Homework Assignment #8

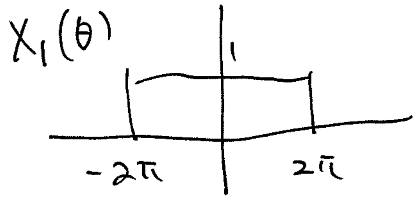
Prob. 1 (b)

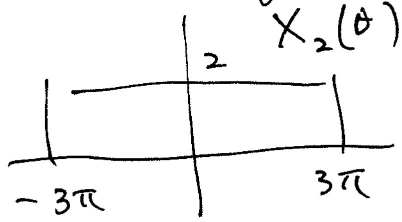
Let
$$x_1(t) = \frac{\sin 2\pi t}{\pi t}, \quad x_2(t) = \frac{2\sin 3\pi t}{\pi t}$$

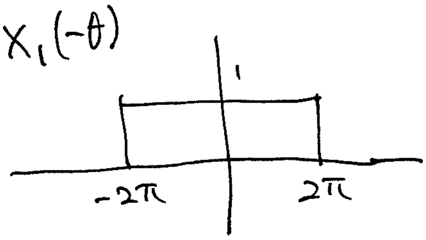
Then
$$X_1(\omega) = \begin{cases} 1 & |\omega| < 2\pi \\ 0 & \text{otherwise} \end{cases}$$
$$X_2(\omega) = 2\begin{cases} 1 & |\omega| < 3\pi \\ 0 & \text{otherwise} \end{cases}$$

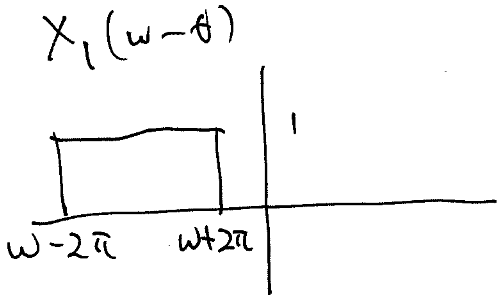
In light of modulation/multiplication property,
$$X(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_2(\theta)\, X_1(\omega-\theta)\, d\theta$$
$$= \frac{1}{2\pi} X_2(\omega) * X_1(\omega)$$

So we carry out convolution in freq. domain.







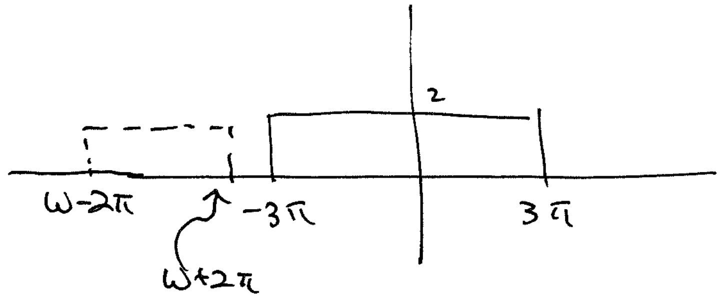


Case 1

$\omega + 2\pi < -3\pi$, i.e.,

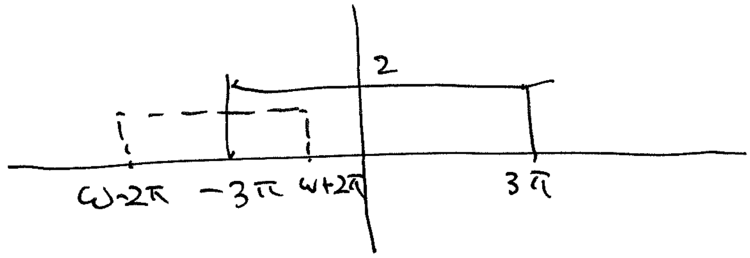
$\omega < -5\pi$,

$X(\omega) = 0$



Case 2

$$\begin{cases} \omega - 2\pi < -3\pi \\ \omega + 2\pi > -3\pi \end{cases} \quad \text{i.e.}$$

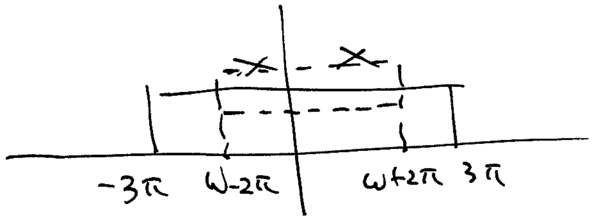


$-5\pi < \omega < -\pi$

$$X(\omega) = \frac{1}{2\pi}\int_{-3\pi}^{\omega+2\pi} 2 \, d\theta = \frac{1}{\pi}(\omega + 5\pi)$$

Case 3

$$\begin{cases} \omega - 2\pi > -3\pi \\ \omega + 2\pi < 3\pi \end{cases} \quad \text{i.e.}$$

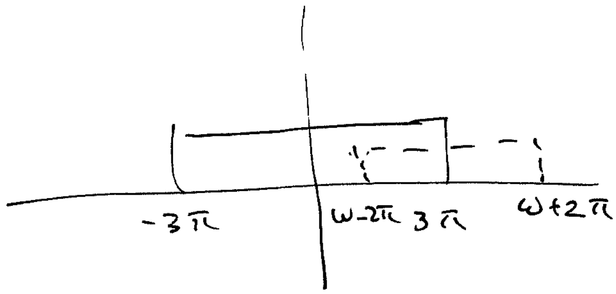


$-\pi < \omega < \pi$

$$X(\omega) = \frac{1}{2\pi}\int_{\omega-2\pi}^{\omega+2\pi} 2 \, d\theta = 4$$

Case 4

$$\begin{cases} \omega - 2\pi < 3\pi \\ \omega + 2\pi > 3\pi \end{cases} \quad \text{i.e.}$$



$\pi < \omega < 5\pi$

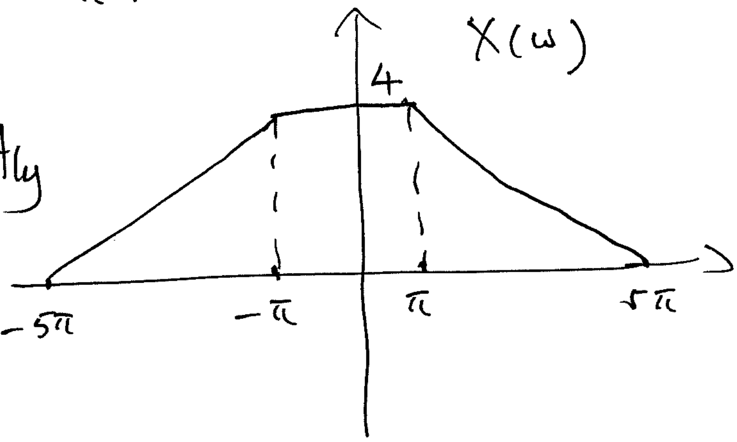
$$X(\omega) = \frac{1}{2\pi}\int_{\omega-2\pi}^{3\pi} 2 \, d\theta = \frac{1}{\pi}(5\pi - \omega)$$

Case 5

$\omega > 5\pi$

$X(\omega) = 0$

Consequently $\Longrightarrow$



2. Prob. 4.32

$$h(t) = \frac{\sin 4(t-1)}{\pi(t-1)}$$

This problem is meant to demonstrate the filtering effect of an LPF (modulo to a time shift).

According to time-shifting property

$$H(w) = e^{-jw}\, \mathcal{F}\left\{\frac{\sin 4t}{\pi t}\right\}$$

$$= e^{-jw} \begin{cases} 1 & |w|<4 \\ 0 & \text{otherwise} \end{cases}$$

(a) $x_1(t) = \cos(6t + \frac{\pi}{2})$

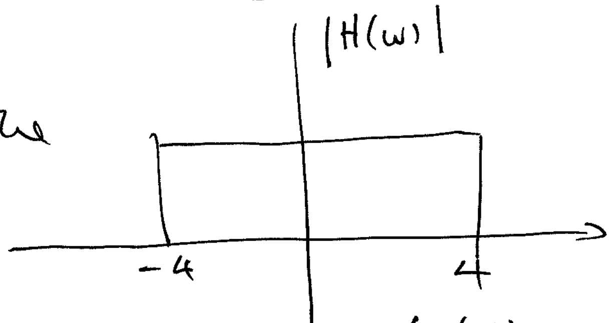
$$= \cos 6t \cos\frac{\pi}{2} - \sin 6t \sin\frac{\pi}{2}$$

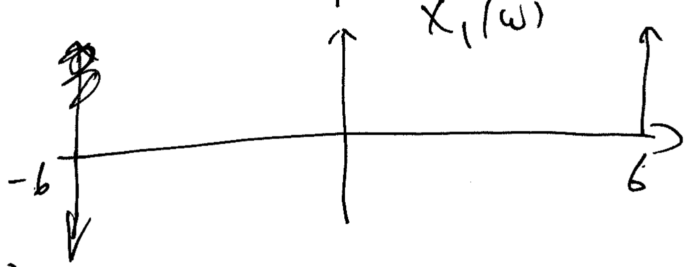
$$= -\sin 6t$$

$$X_1(w) = -\frac{\pi}{j}\left[\delta(w-6) - \delta(w+6)\right]$$

Clearly,

$Y_1(w) = H(w)X_1(w) = 0$, since the two impulses $\delta(w-6)$ and $\delta(w+6)$ lie outside the passing band of the LPF.





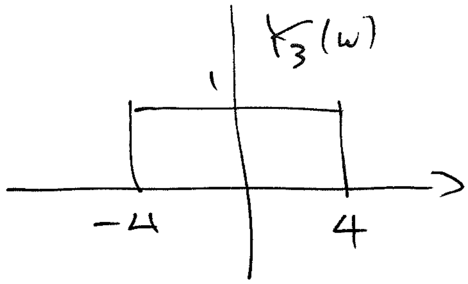
$\Rightarrow y_1(t) \equiv 0$

(c) $x_3(t) = \dfrac{\sin 4(t+1)}{\pi(t+1)}$

$$X_3(w) = e^{jw} \begin{cases} 1 & |w|<4 \\ 0 & \text{otherwise} \end{cases}$$

$$Y_3(\omega) = H(\omega) X_3(\omega)$$

$$= \begin{cases} 1 & |\omega| < 4 \\ 0 & \text{otherwise} \end{cases}$$



$$\Rightarrow y_3(t) = \frac{\sin 4t}{\pi t}$$

4. Prob. 4.36

This problem demostrates the importance of finding the freq. response, which helps us achieve many objectives.

$$x(t) = \left[e^{-t} + e^{-3t}\right]u(t) \Rightarrow X(\omega) = \frac{1}{j\omega+1} + \frac{1}{j\omega+3}$$

$$= \frac{2(j\omega+2)}{(j\omega+1)(j\omega+3)}$$

$$y(t) = \left[2e^{-t} - 2e^{-4t}\right]u(t) \Rightarrow Y(\omega) = 2\left[\frac{1}{j\omega+1} - \frac{1}{j\omega+4}\right]$$

$$= 2\,\frac{3}{(j\omega+1)(j\omega+4)}$$

(a) $$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \cancel{\frac{\frac{2(j\omega+2)}{(j\omega+1)(j\omega+3)}}{\frac{(2)\cdot 3}{(j\omega+1)(j\omega+4)}}}$$

$$= \frac{\frac{\cancel{2}\cdot 3}{\cancel{(j\omega+1)}(j\omega+4)}}{\frac{\cancel{2}(j\omega+2)}{\cancel{(j\omega+1)}(j\omega+3)}} = \frac{3(j\omega+3)}{(j\omega+2)(j\omega+4)}$$

(b) Conduct partial fraction

$$H(\omega) = \frac{A}{j\omega+2} + \frac{B}{j\omega+4}$$

Then $h(t) = ?$

(c) Since $H(\omega) = \dfrac{Y(\omega)}{X(\omega)} = \dfrac{3(j\omega+3)}{(j\omega+2)(j\omega+4)}$

$\Rightarrow$ ~~$Y(\omega)$~~ $\dfrac{Y(\omega)}{X(\omega)} = \dfrac{3j\omega+9}{(j\omega)^2+6(j\omega)+8}$

Cross Multiplication

$$\left[(j\omega)^2+6(j\omega)+8\right]Y(\omega) = \left[3j\omega+9\right]X(\omega)$$

$$\Rightarrow \frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = 3\frac{dx(t)}{dt} + 9x(t)$$

Prob. 5 (iii) (Other parts are similar to prior problems)

$$\frac{Y(\omega)}{X(\omega)} = H(\omega) = \frac{j\omega+1}{(j\omega)^2+8(j\omega)+15}$$

$$\frac{Y(\omega)}{X(\omega)} = \frac{\left(\frac{1}{j\omega}\right)^2 + \frac{1}{j\omega}}{1+8\left(\frac{1}{j\omega}\right)+15\left(\frac{1}{j\omega}\right)^2}$$

Cross multiplication

$$\Rightarrow Y(\omega) + 8\left(\frac{1}{j\omega}\right)Y(\omega) + 15\left(\frac{1}{j\omega}\right)^2 Y(\omega)$$
$$= \left(\frac{1}{j\omega}\right)X(\omega) + \left(\frac{1}{j\omega}\right)^2 X(\omega)$$

$$Y(\omega) = \left(\frac{1}{j\omega}\right)X(\omega) + \left(\frac{1}{j\omega}\right)^2 X(\omega) - 8\left(\frac{1}{j\omega}\right)Y(\omega) - 15\left(\frac{1}{j\omega}\right)^2 Y(\omega)$$