EE 5410 Signal Processing

Solution for Assignment 1

1.
The signal has period $T = 4$ and fundamental frequency $\omega_0 = \pi/2$. Consider the period from $t = -2$ to $t = 2$ and use (2.5):

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt = \frac{1}{4}\left[\int_{-2}^{0} e^{-j0.5k\pi t}dt + \int_{0}^{2} -2e^{-j0.5k\pi t}dt\right]$$

For $k = 0$,

$$a_0 = \frac{1}{4}\left[\int_{-2}^{0} 1dt + \int_{0}^{2} 2dt\right] = \frac{1}{4}[2+4] = 1.5$$

For $k \neq 0$,

$$\begin{aligned}
a_k &= \frac{1}{4}\left[\int_{-2}^{0} e^{-j0.5k\pi t}dt + \int_{0}^{2} 2e^{-j0.5k\pi t}dt\right] \\
&= \frac{1}{4}\left[\frac{1}{-j0.5k\pi}e^{-j0.5k\pi t}\Big|_{-2}^{0} + \frac{2}{-j0.5k\pi}e^{-j0.5k\pi t}\Big|_{0}^{2}\right] \\
&= -\frac{1}{2jk\pi}\left[1 - e^{jk\pi} + 2e^{-jk\pi} - 2\right] \\
&= \frac{1}{2jk\pi}\left[1 + e^{jk\pi} - 2e^{-jk\pi}\right]
\end{aligned}$$

Combining the results, we have:

$$a_k = \begin{cases} 1.5, & k = 0 \\ \frac{1}{2jk\pi}\left[1 + e^{jk\pi} - 2e^{-jk\pi}\right] & k \neq 0 \end{cases}$$

2. Following Example 2.6, we get:

$$H(j\Omega) = \frac{1}{R/L + j\Omega} = \frac{R/L - j\Omega}{R^2/L^2 + \Omega^2}$$

$$|H(j\Omega)| = \frac{1}{\sqrt{R^2/L^2 + \Omega^2}}$$

and

$$\angle(H(j\Omega)) = -\tan^{-1}\left(\frac{\Omega L}{R}\right)$$

The real and imaginary parts, respectively, are then:

$$\frac{R/L}{R^2/L^2 + \Omega^2} \qquad \text{and} \qquad \frac{-\Omega}{R^2/L^2 + \Omega^2}$$

3.

$$
\begin{aligned}
y(t) = x(t) \otimes h(t) &= \int_{-\infty}^{\infty} h(\lambda) x(t-\lambda) d\lambda \\
&= \int_{-\infty}^{\infty} \left[-2\delta(\lambda-2) + \delta(\lambda-10)\right] x(t-\lambda) d\lambda \\
&= \int_{-\infty}^{\infty} -2\delta(\lambda-2) x(t-\lambda) d\lambda + \int_{-\infty}^{\infty} \delta(\lambda-10) x(t-\lambda) d\lambda \\
&= -2x(t-2) + x(t-10)
\end{aligned}
$$

The system is **stable** because if $x(t)$ is bounded, $y(t)$ will also be bounded. (Or, the system is **stable** because $\int_{-\infty}^{\infty} |h(t)|\, dt = 3 < \infty$, that is, the impulse response is absolutely summable.)

The system is **causal** because the output $y(t)$ does not depend on any future input values. (Or, the system is **causal** because $h(t) = 0$ for $t < 0$.)

The system is **not memoryless** because the output at time t does not only depend on the input at time t.

4.
The system is **memoryless** because the output at time n only depends on the input at time n.

The system is **not stable**. It is because for a bounded input $x[n] = 0$, the output will be unbounded.

The system is **causal** because the output does not depend on the future input value.

The system is **not linear**. The proof is as follows:

Let $y_1[n] = T\{x_1[n]\}$, $y_2[n] = T\{x_2[n]\}$ and $y_3[n] = T\{x_3[n]\}$ with $x_3[n] = a \cdot x_1[n] + b \cdot x_2[n]$.

The system outputs for $x_1[n]$ and $x_2[n]$ are:

$$
y_1[n] = x_1[n] + 1/x_1[n] \qquad \text{and} \qquad y_2[n] = x_2[n] + 1/x_2[n]
$$

The system output for $x_3[n]$ is then:

$$
\begin{aligned}
y_3[n] = x_3[n] + 1/x_3[n] &= ax_1[n] + bx_2[n] + 1/\left(ax_1[n] + bx_2[n]\right) \\
&\neq ax_1[n] + bx_2[n] + 1/(ax_1[n]) + 1/(bx_2[n]) = ay_1[n] + by_2[n]
\end{aligned}
$$

The system is **time-invariant**. The proof is as follows:

First, we have $y[n-n_0] = x[n-n_0] + 1/x[n-n_0]$

Consider $x_1[n] = x[n-n_0]$, its system output is

$$
y_1[n] = x_1[n] + 1/x_1[n] = x[n-n_0] + 1/x[n-n_0] = y[n-n_0]
$$

5.(a)

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] = \sum_{m=-\infty}^{\infty} u[-m-1]\cdot(0.5)^{n-m}u[n-m]$$
$$= \sum_{m=-\infty}^{-1} (0.5)^{n-m}u[n-m]$$
$$= \sum_{l=1}^{\infty} (0.5)^{n+l}u[n+l]$$

For $n \geq -1$, all $\{u[n+l]\}$ correspond to 1 and we have:

$$y[n] = \sum_{l=1}^{\infty} (0.5)^{n+l} = (0.5)^n \sum_{l=1}^{\infty} (0.5)^l = (0.5)^n \cdot \frac{0.5}{1-0.5} = (0.5)^n$$

For $n < -1$, $u[n+l] = 1$ when $n + l \geq 0$ or $l \geq -n$

$$y[n] = \sum_{l=-n}^{\infty} (0.5)^{n+l} = (0.5)^n \sum_{l=-n}^{\infty} (0.5)^l = (0.5)^n \cdot \frac{(0.5)^{-n}}{1-0.5} = 2$$

Combining the results, we have:

$$y[n] = \begin{cases} (0.5)^n, & n \geq -1 \\ 2, & n < -1 \end{cases}$$

5.(b)
The z transforms of $x[n] = u[-n-1]$ and $h[n] = (0.5)^n u[n]$ are

$$X(z) = -\frac{1}{1-z^{-1}}, \quad |z| < 1 \qquad \text{and} \qquad H(z) = \frac{1}{1-0.5z^{-1}}, \quad |z| > 0.5$$

So we have:

$$Y(z) = -\frac{1}{1-z^{-1}} \cdot \frac{1}{1-0.5z^{-1}} = \frac{-2}{1-z^{-1}} + \frac{1}{1-0.5z^{-1}}, \quad 0.5 < |z| < 1$$

Taking the inverse z transform yields:

$$y[n] = 2u[-n-1] + (0.5)^n u[n]$$

6.
$x[1] = 1.6180$, $x[2] = -1.1756$, $x[3] = -1.6180$ and $x[4] = 1.1756$ and $x[5] = 1.6180$.

Yes. $x[n]$ is a periodic signal.

7.(a)

$$h[n] = h_1[n] \otimes \left(h_2[n] + h_3[n] \otimes h_4[n]\right)$$

7.(b)
First we note that $h_3[n] \otimes h_4[n] = u[n-2]$. The overall impulse response is then:

$$h[n] = (\delta[n] + \delta[n-1]) \otimes \left(u[n] + u[n-2]\right) = u[n] + u[n-1] + u[n-2] + u[n-3]$$

7.(c)
$y[n] = u[n+2] + u[n+1] + u[n] + u[n-1] + 3u[n-1] + 3u[n-2] + 3u[n-3] + 3u[n-4]$

8.
Let $y[n]$ be the convolution output. Starting from $n = -2$, $y[n]$ = -12, -9, -2, 0, -10, -8, -6, -3. At other time instants, the output is 0. As the lengths of two signals to be convoluted are 5 and 4, the resultant length should be 5+4-1=8.

9.(a)
The z transforms for $x[n]$ and $y[n]$ are:

$$X(z) = \frac{1}{1-0.5z^{-1}} - \frac{1}{1-2z^{-1}} = \frac{-1.5z^{-1}}{(1-0.5z^{-1})(1-2z^{-1})}, \quad 0.5 < |z| < 2$$

and

$$Y(z) = \frac{6}{1-0.5z^{-1}} - \frac{6}{1-0.75z^{-1}} = \frac{-1.5z^{-1}}{(1-0.5z^{-1})(1-0.75z^{-1})}, \quad |z| > 0.75$$

As a result,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-2z^{-1}}{1-0.75z^{-1}}, \quad |z| > 0.75$$

Note that the ROC contains at least the intersection of the ROCs of $x[n]$ and $y[n]$.

9.(b)
There is one pole at $z = 0.75$ and one zero at $z = 2$.

9.(c)

$$H(z) = \frac{1-2z^{-1}}{1-0.75z^{-1}} = \frac{1}{1-0.75z^{-1}} - \frac{2z^{-1}}{1-0.75z^{-1}}, \quad |z| > 0.75$$

Taking the inverse z transform, we have:

$$h[n] = (0.75)^n u[n] - 2(0.75)^{n-1} u[n-1]$$

9.(d)
As the ROC includes the unit circle, the DTFT exists and it is computed as:

$$H(e^{j\omega}) = \frac{1-2e^{-j\omega}}{1-0.75e^{-j\omega}}$$

9.(e)

$$\frac{Y(z)}{X(z)} = \frac{1-2z^{-1}}{1-0.75z^{-1}} \quad \Rightarrow \quad Y(z)(1-0.75z^{-1}) = X(z)(1-2z^{-1})$$
$$\Rightarrow \quad y[n] - 0.75y[n-1] = x[n] - 2x[n-1]$$

9.(f)
As the ROC includes the unit circle, the system is stable.

9.(g)
As $h[n] = 0$ for $n < 0$, the system is causal.