In-Class Exercise 8

1. Let $h(t)$ be the impulse response of a linear time-invariant (LTI) continuous-time system and it has the form of:

$$h(t) = \begin{cases} e^{-at}, & t \geq 0, \quad a > 0 \\ 0, & t < 0 \end{cases}$$

   (a) Determine the Fourier transform of $h(t)$, $H(j\Omega)$.

   (b) The $h(t)$ is sampled with a sampling period of T to produce the discrete-time signal $h[n]$. Determine the discrete-time Fourier transform (DTFT) of $h[n]$, $H(e^{j\omega})$.

   (c) Find the maximum values for $|H(j\Omega)|$ and $|H(e^{j\omega})|$.

2. When the input to a discrete-time LTI system is

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[-n-1]$$

The corresponding output is

$$y[n] = 6\left(\frac{1}{2}\right)^n u[n] - 6\left(\frac{3}{4}\right)^n u[n]$$

(a) Find the transfer function $H(z)$ of the system and specify its ROC.

(b) Determine the pole(s) and zero(s) of $H(z)$.

(c) Find the impulse response $h[n]$ of the system.

(d) Determine the DTFT of $h[n]$.

(e) Write down the difference equation that relates $x[n]$ and $y[n]$.

(f) Is the system stable? Why?

(g) Is the system causal? Why?

3. Given a discrete-time LTI system with impulse response $h[n]=b^n u[n]$ where $|b|\geq 1$. Use z transform or by other means to compute the output $y[n]$ if the input is $x[n]=a^n u[n]$ with $|a|<1$. Is the system stable? Why? Is the system causal? Why?

4. Consider a stable and causal discrete-time LTI system with impulse response $h[n]$ and rational transfer function $H(z)$. It is known that $H(z)$ contains a pole at $z=0.5$ and a zero somewhere on the unit circle. The numbers and locations of all other poles and zeros are unknown.

   Discuss if each of the following statements is "true", "false" or "cannot be determined":

   (a) The DTFT of $(0.5)^n h[n]$ exists.
   (b) $H(e^{j\omega})=0$ for some ω.
   (c) $h[n]$ is of finite duration.
   (d) $h[n]$ is a real-valued signal.

Solution

1.(a)

$$H(j\Omega) = \int_0^{\infty} e^{-at} e^{-j\Omega t} dt = -\frac{1}{a+j\Omega} e^{-(a+j\Omega)t} \Big|_0^{\infty} = \frac{1}{a+j\Omega}$$

1.(b)

$$h[n] = h(nT) = e^{-anT} u[n]$$

Hence

$$H(z) = \frac{1}{1 - e^{-aT} z^{-1}}, \quad |z| > |e^{-aT}|$$

For positive a and T, $|e^{-aT}| < 1$ and thus ROC includes the unit circle. DTFT exists and is obtained by putting $z = e^{j\omega}$:

$$H(e^{j\omega}) = \frac{1}{1 - e^{-aT} e^{-j\omega}}$$

1.(c)

Using (2.4), we have:

$$|H(j\Omega)| = \frac{1}{\sqrt{a^2+\Omega^2}}$$

It is clear that $|H(j\Omega)|$ is maximum when $\Omega = 0$, that is, the maximum value of $|H(j\Omega)|$ is:

$$|H(j\Omega)| = \frac{1}{a}$$

Similarly, we have

$$|H(e^{j\omega})|^2 = \frac{1}{1-e^{-aT}e^{-j\omega}} \cdot \frac{1}{1-e^{-aT}e^{j\omega}}$$

As a result, we obtain:

$$|H(e^{j\omega})| = \frac{1}{\sqrt{1 - 2e^{-aT}\cos(\omega) + e^{-2aT}}}$$

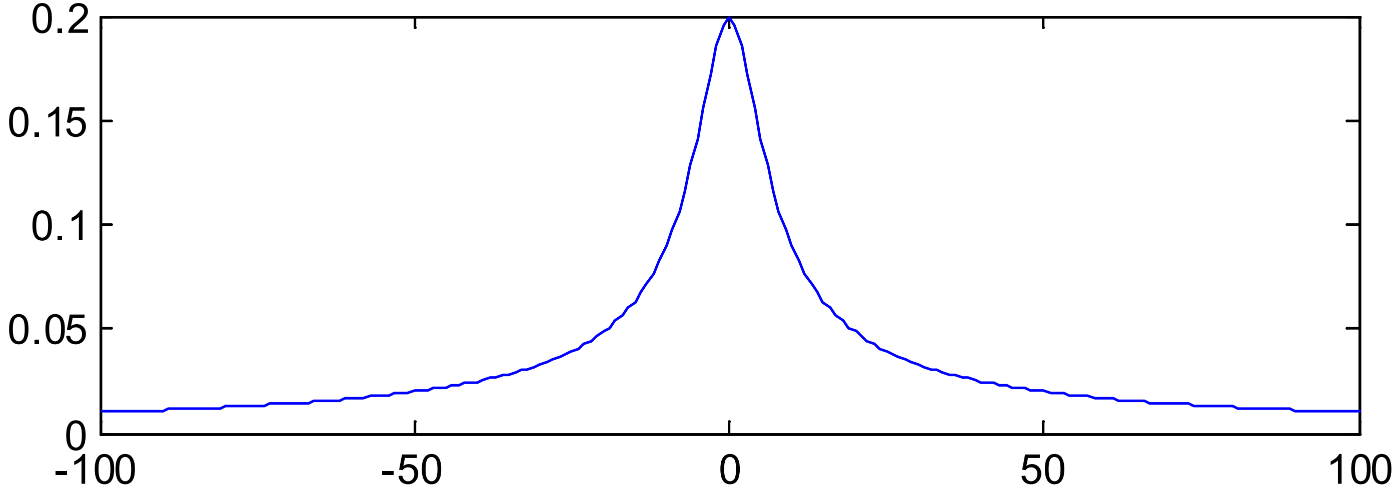
The expression will be maximized if the denominator of the RHS is minimized. As $e^{-aT} > 0$, the denominator has minimum value if $2e^{-aT}\cos(\omega)$ is maximized, that is, when $\cos(\omega) = 1$ or $\omega = 0$.

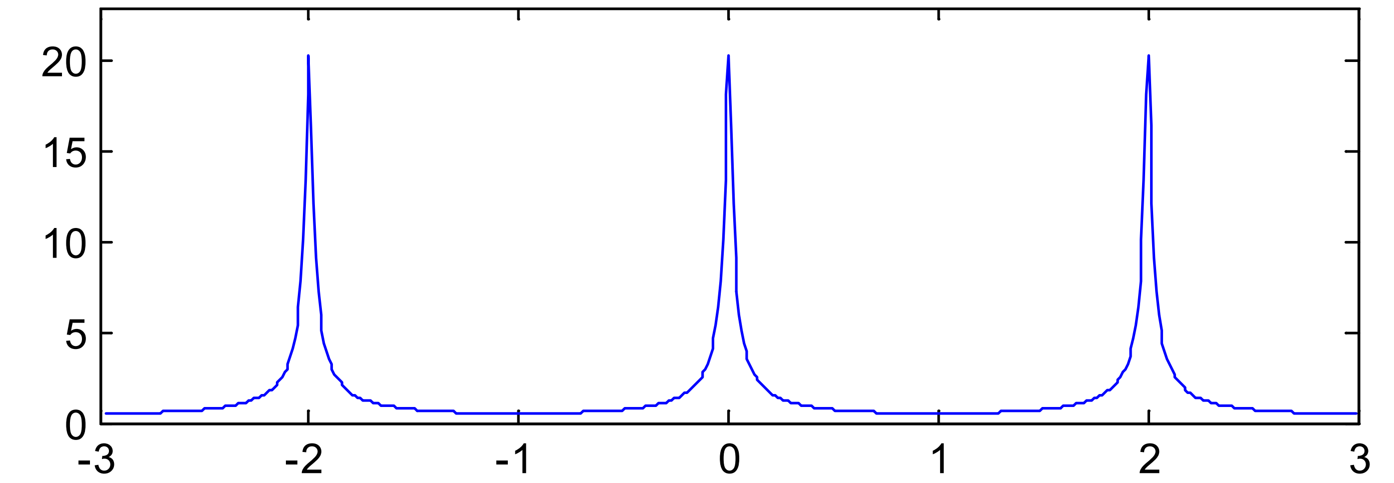
Hence the maximum value of $|H(e^{j\omega})|$ is:

$$|H(e^{j\omega})| = \frac{1}{\sqrt{1 - 2e^{-aT} + e^{-2aT}}} = \frac{1}{\sqrt{(1 - e^{-aT})^2}} = \frac{1}{1 - e^{-aT}}$$

MATLAB illustration:

```
a=5;
T=0.01;
W=-100:1:100;
Hc=1./sqrt(a^2+W.^2);
subplot(2,1,1)
plot(W,Hc)
w=-3*pi+0.1:0.01*pi:3*pi;
H=1./sqrt(1-2*exp(-a*T).*cos(w)+exp(-2*a*T));
subplot(2,1,2)
plot(w./pi,H)
```


0.2
0.15
0.1
0.05
0
-100
-50
0
50
100


20
15
10
5
0
-3
-2
-1
0
1
2
3

2.(a)

The z transforms of $x[n]$ and $y[n]$ are:

$$X(z) = \frac{1}{1-0.5z^{-1}} - \frac{1}{1-2z^{-1}} = \frac{-1.5z^{-1}}{(1-0.5z^{-1})(1-2z^{-1})}, \quad 0.5 < |z| < 2$$

and

$$Y(z) = \frac{6}{1-0.5z^{-1}} - \frac{6}{1-0.75z^{-1}} = \frac{-1.5z^{-1}}{(1-0.5z^{-1})(1-0.75z^{-1})}, \quad |z| > 0.75$$

As a result,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-2z^{-1}}{1-0.75z^{-1}}, \quad |z| > 0.75$$

Note that the ROC of $Y(z)$ contains at least the intersection of the ROCs of $X(z)$ and $H(z)$. Hence it is impossible for $H(z)$ to have a ROC of $|z| < 0.75$.

2.(b)

There is one pole at $z = 0.75$ and one zero at $z = 2$.

2.(c)

$$H(z) = \frac{1 - 2z^{-1}}{1 - 0.75z^{-1}} = \frac{1}{1 - 0.75z^{-1}} - \frac{2z^{-1}}{1 - 0.75z^{-1}}, \quad |z| > 0.75$$

Taking the inverse z transform, we have:

$$h[n] = (0.75)^n u[n] - 2(0.75)^{n-1} u[n-1]$$

2.(d)

As the ROC includes the unit circle, the DTFT exists and it is computed as:

$$H(e^{j\omega}) = \frac{1 - 2e^{-j\omega}}{1 - 0.75e^{-j\omega}}$$

2.(e)

$$\frac{Y(z)}{X(z)} = \frac{1-2z^{-1}}{1-0.75z^{-1}} \Rightarrow Y(z)(1-0.75z^{-1}) = X(z)(1-2z^{-1})$$
$$\Rightarrow y[n] - 0.75y[n-1] = x[n] - 2x[n-1]$$

2.(f)
As the ROC includes the unit circle, the system is stable.

2.(g)
As $h[n]=0$ for $n<0$, the system is causal.

Note that 2.(f) can also be addressed by investigating $h[n]$ obtained in 2.(c). While 2.(g) can be addressed by investigating $H(z)$ and its ROC.

3.

Taking z transform on $x[n]$ and $h[n]$, we get:

$$x[n] = a^n u[n] \leftrightarrow X(z) = \frac{1}{1 - az^{-1}}, \quad |z| > |a|$$

$$h[n] = b^n u[n] \leftrightarrow H(z) = \frac{1}{1 - bz^{-1}}, \quad |z| > |b|$$

Hence the z transform of $y[n]$ is:

$$Y(z) = X(z)H(z) = \frac{1}{1 - az^{-1}} \cdot \frac{1}{1 - bz^{-1}}, \quad |z| > |b|$$

Performing partial fraction expansion on $Y(z)$ yields:

$$Y(z) = \left(\frac{a}{a-b}\right) \frac{1}{1 - az^{-1}} - \left(\frac{b}{a-b}\right) \frac{1}{1 - bz^{-1}}, \quad |z| > |b|$$

Applying inverse z transform, we then obtain:

$$\begin{aligned} y[n] &= \left(\frac{a}{a-b}\right) a^n u[n] - \left(\frac{b}{a-b}\right) b^n u[n] \\ &= \frac{a^{n+1} - b^{n+1}}{a-b} u[n] \end{aligned}$$

The system is causal because $h[n] = 0$ for $n < 0$. Since the ROC of $H(z)$ is $|z| > |b|$ which does not include the unit circle, the system is not stable.

Note that $y[n]$ can also be obtained using the time-domain approach, i.e., convolution, as in Example 3.14, and partial fraction will not be involved.

$$
\begin{aligned}
y[n] &= \sum_{m=-\infty}^{\infty} x[m]h[n-m] \\
&= \sum_{m=-\infty}^{\infty} a^m u[m] b^{n-m} u[n-m] = \sum_{m=0}^{\infty} a^m b^{n-m} u[n-m] \\
&= \sum_{k=n}^{-\infty} a^{n-k} b^k u[k], \quad k = n-m \\
&= a^n \sum_{k=-\infty}^{n} (a^{-1}b)^k u[k]
\end{aligned}
$$

For $n<0$, $y[n]=0$. For $n \geq 0$, we obtain:

$$
y[n] = a^n \sum_{k=0}^{n} (a^{-1}b)^k = a^n \frac{1-(a^{-1}b)^{n+1}}{1-a^{-1}b}
$$

4.(a)

It is true and the explanation is given as follows. Recall:

$$X(z)|_{z=re^{j\omega}} = X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} \left(x[n]r^{-n}\right) e^{-j\omega n}$$

The DTFT of $(0.5)^n h[n]$ corresponds to the z transform of $h[n]$ at $z=2$. Since the system is causal and stable, the ROC must be of the form $|z|>|a|$ where $|a|<1$, which includes the point at $z=2$.

4.(b)

It is true because there is a zero on the unit circle. Note that at $z=e^{j\omega}$, the z transform is the DTFT.

4.(c)

It is false. There is at least one pole at $z=0.5$, which corresponds to a right-sided sequence, which is of infinite duration.

4.(d)

It cannot be determined. According to (8.21), it is true only when $H(z)=H^*(z^*)$. But we do not have sufficient information to get this result. For example, if there is another pole which is a complex number, then $h[n]$ is complex-valued.