EE2302 Foundations of Information Engineering

Assignment 7 (Solution)

1.

a) No. Choose $x = (1,0)$, $y = (0,1)$, and $\alpha = \beta = 1$. Then, $f(\alpha x + \beta y) = 1$ but $\alpha f(x) + \beta f(y) = 2$, which shows that superposition fails.

b) Yes. a is the vector whose first component is –1, the last component is 1, and all other components are 0, i.e., $a = (-1, 0, 0, \dots, 0, 1)$.

2.

a) A vector (a, b, c) reflecting through the x-y plane becomes the vector $(a, b, -c)$. That means, the x-coordinate and the y-coordinate remain unchanged while the z-coordinate is multiplied by -1. The corresponding matrix is given by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

It can be checked that

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ b \\ -c \end{bmatrix}.$$

b) Since only the x- and y-coordinates are rotated while the z-axis remain unchanged, the matrix must be of the form

$$\begin{bmatrix} ? & ? & 0 \\ ? & ? & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where the 2×2 submatrix in the upper left corner is the rotation matrix with angle = 90^o. By the formula for the rotation matrix (given in the lecture notes), we obtain the transformation matrix as follows:

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

3.

a) $a = (10, 10)$, so the slope is 1. Consider a right-angled triangle with base 1, height 1, and hypotenuse $\sqrt{2}$. Therefore, $\cos\theta = \frac{1}{\sqrt{2}}$ and $\sin\theta = \frac{1}{\sqrt{2}}$.

The projection matrix is

$$P = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

b) The vector after projection is

$$\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}.$$

4.

a) The distance between x and y is

$$\|x - y\| = \sum_{i=1}^{n} (x_i - y_i)^2.$$

We want to minimize the distance. If $x_i \geq 0$, obviously we should simply let y_i be equal to x_i. But if $x_i < 0$, we cannot do so because y_i must be non-negative (as stated in the question). To minimize $(x_i - y_i)^2$, the best we can do is to let y_i be zero. Hence,

$$y_i = \begin{cases} x_i & \text{if } x_i \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

b) By the definition of z, we obtain

$$z_i = y_i - x_i = \begin{cases} 0 & \text{if } x_i \geq 0, \\ -x_i & \text{otherwise.} \end{cases}$$

Since each component of z is non-negative, z is a non-negative vector.

c) To determine the inner product, we separate it into two summations, depending on whether x_i is non-negative or not. If $x_i \geq 0$, then $z_i = 0$. Otherwise, $z_i = -x_i$. Hence,

$$z^T y = \sum_i z_i y_i = \sum_{i:\, x_i \geq 0} 0 \cdot x_i + \sum_{i:\, x_i < 0} -x_i \cdot 0 = 0.$$

5. Choose b = **1**. Then $a^T b = a_1 + a_2 + \cdots + a_n \leq \|a\|\|b\| = \sqrt{a_1^2 + a_2^2 + \cdots a_n^2}\sqrt{n}$. Squaring both sides, we obtain

$$(a_1 + a_2 + \cdots + a_n)^2 \leq n(a_1^2 + a_2^2 + \cdots a_n^2).$$