# EE3301
# Final Exam Solutions

# Question 1

**1.1. Optimizing the order of answering questions in an exam**

$W$ is the total time available for the exam.

$N$ is the number of questions in the exam.

$v_i$ is the number of points of the $i$-th question, $i = 1, 2, \ldots, N$.

$w_i$ is the expected time required for the $i$-th question, $i = 1, 2, \ldots, N$.

$x_i$ is the proportion of the $i$-th question that the student will do.

The objective is to maximize the total mark in the exam.

**1.2.** We will use here the unique 10-digit number: 4837871395

$v_1 = 3$. (The first digit is not more than 5.)

$v_2 = 2$. (The second digit is more than 5.)

$v_3 = 1$. (The third digit is not more than 5.)

$w_1 = 1, w_2 = 2$, and $w_3 = 3$.

We obtain the following 0,1 Knapsack problem (relevant for **1.4**)

$$\max_{\{x_1, x_2, x_3\}} 3x_1 + 2x_2 + x_3$$

Subject to:

$$x_1 + 2x_2 + 3x_3 \leq 5$$

and $x_1, x_2, x_3$ are all binary numbers (0, 1 integers).

## 1.2. and 1.3. LP Problem

$$\max_{\{x_1,x_2,x_3\}} 3x_1 + 2x_2 + x_3$$

Subject to:

$$x_1 + 2x_2 + 3x_3 \leq 5$$

and $1 \geq x_i \geq 0, i = 1, 2, 3.$

## Solution by a greedy algorithm:

$\frac{v_1}{w_1} = \frac{3}{1} = 3;$ $\frac{v_2}{w_2} = \frac{2}{3};$ $\frac{v_3}{w_3} = \frac{1}{3}.$

Thus we start with setting $x_1$=1 that has the highest value to weight ratio. This leaves weight of 4. Then setting $x_2$=1, which leaves remaining weight of 2, so $x_3=\frac{2}{3}$ to completely utilize the remaining available weight.

The optimal value of the objective function is $5\frac{2}{3}$ .

## 1.3. LP Problem (cont'd)

$$\max_{\{x_1,x_2,x_3\}} 3x_1 + 2x_2 + x_3$$

Subject to:

$$x_1 + 2x_2 + 3x_3 \leq 5$$

and 1$\geq x_i \geq 0, i = 1, 2, 3.$

## Solution by Excel Solver:

The solution obtained by Excel Solver (see attached file knapsac_LP_ Q_1_4) is consistent with the results of the greedy algorithm. The solution obtained is:

$x_1$=1, $x_2$=1, $x_3$=0.666666667

The optimal value of the objective function is 5.666666667 .

**1.4.** Consider the following 0,1 Knapsack problem

$$\max_{\{x_1, x_2, x_3\}} v_1 x_1 + v_2 x_2 + v_3 x_3$$

Subject to:

$$w_1 x_1 + w_2 x_2 + w_3 x_3 \leq 5$$

and $x_1, x_2, x_3$ are all binary numbers (0, 1 integers).

Provide Dynamic Programming (DP) formulation for this problem.

# 1.4. DP formulation

## Optimal value function

*S(k,w) =* the best (maximum) value of all possible solutions of weight less than or equal to *w* consisting only of 0 or 1 items of types *k, …, 3.*

## Recurrence relation

$$S(k,w) = \max_{x_k=0,1\ (\text{weight} \leq w)} [x_k\, v_k + S(k+1, w - x_k w_k)].$$

## Boundary conditions

$$S(4,w) = 0, \qquad w \geq 0.$$

$$S(k,w) = -\infty, \qquad w < 0 \text{ for all } k.$$

Credit: S. E. Dreyfus and A. M. Law, *The art and theory of dynamic programming,* Academic Press, 1977.

# 1.5. DP Solution

- **Stage 3 (*k* = 3)**
- $S(3,0) = 0$
- *P*(3,*0*) = 0 (For *w* = 0, at stage 3, you add nothing to the knapsack, $x_3$ = 0.)
- $S(3,1) = 0$, $S(3,2) = 0$, $S(3,3) = 1$, $S(3,4) = 1$, $S(3,5) = 1$.
- *P*(3,*w*) = 0 (For *w* =0, 1, 2, at stage 3, you cannot add item 3, which has weight 3 to the knapsack, $x_3$ = 0.)
- *P*(3,*w*) = 1 (For *w* =3, 4, 5, at stage 3, you add item 3, which has weight 3 and value 1, to the knapsack, $x_3$ = 1.)

# 1.5. DP Solution (cont'd)

- **Stage 2 (*k* = 2)**

$S(2,0) = 0$

$P(2,0) = 0$

(For $w = 0$, at stage 2, you add nothing to the knapsack, $x_2 = 0$.)

$S(2,1) = 0 + S(3,1) = 0$. $P(2,1) = 0$.

$S(2,2) = \max[0 + S(3,2), 2 + S(3,0)] = \max[0+0, 2+0] = 2$. $P(2,2) = 1$, ($x_2 = 1$).

$S(2,3) = \max[0 + S(3,3), 2 + S(3,1)] = \max[0+1, 2+0] = 2$. $P(2,3) = 1$.

$S(2,4) = \max[0 + S(3,4), 2 + S(3,2)] = \max[0+1, 2+0] = 2$. $P(2,4) = 1$.

$S(2,5) = \max[0 + S(3,5), 2 + S(3,3)] = \max[0+1, 2+1] = 3$. $P(2,5) = 1$

# 1.5. DP Solution (cont'd)

**Stage 1 (*k* = 1)**

$S(1{,}5) = \max[0 + S(2{,}5), 3 + S(2{,}4)]$ = max[0+3,3+2] = 3.

$P(1,5) = 1$.

**Optimal solution:**

$P(1,5) = 1$, so $x_1 = 1$; $P(2,4) = 1$, so $x_2 = 1$; $P(3,2) = 0$, so $x_3 = 0$.

# Question 2

**2.1**.

The optimization problem is formulated as follows.

Maximize $U_1(x_1) + U_2(x_2) = 1 - (1.5)^{-x_1} + 1 - (50)^{-x_2+0.5}$

Subject to:

$x_1 + x_2 \leq 2$

$x_1 \geq 0$

$x_2 \geq 0.5$

# 2.1. (Cont'd)

This optimization problem was solved by Excel Solver (see the Attached Excel file

"Utility_Q_2" Sheet: "capacity2".

The **optimal results** are:

$x_1 = 0.83411$

$x_2 = 1.16589$

$U_1(x_1) = 0.286949$

$U_2(x_2)$ =0.926095

$U_1(x_1) + U_2(x_2)$= 1.213044

# 2.2.

The optimization problem (with capacity limitation changed to 1) was solved by Excel Solver (see the Attached Excel file

“Utility_Q_2” Sheet: “capacity1”).

The **optimal results** are:

$x_1 = 0$

$x_2 = 1$

$U_1(x_1) = 0$

$U_2(x_2)$ =0.858579

$U_1(x_1) + U_2(x_2)$= 0.858579

# 2.2. (Cont'd)

The optimization problem (with capacity limitation changed to 3) was solved by Excel Solver (see the Attached Excel file

"Utility_Q_2"; Sheet: "capacity3").

The **optimal results** are:

$x_1 = 1.740198$

$x_2 = 1.259802$

$U_1(x_1) = 0.506183$

$U_2(x_2)$ =0.948818

$U_1(x_1) + U_2(x_2)$= 1.455

# 2.2. (Cont'd)

The utility of User 1 start to increase from x = 0. The domain of User 2 only start from x = 0.5, but its utility increase from x = 0.5 is much steeper than that of User 1 that starts from x = 0. Accordingly, the second user can be viewed as more "demanding". Therefore, when capacity is tightest, at C=1, the more "demanding" user obtains all of it. Then when the capacity increases to 2, the allocation is more balanced but still significantly in favor of the more demanding user.  Then at C = 3, the capacity requirement of the demanding User 2 is already satisfied and User 1 obtain more capacity than User 2 because its utility function rate of increase is high at the higher range of x than that of User 1.

The behavior of User 2 - the more demanding user - is similar to that of a UDP (real-time) user. While if you send email or a file using TCP, you normally do not mind very much if you receive lower rate so the service is completed in few seconds, but a real-time service is more delay-sensitive and even delays in the order of less than 0.5 second may adversely affect quality of service. Therefore, the server will give higher rate to real time services until their capacity requirements are met.

# Question 3

## 3.1

Routes of OD pair (1,2)

1,2

1,3,2

1,4,2

1,3,4,2

1,4,3,2

Routes of OD pair (1,3)

1,3

1,2,3

1,4,3

1,2,4,3

1,4,2,3

# 3.1 (Cont'd)

| Routes of OD pair (1,4) | Routes of OD pair (2,3) |
|---|---|
| 1,4 | 2,3 |
| 1,3,4 | 2,1,3 |
| 1,2,4 | 2,4,3 |
| 1,3,2,4 | 2,1,4,3 |
| 1,2,3,4 | 2,4,1,3 |

# 3.1 (Cont'd)

Routes of OD pair (2,4)

2,4

2,1,4

2,3,4

2,1,3,4

2,3,1,4

Routes of OD pair (3,4)

3,4

3,1,4

3,2,4

3,1,2,4

3,2,1,4

# 3.2

The objective is to minimize the total cost. Accordingly, the objective is:

Minimize: $P = X_{12} + X_{13} + X_{23} + X_{14} + X_{24} + X_{34}$
$+2X_{132} + 2X_{142} + 2X_{123} + 2X_{143} + 2X_{124} + 2X_{134} + 2X_{213} + 2X_{243}$
$+2X_{214} + 2X_{234} + 2X_{314} + 2X_{324}$
$+3X_{1342} + 3X_{1432} + 3X_{1243} + 3X_{1423} + 3X_{1234} + 3X_{1324} + 3X_{2143} + 3X_{2413}$
$+3X_{2134} + 3X_{2314} + 3X_{3124} + 3X_{3214}$

# 3.2 (cont'd)

The demands between every OD pair must be met. Accordingly, the following constraints must be satisfied.

$$X_{12}+X_{132}+X_{142}+X_{1342}+X_{1432}=D_{12}$$
$$X_{13}+X_{123}+X_{143}+X_{1243}+X_{1423}=D_{13}$$
$$X_{14}+X_{124}+X_{134}+X_{1234}+X_{1324}=D_{14}$$
$$X_{23}+X_{213}+X_{243}+X_{2143}+X_{2413}=D_{23}$$
$$X_{24}+X_{214}+X_{234}+X_{2134}+X_{2314}=D_{24}$$
$$X_{34}+X_{314}+X_{324}+X_{3124}+X_{3214}=D_{34}$$

# 3.2 (cont'd)

The traffic on every link must not exceed the link capacity limitations. Accordingly, all the link capacity constraints must be satisfied.
In this exam, it is only required to provide the link capacity constraint for link (1,2) and (2,3). Here are the constraints.

$$X_{12}+X_{123}+X_{124}+X_{1234}+X_{1243}+X_{213}+X_{214}+X_{2134}+X_{2143}+X_{3124}+X_{3214} \leq \mathrm{B}_{12}$$

$$X_{23}+X_{123}+X_{234}+X_{1234}+X_{2314}+X_{1423}+X_{132}+X_{324}+X_{1324}+X_{3214}+X_{1432} \leq \mathrm{B}_{23}$$

$$X_{13}+X_{132}+X_{134}+X_{1324}+X_{1342}+X_{2134}+X_{314}+X_{3124}+X_{2314}+X_{213}+X_{2413} \leq \mathrm{B}_{13}$$

In addition, let $R$ be the set of routes. Then, for any route $r \in R$, we also have the non-negativity constraint $X_r \geq 0$.