In-Class Exercise 7

1. Consider two random variables X and Y with joint probability mass function (PMF) given in the following table:

	$Y=0$	$Y=1$	$Y=2$
$X=0$	0.01	0	0
$X=1$	0.09	0.09	0
$X=2$	0	0	0.81

   (a) Compute the correlation of X and Y, i.e., $r_{X,Y} = \mathbb{E}\{XY\}$.
   (b) Compute $\mathrm{cov}(X,Y)$.
   (c) Compute correlation coefficient $\rho_{X,Y}$.

2. A random variable X is transformed to another random variable $Y = aX + b$ where a and b are constants. Suppose $a < 0$. Determine $\rho_{X,Y}$.

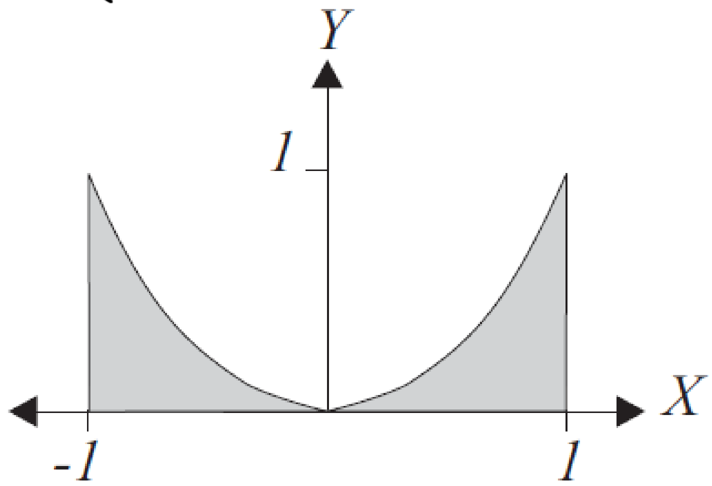
3. Prove the following property of correlation coefficient of random variables X and Y with variances σ_X^2 and σ_Y^2:

$$-1 \leq \rho_{X,Y} = \frac{\mathrm{cov}(X,Y)}{\sqrt{\mathrm{var}(X)}\sqrt{\mathrm{var}(Y)}} = \frac{\mathrm{cov}(X,Y)}{\sigma_X \sigma_Y} \leq 1$$

Hint: Let $W = X - aY$ where $a \in \mathbb{R}$ and then consider $\mathrm{var}(W)$ with suitable values of a. Also apply (3.23).

4. Random variables X and Y have the following joint probability density function (PDF):

$$P_{XY}(x,y) = \begin{cases} 5x^2/2, & -1 \le x \le 1; 0 \le y \le x^2 \\ 0, & \text{otherwise} \end{cases}$$



(a) Compute $\mathbb{E}\{X\}$ and $\mathrm{var}(X)$.

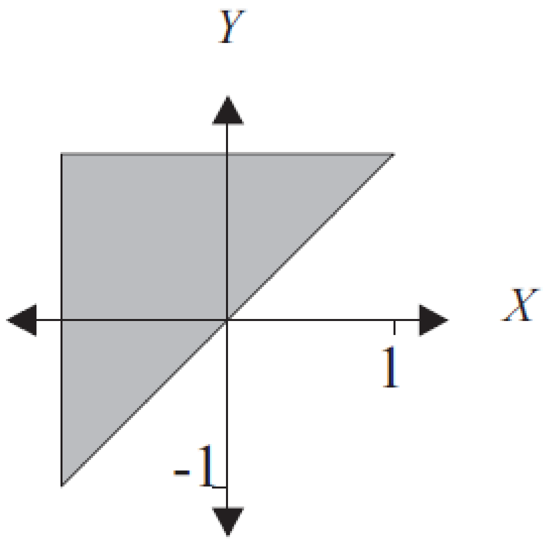
(b) Compute $\mathbb{E}\{Y\}$ and $\mathrm{var}(Y)$.

(c) Compute $\mathrm{cov}(X,Y)$.

(d) Compute $\mathbb{E}\{X+Y\}$ and $\mathrm{var}(X+Y)$.

5. The joint probability density function (PDF) of random variables X and Y is given as:

$$P_{XY}(x,y) = \begin{cases} 1/2, & -1 \le x \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$



(a) Compute $\mathbb{E}\{XY\}$.
(b) Compute $\mathbb{E}\{e^{X+Y}\}$.

6. Consider an observation x which is of the form:

$$x = A + n$$

where A is a constant to be estimated and $n \sim \mathcal{N}(0, \sigma^2)$. It is suggested to estimate A using $\hat{A}$:

$$\hat{A} = x$$

Compute the mean of the estimate $\mathbb{E}\{\hat{A}\}$ and mean square error (MSE) $\mathbb{E}\{(\hat{A} - A)^2\}$.

Suppose now the noise is changed to $n \sim \mathcal{U}(0, 1)$. Determine an unbiased estimate of A and then compute the corresponding MSE.

7. The joint PDF of random variables X and Y is given as:

$$P_{XY}(x,y) = ce^{-\frac{x^2}{8}-\frac{y^2}{18}}$$

where c is a constant.

(a) Are X and Y independent? Briefly explain your answer.
(b) Determine the distributions of X and Y, and then find their marginal PDFs.
(c) Find the value of c.

Solution

1.(a)

$$r_{X,Y} = \mathbb{E}\{XY\} = \sum_{x=0}^{2}\sum_{y=0}^{2} xyP_{XY}(x,y)$$
$$= (1)(1)(0.09) + (2)(2)(0.81) = 3.33$$

1.(b)

The marginal PMFs are:

$$p(x) = \begin{cases} 0.01, & x = 0 \\ 0.18, & x = 1 \\ 0.81, & x = 2 \\ 0, & \text{otherwise} \end{cases} \qquad p(y) = \begin{cases} 0.1, & y = 0 \\ 0.09, & y = 1 \\ 0.81, & y = 2 \\ 0, & \text{otherwise} \end{cases}$$

Then we compute the expected values:

$$\mathbb{E}\{X\} = (1)(0.18) + (2)(0.81) = 1.8$$
$$\mathbb{E}\{Y\} = (1)(0.09) + (2)(0.81) = 1.71$$

Using (3.21), we have:

$$\mathrm{cov}(X,Y) = \mathbb{E}\{XY\} - \mathbb{E}\{X\}\mathbb{E}\{Y\} = 0.252$$

1.(c)

$$\begin{aligned}\mathbb{E}\{X^2\} &= (1)^2(0.18) + (2)^2(0.81) = 3.42 \\ \mathbb{E}\{Y^2\} &= (1)^2(0.09) + (2)^2(0.81) = 3.33\end{aligned}$$

Applying (2.23) yields:

$$\begin{aligned}\mathrm{var}(X) &= \mathbb{E}\{X^2\} - (\mathbb{E}\{X\})^2 = 0.18 \\ \mathrm{var}(Y) &= \mathbb{E}\{Y^2\} - (\mathbb{E}\{Y\})^2 = 0.4059\end{aligned}$$

Using (3.25), we have

$$\rho_{X,Y} = \frac{\mathrm{cov}(X,Y)}{\sqrt{\mathrm{var}(X)}\sqrt{\mathrm{var}(Y)}} = \frac{0.252}{\sqrt{0.18}\sqrt{0.4059}} = 0.9323$$

2.

From the results of Question 7 in In-Class Exercise 5, we have:

$$\mathbb{E}\{Y\} = \mu_y = \mathbb{E}\{aX + b\} = \mathbb{E}\{aX\} + \mathbb{E}\{b\} = a\mathbb{E}\{X\} + b = a\mu_x + b$$

$$\mathrm{var}(Y) = \sigma_y^2 = \mathbb{E}\{(Y - \mu_y)^2\} = a^2\mathrm{var}(X) = a^2\sigma_x^2$$

According to (3.21), we obtain:

$$\begin{aligned}\mathrm{cov}(X, Y) &= \mathbb{E}\{(X - \mu_x)(aX + b - a\mu_x - b)\} \\ &= \mathbb{E}\{(X - \mu_x)(aX - a\mu_x)\} = a\mathbb{E}\{(X - \mu_x)^2\} = a\sigma_x^2\end{aligned}$$

From (3.25), the correlation coefficient is:

$$\rho_{X,Y} = \frac{\mathrm{cov}(X, Y)}{\sqrt{\mathrm{var}(X)}\sqrt{\mathrm{var}(Y)}} = \frac{a\sigma_x^2}{\sqrt{\sigma_x^2}\sqrt{a^2\sigma_x^2}} = \frac{a}{|a|}$$

When $a < 0$, we get $\rho_{X,Y} = -1$.

3.

Let $W = X - aY$ where $a \in \mathbb{R}$. Applying the result in (3.23) (or Question 4 in In-Class Exercise 6), the variance of W is:

$$\mathrm{var}(W) = \mathrm{var}(X) + a^2\mathrm{var}(Y) - 2a\mathrm{cov}(X,Y)$$

As variance must be nonnegative, we have:

$$\begin{aligned}\mathrm{var}(W) \geq 0 \;&\Rightarrow\; \mathrm{var}(X) + a^2\mathrm{var}(Y) \geq 2a\mathrm{cov}(X,Y)\\ &\Rightarrow\; \sigma_X^2 + a^2\sigma_Y^2 \geq 2a\mathrm{cov}(X,Y)\end{aligned}$$

Note that the inequality holds for all $a \in \mathbb{R}$. Set $a = \sigma_X/\sigma_Y > 0$:

$$2\sigma_X^2 \geq 2\sigma_X/\sigma_Y \cdot \mathrm{cov}(X,Y) \Rightarrow \sigma_X\sigma_Y \geq \mathrm{cov}(X,Y) \Rightarrow \rho_{X,Y} \leq 1$$

We then set $a = -\sigma_X/\sigma_Y < 0$:

$$-2\sigma_X^2 \leq 2\sigma_X/\sigma_Y \cdot \mathrm{cov}(X,Y) \Rightarrow -\sigma_X\sigma_Y \leq \mathrm{cov}(X,Y) \Rightarrow \rho_{X,Y} \geq -1$$

Combining the results yields $1 \geq \rho_{X,Y} \geq -1$.

4.(a)

$$\mathbb{E}\{X\} = \int_{-1}^{1}\int_{0}^{x^2} x\frac{5x^2}{2}dydx = \int_{-1}^{1}\frac{5x^5}{2}dx = -\frac{5x^6}{12}\bigg|_{-1}^{1} = 0$$

Since $\mathbb{E}\{X\}=0$, $\mathrm{var}(X)=\mathbb{E}\{X^2\}$:

$$\mathrm{var}(X) = \mathbb{E}\{X^2\} = \int_{-1}^{1}\int_{0}^{x^2} x^2\frac{5x^2}{2}dydx = \int_{-1}^{1}\frac{5x^6}{2}dx = \frac{5x^7}{14}\bigg|_{-1}^{1} = \frac{5}{7}$$

4.(b)

$$\mathbb{E}\{Y\} = \int_{-1}^{1}\int_{0}^{x^2} y\frac{5x^2}{2}dydx = \frac{5}{14}$$

$$\mathbb{E}\{Y^2\} = \int_{-1}^{1}\int_{0}^{x^2} y^2\frac{5x^2}{2}dydx = \frac{5}{27}$$

Hence

$$\mathrm{var}(Y) = \mathbb{E}\{Y^2\} - (\mathbb{E}\{Y\})^2 = 0.0576$$

4.(c)

Since $\mathbb{E}\{X\}=0$, we have:

$$\text{cov}(X,Y)=\mathbb{E}\{XY\}-\mathbb{E}\{X\}\mathbb{E}\{Y\}=\mathbb{E}\{XY\}$$

Hence

$$\text{cov}(X,Y)=\mathbb{E}\{XY\}=\int_{-1}^{1}\int_{0}^{x^2} xy\frac{5x^2}{2}dydx=\int_{-1}^{1}\frac{5x^7}{4}dx=0$$

4.(d)

$$\mathbb{E}\{X+Y\}=\mathbb{E}\{X\}+\mathbb{E}\{Y\}=\frac{5}{14}$$

$$\text{var}(X+Y)\;=\;\text{var}(X)+\text{var}(Y)+2\text{cov}(X,Y)=0.7719$$

5.(a)

$$\mathbb{E}\{XY\} = \int_{-1}^{1}\int_{x}^{1}\frac{xy}{2}dydx = \int_{-1}^{1}\frac{x(1-x^2)}{4}dx = \frac{x^2}{8}-\frac{x^4}{16}\bigg|_{-1}^{1} = 0$$

5.(b)

$$\begin{aligned}\mathbb{E}\{e^{X+Y}\} &= \int_{-1}^{1}\int_{x}^{1}\frac{e^x e^y}{2}dydx\\ &= \int_{-1}^{1}\frac{e^x(e-e^x)}{2}dx\\ &= \frac{e^{1+x}}{2}-\frac{e^{2x}}{4}\bigg|_{-1}^{1}\\ &= \frac{e^2}{4}+\frac{e^{-2}}{4}-\frac{1}{2}\end{aligned}$$

6.

We can follow Example 3.18 to obtain the results:

$$\mathbb{E}\{\hat{A}\} = \mathbb{E}\{x\} = \mathbb{E}\{A+n\} = A + \mathbb{E}\{n\} = A + 0 = A$$

$$\mathrm{MSE}(\hat{A}) = \mathbb{E}\{(x-A)^2\} = \mathbb{E}\{(A+n-A)^2\} = \mathbb{E}\{n^2\} = \sigma^2$$

Alternatively, we notice that $\hat{A} = x$ is also a random variable, that is, $x \sim \mathcal{N}(A, \sigma^2)$:

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-A)^2}$$

The PDF reaches the maximum value at $x = A$, implying that $\hat{A} = x$ is a reasonable choice to estimate A. From $x \sim \mathcal{N}(A, \sigma^2)$, we directly obtain $\mathbb{E}\{\hat{A}\} = \mathbb{E}\{x\} = A$ and $\mathrm{MSE}(\hat{A}) = \mathrm{var}(\hat{A}) = \mathrm{var}(x) = \sigma^2$.

For $n \sim \mathcal{U}(0,1)$, it has a mean of 0.5. Hence an unbiased estimate of A is:

$$\hat{A} = x - 0.5$$

We can easily check that

$$\mathbb{E}\{\hat{A}\} = \mathbb{E}\{x - 0.5\} = \mathbb{E}\{A + n - 0.5\} = A + \mathbb{E}\{n\} - 0.5 = A + 0.5 - 0.5 = A$$

$$\mathrm{MSE}(\hat{A}) = \mathbb{E}\{(x - 0.5 - A)^2\} = \mathbb{E}\{(A + n - 0.5 - A)^2\} = \mathbb{E}\{(n - 0.5)^2\}$$

If we write $m = n - 0.5$, it is clear that $m \sim \mathcal{U}(-0.5, 0.5)$. That is, the MSE is the second moment of m. Recalling Example 2.22, $\mathbb{E}\{m^2\}$ is:

$$\mathbb{E}\{m^2\} = \int_{-0.5}^{0.5} x^2 dx = \left.\frac{x^3}{3}\right|_{-0.5}^{0.5} = \frac{1}{12}$$

7.(a)

We observe that the joint PDF can be factorized as:

$$P_{XY}(x,y) = ce^{-\frac{x^2}{8}-\frac{y^2}{18}} = c_1 e^{-\frac{x^2}{8}} \cdot c_2 e^{-\frac{y^2}{18}}, \quad c = c_1 \cdot c_2$$

where

$$P_X(x) = c_1 e^{-\frac{x^2}{8}}, \quad P_Y(y) = c_2 e^{-\frac{y^2}{18}}$$

As $P_{XY}(x,y) = P_X(x)P_Y(y)$, X and Y are independent.

7.(b)

The forms of $P_X(x)$ and $P_Y(y)$ correspond to Gaussian random variables, e.g.,

$$P_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{1}{2\sigma_x^2}(x-\mu_x)^2}$$

Equating

$$P_X(x) = c_1 e^{-\frac{x^2}{8}} = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{1}{2\sigma_x^2}(x-\mu_x)^2}$$

we easily obtain $\mu_x = 0$ and $\sigma_x = 2$.

Similarly, we get $\mu_y = 0$ and $\sigma_y = 3$.

The values of c_1 and c_2 are:

$$c_1 = \frac{1}{2\sqrt{2\pi}}, \quad c_2 = \frac{1}{3\sqrt{2\pi}} \Rightarrow P_X(x) = \frac{1}{2\sqrt{2\pi}} e^{-\frac{1}{8}x^2}, \quad P_Y(y) = \frac{1}{3\sqrt{2\pi}} e^{-\frac{1}{18}y^2}$$

7.(c)

$$c = c_1 \cdot c_2 = \frac{1}{12\pi}$$