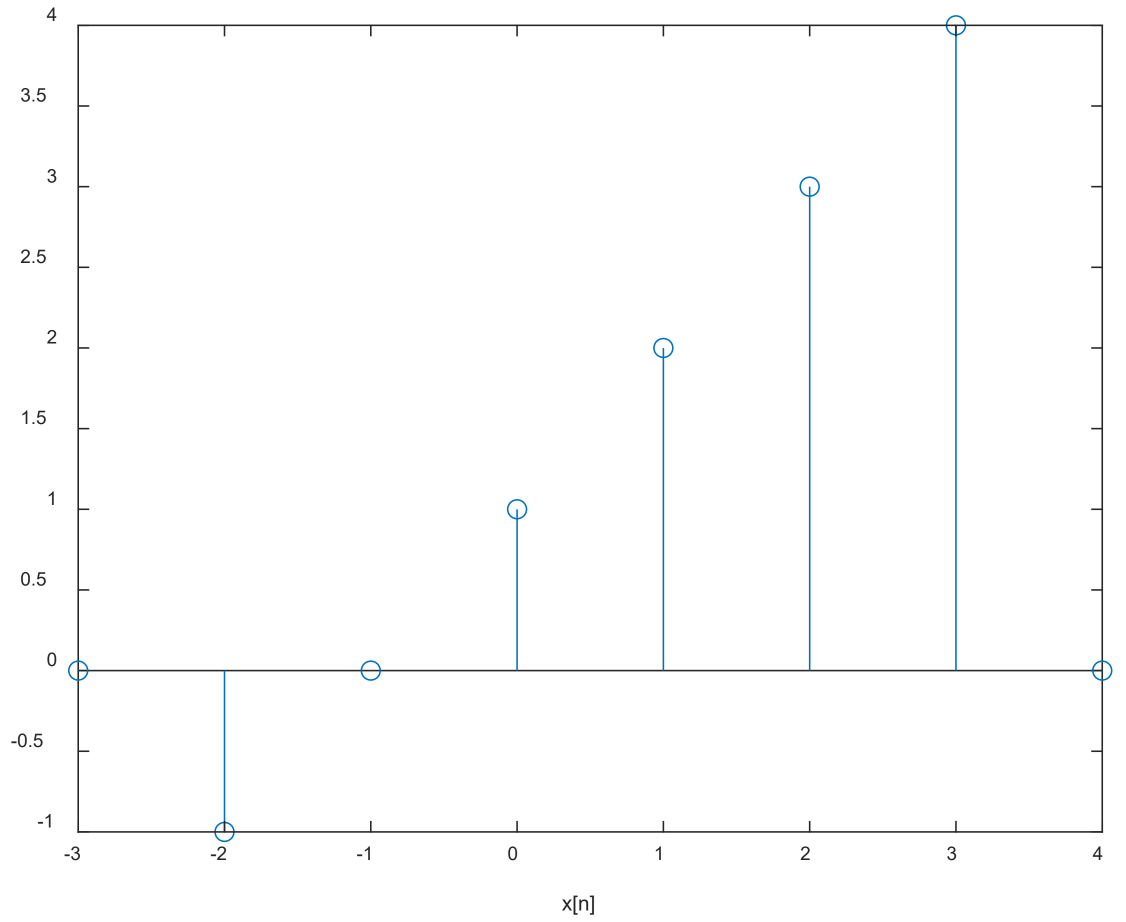
<u>Solution</u>

1.(a)

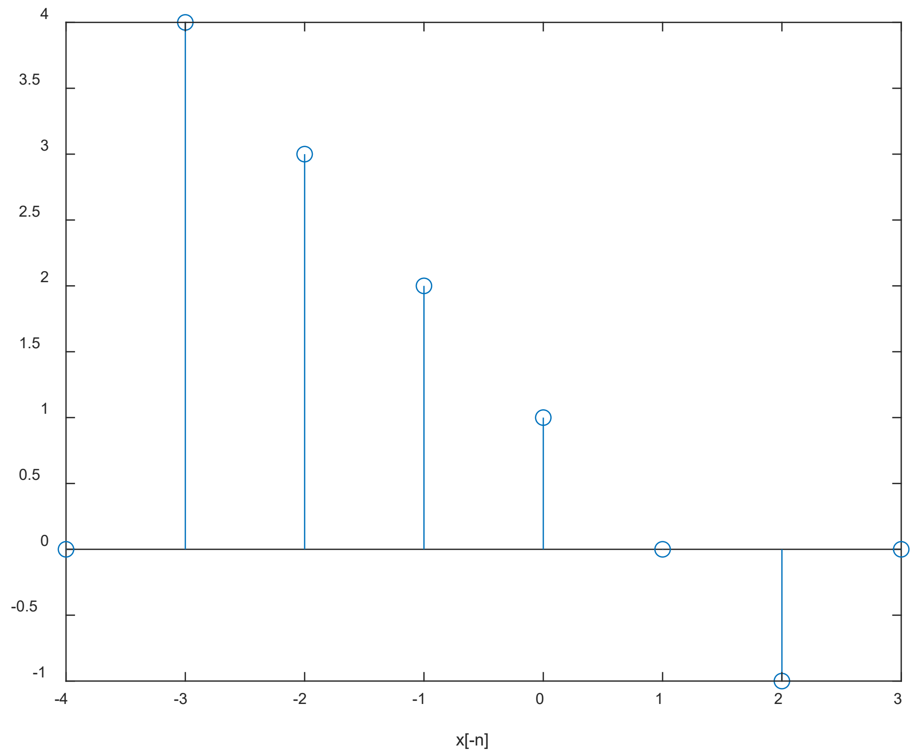


1.(b)

The even and odd components can be determined from:

$$x_e[n] = \frac{1}{2}[x[n] + x[-n]] \quad \text{and} \quad x_o[n] = \frac{1}{2}[x[n] - x[-n]]$$

$x[-n]$ has the form:



Hence we have:

$$x_e[n] = \begin{cases} 2, & n = -3 \\ 1, & n = -2 \\ 1, & n = -1 \\ 1, & n = 0 \\ 1, & n = 1 \\ 1, & n = 2 \\ 2, & n = 3 \\ 0, & \text{otherwise} \end{cases} \qquad x_o[n] = \begin{cases} -2, & n = -3 \\ -2, & n = -2 \\ -1, & n = -1 \\ 0, & n = 0 \\ 1, & n = 1 \\ 2, & n = 2 \\ 2, & n = 3 \\ 0, & \text{otherwise} \end{cases}$$

2.(a)

$x(t)$ is a periodic signal.

2.(b)

$x(t)$ is not an energy signal.

2.(c)

It is clear that $x(t)$ has a fundamental period of $T = 2\pi/B$.

Moreover, the squared magnitude is

$$A_1^2 \sin^2(Bt + C_1) + A_2^2 \cos^2(Bt + C_2) + 2A_1A_2 \sin(Bt + C_1)\cos(Bt + C_2)$$

As $x(t)$ is periodic, we can find the power first. The power can be computed from each of the above terms and then sum together.

For the first term, we note that:

$$A_1^2 \sin^2(Bt + C_1) = \frac{A_1^2(1 - \cos(2Bt + 2C_1))}{2}$$

The power corresponding to this component is:

$$\begin{aligned} P_1 &= \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} A_1^2 \sin^2(Bt + C_1) dt \\ &= \frac{A_1^2}{T} \int_0^T \frac{(1 - \cos(2Bt + 2C_1))}{2} dt \\ &= \frac{A_1^2 B}{2\pi} \int_0^{2\pi/B} \frac{1}{2} [1 - \cos(2Bt + 2C_1)] dt \\ &= \frac{B}{2\pi} \cdot \frac{A_1^2}{2} \cdot \frac{2\pi}{B} = \frac{A_1^2}{2} \end{aligned}$$

The second term is:

$$A_2^2 \cos^2(Bt + C_2) = \frac{A_2^2(1 + \cos(2Bt + 2C_2))}{2}$$

Similarly, the power corresponding to this component is:

$$P_2 = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} A_2^2 \cos^2(Bt + C_2) dt = \frac{A_2^2}{2}$$

Finally, the last term is:

$$2A_1 A_2 \sin(Bt + C_1) \cos(Bt + C_2) = A_1 A_2 \left[\sin(2Bt + C_1 + C_2) + \sin(C_1 - C_2)\right]$$

The power in this component is easily computed as

$$P_3 = A_1 A_2 \sin(C_1 - C_2)$$

Combining the above results, the power of $x(t)$ is:

$$P_x = P_1 + P_2 + P_3 = \frac{A_1^2}{2} + \frac{A_2^2}{2} + A_1 A_2 \sin(C_1 - C_2)$$

As it has finite power, its energy should be $E_x = \infty$.

3.(a)

As the signal is periodic and continuous in time, we can compute the power using one period.

Given that the fundamental period is $T=4$, the power is computed as:

$$
\begin{aligned}
P &= \frac{1}{T}\int_0^T x^2(t)dt \\
&= \frac{1}{4}\left[\int_0^1 (-4)^2 dt + \int_2^4 (-2)^2 dt\right] \\
&= \frac{1}{4}\left[1\cdot 16 + 2\cdot 4\right] \\
&= 6
\end{aligned}
$$

3.(b)

As $T = 4$, then the fundamental frequency is $\Omega_0 = \pi/2$. The Fourier series coefficients are then computed from

$$a_k = \frac{1}{T}\int_0^T x(t)e^{-jk\Omega_0 t}dt = -\frac{1}{4}\left(\int_0^1 4e^{-jk(\pi/2)t}dt + \int_2^4 2e^{-jk(\pi/2)t}dt\right)$$

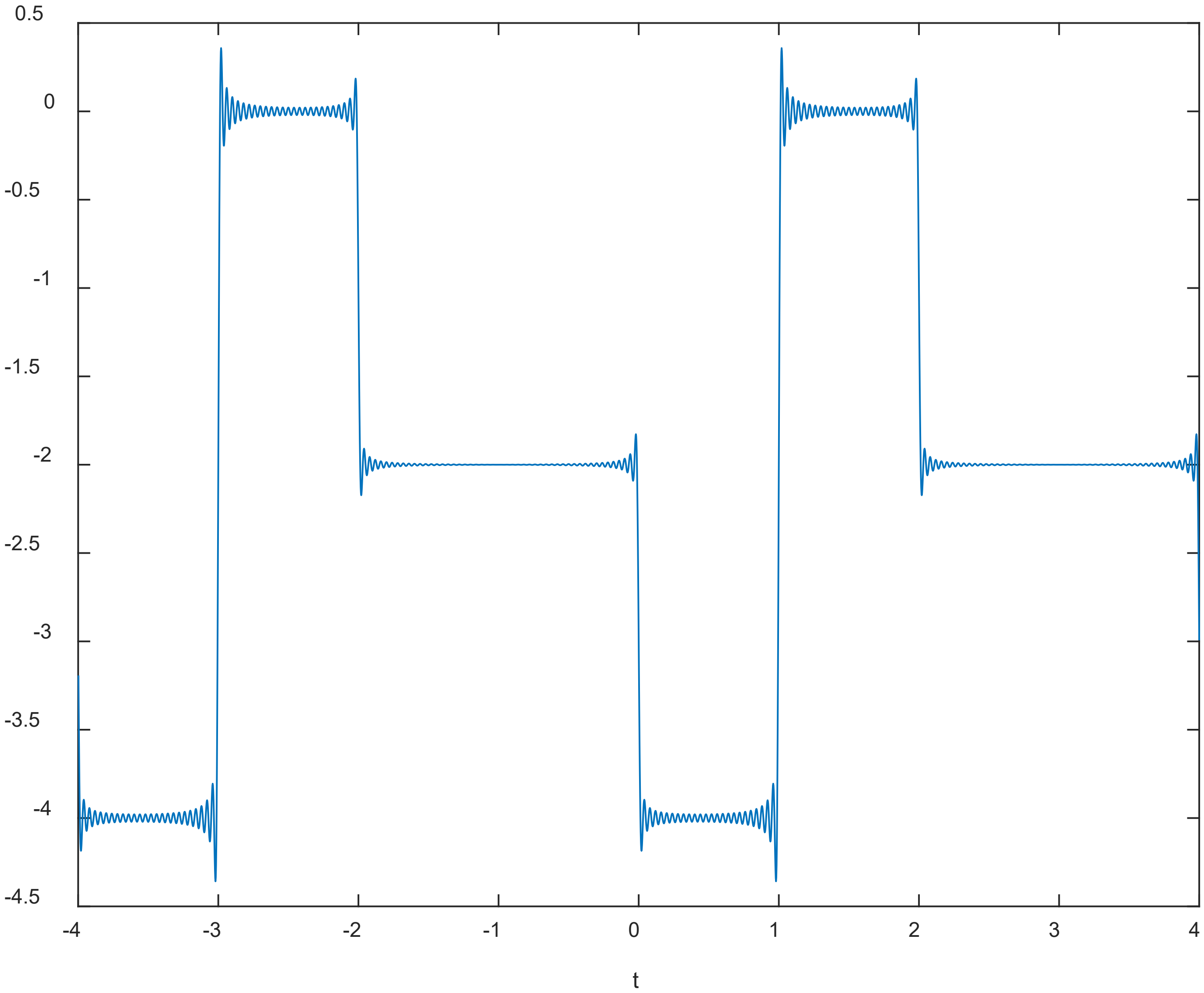
For $k = 0$:

$$a_0 = -\frac{1}{4}\int_0^1 4dt - \frac{1}{4}\int_2^4 2dt = -2$$

For $k \neq 0$:

$$a_k = -\int_0^1 e^{-jk(\pi/2)t}dt - \frac{1}{2}\int_2^4 e^{-jk(\pi/2)t}dt = \frac{-1 + 2e^{-jk(\pi/2)} - e^{-jk\pi}}{jk\pi}$$

```
K=100;
a_p = (-1+2.*exp(-j.*[1:K].*pi/2)-exp(-
j.*[1:K].*pi))./(j.*[1:K].*pi); % +ve a_k
a_n = (-1+2.*exp(-j.*[-K:-1].*pi/2)-exp(-j.*[-K:-
1].*pi)) ./(j.*[-K:-1].*pi);
%-ve a_k
a = [a_n -2 a_p]; %construct vector of a_k
for n=1:4000
 t=(n-2000)/500; %time interval of (-4,4);
      %small sampling interval of 1/500 to approximate
   x(t);
 e = (exp(j.*[-K:K].*pi./2.*t)).'; %construct exponential
vector
 x(n) = a*e;
end
x=real(x); %remove imaginary parts due to precision error
n=1:4000;
t=(n-2000)./500;
plot(t,x)
xlabel('t')
```


0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
-3.5
-4
-4.5
-4
-3
-2
-1
0
1
2
3
4
t

4.

$$
\begin{aligned}
y[n] &= \sum_{m=-\infty}^{\infty} h[m]x[n-m] \\
&= h[-2]x[n+2] + h[-1]x[n+1] + h[0]x[n] \\
&= x[n+2] + x[n+1] - 2x[n]
\end{aligned}
$$

$$
\begin{gathered}
y[-2] = x[0] + x[-1] - 2x[-2] = 1 \\
y[-1] = x[1] + x[0] - 2x[-1] = 1 - 1 = 0 \\
y[0] = x[2] + x[1] - 2x[0] = 3 - 1 - 2 \times 1 = 0 \\
y[1] = x[3] + x[2] - 2x[1] = -3 + 3 - 2 \times -1 = 2 \\
y[2] = x[4] + x[3] - 2x[2] = -3 - 2 \times 3 = -9 \\
y[3] = x[5] + x[4] - 2x[3] = -2 \times -3 = 6
\end{gathered}
$$

$$y[n] = \begin{cases} 1, & n = -2 \\ 2, & n = 1 \\ -9, & n = 2 \\ 6, & n = 3 \\ 0, & \text{otherwise} \end{cases}$$

Note that the length of $y[n]$ is $4+3-1=6$.

$$y[n] = \sum_{k=-\infty}^{n+1} \alpha^{n+1-k} x[k] = \cdots + \alpha^2 x[n-1] + \alpha x[n] + x[n+1], \quad |\alpha| < 1$$

5.(a)

The system is not memoryless.

It is because $y[n]$ does not depend on $x[n]$ only.

5.(b)

The system is invertible. The reason is given as follows.

$$\begin{aligned} y[n] = \sum_{k=-\infty}^{n+1} \alpha^{n+1-k} x[k] &= \left(\cdots + \alpha^2 x[n-1] + \alpha x[n]\right) + x[n+1] \\ &= \alpha y[n-1] + x[n+1] \end{aligned}$$

Hence we have:

$$y[n-1]=\alpha y[n-2]+x[n] \Rightarrow x[n]=y[n-1]-\alpha y[n-2]$$

If we pass $y[n]$ through the above system, $x[n]$ can be obtained.

5.(c)
The system is linear. The reason is given as follows.

Let

$$y_i[n]=\mathcal{T}\{x_i[n]\}, \quad i=1,2,3$$

with

$$x_3[n]=ax_1[n]+bx_2[n]$$

The outputs of $x_1[n]$ and $x_2[n]$ are:

$$y_1[n] = \sum_{k=-\infty}^{n+1} \alpha^{n+1-k} x_1[k] \quad \text{and} \quad y_2[n] = \sum_{k=-\infty}^{n+1} \alpha^{n+1-k} x_2[k]$$

Then the output of $x_3[n] = ax_1[n] + bx_2[n]$ is:

$$\begin{aligned} y_3[n] &= \sum_{k=-\infty}^{n+1} \alpha^{n+1-k} x_3[k] \\ &= \sum_{k=-\infty}^{n+1} \alpha^{n+1-k} \left(ax_1[k] + bx_2[k] \right) \\ &= a \sum_{k=-\infty}^{n+1} \alpha^{n+1-k} x_1[k] + b \sum_{k=-\infty}^{n+1} \alpha^{n+1-k} x_2[k] \\ &= ay_1[n] + by_2[n] \end{aligned}$$

5.(d)

The system is time-invariant. The reason is given as follows.

From the given input-output relationship, $y[n-n_0]$ is:

$$y[n-n_0] = \sum_{k=-\infty}^{n+1-n_0} \alpha^{n+1-n_0-k} x[k]$$

Let $x_1[n] = x[n-n_0]$, its system output is:

$$\begin{aligned} y_1[n] &= \sum_{k=-\infty}^{n+1} \alpha^{n+1-k} x_1[k] = \sum_{k=-\infty}^{n+1} \alpha^{n+1-k} x[k-n_0], \quad l = k - n_0 \\ &= \sum_{l=-\infty}^{n+1-n_0} \alpha^{n+1-n_0-l} x[l] = y[n-n_0] \end{aligned}$$

5.(e)

The system is not causal. It is because the system output depends on future system input.

Alternatively, since this system is LTI. We can check its non-causality from its impulse response, i.e., $h[-1] = 1 \neq 0$.

5.(f)

When $x[n] = \delta[n]$, we have:

$$y[n] = \sum_{k=-\infty}^{n+1} \alpha^{n+1-k}\delta[k] = \cdots + \alpha^2\delta[n-1] + \alpha\delta[n] + \delta[n+1] = \alpha^{n+1}u[n+1]$$

Alternatively, since the system is LTI, we can compute the impulse response $h[n] = \alpha^{n+1}u[n+1]$ to obtain the same result.

It is because when $x[n]=\delta[n]$, $y[n]=h[n]$.

5.(g)
By comparing with the convolution:

$$
\begin{aligned}
y[n]=x[n]\otimes h[n] &= \sum_{m=-\infty}^{\infty} h[m]x[n-m] \\
&= \cdots h[-2]x[n+2]+h[-1]x[n+1]+ \\
&\quad h[0]x[n]+h[1]x[n-1]+h[2]x[n-2]+\cdots
\end{aligned}
$$

We get $h[-2]=h[-3]=\cdots=0$, $h[-1]=1$, $h[0]=\alpha$, $h[1]=\alpha^2,\cdots$, hence the impulse response is:

$$h[n]=\alpha^{n+1}u[n+1]$$

For $x[n]=u[n]$, the output is computed as:

$$\begin{aligned} y[n] &= \sum_{m=-\infty}^{\infty} x[m]h[n-m] = \sum_{m=-\infty}^{\infty} u[m]\alpha^{n-m+1}u[n-m+1] \\ &= \sum_{m=0}^{\infty} \alpha^{n-m+1}u[n-m+1] = \sum_{k=n+1}^{-\infty} \alpha^k u[k], \quad k=n-m+1 \\ &= \sum_{k=-\infty}^{n+1} \alpha^k u[k] \end{aligned}$$

When $n+1<0$ or $n<-1$, we see that $y[n]=0$.

While for $n \geq -1$, the output is:

$$y[n] = \sum_{k=0}^{n+1} \alpha^k = 1 + \alpha + \cdots + \alpha^{n+1} = \frac{1-\alpha^{n+2}}{1-\alpha}$$

Combining the result, we have:

$$y[n] = \frac{1-\alpha^{n+2}}{1-\alpha} u[n+1]$$

6.(a)

Taking the Fourier transform on both sides yields

$$Y(j\Omega)\left(1 - aj\Omega\right) = X(j\Omega)\left(1 - \frac{1}{a}j\Omega\right) \Rightarrow H(j\Omega) = \frac{a - j\Omega}{a(1 - ja\Omega)}$$

6.(b)

$$|H(j\Omega)|^2 = H(j\Omega)H^*(j\Omega) = \frac{a - j\Omega}{a(1 - ja\Omega)} \cdot \frac{a + j\Omega}{a(1 + ja\Omega)} = \frac{a^2 + \Omega^2}{a^2(1 + a^2\Omega^2)}$$

Hence the magnitude response is:

$$|H(j\Omega)| = \sqrt{\frac{a^2 + \Omega^2}{a^2(1 + a^2\Omega^2)}}$$

To find the phase response, we use:

$$H(j\Omega) = \frac{a - j\Omega}{a(1 - ja\Omega)} \cdot \frac{1 + ja\Omega}{1 + ja\Omega} = \frac{a(1 + \Omega^2) + j\Omega(a^2 - 1)}{a(1 + a^2\Omega^2)}$$

Hence the phase response is:

$$\angle H(j\Omega) = \tan^{-1}\left(\frac{\Omega(a^2 - 1)}{a(1 + \Omega^2)}\right)$$