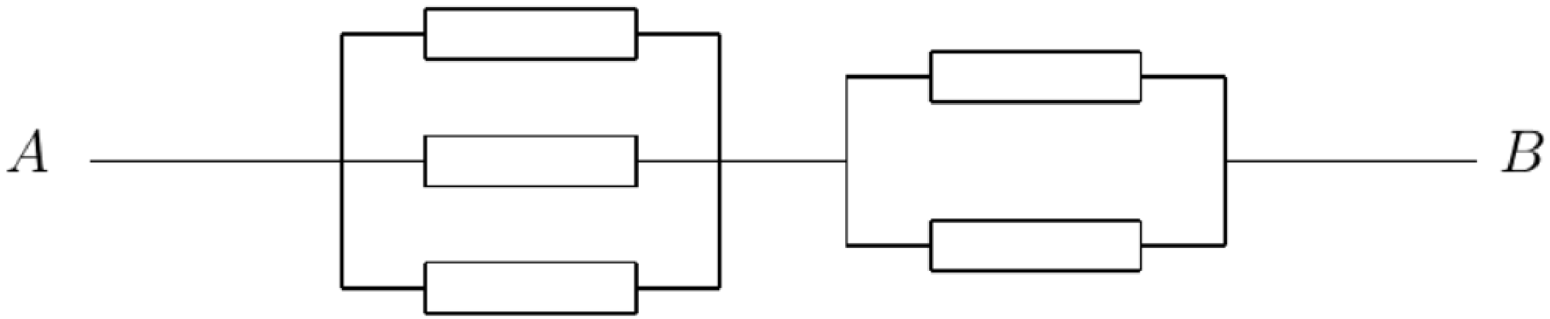
In-Class Exercise 2

1. Consider an experiment of drawing a card from a deck of 52 playing cards. Define the event with ♥ suit as A and the event with "K" as B.
   (a) Determine if A and B are independent or not.
   (b) Find the probability that the card is either ♥ or "K".
   (c) If two events are independent, should they be mutually exclusive or mutually non-exclusive?

2. Three chairs are arranged in a line, and three people randomly take seats. What is the probability that the person with middle height ends up in the middle seat?

3. The Mark Six Lottery is a 6 out of 49 lotto game. The 1st prize will be obtained if all 6 drawn numbers are picked correctly from {1,2, …, 49}.
   (a) Compute the probability of getting the 1st prize.
   (b) The unit investment for each Mark Six Entry is $10. If you want to get the 1st prize with probability 1, how much you need to invest?

4. In the following system, each component fails with probability of 0.3 independently of other components. Compute the reliability of this system.



5. Blackjack is a popular casino game whose goal is to get a total closer to 21 than the dealer, without going over 21. Aces count “1” or “11” at the player’s discretion. Face (i.e., king, queen, jack) cards count “10” and all other cards count their rank.

   In a double-deck game (with two sets of playing cards), suppose that you have been dealt with two “10”, for a total of 20. While the dealer’s first card is an ace. What is the probability that his second card will be a 10-count card that brings his total to 21?

6. Consider that 10 students are interested in forming a tennis club in the university. However, the club is allowed to have 7 committee members including 1 president, 2 (equivalent) vice-presidents, and 4 (equivalent) regular members. Determine the number of possible ways in forming the club.

7. Consider that there are 5 chairs, and 5 people randomly take seats.
   (a) Assume that the chairs are arranged in a line. Compute the probability that they end up in the order of decreasing height, from left to right.
   (b) Assume that the chairs are arranged in a circle. Compute the probability that they end up in the order of decreasing height, in a clockwise manner.

8. Consider that there are 6 chairs arranged in a line, and 3 girls and 3 boys randomly take seats. What is the probability that the 3 girls end up in the three leftmost seats?

9. Consider that you have 5 HKD100 notes and you want to divide among four people, say, A, B, C, and D. Determine the number of ways in doing this. Note that each person can receive 0 to 5 notes.

Solution

1(a)

There are 13 Heart and 4 King cards. Hence we have:

$$P(A) = \frac{13}{52} = \frac{1}{4} \quad \text{and} \quad P(B) = \frac{4}{52} = \frac{1}{13}$$

And there is only one Heart King, hence:

$$P(A \cap B) = P(AB) = \frac{1}{52}$$

It is clear that:

$$P(A \cap B) = P(AB) = P(A)P(B)$$

According to (1.5), A and B are independent.

1(b)
Using

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

We get:

$$P(A \cup B) = \frac{1}{4} + \frac{1}{13} - \frac{1}{52} = \frac{4}{13}$$

1(c)
For the nominal independent cases where $P(A) > 0$ and $P(B) > 0$, $P(A \cap B) = P(AB) = P(A)P(B) \neq 0$.

When A and B are mutually exclusive, this means:

$$P(A \cup B) = P(A) + P(B)$$

or $P(A \cap B) = 0$, which contradicts with independence. Hence, when two events are independent, they should not be mutually exclusive. In fact, they should be mutually non-exclusive.

2.

Consider that the person with middle height choose first. The probability of each seat is the same. Hence the probability should be 1/3.

Alternatively, Let the people from tallest to shortest be labeled as 1, 2, and 3. Then all possible arrangements are:

$$123, 132, 213, 231, 312, 321$$

Among the 6 combinations, 2 appears twice in the middle. Hence we get the same answer of 1/3.

3(a)

The task is to correctly select the 6 drawn numbers. Consider picking the number one by one. We have 6 choices in the first round, 5 in the second and so on. As a result, there are 6x5x4x3x2x1 or $6! = P(6,6)$ combinations.

Similarly, if we choose 6 numbers from numbers 1 to 49, there are 49x48x47x46x45x44 or $P(52,6)$ combinations.

As a result, the probability can be computed as:

$$P(win) = \frac{6}{49} \cdot \frac{5}{48} \cdot \frac{4}{47} \cdot \frac{3}{46} \cdot \frac{2}{45} \cdot \frac{1}{44} = \frac{1}{13983816}$$

Can we interpret as 6 independent events?

Alternatively, to select 6 from 49 without replacement, the number of combinations is:

$$C(49,6) = 13983816$$

Among all, only one combination corresponds to the first prize, hence

$$P(win) = \frac{1}{13983816}$$

3(b)

To ensure obtaining the first prize, we need to invest all combinations, hence the required amount is \$13983816x10 = \$139,838,160.

Note that the first prize from Hong Kong Jockey Club is far less than this bet amount.

4.
We can view it as two parallel sub-systems connected in series. We can follow Examples 1.8 and 1.9 to solve it.

For the first one, the probability that it works is:

$$P(\mathrm{OK}_1) = 1 - (0.3)^3 = 0.9730$$

For the second one, the probability that it works is:

$$P(\mathrm{OK}_2) = 1 - (0.3)^2 = 0.9100$$

Hence the system reliability is:

$$P(\mathrm{OK}) = P(\mathrm{OK}_1)P(\mathrm{OK}_2) = 0.9730 \cdot 0.9100 = 0.8854$$

5.

A double-deck consists of 104 cards. There are 101 cards as yet unaccounted for. Each deck starts out with 16 10-count (10, king, queen, jack) cards, for a total of 32 at the beginning.

2 are in your hand and there are 30 in the deck. Hence:

$$P(Dealer\ 21) = \frac{30}{101} = 0.297$$

6.

To choose 7 from 10, there are $P(10,7)=10\times 9\times\cdots\times 4$ ordered ways.

Since the 2 vice-presidents are equivalent, which correspond to 2! arrangements, and the 4 equivalent regular members correspond to 4! arrangements, we need to divide $P(10,7)$ by the product of 2! and 4! to avoid over-counting.

Hence the number of possible ways is:

$$\frac{P(10,7)}{4!2!}=12600$$

Using MATLAB, the command for factorial $N!$ is `factorial(N)`:

```
>> factorial(10)/(factorial(3)*factorial(4)*factorial(2))
ans = 12600
```

There are other ways of thinking to solve the problem. Suppose picking the president first from 10, there are $C(10,1)=10$ ways.

After the president post is fixed, choosing 2 vice-presidents from the remaining 9 corresponds to $C(9,2)$ ways.

Finally, choosing 4 regular members from the remaining 7 corresponds to $C(7,4)$ ways. Hence the solution is:

$$C(10,1)\cdot C(9,2)\cdot C(7,4)=\binom{10}{1}\cdot\binom{9}{2}\cdot\binom{7}{4}=\frac{10}{1!}\cdot\frac{9\cdot 8}{2!}\cdot\frac{7\cdot 6\cdot 5\cdot 4}{4!}=12600$$

```
>> nchoosek(10,1)*nchoosek(9,2)*nchoosek(7,4)
ans = 12600
```

Starting with the 4 regular members, then president, finally 2 vice-presidents, yields the same result:

$$C(10,4)\cdot C(6,1)\cdot (5,2)=\binom{10}{4}\cdot\binom{6}{1}\cdot\binom{5}{2}=12600$$

```
>> nchoosek(10,4)*nchoosek(6,1)*nchoosek(5,2)
ans = 12600
```

As a result, choosing which position first, second, and third does not affect the final result.

7.(a)

There are $5! = 120$ possible arrangements of 5 people in 5 chairs. Since there is only 1 arrangement which sorts from the tallest to the shortest, from left to right. Hence the probability is 1/120.

Alternatively, we may start with the tallest person. He has the chance of 1/5 to choose the leftmost chair. After taking the chair, the second tallest person has the chance of 1/4 to choose the second leftmost chair. Likewise, the remaining 3 people have the chances of 1/3, 1/2, and 1. Finally, the product of these 5 probabilities is 1/120.

It is based on the viewpoint of conditional probability:

$$\begin{aligned} &P(A \cap B \cap C \cap D \cap E) \\ &= P(A) \cdot P(B|A) \cdot P(C|A \cap B) \cdot P(D|A \cap B \cap C) \cdot P(E|A \cap B \cap C \cap D) \end{aligned}$$

7.(b)

As in 7.(a), there are 120 possible arrangements. But among them, 5 will end up with the required arrangement because the tallest person can take any of the 5 chairs. As a result, the probability is 5/120=1/24.

Alternatively, we can use the viewpoint of conditional probability. As it is a circle, the tallest person can be seated in any of the 5 chairs. The second tallest person has a probability of 1/4 to seat next to the tallest in the clockwise. After the second tallest takes the seat, the middle one has a chance of 1/3 to seat next to him. Likewise, the remaining 2 people have the chances of 1/2 and 1. Finally, the product of these 4 probabilities is 1/24.

8.
The total number of possible seat arrangements is $6! = 720$.

If 3 girls are arranged in the three leftmost seats, there are $3! = 6$ possible arrangements.

Likewise, there are $3! = 6$ possible arrangements for the 3 boys in the rightmost.

Hence the number of possible arrangements in the given setting is $3!3! = 36$. As a result, the probability is 36/720=1/20.

Alternatively, again we can use the concept in conditional probability. Considering that the girls choose the seats first, the first girl has a chance of 3/6 in getting one of the three leftmost seats. After she picks the seat, the second girl then has a chance of 2/5 to get the right seat. Finally, the third girl has a chance of 1/4 to get the right seat.

As a result, the probability is the product: 3/6x2/5x1/4=1/20.

The third approach is to apply combination (counting without ordering). Consider choosing 3 people from 6 for the three leftmost seats, there are $C(6,3) = 20$ ways. As there is only 1 way that they are all girls, the probability is thus 1/20.

9.

Since 0 note can be obtained, a possible arrangement for A, B, C, and D, can be 5,0,0,0 (AAAAA). Note also that this is not identical to 0,0,0,5 (DDDDD).

Hence it can be viewed as calculating the number of combinations with replacement. That is, selecting $k = 5$ from $N = 4$ when the order does not matter (e.g., AABBB = ABBBA):

$$C_r(4,5) = \frac{(4+5-1)!}{(4-1)!5!} = 56$$