In-Class Exercise 3

1. A continuous-time periodic real-valued signal $x(t)$ has a fundamental period of $T=8$. Its Fourier coefficients are:

$$a_k = \begin{cases} -4j, & k=-3 \\ 2, & k=-1 \\ 2, & k=1 \\ 4j, & k=3 \\ 0, & \text{otherwise} \end{cases}$$

Express $x(t)$ in the form:

$$x(t) = \sum_{k=0}^{\infty} A_k \cos(\omega_k t + \phi_k)$$

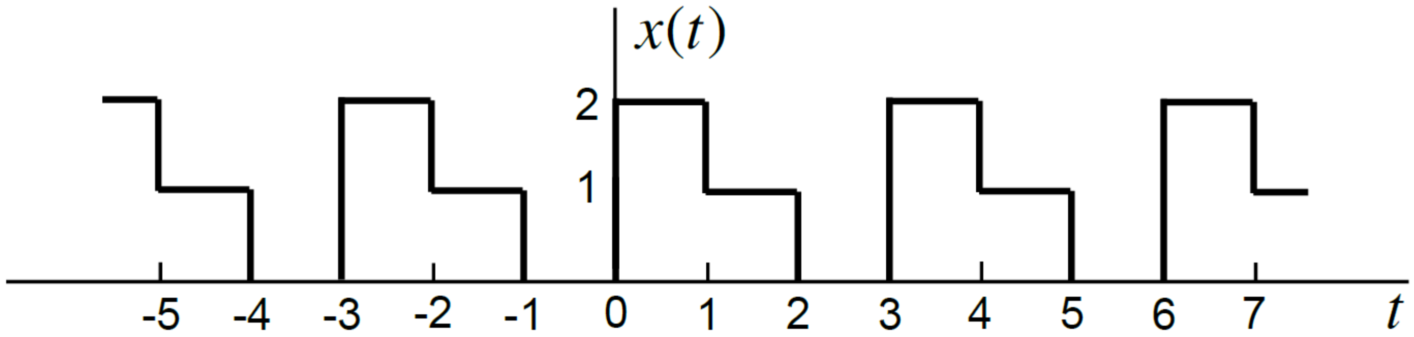
2. Prove the conjugation property of Fourier series:

$$x(t) \leftrightarrow a_k \Rightarrow x^*(t) \leftrightarrow a^*_{-k}$$

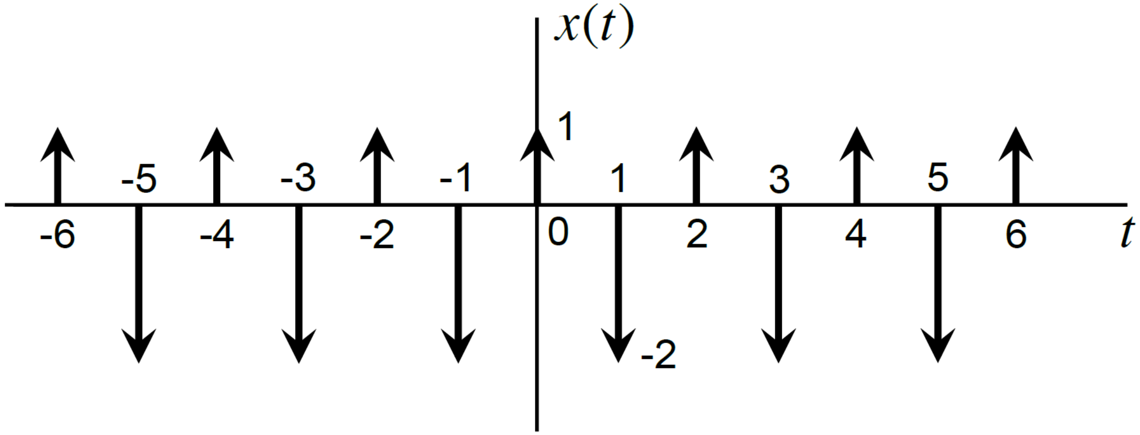
Then show that if $x(t)$ is real-valued, then the magnitudes of Fourier series coefficients are symmetric around $k = 0$:

$$|a_k| = |a_{-k}|$$

3. Determine the Fourier series coefficients of the following continuous-time periodic signal $x(t)$:



4. Determine the Fourier series coefficients of the following continuous-time periodic signal $x(t)$:



5. Consider a continuous-time linear time-invariant system whose frequency response is:

$$H(j\Omega) = \int_{-\infty}^{\infty} h(t)e^{-j\Omega t}dt = \frac{\sin(4\Omega)}{\Omega}$$

where $h(t)$ is the impulse response.

If a periodic signal $x(t)$ with fundamental period $T=8$ and within the interval $(0,8)$, $x(t)$ is:

$$x(t)=\begin{cases}1, & 0<t<4\\ -1, & 4<t<8\end{cases}$$

Determine the system output $y(t)=x(t)\otimes h(t)$.

Solution

1.

The fundamental period is:

$$\Omega_0 = \frac{2\pi}{T} = \frac{\pi}{4}$$

According to (4.3), we have:

$$\begin{aligned}
x(t) &= a_{-3}e^{-3j\Omega_0 t} + a_{-1}e^{-j\Omega_0 t} + a_1 e^{j\Omega_0 t} + a_3 e^{3j\Omega_0 t} \\
&= -4je^{-3j(\pi/4)t} + 2e^{-j(\pi/4)t} + 2e^{j(\pi/4)t} + 4je^{3j(\pi/4)t} \\
&= (4je^{3j(\pi/4)t})^* + (2e^{j(\pi/4)t})^* + 2e^{j(\pi/4)t} + 4je^{3j(\pi/4)t} \\
&= (4e^{j\pi/2}e^{3j(\pi/4)t})^* + (2e^{j(\pi/4)t})^* + 2e^{j(\pi/4)t} + 4e^{j\pi/2}e^{3j(\pi/4)t} \\
&= 2\Re\{2e^{j(\pi/4)t}\} + 2\Re\{4e^{j\pi/2}e^{3j(\pi/4)t}\} \\
&= 4\cos\left(\frac{\pi}{4}t\right) + 8\cos\left(\frac{3\pi}{4}t + \pi/2\right)
\end{aligned}$$

2.
Recall (4.3):

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t}$$

Taking the complex conjugate operation on both sides:

$$x^*(t) = \left(\sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t} \right)^* = \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\Omega_0 t} = \sum_{l=-\infty}^{\infty} a_{-l}^* e^{jl\Omega_0 t}, \quad l = -k$$

Hence we have $x^*(t) \leftrightarrow a_{-k}^*$.

If $x(t)$ is real-valued, then $x(t) = x^*(t)$. Their Fourier series coefficients should be identical, i.e.,

$$a_k = a_{-k}^* \Rightarrow |a_k| = |a_{-k}|$$

3.

Noting that the fundamental period is $T=3$ and the fundamental frequency is $\Omega_0 = 2\pi/3$. In the period at the interval $[0,3]$, $x(t)$ is expressed as:

$$x(t) = \begin{cases} 2, & 0<t<1 \\ 1, & 1<t<2 \\ 0, & 2<t<3 \end{cases}$$

Using (4.4), we get:

$$a_k = \frac{1}{T}\int_0^T x(t)e^{-jk\Omega_0 t}dt = \frac{1}{3}\left(\int_0^1 2e^{-jk(2\pi/3)t}dt + \int_1^2 e^{-jk(2\pi/3)t}dt\right)$$

For $k = 0$:

$$a_0 = \frac{1}{3}\int_0^1 2dt + \frac{1}{3}\int_1^2 dt = 1$$

For $k \neq 0$:

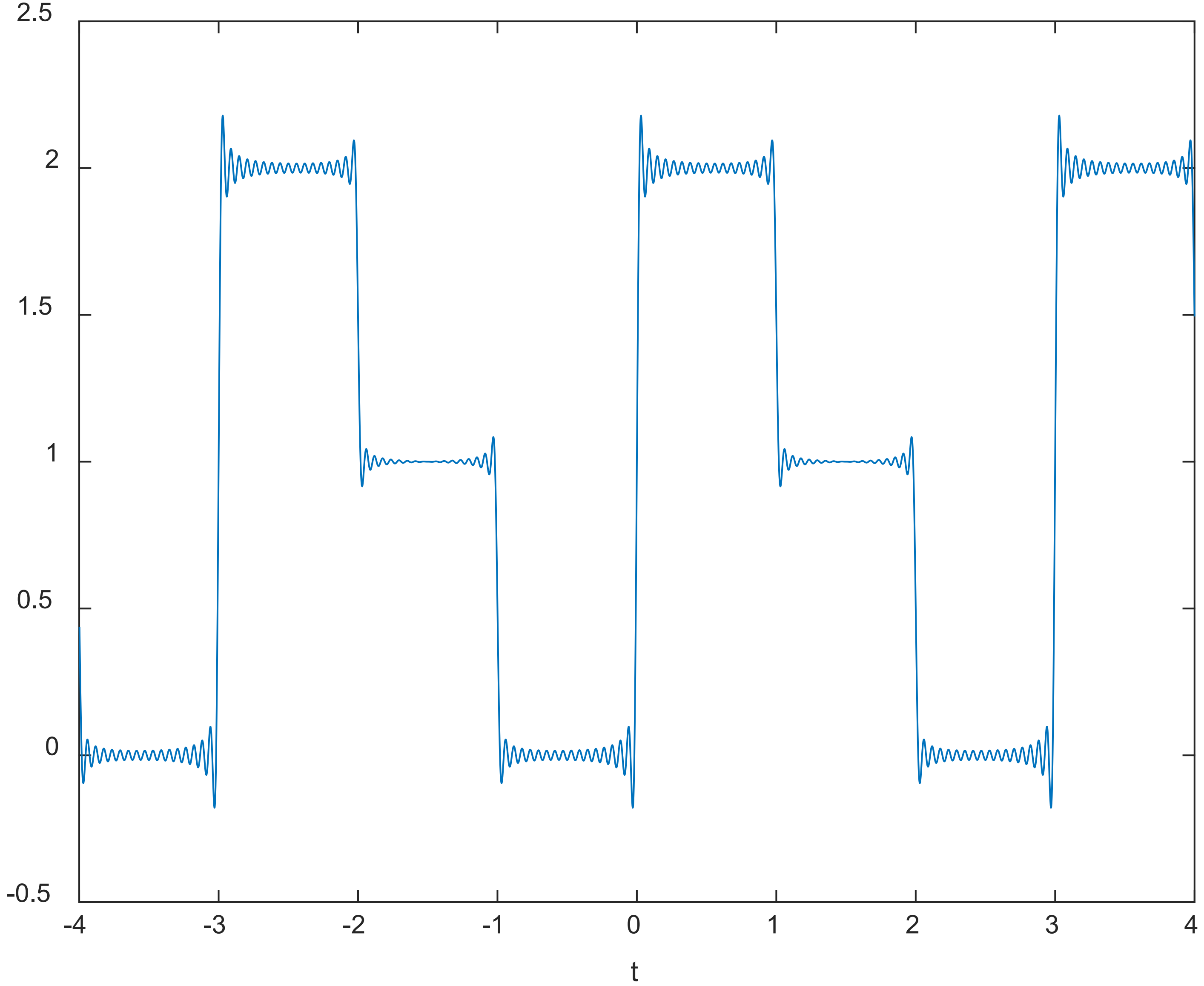
$$a_k = \frac{2}{3}\int_0^1 e^{-jk(2\pi/3)t}dt + \frac{1}{3}\int_1^2 e^{-jk(2\pi/3)t}dt = \frac{2 - e^{-jk2\pi/3} - e^{-jk4\pi/3}}{jk2\pi}$$

Nevertheless, for $k = 0$, we can use $\mathrm{L'H\hat{o}pital}$'s rule:

$$\lim_{k \rightarrow 0} \frac{2 - e^{-jk2\pi/3} - e^{-jk4\pi/3}}{jk2\pi} = 1$$

Setting $K = 50$, we may use the following code:

```
K=50;
a_p=(2-exp(-j.*[1:K].*2*pi/3)-exp(-
j.*[1:K].*4*pi/3))./(j.*[1:K].*2*pi); % +ve a_k
a_n=(2-exp(-j.*[-K:-1].*2*pi/3)-exp(-j.*[-K:-
1].*4*pi/3))./(j.*[-K:-1].*2*pi); %-ve a_k
a = [a_n 1 a_p]; %construct vector of a_k
for n=1:4000
 t=(n-2000)/500; %time interval of (-8,8);
%small sampling interval of 1/500 to approximate x(t);
 e = (exp(j.*[-K:K].*2*pi/3.*t)); %construct exponential
vector
 x(n) = a*e.';
end
x=real(x); %remove imaginary parts due to precision error
n=1:4000;
t=(n-2000)./500;
plot(t,x)
xlabel('t')
```


2.5
2
1.5
1
0.5
0
-0.5
-4
-3
-2
-1
0
1
2
3
4
t

4.

Noting that the fundamental period is $T=2$ and the fundamental frequency is $\Omega_0 = 2\pi/2 = \pi$. In the period at the interval $[-0.5, 1.5]$ where there will be no impulses at the integration limits, $x(t)$ is expressed as:

$$x(t) = \delta(t) - 2\delta(t-1)$$

Using (4.4), we get:

$$\begin{aligned} a_k &= \frac{1}{2}\int_{-1/2}^{3/2} \left[\delta(t) - 2\delta(t-1)\right] e^{-jk\pi t} dt \\ &= \frac{1}{2}\int_{-1/2}^{3/2} \delta(t) e^{-jk\pi t} dt - \int_{-1/2}^{3/2} \delta(t-1) e^{-jk\pi t} dt \\ &= \frac{1}{2} - e^{-jk\pi} = \frac{1}{2} - \left(e^{-j\pi}\right)^k = \frac{1}{2} - (-1)^k \end{aligned}$$

5.

With the use of the results in Example 4.4 and applying the linearity and time scaling properties of Fourier series, the Fourier series coefficients are:

$$a_k = \begin{cases} 0, & k = 0 \\ \dfrac{1}{jk\pi}\left[1 - \cos(k\pi)\right], & k \neq 0 \end{cases}$$

Here, $\Omega_0 = 2\pi/8 = \pi/4$. According to (4.18), we only need to consider the values of $H(j\Omega)$ at $k = 0, \pm\Omega_0, \pm 2\Omega_0, \cdots$.

Since

$$\frac{\sin(4(\pi/4)k)}{(\pi/4)} = 0, \quad k = \pm 1, \pm 2, \cdots$$

and $a_0 = 0$, we get $y(t) = 0$.