Fourier Series

Chapter Intended Learning Outcomes:

(i) Represent continuous-time periodic signals using Fourier series

(ii) Understand the properties of Fourier series

(iii) Understand the relationship between Fourier series and linear time-invariant system

Periodic Signal Representation in Frequency Domain

Fourier series can be considered as the frequency domain representation of a continuous-time periodic signal.

Recall (2.6) that $x(t)$ is said to be periodic if there exists $T_p > 0$ such that

$$x(t) = x(t + T_p), \qquad t \in (-\infty, \infty) \tag{4.1}$$

The smallest T_p for which (4.1) holds is called the fundamental period.

Using (2.26), the fundamental frequency is related to T_p as:

$$\Omega_0 = \frac{2\pi}{T_p} \tag{4.2}$$

According to Fourier series, $x(t)$ is represented as:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t}, \qquad t \in (-\infty, \infty) \tag{4.3}$$

where

$$a_k = \frac{1}{T_p} \int_{-T_p/2}^{T_p/2} x(t) e^{-jk\Omega_0 t} dt, \quad k = \cdots - 1, 0, 1, 2, \cdots \tag{4.4}$$

are called Fourier series coefficients. Note that the integration can be done for any period, e.g., $(0, T_p)$, $(-T_p, 0)$.

That is, every periodic signal can be expressed as a sum of harmonically related complex sinusoids with frequencies $\cdots - \Omega_0, 0, \Omega_0, 2\Omega_0, 3\Omega_0, \cdots$, where the fundamental frequency Ω_0 is called the first harmonic, $2\Omega_0$ is called the second harmonic, and so on.

This means that $x(t)$ only contains frequencies $\cdots - \Omega_0, 0, \Omega_0, 2\Omega_0, \cdots$ with 0 being the DC component.

Note that the sinusoids are complex-valued with both positive and negative frequencies.

Note also that a_k is generally complex and we can also use magnitude and phase for its representation:

$$|a_k| = \sqrt{(\Re\{a_k\})^2 + (\Im\{a_k\})^2} \tag{4.5}$$

and

$$\angle(a_k) = \tan^{-1}\left(\frac{\Im\{a_k\}}{\Re\{a_k\}}\right) \tag{4.6}$$

From (4.3), $\{a_k\}$ can be used to represent $x(t)$.

Example 4.1

Find the Fourier series coefficients for $x(t)=\cos(10\pi t)+\cos(20\pi t)$.

It is clear that the fundamental frequency of $x(t)$ is $\Omega_0=10\pi$. According to (4.2), the fundamental period is thus equal to $T_p=2\pi/\Omega_0=1/5$, which is validated as follows:

$$\begin{aligned} x\left(t+\frac{1}{5}\right) &= \cos\left(10\pi\left(t+\frac{1}{5}\right)\right)+\cos\left(20\pi\left(t+\frac{1}{5}\right)\right) \\ &= \cos(10\pi t+2\pi)+\cos(20\pi t+4\pi) \\ &= \cos(10\pi t)+\cos(20\pi t) \end{aligned}$$

With the use of Euler formula in (2.29):

$$\cos(u)=\frac{e^{ju}+e^{-ju}}{2}$$

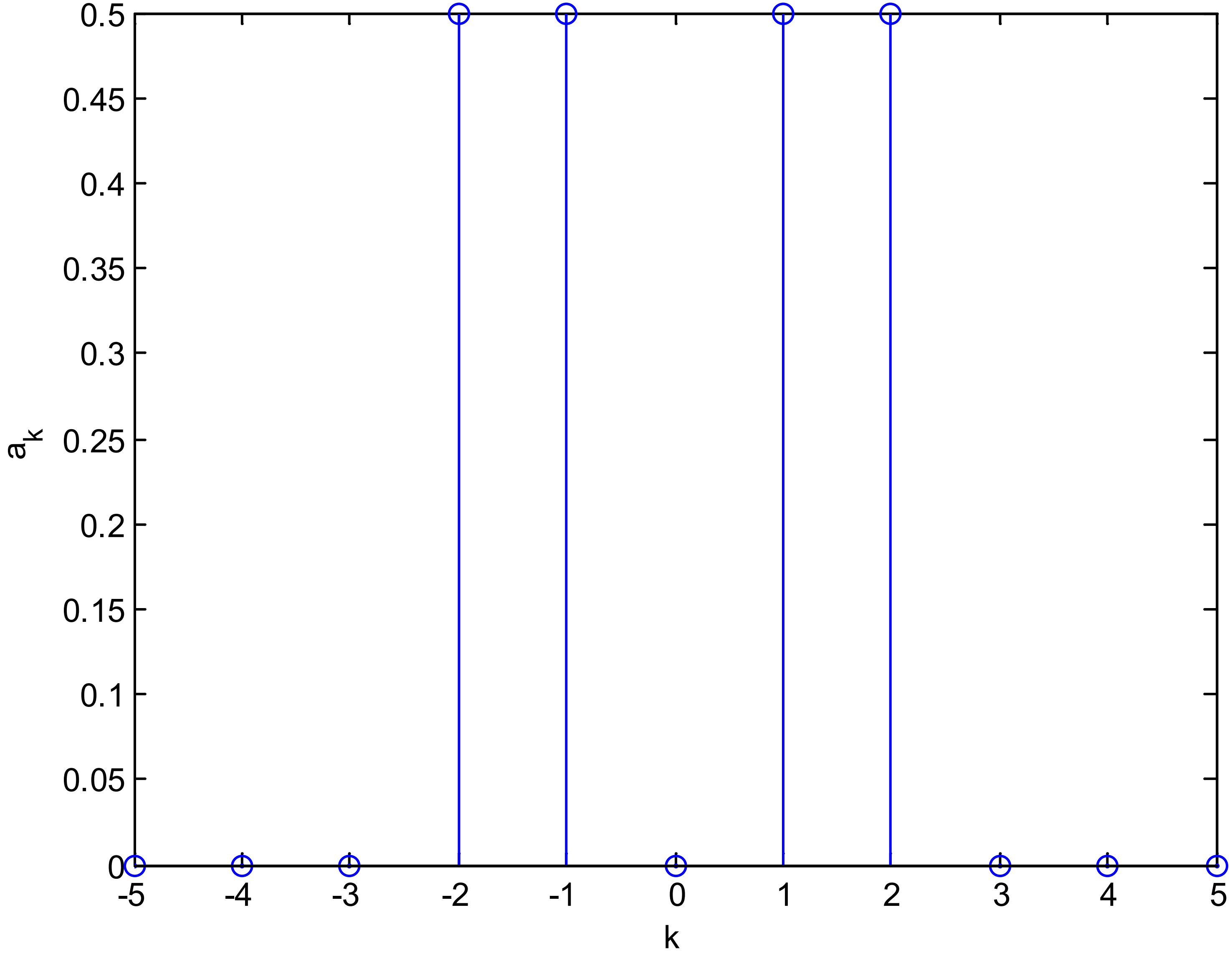
We can express $x(t)$ as:

$$
\begin{aligned}
x(t) &= \cos(10\pi t) + \cos(20\pi t) \\
&= \frac{e^{j10\pi t} + e^{-j10\pi t}}{2} + \frac{e^{j20\pi t} + e^{-j20\pi t}}{2} \\
&= \frac{1}{2}e^{-j20\pi t} + \frac{1}{2}e^{-j10\pi t} + \frac{1}{2}e^{j10\pi t} + \frac{1}{2}e^{j20\pi t}
\end{aligned}
$$

which only contains four frequencies. Comparing with (4.3):

$$
a_k = \begin{cases} 0.5, & k = -2 \\ 0.5, & k = -1 \\ 0.5, & k = 1 \\ 0.5, & k = 2 \\ 0, & \text{otherwise} \end{cases}
$$

Can we use (4.4)? Why?


0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
a_k
-5
-4
-3
-2
-1
0
1
2
3
4
5
k

Example 4.2

Find the Fourier series coefficients for $x(t) = 1 + \sin(\Omega_0 t) + 2\cos(\Omega_0 t) + \cos(3\Omega_0 t + \pi/4)$.

With the use of Euler formulas in (2.29)-(2.30), $x(t)$ can be written as:

$$
\begin{aligned}
x(t) &= 1 + \left(1 + \frac{1}{2j}\right) e^{j\Omega_0 t} + \left(1 - \frac{1}{2j}\right) e^{-j\Omega_0 t} + \frac{1}{2} e^{j\pi/4} e^{3j\Omega_0 t} + \frac{1}{2} e^{-j\pi/4} e^{-3j\Omega_0 t} \\
&= \frac{\sqrt{2}}{4}(1 - j) e^{-3j\Omega_0 t} + \left(1 + j\frac{1}{2}\right) e^{-j\Omega_0 t} + 1 + \left(1 - j\frac{1}{2}\right) e^{j\Omega_0 t} \\
&\quad + \frac{\sqrt{2}}{4}(1 + j) e^{3j\Omega_0 t}
\end{aligned}
$$

Again, comparing with (4.3) yields:

$$a_k = \begin{cases} \dfrac{\sqrt{2}}{4}(1-j), & k=-3 \\ 1+\dfrac{j}{2}, & k=-1 \\ 1, & k=0 \\ 1-\dfrac{j}{2}, & k=1 \\ \dfrac{\sqrt{2}}{4}(1+j), & k=3 \\ 0, & \text{otherwise} \end{cases}$$

To plot $\{a_k\}$, we may compute $|a_k|$ and $\angle(a_k)$ for all k, e.g.,

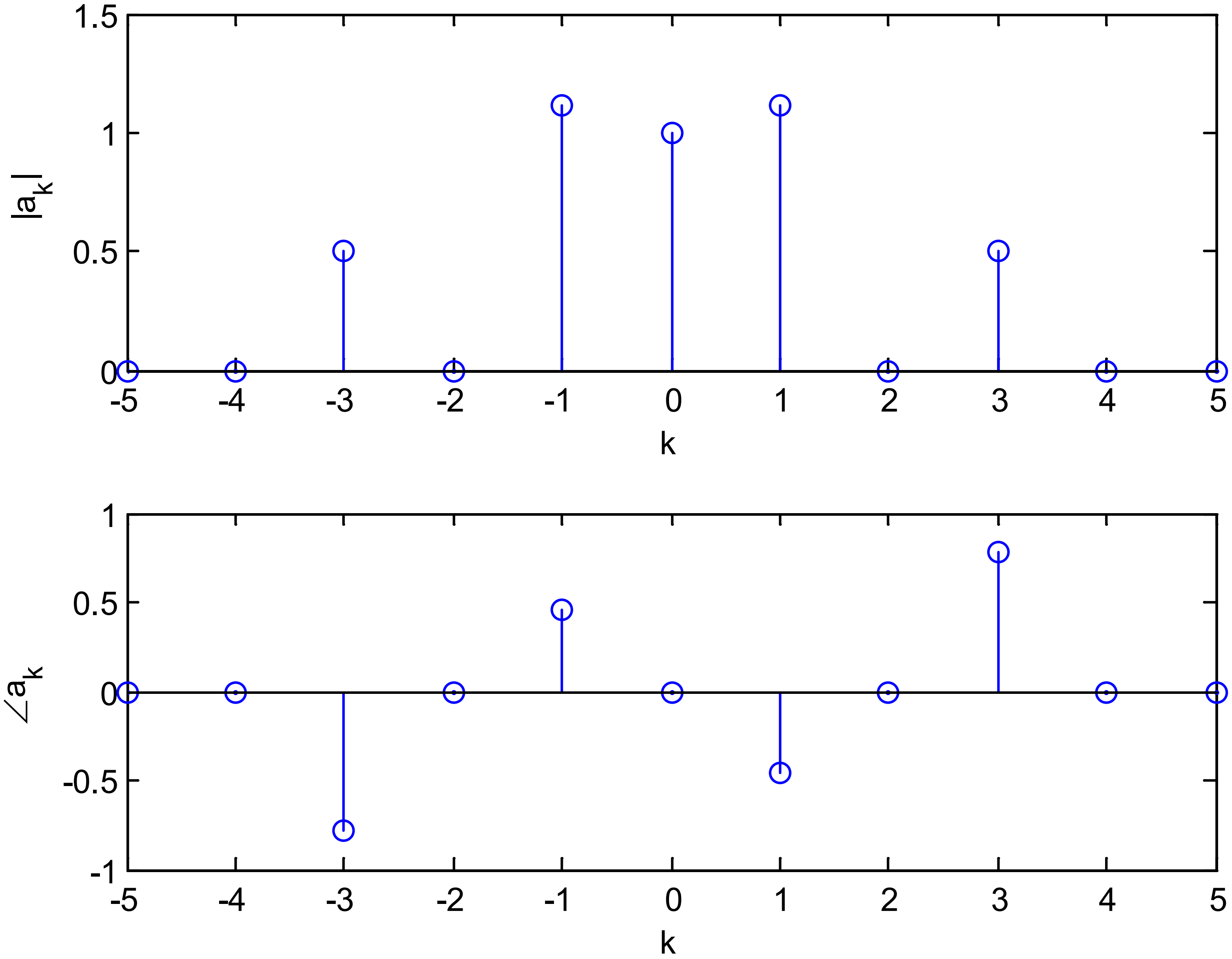
$$|a_{-3}| = \sqrt{\left(\frac{\sqrt{2}}{4}\right)^2 + \left(-\frac{\sqrt{2}}{4}\right)^2} = \frac{1}{2}$$

and

$$\angle(a_{-3}) = \tan^{-1}(-1) = -\frac{\pi}{4}$$

We can also use MATLAB commands `abs` and `angle` to compute the magnitude and phase, respectively. After constructing a vector `x` containing $\{a_k\}$, we can plot $|a_k|$ and $\angle(a_k)$ using:

```
subplot(2,1,1)
stem(n,abs(x))
xlabel('k')
ylabel('|a_k|')
subplot(2,1,2)
stem(n,angle(x))
xlabel('k')
ylabel('\angle{a_k}')
```

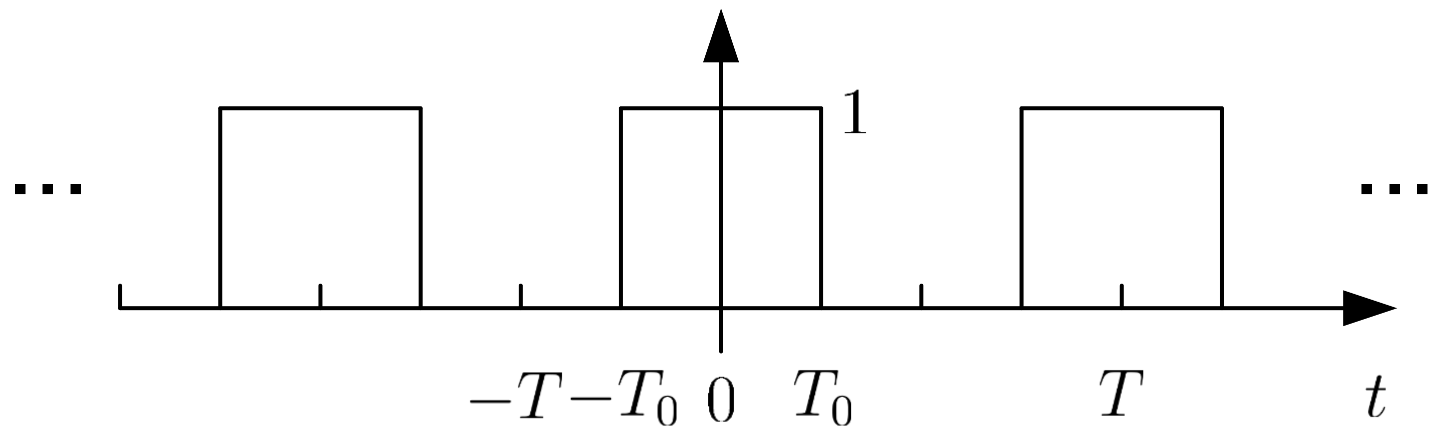

1.5
1
0.5
0
$|a_k|$
-5
-4
-3
-2
-1
0
1
2
3
4
5
k
1
0.5
0
-0.5
-1
$\angle a_k$
k

Example 4.3

Find the Fourier series coefficients for $x(t)$, which is a periodic continuous-time signal of fundamental period T and is a pulse with a width of $2T_0$ in each period. Over the specific period from $-T/2$ to $T/2$, $x(t)$ is:

$$x(t) = \begin{cases} 1, & -T_0 < t < T_0 \\ 0, & \text{otherwise} \end{cases}$$

with $T > 2T_0$.



Noting that the fundamental frequency is $\Omega_0 = 2\pi/T$ and using (4.4), we get:

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\Omega_0 t}dt = \frac{1}{T}\int_{-T_0}^{T_0} e^{-jk\Omega_0 t}dt$$

For $k = 0$:

$$a_0 = \frac{1}{T}\int_{-T_0}^{T_0} 1dt = \frac{2T_0}{T}$$

For $k \neq 0$:

$$a_k = \frac{1}{T}\int_{-T_0}^{T_0} e^{-jk\Omega_0 t}dt = -\frac{1}{jk\Omega_0 T}e^{-jk\Omega_0 t}\Big|_{-T_0}^{T_0} = \frac{\sin(k\Omega_0 T_0)}{k\pi} = \frac{\sin\left(2\pi k T_0/T\right)}{k\pi}$$

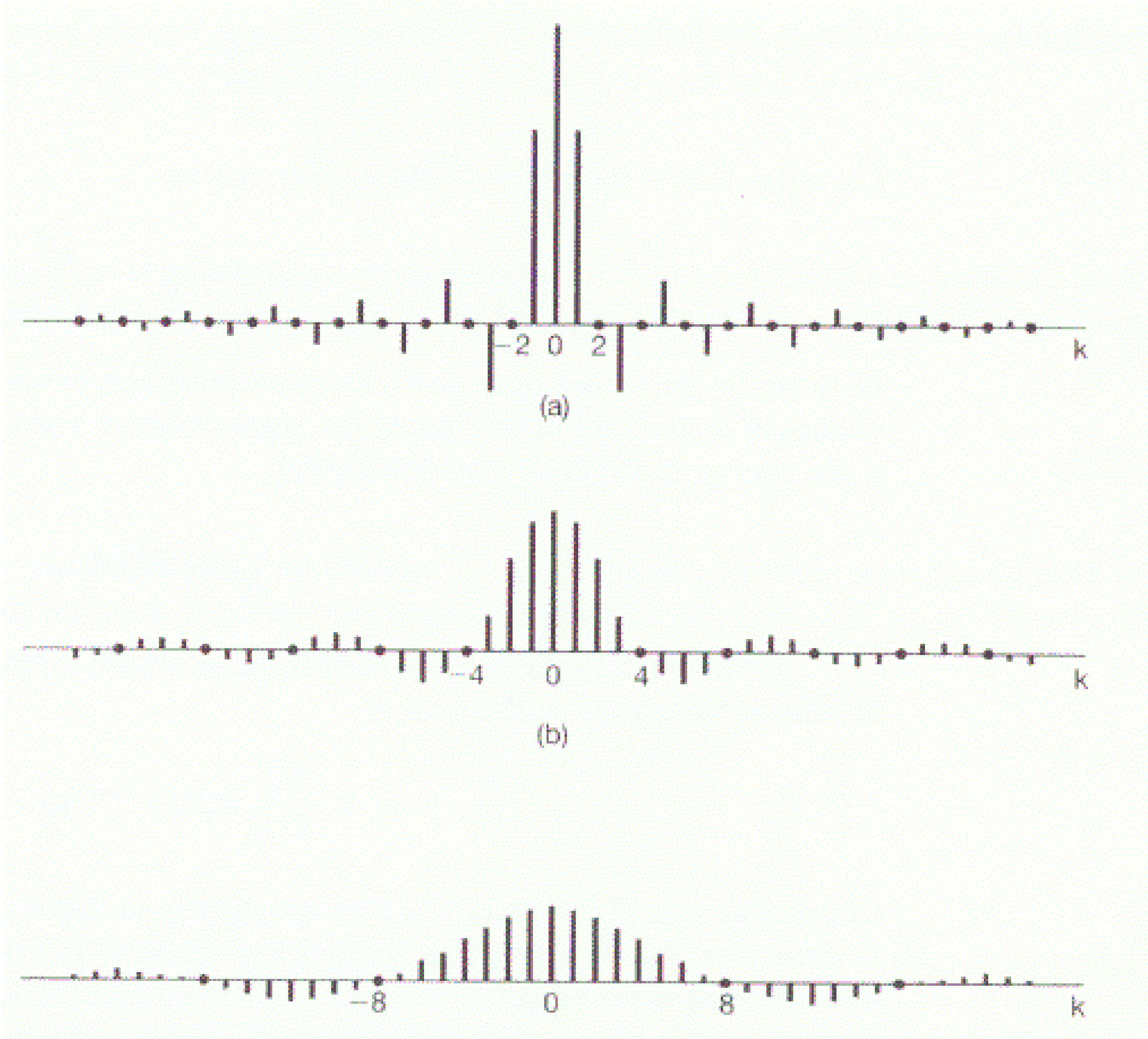
The reason of separating the cases of $k=0$ and $k \neq 0$ is to facilitate the computation of a_0 , whose value is not straightforwardly obtained from the general expression which involves "0/0".

Nevertheless, using $\mathrm{L'Hôpital}$'s rule:

$$\lim_{k \to 0} \frac{\sin(2\pi k T_0/T)}{k\pi} = \lim_{k \to 0} \frac{\frac{d \sin(2\pi k T_0/T)}{dk}}{\frac{dk\pi}{dk}} = \lim_{k \to 0} \frac{2\pi T_0/T \cos((2\pi k T_0/T))}{\pi} = \frac{2T_0}{T}$$

An investigation on the values of $\{a_k\}$ with respect to relative pulse width T_0/T is performed as follows.

We see that when T_0/T decreases, $\{a_k\}$ seem to be stretched.



$$T = 4T_0$$

$$T = 8T_0$$

$$T = 16T_0$$

Example 4.4

Find the Fourier series coefficients for the following continuous-time periodic signal $x(t)$:

$$x(t) = \begin{cases} 1.5, & 0 < t < 1 \\ -1.5, & 1 < t < 2 \end{cases}$$

where the fundamental period is $T_p = 2$ and fundamental frequency is $\Omega_0 = \pi$.

Using (4.4) with the period from $t = -1$ to $t = 1$:

$$\begin{aligned} a_k &= \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jk\Omega_0 t} dt \\ &= \frac{1}{2}\int_{-1}^{0} (-1.5) e^{-jk\pi t} dt + \frac{1}{2}\int_{0}^{1} 1.5 e^{-jk\pi t} dt \end{aligned}$$

For $k = 0$:

$$a_k = \frac{1}{2}\int_{-1}^{0}(-1.5)dt + \frac{1}{2}\int_{0}^{1} 1.5dt = \frac{1}{2}(-1.5+1.5) = 0$$

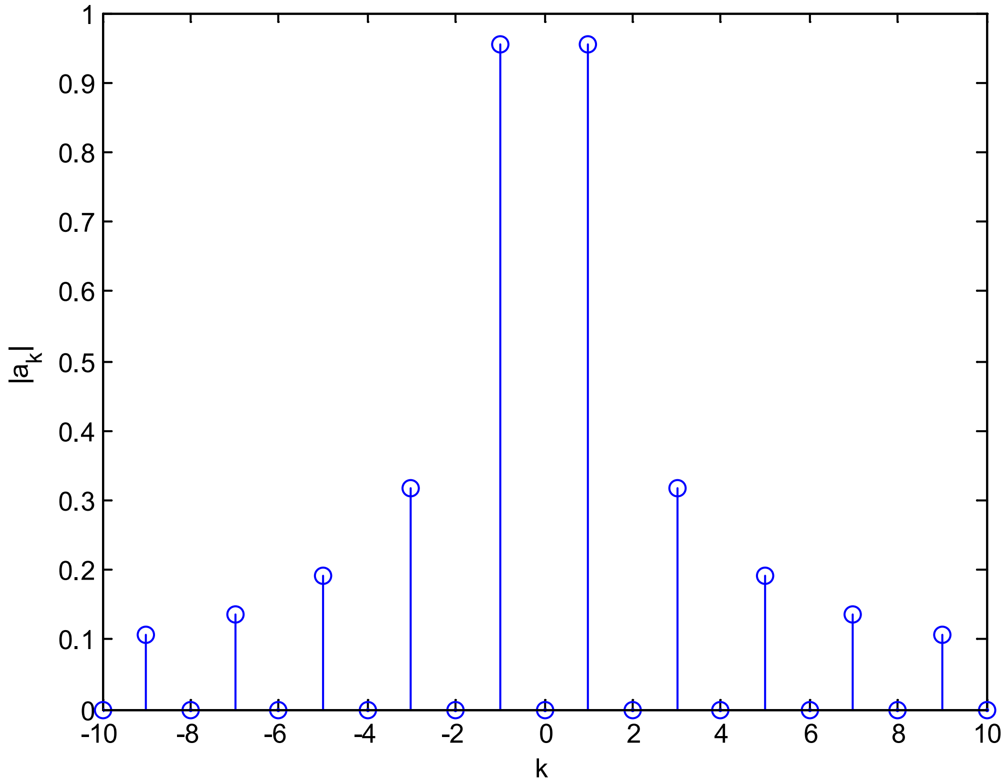
For $k \neq 0$:

$$\begin{aligned}
a_k &= \frac{1}{2}\int_{-1}^{0}(-1.5)e^{-jk\pi t}dt + \frac{1}{2}\int_{0}^{1} 1.5e^{-jk\pi t}dt \\
&= \frac{3}{4}\left[\int_{-1}^{0} -e^{-jk\pi t}dt + \int_{0}^{1} e^{-jk\pi t}dt\right] \\
&= \frac{3}{4}\left[-\frac{1}{-jk\pi}\, e^{-jk\pi t}\Big|_{-1}^{0} + \frac{1}{-jk\pi}\, -e^{-jk\pi t}\Big|_{0}^{1}\right] \\
&= \frac{3}{4jk\pi}\left[1 - e^{jk\pi} - e^{-jk\pi} + 1\right] \\
&= \frac{3}{2jk\pi}\left[1 - \cos(k\pi)\right]
\end{aligned}$$

MATLAB can be used to validate the answer. First we have:

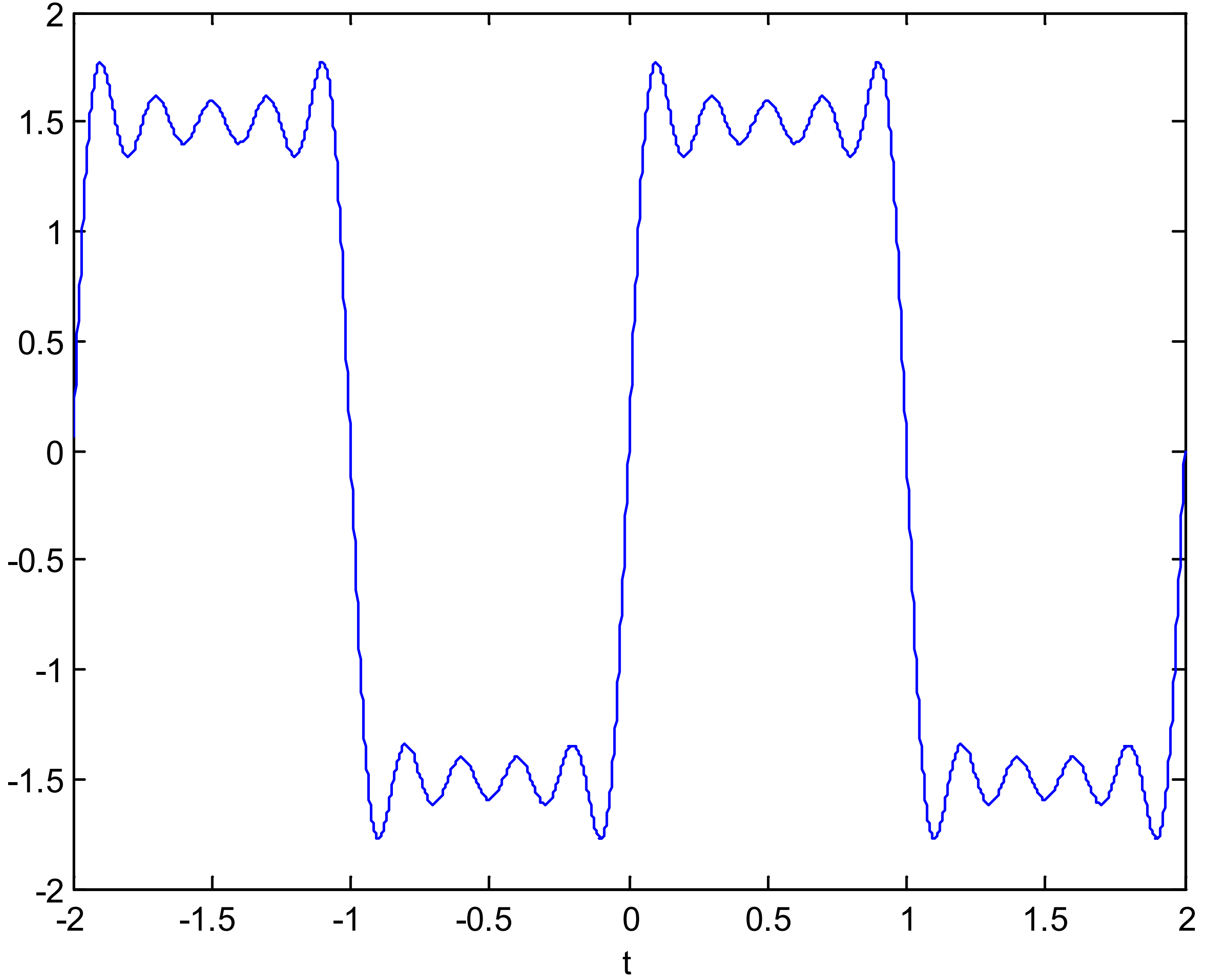
$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t} \approx \sum_{k=-K}^{K} a_k e^{jk\Omega_0 t}$$

for sufficiently large K because $|a_k|$ is decreasing with k

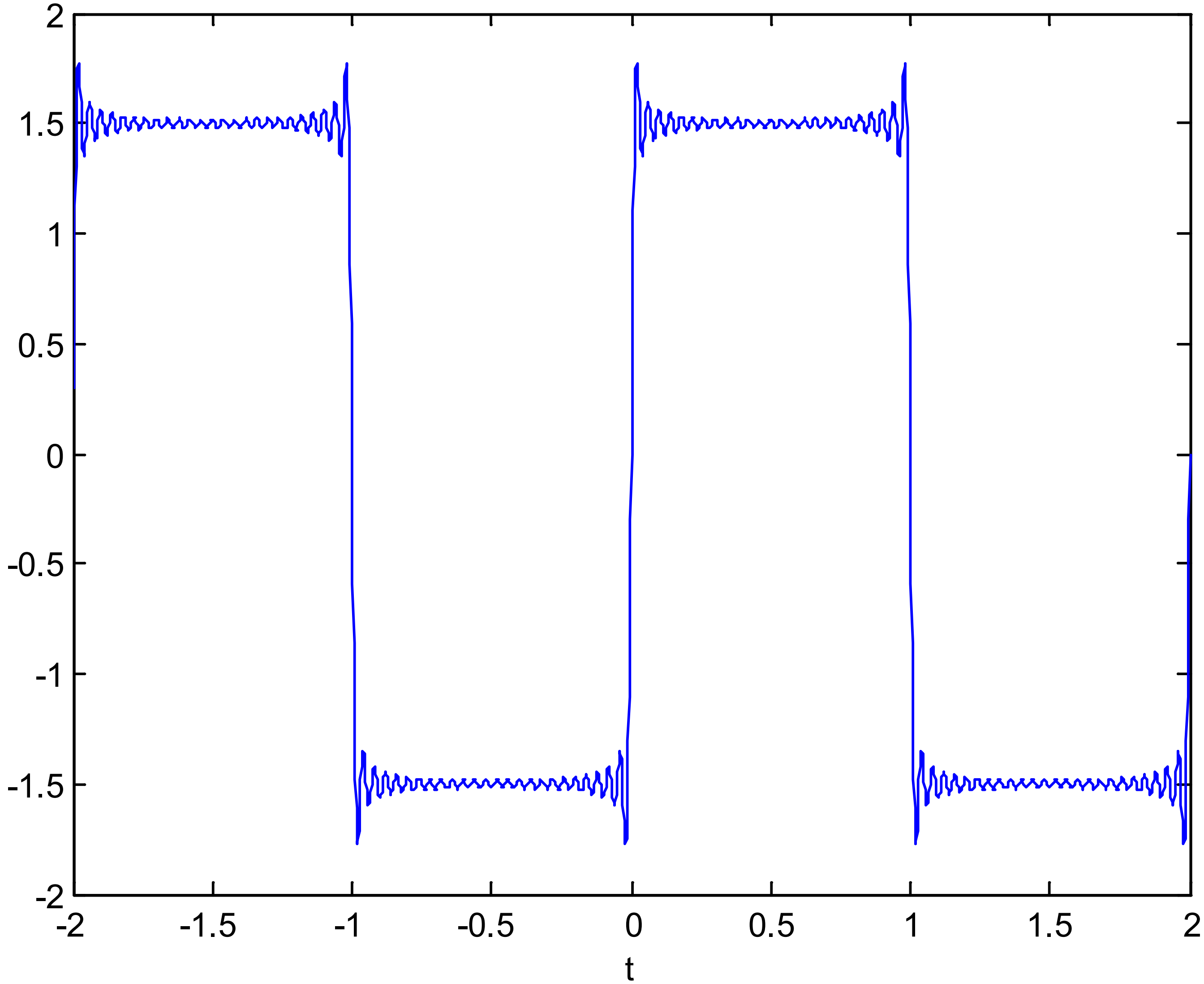


Setting $K=10$, we may use the following code:

```
K=10;
a_p = 3./(j.*2.*[1:K].*pi).*(1-cos([1:K].*pi)); % +ve a_k
a_n = 3./(j.*2.*[-K:-1].*pi).*(1-cos([-K:-1].*pi)); %-ve a_k
a = [a_n 0 a_p]; %construct vector of a_k
for n=1:2000
 t=(n-1000)/500; %time interval of (-2,2);
     %small sampling interval of 1/500 to approximate x(t);
 e = (exp(j.*[-K:K].*pi.*t)).'; %construct exponential vector
 x(n) = a*e;
end
x=real(x); %remove imaginary parts due to precision error
n=1:2000;
t=(n-1000)./500;
plot(t,x)
xlabel('t')
```


2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
t

For $K = 50$:



In summary, if $x(t)$ is periodic, it can be represented as a linear combination of complex harmonics with amplitudes $\{a_k\}$.

That is, $\{a_k\}$ correspond to the frequency domain representation of $x(t)$ and we may write:

$$x(t) \leftrightarrow X(j\Omega) \quad \text{or} \quad x(t) \leftrightarrow a_k \tag{4.7}$$

where $X(j\Omega)$, a function of frequency Ω, is characterized by $\{a_k\}$.

Both $x(t)$ and $X(j\Omega)$ represent the same signal: we observe the former in time domain while the latter in frequency domain.

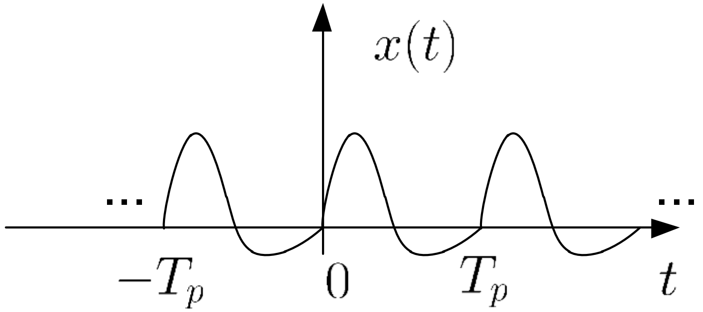
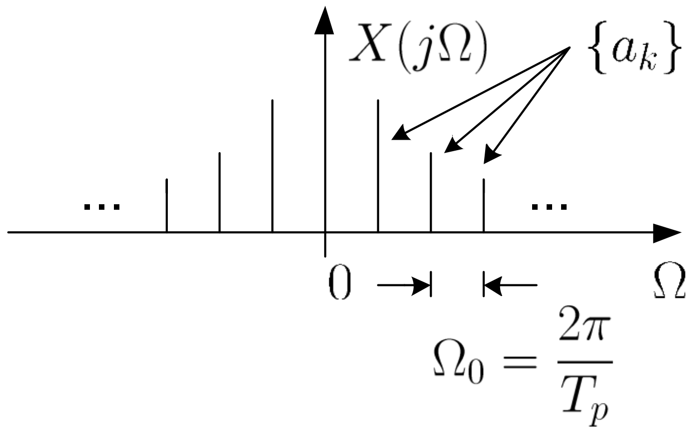
time domain	frequency domain
	
$$a_k = \frac{1}{T_p}\int_{-T_p/2}^{T_p/2} x(t)e^{-jk\Omega_0 t}dt \Longrightarrow$$	
	$$\Longleftarrow x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t}$$
continuous and periodic	discrete and aperiodic

Fig.4.1: Illustration of Fourier series

Properties of Fourier Series

Linearity

Let $x(t) \leftrightarrow a_k$ and $y(t) \leftrightarrow b_k$ be two Fourier series pairs with the same period of T_p. We have:

$$Ax(t) + By(t) \leftrightarrow Aa_k + Bb_k \tag{4.8}$$

This can be proved as follows. As $x(t)$ and $y(t)$ have the same fundamental period of T_p or fundamental frequency Ω_0, we can write:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t}, \qquad y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\Omega_0 t}$$

Multiplying $x(t)$ and $y(t)$ by A and B, respectively, yields:

$$Ax(t) = A \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t}, \qquad By(t) = B \sum_{k=-\infty}^{\infty} b_k e^{jk\Omega_0 t}$$

Summing $Ax(t)$ and $By(t)$, we get:

$$Ax(t) + By(t) = \sum_{k=-\infty}^{\infty} (Aa_k + Bb_k) e^{jk\Omega_0 t} \leftrightarrow Aa_k + Bb_k$$

Time Shifting

A shift of t_0 in $x(t)$ causes a multiplication of $e^{-jk\Omega_0 t_0}$ in a_k:

$$x(t) \leftrightarrow a_k \Rightarrow x(t - t_0) \leftrightarrow e^{-jk\Omega_0 t_0} a_k = e^{-jk(2\pi)/T_p t_0} a_k \qquad (4.9)$$

Time Reversal

$$x(t) \leftrightarrow a_k \Rightarrow x(-t) \leftrightarrow a_{-k} \qquad (4.10)$$

(4.9) and (4.10) are proved as follows.

Recall (4.3):

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t}$$

Substituting t by $t - t_0$, we obtain:

$$x(t-t_0) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 (t-t_0)} = \sum_{k=-\infty}^{\infty} \left(e^{-jk\Omega_0 t_0} a_k\right) e^{jk\Omega_0 t} \leftrightarrow e^{-jk\Omega_0 t_0} a_k$$

Substituting t by $-t$ yields:

$$x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 (-t)} = \sum_{l=-\infty}^{\infty} a_{-l} e^{jl\Omega_0 t} = \sum_{k=-\infty}^{\infty} a_{-k} e^{jk\Omega_0 t} \leftrightarrow a_{-k}$$

Time Scaling

For a time-scaled version of $x(t)$, $x(\alpha t)$ where $\alpha > 0$ is a real number, we have:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t} \Rightarrow x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(\alpha\Omega_0)t} \tag{4.11}$$

Multiplication

Let $x(t) \leftrightarrow a_k$ and $y(t) \leftrightarrow b_k$ be two Fourier series pairs with the same period of T_p. We have:

$$x(t)y(t) \leftrightarrow \sum_{l=-\infty}^{\infty} a_l b_{k-l} = a_k \otimes b_k \tag{4.12}$$

(4.12) is proved as follows.

Applying (4.3) again, the product of $x(t)$ and $y(t)$ is:

$$
\begin{aligned}
x(t)y(t) &= \sum_{l=-\infty}^{\infty} a_l e^{jl\Omega_0 t} \sum_{n=-\infty}^{\infty} b_n e^{jn\Omega_0 t} \\
&= \sum_{l=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} a_l b_n e^{j(l+n)\Omega_0 t} \\
&= \sum_{l=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} a_l b_{k-l} e^{jk\Omega_0 t}, \quad k = l + n \\
&= \sum_{k=-\infty}^{\infty} \left(\sum_{l=-\infty}^{\infty} a_l b_{k-l} \right) e^{jk\Omega_0 t} \leftrightarrow \sum_{l=-\infty}^{\infty} a_l b_{k-l}
\end{aligned}
$$

Conjugation

$$x(t) \leftrightarrow a_k \Rightarrow x^*(t) \leftrightarrow a^*_{-k} \tag{4.13}$$

Parseval's Relation

The Parseval's relation addresses the power of $x(t)$:

$$\frac{1}{T_p}\int_{-T_p/2}^{T_p/2} |x(t)|^2 \, dt = \sum_{k=-\infty}^{\infty} |a_k|^2 \tag{4.14}$$

That is, we can compute the power in either the time domain or frequency domain.

Example 4.5

Prove the Parseval's relation.

Using (4.3), we have:

$$
\begin{aligned}
\frac{1}{T_p}\int_{-T_p/2}^{T_p/2} |x(t)|^2\,dt &= \frac{1}{T_p}\int_{-T_p/2}^{T_p/2} \left(\sum_{m=-\infty}^{\infty} a_m e^{jm\Omega_0 t}\right)\left(\sum_{n=-\infty}^{\infty} a_n e^{jn\Omega_0 t}\right)^* dt \\
&= \frac{1}{T_p}\int_{-T_p/2}^{T_p/2} \left(\sum_{m=-\infty}^{\infty} a_m e^{jm\Omega_0 t}\right)\left(\sum_{n=-\infty}^{\infty} a_n^* e^{-jn\Omega_0 t}\right) dt \\
&= \frac{1}{T_p}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} a_m a_n^* \int_{-T_p/2}^{T_p/2} e^{j(m-n)\Omega_0 t} dt \\
&= \frac{1}{T_p}\sum_{m=-\infty}^{\infty} |a_m|^2 \int_{-T_p/2}^{T_p/2} dt = \sum_{m=-\infty}^{\infty} |a_m|^2
\end{aligned}
$$

Linear Time-Invariant System with Periodic Input

Recall for a linear time-invariant (LTI) system, the input-output relationship is characterized by convolution in (3.17):

$$\begin{aligned} y(t) &= x(t) \otimes h(t) \\ &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \end{aligned} \qquad (4.15)$$

If the input to the system with impulse response $h(t)$ is $x(t) = e^{j\Omega_0 t}$, then the output is:

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} h(\tau)e^{j\Omega_0(t-\tau)}d\tau \\ &= e^{j\Omega_0 t}\int_{-\infty}^{\infty} h(\tau)e^{-j\Omega_0\tau}d\tau \end{aligned} \qquad (4.16)$$

Note that $\int_{-\infty}^{\infty} h(\tau)e^{-j\Omega_0\tau}d\tau$ is independent of t but a function of Ω_0 and we may denote it as $H(j\Omega_0)$:

$$y(t) = e^{j\Omega_0 t}H(j\Omega_0) = H(j\Omega_0)x(t) \quad (4.17)$$

If we input a sinusoid through a LTI system, there is no change in frequency in the output but amplitude and phase are modified.

Generalizing the result to any periodic signal in (4.3) yields:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t} \rightarrow y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\Omega_0)e^{jk\Omega_0 t} \quad (4.18)$$

where only the Fourier series coefficients are modified.

Note that discrete Fourier series is used to represent discrete periodic signal in (2.7) but it will not be discussed.