# EE3210
# Signals and Systems

## Part 8: Discrete-Time Fourier Series

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# Changes of Part6_v1 Lecture Notes

- Page 12, change
  - ... assume that the integral ...

  to

  - ... assume that the summation ...

# Changes of Part7_v1 Lecture Notes

- Pages 5–8 have been revised. The purpose is to show, in a more rigorous way, why we will obtain the same result for the analysis formula of the continuous-time Fourier series if the integration is performed over any interval of length $T$.

# Changes of Part7_v1 Lecture Notes (cont.)

- Page 26, change the figure

$$x(t) = \frac{dg(t)}{dt}$$

to

$$g(t) = \frac{dx(t)}{dt}$$

# Discrete-Time Periodic Complex Exponentials

- In contrast to continuous-time complex exponentials, a discrete-time complex exponential of the form $e^{j\Omega n}$ is periodic only if $|\Omega|/(2\pi)$ is a rational number.
  - The fundamental period $N_0$ of $e^{j\Omega n}$, if it is periodic, is $N_0 = 2\pi m/|\Omega|$ given that the pair of integers $N_0$ and $m$ have no factors in common.

- A harmonically related set of discrete-time complex exponentials, all of which have a common period $N$ with fundamental frequency $\Omega_0 = 2\pi/N$, is defined as

$$\phi_k[n] = e^{jk\Omega_0 n} = e^{jk(2\pi/N)n}, \quad k = 0, \pm 1, \pm 2, \ldots \tag{1}$$

# Discrete-Time Periodic Complex Exponentials (cont.)

- Since

$$\phi_{k+N}[n] = e^{j(k+N)(2\pi/N)n} = e^{jk(2\pi/N)n} e^{j2\pi n} = \phi_k[n]$$

  this implies that there are only $N$ distinct members in the set $\phi_k[n]$ defined by (1).

  - For example, $\phi_0[n] = 1$, $\phi_1[n] = e^{j2\pi n/N}$, $\ldots$, and $\phi_{N-1}[n] = e^{j2\pi(N-1)n/N}$ are all distinct. Any other $\phi_k[n]$ is identical to one of these (e.g., $\phi_N[n] = \phi_0[n]$).
  - Note: This differs from the situation in continuous time in which we have an infinite number of harmonically related complex exponentials that are all different from one another.

# Discrete-Time Periodic Complex Exponentials (cont.)

- Thus, in forming a linear combination of harmonically related discrete-time complex exponentials, we only need to consider a set of $\phi_k[n]$ over any range of $N$ successive values of $k$, which is of the form

$$\sum_{k=\langle N\rangle} a_k\phi_k[n] = \sum_{k=\langle N\rangle} a_k e^{jk\Omega_0 n} = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n} \qquad (2)$$

  - Note: $\sum_{k=\langle N\rangle}$ is a shorthand notation, which has the same effect as $\sum_{k=q}^{q+N-1}$ for any integer number $q$.

# Fourier Series Representation of Discrete-Time Periodic Signals

- Consider a discrete-time periodic signal $x[n]$ with fundamental period $N_0 = N$.
- Assume $x[n]$ can be represented with the series of (2):

$$\boxed{x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\Omega_0 n} = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}} \qquad (3)$$

  with fundamental frequency $\Omega_0 = 2\pi/N$.
  - (3) is known as the synthesis formula of the discrete-time Fourier series.

# Fourier Series Representation of Discrete-Time Periodic Signals (cont.)

- Now, we need a procedure for determining the Fourier series coefficients $a_k$ in (3).

- Multiplying both sides of (3) by $e^{-jr\Omega_0 n}$ for an arbitrary integer $r$, we obtain

$$\begin{aligned} x[n]e^{-jr\Omega_0 n} &= \sum_{k=\langle N\rangle} a_k e^{jk\Omega_0 n} e^{-jr\Omega_0 n} \\ &= \sum_{k=\langle N\rangle} a_k e^{j(k-r)\Omega_0 n} \end{aligned} \tag{4}$$

# Fourier Series Representation of Discrete-Time Periodic Signals (cont.)

- Summing both sides of (4) with the limits of the summation as $n = \langle N \rangle$, we have

$$\begin{aligned} \sum_{n=\langle N \rangle} x[n] e^{-jr\Omega_0 n} &= \sum_{n=\langle N \rangle} \sum_{k=\langle N \rangle} a_k e^{j(k-r)\Omega_0 n} \\ &= \sum_{k=\langle N \rangle} a_k \sum_{n=\langle N \rangle} e^{j(k-r)\Omega_0 n} \end{aligned} \tag{5}$$

# Fourier Series Representation of Discrete-Time Periodic Signals (cont.)

- We observe in the right-hand side of (5) that, if $r$ is chosen from the same range of $N$ successive values as that over which $k$ varies, then:
  - For $k = r$, we have $\displaystyle\sum_{n=\langle N\rangle} e^{j(k-r)\Omega_0 n} = \sum_{n=q}^{q+N-1} 1 = N$
  - For $k \neq r$, we have

$$\sum_{n=\langle N\rangle} e^{j(k-r)\Omega_0 n} = \sum_{n=q}^{q+N-1} \left[e^{j(k-r)\Omega_0}\right]^n$$

$$= \frac{e^{j(k-r)\Omega_0 q}\left[1 - e^{j(k-r)\Omega_0 N}\right]}{1 - e^{j(k-r)\Omega_0}} = \frac{e^{j(k-r)\Omega_0 q}\left[1 - e^{j(k-r)2\pi}\right]}{1 - e^{j(k-r)\Omega_0}} = 0$$

## Fourier Series Representation of Discrete-Time Periodic Signals (cont.)

- Thus,

$$\sum_{k=\langle N\rangle} a_k \sum_{n=\langle N\rangle} e^{j(k-r)\Omega_0 n} = a_r N \quad (6)$$

- By (5) and (6), we obtain $\sum_{n=\langle N\rangle} x[n]e^{-jr\Omega_0 n} = a_r N$

and hence

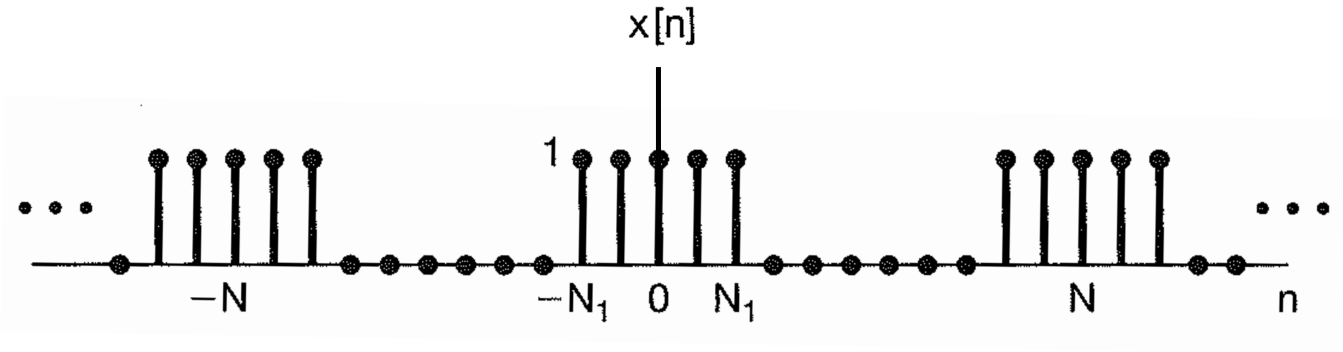
$$\boxed{a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk\Omega_0 n} = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk(2\pi/N)n}} \quad (7)$$

  - (7) is known as the analysis formula of the discrete-time Fourier series, and $a_k$ is periodic with period $N$.

# Convergence of Discrete-Time Fourier Series

- The Fourier series representation of a discrete-time periodic signal in (3) is a finite series.
  - This is in contrast to the infinite series representation required for continuous-time periodic signals.
  - As a consequence, there are no mathematical issues of convergence for discrete-time Fourier series.

# An Example




- Consider the discrete-time periodic square wave $x[n]$ with fundamental period $N_0 = N$.
  - The fundamental frequency of its Fourier series representation is $\Omega_0 = 2\pi/N$.
  - Because $x[n] = 1$ for $-N_1 \leq n \leq N_1$, it is convenient to choose the limits of the summation in (7) to include the range $-N_1 \leq n \leq N_1$.

# An Example (cont.)

- Then, in this case, we can express (7) as

$$a_k = \frac{1}{N} \sum_{n=-N_1}^{N_1} e^{-jk\Omega_0 n} \tag{8}$$

- Solving (8) for $k = 0$, we have

$$a_0 = \frac{1}{N} \sum_{n=-N_1}^{N_1} 1 = \frac{2N_1 + 1}{N} \tag{9}$$

# An Example (cont.)

- Solving (8) for $k \neq 0$ and letting $m = n + N_1$, we have

$$
\begin{aligned}
a_k &= \frac{1}{N}\sum_{m=0}^{2N_1} e^{-jk\Omega_0(m-N_1)} = \frac{1}{N}e^{jk\Omega_0 N_1}\sum_{m=0}^{2N_1} e^{-jk\Omega_0 m} \\
&= \frac{1}{N}e^{jk\Omega_0 N_1}\left[\frac{1-e^{-jk\Omega_0(2N_1+1)}}{1-e^{-jk\Omega_0}}\right] \\
&= \frac{e^{-jk\Omega_0/2}\left[e^{jk\Omega_0(N_1+1/2)} - e^{-jk\Omega_0(N_1+1/2)}\right]}{Ne^{-jk\Omega_0/2}\left[e^{jk\Omega_0/2} - e^{-jk\Omega_0/2}\right]} \\
&= \frac{\sin[k\Omega_0(N_1+1/2)]}{N\sin(k\Omega_0/2)} = \frac{\sin[2k\pi(N_1+1/2)/N]}{N\sin(k\pi/N)}
\end{aligned}
\tag{10}
$$

# Properties of Discrete-Time Fourier Series

- Here, we will describe several important properties, including: 1) linearity, 2) time shift, 3) time reversal, 4) multiplication, 5) first difference, and 6) Parseval's relation.
  - A summary of these and other important properties of discrete-time Fourier series can be found in Table 3.2 on Page 223 of the textbook.
- For notational convenience, we will use $x[n] \leftrightarrow a_k$ to indicate the relationship between a periodic signal $x[n]$ and its Fourier series coefficients $a_k$.

# Linearity

- Given that $x[n]$ and $y[n]$ are both periodic with period $N$ and that $x[n] \leftrightarrow a_k$, $y[n] \leftrightarrow b_k$, then $Ax[n] + By[n]$ is also periodic with period $N$ and

$$Ax[n] + By[n] \leftrightarrow Aa_k + Bb_k$$

where $A$ and $B$ are arbitrary constants.

# Time Shift

- Given that $x[n]$ is periodic with period $N$ and that $x[n] \leftrightarrow a_k$, then $x[n-n_0]$ is also periodic with period $N$ and

$$x[n-n_0] \leftrightarrow \left[e^{-jk(2\pi/N)n_0}\right] a_k$$

# Time Reversal

- Given that $x[n]$ is periodic with period $N$ and that $x[n] \leftrightarrow a_k$, then $x[-n]$ is also periodic with period $N$ and

$$x[-n] \leftrightarrow a_{-k}$$

- Thus:
  - If $x[n]$ is even, i.e., $x[-n] = x[n]$, then its Fourier series coefficients are also even, i.e., $a_{-k} = a_k$.
  - If $x[n]$ is odd, i.e., $x[-n] = -x[n]$, then its Fourier series coefficients are also odd, i.e., $a_{-k} = -a_k$.

# Multiplication

- Given that $x[n]$ and $y[n]$ are both periodic with period $N$ and that $x[n] \leftrightarrow a_k$, $y[n] \leftrightarrow b_k$, then the product $x[n]y[n]$ is also periodic with period $N$ and the Fourier series coefficients $h_k$ of $x[n]y[n]$ can be obtained as

$$h_k = \sum_{l=\langle N \rangle} a_l b_{k-l}$$

# First Difference

- Given that $x[n]$ is periodic with period $N$ and that $x[n] \leftrightarrow a_k$, then $x[n]-x[n-1]$ is also periodic with period $N$ and

$$x[n] - x[n-1] \leftrightarrow \left[1 - e^{-jk(2\pi/N)}\right] a_k$$

# Parseval's Relation

- Given that $x[n]$ is periodic with period $N$ and that $x[n] \leftrightarrow a_k$, then Parseval's relation states that

$$\frac{1}{N} \sum_{n=\langle N \rangle} |x[n]|^2 = \sum_{k=\langle N \rangle} |a_k|^2$$

- Also, we have

$$\frac{1}{N} \sum_{n=\langle N \rangle} \left| a_k e^{jk\Omega_0 n} \right|^2 = |a_k|^2$$

- Thus, the total average power in $x[n]$ equals the sum of the average powers in all of its harmonic components.

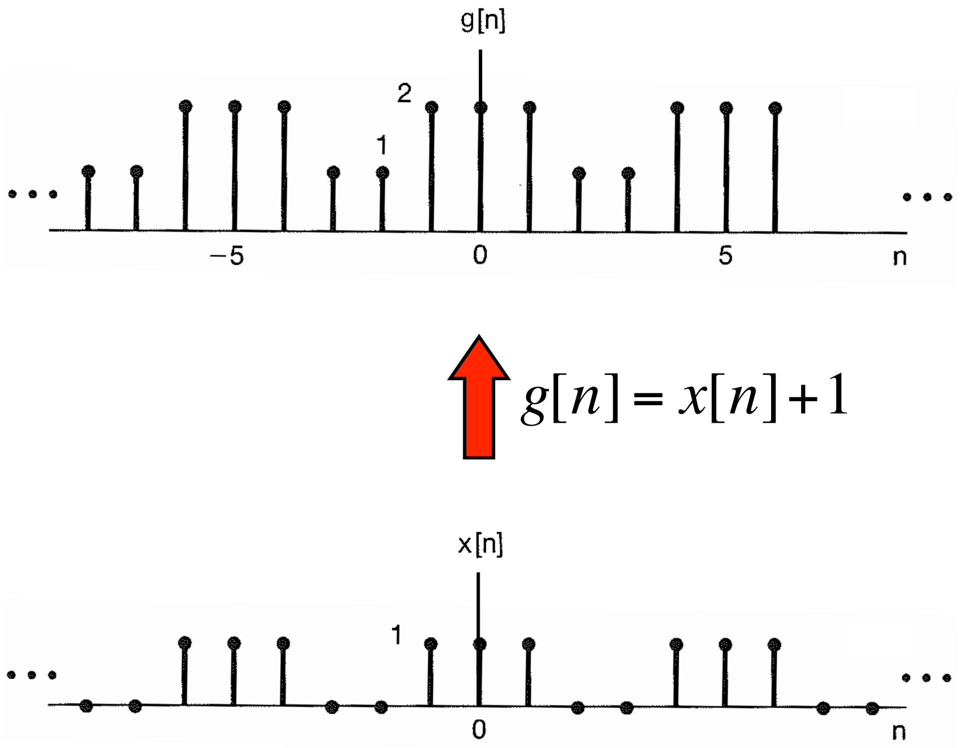
# An Example

- Consider the signal $g[n]$ with period $N = 5$.

- Recall the discrete-time periodic square wave $x[n]$ discussed on pages 13–15 with $N = 5$ and $N_1 = 1$.

- It is clear that $g[n]$ can be obtained from $x[n]$ as

$$g[n] = x[n] + 1$$



# An Example (cont.)

- Using the results of (9) and (10) on pages 14–15, we have in this case the Fourier series coefficients $a_k$ of $x[n]$ as

$$a_k = \begin{cases} 3/5, & k = 0 \\ \dfrac{\sin(3k\pi/5)}{5\sin(k\pi/5)}, & k = 1, 2, 3, 4 \end{cases}$$

- The Fourier series coefficients $b_k$ of the constant $1$ are simply

$$b_k = \begin{cases} 1, & k = 0 \\ 0, & k = 1, 2, 3, 4 \end{cases}$$

# An Example (cont.)

- Applying the linearity property, the Fourier series coefficients $c_k$ of $g[n]$ can be expressed as

$$c_k = a_k + b_k = \begin{cases} 8/5, & k = 0 \\ \dfrac{\sin(3k\pi/5)}{5\sin(k\pi/5)}, & k = 1, 2, 3, 4 \end{cases}$$