Solutions to EE3210 Tutorial 11 Problems

Problem 1: Applying the Fourier transform to both sides of the differential equation, and using the properties of differentiation in time and linearity, we have

$$(j\omega)^2 Y(\omega) + 4j\omega Y(\omega) + 3Y(\omega) = j\omega X(\omega) + 2X(\omega). \tag{1}$$

Rearranging (1), we obtain the frequency response of the system as

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{j\omega + 2}{(j\omega)^2 + 4j\omega + 3} = \frac{j\omega + 2}{(j\omega + 1)(j\omega + 3)}.$$

Given $x(t) = e^{-t}u(t)$, we derive the Fourier transform of $x(t)$ as

$$X(\omega) = \int_0^{+\infty} e^{-t} e^{-j\omega t} dt = \frac{1}{j\omega + 1}$$

(Alternatively, we can determine $\mathcal{F}\{e^{-t}u(t)\}$ from the table of basic continuous-time Fourier transform pairs available on Page 332, Table 4.2, of the textbook.)

Then, we have

$$Y(\omega) = H(\omega)X(\omega) = \frac{j\omega + 2}{(j\omega + 1)^2(j\omega + 3)}.$$

Making the substitution of v for $j\omega$, we obtain the rational function

$$G(v) = \frac{v + 2}{(v + 1)^2(v + 3)}.$$

The partial-fraction expansion for this function is

$$G(v) = \frac{A_1}{v + 1} + \frac{A_2}{(v + 1)^2} + \frac{A_3}{v + 3}$$

where

$$A_3 = [(v + 3)G(v)]|_{v=-3} = -\frac{1}{4}$$

$$A_2 = [(v + 1)^2 G(v)]|_{v=-1} = \frac{1}{2}$$

$$A_1 = \left\{ \frac{d}{dv} \left[(v + 1)^2 G(v)\right] \right\} \Big|_{v=-1} = \frac{1}{4}$$

Therefore,

$$Y(\omega) = \frac{\frac{1}{4}}{j\omega + 1} + \frac{\frac{1}{2}}{(j\omega+1)^2} + \frac{-\frac{1}{4}}{j\omega + 3}.$$

Taking inverse Fourier transforms by inspection, we get

$$y(t) = \left[\frac{1}{4}e^{-t} + \frac{1}{2}te^{-t} - \frac{1}{4}e^{-3t}\right] u(t).$$

Problem 2: Applying the Fourier transform to both sides of the difference equation, and using the properties of time shift and linearity, we have

$$Y[\Omega] - \frac{3}{4}e^{-j\Omega}Y[\Omega] + \frac{1}{8}e^{-j\Omega 2}Y[\Omega] = 2X[\Omega] \tag{2}$$

Rearranging (2), we obtain the frequency response of the system as

$$H[\Omega] = \frac{Y[\Omega]}{X[\Omega]} = \frac{2}{1 - \frac{3}{4}e^{-j\Omega} + \frac{1}{8}e^{-j\Omega 2}} = \frac{2}{\left(1 - \frac{1}{2}e^{-j\Omega}\right)\left(1 - \frac{1}{4}e^{-j\Omega}\right)}.$$

Given $x[n] = \left(\frac{1}{4}\right)^n u[n]$, we derive the Fourier transform of $x[n]$ as

$$X[\Omega] = \sum_{n=0}^{+\infty} \left(\frac{1}{4}\right)^n e^{-j\Omega n} = \frac{1}{1 - \frac{1}{4}e^{-j\Omega}}.$$

(Alternatively, we can determine $\mathcal{F}\left\{\left(\frac{1}{4}\right)^n u[n]\right\}$ from the table of basic discrete-time Fourier transform pairs available on Page 395, Table 5.2, of the textbook.)

Then, we have

$$Y[\Omega] = H[\Omega]X[\Omega] = \frac{2}{\left(1 - \frac{1}{2}e^{-j\Omega}\right)\left(1 - \frac{1}{4}e^{-j\Omega}\right)^2}.$$

Making the substitution of v for $e^{-j\Omega}$, we obtain the rational function

$$G(v) = \frac{2}{\left(1 - \frac{1}{2}v\right)\left(1 - \frac{1}{4}v\right)^2} = \frac{-64}{(v-2)(v-4)^2}.$$

The partial-fraction expansion for this function is

$$G(v) = \frac{A_1}{v-4} + \frac{A_2}{(v-4)^2} + \frac{A_3}{v-2}$$

where

$$A_3 = [(v-2)\,G(v)]\Big|_{v=2} = -16$$

$$A_2 = \left[(v-4)^2\,G(v)\right]\Big|_{v=4} = -32$$

$$A_1 = \left\{\frac{d}{dv}\left[(v-4)^2\,G(v)\right]\right\}\Big|_{v=4} = 16$$

Therefore,

$$G(v) = \frac{16}{v-4} + \frac{-32}{(v-4)^2} + \frac{-16}{v-2} = \frac{-4}{1-\frac{1}{4}v} + \frac{-2}{\left(1-\frac{1}{4}v\right)^2} + \frac{8}{1-\frac{1}{2}v}$$

and

$$Y[\Omega] = \frac{-4}{1-\frac{1}{4}e^{-j\Omega}} + \frac{-2}{\left(1-\frac{1}{4}e^{-j\Omega}\right)^2} + \frac{8}{1-\frac{1}{2}e^{-j\Omega}}.$$

Taking inverse Fourier transforms by inspection, we get

$$y[n] = \left[-4\left(\frac{1}{4}\right)^n - 2(n+1)\left(\frac{1}{4}\right)^n + 8\left(\frac{1}{2}\right)^n\right]u[n].$$