Solutions to EE3210 Assignment 4

Problem 1:

(a) From this block diagram, utilizing the intermediate signal $w[n]$, we have

$$w[n] = x[n-1] + bw[n-1] - ay[n-2] \tag{1}$$

and

$$y[n] = aw[n] + by[n-1]. \tag{2}$$

From (2), we obtain

$$w[n] = \frac{1}{a}y[n] - \frac{b}{a}y[n-1] \tag{3}$$

and hence

$$w[n-1] = \frac{1}{a}y[n-1] - \frac{b}{a}y[n-2]. \tag{4}$$

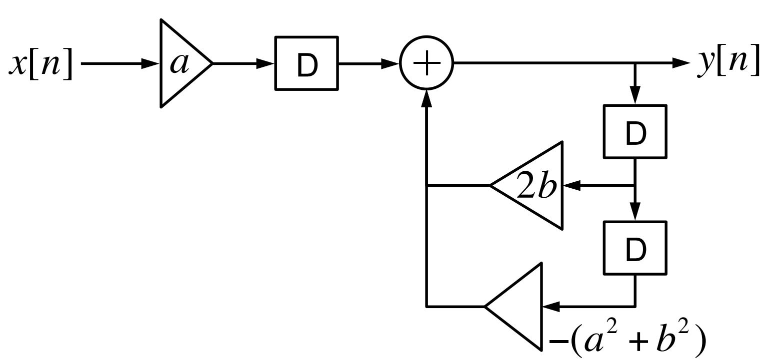
Then, substituting $w[n]$ and $w[n-1]$ in (1) with (3) and (4), respectively, we have

$$\frac{1}{a}y[n] - \frac{b}{a}y[n-1] = x[n-1] + \frac{b}{a}y[n-1] - \frac{b^2}{a}y[n-2] - ay[n-2]. \tag{5}$$

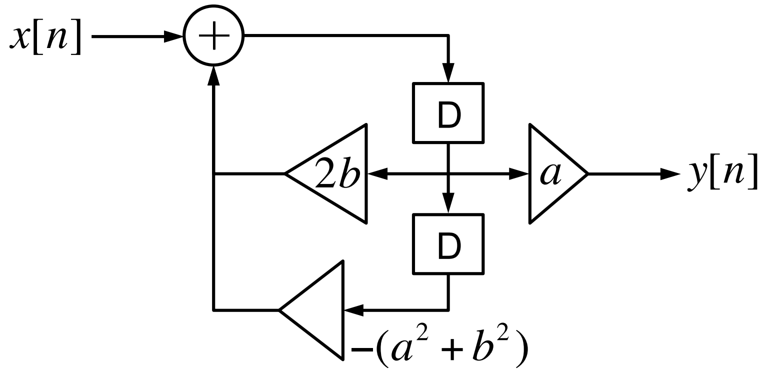
Rearranging (5), we derive the linear constant-coefficient difference equation that describes the relationship between the input $x[n]$ and the output $y[n]$ of the system as

$$y[n] = ax[n-1] + 2by[n-1] - (a^2+b^2)y[n-2].$$

(b) The block diagram representation of the system in direct form I is



(c) The block diagram representation of the system in direct form II is



Problem 2: This signal is periodic with a fundamental period $T = 3$. To determine the Fourier series coefficients a_k, we use the analysis formula of the continuous-time Fourier series, and choose the limits of the integration to include the interval $0 < t < 2$. Within this interval,

$$x(t) = \begin{cases} 2, & 0 < t < 1 \\ 1, & 1 < t < 2. \end{cases}$$

Thus, it follows that:

- For $k = 0$,

$$a_0 = \frac{1}{3}\int_0^1 2dt + \frac{1}{3}\int_1^2 dt = 1.$$

- For $k \neq 0$,

$$\begin{aligned} a_k &= \frac{2}{3}\int_0^1 e^{-jk(2\pi/3)t}dt + \frac{1}{3}\int_1^2 e^{-jk(2\pi/3)t}dt \\ &= \frac{2 - e^{-jk2\pi/3} - e^{-jk4\pi/3}}{jk2\pi}. \end{aligned}$$

  – Note: In this case, we have

$$\lim_{k \to 0} \frac{2 - e^{-jk2\pi/3} - e^{-jk4\pi/3}}{jk2\pi} = 1$$

  following from the l'Hôpital's rule.

Problem 3:

(a) The time shift property of the continuous-time Fourier series indicates that, if $x(t) \leftrightarrow a_k$, the Fourier series coefficients b_k of $x(t-t_0)$ can be expressed as

$$b_k = \left[e^{-jk(2\pi/T)t_0}\right] a_k.$$

Similarly, the Fourier series coefficients c_k of $x(t+t_0)$ can be expressed as

$$c_k = \left[e^{jk(2\pi/T)t_0}\right] a_k.$$

Finally, using the linearity property of the continuous-time Fourier series, the Fourier series coefficients d_k of $x(t-t_0)+x(t+t_0)$ can be obtained as

$$d_k = b_k + c_k = \left[e^{-jk(2\pi/T)t_0} + e^{jk(2\pi/T)t_0}\right] a_k = 2\cos\left(\frac{2\pi k t_0}{T}\right) a_k.$$

(b) Note that

$$\mathcal{E}\{x(t)\} = \frac{1}{2}\left[x(t) + x(-t)\right].$$

The time reversal property of the continuous-time Fourier series indicates that, if $x(t) \leftrightarrow a_k$, the Fourier series coefficients b_k of $x(-t)$ can be expressed as

$$b_k = a_{-k}.$$

Then, using the linearity property of the continuous-time Fourier series, the Fourier series coefficients c_k of $\mathcal{E}\{x(t)\}$ can be obtained as

$$c_k = \frac{a_k + b_k}{2} = \frac{a_k + a_{-k}}{2}.$$

(c) The signal $x(3t-1)$ can be obtained from $x(t)$ in two alternative ways:

- Time scaling first followed by time shift, i.e., $x(t) \Rightarrow x(3t) \Rightarrow x[3(t-1/3)]$.

  In this way, the time scaling property of the continuous-time Fourier series indicates that, given that $x(t)$ is periodic with period T and that $x(t) \leftrightarrow a_k$, the signal $x(3t)$ is periodic with period $T/3$, and the Fourier series coefficients b_k of $x(3t)$ are the same as a_k, i.e.,

  $$b_k = a_k.$$

  Then, the time shift property of the continuous-time Fourier series indicates that, given that $x(3t)$ is periodic with period $T/3$ and that $x(3t) \leftrightarrow b_k$, the Fourier series coefficients c_k of $x[3(t-1/3)]$ can be expressed as

  $$c_k = \left[e^{-jk2\pi/(T/3)/3}\right] b_k = \left[e^{-jk(2\pi/T)}\right] a_k. \tag{6}$$

– Time shift first followed by time scaling, i.e., $x(t) \Rightarrow x(t-1) \Rightarrow x(3t-1)$.

In this way, the time shift property of the continuous-time Fourier series indicates that, given that $x(t)$ is periodic with period T and that $x(t) \leftrightarrow a_k$, the Fourier series coefficients b_k of $x(t-1)$ can be expressed as

$$b_k = \left[e^{-jk(2\pi/T)}\right] a_k.$$

Then, the time scaling property of the continuous-time Fourier series indicates that, given that $x(t-1)$ is periodic with period T and that $x(t-1) \leftrightarrow b_k$, the signal $x(3t-1)$ is periodic with period $T/3$, and the Fourier series coefficients c_k of $x(3t-1)$ are the same as b_k, i.e.,

$$c_k = b_k = \left[e^{-jk(2\pi/T)}\right] a_k.$$

Problem 4: Let $y[n] = x[mn]$.

(a) Given that $x[n]$ is a discrete-time periodic signal with fundamental period N, for an arbitrary positive integer m, we have

$$x[mn + mN] = x[mn].$$

Thus, we have

$$y[n+N] = x[m(n+N)] = x[mn+mN] = x[mn] = y[n].$$

Therefore, by definition, $y[n]$ is a periodic signal with period N.

(b) Determining an expression for the fundamental period of $y[n]$ in this case is complicated and problem-specific.

In general, we know by definition that the fundamental period of $y[n]$ is the smallest positive integer N' such that $y[n+N'] = y[n]$ holds for all values of n. Since

$$y[n+N'] = x[m(n+N')] = x[mn+mN'],$$

this requires that

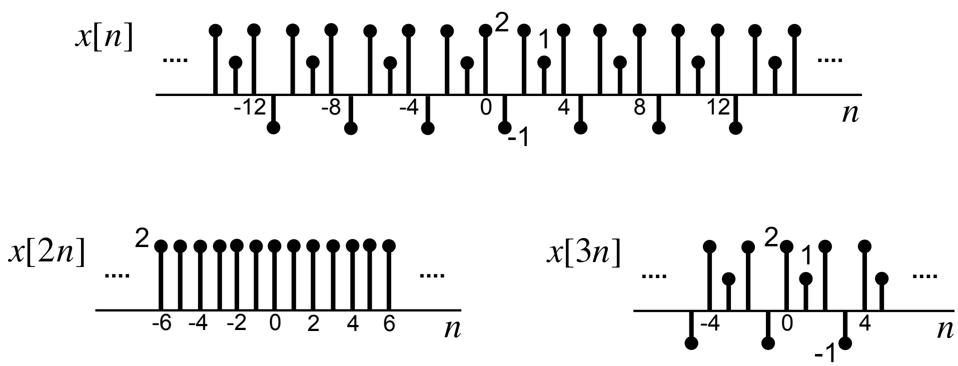
$$mN' = qN$$

where $q \geq 1$ is the smallest positive integer such that $N' = qN/m$ is an integer. Equivalently, N' can be obtained as

$$N' = \frac{\mathrm{lcm}(N,m)}{m} = \frac{N}{\mathrm{gcd}(N,m)} \tag{7}$$

where $\mathrm{lcm}(N, m)$ is the least common multiple of the two integers N and m, $\gcd(N, m)$ is the greatest common divisor of the two integers N and m.

Although (7) is correct in many cases, there exist anomalies such that the actual fundamental period of $x[mn]$ can be smaller than N' derived from (7). This is demonstrated in the example shown in the figure below.



In this example, the fundamental period of $x[n]$ is $N = 4$. The fundamental period of $x[3n]$ (i.e., $m = 3$) is also 4, the same as that obtained from (7). However, the fundamental period of $x[2n]$ (i.e., $m = 2$) is 1, smaller than that obtained from (7). Thus, we can conclude that (7) provides an upper bound on the fundamental period of $x[mn]$.