

In-Class Exercise 6

1. Consider two random variables X and Y with joint probability mass function (PMF) given in the following table:

	$Y = 0$	$Y = 1$	$Y = 2$
$X = 0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$
$X = 1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$

- (a) Find $P(X = 0, Y \leq 1)$.
- (b) Find the marginal PMFs of X and Y .
- (c) Are X and Y independent? Briefly explain your answer.

2. Consider two random variables X and Y with the following joint PMF:

$$P_{XY}(x, y) = \begin{cases} cxy, & x = 1, 2, 4, \ y = 1, 3 \\ 0, & \text{otherwise} \end{cases}$$

where c is an unknown constant.

- (a) Find the value of c .
- (b) Find $P(Y = 3)$.
- (c) Find $P(Y < X)$.
- (d) Find $P(Y > X)$.
- (e) Find $P(X = Y)$.
- (f) Find the marginal PMFs of X and Y .
- (g) Are X and Y independent? Briefly explain your answer.
- (h) Find $\mathbb{E}\{X\}$ and $\mathbb{E}\{Y\}$.
- (i) Find $\text{var}(X)$ and $\text{var}(Y)$.

3. Consider two independent Bernoulli variables X and Y with success probabilities p_1 and p_2 , respectively. Find the joint PMF and cumulative density function (CDF) for X and Y .

4. Show

$$\text{var} \left(\sum_{i=1}^n X_i \right) = \sum_{i=1}^n \text{var}(X_i) + 2 \sum_{i=1}^n \sum_{j < i} \text{cov}(X_i, X_j)$$

for $n = 2$.

5. A bag consists of 100 balls where 40 are blue and 60 are red. 10 balls are randomly chosen at the same time. Let random variables X and Y be the numbers of blue and red balls. Find the joint PMF of X and Y . Which combination of blue and red balls has the highest probability?

Solution

1.(a)

$$\begin{aligned}P(X = 0, Y \leq 1) &= P(X = 0, Y = 0) + P(X = 0, Y = 1) \\&= \frac{1}{6} + \frac{1}{4} = \frac{5}{12}\end{aligned}$$

1.(b)

$$\begin{aligned}P(X = 0) &= P(X = 0, Y = 0) + P(X = 0, Y = 1) + P(X = 0, Y = 1) \\&= \frac{1}{6} + \frac{1}{4} + \frac{1}{8} = \frac{13}{24}\end{aligned}$$

$$\begin{aligned}P(X = 1) &= P(X = 1, Y = 0) + P(X = 1, Y = 1) + P(X = 1, Y = 1) \\&= \frac{1}{8} + \frac{1}{6} + \frac{1}{6} = \frac{11}{24}\end{aligned}$$

Combining the results yields:

$$p(x) = \begin{cases} \frac{13}{24}, & x = 0 \\ \frac{11}{24}, & x = 1 \\ 0, & \text{otherwise} \end{cases}$$

Similarly,

$$p(y) = \begin{cases} \frac{7}{24}, & y = 0 \\ \frac{5}{12}, & y = 1 \\ \frac{7}{24}, & y = 2 \\ 0, & \text{otherwise} \end{cases}$$

1.(c)

If X and Y are independent, this means:

$$p(x, y) = p(x)p(y)$$

for all x and y . Consider when $x = 0$ and $y = 0$, the joint PMF is:

$$p(0, 0) = \frac{1}{6}$$

which cannot be equal to the product of

$$P_X(0) = \frac{13}{24} \quad \text{and} \quad P_Y(0) = \frac{7}{24}$$

Hence X and Y are not independent.

2.(a)

As the sum of all PMFs is equal to 1, we have:

$$\sum_{x=1,2,4} \sum_{y=1,3} cxy = c \sum_{x=1,2,4} x \sum_{y=1,3} y = c[1(1+3) + 2(1+3) + 4(1+3)] = 1$$
$$\Rightarrow 28c = 1 \Rightarrow c = \frac{1}{28}$$

2.(b)

$$P(Y = 3) = \frac{1}{28} \sum_{y=3} y \sum_{x=1,2,4} x = \frac{1}{28} 3(1+2+4) = \frac{3}{4}$$

2.(c)

$$P(Y < X) = \frac{1}{28} \sum_{x=1,2,4} x \sum_{y < x} y = \frac{[1(0) + 2(1) + 4(1+3)]}{28} = \frac{9}{14}$$

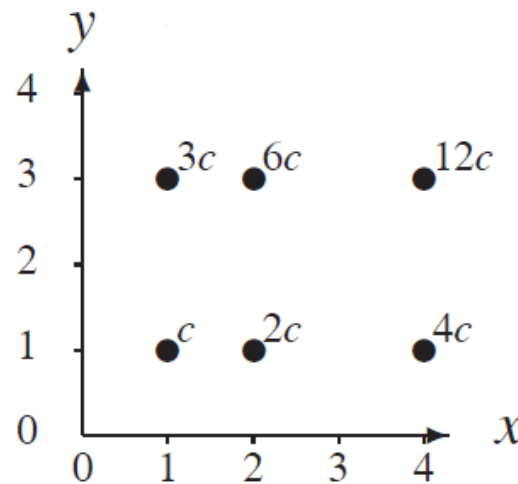
2.(d)

$$P(Y > X) = \frac{1}{28} \sum_{x=1,2,4} x \sum_{y>x} y = \frac{[1(3) + 2(3) + 4(0)]}{28} = \frac{9}{28}$$

2.(e)

$$P(Y = X) = 1 - P(Y < X) - P(Y > X) = 1 - \frac{9}{14} - \frac{9}{28} = \frac{1}{28}$$

Alternatively, the problem can be solved by first showing the nonzero PMFs in a 2-D plot:



2.(f)

From the plot, we can easily obtain:

$$p(x) = \begin{cases} \frac{1}{7}, & x = 1 \\ \frac{2}{7}, & x = 2 \\ \frac{4}{7}, & x = 3 \\ 0, & \text{otherwise} \end{cases}$$

Similarly, we also obtain:

$$p(y) = \begin{cases} \frac{1}{4}, & y = 1 \\ \frac{3}{4}, & y = 3 \\ 0, & \text{otherwise} \end{cases}$$

2.(g)

As $p(x, y) = p(x)p(y)$ holds for all pairs of x and y , X and Y are independent.

It can also be observed from

$$P_{XY}(x, y) = \frac{xy}{28} = \frac{x}{7} \cdot \frac{y}{4} = P_X(x) \cdot P_Y(y), \quad x = 1, 2, 4, \quad y = 1, 3$$

2.(h)

Using (2.19), the expected values are computed as:

$$\mathbb{E}\{X\} = \sum_{x=1,2,4} xp(x) = (1)\frac{1}{7} + (2)\frac{2}{7} + (4)\frac{4}{7} = 3$$

$$\mathbb{E}\{Y\} = \sum_{y=1,3} yp(y) = (1)\frac{1}{4} + (3)\frac{3}{4} = \frac{5}{2}$$

2.(i)

$$\mathbb{E}\{X^2\} = \sum_{x=1,2,4} x^2 p(x) = (1)^2 \frac{1}{7} + (2)^2 \frac{2}{7} + (4)^2 \frac{4}{7} = \frac{73}{7}$$

$$\mathbb{E}\{Y^2\} = \sum_{y=1,3} y^2 p(y) = (1)^2 \frac{1}{4} + (3)^2 \frac{3}{4} = 7$$

Applying (2.23) yields:

$$\text{var}(X) = \mathbb{E}\{X^2\} - (\mathbb{E}\{X\})^2 = \frac{10}{7}$$

$$\text{var}(Y) = \mathbb{E}\{Y^2\} - (\mathbb{E}\{Y\})^2 = \frac{3}{4}$$

3.

We know that both X and Y only have values 0 or 1. Hence the PMF is:

$$P(X = 0, Y = 0) = (1 - p_1)(1 - p_2)$$

$$P(X = 1, Y = 0) = p_1(1 - p_2)$$

$$P(X = 0, Y = 1) = (1 - p_1)p_2$$

$$P(X = 1, Y = 1) = p_1p_2$$

Note that when $p = p_1 = p_2$, it will reduce to the PMF of binomial distribution with $n = 2$.

For the CDF, it is clear that

$$F(x, y) = 0, \quad x < 0$$

$$F(x, y) = 0, \quad y < 0$$

$$F(x, y) = 1, \quad x \geq 1, y \geq 1$$

For $0 \leq x < 1$ and $y \leq 1$, we have:

$$P(X = 0, Y \leq 1) = (1 - p_1)(1 - p_2) + (1 - p_1)p_2 = 1 - p_1$$

For $x \leq 1$ and $0 \leq y < 1$, we have:

$$P(X \leq 1, Y = 0) = (1 - p_1)(1 - p_2) + p_1(1 - p_2) = 1 - p_2$$

For $0 \leq x < 1$ and $0 \leq y < 1$, we have:

$$P(X = 0, Y = 0) = (1 - p_1)(1 - p_2)$$

Combining them yields:

$$F(x, y) = \begin{cases} 0, & x < 0 \\ 0, & y < 0 \\ 1 - p_1, & 0 \leq x < 1, y \leq 1 \\ 1 - p_2, & x \leq 1, 0 \leq y < 1 \\ (1 - p_1)(1 - p_2), & 0 \leq x < 1, 0 \leq y < 1 \\ 1, & x \geq 1, y \geq 1 \end{cases}$$

4.

$$\begin{aligned}\text{cov}\left(\sum_{i=1}^2 X_i, \sum_{j=1}^2 X_j\right) &= \sum_{i=1}^2 \sum_{j=1}^2 \text{cov}(X_i, X_j) \\ &= \text{cov}(X_1, X_1) + \text{cov}(X_2, X_2) \\ &\quad + \text{cov}(X_1, X_2) + \text{cov}(X_2, X_1) \\ &= \text{var}(X_1) + \text{var}(X_2) + 2\text{cov}(X_2, X_1) \\ &= \sum_{i=1}^2 \text{var}(X_i) + 2 \sum_{i=2}^2 \sum_{j<i} \text{cov}(X_i, X_j) \\ &= \sum_{i=1}^2 \text{var}(X_i) + 2 \sum_{i=1}^2 \sum_{j<i} \text{cov}(X_i, X_j)\end{aligned}$$

5.

Here, there is a constraint of $X + Y = 10$. Hence the combination of (X, Y) includes $(0, 10), (1, 9), \dots, (10, 0)$. For example, the probability of having 0 blue ball and 10 red balls is:

$$\frac{\binom{60}{10}}{\binom{100}{10}}$$

Similarly, the probability of having 1 blue ball and 9 red balls is:

$$\frac{\binom{40}{1} \cdot \binom{60}{9}}{\binom{100}{10}}$$

We can write the joint PMF as:

$$p(i, j) = \begin{cases} \frac{\binom{40}{i} \cdot \binom{60}{j}}{\binom{100}{10}}, & i + j = 10, 10 \geq i \geq 0, 10 \geq j \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Intuitively, the probability of getting a blue ball is 4/10 while that of red ball is 6/10. Hence we expect the highest chance is:

$$p(i, j) = \frac{\binom{40}{4} \cdot \binom{60}{6}}{\binom{100}{10}} = 0.2643$$

We can compute all PMF values to check it. If we vary the value of i in the range of $10 \geq i \geq 0$, then the value of j is fixed because of the relationship of $i + j = 10$.

```
>> for i=0:10
    (nchoosek(40,i)*nchoosek(60,10-i))/nchoosek(100,10)
end
ans = 0.0044
ans = 0.0342
ans = 0.1153
ans = 0.2204
ans = 0.2643
ans = 0.2076
ans = 0.1081
ans = 0.0369
ans = 0.0079
ans = 9.4778e-04
ans = 4.8969e-05
```