Homework #7 : Solutions and Keys

Prob. 1

(iii) $e^{-an}\cos\Omega n\, u(n)$, $a>0$

Use Euler identity,

$$\cos\Omega n = \frac{e^{j\Omega n}+e^{-j\Omega n}}{2}$$

(iv) $7\left(\frac{1}{2}\right)^n \cos\left[\frac{2\pi n}{6}+\frac{\pi}{4}\right]u(n)$

Use trig. identity

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

In other words,

$$\cos\left[\frac{2\pi n}{6}+\frac{\pi}{4}\right] = \cos\frac{2\pi n}{6}\cos\frac{\pi}{4} - \sin\frac{2\pi n}{6}\sin\frac{\pi}{4}$$

$$= \frac{\sqrt{2}}{2}\left(\cos\frac{2\pi n}{6} - \sin\frac{2\pi n}{6}\right)$$

Prob. 2

(ii)
$$X(z) = \frac{z^{-1}(0.5-z^{-1})}{(1-0.5z^{-1})(1-0.8z^{-1})^2}$$

$$= \frac{z(0.5z-1)}{(z-0.5)(z-0.8)^2}$$

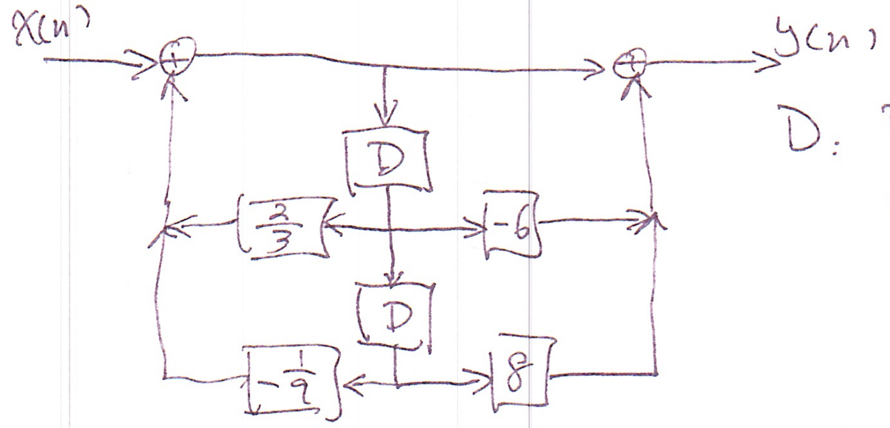
$$= \frac{1}{2}\,\frac{z(z-2)}{(z-0.5)(z-0.8)^2}$$

Do partial fraction on

$$\frac{X(z)}{z} = \frac{1}{2}\,\frac{(z-2)}{(z-0.5)(z-0.8)^2} = \frac{A}{z-0.5} + \frac{B_1}{z-0.8} + \frac{B_2}{(z-0.8)^2}$$

A Problem for Tutorial, Wednesday, 23/11

Prob 10.18. An LTI system is given by the following block diagram

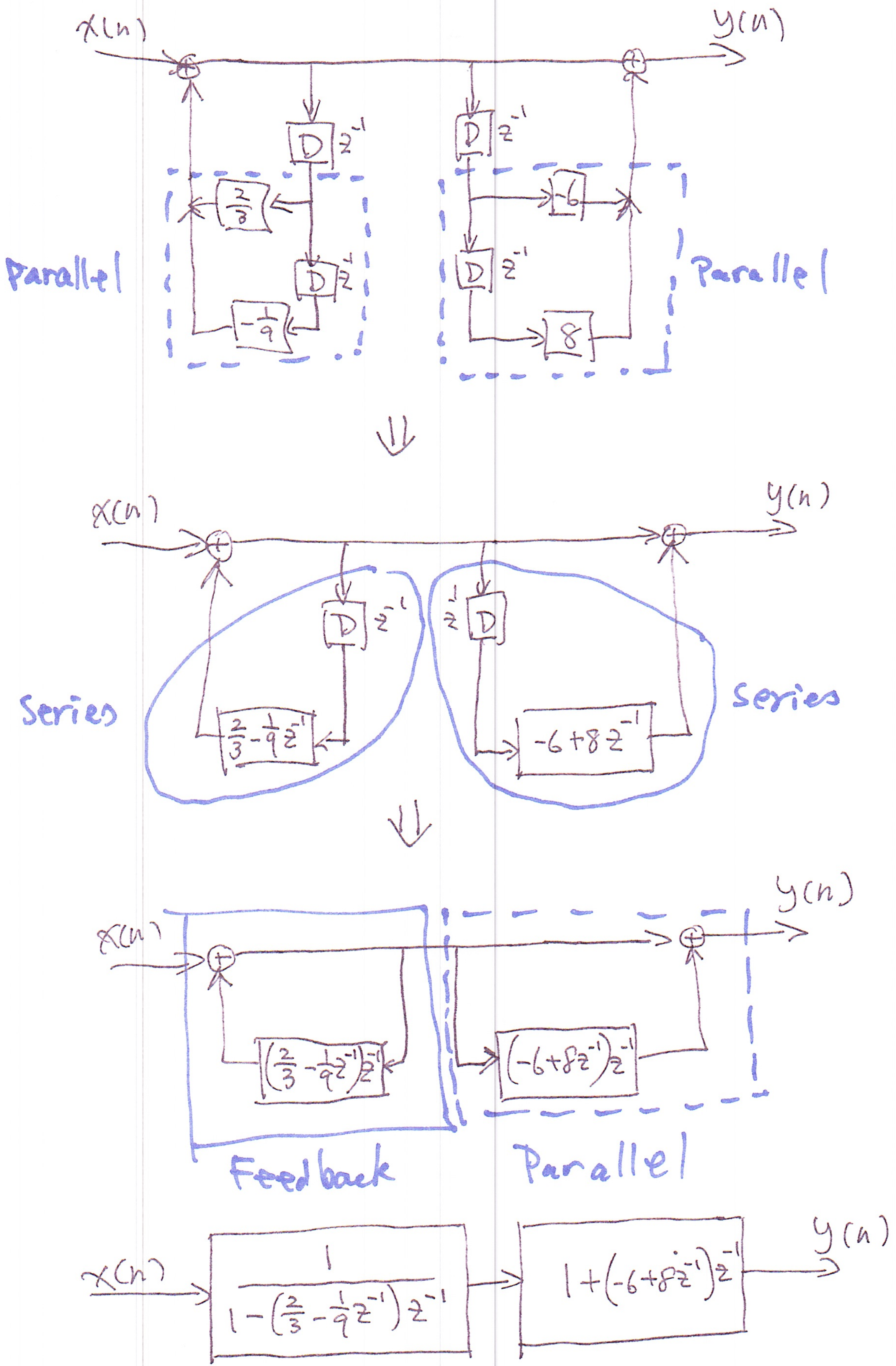


D: Delay element, with a transfer function $\frac{1}{z}$.

During the lectures, we have discussed how to use transfer function to calculate output response, and to obtain a realization. In the discussion of Fourier transform, we discussed the central role of the transfer function in system analysis, including its roles in solving diff. equations, in calculating output response, ~~&~~ and impulse response, in obtain realizations, and in obtaining differential equation descriptions. This problem is meant to use transfer function to achieve the following objectives

(i) Find a difference equation relating $x(n)$ and $y(n)$,

(ii) Find the impulse response of the system.



x(n)
y(n)
D
z^{-1}
$\frac{2}{3}$
$-\frac{1}{9}$
-6
8
Parallel
Parallel
Series
Series
$\frac{2}{3}-\frac{1}{9}z^{-1}$
$-6+8z^{-1}$
$\left(\frac{2}{3}-\frac{1}{9}z^{-1}\right)z^{-1}$
$\left(-6+8z^{-1}\right)z^{-1}$
Feedback
Parallel
$\frac{1}{1-\left(\frac{2}{3}-\frac{1}{9}z^{-1}\right)z^{-1}}$
$1+\left(-6+8z^{-1}\right)z^{-1}$

$$\frac{Y(z)}{X(z)} = H(z) = \frac{1+(-6+8z^{-1})z^{-1}}{1-\left(\frac{2}{3}-\frac{1}{9}z^{-1}\right)z^{-1}}$$

$$= \frac{1-6z^{-1}+8z^{-2}}{1-\frac{2}{3}z^{-1}+\frac{1}{9}z^{-2}}$$

$$\left(1-\frac{2}{3}z^{-1}+\frac{1}{9}z^{-2}\right)Y(z) = \left(1-6z^{-1}+8z^{-2}\right)X(z)$$

(i) Difference Equation

$$y(n)-\frac{2}{3}y(n-1)+\frac{1}{9}y(n-2)$$
$$= x(n)-6x(n-1)+8x(n-2)$$

(ii) Impulse Response

$$H(z) = \frac{1-6z^{-1}+8z^{-2}}{1-\frac{2}{3}z^{-1}+\frac{1}{9}z^{-2}}$$

$$= \frac{z^2-6z+8}{z^2-\frac{2}{3}z+\frac{1}{9}}$$

$$= \frac{z^2-6z+8}{\left(z-\frac{1}{3}\right)^2}$$

$$= A_0 + \frac{A_1}{z-\frac{1}{3}} + \frac{A_2}{\left(z-\frac{1}{3}\right)^2}$$

$$A_0 = 1$$

$$A_2 = \left(z-\tfrac{1}{3}\right)^2 H(z)\Big|_{z=\frac{1}{3}} = \left(z^2-6z+8\right)\Big|_{z=\frac{1}{3}}$$

$$= \frac{1}{9} - 2 + 8 = 6 + \frac{1}{9} = \frac{55}{9}$$

$$A_1 = \frac{d}{dz}\left(z-\tfrac{1}{3}\right)^2 H(z)\Big|_{z=\frac{1}{3}} = \frac{d}{dz}\left(z^2-6z+8\right)\Big|_{z=\frac{1}{3}}$$

$$= 2z-6\Big|_{z=\frac{1}{3}} = \frac{2}{3} - 6 = -\frac{16}{3}$$

$$\cancel{H(z) = A_0 + \quad 1 +}$$

$$H(z) = 1 - \frac{16}{3} z^{-1} \underbrace{\frac{z}{z-\frac{1}{3}}}_{\left(\frac{1}{3}\right)^n u(n)} + \frac{55}{9} z^{-1} \cdot 3 \underbrace{\frac{\frac{1}{3}z}{\left(z-\frac{1}{3}\right)^2}}_{n\left(\frac{1}{3}\right)^n u(n)}$$

$$h(n) = \delta(n) - \left(\tfrac{1}{3}\right)^{n-1} u(n-1) + \frac{55}{3}(n-1)\left(\tfrac{1}{3}\right)^{n-1} u(n-1)$$

$$= \delta(n) + \left[\frac{55}{3}(n-1) - 1\right]\left(\tfrac{1}{3}\right)^{n-1} u(n-1)$$

$$= \delta(n) + \left[\frac{55}{3}n - \frac{58}{3}\right]\left(\tfrac{1}{3}\right)^{n-1} u(n-1)$$

* For both problems, the key is to find $H(z)$ first !!

Problem 10.20

Solution:

$$y(n-1)+2y(n)=x(n)$$

Taking z-transform, we obtain

$$z^{-1}Y(z)+y(-1)+2Y(z)=X(z)$$

$$(2+z^{-1})Y(z)=X(z)-y(-1)$$

$$Y(z)=\frac{1}{2+z^{-1}}X(z)-\frac{y(-1)}{2+z^{-1}}$$

(a) zero-input response: $y(-1)=2$

$$Y_{zI}(z)=-\frac{2}{2+z^{-1}}=-\frac{1}{1+\frac{1}{2}z^{-1}}$$

$$y_{zI}(n)=-\left(-\frac{1}{2}\right)^n u(n)$$

(b) zero-state response: $x(n)=\left(\frac{1}{4}\right)^n u(n)$

$$X(z)=\frac{1}{1-\frac{1}{4}z^{-1}}$$

$$Y_{zs}(z)=\frac{1}{(2+z^{-1})(1-\frac{1}{4}z^{-1})}=\frac{z^2}{(2z+1)(z-\frac{1}{4})}$$

$$=\frac{1}{2}\frac{z^2}{(z+\frac{1}{2})(z-\frac{1}{4})}$$

Consider $\frac{Y_{zs}(z)}{z}=\frac{1}{2}\frac{z}{(z+\frac{1}{2})(z-\frac{1}{4})}=\frac{A}{z+\frac{1}{2}}+\frac{B}{z-\frac{1}{4}}$

$$\Rightarrow Y_{zs}(z)=A\frac{z}{z+\frac{1}{2}}+B\frac{z}{z-\frac{1}{4}}$$

$$y_{zs}(n)=\left[A\left(-\frac{1}{2}\right)^n+B\left(\frac{1}{4}\right)^n\right]u(n)$$

Total response

$$y(n) = y_{ZI}(n) + y_{ZS}(n)$$
$$= \left[-\left(-\tfrac{1}{2}\right)^n + A\left(-\tfrac{1}{2}\right)^n + B\left(\tfrac{1}{4}\right)^n \right] u(n)$$

Problem 10.36

$$y(n-1) - \frac{10}{3} y(n) + y(n+1) = x(n)$$

$$z^{-1} Y(z) - \frac{10}{3} Y(z) + z Y(z) = X(z)$$

$$Y(z) = \frac{1}{z^{-1} - \frac{10}{3} + z} X(z)$$

$$= \frac{z}{z^2 - \frac{10}{3} z + 1} X(z)$$

$$H(z) = \frac{z}{z^2 - \frac{10}{3} z + 1}$$

Do partial fraction on $\frac{H(z)}{z}$.