Derivation of Power Spectrums of Digital Baseband & Bandpass Modulated Signals

1. Power Spectrum of Digital Baseband Modulated Signal

For a digital baseband modulated signal

$$s(t) = \sum_{n=-\infty}^{\infty} Z_n \cdot v(t - n\tau),$$

where $v(t)$ is a baseband signal and $Z_n = Z$ is a discrete random variable, the power spectrum of $s(t)$ is

$$G_s(f) = \frac{1}{\tau} |V(f)|^2 \left(\sigma_Z^2 + \frac{\mu_Z^2}{\tau} \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{\tau} \right) \right)$$

The mean and autocorrelation of $s(t)$ are given by

$$E[s(t)] = \sum_{n=-\infty}^{\infty} E[Z_n]v(t-n\tau) = \mu_z \sum_{-\infty}^{\infty} v(t-n\tau)$$

and

$$R_s(t+x,t) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} E[Z_m Z_n] \cdot v(t-m\tau)v(t+x-n\tau)$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} R_Z(n-m) \cdot v(t-m\tau)v(t+x-n\tau)$$

$$= \sum_{n=-\infty}^{\infty} R_Z(n) \sum_{m=-\infty}^{\infty} v(t-m\tau)v(t+x-m\tau-n\tau)$$

respectively. Note that both of them are periodic with period τ. Therefore, $s(t)$ is a cyclostationary process.

We have known that the power spectrum of a cyclostationary process with period τ is given by

$$G_s(f) \Leftrightarrow \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} R_s(t+x,t)\,dt$$

According to

$$\frac{1}{\tau}\int_{-\tau/2}^{\tau/2} R_s(t+x,t)\,dt = \frac{1}{\tau}\sum_{n=-\infty}^{\infty} R_Z(n)\sum_{m=-\infty}^{\infty}\int_{-\tau/2}^{\tau/2} v(t-m\tau)v(t+x-m\tau-n\tau)\,dt$$

$$= \frac{1}{\tau}\sum_{n=-\infty}^{\infty} R_Z(n)\sum_{m=-\infty}^{\infty}\int_{-m\tau-\tau/2}^{-m\tau+\tau/2} v(t)v(t+x-n\tau)\,dt$$

$$= \frac{1}{\tau}\sum_{n=-\infty}^{\infty} R_Z(n)\int_{-\infty}^{-\infty} v(t)v(t+x-n\tau)\,dt \overset{\text{denoted as}}{=} \overline{R}_s(x)$$

we have

$$G_s(f) = \int_{-\infty}^{\infty} \overline{R}_s(x)e^{-j2\pi fx}\,dx$$

$$= \frac{1}{\tau}\sum_{n=-\infty}^{\infty} R_Z(n)e^{-j2\pi fn\tau}\int_{-\infty}^{\infty} v(t)\left(\int_{-\infty}^{\infty} v(t+x-n\tau)e^{-j2\pi f(t+x-n\tau)}\,dx\right)e^{j2\pi ft}\,dt$$

$$= \frac{1}{\tau}G_Z(f)\,|V(f)|^2$$

Recall that $\{Z_n\}$ is a random sequence with $Z_n = Z$. The auto-correlation function of $\{Z_n\}$ is then

$$R_Z(n) = \begin{cases} \sigma_Z^2 + \mu_Z^2, & n = 0 \\ \mu_Z^2, & n \neq 0 \end{cases}$$

The power spectrum of Z_n can be obtained as

$$G_Z(f) = \sum_{n=-\infty}^{\infty} R_Z(n) e^{-j2\pi fn\tau}$$

$$= \sigma_Z^2 + \mu_Z^2 \sum_{n=-\infty}^{\infty} e^{-j2\pi fn\tau} = \sigma_Z^2 + \frac{\mu_Z^2}{\tau} \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{\tau}\right)$$

Finally, the power spectrum of the digital baseband modulated signal $s(t)$ can be written as

$$\begin{aligned} G_s(f) &= \frac{1}{\tau} G_Z(f) |V(f)|^2 \\ &= \frac{1}{\tau} |V(f)|^2 \left(\sigma_Z^2 + \frac{\mu_Z^2}{\tau} \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{\tau} \right) \right) \end{aligned}$$

2. Power Spectrum of Digital Bandpass Modulated Signal

For a digital bandpass modulated signal

$$u(t) = s(t)\cos 2\pi f_c t,$$

where $s(t)$ is a digital baseband modulated signal, the power spectrum of $u(t)$ is

$$G_u(f) = \frac{1}{4}\left[G_s(f - f_c) + G_s(f + f_c)\right]$$

It can be easily shown that $u(t)$ is also a cyclostaionary process with period τ, and the autocorrelation function of $u(t)$ can be obtained as

$$R_u(t+x,t) = E[u(t+x)u(t)]$$
$$= E[s(t+x)s(t)]\cos 2\pi f_c t \cdot \cos 2\pi f_c (t+x)$$
$$= \frac{1}{2} R_s(t+x,t)[\cos 2\pi f_c x + \cos 2\pi f_c (2t+x)]$$

We have

$$\frac{1}{\tau}\int_{-\tau/2}^{\tau/2} R_u(t+x,t)dt = \frac{1}{2}\cdot\frac{1}{\tau}\int_{-\tau/2}^{\tau/2} R_s(t+x,t)[\cos 2\pi f_c x + \cos 2\pi f_c (2t+x)]dt$$
$$= \frac{1}{2}\bar{R}_s(x)\cos 2\pi f_c x$$

Accordingly,

$$G_u(f) \Leftrightarrow \frac{1}{2}\bar{R}_s(x)\cos 2\pi f_c x$$
$$= \frac{1}{4}\left[G_s(f-f_c) + G_s(f+f_c)\right]$$