

# Lecture 2. Deterministic Signal Analysis

- Time-Domain Analysis
- Frequency-Domain Analysis

# Time-Domain Analysis

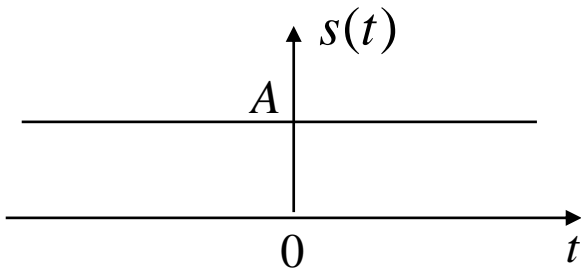
- Signal Representation in Time Domain
- Signal Energy and Signal Power
- Signal Transmission Through an LTI System

## Signals in Time Domain

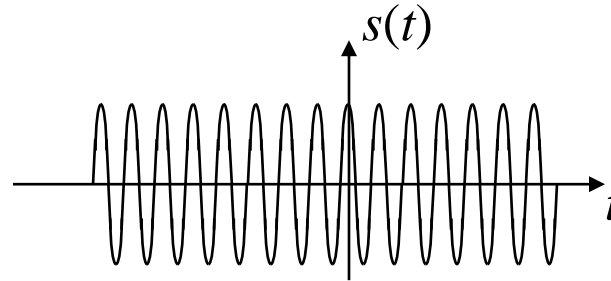
- A signal is a set of data or information, which can be represented as a function of **time**  $t$ :  $s(t)$
- **Deterministic Signal vs. Random Signal**
  - ✓ Deterministic signal is a signal whose physical description is known completely, either in a mathematical form or a graphical form: Expression of  $s(t)$  is known. **Lecture 2**
  - ✓ Random signal is a signal that cannot be predicted precisely, but known in terms of probabilistic description:  $s(t)$  is a random process. **Lecture 5**
- Other Classification of Signals:
  - ✓ Periodic signal vs. Aperiodic signal
  - ✓ Continuous-time vs. Discrete-time signal
  - ✓ Analog signal vs. Digital signal

## Examples of Deterministic Signals

- Constant signal:  $s(t) = A$

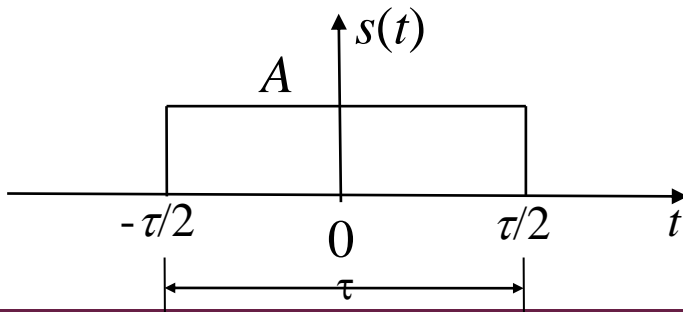


- Sinusoidal signal:  $s(t) = \cos(2\pi f_0 t)$

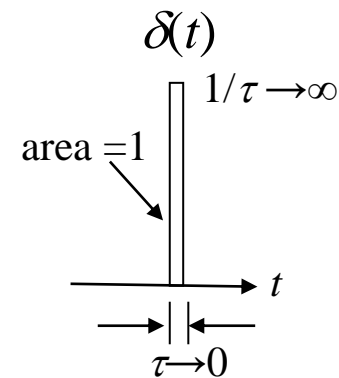
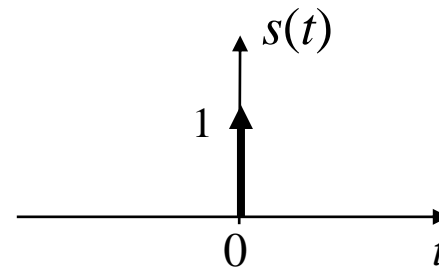


- Rectangular pulse:

$$s(t) = \begin{cases} A & -\tau/2 \leq t \leq \tau/2 \\ 0 & \text{otherwise} \end{cases}$$

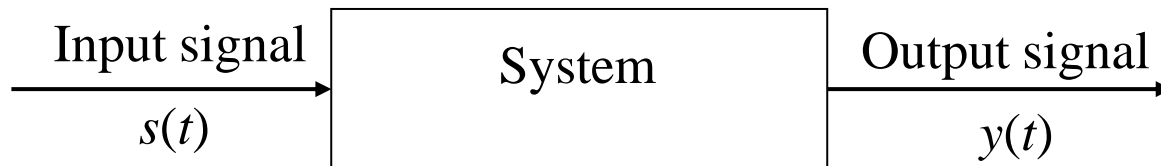


- Unit impulse:**  $s(t) = \delta(t)$
- $$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t) dt = 1$$



## Linear Time Invariant (LTI) System

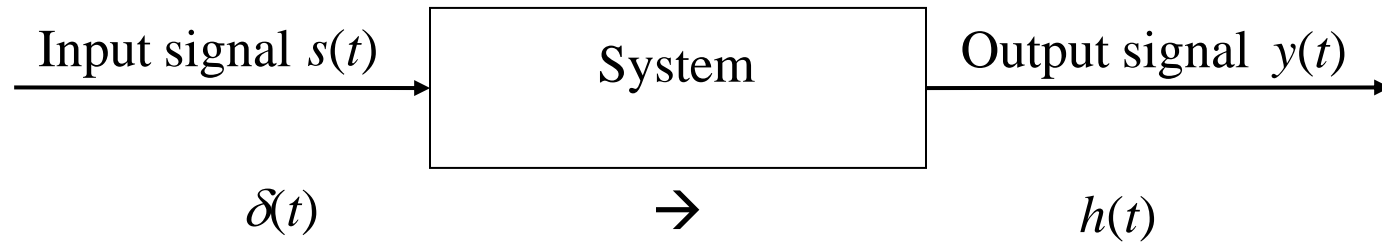
- System: A system is an entity that processes a set of signals (inputs) to yield another set of signals (outputs).



- Linear and Time Invariant:
  - ✓ Linear: If  $s_1(t) \rightarrow y_1(t)$ , and  $s_2(t) \rightarrow y_2(t)$ , then  $a_1 s_1(t) + a_2 s_2(t) \rightarrow a_1 y_1(t) + a_2 y_2(t)$  for coefficients  $a_1$  and  $a_2$ .
  - ✓ Time Invariant : If  $s(t) \rightarrow y(t)$ , then  $s(t-T) \rightarrow y(t-T)$  for delay  $T$ .
- Impulse response  $h(t)$ : If the input signal is a unit impulse  $\delta(t)$ , then the output signal is called the impulse response of the system:  $\delta(t) \rightarrow h(t)$

## Linear Time Invariant (LTI) System

- For an LTI system: the output signal  $y(t)$  is the **convolution** of the input signal  $s(t)$  and the impulse response of the system  $h(t)$ .

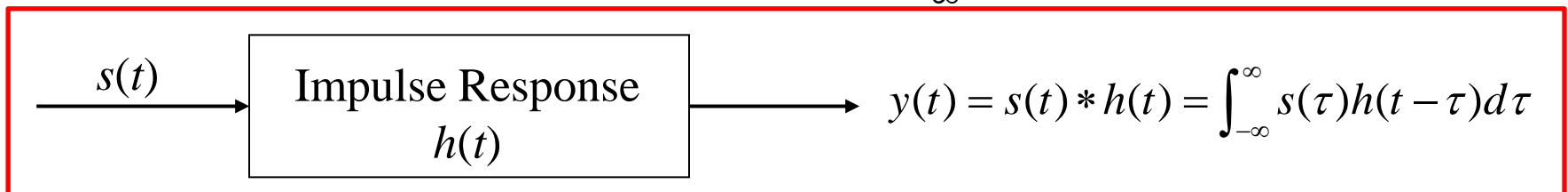


$$\delta(t - n\Delta\tau) \rightarrow h(t - n\Delta\tau)$$

$$s(n\Delta\tau)\delta(t - n\Delta\tau)\Delta\tau \rightarrow s(n\Delta\tau)h(t - n\Delta\tau)\Delta\tau$$

$$s(t) = \lim_{\Delta\tau \rightarrow 0} \sum_{n=-\infty}^{\infty} s(n\Delta\tau)\delta(t - n\Delta\tau)\Delta\tau \rightarrow \lim_{\Delta\tau \rightarrow 0} \sum_{n=-\infty}^{\infty} s(n\Delta\tau)h(t - n\Delta\tau)\Delta\tau = y(t)$$

$$= \int_{-\infty}^{\infty} s(\tau)h(t - \tau)d\tau = \mathbf{s(t) * h(t)}$$



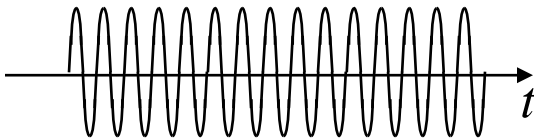
# Frequency-Domain Analysis

- Fourier Transform
- Energy Spectrum, Power Spectrum and Signal Bandwidth
- Signal Transmission Through an LTI System

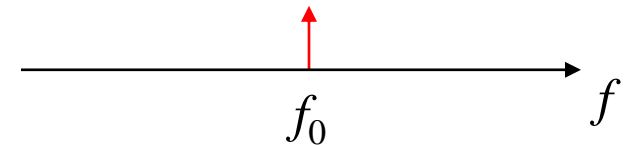
# Signals in Frequency Domain

## Time domain

- $\cos(2\pi f_0 t)$

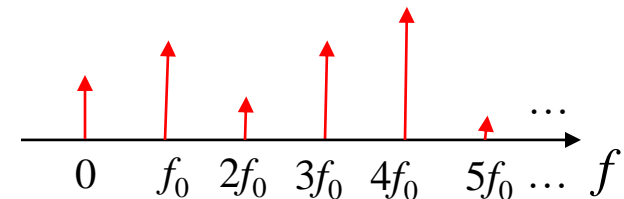
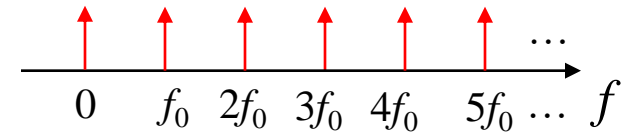


- $\sum_{n=0}^{\infty} \cos(2\pi n f_0 t) \quad \sum_{n=0}^{\infty} \cos(2\pi n f_0 t + \theta_n)$



- $$a_0 + \sum_{n=1}^{\infty} a_n \cos(2\pi n f_0 t) + \sum_{n=1}^{\infty} b_n \sin(2\pi n f_0 t)$$

$$= \sum_{n=-\infty}^{\infty} s_n e^{j2\pi n f_0 t}$$



$$s_0 = a_0, s_n = (a_n - j b_n)/2, s_{-n} = (a_n + j b_n)/2.$$

$$\cos(2\pi f_0 t) = \frac{1}{2}(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t})$$

$$\sin(2\pi f_0 t) = \frac{1}{2j}(e^{j2\pi f_0 t} - e^{-j2\pi f_0 t})$$



## Fourier Series

- $s(t) = \sum_{n=-\infty}^{\infty} s_n e^{j2\pi n f_0 t}$ . Let  $T_0 = 1/f_0$ .

$$s(t + T_0) = \sum_{n=-\infty}^{\infty} s_n e^{j2\pi n f_0 (t+T_0)} = \sum_{n=-\infty}^{\infty} s_n e^{j2\pi n f_0 t + j2\pi n} = s(t)$$

$s(t)$  is a periodic signal with period  $T_0$ !

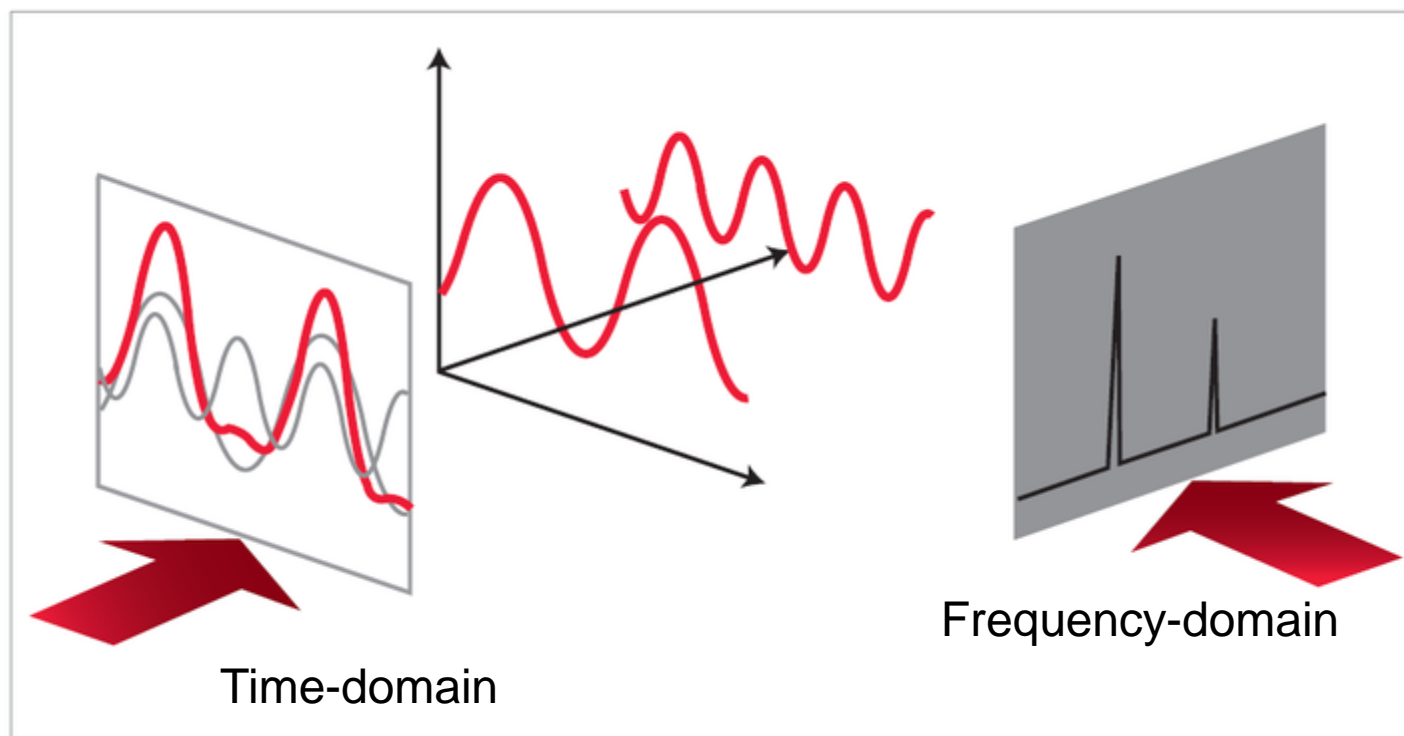
- Fourier Series:  $s(t) = \sum_{n=-\infty}^{\infty} s_n e^{j2\pi n f_0 t}$       $s_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} s(t) e^{-j2\pi n f_0 t} dt$

✓ Any **periodic** signal with period  $T_0$  can be expressed as a sum of sinusoidal signals, each with the frequency an integer number of  $1/T_0$ .

✓ To obtain  $s_n$ :  $\int_{-T_0/2}^{T_0/2} s(t) e^{-j2\pi m f_0 t} dt = \sum_{n=-\infty}^{\infty} s_n \int_{-T_0/2}^{T_0/2} e^{j2\pi(n-m)f_0 t} dt$

$$\begin{aligned}
 &= s_m T_0 \\
 \Rightarrow \quad s_n &= \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} s(t) e^{-j2\pi n f_0 t} dt
 \end{aligned}$$

$$\int_{-T_0/2}^{T_0/2} e^{j2\pi m f_0 t} dt = \begin{cases} T_0 & m = 0 \\ 0 & m \neq 0 \end{cases}$$



## From Fourier Series to Fourier Transform

For an **aperiodic** signal  $s(t)$ , construct a periodic signal  $s'(t)$  by repeating the signal  $s(t)$  every  $T_0$  seconds:  $s(t) = \lim_{T_0 \rightarrow \infty} s'(t)$ .

✓ **Fourier Series:**  $s'(t) = \sum_{n=-\infty}^{\infty} s'_n e^{j2\pi n f_0 t}$ ,  $s'_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} s'(t) e^{-j2\pi n f_0 t} dt$

✓ **Let**  $S'(f) = \int_{-T_0/2}^{T_0/2} s'(t) e^{-j2\pi f t} dt$ . **Then**  $s'_n = \frac{S'(n f_0)}{T_0}$

$$s'(t) = \sum_{n=-\infty}^{\infty} \frac{S'(n f_0)}{T_0} e^{j2\pi n f_0 t} = \sum_{n=-\infty}^{\infty} S'(n f_0) f_0 e^{j2\pi n f_0 t}$$

✓ **With**  $T_0 \rightarrow \infty, f_0 \rightarrow 0$ .

$$s(t) = \lim_{T_0 \rightarrow \infty} s'(t) = \lim_{f_0 \rightarrow 0} \sum_{n=-\infty}^{\infty} S'(n f_0) f_0 e^{j2\pi n f_0 t} = \int_{-\infty}^{\infty} S(f) e^{j2\pi f t} df$$

$$S(f) = \lim_{T_0 \rightarrow \infty} S'(f) = \lim_{T_0 \rightarrow \infty} \int_{-T_0/2}^{T_0/2} s'(t) e^{-j2\pi f t} dt = \int_{-\infty}^{\infty} s(t) e^{-j2\pi f t} dt$$

**Fourier Transform**

## Fourier Transform

Given a time-domain signal  $s(t)$ , its Fourier transform is defined as follows.

**Fourier transform:**

$$S(f) = \int_{-\infty}^{\infty} s(t) e^{-j2\pi ft} dt$$

The time-domain signal  $s(t)$  can be expressed by  $S(f)$  using an inverse transform.

**Inverse Fourier transform:**

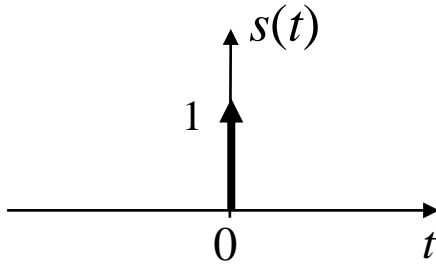
$$s(t) = \int_{-\infty}^{\infty} S(f) e^{j2\pi ft} df$$

- (Fourier) spectrum of  $s(t)$ :  $S(f)$

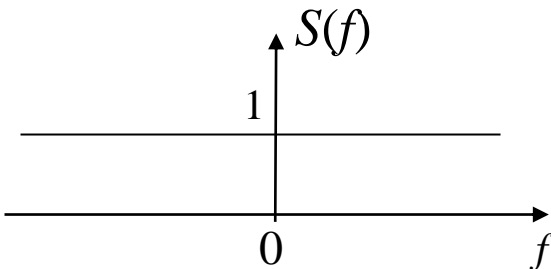
$$s(t) \Leftrightarrow S(f)$$

- Magnitude spectrum of  $s(t)$ :  $|S(f)|$

## Example 1: Spectrum of Unit Impulse $\delta(t)$

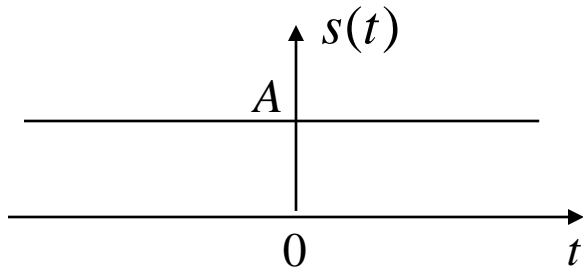


$$s(t) = \delta(t): \quad \delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t) dt = 1$$

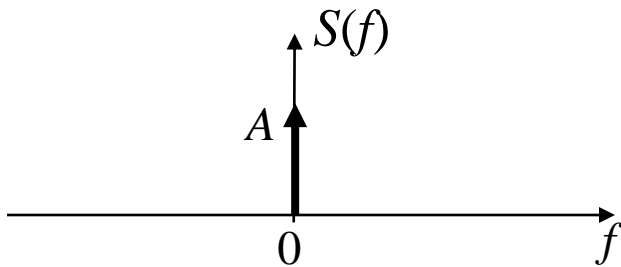


$$S(f) = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi ft} dt = 1$$

## Example 2: Spectrum of Constant Signal



$$s(t) = A$$



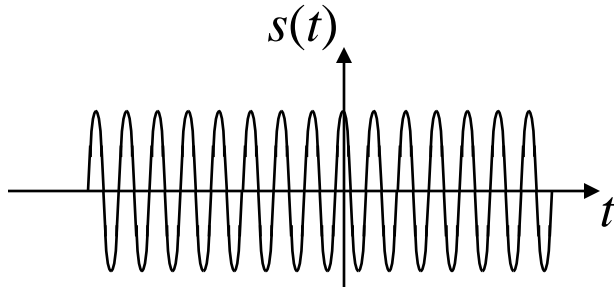
$$S(f) = A \int_{-\infty}^{\infty} e^{-j2\pi ft} dt$$

$$S(0) = A \int_{-\infty}^{\infty} e^{-j2\pi 0t} dt = A \int_{-\infty}^{\infty} 1 dt = \infty$$

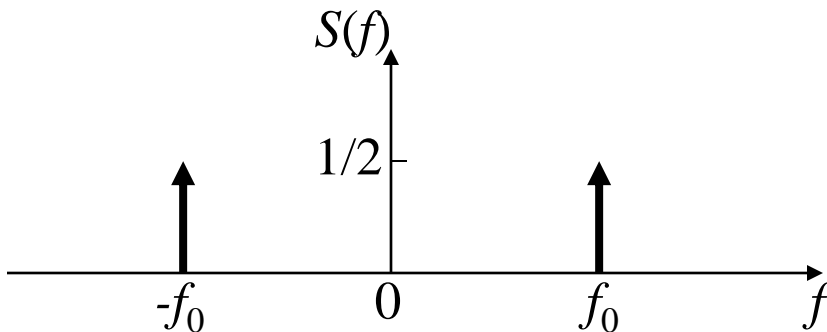
$$S(f) = A \int_{-\infty}^{\infty} e^{-j2\pi ft} dt = 0 \quad \text{for } f \neq 0$$

$$S(f) = A\delta(f)$$

## Example 3: Spectrum of Sinusoidal Signal



$$s(t) = \cos(2\pi f_0 t)$$

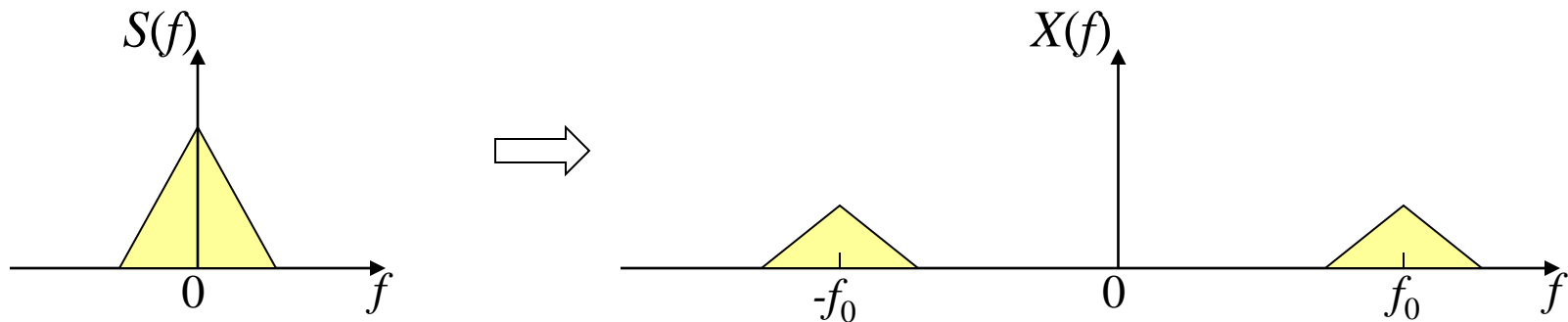


$$\begin{aligned}
 S(f) &= \int_{-\infty}^{\infty} \cos 2\pi f_0 t \cdot e^{-j2\pi f t} dt \\
 &= \int_{-\infty}^{\infty} \frac{1}{2} (e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}) \cdot e^{-j2\pi f t} dt \\
 &= \frac{1}{2} \int_{-\infty}^{\infty} e^{-j2\pi(f-f_0)t} dt + \frac{1}{2} \int_{-\infty}^{\infty} e^{-j2\pi(f+f_0)t} dt \\
 &= \underline{\underline{\frac{1}{2} (\delta(f-f_0) + \delta(f+f_0))}}
 \end{aligned}$$

## Example 4: Spectrum of $s(t)\cos(2\pi f_0 t)$

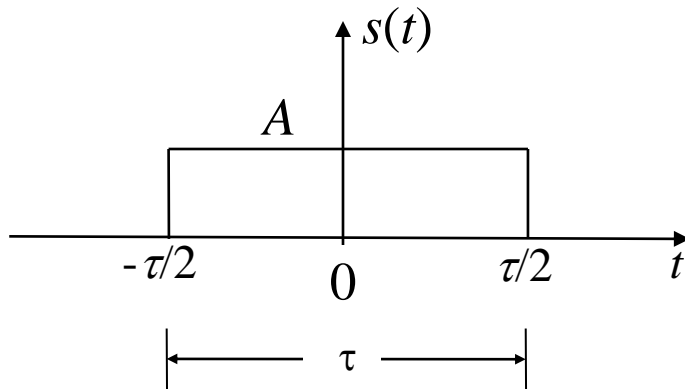
$$x(t) = s(t) \cos(2\pi f_0 t)$$

$$\begin{aligned}
 X(f) &= \int_{-\infty}^{\infty} s(t) \cos(2\pi f_0 t) \cdot e^{-j2\pi f t} dt = \int_{-\infty}^{\infty} s(t) \cdot \frac{1}{2} (e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}) \cdot e^{-j2\pi f t} dt \\
 &= \frac{1}{2} \int_{-\infty}^{\infty} s(t) \cdot e^{-j2\pi(f-f_0)t} dt + \frac{1}{2} \int_{-\infty}^{\infty} s(t) \cdot e^{-j2\pi(f+f_0)t} dt = \underline{\underline{\frac{1}{2}[S(f-f_0) + S(f+f_0)]}}
 \end{aligned}$$

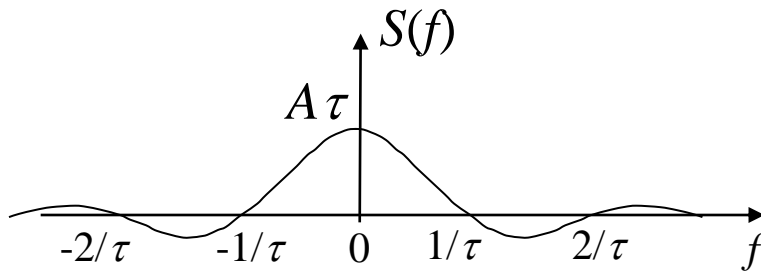




## Example 5: Spectrum of Single Rectangular Pulse



$$s(t) = \begin{cases} A & -\tau/2 \leq t \leq \tau/2 \\ 0 & \text{otherwise} \end{cases}$$



$$\begin{aligned}
 S(f) &= A \int_{-\tau/2}^{\tau/2} e^{-j2\pi ft} dt = A \cdot \frac{e^{-j\pi f\tau} - e^{+j\pi f\tau}}{-j2\pi f} \\
 &= A\tau \cdot \frac{\sin(\pi f\tau)}{\pi f\tau} = A\tau \text{sinc}(f\tau)
 \end{aligned}$$

$$\lim_{x \rightarrow 0} \sin x / x = 1$$

$$\text{sinc}(x) \triangleq \frac{\sin(\pi x)}{\pi x}$$

- sinc function is an even, oscillating function with a decreasing magnitude.
- It has unit peak at  $x=0$ , and zero crossing points at  $x = \text{non-zero integers}$ .

## Properties of Fourier Transform

$$\alpha s_1(t) + \beta s_2(t) \quad \Leftrightarrow \quad \alpha S_1(f) + \beta S_2(f) \quad \text{Linearity}$$

$$s_1(t)s_2(t) \quad \Leftrightarrow \quad S_1(f) * S_2(f) \quad \text{Convolution}$$

$$s_1(t) * s_2(t) \quad \Leftrightarrow \quad S_1(f) \cdot S_2(f)$$

$$S(t) \quad \Leftrightarrow \quad s(-f) \quad \text{Duality}$$

$$s(t - \tau) \quad \Leftrightarrow \quad S(f)e^{-j2\pi f\tau} \quad \text{Time shift}$$

$$s(t)e^{-j2\pi f_0 t} \quad \Leftrightarrow \quad S(f + f_0) \quad \text{Frequency shift}$$

$$s(t)\cos(2\pi f_0 t) \quad \Leftrightarrow \quad \frac{1}{2}[S(f - f_0) + S(f + f_0)] \quad \text{Modulation}$$

$$s(at) \quad \Leftrightarrow \quad \frac{1}{|a|} S\left(\frac{f}{a}\right) \quad \text{Time scale}$$

(for any real  $a \neq 0$ )

## Review Examples 2 & 4

$$s(t) = \delta(t) \Leftrightarrow S(f) = 1$$

Duality:  $S(t) \Leftrightarrow s(-f)$

$$s(t) = 1 \Leftrightarrow S(f) = \delta(f)$$

$$x(t) = s(t) \cos(2\pi f_0 t)$$

$$\Leftrightarrow$$

$$\begin{aligned} X(f) &= S(f) * \left[ \frac{1}{2} (\delta(f - f_0) + \delta(f + f_0)) \right] \\ &= \frac{1}{2} [S(f - f_0) + S(f + f_0)] \end{aligned}$$

Modulation:

$$s_1(t) \cos(2\pi f_0 t)$$

$$\Leftrightarrow$$

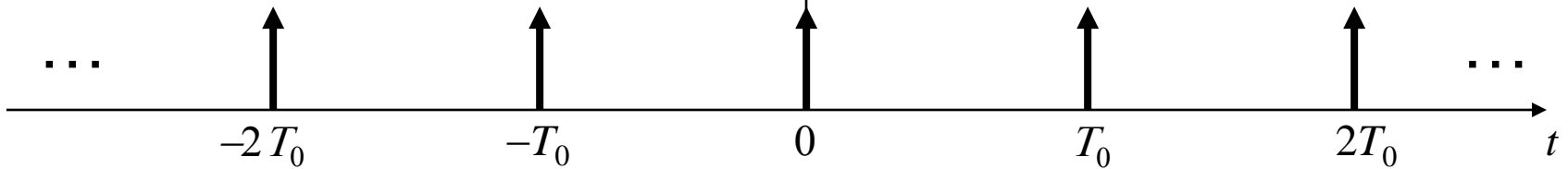
$$\frac{1}{2} [S(f - f_0) + S(f + f_0)]$$

Convolution:

$$s_1(t)s_2(t) \Leftrightarrow S_1(f) * S_2(f)$$

## Example 6: Spectrum of Impulse Train

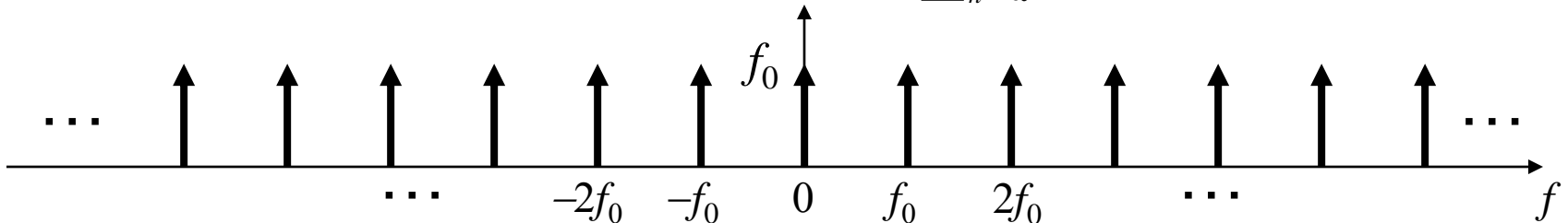
$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0) = \sum_{n=-\infty}^{\infty} s_n e^{j2\pi n f_0 t} \quad f_0 = \frac{1}{T_0}$$



$$S(f) = \sum_{n=-\infty}^{\infty} s_n \delta(f - n f_0)$$

$$s_n = \frac{1}{T_0} \int_0^{T_0} s(t) e^{-j2\pi n f_0 t} dt = f_0 \int_0^{1/f_0} \delta(t) e^{-j2\pi n f_0 t} dt = f_0$$

$$S(f) = f_0 \sum_{n=-\infty}^{\infty} \delta(f - n f_0)$$



## Example 7: Spectrum of Periodic Signal

- For periodic signal  $s(t)$  with period  $T_0$ , define  $s_{T_0}(t)$  as

$$s_{T_0}(t) = \begin{cases} s(t) & -T_0/2 < t < T_0/2 \\ 0 & \text{otherwise} \end{cases}$$

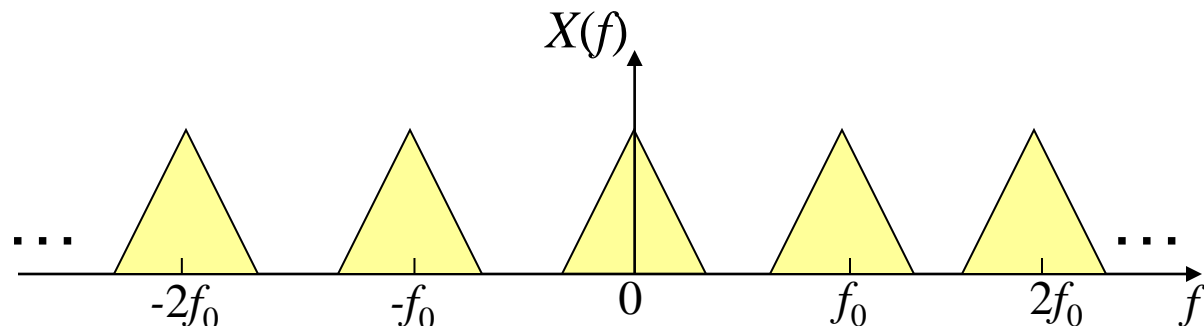
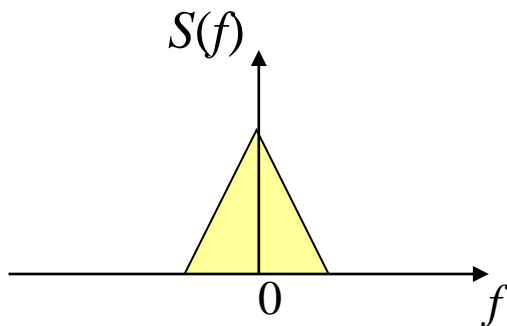
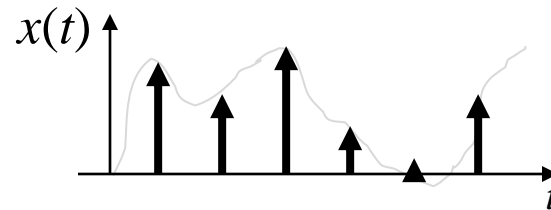
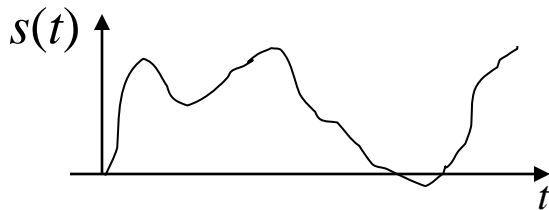
- $s(t) = \sum_{n=-\infty}^{\infty} s_{T_0}(t - nT_0) = s_{T_0}(t) * \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$

- 
- $S(f) = S_{T_0}(f) \cdot f_0 \sum_{n=-\infty}^{\infty} \delta(f - nf_0) = f_0 \sum_{n=-\infty}^{\infty} S_{T_0}(nf_0) \delta(f - nf_0)$

## Example 8: Spectrum of Sampled Signal

$$x(t) = s(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$$

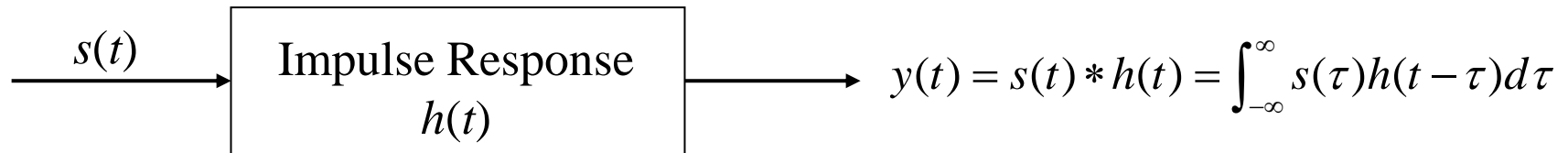
$$X(f) = S(f) * f_0 \sum_{n=-\infty}^{\infty} \delta(f - nf_0) = f_0 \sum_{n=-\infty}^{\infty} S(f - nf_0)$$



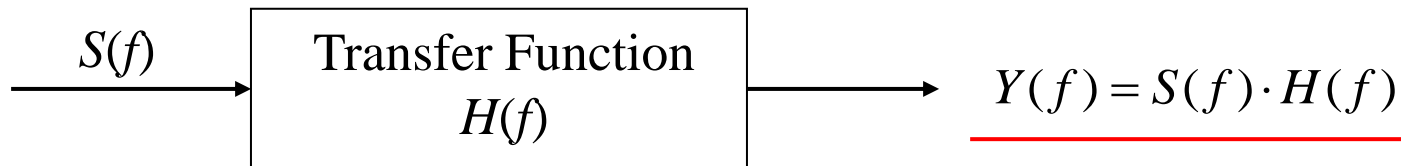
# Signal Transmission through an LTI System

## Signal Transmission over an LTI System

- Time Domain:



- Frequency Domain:



$$s(t) \Leftrightarrow S(f)$$

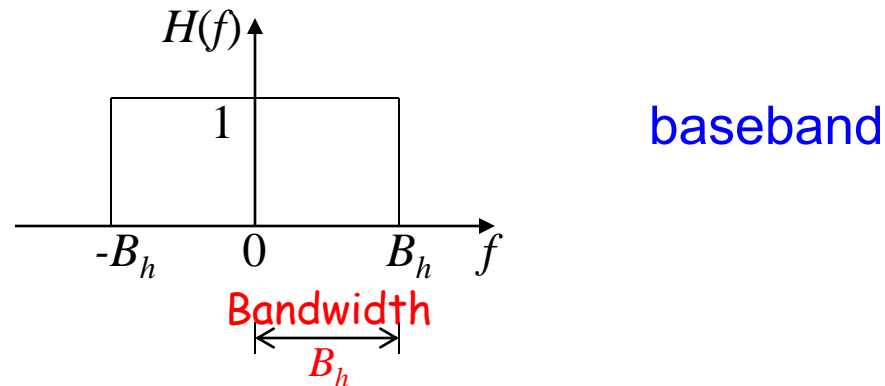
$$h(t) \Leftrightarrow H(f)$$

$$y(t) \Leftrightarrow Y(f)$$

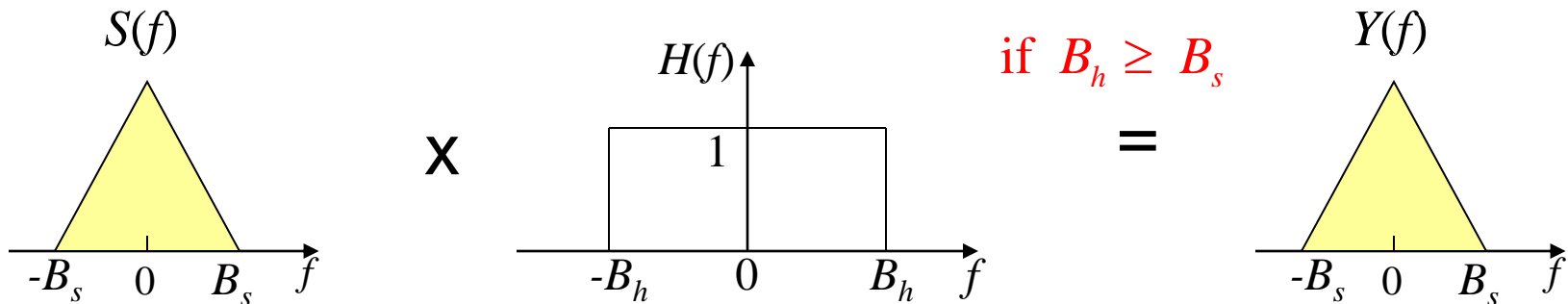


## Ideal Lowpass System

- Transfer Function  $H(f)$  of an ideal lowpass system:

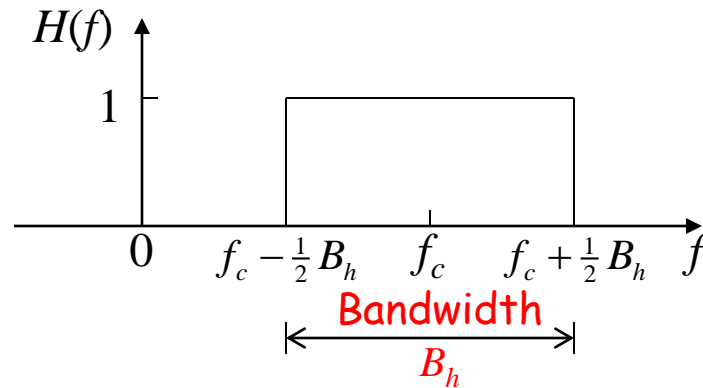


- For a baseband input signal with bandwidth  $B_s$ :

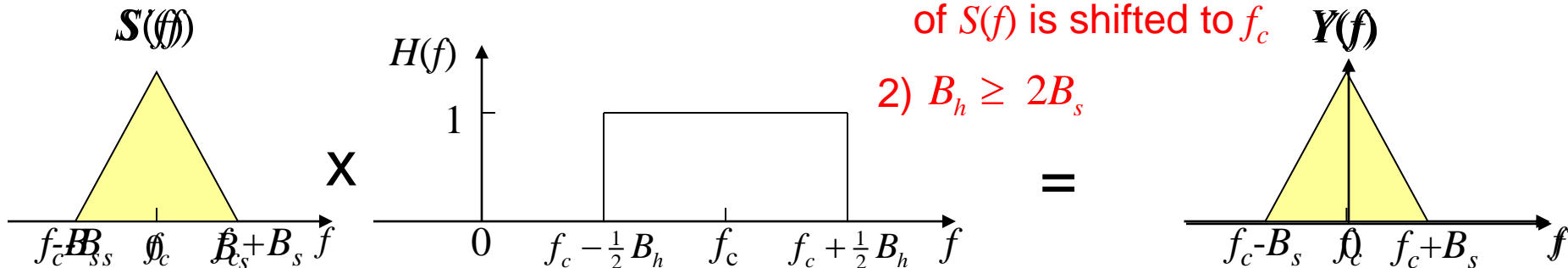


## Ideal Bandpass System

- Transfer Function  $H(f)$  of an ideal bandpass system:



- For a baseband input signal with bandwidth  $B_s$ :



If: 1) the center frequency of  $S(f)$  is shifted to  $f_c$

2)  $B_h \geq 2B_s$

## Baseband Channel and Bandpass Channel

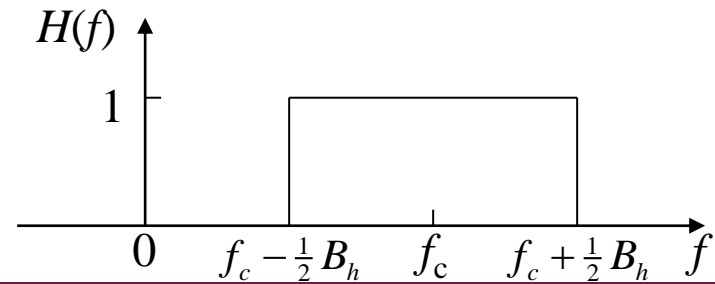
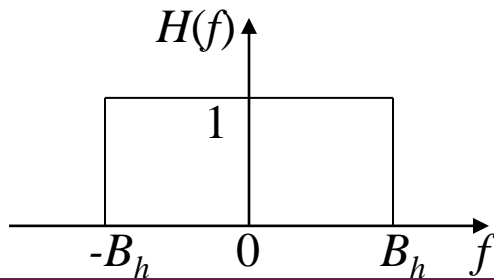
- Baseband channel

- A baseband channel efficiently passes frequency components from dc (zero) to the cutoff frequency  $B_h$  Hz.
- Examples: coaxial cable

- Bandpass channel

- A bandpass channel efficiently passes frequency components within a certain band, say, between  $f_c - \frac{1}{2} B_h$  and  $f_c + \frac{1}{2} B_h$  Hz.
- Examples: EM wave, fibre

In this course, a baseband channel and a bandpass channel are modeled as an ideal low-pass LTI system and an ideal bandpass LTI system, respectively.



# Energy Spectrum, Power Spectrum

## Energy-type Signal and Power-type Signal

- Energy-type Signal: A signal is an energy-type signal if and only if its energy is positive and finite.

✓  $s(t)$  is an energy-type signal if and only if  $0 < E_s = \int_{-\infty}^{\infty} |s(t)|^2 dt < \infty$ .

- Power-type Signal: A signal is a power-type signal if and only if its power is positive and finite.

✓  $s(t)$  is a power-type signal if and only if  $0 < P_s = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} |s(t)|^2 dt < \infty$ .

How to determine if a signal is an energy-type signal or a power-type signal from the frequency domain?

## Energy and Energy Spectrum

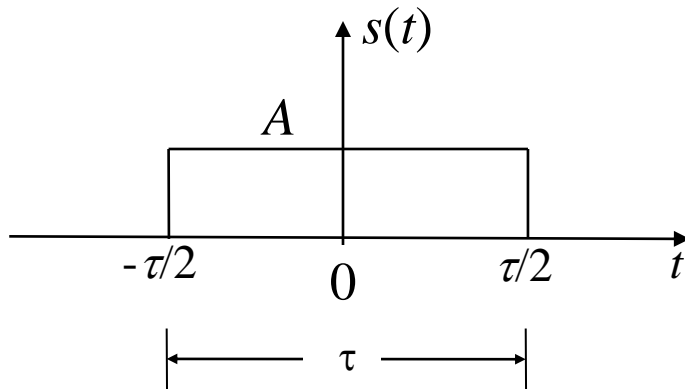
- Energy of energy-type signal  $s(t)$ :

$$\begin{aligned} E_s &= \int_{-\infty}^{\infty} |s(t)|^2 dt = \int_{-\infty}^{\infty} s(t)s^*(t)dt = \int_{-\infty}^{\infty} s(t) \left[ \int_{-\infty}^{\infty} S^*(f)e^{-j2\pi ft} df \right] dt \\ &= \int_{-\infty}^{\infty} S^*(f) \left[ \int_{-\infty}^{\infty} s(t)e^{-j2\pi ft} dt \right] df = \int_{-\infty}^{\infty} S^*(f)S(f)df = \int_{-\infty}^{\infty} |S(f)|^2 df \\ &= \int_{-\infty}^{\infty} U_s(f)df \end{aligned}$$

Parseval's Theorem:  $E_s = \int_{-\infty}^{\infty} |s(t)|^2 dt = \int_{-\infty}^{\infty} |S(f)|^2 df$

- Energy spectrum:  $U_s(f) \triangleq |S(f)|^2$

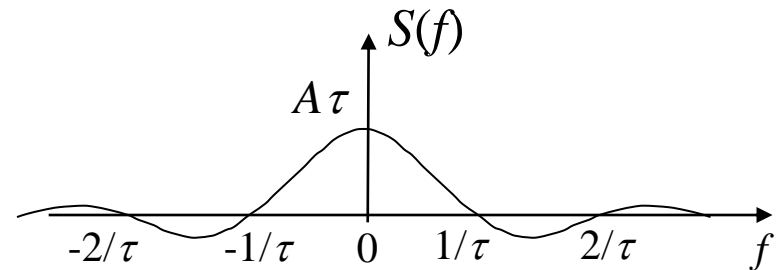
## Example 9: Energy Spectrum of Single Rectangular Pulse



$$s(t) = \begin{cases} A & -\tau/2 \leq t \leq \tau/2 \\ 0 & \text{otherwise} \end{cases}$$

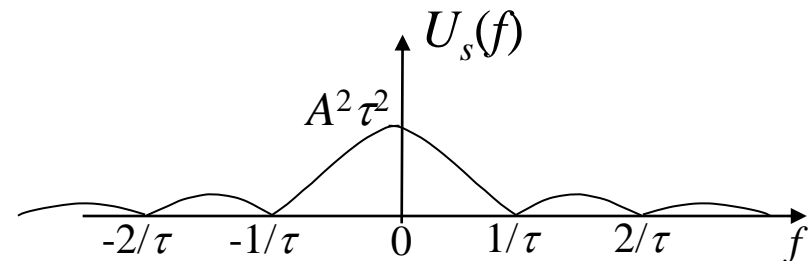
- Fourier spectrum:

$$S(f) = A\tau \text{sinc}(f\tau)$$



- Energy spectrum:

$$U_s(f) = |S(f)|^2 = A^2\tau^2 \text{sinc}^2(f\tau)$$



## Power and Power Spectrum

- Power of power-type signal  $s(t)$ :

$$s_T(t) \triangleq \begin{cases} s(t) & -T/2 < t < T/2 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned}
 P_s &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} |s(t)|^2 dt = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} |s_T(t)|^2 dt \\
 &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\infty}^{\infty} |S_T(f)|^2 df = \int_{-\infty}^{\infty} \lim_{T \rightarrow \infty} \frac{1}{T} |S_T(f)|^2 df = \int_{-\infty}^{\infty} G_s(f) df
 \end{aligned}$$

- Power spectrum:

when  $S(t) = A \cos(2\pi f_0 t)$   
 Power =  $A^2/2$

$$G_s(f) \triangleq \lim_{T \rightarrow \infty} \frac{1}{T} |S_T(f)|^2$$

$$G_s(f) \Leftrightarrow \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} s(t + \tau) s^*(t) dt$$

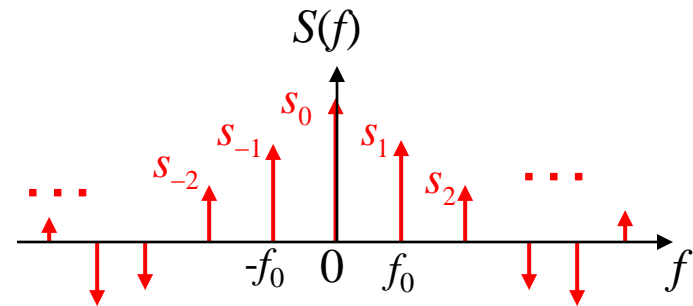


## Example 10: Power Spectrum of Periodic Signal

For periodic signal  $s(t)$  with period  $T_0$ :  $s(t) = \sum_{n=-\infty}^{\infty} s_n e^{j2\pi n f_0 t}$   $f_0 = 1/T_0$

- Fourier spectrum:

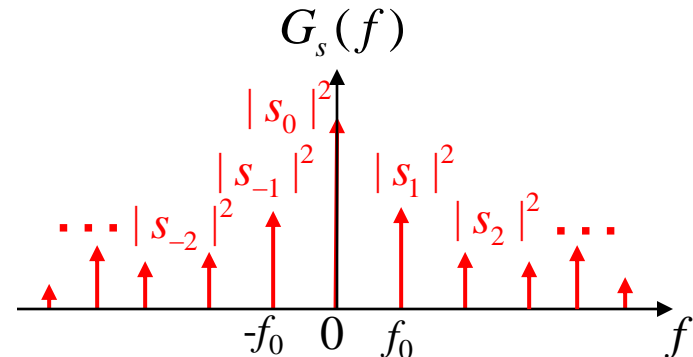
$$S(f) = \sum_{n=-\infty}^{\infty} s_n \delta(f - n f_0)$$



- Power spectrum:

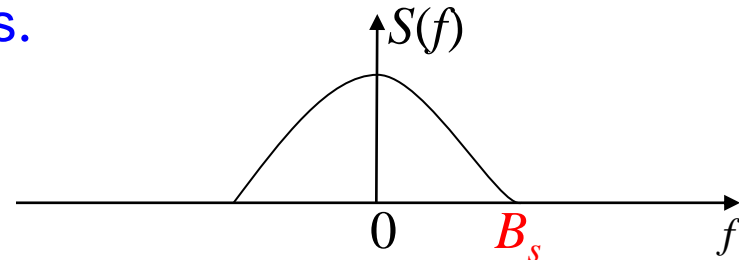
$$G_s(f) \Leftrightarrow \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} s(t + \tau) s^*(t) dt = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} s(t + \tau) s^*(t) dt = \sum_{n=-\infty}^{\infty} |s_n|^2 e^{j2\pi n f_0 \tau}$$

$$G_s(f) = \sum_{n=-\infty}^{\infty} |s_n|^2 \delta(f - n f_0)$$



## Signal Bandwidth

- Bandwidth of signal  $s(t)$ : the amount of **positive** frequency spectrum that signal  $s(t)$  occupies.



- Effective Bandwidth:  $x\%$  of the signal's power (energy) are included.

