In-Class Exercise 1

1. Consider an experiment of rolling 3 fair dice where the sum of the faces is recorded.
   (a) Determine the sample space.
   (b) Determine the number of possible events.
   (c) Find the probability that the sum is (i) 3; (ii) 10; and (iii) 9.

2. Consider an experiment of tossing a fair coin. Find the probability that a tail will be resulted in 10 trials. That is, it includes the cases that a tail is obtained in the 1st trial up to the 10th trial. Also, when a tail appears in the 1st trial, then the experiment stops.

3. Two events A and B are related as $B \subset A$. Prove

$$P(B) \leq P(A)$$

Hint: Express A in terms of B and $(\overline{B} \cap A)$ and apply Axioms.

4. For any events A and B, prove

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

5. 3 fair dice are rolled. What is the probability of obtaining at least one "6"?

6. Extend the result of $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ to the case of 3 events A, B, and C: $P(A \cup B \cup C)$.

Solution

1(a)

Since the minimum and maximum number are 3 and 18, respectively, the sample space is:

$$\{3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18\}$$

1(b)

Since there are 16 possible outcomes in the sample space, the number of possible events is $2^{16} = 65536$.

1(c)(i)

The chance of having 3 only when all faces are 1. Since each face has the same probability of 1/6, and there is only one combination for 1+1+1, we have:

$$P(\{3\}) = \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{216}$$

1(c)(ii)

Recall:

10= 6+3+1 = 6+2+2 = 5+4+1 = 5+3+2 = 4+4+2 = 4+3+3

For "6+3+1", there are in fact 6 combinations with "6+1+3", "3+1+6", "3+6+1", "1+3+6", "1+6+3".

There are also 6 combinations for "5+4+1" and "5+3+2".

While for "6+2+2", "4+4+2" and "4+3+3", there are only 3 combinations. As a result,

$$P(\{10\}) = \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot [6+3+6+6+3+3] = \frac{27}{216} = \frac{1}{8}$$

1(c)(iii)

9= 6+2+1 = 5+3+1 = 5+2+2 = 4+4+1 = 4+3+2 = 3+3+3

$$P(\{9\}) = \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot [6+6+3+3+6+1] = \frac{25}{216}$$

2.

Denote head and tail as H and T. For fair coin, we know:

$$P(H) = P(T) = \frac{1}{2}$$

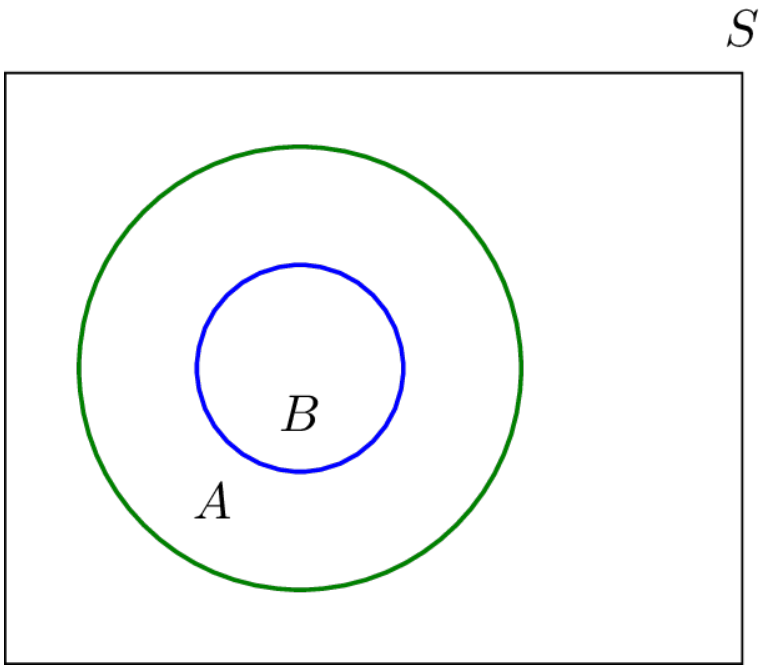
From Example 1.2, possible outcomes are:

x_1: T
x_2: H,T
x_3: H,H,T
...

The required probability should be:

$$\sum_{n=1}^{10} P(x_n) = \sum_{n=1}^{10} 0.5^n = 0.5\frac{1-0.5^{10}}{1-0.5} = 0.999$$

3.



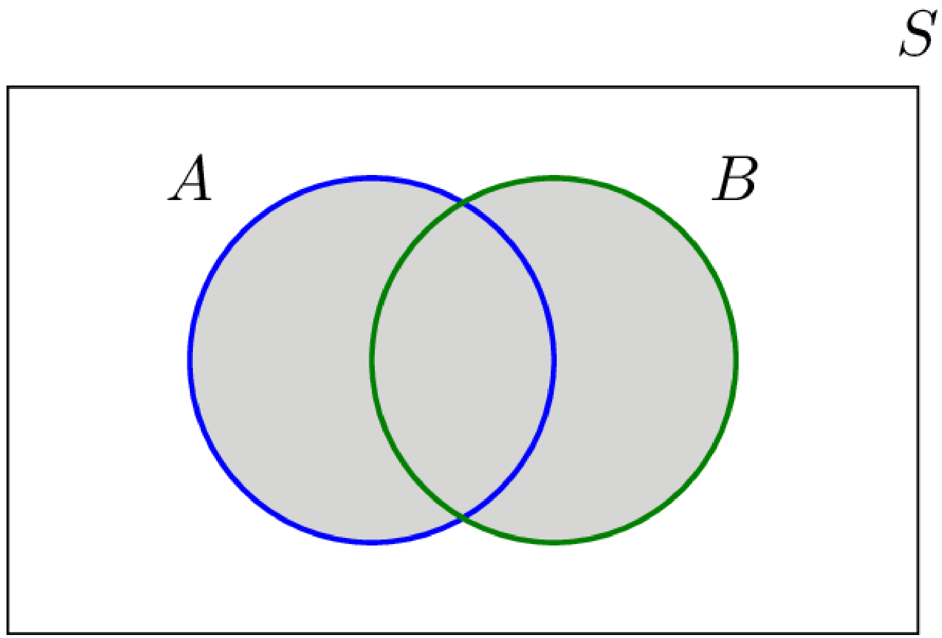
As $B \subset A$, we can express

$$A = (B \cap A) \cup (\overline{B} \cap A) = B \cup (\overline{B} \cap A), \quad B \cap (\overline{B} \cap A) = \emptyset$$

Applying Axioms 1 and 3 yields:

$$P(A) = P(B) + P(\overline{B} \cap A) \geq P(B)$$

4.



$$A \cup B = A \cup (\overline{A} \cap B), \quad A \cap (\overline{A} \cap B) = \emptyset$$

$$B = (A \cap B) \cup (\overline{A} \cap B), \quad (A \cap B) \cap (\overline{A} \cap B) = \emptyset$$

Applying Axiom 3 yields:

$P(A \cup B) = P(A) + P(\overline{A} \cap B)$ and $P(B) = P(A \cap B) + P(\overline{A} \cap B)$

The proof is then obtained from the two equations.

5.

At least one “6” can mean one, two or three “6”. One direction is to compute the probabilities of these 3 cases and sum together. On the other hand, we can make use of the complement rule in (1.1):

$$\overline{\bigcup_{n=1}^{N} A_n} = \overline{A_1 \cup A_2 \cup \cdots \cup A_N} = \overline{A}_1 \cap \overline{A}_2 \cap \cdots \cap \overline{A}_N = \bigcap_{n=1}^{N} \overline{A}_n$$

That is, we first compute the probability that there is no “6”, and then subtract this probability from 1:

$$P(\geq 1 \text{ ''6''}) = 1 - \left(\frac{5}{6}\right)^3 = \frac{91}{216}$$

6.

With the use of the Venn diagram or by other means, we get:

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C)$$
$$-P(B \cap C) + P(A \cap B \cap C)$$