In-Class Exercise 8

1. Consider the experiment of rolling a fair dice and let random variable X denote the outcome which is the face number. Find the conditional probability mass function (PMF) of X given that we know the observed number is less than 5.

2. Consider tossing a coin and the probability of getting head is p. The coin is repeatedly tossed until two consecutive heads occur. Let X be the total number of coin tosses. Based on (2.6), it is suggested that the PMF of X equal to:

$$p(r) = P(X = r) = (1-p)^{r-2}p^2, \quad 2 \le r < \infty$$

   Do you agree? Explain your answer.

3. Consider tossing a coin and the probability of getting head is p. The coin is repeatedly tossed until two consecutive heads occur. Let X be the total number of coin tosses. Determine $\mathbb{E}\{X\}$.

4. Consider two random variables X and Y with joint probability density function (PDF):

$$P_{XY}(x,y) = \begin{cases} \dfrac{1}{\pi r^2}, & x^2 + y^2 \leq r^2 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find $P_{Y|X}(y|x)$.

(b) Determine $\mathbb{E}\{Y|X=x\}$.

5. Consider two random variables X and Y with joint PMF given in the following table:

	$Y=0$	$Y=1$
$X=0$	$\frac{1}{5}$	$\frac{2}{5}$
$X=1$	$\frac{2}{5}$	0

(a) Find all conditional PMFs of X given Y.

(b) Let $Z=\mathbb{E}\{X|Y\}$. Find the PMF of Z.

(c) Compute $\mathbb{E}\{Z\}$.

(d) Let $V=\mathrm{var}(X|Y)$. Find the PMF of V.

(e) Compute $\mathbb{E}\{V\}$.

6. Consider a random variable $R \sim \mathcal{U}(0,1)$. Given $R = r$, another random variable is generated as $X \sim \mathcal{U}(0,r)$. Find $P_{R|X}(r|x)$.

   Note: $\int du/u = \ln(u) + C$

7. Consider two independent geometric random variables X and Y with parameter p. That is, they have the same PMF:

$$P(X = r) = P(Y = r) = (1-p)^{r-1}p, \quad 1 \leq r < \infty$$

   Let $Z = X - Y$. Determine the PMF of Z.

Solution

1.

For a fair dice, the probability is the same for all face numbers:

$$P(X=1)=P(X=2)=\cdots=P(X=6)=\frac{1}{6}$$

On the other hand, the given information corresponds to an event, say, $A=\{X<5\}$ or $A=\{X=1,2,3,4\}$.

Hence we have:

$$P(A)=\frac{4}{6}=\frac{2}{3}$$

The conditional PMF is thus:

$$P_{X|A}(x)=\begin{cases}(1/6)/(4/6)=0.25, & x=1,2,3,4\\ 0, & \text{otherwise}\end{cases}$$

2.

The suggested PMF assumes that the first $(r-2)$th trials must correspond to tails only. However, non-consecutive heads are allowed, and this probability is not included. Hence the PMF is not correct.

That is, we can see the possibilities include:

HH + THH + HTHH + TTHH + HTTHH + …

3.

Let $\mu = \mathbb{E}\{X\}$. We can follow Example 4.6 and let the events of having a head and tail be H and T, respectively. Clearly, $P(H) = p$ and $P(T) = 1 - p$.

We first condition on the result of the first coin toss:

$$
\begin{aligned}
\mathbb{E}\{X\} &= \mathbb{E}\{X|H\}P(H) + \mathbb{E}\{X|T\}P(T) \\
&= pE\{X|H\} + (\mu + 1) \cdot (1 - p) \Rightarrow p\mu = pE\{X|H\} + (1 - p)
\end{aligned}
$$

To find $\mathbb{E}\{X|H\}$, we need to condition on the result of the second coin toss:

$$
\begin{aligned}
\mathbb{E}\{X|H\} &= \mathbb{E}\{X|HH\}P(H) + \mathbb{E}\{X|HT\}P(T) \\
&= 2p + (2 + \mu)(1 - p) = 2 + (1 - p)\mu
\end{aligned}
$$

Note that for $\mathbb{E}\{X|HT\}$, because the first two tosses are HT, we have wasted two coin tosses and we start over at the third toss, resulting in 2 and $\mu = \mathbb{E}\{X\}$.

As a result, we get:

$$\mu = p(2 + (1-p)\mu) + (1-p) \Rightarrow \mu = \mathbb{E}\{X\} = \frac{1+p}{p^2}$$

4.(a)

First we compute the marginal PDF:

$$P_X(x) = \int_{-\infty}^{\infty} P_{XY}(x,y)dy = \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \frac{1}{\pi r^2} dy = \frac{1}{\pi r^2} \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} dy$$

$$= \begin{cases} \dfrac{2\sqrt{r^2-x^2}}{\pi r^2}, & -r \le x \le r \\ 0, & \text{otherwise} \end{cases}$$

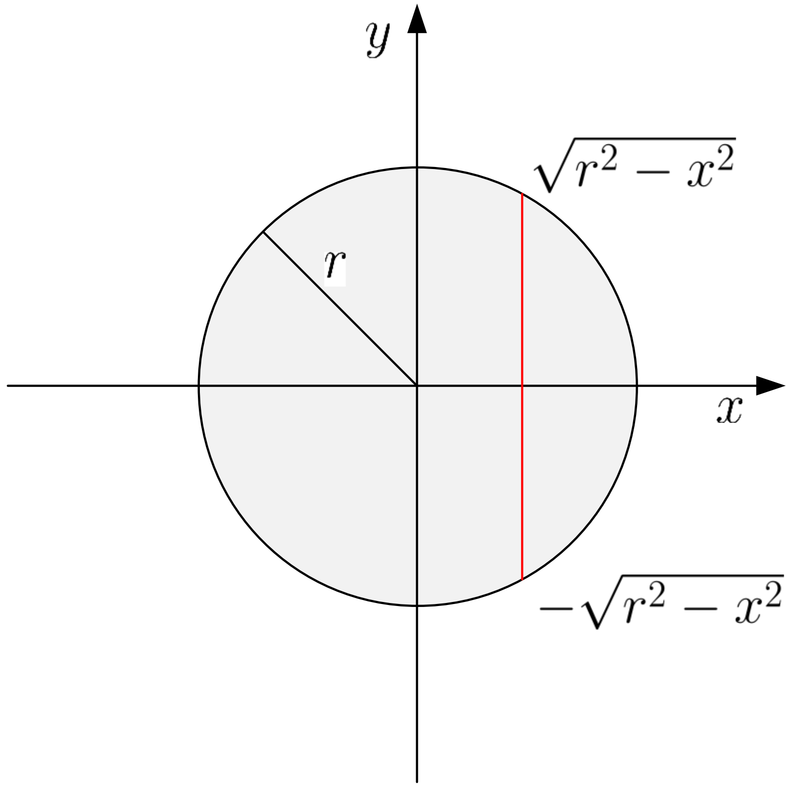
Using (4.14), $P_{Y|X}(y|x)$ is obtained as:

$$P_{Y|X}(y|x) = \frac{P_{XY}(x,y)}{P_X(x)} = \begin{cases} \dfrac{1}{2\sqrt{r^2-x^2}}, & y^2 \le r^2 - x^2 \\ 0, & \text{otherwise} \end{cases}$$

4.(b)

Given $X = x$, it can be seen that $Y \sim \mathcal{U}(-\sqrt{r^2-x^2}, \sqrt{r^2-x^2})$, indicating that $\mathbb{E}\{Y|X=x\} = 0$ for any values of X.

You may also easily obtain the results using the following graphical illustration:



When x is fixed, y can only has values between $-\sqrt{r^2 - x^2}$ and $\sqrt{r^2 - x^2}$.

It is also clear that the mean value of y is 0.

5.(a)

Applying (4.14), we get:

$$P_{X|Y}(0|0) = P(X=0|Y=0) = \frac{P(X=0,Y=0)}{P(Y=0)}$$

$$= \frac{1/5}{3/5} = \frac{1}{3}$$

$$P_{X|Y}(1|0) = 1 - P(X=0|Y=0) = \frac{2}{3}$$

Similarly,

$$P_{X|Y}(0|1) = P(X=0|Y=1) = \frac{P(X=0,Y=1)}{P(Y=1)}$$

$$= \frac{2/5}{2/5} = 1$$

$$P_{X|Y}(1|1) = 1 - P(X=0|Y=1) = 0$$

5.(b)

Since there are two possible values of Y, we need to compute two values of $Z=\mathbb{E}\{X|Y\}$. Applying (2.59), we get:

$$\mathbb{E}\{X|Y=0\}=0\cdot P_{X|Y}(0|0)+1\cdot P_{X|Y}(1|0)=\frac{2}{3}$$

$$\mathbb{E}\{X|Y=1\}=0\cdot P_{X|Y}(0|1)+1\cdot P_{X|Y}(1|1)=0$$

Recall $P(Y=0)=3/5$ and $P(Y=1)=2/5$, the PMF of Z can be written as:

$$P_Z(z)=\begin{cases}\dfrac{2}{5}, & z=0\\[2ex] \dfrac{3}{5}, & z=\dfrac{2}{3}\\ 0 & \text{otherwise}\end{cases}$$

5.(c)

We use (2.19) to obtain $\mathbb{E}\{Z\}$:

$$\mathbb{E}\{Z\} = \frac{2}{5} \cdot 0 + \frac{3}{5} \cdot \frac{2}{3} = \frac{2}{5}$$

Note that

$$P_X(x) = \begin{cases} \frac{3}{5}, & x = 0 \\ \frac{2}{5}, & x = 1 \\ 0 & \text{otherwise} \end{cases} \Rightarrow \mathbb{E}\{X\} = \frac{3}{5} \cdot 0 + \frac{2}{5} \cdot 1 = \frac{2}{5}$$

which aligns with:

$$\mathbb{E}\{X\} = \mathbb{E}\{Z\} = \mathbb{E}\left\{\mathbb{E}\{X|Y\}\right\}$$

5.(d)
Similarly, we need to compute two values of $V = \mathrm{var}(X|Y)$. We consider $Y = 0$ first.

$$\mathbb{E}\{X^2|Y=0\} = 0^2 \cdot P_{X|Y}(0|0) + 1^2 \cdot P_{X|Y}(1|0) = \frac{2}{3}$$

Recall $\mu_{X|Y}(0) = \mathbb{E}\{X|Y=0\} = \frac{2}{3}$. Applying (4.25), we get:

$$\mathrm{var}(X|Y=0) = \mathbb{E}\{X^2|Y=0\} - (\mu_{X|Y}(0))^2 = \frac{2}{9}$$

Also,

$$\mathbb{E}\{X^2|Y=1\} = 0^2 \cdot P_{X|Y}(0|1) + 1^2 \cdot P_{X|Y}(1|1) = 0$$

and recall $\mu_{X|Y}(1) = \mathbb{E}\{X|Y=1\} = 0$, we have:

$$\mathrm{var}(X|Y=1) = 0$$

Recall $P(Y=0)=3/5$ and $P(Y=1)=2/5$, the PMF of V can be written as:

$$P_V(v)=\begin{cases}\frac{2}{5}, & v=0\\ \\ \frac{3}{5}, & v=\frac{2}{9}\\ 0 & \text{otherwise}\end{cases}$$

5.(e)
We use (2.19) to obtain $\mathbb{E}\{V\}$:

$$\mathbb{E}\{V\}=\frac{2}{5}\cdot 0+\frac{3}{5}\cdot\frac{2}{9}=\frac{2}{15}$$

6.

We first have:

$$P_R(r) = \begin{cases} 1, & 0 < r < 1 \\ 0, & \text{otherwise} \end{cases}, \quad P_{X|R}(x|r) = \begin{cases} 1/r, & 0 < x < r \\ 0, & \text{otherwise} \end{cases}$$

With the use of (4.16), we get:

$$P_{RX}(r,x) = P_{X|R}(x|r)P_R(r) = \begin{cases} 1/r, & 0 < x < r < 1 \\ 0, & \text{otherwise} \end{cases}$$

Then we can compute $P_X(x)$ for $x < r < 1$ using (3.8):

$$P_X(x) = \int_{-\infty}^{\infty} P_{RX}(r,x)dr = \int_x^1 \frac{dr}{r} = -\ln(x)$$

Finally, $P_{R|X}(r|x)$ is obtained using (4.14) as:

$$P_{R|X}(r|x) = \frac{P_{RX}(r,x)}{P_X(x)} = \begin{cases} -\dfrac{1}{r\ln(x)}, & x < r < 1 \\ 0, & \text{otherwise} \end{cases}$$

7.

Since the range of both X and Y is {1,2, …}, then the range of Z is {…, -1,0,1, …}. Let $q = 1 - p$, then:

$$P(X = r) = P(Y = r) = q^{r-1}p, \quad 1 \leq r < \infty$$

$$\begin{aligned}
P_Z(k) &= P(Z = k) = P(X - Y = k) = P(X = Y + k) \\
&= \sum_{j=1}^{\infty} P(X = Y + k | Y = j) P(Y = j) \\
&= \sum_{j=1}^{\infty} P(X = j + k | Y = j) P(Y = j) \\
&= \sum_{j=1}^{\infty} P(X = j + k) P(Y = j), \quad X \text{ and } Y \text{ are independent} \\
&= \sum_{j=1}^{\infty} P_X(j + k) P_Y(j)
\end{aligned}$$

Because $j+k$ can be outside the range of X, we need to consider two cases, $k \geq 0$ and $k < 0$.

For $k \geq 0$:

$$
\begin{aligned}
P_Z(k) &= \sum_{j=1}^{\infty} P_X(j+k) P_Y(j) \\
&= \sum_{j=1}^{\infty} pq^{(j+k-1)} \cdot pq^{(j-1)} \\
&= p^2 q^k \sum_{j=1}^{\infty} q^{2(j-1)} \\
&= p^2 q^k \frac{1}{1-q^2} \\
&= \frac{p(1-p)^k}{2-p}
\end{aligned}
$$

For $k < 0$:

$$
\begin{aligned}
P_Z(k) &= \sum_{j=1}^{\infty} P_X(j+k) P_Y(j) \\
&= \sum_{j=-k+1}^{\infty} P_X(j+k) P_Y(j), \quad P_X(j+k) = 0 \text{ if } j+k < 1 \\
&= \sum_{j=-k+1}^{\infty} pq^{(j+k-1)} \cdot pq^{(j-1)} \\
&= p^2 \sum_{j=-k+1}^{\infty} q^{k+2(j-1)} \\
&= p^2(q^{-k} + q^{-k+2} + \cdots) = p^2 q^{-k}(1 + q^2 + \cdots) \\
&= p^2 q^{-k} \frac{1}{1-q^2} \\
&= \frac{p}{(1-p)^k(2-p)}
\end{aligned}
$$

Combining the results, we have:

$$P_Z(k) = \begin{cases} \dfrac{p(1-p)^k}{2-p}, & k \geq 0 \\ \\ \dfrac{p}{(1-p)^k(2-p)}, & k < 0 \end{cases}$$