In-Class Exercise 3

1. Suppose 0.01% of bits transmitted through a digital transmission are received in error. Bits are transmitted until the first error occurs. Find the probability that there is no error in the first 10000 bit transmission.

2. With the use of the binomial theorem:

$$(\alpha+\beta)^n = \sum_{k=0}^{n} \binom{n}{k} \alpha^k \beta^{n-k} = \sum_{k=0}^{n} \binom{n}{k} \alpha^{n-k} \beta^k$$

   Prove that the sum of all probability mass functions (PMFs) of a binomial random variable with parameters n and p is 1.

3. Consider the binomial distribution with parameters n and p. When n is fixed, what is the value of p in terms of n if the probability of 0 success is equal to the probability of 1 success?

4. A discrete random variable (RV) K has the following cumulative density function (CDF):

$$F(k) = \begin{cases} 0, & k < 1 \\ 0.2, & 1 \leq k < 2 \\ 0.4, & 2 \leq k < 3 \\ 0.6, & 3 \leq k < 4 \\ 0.8, & 4 \leq k < 5 \\ 1, & k \geq 5 \end{cases}$$

Determine and sketch the PMF of K.

5. Given that the PMF $P_R(r)$ of a discrete RV R has the form:

$$P_R(r) = \begin{cases} \alpha p^r, & r = 2, 3, \cdots \\ 0, & \text{otherwise} \end{cases}$$

Find all possible values for α and p.

6. The number of buses that arrive at a bus stop in T minutes is a Poisson RV B, and its average value is $T/5$.
   (a) Determine the PMF of B.
   (b) Find the probability that 3 buses arrive in a 2-minute interval.
   (c) Find the probability that no buses arrive in a 10-minute interval.
   (d) How much time is allowed for at least 1 bus arriving with a probability of 0.99?

7. An airline knows 5% of the people making reservations on a certain flight will not show up. Consequently, their policy is to sell 52 tickets for a flight that can hold only 50 passengers. What is the probability that there will be a seat available for every passenger who shows up?

8. Consider a shop is selling 100 blind boxes (盲盒) of toy figures. Among these 100 boxes, it is stated that 2 of them are special editions. Suppose you want to get one special edition and you plan to buy at most 5 boxes. That is, you will buy one by one. For example, after you buy one, you will immediately open the box to see if it is a special-edition figure. What is the probability that you can get one special edition within 5 purchases? Which distribution can you apply to approximate this probability?

9. Consider throwing n balls randomly into $b < n$ boxes. What is the probability, denoted by $P(k)$, that a given box has exactly $k \leq n$ balls in it? Can you guess at what value of k, $P(k)$ will be maximized?

Solution

1.

Let the probability of error be p. Here, $p = 0.0001$ and the probability that there is an error in the first or second or up to the 10000th bit is:

$$p(1) + \cdots + p(10000)$$

We may make use of the CDF of geometric RV:

$$F(r) = P(X \leq r) = 1 - (1-p)^r$$

Then the required probability is:

$$1 - F(10000) = (1-p)^{10000} = 0.9999^{10000} = 0.3679$$

```
>> (0.9999)^(10000)
ans = 0.3679
```

2.

Recall the PMF of binomial RV:

$$p(r) = P(X = r) = C(n, r)p^r(1-p)^{n-r}, \quad 0 \leq r \leq n$$

The sum of all PMFs is then:

$$\sum_{r=0}^{n} C(n, r)p^r(1-p)^{n-r}$$

Applying the binomial theorem with $\alpha = p$ and $\beta = 1 - p$ yields:

$$(p + 1 - p)^n = 1$$

3.

Recall the PMF of binomial RV:

$$p(r) = P(X = r) = C(n, r)p^r(1-p)^{n-r}, \quad 0 \le r \le n$$

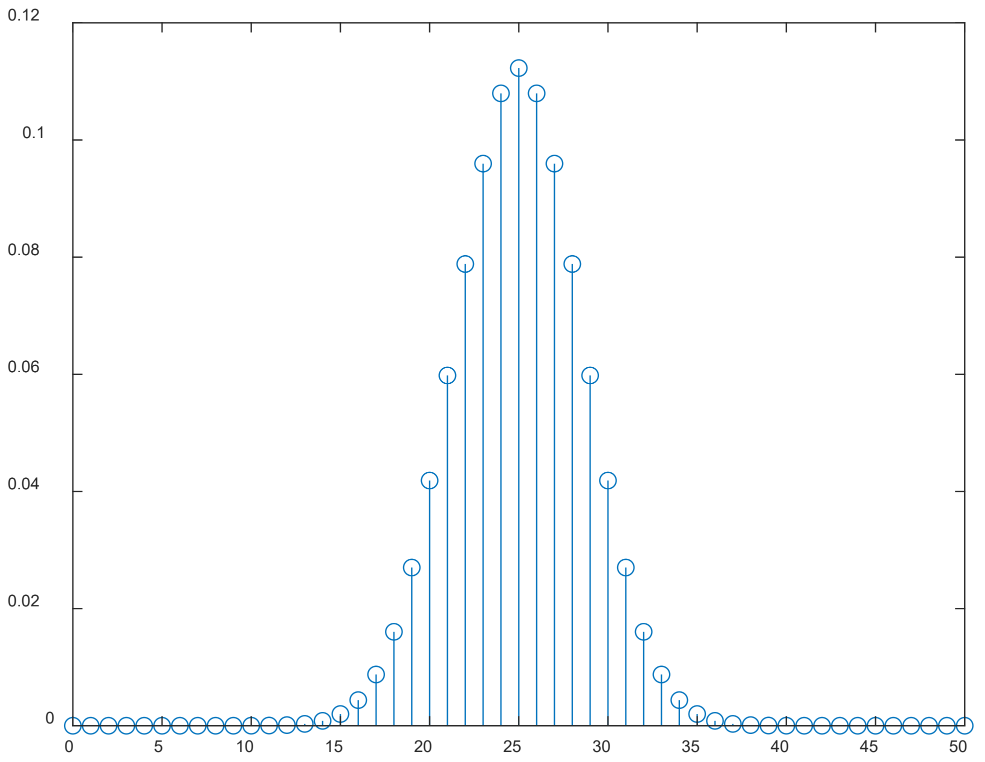
Now we need $p(0) = p(1)$:

$$C(n, 0)p^0(1-p)^{n-0} = C(n, 1)p^1(1-p)^{n-1} \Rightarrow 1 - p = np \Rightarrow p = \frac{1}{n+1}$$

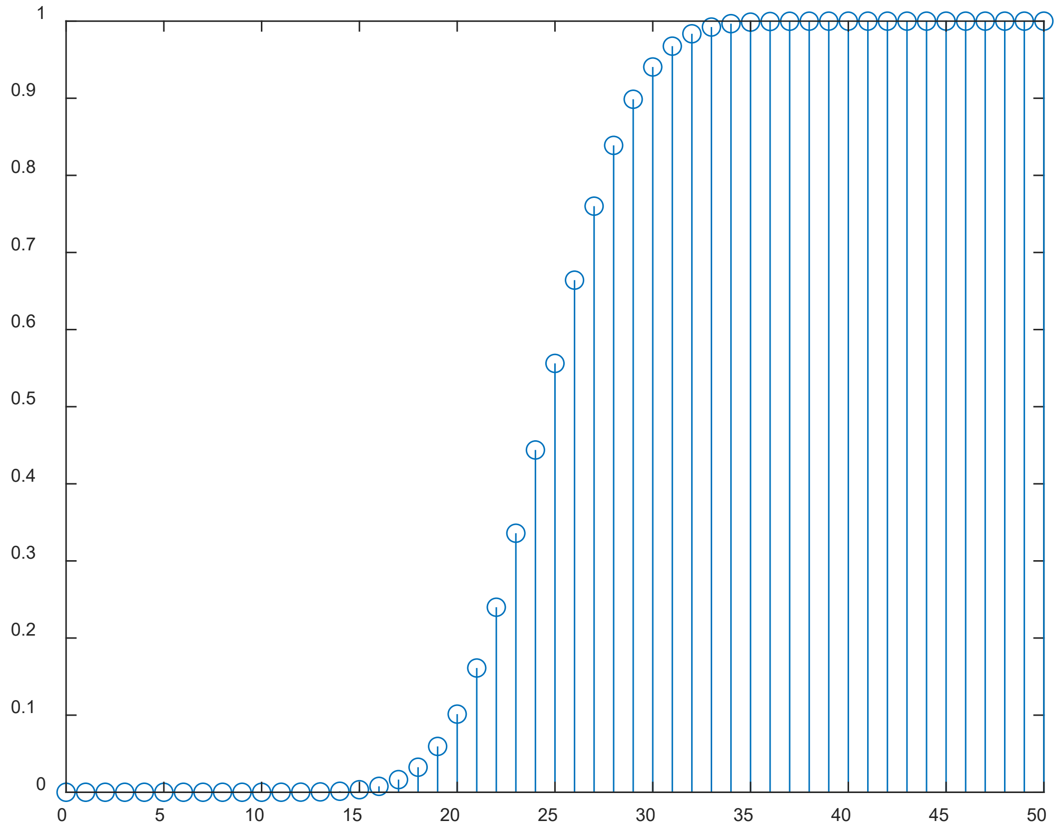
When n is larger, we need smaller value of probability of success to make $p(0) = p(1)$.

We may also realize that the PMF plot of binomial distribution can have different shapes by considering different values of p. Note the value of r when the PMF reaches its maximum.

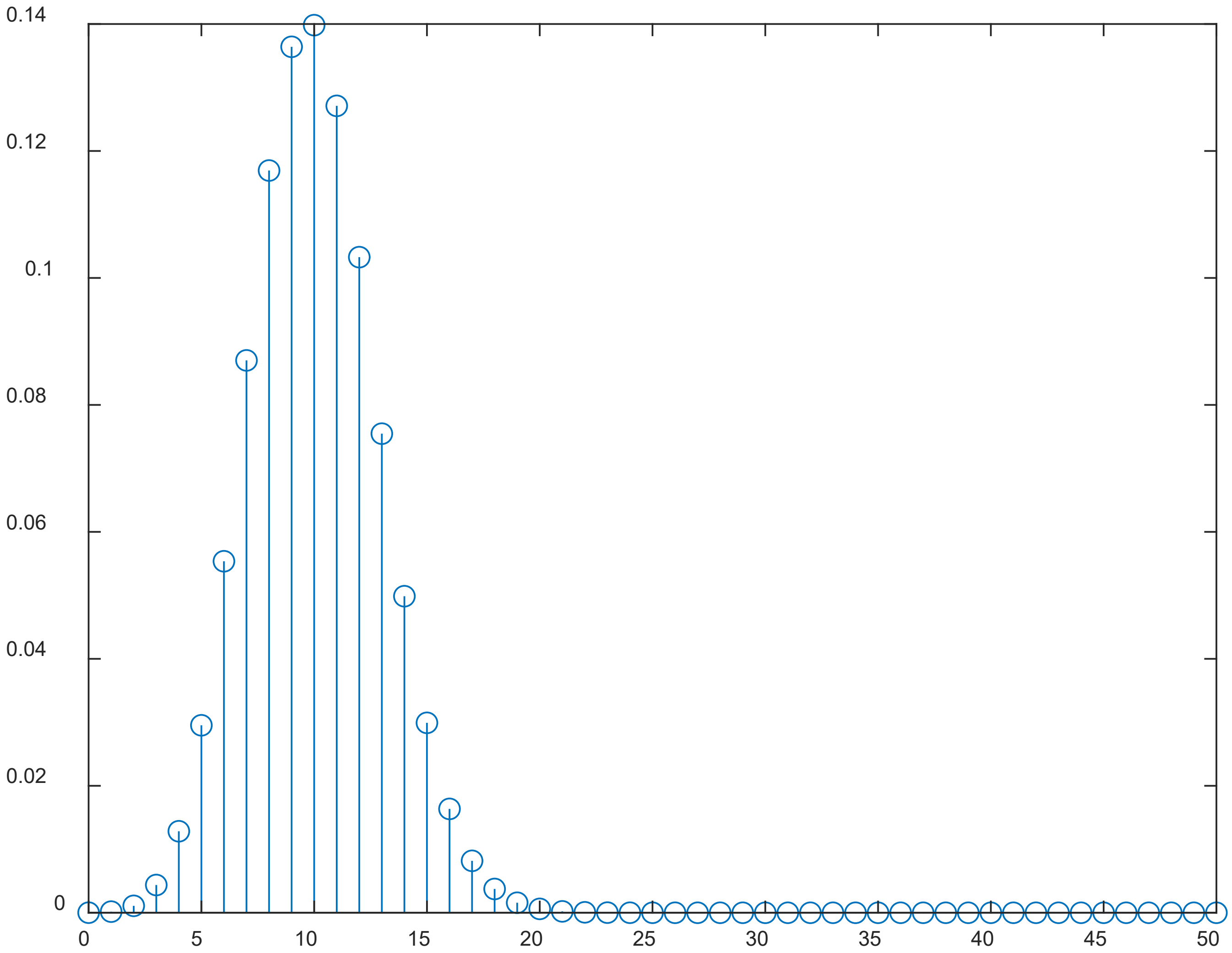
```
n=50;
p=0.5;
for k=0:50;
    P(k+1)=nchoosek(n,k)*p^k*(1-p)^(n-k);
end
stem(0:50,P)
```



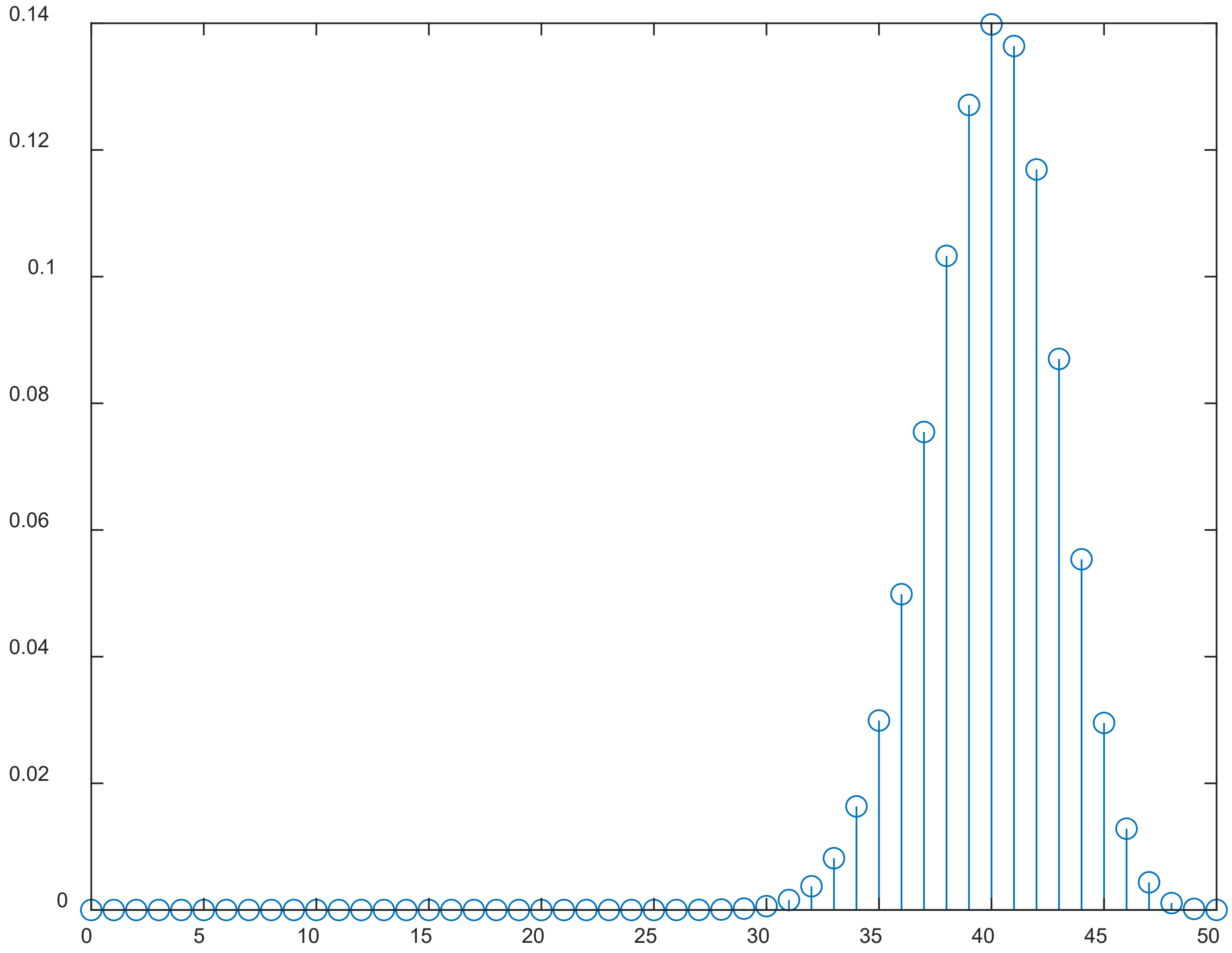
```
C(1)=P(1);
for k=1:50;
    C(k+1)=C(k)+P(k+1);
end
stem(0:50,C)
```



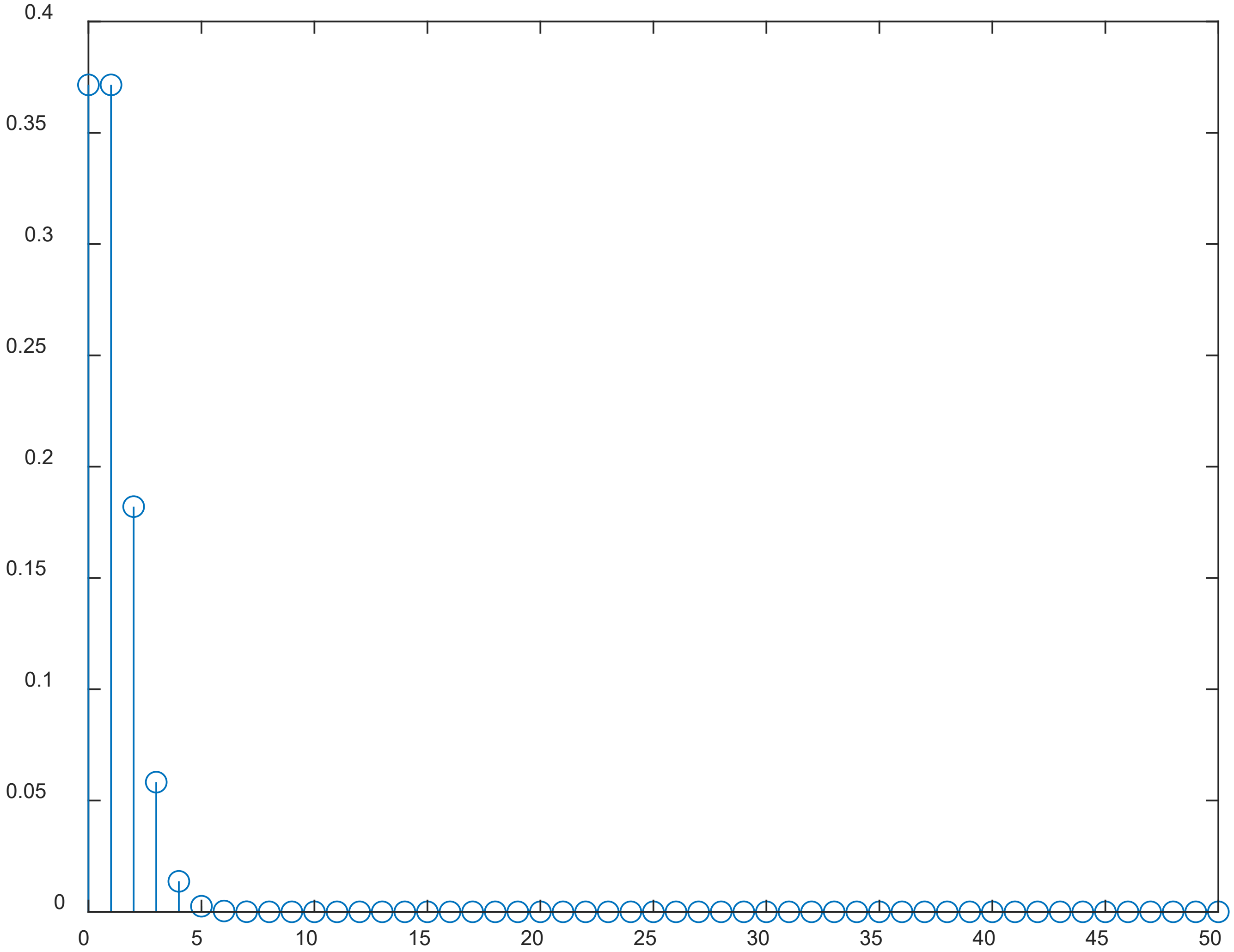
$n = 50$ and $p = 0.2$



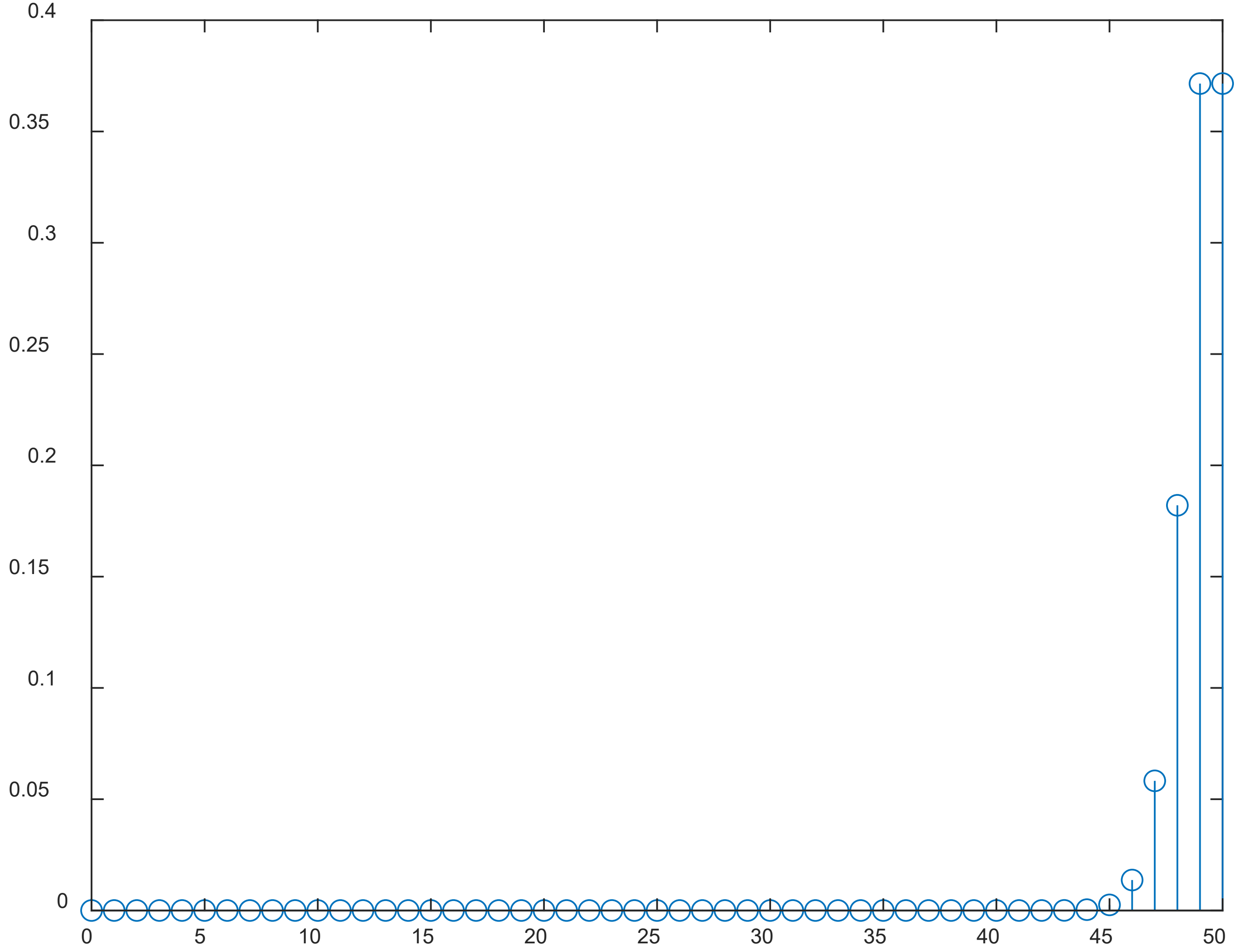
$n = 50$ and $p = 0.8$



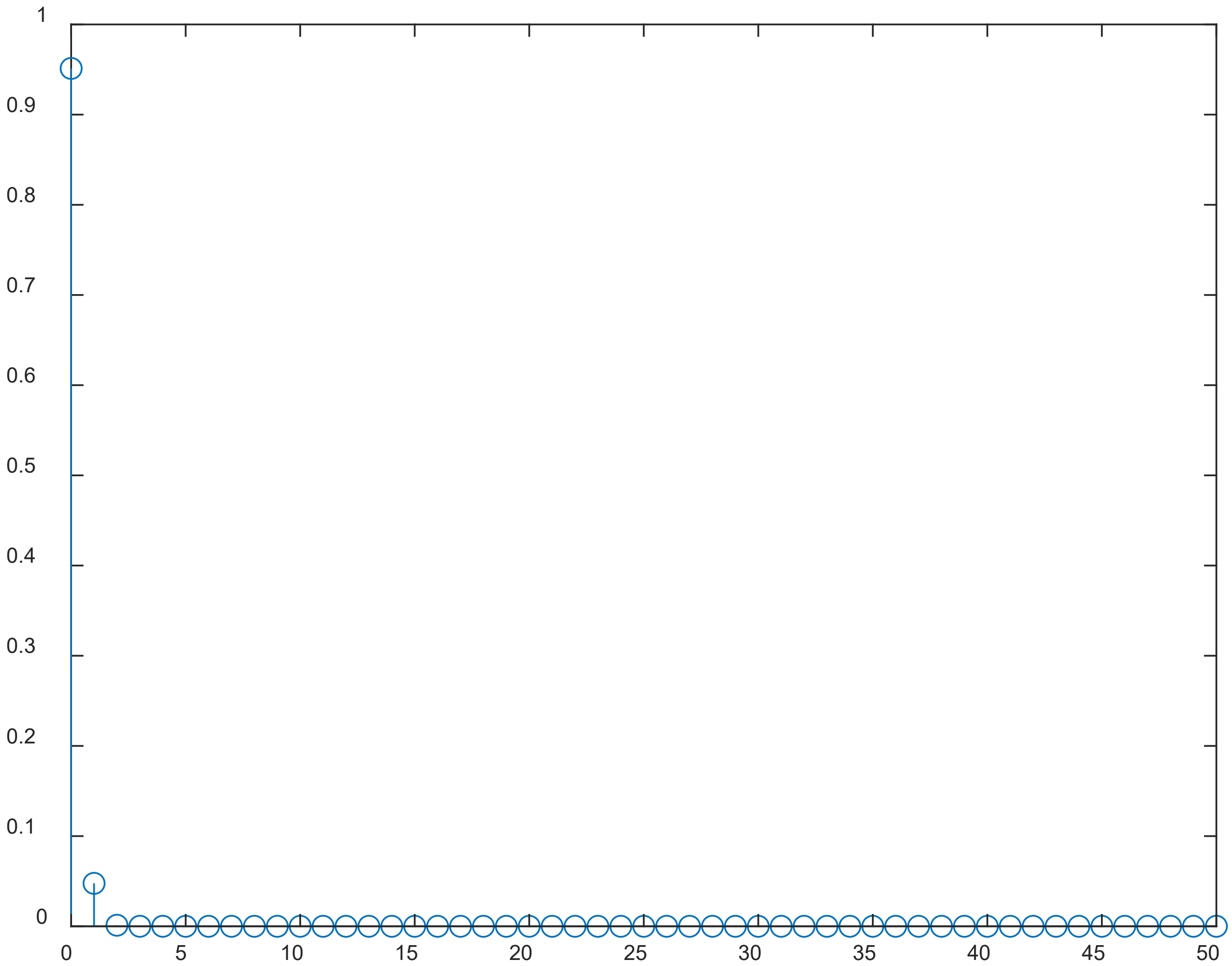
$n = 50$ and $p = 1/51$



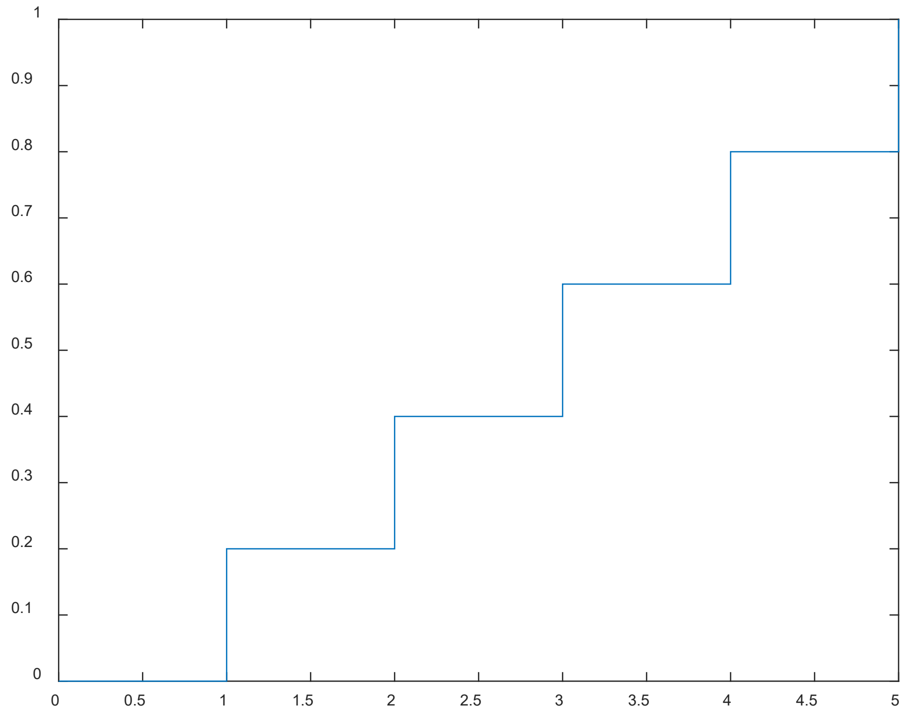
$n = 50$ and $p = 50/51$



$n = 50$ and $p = 0.001$

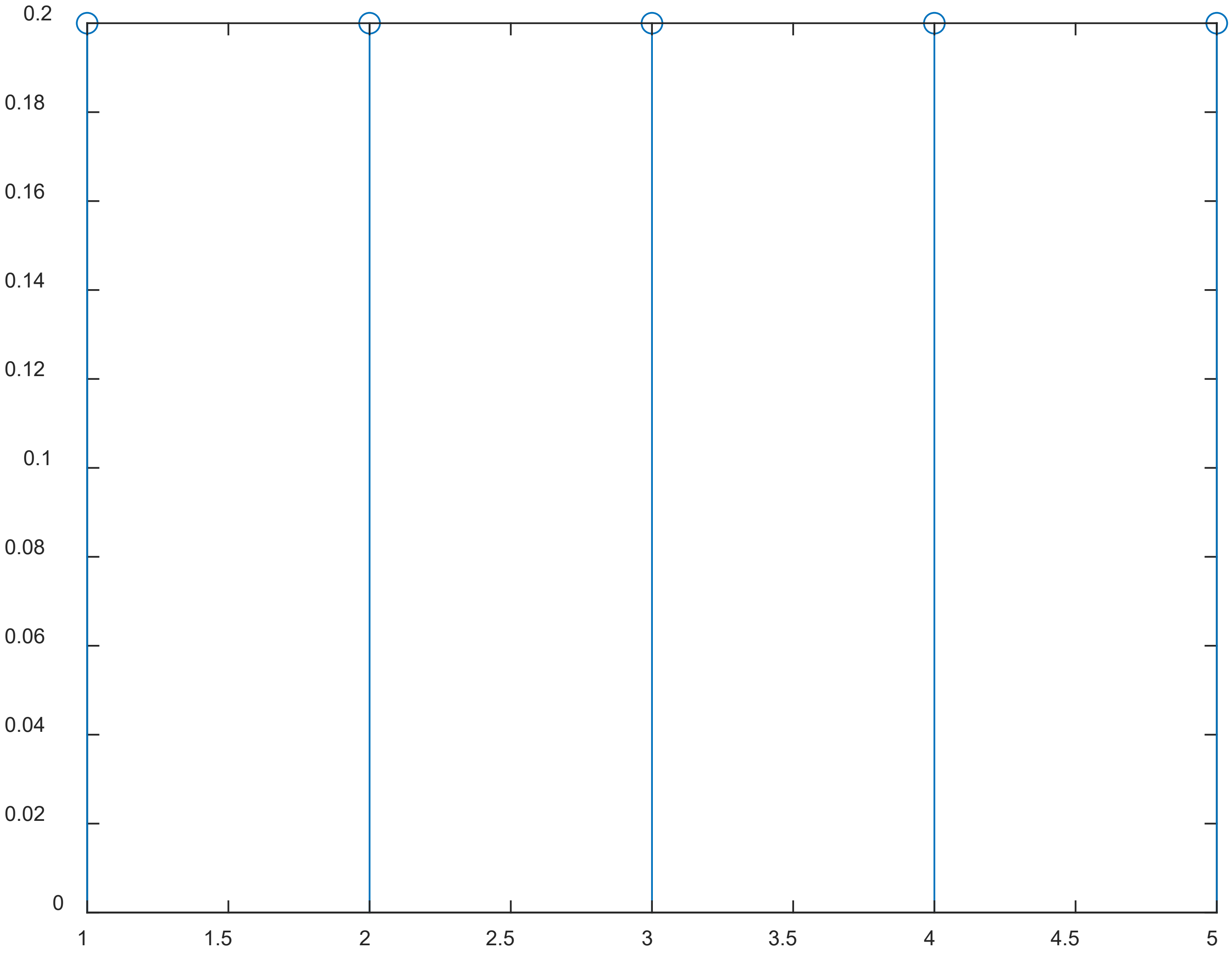


4.
Graphically, the CDF is:



It can be easily deduced that

$$P_K(k) = \begin{cases} 0.2, & k = 1, 2, 3, 4, 5 \\ 0, & \text{otherwise} \end{cases}$$


0.2
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
1
1.5
2
2.5
3
3.5
4
4.5
5

5.

The sum of all PMFs should be equal to 1:

$$\sum_{r=2}^{\infty} p(r) = \sum_{r=2}^{\infty} \alpha p^r = \alpha p^2 [1 + p + p^2 + \cdots] = 1$$

First, the geometric sum must converge and hence $|p| < 1$. Together with the fact that the PMF must be nonnegative, we have $0 < p < 1$.

When the geometric sum converges, we have:

$$\alpha p^2 [1 + p + p^2 + \cdots] = \alpha p^2 \frac{1}{1-p} = \frac{\alpha p^2}{1-p} = 1 \Rightarrow \alpha = \frac{1-p}{p^2}$$

Hence their possible values are:

$$0 < p < 1, \quad \alpha = \frac{1-p}{p^2}$$

6.(a)
With $\lambda = T/5$, according to (2.7), we have:

$$P_B(b) = \begin{cases} e^{-T/5}\dfrac{(T/5)^b}{b!}, & b = 0, 1, \cdots \\ 0, & \text{otherwise} \end{cases}$$

6.(b)
For $T = 2$, the probability that 3 buses arrive is:

$$P_B(3) = e^{-2/5}\frac{(2/5)^3}{3!} = 0.0072$$

6.(c)

For $T = 10$, the probability of no bus arrival is:

$$P_B(0) = e^{-2}\frac{2^0}{0!} = 0.1353$$

6.(d)

The probability that at least one bus arrives in T minutes is:

$$P(B \geq 1) = 1 - P(B = 0) = 1 - e^{-T/5} = 0.99 \Rightarrow 0.01 = e^{-T/5} \Rightarrow T = 23$$

That is, 23 minutes are needed.

7.

We can apply the binomial distribution for the probability computation. Let $p=0.95$ be the probability of success or show up. It is required that the number of passengers should be 0 to 50. That is, the probability of having 51 or 52 is not allowed, implying the required probability is:

$$1-C(52,0)0.95^{52}-C(52,1)0.95^{51}(0.05)=0.7405$$

```
>> 1-(0.95)^(52)-52*(0.95)^(51)*0.05

ans = 0.7405
```

8.

There are 5 chances to get a special-edition figure. You may get in the first try. If not, you need to have a second try, and so on. Hence the probability is:

2/100+
98/100*2/99+
98/100*97/99*2/98+
98/100*97/99*96/98*2/97+
98/100*97/99*96/98*95/97*2/96 = 0.0980

We may use geometric distribution to perform approximation. Applying the CDF of geometric RV:

$$F(r) = P(X \leq r) = 1 - (1-p)^r$$

Now we have $r = 5$ and $p = 0.02$, $1 - (1-p)^r = 0.0961$.

In fact, this approximation will become even more accurate when the number of boxes is larger, say, 1000:

20/1000 +
980/1000*20/999 +
980/1000*979/999*20/998 +
980/1000*979/999*978/998*20/997 +
980/1000*979/999*978/998*977/997*20/996=0.0963

9.

As there are b boxes, each ball has a chance of $1/b$ throwing into one of the boxes. Hence we may apply binomial distribution and employ $p = 1/b$ as the probability of success or the probability of throwing into a given box. Hence the probability is:

$$P(k) = \binom{n}{k} \left(\frac{1}{b}\right)^k \left(1 - \frac{1}{b}\right)^{n-k}$$

Since there are n balls and b boxes, the most probable case might be when each box gets n/b balls. Analogously, when flipping 50 fair coins, we expect that the most probable case corresponds to 25 heads and 25 tails.

Hence, $P(k)$ will reach its maximum value when k is the integer nearest to n/b.

$n = 1000$, $b = 100 \Rightarrow k = 10$ and $p = 0.01$