Lecture 5. Random Signal Analysis

- Random Variables and Random Processes
- Signal Transmission through a Linear System

Discrete Random Variables

- A discrete random variable takes on a countable number of possible values.

Suppose that a discrete random variable X takes on one of the values $x_1, \ldots, x_n$.

✓ Distribution functions:

$$\boxed{\sum_{i=1}^{n} p(x_i) = 1}$$

Probability Mass Function (pmf): $p(x_i) = \Pr\{X = x_i\}$

Cumulative Distribution Function (cdf): $F(a) = \Pr\{X \le a\} = \sum_{x_i \le a} p(x_i)$

✓ Moments:

Expected Value, or Mean: $\mu_X = E[X] = \sum_{i=1}^{n} x_i p(x_i)$

The m-th Moment: $E[X^m] = \sum_{i=1}^{n} x_i^m p(x_i), \ m = 1, 2, \ldots$

Continuous Random Variables

- A continuous random variable has an uncountable set of possible values.

X is a continuous random variable if there exists a nonnegative function f, defined for all real $x \in (-\infty, \infty)$, having the property that for any set B of real numbers, $\Pr\{X \in B\} = \int_B f(x)dx$.

$$\boxed{\int_{-\infty}^{\infty} f_X(x)dx = 1}$$

- ✓ f is called the probability density function (pdf) of X, denoted as: $f_X(x)$
- ✓ Cumulative Distribution Function (cdf): $F_X(a) = \int_{-\infty}^{a} f_X(x)dx$
- ✓ Expected Value, or Mean: $\mu_X = E[X] = \int_{-\infty}^{\infty} x f_X(x)dx$
- ✓ The m-th Moment: $E[X^m] = \int_{-\infty}^{\infty} x^m f_X(x)dx, \quad m = 1, 2, \ldots$

Variance

- Define *variance* of a random variable X as:

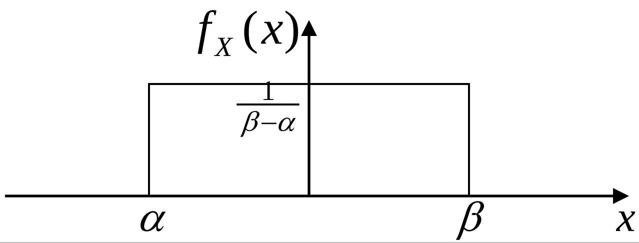
$$\mathrm{Var}[X] = E\left[(X - E[X])^2\right]$$

  - ✓ $\mathrm{Var}[X]$ describes how far apart X is from its mean on the average.
  - ✓ $\mathrm{Var}[X]$ can be also obtained as: $\mathrm{Var}[X] = E[X^2] - \left(E[X]\right)^2$
  - ✓ $\mathrm{Var}[X]$ is usually denoted as σ_X^2 .
  - ✓ The square root of Var[X], σ_X , is called the *standard deviation* of X.

Example 1. Uniform Distribution

- X is a uniform random variable on the interval (α, β) if its pdf is given by

$$f_X(x) = \begin{cases} \dfrac{1}{\beta-\alpha} & \alpha \le x \le \beta \\ 0 & otherwise \end{cases}$$



✓ cdf:

$$F_X(a) = \int_{-\infty}^{a} f_X(x)dx = \begin{cases} 0 & a \le \alpha \\ \dfrac{a-\alpha}{\beta-\alpha} & \alpha < a < \beta \\ 1 & a \ge \beta \end{cases}$$

✓ Mean: $\mu_X = \int_{-\infty}^{\infty} x f_X(x)dx = \dfrac{\beta+\alpha}{2}$

✓ The second moment:

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)dx = \frac{\beta^2+\alpha\beta+\alpha^2}{3}$$

✓ Variance:

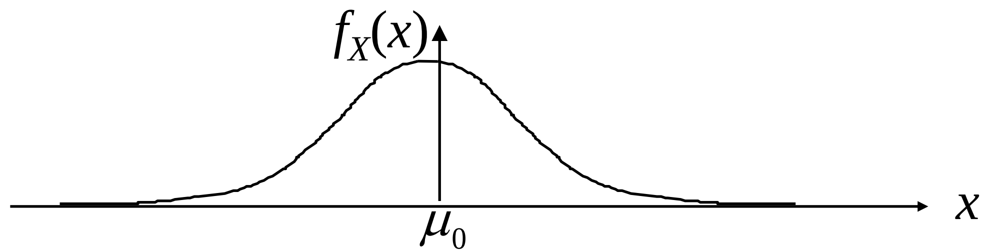
$$\sigma_X^2 = E[X^2] - \mu_X^2$$

$$= \frac{(\beta-\alpha)^2}{12}$$

Example 2. Gaussian (Normal) Distribution

- X is a Gaussian random variable with parameters μ_0 and σ_0^2 if its pdf is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_0} \exp\left(-\frac{(x-\mu_0)^2}{2\sigma_0^2}\right)$$



- X is denoted as $X \sim N\ (\mu_0, \sigma_0^2)$

✓ Mean: $\mu_X = \mu_0$

✓ Variance: $\sigma_X^2 = \sigma_0^2$

✓ cdf: $F_X(a) = \int_{-\infty}^{a} f_X(x)dx = 1 - \int_a^\infty \frac{1}{\sqrt{2\pi}\sigma_0} \exp\left\{-\frac{(x-\mu_0)^2}{2\sigma_0^2}\right\} dx$

$$\stackrel{z=\frac{x-\mu_0}{\sigma_0}}{=} 1 - \int_{\frac{a-\mu_0}{\sigma_0}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{z^2}{2}\right\} dz = 1 - Q\left(\frac{a-\mu}{\sigma_0}\right)$$

$$\boxed{Q(\alpha) \triangleq \int_\alpha^\infty \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\} dx}$$

More about Q Function

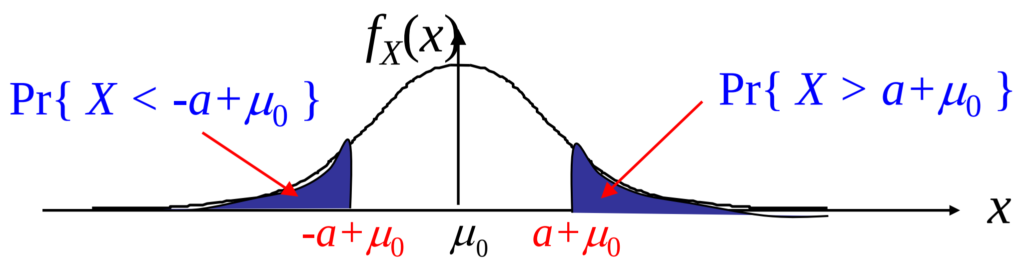
$$Q(\alpha) = \int_{\alpha}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\} dx$$

- $Q(\alpha)$ is a decreasing function of α.
- For $X \sim N(\mu_0, \sigma_0^2)$,

$$\Pr\{X>a+\mu_0\}=\int_{a+\mu_0}^{\infty} f_X(x)dx=\int_{a+\mu_0}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_0} \exp\left\{-\frac{(x-\mu_0)^2}{2\sigma_0^2}\right\} dx = Q\left(\frac{a}{\sigma_0}\right)$$

$$\Pr\{X<-a+\mu_0\}=\int_{-\infty}^{-a+\mu_0} f_X(x)dx=\int_{-\infty}^{-a+\mu_0} \frac{1}{\sqrt{2\pi}\sigma_0} \exp\left\{-\frac{(x-\mu_0)^2}{2\sigma_0^2}\right\} dx =$$

$$\int_{a-\mu_0}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_0} \exp\left\{-\frac{(x+\mu_0)^2}{2\sigma_0^2}\right\} dx = Q\left(\frac{a}{\sigma_0}\right)$$



Read the textbook Sec. 4.1.4 for more discussion about Q function.

More about Q Function

TABLE I.1 Values of $Q(x)$ versus x.

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641
0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
2.3	.0107	.0104	.0102	.00990	.00964	.00939	.00914	.00889	.00866	.00842
2.4	.00820	.00798	.00776	.00755	.00734	.00714	.00695	.00676	.00657	.00639
2.5	.00621	.00604	.00587	.00570	.00554	.00539	.00523	.00508	.00494	.00480
2.6	.00466	.00453	.00440	.00427	.00415	.00402	.00391	.00379	.00368	.00357
2.7	.00347	.00336	.00326	.00317	.00307	.00298	.00289	.00280	.00272	.00264
2.8	.00256	.00248	.00240	.00233	.00226	.00219	.00212	.00205	.00199	.00193
2.9	.00187	.00181	.00175	.00169	.00164	.00159	.00154	.00149	.00144	.00139

Random Processes

- Sample values of a random process at time t_1, t_2, …, are a collection of random variables $\{X(t_1), X(t_2), \ldots\}$.
  - Continuous-time random process: $t \in \mathrm{R}$ (set of real numbers)
  - Discrete-time random process: $t \in \mathrm{Z}$ (set of integers)

- Statistical description of random process X(t)
  - A complete statistical description of a random process $X(t)$ is known if for any integer n and any choice of $(t_1, \ldots, t_n) \in \mathrm{R}^n$, the joint PDF of $(X(t_1), \ldots, X(t_n))$ is given.

Difficult to be obtained!

Statistical Averages

- The mean of the random process X(t):

$$\mu_X(t_k) = E[X(t_k)] = \int_{-\infty}^{\infty} x f_{X(t_k)}(x)\,dx$$

$X(t_k)$ is the random variable obtained by observing the random process $X(t)$ at time t_k, with the pdf $f_{X(t_k)}(x)$.

- The autocorrelation function of the random process X(t):

$$R_X(t_1, t_2) = E[X(t_1)X(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{X(t_1),X(t_2)}(x_1, x_2)\,dx_1 dx_2$$

$f_{X(t_1),X(t_2)}(x_1, x_2)$ is the joint pdf of $X(t_1)$ and $X(t_2)$.

Power and Power Spectrum of Random Signal

Time Domain

Deterministic signal s(t): $P_s = \lim\limits_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |s(t)|^2 \, dt$

Random signal (described as a random process X(t)):

$$P_X = E\left[\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} X^2(t) dt\right] = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} R_X(t,t) dt$$

Frequency Domain

Deterministic signal:

$$P_s = \int_{-\infty}^{\infty} G_s(f) df, \quad G_s(f) \triangleq \lim_{T\to\infty} \frac{1}{T} |S_T(f)|^2 \Leftrightarrow \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} s(t+\tau) s^*(t) dt$$

Random signal:

$$P_X = \int_{-\infty}^{\infty} G_X(f) df, \quad G_X(f) \triangleq \lim_{T\to\infty} \frac{1}{T} E|X_T(f)|^2 \Leftrightarrow \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} R_X(t+\tau, t) dt$$

Example 1. Wide-Sense Stationary (WSS) Processes

- A random process $X(t)$ is wide-sense stationary (WSS) if the following conditions are satisfied:
  - $\mu_X(t) = E[X(t)]$ is independent of t;
  - $R_X(t_1, t_2)$ depends only on the time difference $\tau = t_1 - t_2$, and not on t_1 and t_2 individually.

✓ Power: $P_X = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} R_X(t,t)dt = R_X(0)$

✓ Power spectrum: $G_X(f) \Leftrightarrow \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} R_X(t+\tau,t)dt = R_X(\tau)$

Example 2. Cyclostationary Processes

- A random process $X(t)$ with mean $\mu_X(t)$ and autocorrelation function $R_X(t+\tau, t)$ is called **cyclostationary**, if both the mean and the autocorrelation are periodic in t with some period T_0, i.e., if
  - $\mu_X(t+T_0) = \mu_X(t)$
  - $R_X(t+\tau+T_0, t+T_0) = R_X(t+\tau, t)$

✓ Power: $P_X = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} R_X(t,t)dt = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} R_X(t,t)dt$

✓ Power spectrum: $G_X(f) \Leftrightarrow \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} R_X(t+\tau,t)dt = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} R_X(t+\tau,t)dt$

Signal Transmission through a Linear System

Linear Time Invariant (LTI) System

Random signal $X(t)$ → Impulse Response $h(t)$ → $Y(t) = X(t) * h(t) = \int_{-\infty}^{\infty} X(\tau) h(t-\tau) d\tau$

- If a WSS random process $X(t)$ passes through an LTI system with impulse response $h(t)$, the output process $Y(t)$ will be also WSS with mean

$$\mu_Y = \mu_X \int_{-\infty}^{\infty} h(t)dt = \mu_X H(0)$$

autocorrelation

$$R_Y(\tau) = R_X(\tau) * h(\tau) * h(-\tau)$$

and power spectrum

$$G_Y(f) = G_X(f) \,|\, H(f) \,|^2$$

Example 3. Gaussian Processes

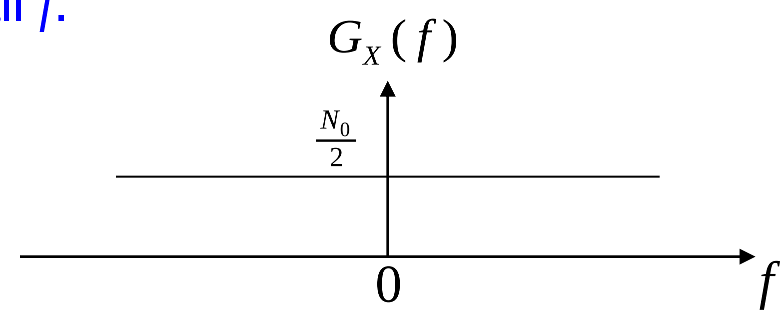
- A random process $X(t)$ is a **Gaussian** process if for all n and all $(t_1, \ldots, t_n)$, the random variables $\{X(t_i)\}_{i=1}^{n}$ have a jointly Gaussian pdf.

✓ For Gaussian processes, knowledge of the mean and auto-correlation gives a complete statistical description of the process.

✓ If a Gaussian process $X(t)$ is passed through an LTI system, the output process $Y(t)$ will also be a Gaussian process.

Example 4. White Processes

- A random process $X(t)$ is called a **white** process if it has a flat spectral density, i.e., if $G_X(f)$ is a constant for all f.



$\frac{N_0}{2}$: two-sided power spectral density

✓ Power: $P_X = \int_{-\infty}^{\infty} G_X(f)df = \infty$

✓ Autocorrelation:

$$G_X(f) \Leftrightarrow R_X(\tau) = \frac{N_0}{2}\delta(\tau)$$

✓ If a white process $X(t)$ passes through an LTI system with impulse response $h(t)$, the output process $Y(t)$ will not be white any more.

Power spectrum of $Y(t)$: $G_Y(f) = \frac{N_0}{2}|H(f)|^2$

Power of $Y(t)$: $P_Y = \frac{N_0}{2}\int_{-\infty}^{\infty} |H(f)|^2 df = \frac{N_0}{2}\int_{-\infty}^{\infty} h^2(t)dt$