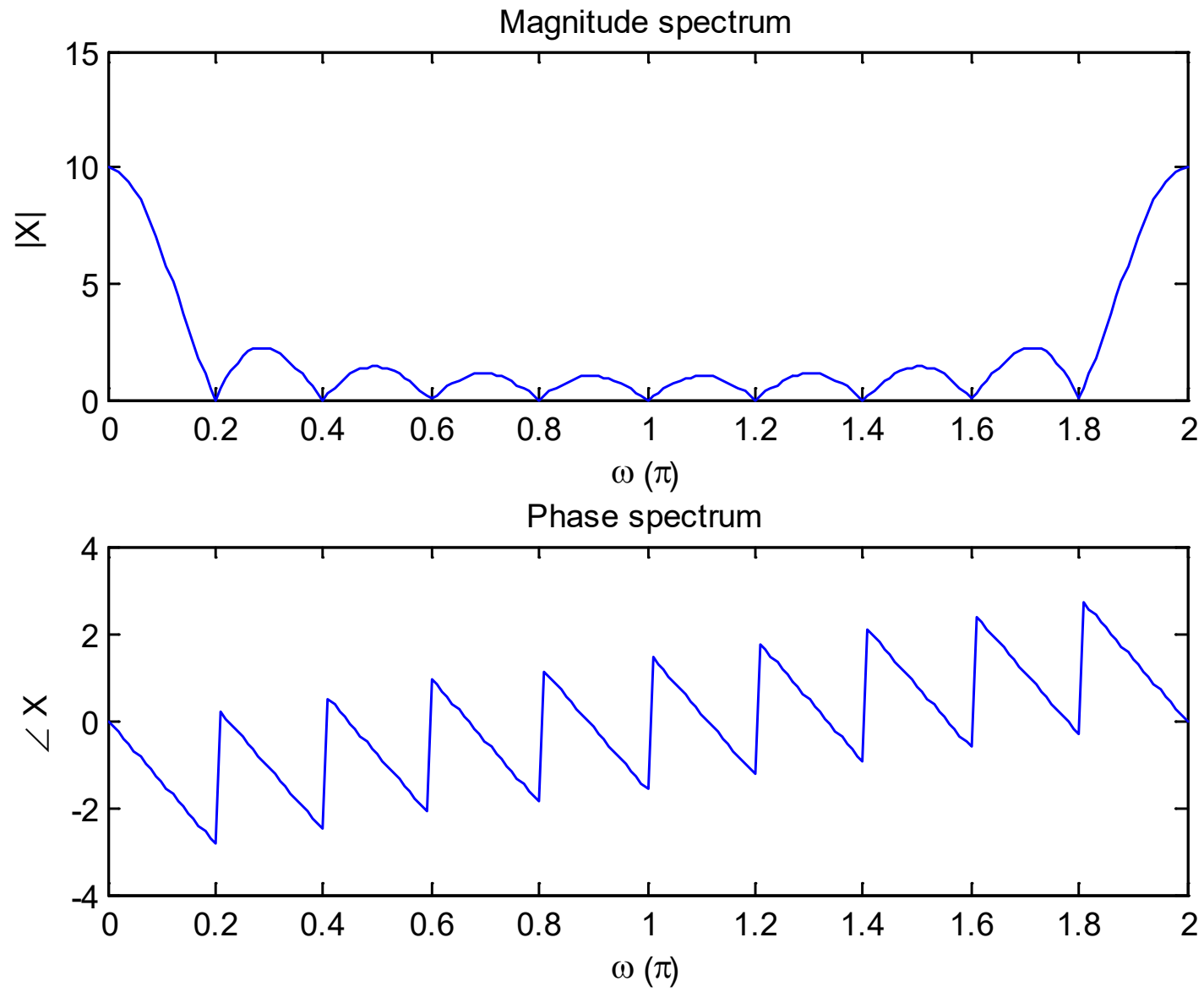
Phase spectrum

Representing $X(e^{j\omega})$ as a rational function of $e^{-j\omega}$:

$$X(e^{j\omega}) = \frac{1 - (e^{-j\omega})^N}{1 - e^{-j\omega}}$$

we can use the command `freqz` to plot the magnitude and phase spectra.

The corresponding MATLAB code is:

```
N=10;                      %N=10
a=[1,-1];                  %vector for denominator
b=[1,zeros(1,N-1),-1];     %vector for numerator
freqz(b,a)                 %plot magnitude (dB) & phase
```
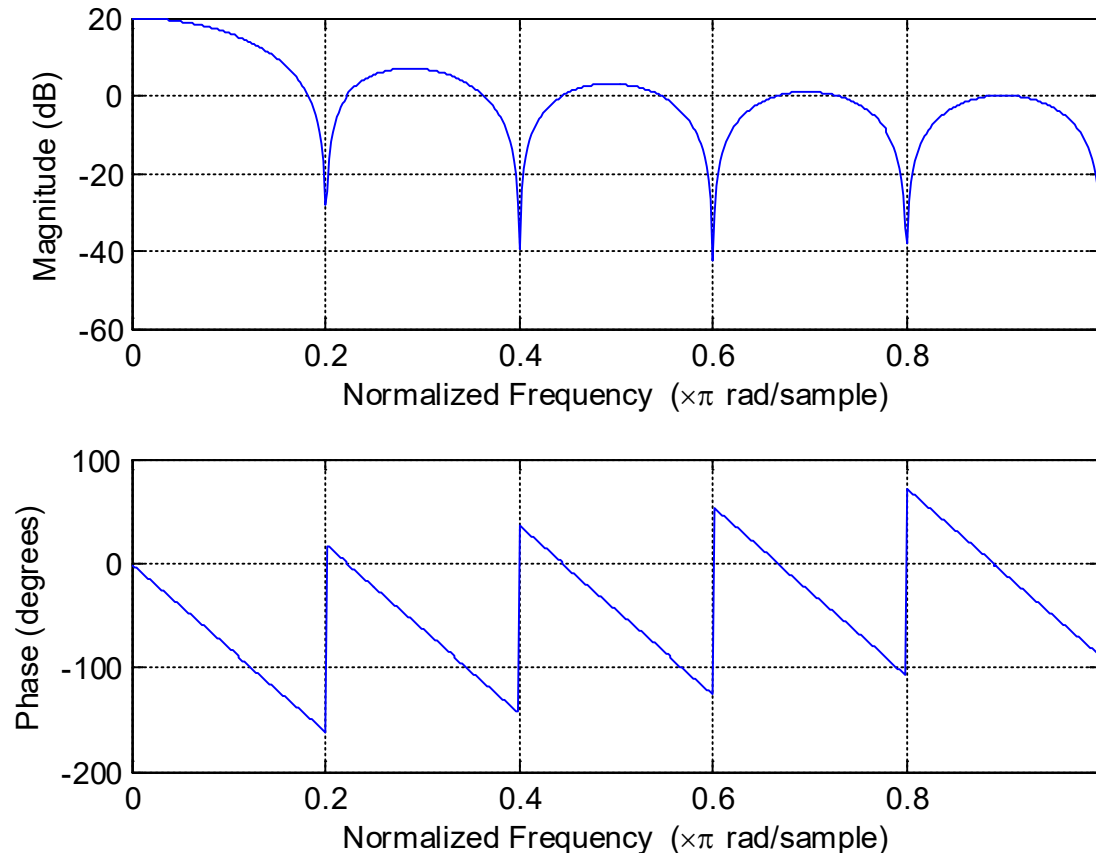
- At $\omega = 0$, $|X(e^{j\omega})| = 10$ corresponds to $20\log_{10} 10 = 20$ dB.

- Units of phase spectra in the two figures are radian and degree, respectively. To make phase values in both plots identical, we need to take care of the 2π phase ambiguity.

Example 6.4

Find the inverse DTFT of $X(e^{j\omega})$ which is a rectangular pulse. Within the period of $[-\pi, \pi]$, $X(e^{j\omega})$ has the form of:

$$X(e^{j\omega}) = \begin{cases} 1, & -w_0 < \omega < w_0 \\ 0, & \text{otherwise} \end{cases}$$

where $0 < w_0 < \pi$.

Using (6.6), we get:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega = \frac{1}{2\pi} \int_{-w_0}^{w_0} e^{j\omega n} d\omega = \frac{\sin(w_0 n)}{\pi n} = \frac{w_0}{\pi} \text{sinc}\left(\frac{w_0 n}{\pi}\right)$$

That is, $x[n]$ is an infinite-duration sequence whose values are drawn from a scaled sinc function.

Note also that $x[n]$ corresponds to the discrete-time version in Example 5.2.

Example 6.5

Given a discrete-time finite-duration sinusoid:

$$x[n] = 2\cos(0.5\pi n + 1), \quad n = 0, 1, \cdots, 20$$

Find the tone frequency using DTFT.

Consider the continuous-time case first. According to (5.10), the Fourier transform pair for a complex continuous-time tone of frequency Ω_0 is:

$$e^{j\Omega_0 t} \leftrightarrow 2\pi\delta(\Omega - \Omega_0)$$

That is, Ω_0 can be found by locating the peak of the Fourier transform. Moreover, a real-valued tone $\cos(\Omega_0 t)$ is:

$$\cos(\Omega_0 t) = \frac{e^{j\Omega_0 t} + e^{-j\Omega_0 t}}{2}$$

This means that Ω_0 and $-\Omega_0$ can be found from the two impulses of the Fourier transform of $\cos(\Omega_0 t)$.

Analogously, we expect that there are two peaks which correspond to frequencies 0.5π and -0.5π in the DTFT for $x[n]$.

The MATLAB code is

```
N=21;                        %number of samples is 21
A=2;                         %tone amplitude is 2
w=0.5*pi;                    %frequency is 0.5*pi
p=1;                         %phase is 1
n=0:N-1;                     %define a vector of size N
x=A*cos(w*n+p);    %generate tone
for k=1:2001         %frequency index k
w=(k-1)*pi/1000; %frequency interval of [0,2pi];
%compute DTFT at frequency points w only
e=(exp(j.*w.*n)).'; %construct exponential vector
X(k) = x*e;
end
```

```
X=abs(X); %compute magnitude
k=1:2001;
f=(k-1)./1000;
plot(f,X)
```

Note that $X(e^{j\omega})$ is continuous in ω and we cannot compute all points. Instead, here we only compute $X(e^{j\omega})$ at $\omega_k = 2\pi k/1000$ for $k = 0, 1, \cdots, 2000$. That is, `k` corresponds to frequency `w=(k-1)*pi/1000`.

With the use of `max(abs(X))`, we find that the peak magnitude corresponds to the index `k=501`, then the signal frequency is correctly determined as:

$$\frac{501 - 1}{1000}\pi = 0.5\pi$$
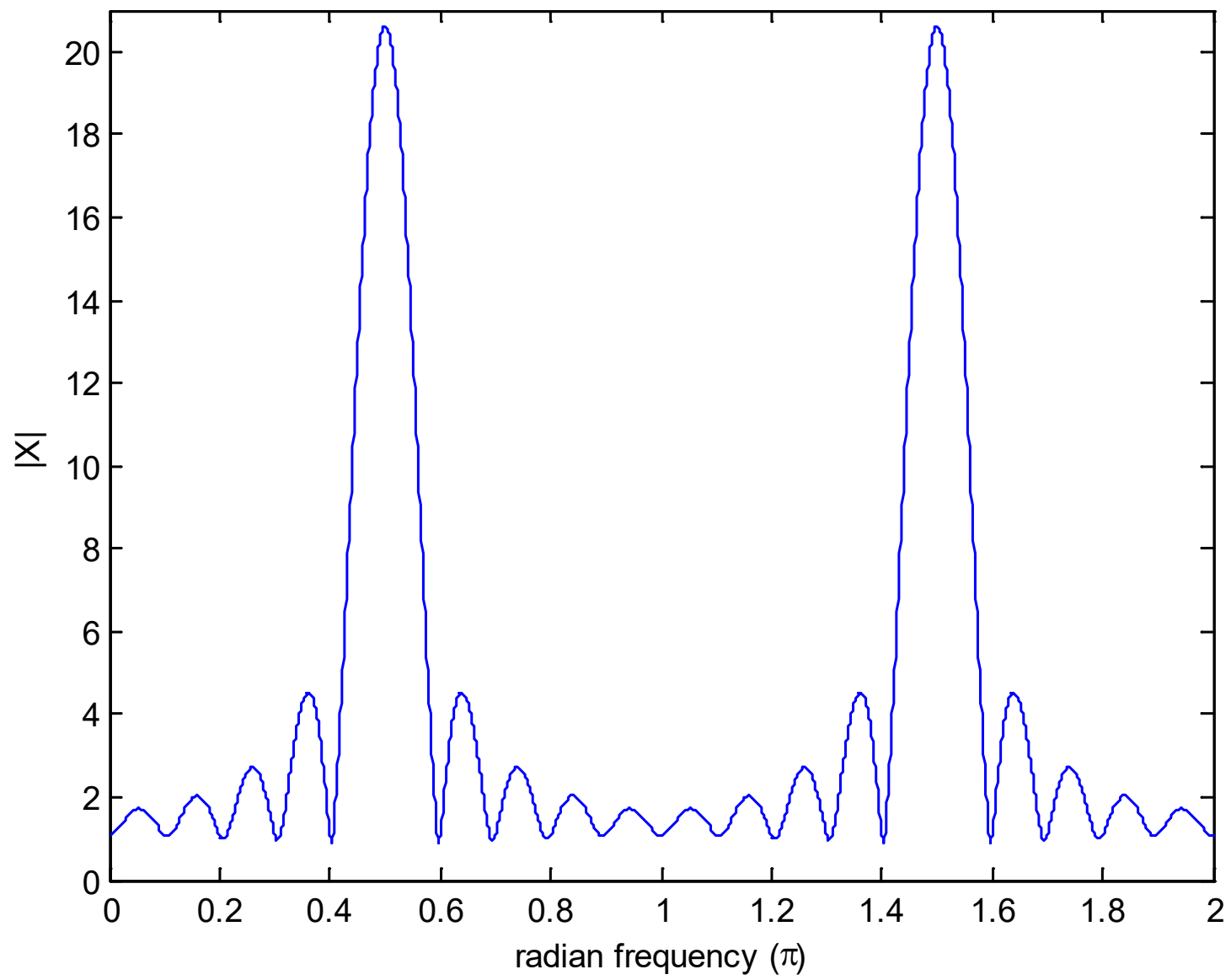
Properties of DTFT

Linearity

If $x_1[n] \leftrightarrow X_1(e^{j\omega})$ and $x_2[n] \leftrightarrow X_2(e^{j\omega})$ are two DTFT pairs, then:

$$ax_1[n] + bx_2[n] \leftrightarrow aX_1(e^{j\omega}) + bX_2(e^{j\omega}) \qquad (6.10)$$

Time Shifting

A shift of n_0 in $x[n]$ causes a multiplication of $e^{-j\omega n_0}$ in $X(e^{j\omega})$:

$$x[n] \leftrightarrow X(e^{j\omega}) \Rightarrow x[n - n_0] \leftrightarrow e^{-j\omega n_0}X(e^{j\omega}) \qquad (6.11)$$

Time Reversal

The DTFT pair for $x[-n]$ is given as:

$$x[n] \leftrightarrow X(e^{j\omega}) \Rightarrow x[-n] \leftrightarrow X(e^{-j\omega}) \qquad (6.12)$$

Multiplication

Multiplication in the time domain corresponds to convolution in the frequency domain:

$$x_1[n] \cdot x_2[n] \leftrightarrow X_1(e^{j\omega}) \tilde{\otimes} X_2(e^{j\omega}) = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} X_1(e^{j\tau}) X_2(e^{j(\omega-\tau)}) d\tau \quad (6.13)$$

where $\tilde{\otimes}$ denotes convolution within one period.

Conjugation

The DTFT pair for $x^*[n]$ is given as:

$$x[n] \leftrightarrow X(e^{j\omega}) \Rightarrow x^*[n] \leftrightarrow X^*(e^{-j\omega}) \quad (6.14)$$

Multiplication by an Exponential Sequence

Multiplying $x[n]$ by $e^{j\omega_0 n}$ in time domain corresponds to a shift of ω_0 in the frequency domain:

$$x[n] \leftrightarrow X(e^{j\omega}) \Rightarrow e^{j\omega_0 n}x[n] \leftrightarrow X(e^{j(\omega-\omega_0)}) \qquad (6.15)$$

Differentiation

Differentiating $X(e^{j\omega})$ with respect to ω corresponds to multiplying $x[n]$ by n:

$$x[n] \leftrightarrow X(e^{j\omega}) \Rightarrow nx[n] \leftrightarrow j\frac{dX(e^{j\omega})}{d\omega} \qquad (6.16)$$

Parseval's Relation

The Parseval's relation addresses the energy of $x[n]$:

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| X(e^{j\omega}) \right|^2 d\omega \qquad (6.17)$$

With the use of (6.6), (6.17) is proved as:

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=-\infty}^{\infty} x[n] x^*[n]$$

$$= \sum_{n=-\infty}^{\infty} x[n] \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \right)^*$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X^*(e^{j\omega}) \left(\sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \right) d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| X(e^{j\omega}) \right|^2 d\omega \qquad\qquad (6.18)$$

<u>Convolution</u>

If $x_1[n] \leftrightarrow X_1(e^{j\omega})$ and $x_2[n] \leftrightarrow X_2(e^{j\omega})$ are two DTFT pairs, then:

$$x_1[n] \otimes x_2[n] \leftrightarrow X_1(e^{j\omega})X_2(e^{j\omega}) \qquad (6.19)$$

which can be derived as:

$$\sum_{n=-\infty}^{\infty} x_1[n] \otimes x_2[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x_1[m]x_2[n-m]e^{-j\omega n}$$

$$= \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x_1[m]x_2[k]e^{-j\omega(k+m)}, \quad k = n - m$$

$$= \left[\sum_{m=-\infty}^{\infty} x_1[m]e^{-j\omega m} \right] \cdot \left[\sum_{k=-\infty}^{\infty} x_2[k]e^{-j\omega k} \right]$$

$$= X_1(e^{j\omega}) \cdot X_2(e^{j\omega}) \qquad (6.20)$$

DTFT and Linear Time-Invariant System

Recall for a discrete-time LTI system, the input-output relationship is characterized by convolution in (3.11):

$$y[n] = x[n] \otimes h[n]$$
$$= \sum_{m=-\infty}^{\infty} x[m]h[n-m] = \sum_{m=-\infty}^{\infty} h[m]x[n-m] \qquad (6.21)$$

Using (6.19), we can consider (6.21) in frequency domain:

$$y[n] = x[n] \otimes h[n] \leftrightarrow Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) \qquad (6.22)$$

This suggests apart from computing the output using time-domain approach via convolution, we can convert the input and impulse response to frequency domain, then $y[n]$ is computed from inverse DTFT of $X(e^{j\omega})H(e^{j\omega})$.

In fact, $H(e^{j\omega})$ represents the LTI system in the frequency domain, is called the system frequency response.

Recall (3.22) that the input and output of a discrete-time LTI system satisfy the difference equation:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \qquad (6.23)$$

Taking the DTFT and using the linearity and time shifting properties, we get:

$$Y(e^{j\omega}) \sum_{k=0}^{N} a_k e^{-j\omega k} = X(e^{j\omega}) \sum_{k=0}^{M} b_k e^{-j\omega k} \qquad (6.24)$$

The system frequency response can also be computed as:

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\displaystyle\sum_{k=0}^{M} b_k e^{-j\omega k}}{\displaystyle\sum_{k=0}^{M} a_k e^{-j\omega k}} \qquad (6.25)$$

Example 6.6

Determine the system frequency response for a causal LTI system described by the following difference equation:

$$y[n] = 0.1y[n-1] + x[n] + x[n-1]$$

Applying (6.25), we easily obtain:

$$Y(e^{j\omega})\left(1 - 0.1e^{-j\omega}\right) = X(e^{j\omega})\left(1 + e^{-j\omega}\right) \Rightarrow H(e^{j\omega}) = \frac{1 + e^{-j\omega}}{1 - 0.1e^{-j\omega}}$$

Example 6.7

The moving average (MA) is in fact a LTI system. Consider the close price of Dow Jones Industrial Average (DJIA) index as input $x[n]$ and the output $y[n]$ is the 20-day MA. Establish the input-output relationship using a difference equation. Then compute the system impulse response and frequency response. Plot the system magnitude frequency response.

In stock market (or other applications), future data are unavailable. The best we can do is to use the today value and close prices of previous 19 trading days in MA calculation, that is:

$$y[n] = \frac{1}{20} \sum_{k=0}^{19} x[n-k]$$

Following Example 3.18, we can easily deduce the impulse response as:

$$h[n] = \frac{1}{20}\left(\delta[n] + \delta[n-1] + \cdots + \delta[n-19]\right) = \frac{1}{20}\sum_{k=0}^{19}\delta[n-k]$$

Applying (6.25), the system frequency response is:

$$H(e^{j\omega}) = \frac{1}{20}\sum_{k=0}^{19}e^{-j\omega k}$$

From the magnitude plot, the frequency is concentrated around the DC. It is called a lowpass filter (also for Example 6.3).

From Fig. 1.11, we see that low-frequency components (smooth part) are kept while high-frequency components (fluctuations) are suppressed in the system output.
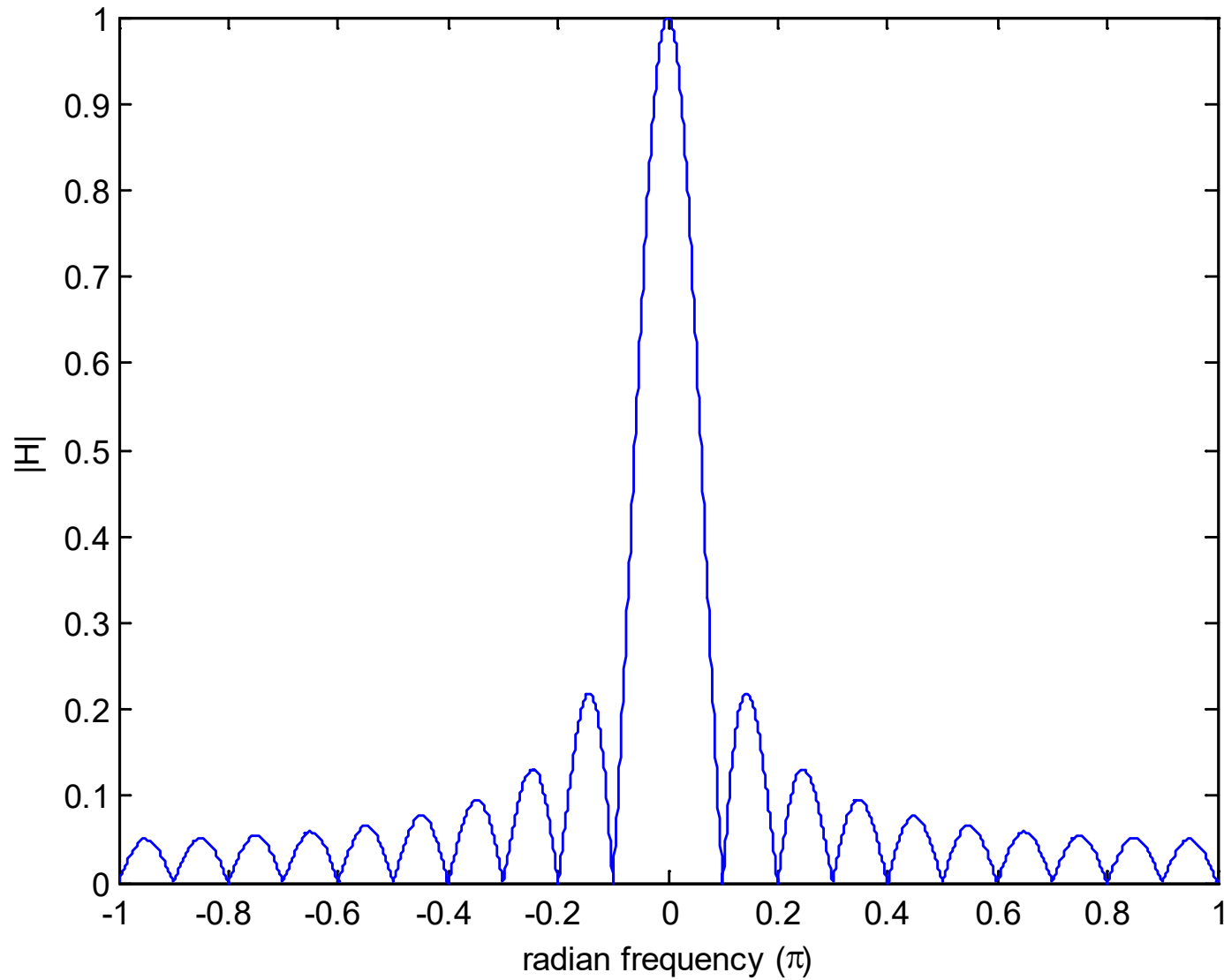
The MATLAB code for the plot is provided as `ex6_7.m`.