EE3210 Signals & Systems

Due on Midnight, April 2, 2020

Homework #2

1. Total mark is 20 points (= 4 points per problem $\times$ 5 problems)
2. Solution will be posted on April 3rd on Canvas website
3. Submission due by April 2, 2020, midnight. We will not accept late submission.
4. Online submission through Canvas
   - Scan or taking a photo of your anwser sheet, then upload to Canvas
   - After initial submission to Canvas, you can resubmit through email to yjchun@cityu.edu.hk
     - For revision purpose or if the submitted file is corrupted

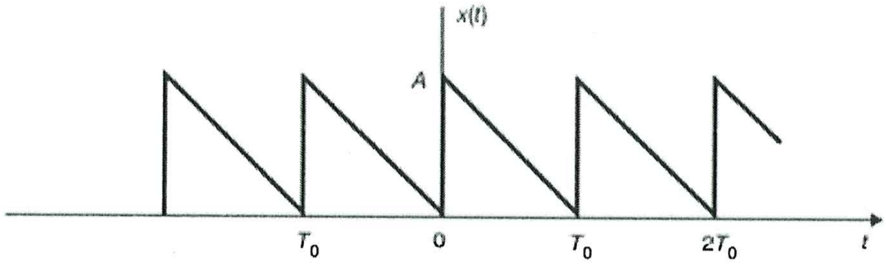
Problem 1 (4 pts)

Let's consider the triangular wave $x(t)$ as shown below.

$$x(t) = A\left(1 - \frac{t}{T_0}\right), \quad 0 \le t < T_0, \quad \text{and } x(t + T_0) = x(t)$$

(2pts) a) Find the complex exponential Fourier series of $x(t)$

(2pts) b) Find the triangular Fourier series of $x(t)$



(a)

Solution)

a) $C_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\,dt = \frac{A}{2}$

$$C_k = \frac{A}{T_0}\int_0^{T_0}\left(1 - \frac{t}{T_0}\right)e^{-jk\omega_0 t}\,dt = \frac{A}{T_0^2}\int_0^{T_0} \tau\, e^{jk\omega_0\tau}\,d\tau \cdot e^{-jk\omega_0 T_0}$$

Change of variable
$\tau = T_0 - t$
$d\tau = -dt$

- $\omega_0 T_0 = 2\pi$ and $e^{-jk\omega_0 T_0} = \left(e^{-j2\pi}\right)^k = 1$
- $\int_0^{T_0} \tau\, e^{jk\omega_0\tau}\,d\tau = \frac{\tau e^{jk\omega_0\tau}}{jk\omega_0}\Big|_0^{T_0} - \frac{1}{jk\omega_0}\cancel{\int_0^{T_0} e^{jk\omega_0\tau}\,d\tau} = \frac{T_0}{jk\omega_0}$

Hence, $C_0 = \frac{A}{2}$, $C_k = \frac{A}{jk(2\pi)}$, for $k \ne 0$

b) $a_0 = 2C_0 = A$, $a_k = 2\,\mathrm{Re}[C_k] = 0$ for $k \ne 0$

$b_k = -2\,\mathrm{Im}[C_a] = \frac{A}{k\pi}.$

Problem 2 (4pts)

Find the Fourier transform of the following signals ($\alpha > 0$)

(2pts) a) $x(t) = e^{-\alpha t^2}$

(2pts) b) $x(t) = e^{-\alpha|t|}$

Solution)

a) $$X(f) = \int_{-\infty}^{\infty} e^{-\alpha t^2} e^{-j2\pi f t}\, dt = \int_{-\infty}^{\infty} e^{-\alpha\left[t^2 + \frac{j2\pi f}{\alpha}t + \left(\frac{j\pi f}{\alpha}\right)^2 - \left(\frac{j\pi f}{\alpha}\right)^2\right]} dt$$

$$= \int_{-\infty}^{\infty} e^{-\alpha\left(t + \frac{j\pi f}{\alpha}\right)^2} dt \cdot e^{-\frac{(\pi f)^2}{\alpha}} \leftarrow$$

change of variable $t + \frac{j\pi f}{\alpha} = \tau$, $dt = d\tau$

$$= \int_{-\infty}^{\infty} e^{-\alpha\tau^2} d\tau \cdot e^{-\frac{(\pi f)^2}{\alpha}} \leftarrow \int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$$

$$= \sqrt{\frac{\pi}{a}}\, e^{-\frac{\pi^2}{\alpha} f^2}$$

b) $$x(t) = e^{-\alpha|t|} = e^{\alpha t} u(-t) + e^{-\alpha t} u(t)$$

$$\updownarrow \mathcal{F} \qquad \updownarrow \mathcal{F}$$

$$X(f) = \mathcal{F}(x(t)) = \frac{1}{\alpha - j2\pi f} + \frac{1}{\alpha + j2\pi f} = \frac{2\alpha}{\alpha^2 + (2\pi f)^2}$$

Problem 3 (4 pts)

Consider a continuous time LTI system where the input and the output are related by the following differential equations

$$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = 2x(t)$$

(2pts) a) Find the impulse response of this system.

(2pts) b) Find the output of this system if $x(t) = e^{-4t}u(t) - te^{-4t}u(t)$.

(Solution)

a) The FT of the differential equation is given by

$$[(j2\pi f)^2 + 6(j2\pi f) + 8]\,Y(f) = 2X(f)$$

$$H(f) = \frac{Y(f)}{X(f)} = \frac{2}{(4+j2\pi f)(2+j2\pi f)} = \frac{1}{2+j2\pi f} - \frac{1}{4+j2\pi f}$$

By using the FT table

$$h(t) = e^{-2t}u(t) - e^{-4t}u(t)$$

b) $$X(f) = \mathcal{F}(x(t)) = \frac{1}{4+j2\pi f} - \frac{1}{(4+j2\pi f)^2} = \frac{3+j2\pi f}{(4+j2\pi f)^2}$$

$$Y(f) = H(f)X(f) = \frac{2(3+j2\pi f)}{(4+j2\pi f)^3(2+j2\pi f)} = \frac{1}{(4+j2\pi f)^3} + \frac{-\frac{1}{2}}{(4+j2\pi f)^2}$$

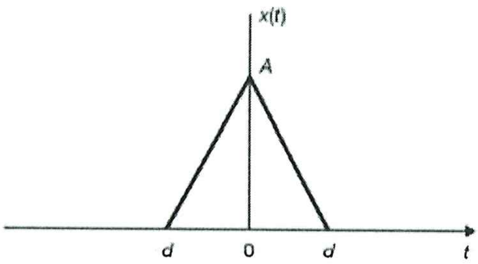
$$+ \frac{-\frac{1}{4}}{(4+j2\pi f)} + \frac{1/4}{(2+j2\pi f)}$$

Based on the FT table

$$y(t) = \frac{1}{2}t^2e^{-4t}u(t) - \frac{1}{2}te^{-4t}u(t) - \frac{1}{4}e^{-4t}u(t) + \frac{1}{4}e^{-2t}u(t)$$

Problem 4 (4pts)

(1pts) a) Find the Fourier transform of the triangular pulse signal shown below



(1pts) b) Find the inverse Fourier transform of

$$X(f) = \frac{1}{2 - f^2 + j3f}$$

(2pts) c) Find the 80 percent energy containment bandwidth for the signal

$$x(t) = \frac{1}{t^2 + a^2}, \quad a > 0$$

a) $x(t) = A\left(1 - \frac{|t|}{d}\right) = A \cdot \mathrm{tri}\left(\frac{t}{d}\right) = \frac{A}{d} \cdot \mathrm{rect}\left(\frac{t}{d}\right) * \mathrm{rect}\left(\frac{t}{d}\right)$

Since $\mathcal{F}\left(\mathrm{rect}\left(\frac{t}{d}\right)\right) = d\,\mathrm{Sinc}(f \cdot d)$,

$X(f) = \frac{A}{d} \cdot d^2\, \mathrm{Sinc}^2(f \cdot d) = (A \cdot d)\, \mathrm{Sinc}^2(f \cdot d)$

b) $X(f) = \frac{1}{(2+jf)(1+jf)} = 2\pi\left[\frac{1}{2\pi + j2\pi f} - \frac{1}{4\pi + j2\pi f}\right] \rightarrow x(t) = 2\pi\left[e^{-2\pi t} - e^{-4\pi t}\right]u(t)$

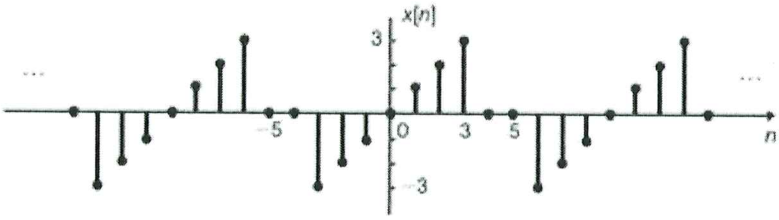
c) $x(t) = \frac{(2\pi)^2}{(2\pi t)^2 + (2\pi a)^2} \longrightarrow X(f) = \frac{\pi}{a} e^{-2\pi a|f|}$

$\int_{-f_{80\%}}^{f_{80\%}} |X(f)|^2 df = 0.8 \int_{-\infty}^{\infty} |X(f)|^2 df \Rightarrow 1 - e^{-4\pi a f_{80\%}} = 0.8$

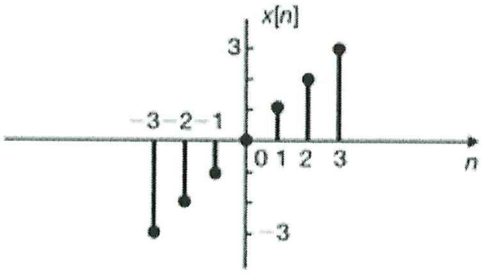
$\Rightarrow f_{80\%} = -\frac{\ln(0.2)}{4\pi a}$

Problem 5 (4 pts)

(2 pts) a) Find the discrete-time Fourier series of the sequence $x[n]$ as plotted below



(2 pts) b) Find the discrete-time Fourier transform of the sequence $x[n]$ as shown below



a) The period is $N = 9$, $\Omega_0 = \frac{2\pi}{N} = \frac{2\pi}{9}$

$$c_k = \frac{1}{9}\sum_{n=-4}^{4} x[n]\, e^{-jk\Omega_0 n} = \frac{1}{9}\Big[-3\cdot e^{j3k\Omega_0} + 3\, e^{-j3k\Omega_0} - 2\, e^{j2k\Omega_0} + 2\, e^{-j2k\Omega_0}$$
$$- e^{jk\Omega_0} + e^{-jk\Omega_0}\Big] = \frac{-2j}{9}\Big[3\cdot\sin\Big(\frac{6\pi}{9}k\Big) + 2\sin\Big(\frac{4\pi}{9}k\Big) + \sin\Big(\frac{2\pi}{9}k\Big)\Big]$$

b) $$X(f) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j2\pi f n} = \sum_{n=-3}^{3} x[n]\cdot e^{-j2\pi f n}$$
$$= \Big[-3\, e^{j6\pi f \cancel{n}} + 3\, e^{-j6\pi f} - 2\, e^{j4\pi f} + 2\, e^{-j4\pi f} - e^{j2\pi f} + e^{-j2\pi f}\Big]$$
$$= -2j\Big[3\sin(6\pi f) + 2\sin(4\pi f) + \sin(2\pi f)\Big]$$