EE3210
Signals and Systems

Part 10: Frequency-Domain Analysis of LTI Systems

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Changes of Introduction_v1 Lecture Notes

- Page 11, change

Assignment 5	Available in Week 11, due in Week 12

to

Assignment 5	Available in Week 12, due in Week 13

Changes of Part7_v1 Lecture Notes

- Page 12, add:
  - Note: $\lim_{k\to 0} \dfrac{\sin(2\alpha k\pi)}{k\pi} = 2\alpha$ by l'Hôpital's rule.

Frequency Response

- The Fourier transform $H(\omega)$ of the unit impulse response $h(t)$ of a continuous-time LTI system is referred to as the frequency response of the system.

$$H(\omega) = \int_{-\infty}^{+\infty} h(t)e^{-j\omega t}dt$$

- Similarly, the Fourier transform $H[\Omega]$ of the unit impulse response $h[n]$ of a discrete-time LTI system is referred to as the frequency response of the system.

$$H[\Omega] = \sum_{n=-\infty}^{+\infty} h[n]e^{-j\Omega n}$$

Magnitude Response and Phase Response

- The frequency response $H(\omega)$ or $H[\Omega]$ of a continuous-time or discrete-time LTI system, which is a complex-valued function of ω or Ω, can be expressed in polar form as:

$$H(\omega) = |H(\omega)|e^{j\angle H(\omega)} \quad \text{or} \quad H[\Omega] = |H[\Omega]|e^{j\angle H[\Omega]}$$

  - $|H(\omega)|$ or $|H[\Omega]|$ is referred to as the magnitude response of the system.
  - $\angle H(\omega)$ or $\angle H[\Omega]$ is referred to as the phase response of the system.

Example 1

- Consider a discrete-time LTI system known as an ideal delay system, which is defined by $y[n] = x[n-k]$, where k is a fixed positive integer.
  - The unit impulse response of the system is

$$h[n] = \delta[n-k]$$

  - The frequency response of the system is

$$H[\Omega] = \sum_{n=-\infty}^{+\infty} \delta[n-k]e^{-j\Omega n} = e^{-jk\Omega}$$

    - Its magnitude response is $|H[\Omega]| = 1$.
    - Its phase response is $\angle H[\Omega] = -k\Omega$.

Example 2

- Consider a discrete-time LTI system with frequency response given by

$$H[\Omega] = \frac{1 - 2e^{-j\Omega}}{1 - 0.5e^{-j\Omega}}$$

- To determine the magnitude response $|H[\Omega]|$ of this system, it is convenient to use the properties of complex conjugation:
  - $|z|^2 = zz^*$
  - $\left(\frac{z_1}{z_2}\right)^* = \frac{z_1^*}{z_2^*}$

Example 2 (cont.)

- So we have:

$$|H[\Omega]|^2 = H[\Omega]H^*[\Omega] = \left(\frac{1-2e^{-j\Omega}}{1-0.5e^{-j\Omega}}\right)\left(\frac{1-2e^{-j\Omega}}{1-0.5e^{-j\Omega}}\right)^*$$

$$= \frac{\left(1-2e^{-j\Omega}\right)\left(1-2e^{-j\Omega}\right)^*}{\left(1-0.5e^{-j\Omega}\right)\left(1-0.5e^{-j\Omega}\right)^*}$$

$$= \frac{\left(1-2e^{-j\Omega}\right)\left(1-2e^{j\Omega}\right)}{\left(1-0.5e^{-j\Omega}\right)\left(1-0.5e^{j\Omega}\right)}$$

$$= \frac{5-2\left(e^{j\Omega}+e^{-j\Omega}\right)}{1.25-0.5\left(e^{j\Omega}+e^{-j\Omega}\right)} = 4$$

- Therefore, $|H[\Omega]| = 2$.

Frequency Domain Analysis

- Fourier transform makes it possible to simplify the analysis of LTI systems.
  - This simplicity is furnished by analyzing the frequency response of the system.
  - The very foundation for this analysis is the convolution property of Fourier transform.

$$\begin{tikzcd}
x(t) \arrow[r, "*h(t)"] \arrow[d, "\mathcal{F}"'] \arrow[dr, phantom, "\text{Continuous-time LTI system}"] & y(t) \\
X(\omega) \arrow[r, "H(\omega)"'] & Y(\omega) \arrow[u, "\mathcal{F}^{-1}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
x[n] \arrow[r, "*h[n]"] \arrow[d, "\mathcal{F}"'] \arrow[dr, phantom, "\text{Discrete-time LTI system}"] & y[n] \\
X[\Omega] \arrow[r, "H[\Omega]"'] & Y[\Omega] \arrow[u, "\mathcal{F}^{-1}"']
\end{tikzcd}$$

Example 1

- Consider a discrete-time LTI system with input $x[n] = \alpha^n u[n]$ and unit impulse response $h[n] = \beta^n u[n]$.
  - Assume $|\alpha| < 1$, i.e., an input bounded in magnitude, and $|\beta| < 1$, i.e., a stable system.

- Then, we derive the Fourier transform of $x[n]$ and $h[n]$ as:

$$X[\Omega] = \frac{1}{1 - \alpha e^{-j\Omega}} \quad \text{and} \quad H[\Omega] = \frac{1}{1 - \beta e^{-j\Omega}} \tag{1}$$

  - Alternatively, we can determine $X[\Omega]$ and $H[\Omega]$ from the table of basic discrete-time Fourier transform pairs available on Page 395, Table 5.2, of the textbook.

Example 1 (cont.)

- Now, using the convolution property of the Fourier transform, we obtain

$$Y[\Omega] = \mathcal{F}\{x[n] * h[n]\} = X[\Omega]H[\Omega]$$
$$= \frac{1}{(1-\alpha e^{-j\Omega})(1-\beta e^{-j\Omega})}$$

- Recall Assignment 3 Problem 1 that the expression of $y[n]$ in this example has two cases:
  - For $\alpha \neq \beta$: $y[n] = \dfrac{\beta^{n+1} - \alpha^{n+1}}{\beta - \alpha} u[n]$
  - For $\alpha = \beta$: $y[n] = (n+1)\beta^n u[n]$

Solution for $\alpha \neq \beta$

- Given

$$Y[\Omega] = \frac{1}{(1-\alpha e^{-j\Omega})(1-\beta e^{-j\Omega})} \qquad (2)$$

we use the technique of partial fraction expansion for solving $y[n]$ in this case.

- Consider the partial fraction expansion of $Y[\Omega]$ in the form of

$$Y[\Omega] = \frac{A}{1-\alpha e^{-j\Omega}} + \frac{B}{1-\beta e^{-j\Omega}} \qquad (3)$$

- Equating the right-hand sides of (2) and (3), we find that

$$A = \frac{\alpha}{\alpha - \beta} \text{ and } B = \frac{\beta}{\beta - \alpha}$$

Solution for $\alpha \neq \beta$ (cont.)

- Therefore,

$$Y[\Omega] = \frac{\alpha}{\alpha - \beta} \cdot \frac{1}{1 - \alpha e^{-j\Omega}} + \frac{\beta}{\beta - \alpha} \cdot \frac{1}{1 - \beta e^{-j\Omega}}$$

- Then, from (1) and using the linearity property, we can obtain $y[n] = \mathcal{F}^{-1}\{Y[\Omega]\}$ by inspection:

$$y[n] = \frac{\alpha}{\alpha - \beta}\alpha^n u[n] + \frac{\beta}{\beta - \alpha}\beta^n u[n]$$

$$= \frac{\beta^{n+1} - \alpha^{n+1}}{\beta - \alpha} u[n]$$

Solution for $\alpha = \beta$

- Given $Y[\Omega] = \dfrac{1}{(1-\beta e^{-j\Omega})^2}$, we use the properties of differentiation in frequency and time shift for solving $y[n]$ in this case.
- We know that $\dfrac{1}{(1-\beta e^{-j\Omega})^2} = \dfrac{j}{\beta} e^{j\Omega} \dfrac{d}{d\Omega}\left(\dfrac{1}{1-\beta e^{-j\Omega}}\right)$.
- From (1) and using the differentiation in frequency property, we have

$$n\beta^n u[n] \leftrightarrow j\frac{d}{d\Omega}\left(\frac{1}{1-\beta e^{-j\Omega}}\right)$$

Solution for $\alpha = \beta$ (cont.)

- Further, to account for the factor $e^{j\Omega}$, we use the time shift property to obtain

$$(n+1)\beta^{n+1}u[n+1] \leftrightarrow je^{j\Omega}\frac{d}{d\Omega}\left(\frac{1}{1-\beta e^{-j\Omega}}\right)$$

- Finally, accounting for the factor $1/\beta$, we obtain

$$y[n] = (n+1)\beta^n u[n+1] = (n+1)\beta^n u[n]$$

Example 2

- Consider a discrete-time LTI system that is characterized by the difference equation

$$y[n] - ay[n-1] = x[n], \text{ for } |a| < 1 \qquad (4)$$

- Applying Fourier transform to both sides of (4), and using the properties of time shift and linearity, we have

$$Y[\Omega] - ae^{-j\Omega}Y[\Omega] = X[\Omega] \qquad (5)$$

- Rearranging (5), we obtain

$$H[\Omega] = \frac{Y[\Omega]}{X[\Omega]} = \frac{1}{1 - ae^{-j\Omega}} \Rightarrow h[n] = a^n u[n]$$