Solutions to EE3210 Assignment 5

Problem 1:

(a) Applying the Fourier transform to both sides of the differential equation, and using the properties of differentiation in time and linearity, we have

$$(j\omega)^2 Y(\omega) + 6j\omega Y(\omega) + 8Y(\omega) = 3j\omega X(\omega) + 9X(\omega). \tag{1}$$

Rearranging (1), we obtain the frequency response of the system as

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{3j\omega + 9}{(j\omega)^2 + 6j\omega + 8} = \frac{3j\omega + 9}{(j\omega + 2)(j\omega + 4)}. \tag{2}$$

(b) Using the properties of complex conjugation, we obtain

$$\begin{aligned} |H[\omega]|^2 = H[\omega]H^*[\omega] &= \frac{(3j\omega + 9)(3j\omega + 9)^*}{[(j\omega + 2)(j\omega + 2)^*][(j\omega + 4)(j\omega + 4)^*]} \\ &= \frac{81 + 9\omega^2}{(4 + \omega^2)(16 + \omega^2)}. \end{aligned}$$

Therefore, $|H[\omega]| = 3\sqrt{\dfrac{9 + \omega^2}{(4 + \omega^2)(16 + \omega^2)}}$.

(c) Making the substitution of v for $j\omega$ in the right-hand side of (2), we obtain the rational function

$$G(v) = \frac{3v + 9}{(v + 2)(v + 4)}.$$

The partial-fraction expansion for this function is

$$G(v) = \frac{A}{v + 2} + \frac{B}{v + 4}$$

where

$$A = (v + 2)G(v)|_{v=-2} = \frac{3}{2} \quad \text{and} \quad B = (v + 4)G(v)|_{v=-4} = \frac{3}{2}.$$

Therefore,

$$H(\omega) = \frac{\frac{3}{2}}{j\omega + 2} + \frac{\frac{3}{2}}{j\omega + 4}.$$

Taking inverse Fourier transforms by inspection, we get

$$h(t) = \frac{3}{2}\left(e^{-2t} + e^{-4t}\right)u(t).$$

(d) Taking the Fourier transform of $x(t)$, we obtain

$$X(\omega) = \frac{1}{j\omega + 1} + \frac{1}{j\omega + 3} = \frac{2(j\omega + 2)}{(j\omega + 1)(j\omega + 3)}.$$

Then, we have

$$Y(\omega) = H(\omega)X(\omega) = \frac{6}{(j\omega + 1)(j\omega + 4)}.$$

Making the substitution of v for $j\omega$, we obtain the rational function

$$G(v) = \frac{6}{(v+1)(v+4)}.$$

The partial-fraction expansion for this function is

$$G(v) = \frac{A}{v+1} + \frac{B}{v+4}$$

where

$$A = (v+1)G(v)|_{v=-1} = 2 \quad \text{and} \quad B = (v+4)G(v)|_{v=-4} = -2.$$

Therefore,

$$Y(\omega) = \frac{2}{j\omega + 1} - \frac{2}{j\omega + 4}.$$

Taking inverse Fourier transforms by inspection, we get

$$y(t) = 2\left(e^{-t} - e^{-4t}\right)u(t).$$

Problem 2:

(a) Here, we provide two ways in evaluating the Fourier transform of $x[n]$.

1. Applying the Fourier transform directly to $x[n]$, we derive

$$X[\Omega] = \sum_{n=-\infty}^{+\infty} \left(-\frac{1}{2}\right)^n u[n-4]e^{-j\Omega n} = \sum_{l=-\infty}^{+\infty} \left(-\frac{1}{2}e^{-j\Omega}\right)^{l+4} u[l] = \frac{1}{16}e^{-j4\Omega}\sum_{l=0}^{+\infty}\left(-\frac{1}{2}e^{-j\Omega}\right)^l$$
$$= \frac{\frac{1}{16}e^{-j4\Omega}}{1+\frac{1}{2}e^{-j\Omega}}$$

2. We know, e.g., from the table of basic discrete-time Fourier transform pairs, that

$$a^n u[n],\ |a|<1 \ \leftrightarrow\ \frac{1}{1-ae^{-j\Omega}}. \tag{3}$$

Thus, letting $a=-\frac{1}{2}$, we have

$$\left(-\frac{1}{2}\right)^n u[n] \ \leftrightarrow\ \frac{1}{1+\frac{1}{2}e^{-j\Omega}}.$$

Using the time shift property, we have

$$\left(-\frac{1}{2}\right)^{n-4} u[n-4] \ \leftrightarrow\ \frac{e^{-j4\Omega}}{1+\frac{1}{2}e^{-j\Omega}}.$$

Therefore,

$$\left(-\frac{1}{2}\right)^{n} u[n-4] \ \leftrightarrow\ \frac{\frac{1}{16}e^{-j4\Omega}}{1+\frac{1}{2}e^{-j\Omega}}.$$

(b) Here, we provide two ways in deriving the frequency response of the system $H[\Omega]$.

1. Applying the Fourier transform directly to $h[n]$, we obtain

$$H[\Omega] = \sum_{n=-\infty}^{+\infty} 4^n u[2-n]e^{-j\Omega n} = \sum_{l=-\infty}^{+\infty} \left(4e^{-j\Omega}\right)^{2-l} u[l] = 16e^{-j2\Omega}\sum_{l=0}^{+\infty}\left(\frac{1}{4}e^{j\Omega}\right)^l$$
$$= \frac{16e^{-j2\Omega}}{1-\frac{1}{4}e^{j\Omega}}.$$

2. From (3), letting $a=4^{-1}$, we have

$$4^{-n}u[n] \ \leftrightarrow\ \frac{1}{1-\frac{1}{4}e^{-j\Omega}}.$$

Using the time reversal property, we have

$$4^n u[-n] \leftrightarrow \frac{1}{1-\frac{1}{4}e^{j\Omega}}.$$

Then, using the time shift property, we have

$$4^{n-2} u[-(n-2)] \leftrightarrow \frac{e^{-j2\Omega}}{1-\frac{1}{4}e^{j\Omega}}.$$

Therefore,

$$4^n u[-(n-2)] \leftrightarrow \frac{16e^{-j2\Omega}}{1-\frac{1}{4}e^{j\Omega}}.$$

(c) Using the properties of complex conjugation, we obtain

$$|H[\Omega]|^2 = H[\Omega]H^*[\Omega] = \left(\frac{16e^{-j2\Omega}}{1-\frac{1}{4}e^{j\Omega}}\right)\left(\frac{16e^{-j2\Omega}}{1-\frac{1}{4}e^{j\Omega}}\right)^* = \frac{\left(16e^{-j2\Omega}\right)\left(16e^{-j2\Omega}\right)^*}{\left(1-\frac{1}{4}e^{j\Omega}\right)\left(1-\frac{1}{4}e^{j\Omega}\right)^*}$$

$$= \frac{\left(16e^{-j2\Omega}\right)\left(16e^{j2\Omega}\right)}{\left(1-\frac{1}{4}e^{j\Omega}\right)\left(1-\frac{1}{4}e^{-j\Omega}\right)} = \frac{16^2}{\frac{17}{16}-\frac{1}{4}\left(e^{j\Omega}+e^{-j\Omega}\right)} = \frac{16^3}{17-8\cos\Omega}.$$

Therefore, $|H[\Omega]| = \dfrac{64}{\sqrt{17-8\cos\Omega}}$.

(d) We have

$$Y[\Omega] = X[\Omega]H[\Omega] = e^{-j6\Omega}\frac{1}{(1+\frac{1}{2}e^{-j\Omega})(1-\frac{1}{4}e^{j\Omega})} \tag{4}$$

Rearranging (4), we have

$$e^{j6\Omega}Y[\Omega] = \frac{1}{(1+\frac{1}{2}e^{-j\Omega})(1-\frac{1}{4}e^{j\Omega})}. \tag{5}$$

We know by the time shift property that

$$e^{6j\Omega}Y[\Omega] = \mathcal{F}\left\{y[n+6]\right\}. \tag{6}$$

Making the substitution of v for $e^{-j\Omega}$ in the right-hand side of (5), we obtain the rational function

$$G(v) = \frac{1}{(1+\frac{1}{2}v)(1-\frac{1}{4}v^{-1})} = \frac{-4v}{(1+\frac{1}{2}v)(1-4v)} = \frac{2v}{(v+2)(v-\frac{1}{4})}.$$

The partial-fraction expansion for this function is

$$G(v) = \frac{A}{v+2} + \frac{B}{v - \frac{1}{4}}$$

where

$$A = (v+2)G(v)|_{v=-2} = \frac{16}{9} \quad \text{and} \quad B = \left(v - \frac{1}{4}\right) G(v)|_{v=\frac{1}{4}} = \frac{2}{9}.$$

Therefore,

$$G(v) = \frac{\frac{16}{9}}{v+2} + \frac{\frac{2}{9}}{v - \frac{1}{4}} = \frac{\frac{8}{9}}{1 + \frac{1}{2}v} - \frac{\frac{8}{9}}{1 - 4v}.$$

and

$$\frac{1}{(1 + \frac{1}{2}e^{-j\Omega})(1 - \frac{1}{4}e^{j\Omega})} = \frac{\frac{8}{9}}{1 + \frac{1}{2}e^{-j\Omega}} - \frac{\frac{8}{9}}{1 - 4e^{-j\Omega}}. \tag{7}$$

We know that

$$\mathcal{F}^{-1}\left\{\frac{1}{1 + \frac{1}{2}e^{-j\Omega}}\right\} = \left(-\frac{1}{2}\right)^n u[n]. \tag{8}$$

To obtain $\mathcal{F}^{-1}\left\{\frac{1}{1 - 4e^{-j\Omega}}\right\}$, we rewrite

$$\frac{1}{1 - 4e^{-j\Omega}} = \frac{-\frac{1}{4}e^{j\Omega}}{1 - \frac{1}{4}e^{j\Omega}}$$

and note from Part (b) that

$$\mathcal{F}\left\{4^n u[-n]\right\} = \frac{1}{1 - \frac{1}{4}e^{j\Omega}}.$$

Then, using the time shift property, we obtain

$$\mathcal{F}^{-1}\left\{\frac{1}{1 - 4e^{-j\Omega}}\right\} = -4^n u[-n-1]. \tag{9}$$

Thus, from (5), (6), (7), (8) and (9), we derive

$$\begin{aligned} & y[n+6] = \left(\frac{8}{9}\right)\left(-\frac{1}{2}\right)^n u[n] + \left(\frac{8}{9}\right) 4^n u[-n-1] \\ \Rightarrow \quad & y[l] = \left(\frac{8}{9}\right)\left(-\frac{1}{2}\right)^{l-6} u[l-6] + \left(\frac{8}{9}\right) 4^{l-6} u[5-l] \\ \Rightarrow \quad & y[n] = \left(\frac{8^3}{9}\right)\left(-\frac{1}{2}\right)^n u[n-6] + \left(\frac{8}{9}\right)\left(\frac{1}{4}\right)^6 4^n u[5-n]. \end{aligned}$$

Note that some students may obtain

$$y[n] = \left(\frac{8^3}{9}\right)\left(-\frac{1}{2}\right)^n u[n-7] + \left(\frac{8}{9}\right)\left(-\frac{1}{8}\right)^4 4^n u[6-n]$$

which, according to Tutorial 5 Problem 2, is also a valid solution.