EE3210
Signals and Systems

Part 7: Continuous-Time Fourier Series

香港城市大學
City University
of Hong Kong

Instructor: Dr. Jun Guo

DEPARTMENT OF ELECTRONIC ENGINEERING

Continuous-Time Periodic Complex Exponentials

- A continuous-time complex exponential of the form $e^{j\omega t}$ is periodic for any (positive or negative) value of ω.
  - The fundamental period T_0 of $e^{j\omega t}$ is $T_0 = 2\pi/|\omega|$.
    - Thus, the signals $e^{j\omega t}$ and $e^{-j\omega t}$ have the same fundamental period.

- A harmonically related set of continuous-time complex exponentials, all of which have a common period T with fundamental frequency $\omega_0 = 2\pi/T$, is defined as

$$\phi_k(t) = e^{jk\omega_0 t} = e^{jk(2\pi/T)t}, \quad k = 0, \pm 1, \pm 2, \ldots \qquad (1)$$

Continuous-Time Periodic Complex Exponentials (cont.)

- We observe in (1) that:
  - For $k = 0$, $\phi_k(t)$ is a constant, which is periodic for any value of T.
  - For $k \neq 0$, $\phi_k(t)$ is periodic with fundamental period $T/|k|$, which is also periodic with period T.

- Thus, a linear combination of harmonically related continuous-time complex exponentials of the form

$$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t} \tag{2}$$

  is also periodic with period T.

Fourier Series Representation of Continuous-Time Periodic Signals

- Consider a continuous-time periodic signal $x(t)$ with fundamental period $T_0 = T$.
- Assume $x(t)$ can be represented with the series of (2):

$$\boxed{x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}} \qquad (3)$$

  with fundamental frequency $\omega_0 = 2\pi/T$.
  - The representation of $x(t)$ in the form of (3) is referred to as the Fourier series representation.
  - (3) is known as the synthesis formula of the continuous-time Fourier series.

Fourier Series Representation of Continuous-Time Periodic Signals (cont.)

- Note in (3) that:
  - The term for $k = 0$ is simply a_0, which is the constant or dc component of $x(t)$.
  - The two terms for $k = +1$ and $k = -1$ are periodic with fundamental period T and are collectively referred to as the 1st harmonic components.
  - The two terms for $k = +2$ and $k = -2$ are periodic with fundamental period $T/2$ and are referred to as the 2nd harmonic components.
  - In general, the two terms for $k = +N$ and $k = -N$ are referred to as the Nth harmonic components.

Fourier Series Representation of Continuous-Time Periodic Signals (cont.)

- Now, we need a procedure for determining the Fourier series coefficients a_k in (3).
- Multiplying both sides of (3) by $e^{-jn\omega_0 t}$ for an arbitrary integer n, we obtain

$$\begin{aligned} x(t)e^{-jn\omega_0 t} &= \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t} \\ &= \sum_{k=-\infty}^{+\infty} a_k e^{j(k-n)\omega_0 t} \end{aligned} \tag{4}$$

Fourier Series Representation of Continuous-Time Periodic Signals (cont.)

- Integrating both sides of (4) over any interval of length T, i.e., over one fundamental period of $x(t)$, we have

$$\begin{aligned}\int_T x(t)e^{-jn\omega_0 t}dt &= \int_T \sum_{k=-\infty}^{+\infty} a_k e^{j(k-n)\omega_0 t}dt \\ &= \sum_{k=-\infty}^{+\infty} a_k \left[\int_T e^{j(k-n)\omega_0 t}dt\right]\end{aligned} \tag{5}$$

  - Note: $\int_T$ is a shorthand notation, which has the same effect as $\int_\tau^{\tau+T}$ for any real number τ.

Fourier Series Representation of Continuous-Time Periodic Signals (cont.)

- We observe in the right-hand side of (5) that:
  - For $k = n$, we have $\int_T e^{j(k-n)\omega_0 t} dt = \int_\tau^{\tau+T} dt = T$
  - For $k \neq n$, we have

$$\int_T e^{j(k-n)\omega_0 t} dt = \int_\tau^{\tau+T} e^{j(k-n)\omega_0 t} dt$$

$$= \frac{e^{j(k-n)\omega_0(\tau+T)} - e^{j(k-n)\omega_0 \tau}}{j(k-n)\omega_0}$$

$$= \frac{e^{j(k-n)\omega_0 \tau} e^{j(k-n)2\pi} - e^{j(k-n)\omega_0 \tau}}{j(k-n)\omega_0} = 0$$

Fourier Series Representation of Continuous-Time Periodic Signals (cont.)

- Thus,

$$\sum_{k=-\infty}^{+\infty} a_k \left[\int_T e^{j(k-n)\omega_0 t} dt \right] = a_n T \tag{6}$$

- By (5) and (6), we obtain $\int_T x(t) e^{-jn\omega_0 t} dt = a_n T$ and hence

$$\boxed{a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt = \frac{1}{T} \int_T x(t) e^{-jk(2\pi/T)t} dt} \tag{7}$$

  - (7) is known as the analysis formula of the continuous-time Fourier series.

Convergence of Continuous-Time Fourier Series

- Virtually all periodic continuous-time signals that are not pathological in nature have a Fourier series representation.
  - Thus, convergence of continuous-time Fourier series is not a problem in general engineering practice.
- In particular, it is known that, for a periodic signal $x(t)$ with no discontinuities, the Fourier series representation converges and equals $x(t)$ at every value of t.

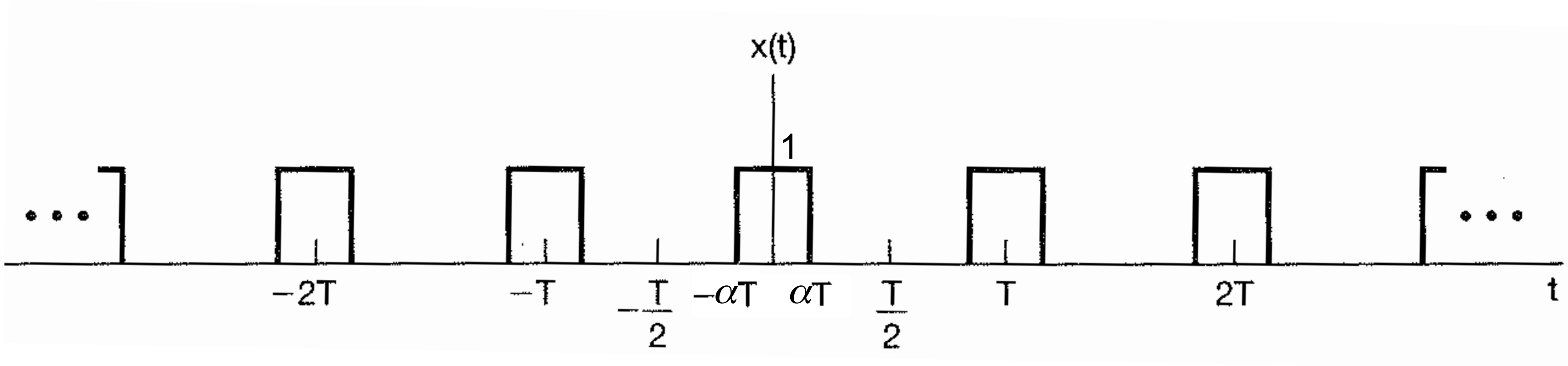
Convergence of Continuous-Time Fourier Series (cont.)

- For $x(t)$ with a finite number of discontinuities in each period T, the Fourier series representation equals $x(t)$ everywhere except at the discontinuities.
  - However, in this case, the difference between $x(t)$ and its Fourier series representation contains no energy.
    - That is, if we define an error signal

$$e(t) = x(t) - \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}, \text{ then } \int_T |e(t)|^2 dt = 0.$$

  - Consequently, the two signals can be thought of as being the same for all practical purposes.

An Example



- Consider the periodic square wave $x(t)$ with $0 < \alpha < \frac{1}{2}$ and fundamental period $T_0 = T$.
  - The fundamental frequency of its Fourier series representation is $\omega_0 = 2\pi/T$.
  - Because of the symmetry of $x(t)$ about $t = 0$ in this case, it is convenient to choose $-T/2 \leq t \leq T/2$ as the interval over which the integration is performed.

An Example (cont.)

- Using (7) with the limits $-T/2$ and $T/2$, we obtain:
  - For $k = 0$,

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)dt = \frac{1}{T}\int_{-\alpha T}^{\alpha T} dt = 2\alpha \tag{8}$$

  - For $k \neq 0$,

$$\begin{aligned} a_k &= \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-\alpha T}^{\alpha T} e^{-jk\omega_0 t}dt \\ &= \frac{e^{j\alpha k\omega_0 T} - e^{-j\alpha k\omega_0 T}}{jk\omega_0 T} = \frac{2j\sin(\alpha k\omega_0 T)}{jk\omega_0 T} = \frac{\sin(2\alpha k\pi)}{k\pi} \end{aligned} \tag{9}$$

    - Note: $\lim_{k\to 0} \frac{\sin(2\alpha k\pi)}{k\pi} = 2\alpha$ by l'Hôpital's rule.

An Example (cont.)

- Thus, the Fourier series representation of the periodic square wave $x(t)$ in this example is

$$x(t) = \sum_{k=-\infty}^{+\infty} \frac{\sin(2\alpha k\pi)}{k\pi} e^{jk(2\pi/T)t}$$

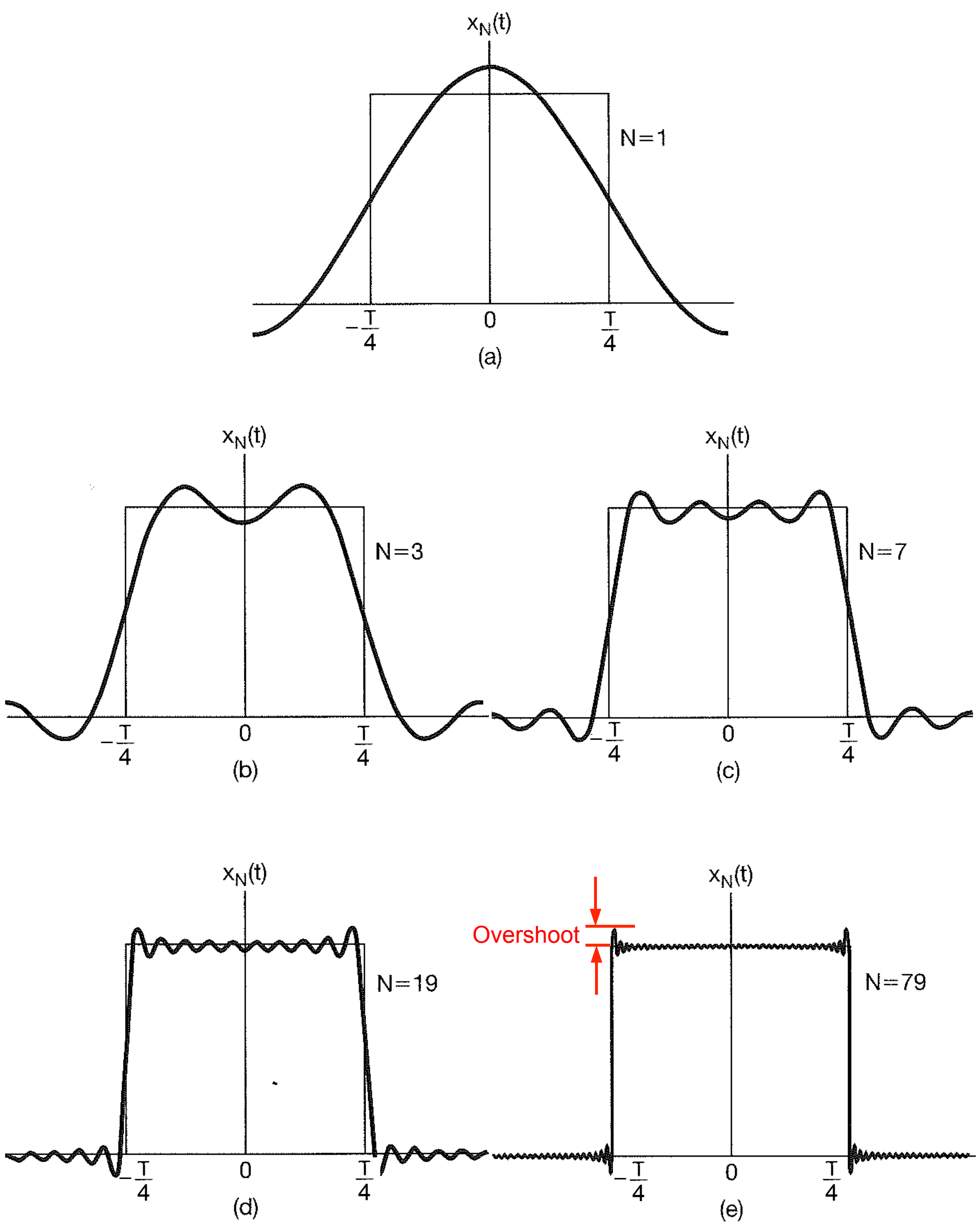
or, equivalently, $x(t) = \lim_{N\to\infty} x_N(t)$, where

$$x_N(t) = \sum_{k=-N}^{+N} \frac{\sin(2\alpha k\pi)}{k\pi} e^{jk(2\pi/T)t}$$

is known as a truncated Fourier series approximation of $x(t)$.

An Example (cont.)

- Here, we show $x_N(t)$ for several values of N for $x(t)$ with $\alpha = 1/4$.
  - Note: An overshoot of 9% of the height of the discontinuity, no matter how large N becomes.
  - Known as the Gibbs phenomenon.



Properties of Continuous-Time Fourier Series

- Here, we will describe several important properties, including: 1) linearity, 2) time shift, 3) time reversal, 4) time scaling, 5) multiplication, 6) differentiation, 7) Parseval's relation.
  - A summary of these and other important properties of continuous-time Fourier series can be found in Table 3.1 on Page 208 of the textbook.
- For notational convenience, we will use $x(t) \leftrightarrow a_k$ to indicate the relationship between a periodic signal $x(t)$ and its Fourier series coefficients a_k.

Linearity

- Given that $x(t)$ and $y(t)$ are both periodic with period T and that $x(t) \leftrightarrow a_k$, $y(t) \leftrightarrow b_k$, then $Ax(t) + By(t)$ is also periodic with period T and

$$Ax(t) + By(t) \leftrightarrow Aa_k + Bb_k$$

where A and B are arbitrary constants.

Time Shift

- Given that $x(t)$ is periodic with period T and that $x(t) \leftrightarrow a_k$, then $x(t - t_0)$ is also periodic with period T and

$$x(t - t_0) \leftrightarrow \left[e^{-jk(2\pi/T)t_0}\right] a_k$$

Time Reversal

- Given that $x(t)$ is periodic with period T and that $x(t) \leftrightarrow a_k$, then $x(-t)$ is also periodic with period T and

$$x(-t) \leftrightarrow a_{-k}$$

- Thus:
  - If $x(t)$ is even, i.e., $x(-t) = x(t)$, then its Fourier series coefficients are also even, i.e., $a_{-k} = a_k$.
  - If $x(t)$ is odd, i.e., $x(-t) = -x(t)$, then its Fourier series coefficients are also odd, i.e., $a_{-k} = -a_k$.

Time Scaling

- Given that $x(t)$ is periodic with period T and that $x(t) \leftrightarrow a_k$, then $x(\alpha t)$, where α is a positive real number, is periodic with period T/α, and

$$x(\alpha t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk(\alpha\omega_0)t}$$

- That is:
  - The fundamental frequency of the Fourier series representation has changed.
  - However, the Fourier series coefficients have not changed.

Multiplication

- Given that $x(t)$ and $y(t)$ are both periodic with period T and that $x(t) \leftrightarrow a_k$, $y(t) \leftrightarrow b_k$, then the product $x(t)y(t)$ is also periodic with period T and the Fourier series coefficients h_k of $x(t)y(t)$ can be obtained as

$$h_k = \sum_{l=-\infty}^{+\infty} a_l b_{k-l}$$

Differentiation

- Given that $x(t)$ is periodic with period T and that $x(t) \leftrightarrow a_k$, then $\frac{dx(t)}{dt}$ is also periodic with period T and

$$\frac{dx(t)}{dt} \leftrightarrow (jk\omega_0)a_k$$

Parseval's Relation

- Given that $x(t)$ is periodic with period T and that $x(t) \leftrightarrow a_k$, then Parseval's relation states that

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

- Also, we have

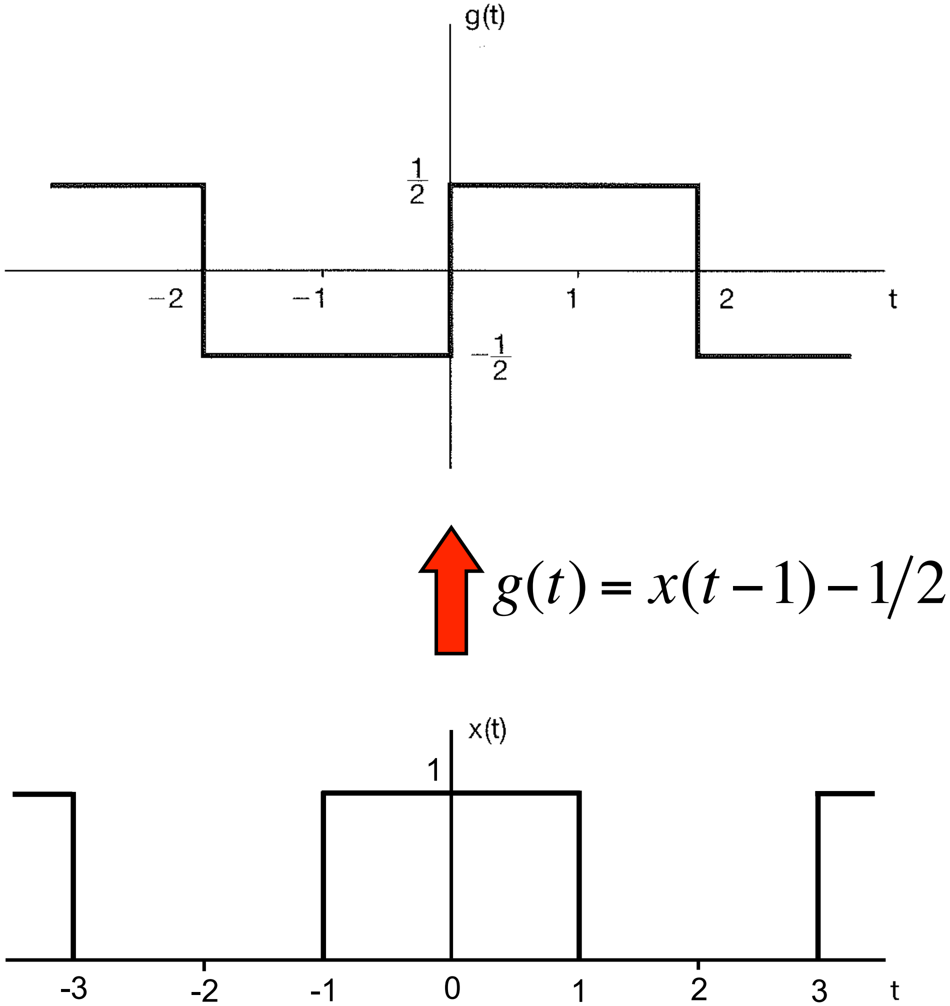
$$\frac{1}{T}\int_T \left|a_k e^{jk\omega_0 t}\right|^2 dt = |a_k|^2$$

- Thus, the total average power in $x(t)$ equals the sum of the average powers in all of its harmonic components.

Example 1

- Consider the signal $g(t)$ with period $T = 4$.
- Recall the periodic square wave $x(t)$ discussed on pages 11–14 with $T = 4$ and $\alpha = 1/4$.
- It is clear that $g(t)$ can be obtained from $x(t)$ as

$$g(t) = x(t-1) - 1/2$$



Example 1 (cont.)

- Using the results of (8) and (9) on Page 12, we have in this case the Fourier series coefficients a_k of $x(t)$ as

$$a_k = \begin{cases} 1/2, & k = 0 \\ \dfrac{\sin(k\pi/2)}{k\pi}, & k \neq 0 \end{cases}$$

- The time shift property of continuous-time Fourier series indicates that, if $x(t) \leftrightarrow a_k$, then the Fourier series coefficients b_k of $x(t-1)$ can be expressed as

$$b_k = a_k e^{-jk\pi/2} = \begin{cases} 1/2, & k = 0 \\ \dfrac{\sin(k\pi/2)}{k\pi} e^{-jk\pi/2}, & k \neq 0 \end{cases}$$

Example 1 (cont.)

- The Fourier series coefficients c_k of the constant $-1/2$ are simply

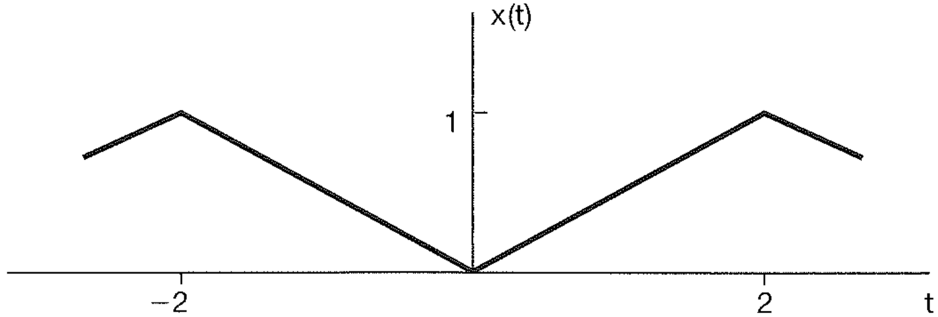
$$c_k = \begin{cases} -1/2, & k = 0 \\ 0, & k \neq 0 \end{cases}$$

- Applying the linearity property, the Fourier series coefficients d_k of $g(t)$ can be expressed as

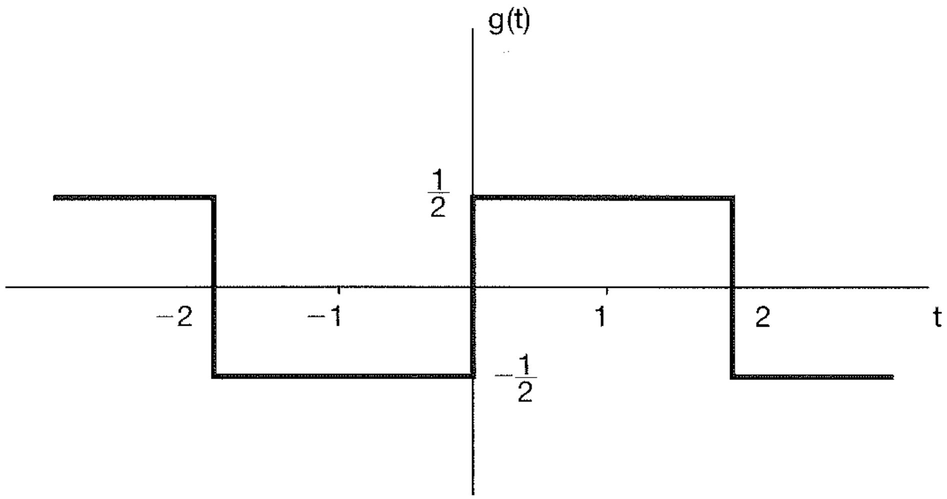
$$d_k = b_k + c_k = \begin{cases} 0, & k = 0 \\ \dfrac{\sin(k\pi/2)}{k\pi} e^{-jk\pi/2}, & k \neq 0 \end{cases} \tag{10}$$

Example 2

- Consider the triangular wave signal $x(t)$ with period $T = 4$ and hence $\omega_0 = \pi/2$.
- The derivative of $x(t)$ is the signal $g(t)$ in Example 1.



$$g(t) = \frac{dx(t)}{dt}$$



Example 2 (cont.)

- The differentiation property of continuous-time Fourier series indicates that, if $g(t) \leftrightarrow d_k$, then the Fourier series coefficients a_k of $x(t)$ can be obtained from

$$d_k = jk(\pi/2)a_k \quad \Rightarrow \quad a_k = \frac{2d_k}{jk\pi}, \quad \text{for} \quad k \neq 0$$

  - Thus, using the results of d_k of $g(t)$ in (10) on Page 25, we obtain

$$a_k = \frac{2\sin(k\pi/2)}{j(k\pi)^2}e^{-jk\pi/2}, \quad \text{for} \quad k \neq 0$$

  - For $k = 0$, a_0 can be obtained from

$$a_0 = \frac{1}{T}\int_0^T x(t)dt = \frac{1}{2}$$