In-Class Exercise 9

1. Let L be the number of flip(s) of a coin in an experiment until the first head occurs. Given a hypothesis $\mathcal{H}$ that the coin is fair, it is proposed to reject $\mathcal{H}$ if $L > r$.

   (a) Does it correspond to a one-tail or two-tail significance test?

   (b) Determine the value of r when the significance level is $\alpha \leq 0.05$.

   (c) What is the shortcoming of this significance test?

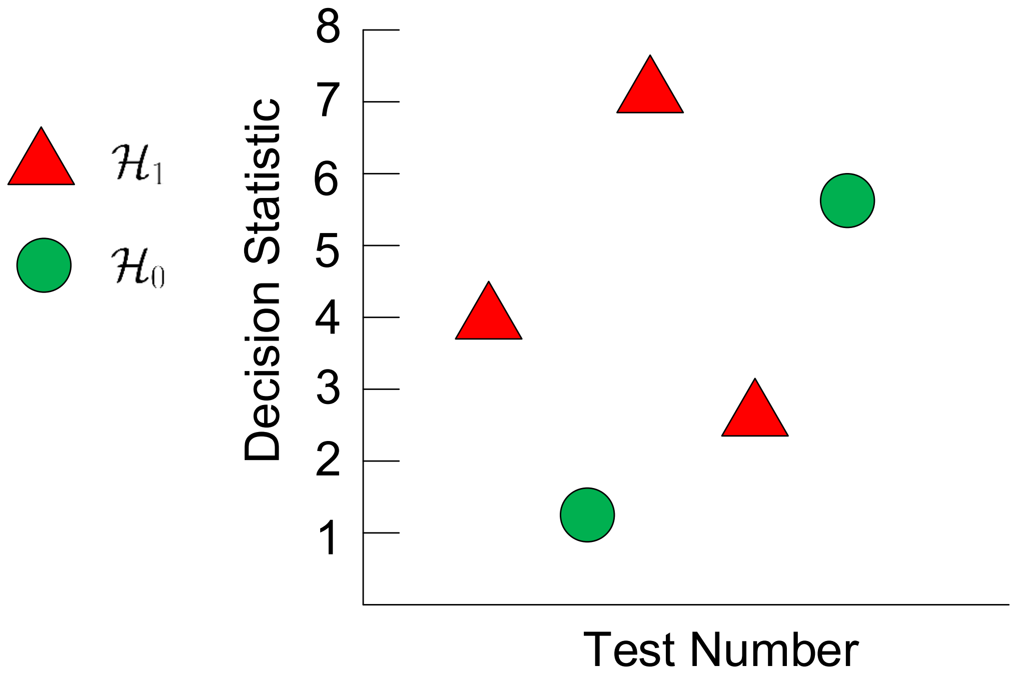
2. Suppose the time duration of a voice call is an exponential random variable T in min. and its probability density function (PDF) is:

$$p(t) = \begin{cases} \frac{1}{3}e^{-t/3}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Data calls tend to be longer than voice calls on average. A significance test is designed as follows. When a call is received, the hypothesis that the call is a voice call, is rejected, if the call duration is greater than t_0 min.

(a) Determine the mean value of voice call duration $\mathbb{E}\{T\}$.
(b) Express the significance level α in terms of t_0.
(c) Find the value of t_0 for $\alpha = 0.05$.

3. Suppose an experiment of binary detection is performed. There are 5 test cases where 3 and 2 correspond to signal presence $\mathcal{H}_1$ and signal absence $\mathcal{H}_0$, respectively. When the detection statistic is greater than a certain threshold, $\mathcal{H}_1$ is chosen, and otherwise $\mathcal{H}_0$ is chosen. Draw the receiver operating characteristic (ROC) curve.



4. For a test on a certain disease, suppose we know:

- 2% of the overall population has the disease.
- If a person has the disease, then the test has a 95% chance of correctly indicating he has it.
- If a person does not have the disease, then the test has a 10% chance of incorrectly indicating he has it.

Denote the event of having the disease as D and the event of test positive as T. Using signal detection interpretation, false alarm refers to incorrectly indicating an uninfected person as infected, while miss corresponds to incorrectly indicating an infected person as uninfected.

(a) Determine the probability of false alarm.
(b) Determine the probability of miss.
(c) Determine the probability of making an incorrect decision (detect uninfected as infected or infected as uninfected).

5. Consider the binary hypothesis testing problem using a single observation x:

$$\mathcal{H}_0 : x \text{ corresponds to } p_0(x) = \lambda_0 e^{-\lambda_0 x},$$

$$\mathcal{H}_1 : x \text{ corresponds to } p_1(x) = \lambda_1 e^{-\lambda_1 x}, \lambda_1 > \lambda_0 > 0,\ x \geq 0$$

That is, we need to choose between two exponential distributions with parameters λ_0 and λ_1.

Based on the Neyman-Pearson theorem, suggest a decision statistic for this binary hypothesis test. It is assumed that λ_0 and λ_1 are unknown.

6. In a communication system, a transmitter sends a binary signal to a receiver. The bits "0" and "1" are sent as $-v$ and v, with probabilities p and $1-p$, respectively. Using a single observation x, the binary signal detection problem is modelled as choosing between the two hypotheses:

$$\mathcal{H}_0 : x = -v + w, \quad w \sim \mathcal{N}(0, \sigma^2)$$

$$\mathcal{H}_1 : x = v + w$$

The error probability defined as:

$$P_{\text{Error}} = P(A_1|\mathcal{H}_0) \cdot P(\mathcal{H}_0) + P(A_0|\mathcal{H}_1) \cdot P(\mathcal{H}_1)$$

where A_0 and A_1 correspond to the events of choosing "0" and "1", respectively. To minimize P_{Error}, it is suggested:

$$x \in A_0 \quad \text{if} \quad \frac{P_{x|\mathcal{H}_0}(x)}{P_{x|\mathcal{H}_1}(x)} \geq \frac{P(\mathcal{H}_1)}{P(\mathcal{H}_0)}, \quad \text{otherwise} \quad x \in A_1$$

Determine the corresponding decision rule.

Solution

1.(a)

It is a one-tail (right-tail) significance test.

1.(b)

Denote H and T as Head and Tail, respectively. The hypothesis of a fair coin means that $p(H) = p(T) = 0.5$.

Rejecting $\mathcal{H}$ if $L > r$ means that there is no head up to the rth trial, i.e., all tosses give T. Given $\mathcal{H}$, this probability is:

$$p(\text{all } r \text{ tosses give } T) = (0.5)^r = \alpha \leq 0.05$$

$$\Rightarrow \log((0.5)^r) \leq \log(0.05) \Rightarrow r \geq 4.32 = 5$$

That is, $\mathcal{H}$ is rejected if the number of tosses is at least 6. Or $\mathcal{H}$ is accepted if $L = 1, 2, 3, 4, 5$.

Note that for discrete random variables, we may not be able to set α exactly equal to an arbitrary value.

The experiment corresponds to geometric distribution. Recall (2.6) and Example 2.7:

$$p(r) = P(R = r) = (1-p)^{r-1}p, \quad 1 \leq r < \infty$$

$$F(r) = P(R \leq r) = \sum_{i=1}^{r} q^{i-1}p = \frac{p(1-q^r)}{1-q} = 1-(1-p)^r$$

With $p = 0.5$, the probability that $\mathcal{H}$ is accepted is:

$$F(5) = 1-(1-0.5)^5 = 1-(0.5)^5 = 0.9688$$

This aligns with

$$\alpha = (0.5)^5 = 0.0313$$

1.(c)

This significance test accepts $\mathcal{H}$ if $L = 1, 2, 3, 4, 5$. When the coin is biased to H with $p > 0.5$, $\mathcal{H}$ has a higher chance to be accepted because $F(5)$ will be larger. As a result, there is a higher probability of accepting $\mathcal{H}$ when it is false.

Note that it corresponds to the limitation of significance test, i.e., it has no control on the probability of accepting the null hypothesis when it is false.

2.(a)

Recall (2.17) and Question 1 of In-Class Exercise 5:

$$p(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$\mathbb{E}\{X\} = \frac{1}{\lambda}$$

Clearly, now $\lambda = 1/3$, hence we obtain:

$$\mathbb{E}\{T\} = 3$$

2.(b)

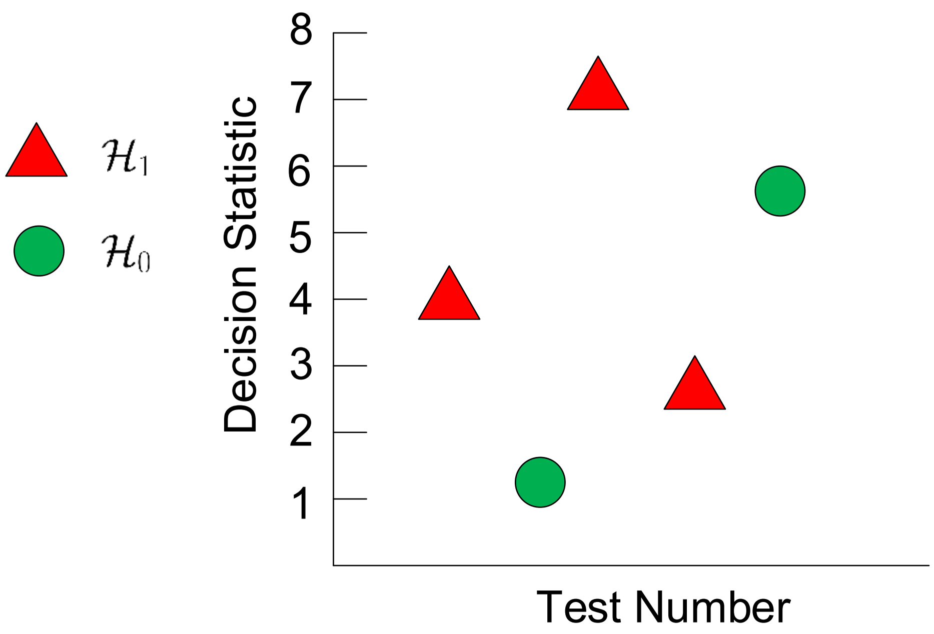
The rejection region corresponds to $T > t_o$. As the significance level α is the probability of rejecting $\mathcal{H}$ when it is true, we have:

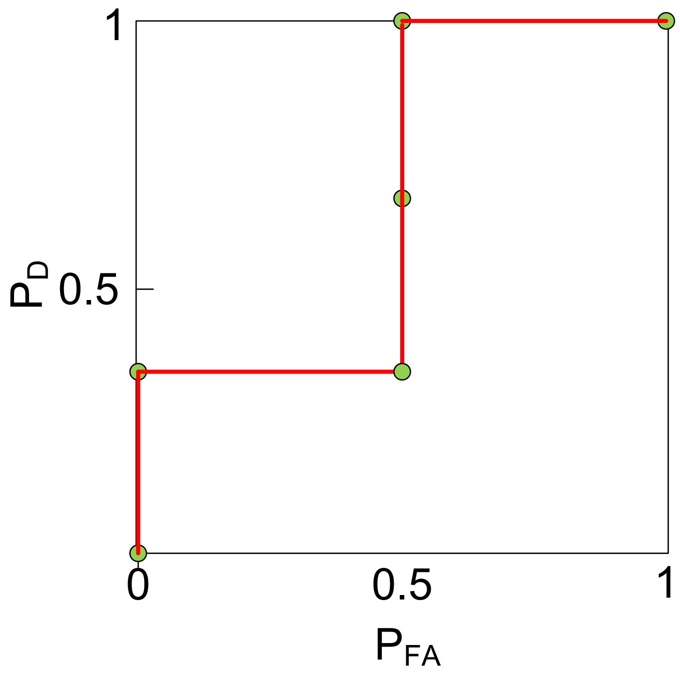
$$\alpha = P(T > t_0) = \int_{t_0}^{\infty} p(t)dt = \int_{t_0}^{\infty} \frac{1}{3}e^{-t/3}dt = e^{-t_0/3}$$

2.(c)

$$\alpha = 0.05 = e^{-t_0/3} \Rightarrow t_0 = -3\ln(0.05) = 8.9872$$

3.





We start with a threshold with minimum value of 0, producing the first ROC point $(P_{\mathrm{FA}}, P_{\mathrm{D}}) = (1, 1)$.

Increasing the threshold gradually, we then have the points: $(0.5, 1)$, $(0.5, 2/3)$, $(0.5, 1/3)$, $(0, 1/3)$, $(0, 0)$.

Connecting these 6 points yields the ROC curve.

4.(a)
The probability of false alarm is:

$$P_{\mathrm{FA}} = P(T|\overline{D}) = 0.1$$

4.(b)
The probability of miss is:

$$P_{\mathrm{M}} = P(\overline{T}|D) = 0.05$$

4.(c)
Combining the false alarm and miss scenarios, the probability of making an error is then:

$$P_{\mathrm{Error}} = P(T|\overline{D}) \cdot P(\overline{D}) + P(\overline{T}|D) \cdot P(D) = 0.099$$

Note that $1 - P_{\mathrm{Error}}$ is the accuracy defined in classification (See Example 1.16).

5.

We apply (5.6) to obtain:

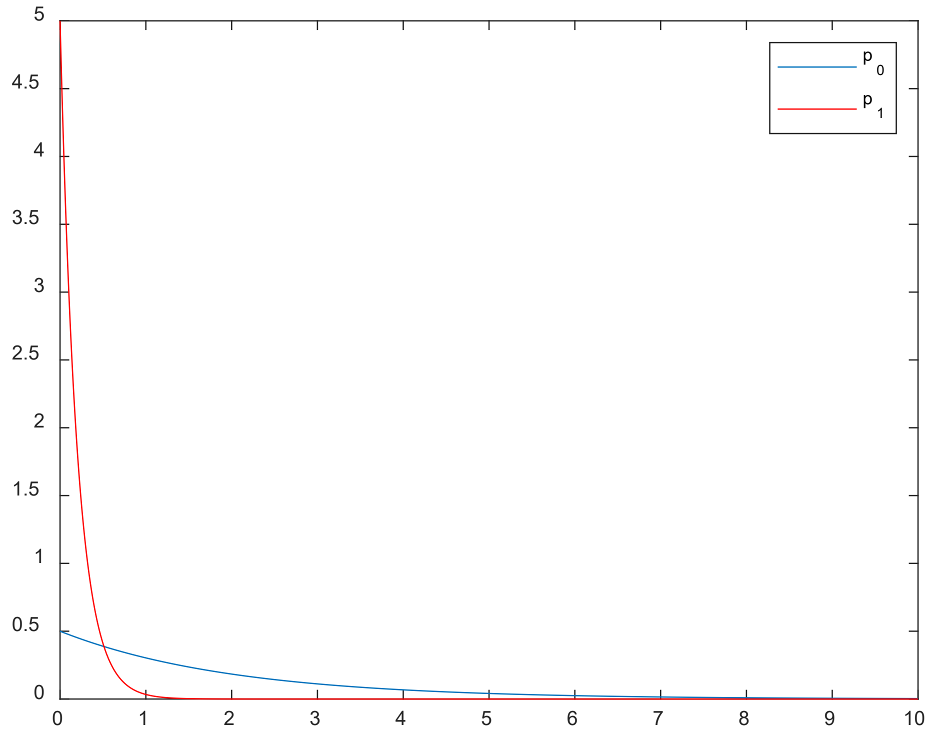
$$L(x) = \frac{p(x;\mathcal{H}_1)}{p(x;\mathcal{H}_0)} = \frac{\lambda_1 e^{-\lambda_1 x}}{\lambda_0 e^{-\lambda_0 x}} = \frac{\lambda_1}{\lambda_0} e^{-(\lambda_1-\lambda_0)x} > \gamma_{\mathrm{NP}}$$

$$\begin{aligned}
&\Rightarrow e^{-(\lambda_1-\lambda_0)x} > \frac{\lambda_0}{\lambda_1}\gamma_{\mathrm{NP}} \\
&\Rightarrow (\lambda_0-\lambda_1)x > \ln(\lambda_0\gamma_{\mathrm{NP}}/\lambda_1) \\
&\Rightarrow x < \frac{\ln(\lambda_0\gamma_{\mathrm{NP}}/\lambda_1)}{\lambda_1-\lambda_0} = \gamma
\end{aligned}$$

That is, we can directly use the measurement x as the decision statistic. If $x < \gamma$, then we choose $\mathcal{H}_1$. Otherwise, $\mathcal{H}_0$ is chosen.

Since $\lambda_1 > \lambda_0$, this means that it is more probable a random variable drawn from $p_0(x)$ is larger than that from $p_1(x)$. Note also that the mean values of the random variables are $1/\lambda_1$ and $1/\lambda_0$, and apparently, $1/\lambda_0 > 1/\lambda_1$.

For example, the PDFs for $\lambda_1 = 5$ and $\lambda_0 = 0.5$ are illustrated as:



6.

We have:

$$P(\mathcal{H}_0) = p$$

$$P(\mathcal{H}_1) = 1 - p$$

$$P_{x|\mathcal{H}_0}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x+v)^2}$$

$$P_{x|\mathcal{H}_1}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-v)^2}$$

Then

$$\frac{e^{-\frac{1}{2\sigma^2}(x+v)^2}}{e^{-\frac{1}{2\sigma^2}(x-v)^2}} = e^{-\frac{1}{2\sigma^2}[(x+v)^2-(x-v)^2]} = e^{-\frac{2v}{\sigma^2}x} \geq \frac{1-p}{p}$$

$$\Rightarrow e^{\frac{2v}{\sigma^2}x} \leq \frac{p}{1-p} \Rightarrow x \leq \frac{\sigma^2}{2v} \ln\left(\frac{p}{1-p}\right)$$

Hence the decision rule is:

$$x \in A_0 \quad \text{if} \quad x \leq \frac{\sigma^2}{2v} \ln\left(\frac{p}{1-p}\right), \quad \text{otherwise} \quad x \in A_1$$

This is known as maximum a posteriori (MAP) test.

For equal probabilities of sending “0” and “1”, we have $P(\mathcal{H}_0) = P(\mathcal{H}_1) = 0.5$. Then we choose $x \in A_0 \quad \text{if} \quad x \le 0$.

For $p > 0.5$ where sending “0” is of higher probability, $\ln(p/(1-p))$ is positive. That is, the decision threshold is shifted to right.