Solution

1.
Apparently, Bernoulli distribution can be applied with probability of success or hit equals 0.2. Let n be the required number of shots, we can establish:

$$P(\geq 1 \text{ hit in } n \text{ shots}) \geq 0.9 \Leftrightarrow P(0 \text{ hit in } n \text{ shots}) < 0.1 \Rightarrow (0.8)^n < 0.1 \Rightarrow n = 11$$

2.
Denote the random variable and its variance be X and σ^2, respectively, we have:

$$X \sim \mathcal{N}(72, \sigma^2)$$

Converting to standard Gaussian RV yields:

$$0.025 = P(X \geq 96) = P(\tfrac{X-72}{\sigma} \geq \tfrac{96-72}{\sigma}) = 1 - F(\tfrac{24}{\sigma}) \Rightarrow F(\tfrac{24}{\sigma}) = 0.975$$

Using MATLAB command `norminv(0.975)`, we have

$$\tfrac{24}{\sigma} = 1.96 \Rightarrow \sigma = 12.2451$$

Using MATLAB command `normcdf`, we obtain:

$$P(60 \leq X \leq 84) = P(\tfrac{60-72}{12.2451} \leq \tfrac{X-72}{12.2451} \leq \tfrac{84-72}{12.2451}) = F(\tfrac{12}{12.2451}) - F(\tfrac{-12}{12.2451}) = 0.6729$$

3.
If the equation has real roots, the discriminant should be non-negative or $X^2 - 4 \geq 0$. Hence the probability is

$$P(X^2 - 4 \geq 0) = P(X \geq 2) + P(X \leq -2) = 0.8$$

4.
To facilitate the probability computation, we assign X as the value of the absolute difference, and we have $0 \leq X \leq 5$. The sample space contains 36 outcomes and the PMF of each value of X is determined as:

$$p(0) = \frac{6}{36} = \frac{1}{6}, \quad p(1) = \frac{10}{36} = \frac{5}{18}, \quad p(2) = \frac{8}{36} = \frac{2}{9}, \quad p(3) = \frac{6}{36} = \frac{1}{6}, \quad p(4) = \frac{4}{36} = \frac{1}{9},$$
$$p(5) = \frac{2}{36} = \frac{1}{18}$$

The required probability is:

$$p(1) + p(3) + p(5) = \frac{1}{2}$$

5.
We have $n = 100$ and $p = 0.1$, assuming that a head corresponds to a success. The required probability is:

$$p(10) = C(100, 10)0.1^{10}(1 - 0.9)^{100-10} = 0.1319$$

Using Poisson approximation, we have $\lambda = np = 10$, the probability is:

$$p(10) = e^{-50}\frac{\lambda^{10}}{10!} = 0.1251$$

It is because the approximation will be accurate for large n and small p. However, in this case, $n = 100$ is large and $p = 0.1$ is small.

6.(a)
Using the fact that the sum of all PMFs should be equal to 1, we have:

$$\left[\frac{1}{\alpha} + \frac{1}{\alpha^2} + \cdots\right] = \frac{1/\alpha}{1 - 1/\alpha} = \frac{1}{\alpha - 1} = 1 - \frac{1}{2} = \frac{1}{2} \Rightarrow 2 = \alpha - 1 \Rightarrow \alpha = 3$$

6.(b)

$$F(0) = \frac{1}{2}$$

For $x \geq 1$, we

$$F(x) = \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{3^x} = \frac{1}{2} + \frac{1}{2}\left[1 - \frac{1}{3^x}\right]$$

Combining the results yields:

$$F(x) = \begin{cases} \frac{1}{2}, & x < 1 \\ \frac{1}{2}\left[2 - \frac{1}{3^x}\right], & x \geq 1 \end{cases}$$

7.(a)

$$\mathbb{E}\{X\} = \tfrac{1}{4}(-2 - 1 + 0 + 1) = -0.5$$

$$\mathbb{E}\{X^2\} = \tfrac{1}{4}((-2)^2 + (-1)^2 + 0 + 1^2) = 1.5$$

Applying (2.23) yields:

$$\mathrm{var}(X) = \mathbb{E}\{X^2\} - (\mathbb{E}\{X\})^2 = 1.25$$

7.(b)
From $Y = |2X - 4|$, we have

$$P(Y = y) = \begin{cases} \frac{1}{4}, & y = 2, 4, 6, 8 \\ 0, & \text{otherwise} \end{cases}$$

Analogous to (a), we obtain:

$$\mathbb{E}\{Y\} = \tfrac{1}{4}(2 + 4 + 6 + 8) = 5$$

$$\mathbb{E}\{Y^2\} = 30 \Rightarrow \text{var}(Y) = \mathbb{E}\{Y^2\} - (\mathbb{E}\{Y\})^2 = 5$$

8.
Define this event as A and its possible combinations include 5+5 or 6+4 or 4+6.

The probability of A in one trial is:

$$\mathrm{P(A)} = 3/36 = 1/12$$

Event A can occur in the 1st, 2nd, 3rd, 4th or 5th trial, hence the probability is

$$\frac{1}{12} + \frac{11}{12}\frac{1}{12} + \left(\frac{11}{12}\right)^2 \frac{1}{12} + \left(\frac{11}{12}\right)^3 \frac{1}{12} + \left(\frac{11}{12}\right)^4 \frac{1}{12} = 0.3528$$

9.(a)
Differentiating the CDF yields the PDF:

$$p(x) = \begin{cases} 0.25, & -2 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

9.(b)

$$\mathbb{E}\{X^4\} = \int_{-\infty}^{\infty} x^4 p(x) dx = \int_{-2}^{2} 0.25 x^4 dx = 3.2$$

10.(a)
With the use of binomial distribution with success probability 0.6, the PMF of X is:

$$p(x) = P(X = x) = C(m, x)(0.6)^x(0.4)^{m-x}, \quad 0 \leq x \leq m$$

10.(b)
Using the result of Example 2.30, we know that $\mathbb{E}\{X\} = m \cdot 0.6 = 0.6m$.

Alternatively, we can define

$$X_j = \begin{cases} 1, & \text{Peter wins the jth game} \\ 0, & \text{otherwise} \end{cases}$$

Then we have $X = X_1 + X_2 + \cdots X_m$. Since $\mathbb{E}\{X_j\} = 0.6$ for all j, we also obtain $\mathbb{E}\{X\} = 0.6m$.

11.(a)
Denote H and T as Head and Tail, respectively. There are four possible outcomes:

$A_1 = \{T, T, T\}$, gain -6,000
$A_2 = \{H, T, T\}$ with any order, gain -3,000
$A_3 = \{H, H, T\}$ with any order, gain 0
$A_4 = \{H, H, H\}$, gain 3,000

Based on the binomial distribution, we have

$$p(k) = \begin{cases} (1-p)^3, & k = -6000 \\ 3p(1-p)^2, & k = -3000 \\ 3p^2(1-p), & k = 0 \\ p^3, & k = 3000 \end{cases}$$

11.(b)
$$\mathbb{E}\{K\} = (1-p)^3 \cdot -6000 + 3p(1-p)^2 \cdot -3000 + p^3 \cdot 3000 = 3000(3p-2)$$

If the gambler will not lose, $\mathbb{E}\{K\} = 3000(3p-2) \geq 0$ is required, yielding:

$$3p - 2 \geq 0 \Rightarrow 1 \geq p \geq 2/3$$

12.(a)
It is clear the admissible value of N is 3, 4 or 5. Let E_n denote the event that Eagles wins the series in n games. Similarly, G_n denotes the event that the Gladiators wins in n games. The Eagles can win the series in 3 games if they win three straight, which occurs with probability:
$$P(E_3) = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

The Eagles can win the series in 4 games if the team wins two out of the first three games and wins the fourth game so that

$$P(E_4) = \binom{3}{2}\left(\frac{1}{2}\right)^3 \cdot \frac{1}{2} = \frac{3}{16}$$

Also, the Eagles can win the series in 5 games if the team wins two out of the first four games and wins the fifth game so that

$$P(E_5) = \binom{4}{2}\left(\frac{1}{2}\right)^4 \cdot \frac{1}{2} = \frac{3}{16}$$

As they have the same chance to win, by symmetry, $P(G_n) = P(E_n)$. Further we observe that the series last n games if either Eagles or Gladiators win the series in n games. Thus,

$$P(N = n) = P(E_n) + P(G_n) = 2P(E_n)$$

Consequently, the PMF of N is:

$$P_N(n) = \begin{cases} \frac{1}{4}, & n = 3 \\ \frac{3}{8}, & n = 4 \\ \frac{3}{8}, & n = 5 \\ 0, & \text{otherwise} \end{cases}$$

12.(b)

It is clear the admissible value of W is 0, 1, 2, or 3. Eagles wins only when $W = 3$.

For $W = 3$, it can mean G_3, G_4 or G_5. Hence

$$P(W = 3) = P(G_3) + P(G_4) + P(G_5) = \frac{1}{8} + \frac{3}{16} + \frac{3}{16} = \frac{1}{2}$$

While $W = 0$ corresponds to E_3, $W = 1$ corresponds to E_4, and $W = 2$ corresponds to E_5. As a result, the PMF of W is:

$$P_W(w) = \begin{cases} \frac{1}{8}, & w = 0 \\ \frac{3}{16}, & w = 1 \\ \frac{3}{16}, & w = 2 \\ \frac{1}{2}, & w = 3 \\ 0, & \text{otherwise} \end{cases}$$

12.(c)

It is clear the admissible value of L is 0, 1, 2, or 3. When there is no loss or $L = 0$, it corresponds to G_3. Similarly, $L = 1$ corresponds to G_4, and $L = 2$ corresponds to G_5. While $L = 3$ means E_3, E_4 or G_5. As a result, the PMF of L is:

$$P_L(l) = \begin{cases} \frac{1}{8}, & l = 0 \\ \frac{3}{16}, & l = 1 \\ \frac{3}{16}, & l = 2 \\ \frac{1}{2}, & l = 3 \\ 0, & \text{otherwise} \end{cases}$$

13.

Let $X \sim \mathcal{U}(a, b)$. According to Example 2.21, we have

$$\mathbb{E}\{X\} = \frac{a+b}{2} = 7 \Rightarrow a + b = 14$$
$$\text{var}(X) = \mathbb{E}\{X^2\} - (\mathbb{E}\{X\})^2 = \frac{(b-a)^2}{12} = 3 \Rightarrow b - a = 6$$

Solving the two equations yields $a = 4$ and $b = 10$. Hence the PDF is:

$$p(x) = \begin{cases} 1/6, & 10 \geq x \geq 4 \\ 0, & \text{otherwise} \end{cases}$$