# EE3210 Signals and Systems

Semester A 2023-2024

## Solution for Assignment 2

1.
Re-expressing $x(t)$ as

$$x(t) = e^{-2|t-1|} = \begin{cases} e^{-2(t-1)}, & t > 1 \\ e^{2(t-1)}, & t < 1 \end{cases}$$

We then apply (5.1) to obtain:

$$\begin{aligned} X(j\Omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt \\ &= \int_{-\infty}^{1} e^{2(t-1)} e^{-j\Omega t} dt + \int_{1}^{\infty} e^{-2(t-1)} e^{-j\Omega t} dt \\ &= \frac{e^{-j\Omega}}{2 - j\Omega} + \frac{e^{-j\Omega}}{2 + j\Omega} \\ &= \frac{4e^{-j\Omega}}{4 + \Omega^2} \end{aligned}$$

2.
Taking the DTFT on both sides of

$$y[n] - \frac{1}{2} y[n-1] = x[n] + 2[n-1] + x[n-2]$$

yields:

$$Y(e^{j\omega})\left(1 - 0.5e^{-j\omega}\right) = X(e^{j\omega})\left(1 + 2e^{-j\omega} + e^{-j2\omega}\right) \Rightarrow H(e^{j\omega}) = \frac{1 + 2e^{-j\omega} + e^{-j2\omega}}{1 - 0.5e^{-j\omega}}$$

3.
Let $X(e^{j\omega})$ and $Y(e^{j\omega})$ be the DTFTs of the system input $x[n]$ and output $y[n]$. We then have:

$$\begin{aligned} & H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1 - \frac{1}{2}e^{-j\omega} + e^{-j3\omega}}{1 + \frac{1}{2}e^{-j\omega} + \frac{3}{4}e^{-j2\omega}} \\ & \Rightarrow Y(e^{j\omega})\left(1 + \frac{1}{2}e^{-j\omega} + \frac{3}{4}e^{-j2\omega}\right) = X(e^{j\omega})\left(1 - \frac{1}{2}e^{-j\omega} + e^{-j3\omega}\right) \\ & \Rightarrow y[n] + \frac{1}{2}y[n-1] + \frac{3}{4}y[n-2] = x[n] - \frac{1}{2}x[n-1] + x[n-3] \end{aligned}$$

Alternatively, we can convert $X(e^{j\omega})$ and $Y(e^{j\omega})$ to $X(z)$ and $Y(z)$ and then apply inverse $z$ transform.

4.(a)
From the figure, we have

$$\begin{aligned} y[n] &= (x[n] + x[n] \otimes h_1[n]) \otimes h_2[n] \\ &= (x[n] \otimes \delta[n] + x[n] \otimes h_1[n]) \otimes h_2[n] \\ &= x[n] \otimes ([\delta[n] + h_1[n]]) \otimes h_2[n] \end{aligned}$$

As a result, the overall impulse response $h[n]$ is:

$$h[n] = ([\delta[n] + h_1[n]]) \otimes h_2[n] = [\delta[n] + \beta\delta[n-1]] \otimes \alpha^n u[n] = \alpha^n u[n] + \beta\alpha^{n-1} u[n-1]$$

4.(b)
Taking the $z$ transform of $h[n]$ yields

$$H(z) = \frac{1}{1-\alpha z^{-1}} + \frac{\beta z^{-1}}{1-\alpha z^{-1}}, \quad |z| > |\alpha|$$

4.(c)
Apply cross-multiplying and perform inverse $z$ transform, we get:

$$\begin{aligned} &H(z) = \frac{Y(z)}{X(z)} = \frac{1+\beta z^{-1}}{1-\alpha z^{-1}} \\ &\Rightarrow Y(z)(1-\alpha z^{-1}) = X(z)(1+\beta z^{-1}) \\ &\Rightarrow y[n] - \alpha y[n-1] = x[n] + \beta x[n-1] \end{aligned}$$

4.(d)
As $h[n] = 0$ for $n < 0$, the system is causal.

4.(e)
The system is stable if the ROC of $H(z)$ includes the unit circle, i.e., $|\alpha| < 1$.

5.(a)

$$\begin{aligned} X(e^{j\omega}) &= \alpha e^{j\phi} \sum_{n=0}^{N-1} e^{j(\omega_0-\omega)n} \\ &= \alpha e^{j\phi} \frac{1-e^{j(\omega_0-\omega)N}}{1-e^{j(\omega_0-\omega)}} \\ &= \alpha e^{j(\phi-(\omega_0-\omega)(N-1)/2)} \frac{\sin(\frac{(\omega_0-\omega)N}{2})}{\sin(\frac{\omega_0-\omega}{2})} \end{aligned}$$

5.(b)
Using the result in Example 6.3, we have

$$\frac{\sin((\omega_0-\omega)N/2)}{\sin((\omega_0-\omega)/2)} = N \cdot \frac{\mathrm{sinc}((\omega_0-\omega)N/(2\pi))}{\mathrm{sinc}((\omega_0-\omega)/(2\pi))}$$

where its maximum appears at $\omega = \omega_0$, with a value of $N$. As a result, the maximum value of $|X(e^{j\omega})|$ is $|\alpha|N$. The value of $\omega$ which maximizes $|X(e^{j\omega})|$ is thus $\omega = \omega_0$.

6.
$x[0]=0,\ x[1]=1,\ x[2]=0,\ x[3]=-1$ and $x[4]=0$.

Yes. $x[n]$ is a periodic signal.

7.
The $z$ transform is:

$$X(z)=\sum_{n=1}^{N} na^n z^{-n}=az^{-1}+2a^2z^{-2}+\cdots Na^Nz^{-N}$$

Clearly, the ROC is $|z|>0$.

Considering $X(z)=X_1(z)+X_2(z)+\cdots+X_N(z)$ where

$$X_1(z)=az^{-1}+a^2z^{-2}+\cdots a^Nz^{-N}=\frac{az^{-1}(1-(az^{-1})^N)}{1-az^{-1}}$$
$$X_2(z)=a^2z^{-2}+a^3z^{-3}+\cdots a^Nz^{-N}=\frac{a^2z^{-2}(1-(az^{-1})^{N-1})}{1-az^{-1}}$$

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$$X_N(z)=a^Nz^{-N}=\frac{a^Nz^{-N}(1-az^{-1})}{1-az^{-1}}$$

As a result, we have:

$$\begin{aligned}
X(z) &= \frac{az^{-1}(1-(az^{-1})^N)}{1-az^{-1}}+\frac{a^2z^{-2}(1-(az^{-1})^{N-1})}{1-az^{-1}}+\cdots+\frac{a^Nz^{-N}(1-az^{-1})}{1-az^{-1}} \\
&= \frac{az^{-1}+a^2z^{-2}+\cdots+a^Nz^{-N}-N(az^{-1})^{N+1}}{1-az^{-1}} \\
&= \frac{\frac{az^{-1}(1-(az^{-1})^N)}{1-az^{-1}}-N(az^{-1})^{N+1}}{1-az^{-1}} \\
&= \frac{az^{-1}-(N+1)(az^{-1})^{N+1}+N(az^{-1})^{N+2}}{(1-az^{-1})^2}, \quad |z|>0
\end{aligned}$$

Alternatively, you can find $X(z)$ by first expressing $x[n]$ as:

$$x[n]=na^n(u[n-1]-u[n-N+1])$$

and then make use of Table 8.1 and time shifting property.

8.(a)
If the system is stable, then the ROC for $H(z)$ is $0.5<|z|<3$. On the other hand, for the unit step input, we have:

$$X(z)=\frac{1}{1-z^{-1}}, \quad |z|>1$$

The $z$ transform for $y[n]$ is $Y(z)=H(z)X(z)$. Using partial fraction expansion, we get:

$$Y(z) = H(z)X(z) = \frac{0.8}{1-0.5z^{-1}} + \frac{0.2}{1-3z^{-1}} - \frac{1}{1-z^{-1}}, \quad 1 < |z| < 3$$

Taking the inverse $z$ transform, we get:

$$y[n] = (0.8)(0.5)^n u[n] - 0.2(3)^n u[-n-1] - u[n]$$

8.(b)
If the system is causal, then the ROC for $H(z)$ is $|z| > 3$. For $x[n] = \delta[n]$, $y[n] = h[n]$. Using partial fraction expansion, we get:

$$Y(z) = H(z) = \frac{-0.4z^{-1}}{1-0.5z^{-1}} + \frac{0.4z^{-1}}{1-3z^{-1}}, \quad |z| > 3$$

Taking the inverse $z$ transform, we get:

$$y[n] = -(0.4)(0.5)^{n-1} u[n-1] + 0.4(3)^{n-1} u[n-1]$$

9.
The $z$ transforms of $x[n] = u[-n-1]$ and $h[n] = (0.5)^n u[n]$ are:

$$X(z) = -\frac{1}{1-z^{-1}}, \quad |z| < 1 \qquad \text{and} \qquad H(z) = \frac{1}{1-0.5z^{-1}}, \quad |z| > 0.5$$

So we have:

$$Y(z) = -\frac{1}{1-z^{-1}} \cdot \frac{1}{1-0.5z^{-1}} = \frac{-2}{1-z^{-1}} + \frac{1}{1-0.5z^{-1}}, \quad 0.5 < |z| < 1$$

Taking the inverse $z$ transform yields:

$$y[n] = 2u[-n-1] + (0.5)^n u[n]$$

10.(a)
In a denoising system, the input contains a signal-of-interest and additive noise, and the system attempts to extract the signal-of-interest while removing/suppressing the noise, thus it is expected that the output is a good approximation of the signal-of-interest. The principle is that, under a certain transform, the signal-of-interest is sparse, meaning that there are only a few non-zero entries and we only need to keep them and ignoring the rest.

10.(b)
In theory, we can transform the continuous-time signal to frequency domain using Fourier transform. The $\cos(100\pi t)$ corresponds to two impulses at $-100\pi$ and $100\pi$ in the frequency domain. We keep only the two components at $-100\pi$ and $100\pi$, and assign the rest to zero, and convert this resultant frequency-domain signal to time domain using the inverse Fourier transform.

Compression is easily achieved. Instead of storing $\cos(100\pi t)$, we only need to store the amplitudes and locations of the two impulses in the frequency-domain signal.