Solution of Midterm Quiz 2

Name:______________

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City University of Hong Kong
Department of Electronic Engineering

EE 3210 Systems and Signals

Midterm Quiz 2

NOTE:

1. This is a 1.5-hour, open book, open notes midterm test.
2. There are 4 problems altogether, which amount to 20 points.
3. In addition to final answers, you need to include necessary steps to show your derivations. An answer without any supporting development will unlikely be awarded with a point. You do receive partial credits if you write down the steps despite that you may not complete your solution.
4. Write your work in the blank space. If necessary, write on the back sheet.
5. Enjoy the test, and good luck!

1. (4 pts) Suppose that $x(t)$ is a periodic signal with a fundamental period T_x. Let

$$y(t) = x(-at + b), \qquad a > 0.$$

Show that the exponential Fourier coefficients for $y(t)$ are given by

$$y_k = x_{-k} e^{-jk\omega_x b},$$

where $\omega_x = 2\pi/T_x$, and x_k are the exponential Fourier coefficients for $x(t)$.

(Students may choose to use directly properties of Fourier series.)

Proof.

1 pt $\quad y_k = \frac{1}{T_y}\int_{T_y} y(t)\, e^{-jk\omega_y t}\, dt.$

Consider

$$y(t+T_y) = x[-a(t+T_y)+b]$$
$$= x(-at+b-aT_y)$$

~~1 pt~~

Set $-aT_y = -T_x$, i.e., $\boxed{T_y = \frac{T_x}{a}}$. Then

$$y(t+T_y) = x(-at+b-T_x) = x(-at+b) = y(t).$$

It follows that

1 pt $\quad \omega_y = \frac{2\pi}{T_y} = \frac{2\pi}{\left(\frac{T_x}{a}\right)} = a\frac{2\pi}{T_x} = a\omega_x$

$$\Rightarrow \quad y_k = \frac{a}{T_x}\int_{T_y} x(-at+b)\, e^{-jka\omega_x t}\, dt$$

Let $\tau = -at + b$ $\quad = \frac{a}{T_x}\int_{T_x} x(\tau)\, e^{-jka\omega_x \frac{\tau - b}{-a}} \left(-\frac{1}{a}\right) d\tau\, (-1)$

1 pt $\quad = e^{-jk\omega_x b}\, \underbrace{\frac{1}{T_x}\int_{T_x} x(\tau)\, e^{-j(-k)\omega_x \tau}\, d\tau}_{x_{-k}}$

1 pt $\quad = x_{-k}\, e^{-jk\omega_x b}$

2. (6 pts) Consider a LTI system with frequency response

$$H(j\omega) = \frac{j\omega + 1}{(j\omega)^2 + 7j\omega + 12}.$$

(i) Determine a differential equation that describes the system.
(ii) Find a block diagram realization consisting of adders, integrators, and coefficient multipliers for this system.
(iii) Suppose that an input signal $x(t) = e^{-2t}u(t)$ is applied. Determine the output response.

Solution:

(i) $$\frac{Y(j\omega)}{X(j\omega)} = H(j\omega) = \frac{j\omega+1}{(j\omega)^2+7j\omega+12}$$

$$\Rightarrow (j\omega)^2 Y(j\omega) + 7(j\omega)Y(j\omega) + 12Y(j\omega) = (j\omega)X(j\omega) + X(j\omega)$$

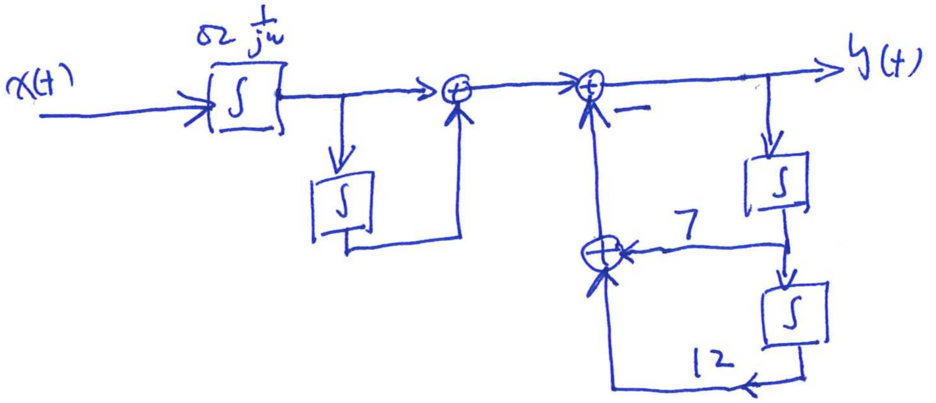
1 pt Inverse transform:

$$\frac{d^2y(t)}{dt^2} + 7\frac{dy(t)}{dt} + 12y(t) = \frac{dx(t)}{dt} + x(t)$$

(ii) $$\frac{Y(j\omega)}{X(j\omega)} = \frac{\left(\frac{1}{j\omega}\right) + \left(\frac{1}{j\omega}\right)^2}{1 + 7\left(\frac{1}{j\omega}\right) + 12\left(\frac{1}{j\omega}\right)^2}$$

1 pt $$Y(j\omega) = X(j\omega)\left[\frac{1}{j\omega} + \left(\frac{1}{j\omega}\right)^2\right] - Y(j\omega)\left[7\frac{1}{j\omega} + 12\left(\frac{1}{j\omega}\right)^2\right]$$

1 pt



(iii) $$Y(j\omega) = H(j\omega)X(j\omega) = \frac{j\omega+1}{(j\omega+2)(j\omega+3)(j\omega+4)}$$

Consider $$Y(s) = \frac{s+1}{(s+2)(s+3)(s+4)}$$

Partial Fraction

$$Y(s) = \frac{A}{s+2} + \frac{B}{s+3} + \frac{C}{s+4}$$

$$A = (s+2)Y(s)\Big|_{s=-2} = \frac{s+1}{(s+3)(s+4)}\Big|_{s=-2} = -\frac{1}{2}$$

2 pts

$$B = (s+3)Y(s)\Big|_{s=-3} = \frac{s+1}{(s+2)(s+4)}\Big|_{s=-3} = 2$$

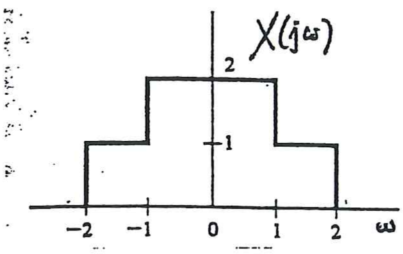
$$C = (s+4)Y(s)\Big|_{s=-4} = \frac{s+1}{(s+2)(s+3)}\Big|_{s=-4} = -\frac{3}{2}$$

Hence,

1 pt $$Y(j\omega) = -\frac{1}{2}\frac{1}{j\omega+2} + 2\frac{1}{j\omega+3} - \frac{3}{2}\frac{1}{j\omega+4}$$

$$y(t) = \left[-\frac{1}{2}e^{-2t} + 2e^{-3t} - \frac{3}{2}e^{-4t}\right]u(t)$$

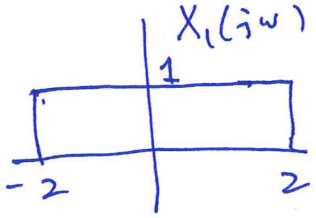
3~~6~~. (5 pts) The Fourier transform of a certain signal $x(t)$ is given in the following figure. Find $x(t)$ without performing any integration.

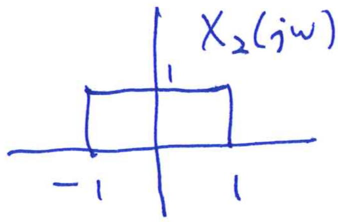


Solution:

1 pt Consider $x_1(t) = \dfrac{\sin 2t}{\pi t} \longleftrightarrow X_1(j\omega) = \begin{cases} 1 & |\omega| < 2 \\ 0 & \text{otherwise} \end{cases}$

1 pt $x_2(t) = \dfrac{\sin t}{\pi t} \longleftrightarrow X_2(j\omega) = \begin{cases} 1 & |\omega| < 1 \\ 0 & \text{otherwise} \end{cases}$





1 pt Then, $X(j\omega) = X_1(j\omega) + X_2(j\omega)$

1 pt $x(t) = x_1(t) + x_2(t)$

$$= \frac{\sin 2t}{\pi t} + \frac{\sin t}{\pi t}$$

1 pt $$= \frac{\sin t}{\pi t}(1 + 2\cos t)$$

4. (5 pts) A continuous-time system is determined by the following integral-differential equation

$$\frac{d^2y(t)}{dt^2} + 3y(t-1) + \int_0^t y(\tau)d\tau = \frac{dx(t)}{dt} - x(t),$$

which relates the input $x(t)$ to the output $y(t)$. Find the transfer function $H(j\omega)$ for this system.

Solution:

1 pt $\mathcal{F}\left\{\frac{d^2y(t)}{dt^2}\right\} = (j\omega)^2 Y(j\omega)$, $\mathcal{F}\left\{\frac{dx(t)}{dt}\right\} = (j\omega)X(j\omega)$

1 pt $\mathcal{F}\{y(t-1)\} = e^{-j\omega}Y(j\omega)$

1 pt $\mathcal{F}\left\{\int_0^t y(\tau)d\tau\right\} = \frac{1}{j\omega}Y(j\omega)$

Hence

1 pt $(j\omega)^2 Y(j\omega) + 3e^{-j\omega}Y(j\omega) + \left(\frac{1}{j\omega}\right)Y(j\omega)$

$= (j\omega)X(j\omega) - X(j\omega)$

1 pt $H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{j\omega - 1}{(j\omega)^2 + 3e^{-j\omega} + \left(\frac{1}{j\omega}\right)}$

$$= \frac{(j\omega)(j\omega - 1)}{(j\omega)^3 + 3(j\omega)e^{-j\omega} + 1}$$