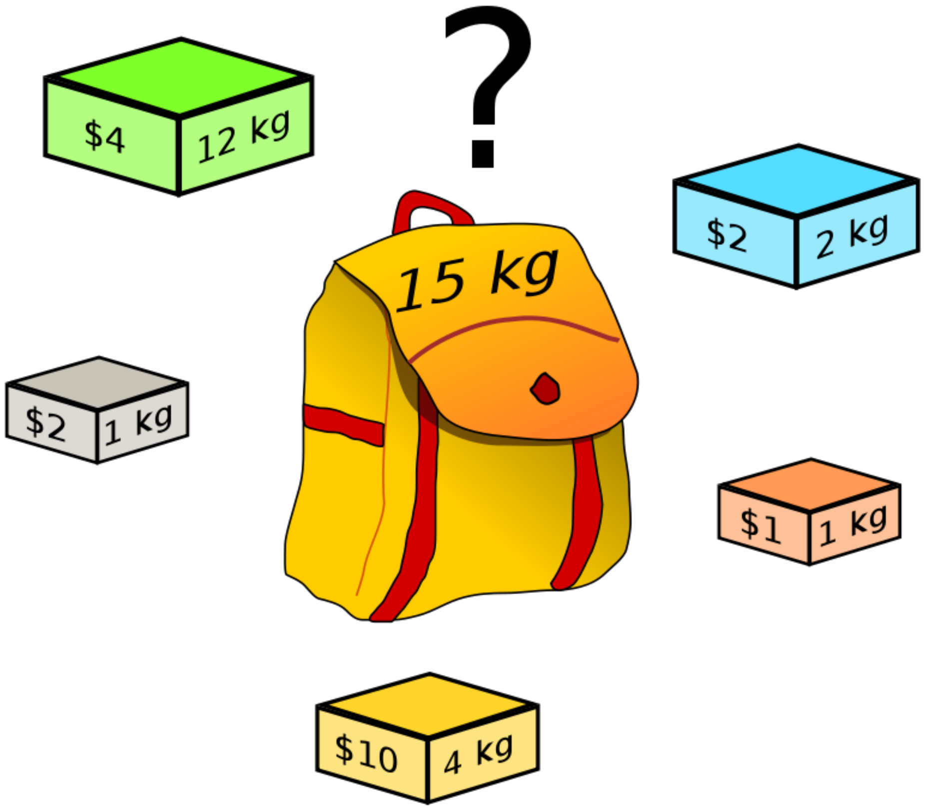
The Knapsack Problem

The 0,1 Knapsack Problem

There are N items. Item i ($i = 1, 2, 3, \ldots N$) has value v_i and weight w_i. The items need to be packed in a knapsack that has a limit of W on the total weight it can carry. Which items will you pack in the knapsack to maximize their total value?



Source: https://commons.wikimedia.org/wiki/File:Knapsack.svg
Author: Dake.
https://en.wikipedia.org/wiki/Knapsack_problem

The 0,1 Knapsack Problem (Cont'd) ILP Formulation

$$\max_{\{x_i\}} \sum_{i=1}^{N} v_i \, x_i$$

Subject to:

$$\sum_{i=1}^{N} w_i \, x_i \leq W$$

(all the w_is are positive integers)

$x_i = 0,$ or 1 $\qquad i = 1, 2, 3, \ldots, N$

Credit: S. E. Dreyfus and A. M. Law, *The art and theory of dynamic programming*, Academic Press, 1977.

LP Version

Suppose that the items are sugar, flour, oil, etc. so the items you carry in the knapsack do not have to be integers. How will you solve the following LP problem? (Think of a greedy algorithm.)

$$\max_{\{x_i\}} \sum_{i=1}^{N} v_i \, x_i$$

Subject to: $\sum_{i=1}^{N} w_i \, x_i \leq W$

$$1 \geq x_i \geq 0, \qquad i = 1, 2, 3, \ldots, N$$

Dynamic Programming (DP) Formulation of the 0,1 knapsack problem

Optimal value function

S(k,w) = the best (maximum) value of all possible solutions of weight less than or equal to *w* consisting only of 0 or 1 items of types *k, k+1, …, N.*

Recurrence relation

$$S(k,w) = \max_{x_k=0,1\ (\text{weight} \leq w)} [x_k\, v_k + S(k+1, w - x_k w_k)].$$

Boundary conditions

$$S(N+1, w) = 0, \qquad w \geq 0.$$

$$S(k,w) = -\infty, \qquad w < 0 \text{ for all } k.$$

Source: S. E. Dreyfus and A. M. Law, *The art and theory of dynamic programming,* Academic Press, 1977.

Example

Item	w_i	v_i
1	3	4
2	1	2
3	2	3
4	3	5
5	1	1

$N = 5$

$W = 6$

ILP Formulation

Maximize: $4X_1+2X_2+3X_3+5X_4+X_5$

Subject to:

$3X_1+X_2+2X_3+3X_4+X_5 \leq W$

X_i = 0 or 1, i = 1, 2, 3, 4, 5.

Homework on the LP Version

Consider the previous ILP problem
and replace the binary constraints
X_i = 0 or 1, i = 1, 2, 3, 4, 5 with the constraints $1 \geq x_i \geq 0$, i = 1, 2, 3, 4, 5. Solve this LP problem in two ways: (1) using Excel (See Excel file: spreadsheet_knapsack), and (2) using a greedy algorithm. Make sure that the results are consistent. Upload your solutions to Canvas/Discussions and discuss. Then, change some parameters and upload your solutions to Canvas.

ILP Solution

Item	x_i
1	0
2	1
3	1
4	1
5	0

Optimal value of the objective function = 10

See Excel file: spreadsheet_knapsack

Homework: Compare the results of ILP versus LP solutions for various input data and discuss differences.

DP Solution

Stage 5 (*k* = 5)

$S(5,0) = 0$

$P(5,0) = 0$ (For $w = 0$, at stage 5, you add nothing to the knapsack, $x_5 = 0$.)

$S(5,1) = 1$, $S(5,2) = 1$, $S(5,3) = 1$, $S(5,4) = 1$, $S(5,5) = 1$, $S(5,6) = 1$.

$P(5,w) = 1$ (For w =1, 2, …, 6, at stage 5, you add item 5, which has weight 1 and value 1, to the knapsack, $x_5 = 1$.)

DP Solution (cont'd)

Stage 4

$S(4,0) = 0$

$P(4,0) = 0$

(For $w = 0$, at stage 4, you add nothing to the knapsack, $x_4 = 0$.)

$S(4,1) = 0 + S(5,1) = 1.$ $P(4,1) = 0.$

$S(4,2) = 0 + S(5,1) = 1.$ $P(4,2) = 0.$

$S(4,3) = \max[0 + S(5,3), 5 + S(5,0)] = \max[0+1, 5+0] = 5.$ $P(4,3) = 1,$ ($x_4 = 1$).

$S(4,4) = \max[0 + S(5,4), 5 + S(5,1)] = \max[0+1, 5+1] = 6.$ $P(4,4) = 1.$

$S(4,5) = \max[0 + S(5,5), 5 + S(5,2)] = \max[0+1, 5+1] = 6.$ $P(4,5) = 1.$

$S(4,6) = \max[0 + S(5,6), 5 + S(5,3)] = \max[0+1, 5+1] = 6.$ $P(4,6) = 1.$

DP Solution (cont'd)

Stage 3

$S(3,0) = 0.$ $P(3,0) = 0.$

$S(3,1) = 0 + S(4,1) = 1.$ $P(3,1) = 0.$

$S(3,2) = \max[0 + S(4,2), 3 + S(4,0)] = \max[0+1,3+0] = 3.$ $P(3,2) = 1.$

$S(3,3) = \max[0 + S(4,3), 3 + S(4,1)] = \max[0+5,3+1] = 5.$ $P(3,3) = 0.$

$S(3,4) = \max[0 + S(4,4), 3 + S(4,2)] = \max[0+6,3+1] = 6.$ $P(3,4) = 0.$

$S(3,5) = \max[0 + S(4,5), 3 + S(4,3)] = \max[0+6,3+5] = 8.$ $P(3,5) = 1.$

$S(3,6) = \max[0 + S(4,6), 3 + S(4,4)] = \max[0+6,3+6] = 9.$ $P(3,6) = 1.$

DP Solution (cont'd)

Stage 2

$S(2,0) = 0.$ $P(2,0) = 0.$

$S(2,1) = \max[0 + S(3,1), 2 + S(3,0)] = \max[0+1,2+0] = 2.$ $P(2,1) = 1.$

$S(2,2) = \max[0 + S(3,2), 2 + S(3,1)] = \max[0+3,2+1] = 3.$ $P(2,2) = 0.$

$S(2,3) = \max[0 + S(3,3), 2 + S(3,2)] = \max[0+5,2+3] = 5.$ $P(2,3) = 0.$

$S(2,4) = \max[0 + S(3,4), 2 + S(3,3)] = \max[0+6,2+5] = 7.$ $P(2,4) = 1.$

$S(2,5) = \max[0 + S(3,5), 2 + S(3,4)] = \max[0+8,2+6] = 8.$ $P(2,5) = 1.$

$S(2,6) = \max[0 + S(3,6), 2 + S(3,5)] = \max[0+9,2+8] = 10.$ $P(2,6) = 1.$

DP Solution (cont'd)

Stage 1

$S(1,6) = \max[0 + S(2,6), 4 + S(2,3)]$ = max[0+10,4+5] = 10.

$P(1,6)$ = 0.

Optimal solution:

$P(1,6)$ = 0, so x_1 = 0; $P(2,6)$ = 1, so x_2 = 1; $P(3,5)$ = 1, so x_3 = 1;

$P(4,3)$ = 1, so x_4 = 1; $P(5,0) = 0$, so x_5 = 0.

This is consistent with the ILP optimal solution.

Note: there may be more than one optimal solution here as in the process, we have made some arbitrary choices when the two values in the max[] are equal.

Homework

1. Repeat the 0,1 knapsack problem solution process for other case(s). Choose $N > 5$ and $W > 6$. Use the ILP approach to verify your solution.
2. Compare the number of computations required by DP versus a brute force solution based on exhaustive search and explain why DP is more computationally efficient.
3. Write a program for the DP procedure for the 0,1 knapsack problem.

Generalizations of the Knapsack Problem

Bounded Knapsack Problem (BKP)

$$\max_{\{x_i\}} \sum_{i=1}^{N} v_i \, x_i$$

Subject to:

$$\sum_{i=1}^{N} w_i x_i \ \leq W$$

$$c_i \geq x_i \geq 0, \qquad i = 1, 2, 3, \dots, N$$

$$x_i \quad \text{integer}, \qquad i = 1, 2, 3, \dots, N$$

Bounded Knapsack Problem (BKP) [Another version]

$$\min_{\{x_i\}} \sum_{i=1}^{N} c_i \, x_i$$

Subject to:

$$\sum_{i=1}^{N} w_i \, x_i \geq W$$

$$c_i \geq x_i \geq 0, \qquad i = 1, 2, 3, \dots, N$$

$$x_i \quad \text{integer,} \qquad i = 1, 2, 3, \dots, N$$

Unbounded Knapsack Problem (UKP)

$$\max_{\{x_i\}} \sum_{i=1}^{N} v_i \, x_i$$

Subject to:

$$\sum_{i=1}^{N} w_i x_i \leq W$$

$$x_i \geq 0, \qquad i = 1, 2, 3, \dots, N$$

$$x_i \quad \text{integer}, \qquad i = 1, 2, 3, \dots, N$$

Unbounded Knapsack Problem (UKP) [Another version]

$$\min_{\{x_i\}} \sum_{i=1}^{N} c_i \, x_i$$

Subject to:

$$\sum_{i=1}^{N} w_i x_i \ \geq W$$

$$x_i \geq 0, \qquad\qquad i = 1, 2, 3, \dots, N$$

$$x_i \quad \text{integer}, \qquad i = 1, 2, 3, \dots, N$$

Two related short papers

M. Zukerman, L. Jia, T. D. Neame and G. J. Woeginger, “A polynomially solvable special case of the unbounded knapsack problem”, Operations Research Letters, vol. 29, no. 1, August 2001, pp. 13-16.

Vladimir G. Deineko and Gerhard J.Woeginger, “A well-solvable special case of the bounded knapsack problem”, Operations Research Letters, vol. 39, 2011, pp. 118-120.