EE3210
Signals and Systems

Part 5: More on LTI Systems

香港城市大學
City University
of Hong Kong

Instructor: Dr. Jun Guo

DEPARTMENT OF ELECTRONIC ENGINEERING

Changes of Assignment 1 Solutions

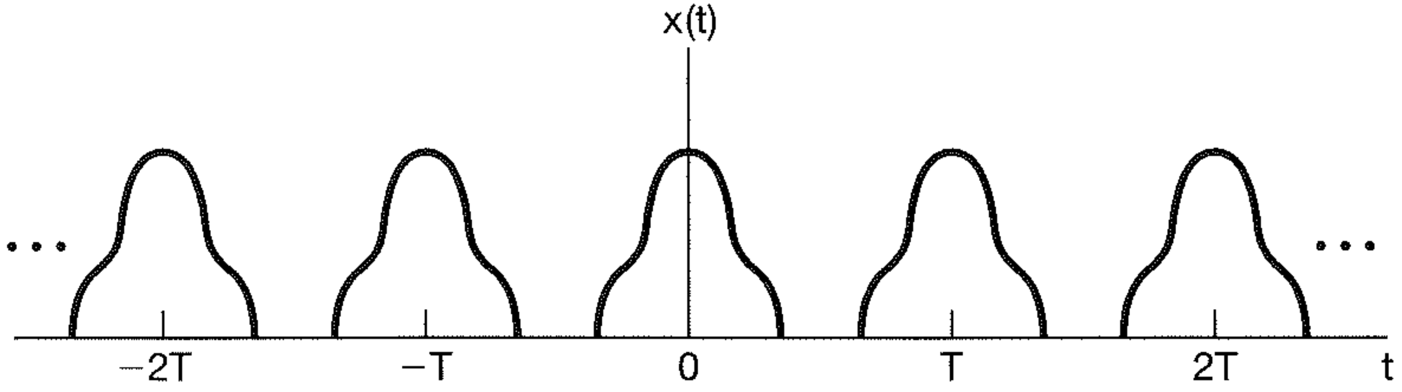
- Now Page 4 and Page 5: Add one alternative solution to Problem 3(c).

Changes of Part2_v2 Lecture Notes

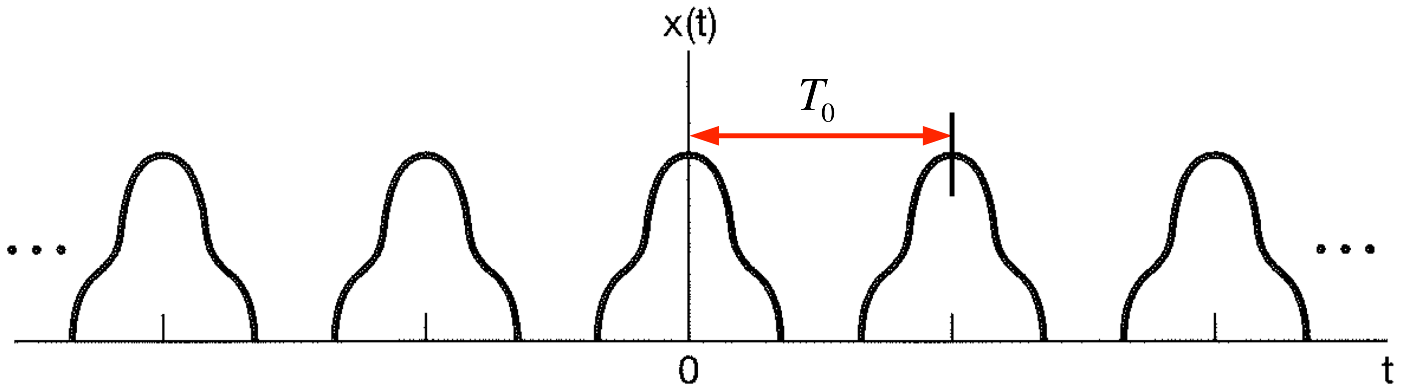
- Page 10, remove:
  - Can we obtain $x[2n]$, $x[n/2]$ from a discrete-time signal $x[n]$?
- Now Page 11: Add one whole slide that provides more information for discussing if we can obtain $x[2n]$ from a discrete-time signal $x[n]$.
- Now Page 12: Add one whole slide that provides more information for discussing if we can obtain $x[n/2]$ from a discrete-time signal $x[n]$.

Changes of Part2_v2 Lecture Notes (cont.)

- Previously Page 15, now Page 17, change the figure



to



Changes of Part2_v2 Lecture Notes (cont.)

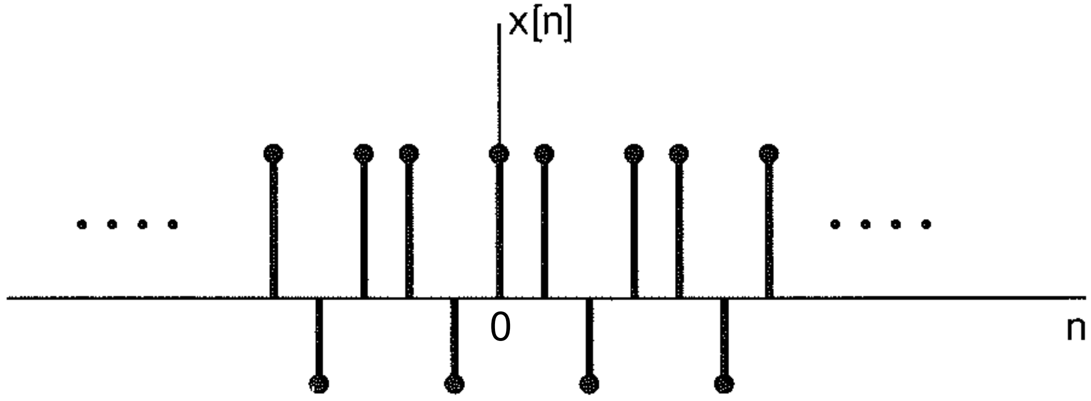
- Previously Page 15, now Page 17, change
  - Fundamental period: The smallest positive value of T for which (1) holds.

to

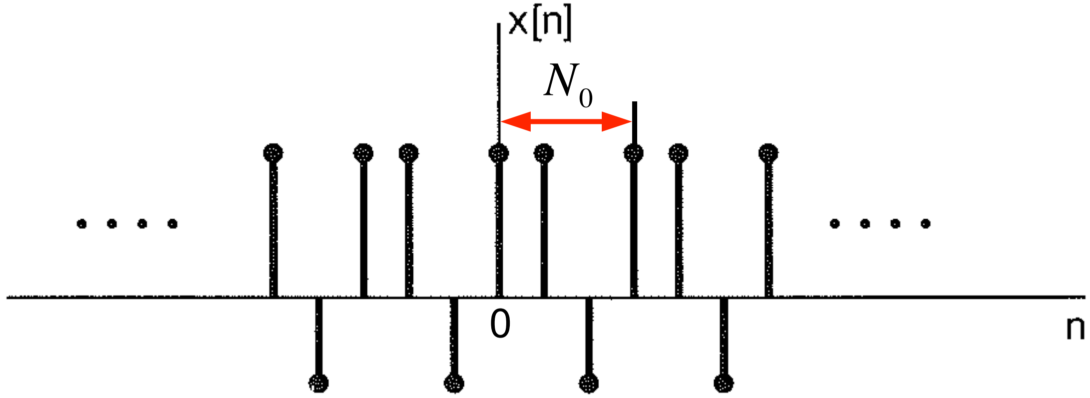
  - Fundamental period T_0: The smallest positive value of T for which (1) holds, hence $T = T_0, 2T_0, 3T_0 \ldots$
    - Example 1: If $T_0 = 1$, then $T = 1, 2, 3, \ldots$
    - Example 2: If $T_0 = \pi$, then $T = \pi, 2\pi, 3\pi, \ldots$

Changes of Part2_v2 Lecture Notes (cont.)

- Previously Page 16, now Page 18, change the figure



to



Changes of Part2_v2 Lecture Notes (cont.)

- Previously Page 16, now Page 18, change
  - Fundamental period: The smallest positive value of N for which (2) holds.

to

  - Fundamental period N_0: The smallest positive value of N for which (2) holds, hence $N = N_0, 2N_0, 3N_0 \dots$
    - Example: If $N_0 = 3$, then $N = 3, 6, 9, \dots$

Changes of Part3_v2 Lecture Notes

- Page 7, change
  - A system is invertible if distinct inputs lead to distinct outputs.

  to

  - A system is invertible if distinct inputs lead to distinct outputs, or, equivalently, if an inverse system exists such that:

Changes of Part4_v1 Lecture Notes

- Page 7 swapped with Page 6, and add:
  - Note that $x[n]$ can be represented in terms of $\delta[n]$ as

$$x[n] = \ldots + x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] \\ + x[1]\delta[n-1] + \ldots$$

Changes of Part4_v1 Lecture Notes (cont.)

- Page 9, now Page 9 and Page 10: This slide regarding convolution sum has been revised.

Changes of Part4_v1 Lecture Notes (cont.)

- Previously Page 10, now Page 11, change
  - For this case, $y[n]$ is simply

to

  - For this case, since $x[k] = 0$ for all values of k other than 0 and 1, we obtain $y[n]$ from (3) on Page 10 simply as

Changes of Part4_v1 Lecture Notes (cont.)

- Previously Pages 12-14, now Pages 13-16: Solution to Example 2 has been revised and more clearly presented.

Changes of Part4_v1 Lecture Notes (cont.)

- Page 25, now Page 27 and Page 28: This slide regarding convolution integral has been revised.

Changes of Part4_v1 Lecture Notes (cont.)

- Previously Pages 26-28, now Pages 29-31: Solution to the example has been revised and more clearly presented.

Properties of LTI Systems

- LTI Systems with and without memory
- Invertibility of LTI systems
- Causality of LTI systems
- Stability of LTI systems

LTI Systems With and Without Memory

- Recall: A system is memoryless if its output at any time depends only on the value of the input at that same time.

- For a discrete-time LTI system represented by

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] \quad (1)$$

the system is memoryless if $h[n] = 0$ for $n \neq 0$. Then:

  - $h[n] = K\delta[n]$ where $K = h[0]$ is a constant.
  - $y[n] = Kx[n]$
    - If $K = 1$, $y[n] = x[n]$ is known as an identity system with $h[n] = \delta[n]$.

LTI Systems With and Without Memory (cont.)

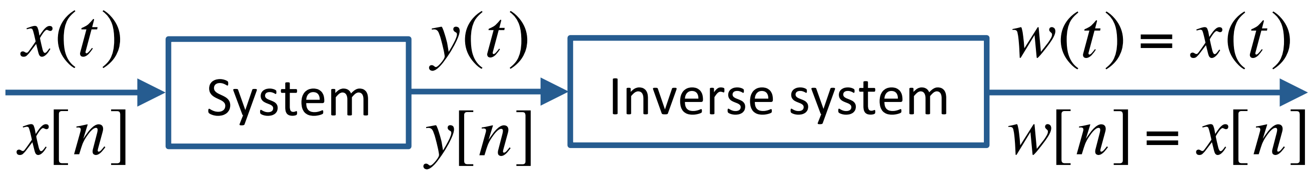
- For a continuous-time LTI system represented by

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau \tag{2}$$

  the system is memoryless if $h(t) = 0$ for $t \neq 0$. Then:
  - $h(t) = K\delta(t)$ where $K = h(0)$ is a constant.
  - $y(t) = Kx(t)$
    - If $K = 1$, $y(t) = x(t)$ is an identity system with unit impulse response $h(t) = \delta(t)$.

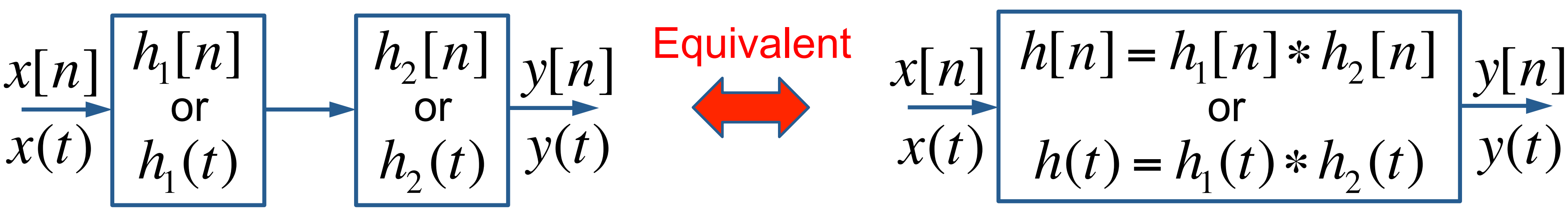
Invertibility of LTI Systems

- Recall: A system is invertible only if an inverse system exists such that:



- We can show that, if an LTI system is invertible, it has an LTI inverse system.

- Recall: A series interconnection of two LTI systems is equivalent to a single system:



Invertibility of LTI Systems (cont.)

- Thus, for a discrete-time LTI system with unit impulse response $h[n]$ to be invertible, it must have an inverse system with unit impulse response $h_1[n]$ such that:

$$h[n] * h_1[n] = \delta[n]$$

- Similarly, for a continuous-time LTI system with unit impulse response $h(t)$ to be invertible, it must have an inverse system with unit impulse response $h_1(t)$ such that:

$$h(t) * h_1(t) = \delta(t)$$

Example 1

- Consider a continuous-time LTI system defined by

$$y(t) = x(t - t_0) \tag{3}$$

  - The unit impulse response $h(t)$ of the system can be obtained from (3) by taking $x(t) = \delta(t)$, so that

$$h(t) = \delta(t - t_0)$$

- We observe in (3) that the output $y(t)$ is a time-shifted version of the input $x(t)$.
  - To recover the input $x(t)$ from the output $y(t)$, all that is required is to shift the output back.

Example 1 (cont.)

- Thus, the system defined by (3) is invertible, and its inverse system can be defined by

$$w(t) = y(t + t_0) \tag{4}$$

  - The unit impulse response $h_1(t)$ of the inverse system can be obtained from (4) by taking $y(t) = \delta(t)$, so that

$$h_1(t) = \delta(t + t_0)$$

- We can verify that

$$h(t) * h_1(t) = \delta(t - t_0) * \delta(t + t_0) = \delta(t)$$

Example 2

- Consider the discrete-time LTI system defined by

$$y[n] = \sum_{k=-\infty}^{n} x[k] \tag{5}$$

  - Comparing (5) with the convolution sum formula (1) on Page 15, we observe that for (5) to hold it requires that

$$h[n-k] = \begin{cases} 1, & k \leq n \\ 0, & k > n \end{cases}$$

    which implies that

$$h[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

Example 2 (cont.)

- Recall from Page 7 of Part 3's lecture notes that this system is invertible with an inverse system defined by

$$w[n] = y[n] - y[n-1] \tag{6}$$

  - The unit impulse response $h_1[n]$ of the inverse system can be obtained from (6) by taking $y[n] = \delta[n]$, so that

$$h_1[n] = \delta[n] - \delta[n-1]$$

- We can verify that

$$h[n] * h_1[n] = u[n] * (\delta[n] - \delta[n-1]) = u[n] - u[n-1] = \delta[n]$$

Causality of LTI Systems

- Recall: A system is causal if the output depends only on the present and past values of the input to the system.
- For a discrete-time LTI system represented by (1) on Page 15 to be causal, $y[n]$ must not depend on $x[k]$ for $k > n$.
  - This requires that the unit impulse response $h[n]$ of the system satisfy the condition

$$h[n] = 0 \text{ for } n < 0$$

which implies that $y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k]$.

Causality of LTI Systems (cont.)

- Similarly, for a continuous-time LTI system represented by (2) on Page 16 to be causal, $y(t)$ must not depend on $x(\tau)$ for $\tau > t$.
  - This requires that the unit impulse response $h(t)$ of the system satisfy the condition

$$h(t) = 0 \text{ for } t < 0$$

which implies that $y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau$.

Initial Rest

- We see that causality of an LTI system is equivalent to the condition of initial rest, i.e.:
  - If the input to a causal LTI system is 0 up to some point in time, then the output must also be 0 up to that time.
- Note: The equivalence of causality and the condition of initial rest applies only to linear systems.
  - For example, the system $y[n] = 2x[n] + 3$ is not linear though causal. If $x[n] = 0$, $y[n] = 3 \neq 0$, which does not satisfy the condition of initial rest.

Causal Signals

- While causality is a property of systems, it is common terminology to refer to a signal $x[n]$ or $x(t)$ as being causal if it is zero for $n < 0$ or $t < 0$.
  - For example, unit step signals $u[n]$ and $u(t)$ are both causal.
- With this definition of a causal signal, causality of an LTI system is also equivalent to its unit impulse response being a causal signal.

Stability of LTI Systems

- Recall: A system is stable if any bounded input produces a bounded output.
- It can be shown that a discrete-time LTI system represented by (1) on Page 15 is stable if and only if its unit impulse response is absolutely summable:

$$\sum_{n=-\infty}^{+\infty} |h[n]| < \infty$$

Stability of LTI Systems (cont.)

- Similarly, we can show that a continuous-time LTI system represented by (2) on Page 16 is stable if and only if its unit impulse response is absolutely integrable:

$$\int_{-\infty}^{+\infty} |h(t)|\, dt < \infty$$

Example 1

- The continuous-time LTI system defined by (3) on Page 19, i.e.,

$$y(t) = x(t - t_0)$$

has an unit impulse response

$$h(t) = \delta(t - t_0)$$

- Since

$$\int_{-\infty}^{+\infty} |h(t)|\, dt = \int_{-\infty}^{+\infty} |\delta(t - t_0)|\, dt = 1$$

we conclude that this system is stable.

Example 2

- The discrete-time LTI system defined by (5) on Page 21, i.e.,

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

has an unit impulse response

$$h[n] = u[n]$$

- Since

$$\sum_{n=-\infty}^{+\infty} |h[n]| = \sum_{n=-\infty}^{+\infty} |u[n]| = \sum_{n=0}^{+\infty} 1 = \infty$$

we conclude that this system is not stable.

Difference Equations

- An important class of discrete-time LTI systems is that for which the input-output relationship can be implicitly described by a general Nth-order linear constant-coefficient difference equation of the form:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \quad (7)$$

  - Note: The order refers to the most delayed version of the output $y[n]$ appearing in (7).
  - If auxiliary conditions (e.g., initial rest) on the system output $y[n]$ are specified, we can solve (7) and obtain an explicit expression for $y[n]$ in terms of the input $x[n]$.

Differential Equations

- Correspondingly, an important class of continuous-time LTI systems is one where the input-output relationship can be implicitly described by a general Nth-order linear constant-coefficient differential equation of the form:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \tag{8}$$

  - Note: The order refers to the highest derivative of the output $y(t)$ appearing in (8).
  - If auxiliary conditions (e.g., initial rest) on the system output $y(t)$ are specified, we can solve (8) and obtain an explicit expression for $y(t)$ in terms of the input $x(t)$.

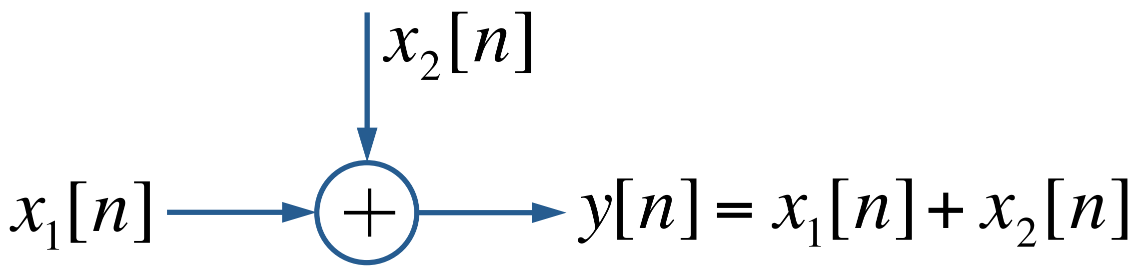
Block Diagram Representation

- LTI systems described by linear constant-coefficient difference and differential equations can be represented in terms of block diagram interconnections of elementary operations.

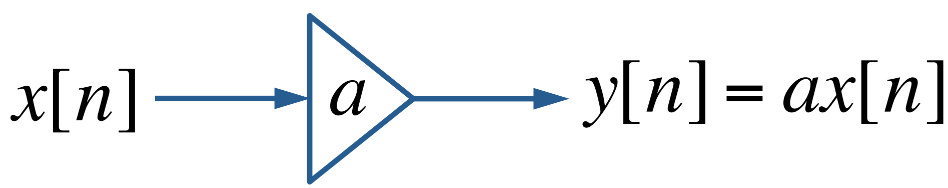
- Why it is useful?
  - It provides a pictorial representation of the system, which can add to our understanding of the behavior and properties of the system.
  - Such a representation is useful for the simulation or implementation of the system.

Basic Elements for Discrete-Time Systems

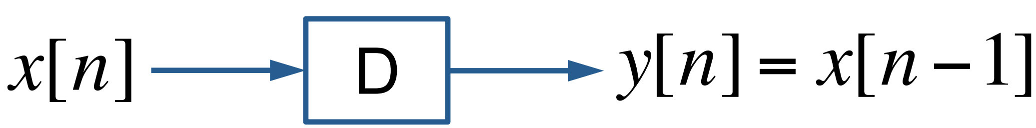
- **Adder**: Addition of two sequences



- **Multiplier**: Multiplication of a sequence by a constant



- **Unit delay**: Delaying a sequence by one sample

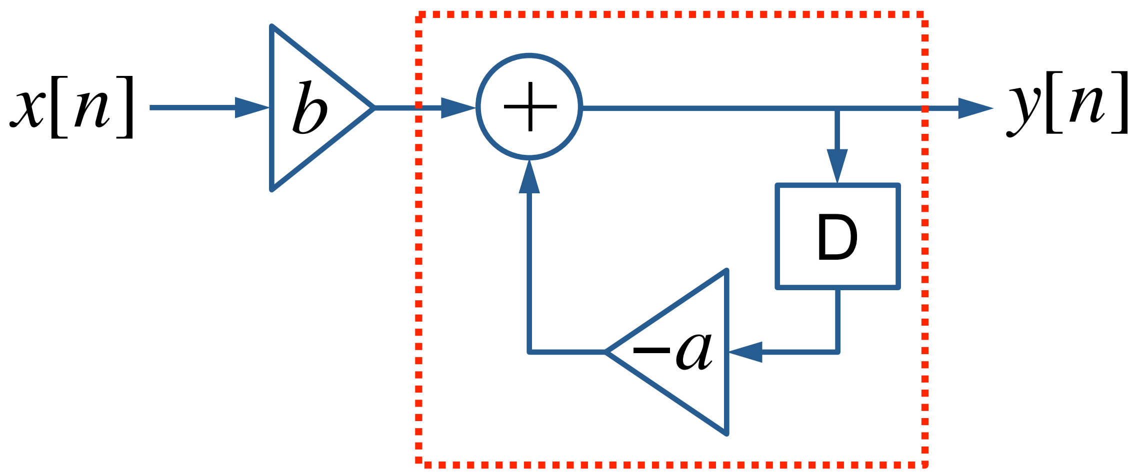


Example 1

- Consider a discrete-time LTI system described by the difference equation:

$$y[n] + ay[n-1] = bx[n] \tag{9}$$

- Rewrite (9) as $y[n] = -ay[n-1] + bx[n]$, which is in the form of a recursive equation.



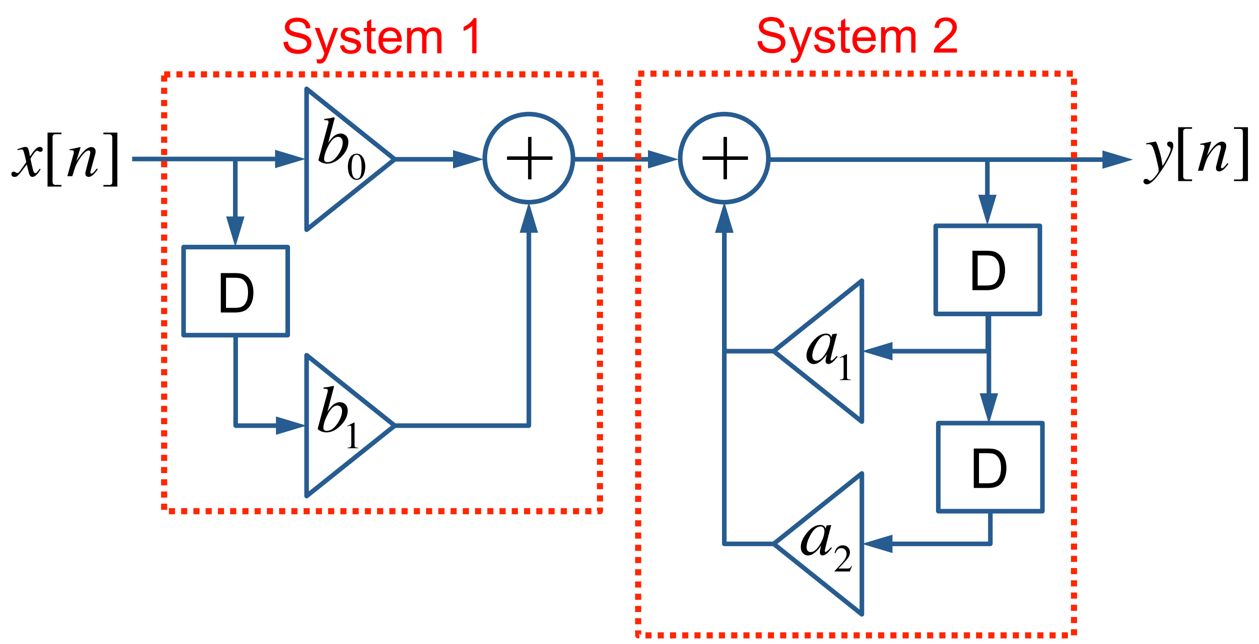
A feedback system

Example 2

- Consider a discrete-time LTI system described by the difference equation:

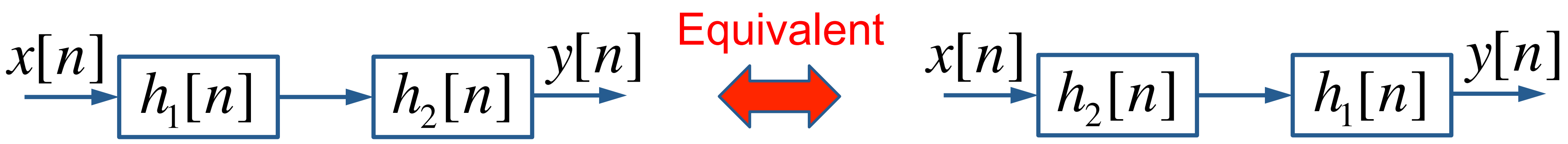
$$y[n] = a_1 y[n-1] + a_2 y[n-2] + b_0 x[n] + b_1 x[n-1]$$

- Block diagram representation: Direct form I

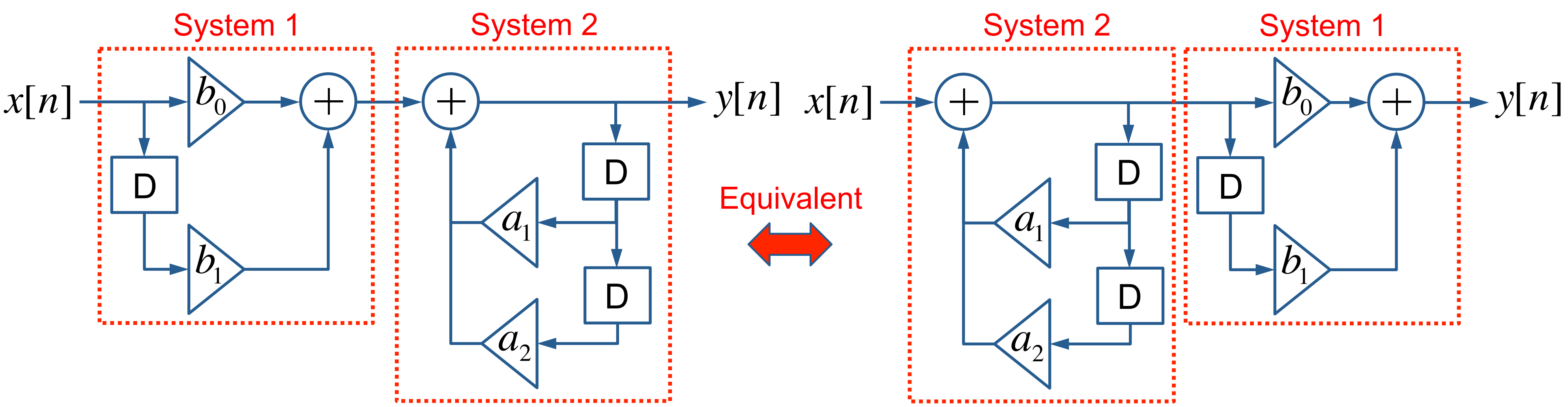


Example 2 (cont.)

- Recall: The impulse response of a series interconnection of two LTI systems is independent of the order in which they are cascaded, i.e.,

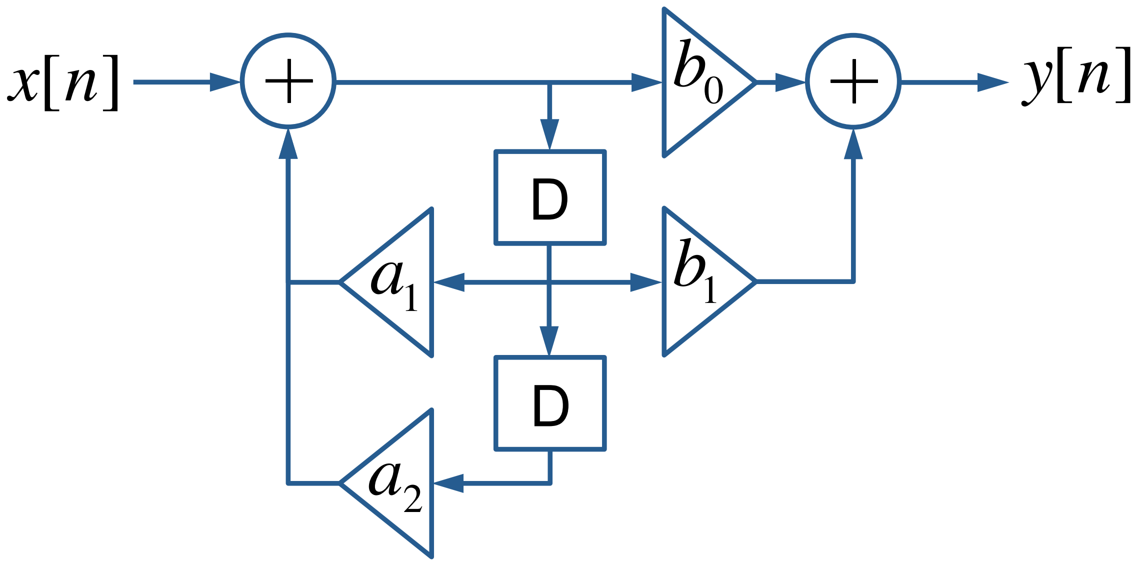


- Therefore:



Example 2 (cont.)

- Block diagram representation: Direct form II



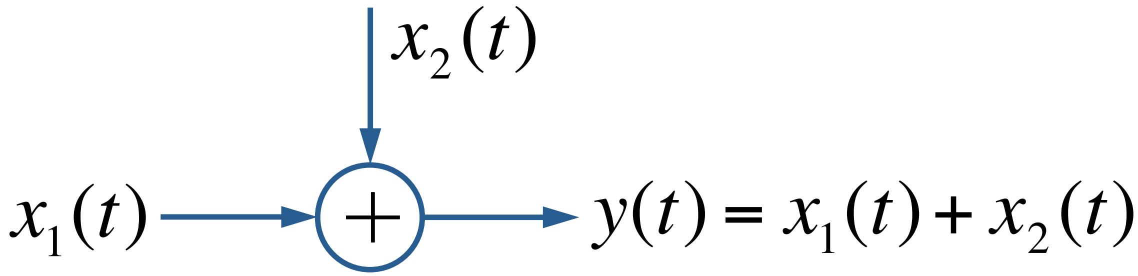
- In this example, we observe:
  - Direct form I requires three unit delays to implement the system.
  - Direct form II requires only two unit delays.

Example 2 (cont.)

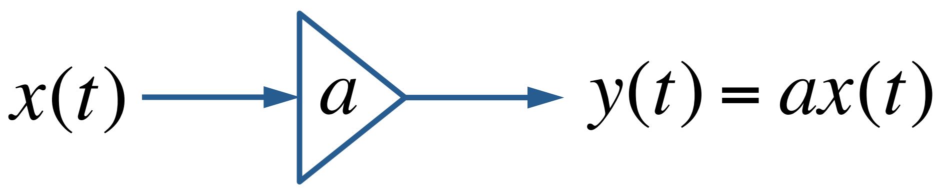
- For the general Nth-order system described by (7) on Page 31:
  - Direct form I requires $(N + M)$ unit delays.
  - Direct form II requires only $\max(N, M)$ unit delays.

Basic Elements for Continuous-Time Systems

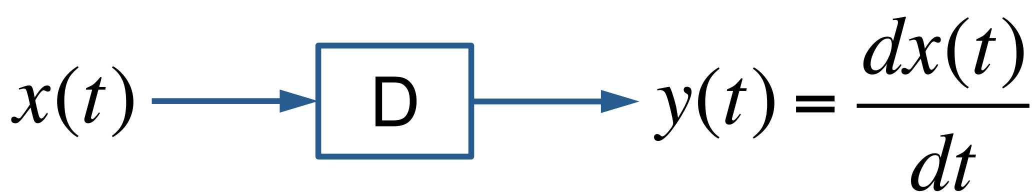
- Adder: Addition of two signals



- Multiplier: Multiplication of a signal by a constant



- Differentiator: Differentiating a signal



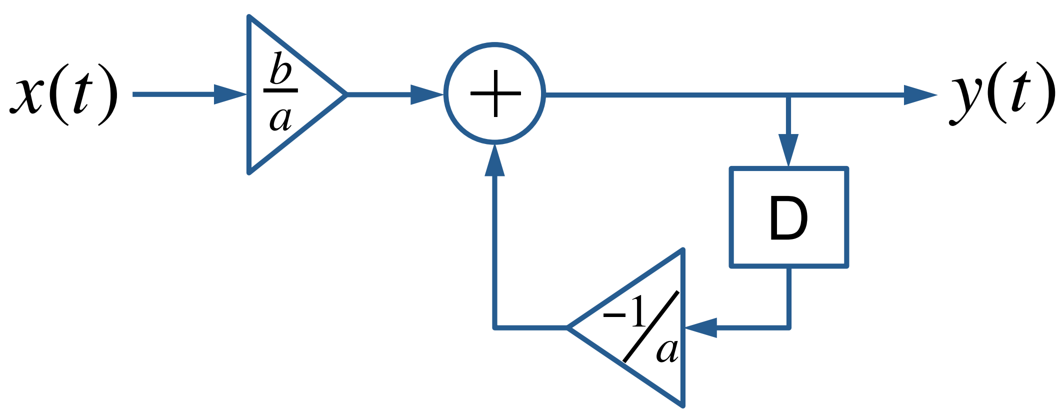
An Example

- Consider a continuous-time LTI system described by the differential equation:

$$\frac{dy(t)}{dt} + ay(t) = bx(t) \tag{10}$$

- Rewrite (10) as

$$y(t) = -\frac{1}{a}\frac{dy(t)}{dt} + \frac{b}{a}x(t)$$



An Example (cont.)

- Note that differentiators are both difficult to implement and extremely sensitive to errors and noise.

- An alternative and practical implementation of (10) is by rewriting it as

$$\frac{dy(t)}{dt} = bx(t) - ay(t)$$

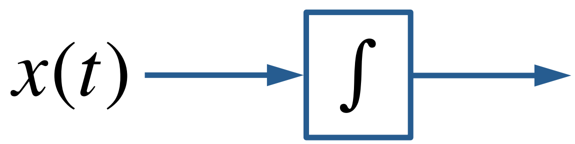
and then we have

$$\int_{-\infty}^{t} \frac{dy(\tau)}{d\tau} d\tau = y(t) = \int_{-\infty}^{t} [bx(\tau) - ay(\tau)] d\tau \qquad (11)$$

with the assumption of $y(-\infty) = 0$.

An Example (cont.)

- Integrator: Integrating a signal



$$y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$$

  - Note that integrators can be readily implemented using operational amplifiers.

- Thus, (11) can be implemented as