Unit 6

Modulo

4-Digit Combination Lock

Any two kids together can open the lock, but any single kid obtains no information (meaning that he/she needs to try all 10,000 combinations).

How can it be done?

Outline of Unit 6

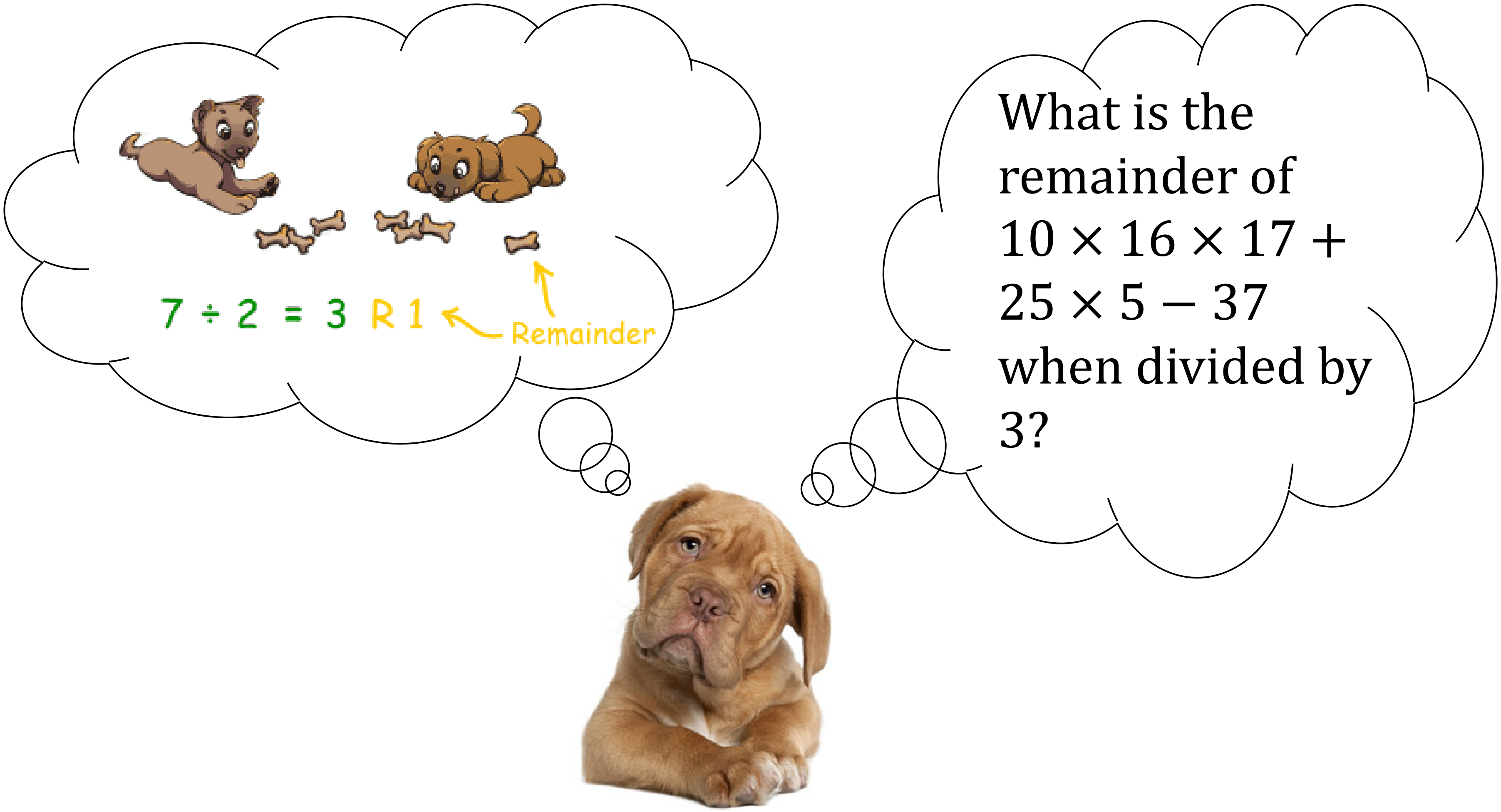
- 6.1 Modular Arithmetic
- 6.2 Diophantine Equations
- 6.3 Modular Division
- 6.4 Modular Exponentiation
- 6.5 Secret Sharing
- 6.6 SageMath: a free math software

Modular arithmetic plays an important role in cryptography.

Unit 6.1

Modular Arithmetic

How Much will be Left?



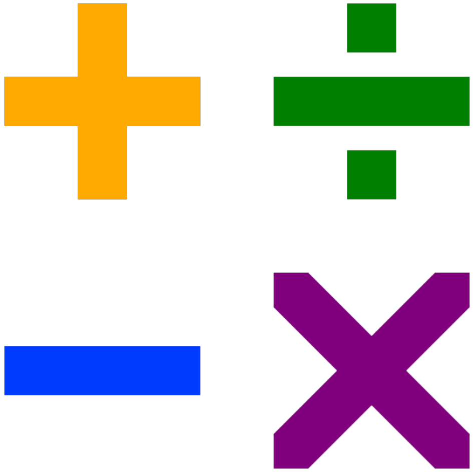
What is the remainder of $10 \times 16 \times 17 + 25 \times 5 - 37$ when divided by 3?

Modulo Arithmetic

- **Definition**: We say that two numbers a and b are congruent modulo n if they have the same remainder when divided by n. We write

$$a \equiv b \pmod{n}.$$

We discussed before that it is an equivalence relation. Now we study it from another perspective.



Arithmetic is all about addition, subtraction, multiplication and division.

Modular Arithmetic

- Suppose $a, b, c, d \in \boldsymbol{Z},\ n > 1$,

$$a \equiv c \pmod{n} \text{ and } b \equiv d \pmod{n}.$$

- Are the following statement true?

a) $a + b \equiv c + d \pmod{n}$ (Addition)

b) $a - b \equiv c - d \pmod{n}$ (Subtraction)

c) $ab \equiv cd \pmod{n}$ (Multiplication)

Nice Properties

- Congruence is preserved under addition, subtraction, and multiplication.

- The dog is now ready to solve its problem:
  - What is the remainder of $10 \times 16 \times 17 + 25 \times 5 - 37$ when divided by 3?

$$\begin{aligned} &10 \times 16 \times 17 + 25 \times 5 - 38 \\ &\qquad \equiv 1 \times 1 \times 2 + 1 \times 2 - 2 \equiv 2 \pmod{3} \end{aligned}$$

Modular Division?

- Suppose we have non-zero number a and another number b.

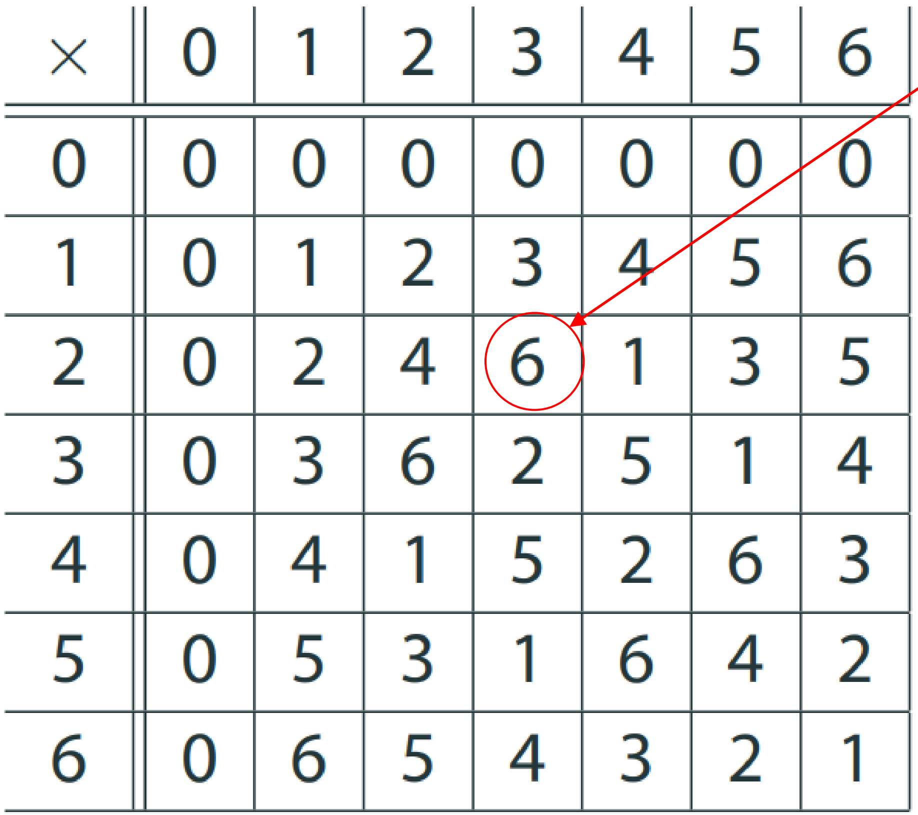
- Is there a number x such that $ax \equiv b \pmod{7}$?

- If so, x can be regarded as b divided by a modulo 7.

$ax \equiv b \pmod 7$?

Multiplication Table

x



×	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

a

If

$2x \equiv 6 \pmod 7$,

then

$x \equiv 3 \pmod 7$.

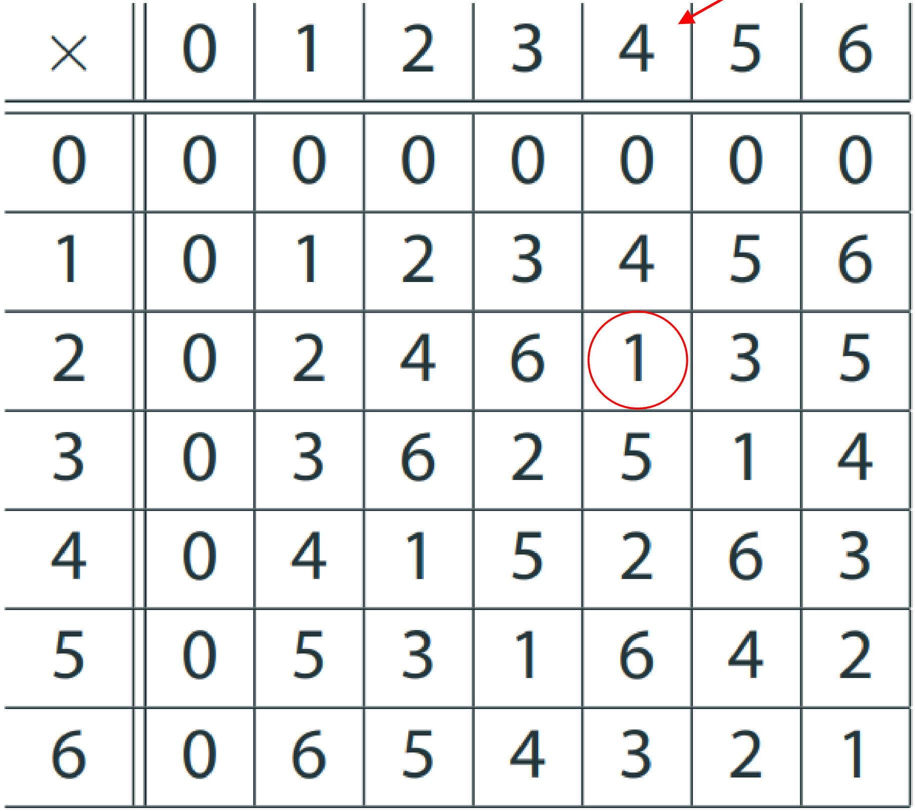
$ax \equiv b \pmod 7$ can always be solved.

Each non-zero row contains all possible remainders!

Another View

x

a



×	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

$$2 \times 4 \equiv 1 \pmod{7}$$

4 is said to be the multiplicative inverse of 2.

If

$$2x \equiv 6 \pmod{7},$$

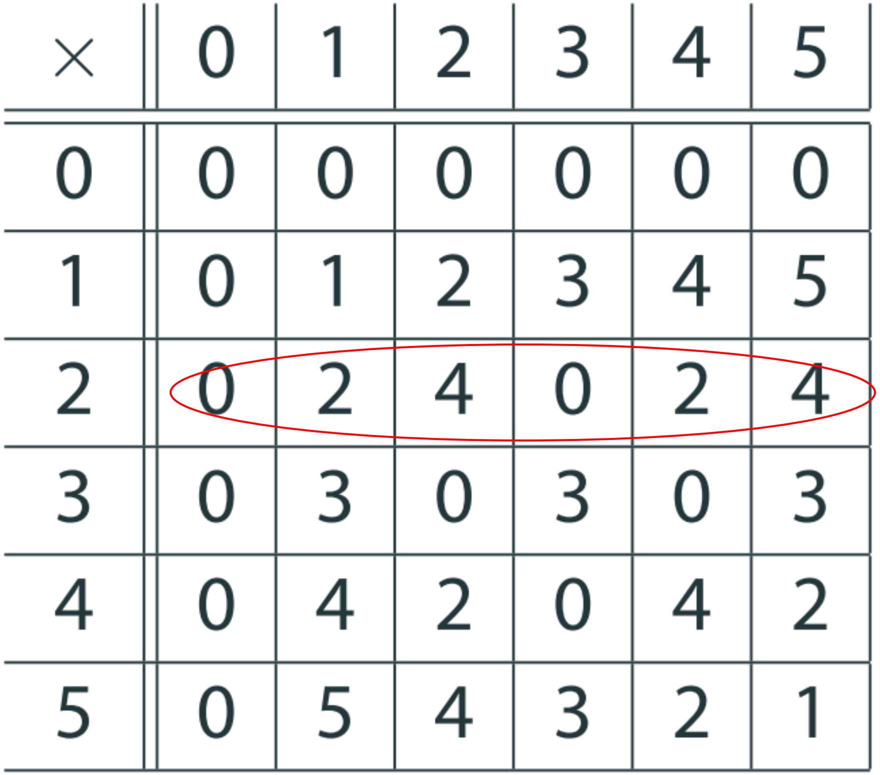
then

$$\begin{aligned} 4(2x) &\equiv 4 \times 6 \pmod{7} \\ x &\equiv 24 \pmod{7} \\ x &\equiv 3 \pmod{7} \end{aligned}$$

Modulo 7

- Given $a \neq 0$ and b, consider $ax \equiv b \pmod{7}$.
- We have seen that x always exists.
  - Equivalently, the multiplicative inverse of a always exists.
- x plays the role of modulo division b/a.
- Everything good? What if modulo 6?

$ax \equiv b \pmod 6$?



×	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

Consider $2x \equiv 5 \pmod 6$.

What is the value of x?

No solution!

The multiplicative inverse of 2 does not exists.

The story has not ended!

Unit 6.2

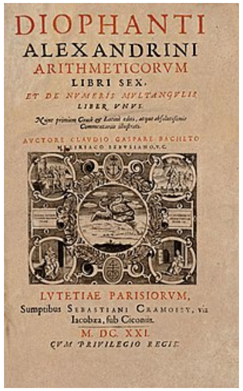
Diophantine Equations

Diophantine Equations

- ❑ Diophantine equation is a polynomial equation whose solutions are restricted to be **integers**.
- ❑ In this section, we consider **linear** Diophantine equation in **two unknowns**:

$$ax + by = c.$$

  - ❍ It is useful when we consider modular division.



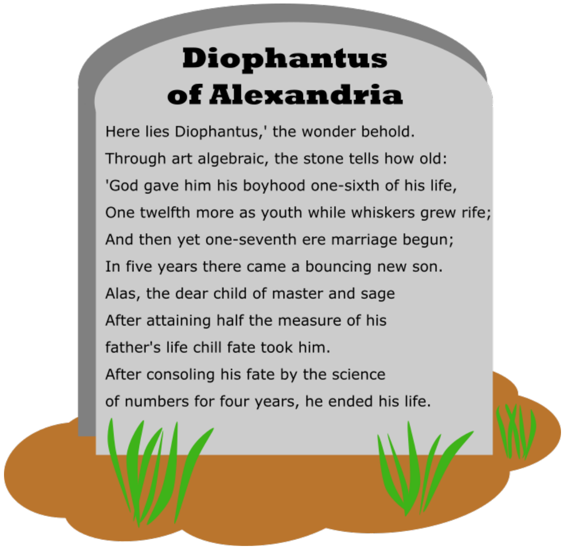
DIOPHANTI
ALEXANDRINI
ARITHMETICORVM
LIBRI SEX.
ET DE NVMERIS MVLTANGVLIS
LIBER VNVS.
Nunc primùm Græcè & Latinè editi, atque absolutissimis Commentariis illustrati.
AVCTORE CLAVDIO GASPARE BACHETO
MEZIRIACO SEBVSIANO, V.C.

LVTETIAE PARISIORVM,
Sumptibus SEBASTIANI CRAMOISY, via
Iacobæa, sub Ciconiis.
M. DC. XXI.
CVM PRIVILEGIO REGIS.

Diophantus of Alexandria, the father of algebra. (200-284 AD) He has written a series of books called the *Arithmetica*.

Diophantus Puzzle (optional, just for fun)

- ❑ Diophantus passed one sixth of his life in childhood, one twelfth in youth, and one seventh more as a bachelor;
- ❑ Five years after his marriage a son was born who died four years before his father at half his final age.
- ❑ How old did Diophantus die?



Diophantus
of Alexandria

Here lies Diophantus,' the wonder behold.
Through art algebraic, the stone tells how old:
'God gave him his boyhood one-sixth of his life,
One twelfth more as youth while whiskers grew rife;
And then yet one-seventh ere marriage begun;
In five years there came a bouncing new son.
Alas, the dear child of master and sage
After attaining half the measure of his
father's life chill fate took him.
After consoling his fate by the science
of numbers for four years, he ended his life.

The puzzle is an epitaph of Diophantus.

Example

- ❑ You want to buy a book that costs $230.
- ❑ You only have $20 notes.
- ❑ The bookseller only has $50 notes.
- ❑ Can you pay the exact price of the book?

$x =$ $y +$

Can you solve the equation?

More Examples

- $187x + 55y = 121,$ where x, y are integers.
  - $x = 3, y = -8$
  - $x = -2, y = 9$
  - Infinitely many solutions!
- $187x + 55y = 45,$ where x, y are integers.
  - No solutions!
- When does a Diophantine equation have solutions?

Existence of Solutions

- **Theorem**: Given integers a, b, c (at least one of a and b is nonzero), the Diophantine equation

$$ax + by = c$$

has a solution if and only if

$$\gcd(a, b) \mid c.$$

Proof (optional, for self study)

Let $d = \gcd(a, b)$.

Solution exists ⇒ $\boldsymbol{d|c}$

Assume there exists integers x, y such that

$$c = ax + by.$$

Write $a = dp$ and $b = dq$,

(where p and q are integers).

Then $c = d(px + qy)$.

Hence, $d|c$.

Solution exists ⇐ $\boldsymbol{d|c}$

Assume $d|c$.

Write $c = td$.

Bézout's identity (in Unit 5):

$$ax' + by' = d$$

Multiply both sides by t;

$$a(tx') + b(ty') = c$$

Hence, (tx', ty') is a solution.

Q.E.D.

A Particular Solution

- $391x + 299y = -69$
  - Extended Euclidean Algorithm gives
    $$391(-3) + 299(4) = \gcd(391, 299) = 23$$
    Multiplying the whole equation by –3,
    $$391(9) + 299(-12) = -69$$
  - Hence, $x = 9, y = -12$ is a solution.
  - But $x = -4, y = 5$ is also a solution.
  - How do we find all solutions?

Extended Euclidean algorithm can find a particular solution.

Example: From one solution to many…

Q: Solve $2x + 3y = 7$.

By direct observation, $x = 2, y = 1$ is a solution.

Consider $x = 2 + 3t, y = 1 - 2t$, where $t \in \mathbb{Z}$.

Substituting them into the given equation,

$$2(2 + 3t) + 3(1 - 2t) = 7.$$

Since t can be any integer, a particular solution gives rise to an infinite number of solutions.

General Solution (proof omitted)

- **Theorem**: If (x_0, y_0) is any particular solution to the Diophantine equation

$$ax + by = c$$

then all the solutions have the form

$$a\left(x_0 + t\frac{b}{d}\right) + b\left(y_0 - t\frac{a}{d}\right) = c,$$

where $d = \gcd(a, b)$ and t is an arbitrary integer.

Example (revisited)

x = y + 230

- The equation is $20x - 50y = 230$.
- Dividing it by $\gcd(20,50) = 10$, we obtain $2x - 5y = 23$.
- We have seen that $x_0 = 14, y_0 = 1$ is a particular solution.
- Therefore, $2(14 - 5t) - 5(1 - 2t) = 23$.
  - This result can also be obtained directly by the formula in the previous slide.

- The equation is $20x - 50y = 230$.
- Dividing it by $\gcd(20,50) = 10$, we obtain

$$2x - 5y = 23.$$

- We have seen that $x_0 = 14, y_0 = 1$ is a particular solution.
- Therefore, $2({\color{red}14 - 5t}) - 5({\color{red}1 - 2t}) = 23$.
  - This result can also be obtained directly by the formula in the previous slide.

Unit 6.3

Modular Division

Multiplicative Inverse

- a^{-1} is said to be a multiplicative inverse of a (mod n) if

$$a\,a^{-1} \equiv 1 \pmod{n}.$$

- If a has an inverse, then we can "divide by a".

Divide by a $\triangleq$ Multiply by a^{-1}

Uniqueness of Inverses

Lemma: If a has a multiplicative inverse modulo n, then it is unique modulo n.

Proof:

Suppose x and y are both inverses of a.

$$x \equiv x(ay) \equiv (xa)y \equiv y \pmod{n}.$$

$\because ay = 1$ $\quad\quad$ $\because xa = 1$

Q.E.D.

Existence of Inverses

Theorem: a has a multiplicative inverse modulo n iff

$$\boxed{\gcd(a, n) = 1.}$$

i.e., a and n are co-prime

Proof:

- $ax \equiv 1 \pmod{n}$ iff $ax + kn = 1$ for some integer k.
- For fixed a and n, this Diophantine equation has an integer solution for x iff $\gcd(a, n) \mid 1$.

Q.E.D.

Modular Division

- If $\gcd(a, n) = 1$, then we can perform "division by a modulo n".
  - i.e., multiply by a^{-1}.

- How to find a^{-1}?

  i. Use extended Euclidean algorithm to find s and t such that $as + nt = 1$.
     - i.e., $as \equiv 1 \pmod{n}$.

  ii. Hence, $a^{-1} = s$.

Example

- Find the value of 3^{-1} mod 11.

- Solution:

We want to solve $3s + 11t = 1$.

11	3		
1	0	11	(a)
0	1	3	(b)
1	–3	2	(c)=(a)–3(b)
–1	4	1	(d)=(b)–1(c)

Hence, $3^{-1} \equiv 4 \pmod{11}$.

Modulo p

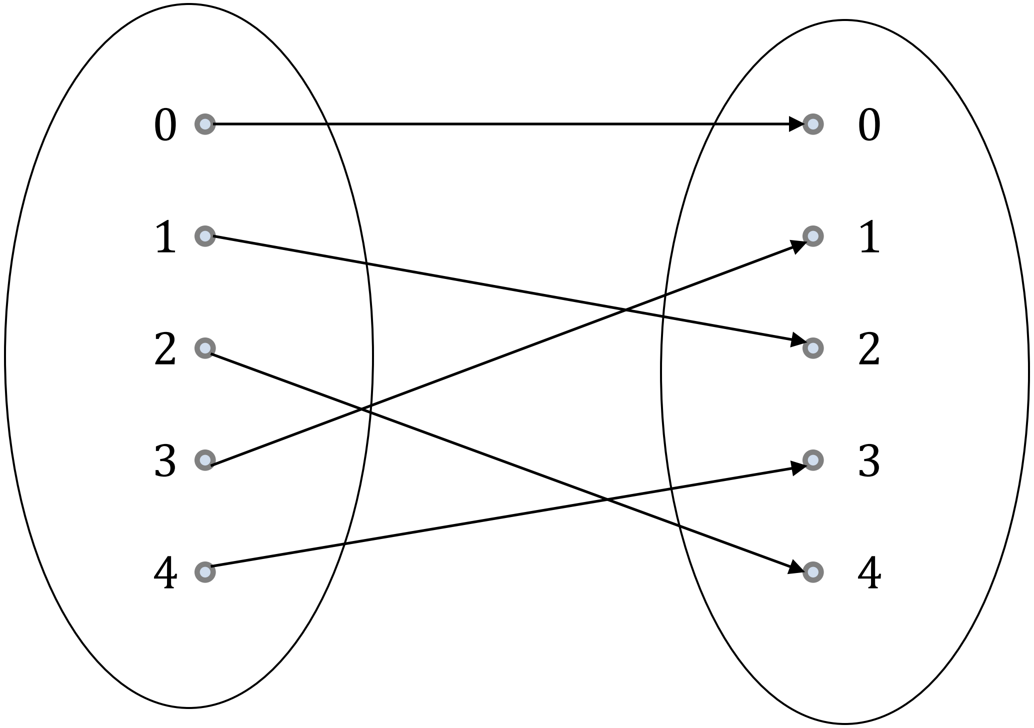
- If p is a prime number and $a \neq 0$, then $\gcd(a, p) = 1$.
- Hence, a has multiplicative inverse modulo p.
- Division $(\mathrm{mod}\ p)$ can be performed for all non-zero elements.

- In fact, multiplication $(\mathrm{mod}\ p)$ is a bijection.
  - Division is the inverse function.

Multiplication (mod p) is a Bijection

- Example: $f(x) = 2x \pmod 5$

Five possible remainders: 0, 1, 2, 3, 4



Bijection!

Division is just the inverse function.

Unit 6.4

Modular Exponentiation

Modular Exponentiation

- How to compute m^e mod n?

- Straightforward?
  - First, compute m^e.
  - Next, divide it by n and obtain the remainder.

- How about 17^{29} mod 35?
  - For cryptography, m, e, and n may have 1000 digits.
  - A fast method is needed.

Square-and-Multiply Method

First, note the following:

- $17 \bmod 35 = 17$
- $17^2 \bmod 35 = 289 \bmod 35$
$= 9$
- $17^4 \bmod 35 = (17^2)^2 \bmod 35$
$= 9^2 \bmod 35$
$= 11$
- $17^8 \bmod 35 = 11^2 \bmod 35$
$= 16$
- $17^{16} \bmod 35 = 16^2 \bmod 35$
$= 11$

Second, do the calculation:

- $17^{29} \bmod 35$
$= 17^{16}\, 17^8\, 17^4\, 17 \bmod 35$
$= (11)\,(16)\,(11)\,(17) \bmod 35$
$= 32912 \bmod 35$
$= 12$

Fermat's Little Theorem

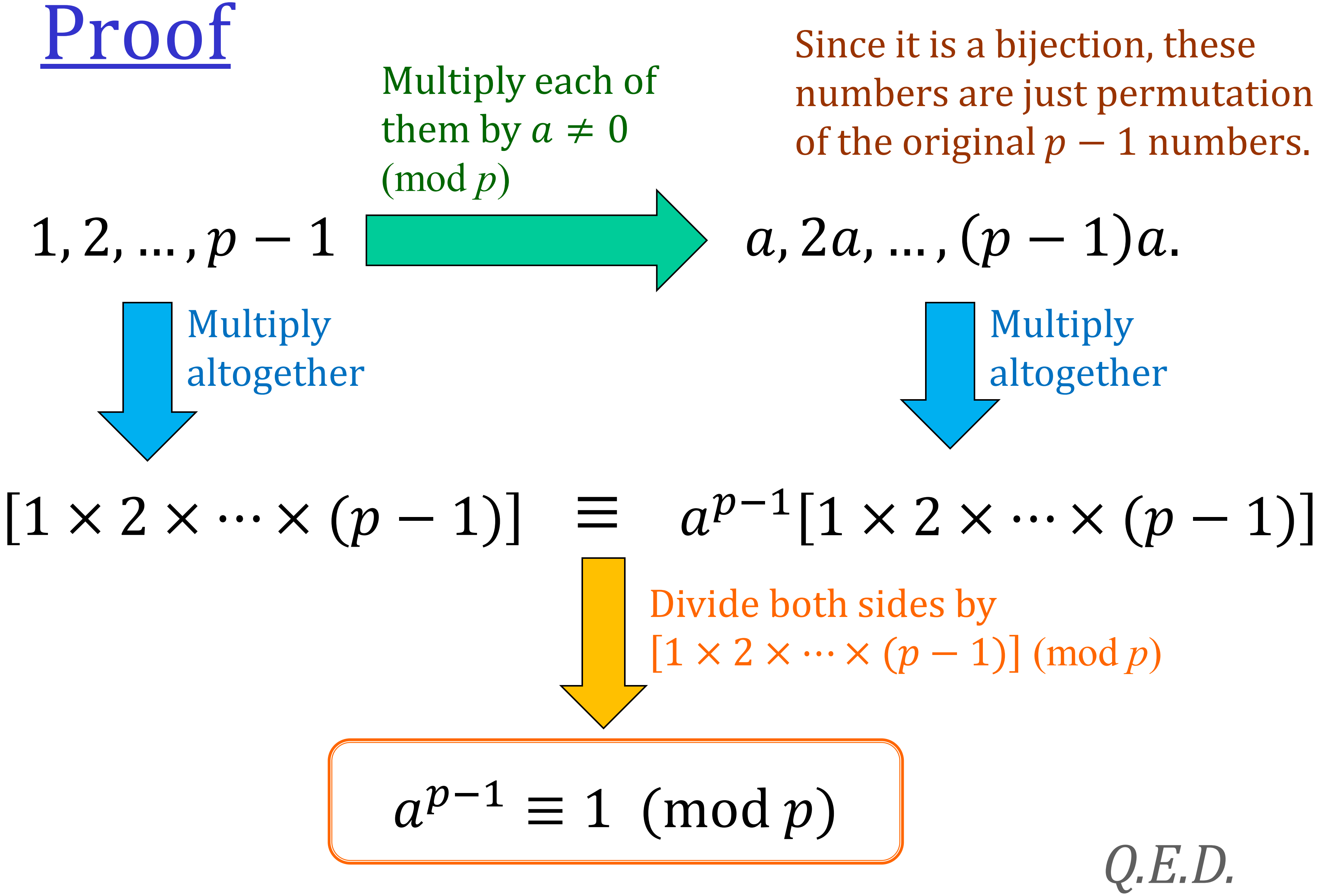
Theorem: If p is prime and $p \nmid a$ (which means p does not divide a), then

$$\boxed{a^{p-1} \equiv 1 \pmod{p}.}$$

- ❑ It can be used to calculate modular exponentiation if if p is a prime.
- ❑ Example: Compute 2^{35} (mod 7).
  - ❍ $2^{35} = (2^6)^5 \cdot 2^5$
  - ❍ $2^{35} \equiv 2^5 \equiv 32 \equiv 4 \pmod{7}$

Pierre de Fermat (1607-1665), the father of modern number theory. Watch this 5-min video to learn more: https://www.youtube.com/watch?v=Ij01HGgxnkA

Proof



Multiply each of them by $a \neq 0$ (mod p)

Since it is a bijection, these numbers are just permutation of the original $p-1$ numbers.

$$1, 2, \ldots, p-1 \quad \longrightarrow \quad a, 2a, \ldots, (p-1)a.$$

Multiply altogether

Multiply altogether

$$[1 \times 2 \times \cdots \times (p-1)] \quad \equiv \quad a^{p-1}[1 \times 2 \times \cdots \times (p-1)]$$

Divide both sides by $[1 \times 2 \times \cdots \times (p-1)]$ (mod p)

$$\boxed{a^{p-1} \equiv 1 \pmod{p}}$$

Q.E.D.

Euler's Theorem

Theorem:

If a is co-prime with n, then

$$\boxed{a^{\phi(n)} \equiv 1 \pmod{n}.}$$

n may not be a prime number.

Proof:

Same as in previous slide, except that we start with the $\phi(n)$ numbers that are co-prime with n.

Unit 6.5

Secret Sharing

4-Digit Combination Lock

Key Idea: Two points determine a line.

Suppose the lock's combination is 6965.

slope = 6965, y-intercept (generated randomly)

Secret Sharing Scheme

- Let $p = 10007$ (any prime number greater than 10,000)
- Pick a random number (mod p), say 30.
- Define the line L by

$$y \equiv 6965x + 30 \pmod{10007}$$

- Give the kids the following points on L:

Alice:	$(1, 6995)$
Bob:	$(2, 3953)$
Claire:	$(3, 911)$
Daniel:	$(4, 7876)$

How to Open the Lock?

- Suppose Alice and Claire want to open the lock.
  - Alice: $(1, 6995)$, Claire: $(3, 911)$
- The slope of the line through their point is

$$\frac{911 - 6995}{3 - 1} \equiv \frac{-6084}{2} \pmod{10007}$$

- Note that $2^{-1} \equiv 5004 \pmod{10007}$.
- Hence,

$$-6084 \times 5004 \equiv 6965 \pmod{10007}$$

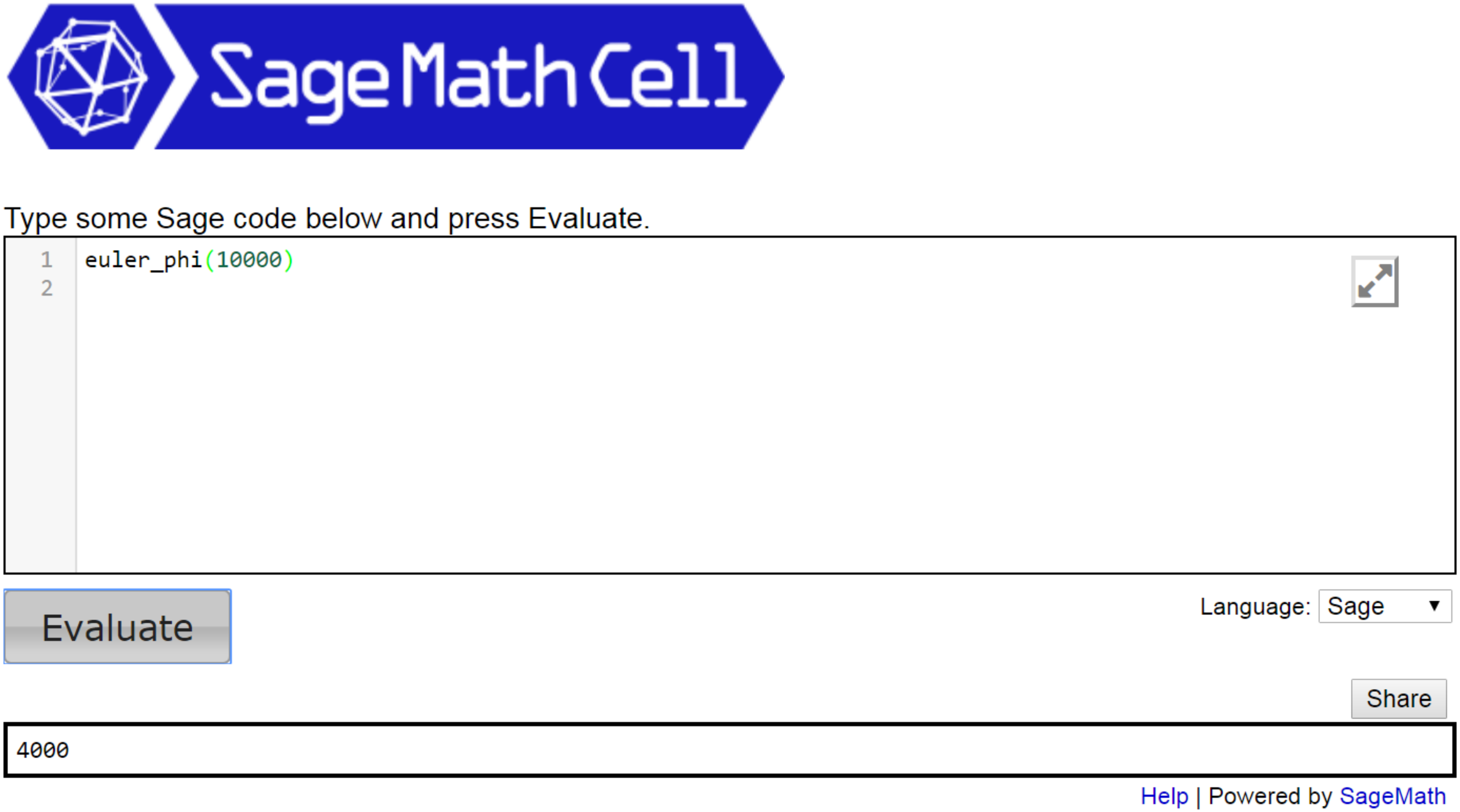
They can open the lock!

Unit 6.6

SageMath: a free math software

SageMath

- A free open-source mathematics software system
  - alternative to Magma, Maple, Mathematica and Matlab.
  - http://www.sagemath.org/
- Built on top of Python
  - You can use python commands in Sage.
- There is a web interface called SageMathCell
  - http://sagecell.sagemath.org/
  - Powerful programmable calculator online
    - Great for small (or even medium-sized) tasks.
    - Accessible by desktop/mobile.
- For large projects, switch to SageMathCloud.
  - You need to open an account (which is free).



Type some Sage code below and press Evaluate.

```
euler_phi(10000)
```

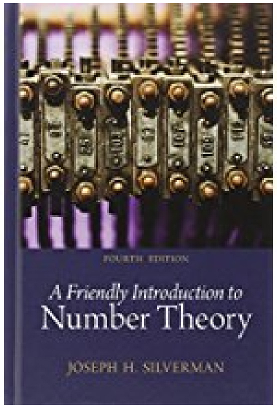
Language: Sage

```
4000
```

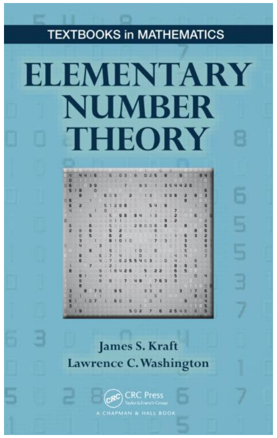
Some Useful Commands

- factor(x) // factorize x
- nth_prime(n) // return the n-th prime
- gcd(a, b)
- xgcd(a, b) // extended Euclidean alg.
- euler_phi(x)
- mod(a, n) (or $a\%n$)

Recommended Reading



- ❑ Chapters 6 and 8 – 10, J. H. Silverman, *A Friendly Introduction to Number Theory*, 4th ed., Pearson, 2013.



- ❑ Chapters 2, 5 and Section 7.8, J. S. Kraft and L. C. Washington, *Elementary Number Theory*, CRC Press, 2015.