Solutions to EE3210 Tutorial 8 Problems

Problem 1:

(a) $x[n] = 1 + \cos\left(\frac{2\pi}{6}n\right)$ is a periodic signal with fundamental period $N = 6$. Using Euler's formula, we can rewrite $x[n]$ as

$$x[n] = 1 + \frac{1}{2}e^{j(2\pi n/6)} + \frac{1}{2}e^{-j(2\pi n/6)}. \tag{1}$$

Comparing the right-hand sides of (1) and the synthesis formula of the discrete-time Fourier series with the limits of the summation chosen to be $-2 \le k \le 3$, i.e.,

$$x[n] = \sum_{k=-2}^{3} a_k e^{jk(2\pi n/6)} \tag{2}$$

we obtain the Fourier series coefficients a_k of $x[n]$ as $a_0 = 1$, $a_{-1} = a_1 = 1/2$, and $a_k = 0$ for $k = -2, 2, 3$.

(b) $y[n] = \sin\left(\frac{2\pi}{6}n + \frac{\pi}{4}\right)$ is a periodic signal with fundamental period $N = 6$. Using Euler's formula, we can rewrite $y[n]$ as

$$y[n] = \frac{1}{2j}e^{j\pi/4}e^{j(2\pi n/6)} - \frac{1}{2j}e^{-j\pi/4}e^{-j(2\pi n/6)}. \tag{3}$$

Comparing the right-hand sides of (3) and (2), we obtain the Fourier series coefficients b_k of $y[n]$ as $b_{-1} = -\frac{1}{2j}e^{-j\pi/4}$, $b_1 = \frac{1}{2j}e^{j\pi/4}$, and $b_k = 0$ for $k = -2, 0, 2, 3$.

(c) The signal $z[n] = x[n]y[n]$ is also periodic with period $N = 6$. Applying the multiplication property of the discrete-time Fourier series, we obtain the Fourier series coefficients c_k of $z[n]$ as

$$c_k = \sum_{l=-2}^{3} a_l b_{k-l} = a_{-1}b_{k+1} + a_0 b_k + a_1 b_{k-1}$$

$$= \begin{cases} a_{-1}b_{-1} = -\dfrac{1}{4j}e^{-j\pi/4}, & k=-2 \\ a_0 b_{-1} = -\dfrac{1}{2j}e^{-j\pi/4}, & k=-1 \\ a_{-1}b_1 + a_1 b_{-1} = \dfrac{1}{4j}e^{j\pi/4} - \dfrac{1}{4j}e^{-j\pi/4} = \dfrac{1}{2}\sin\left(\dfrac{\pi}{4}\right), & k=0 \\ a_0 b_1 = \dfrac{1}{2j}e^{j\pi/4}, & k=1 \\ a_1 b_1 = \dfrac{1}{4j}e^{j\pi/4}, & k=2 \\ 0, & k=3. \end{cases}$$

(d) Through direct evaluation of $z[n]$, we have

$$\begin{aligned} z[n] = x[n]y[n] &= \sin\left(\frac{2\pi}{6}n + \frac{\pi}{4}\right) + \sin\left(\frac{2\pi}{6}n + \frac{\pi}{4}\right)\cos\left(\frac{2\pi}{6}n\right) \\ &= \sin\left(\frac{2\pi}{6}n + \frac{\pi}{4}\right) + \frac{1}{2}\sin\left(\frac{4\pi}{6}n + \frac{\pi}{4}\right) + \frac{1}{2}\sin\left(\frac{\pi}{4}\right). \end{aligned}$$

This implies that the Fourier series coefficients c_k of $z[n]$ are

$$c_k = \begin{cases} -\dfrac{1}{4j}e^{-j\pi/4}, & k=-2 \\ -\dfrac{1}{2j}e^{-j\pi/4}, & k=-1 \\ \dfrac{1}{2}\sin\left(\dfrac{\pi}{4}\right), & k=0 \\ \dfrac{1}{2j}e^{j\pi/4}, & k=1 \\ \dfrac{1}{4j}e^{j\pi/4}, & k=2 \\ 0, & k=3. \end{cases}$$

Problem 2: Using the analysis formula of the continuous-time Fourier transform, we have

$$
\begin{aligned}
X(\omega) &= \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt = \int_{-\infty}^{+\infty} \sum_{k=0}^{+\infty} \alpha^k \delta(t - kT) e^{-j\omega t} dt \\
&= \sum_{k=0}^{+\infty} \alpha^k \int_{-\infty}^{+\infty} \delta(t - kT) e^{-j\omega t} dt = \sum_{k=0}^{+\infty} \alpha^k e^{-jk\omega T} = \sum_{k=0}^{+\infty} \left(\alpha e^{-j\omega T}\right)^k \\
&= \frac{1}{1 - \alpha e^{-j\omega T}}.
\end{aligned}
$$

Problem 3:

(a) Using the synthesis formula of the continuous-time Fourier transform, we have

$$
x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) e^{j\omega t} d\omega = \int_{-\infty}^{+\infty} \delta(\omega - \omega_0) e^{j\omega t} d\omega = e^{j\omega_0 t}.
$$

(b) Using the synthesis formula of the continuous-time Fourier transform, we have

$$
\begin{aligned}
x(t) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) e^{j\omega t} d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{+\infty} 2[\delta(\omega - 1) - \delta(\omega + 1)] e^{j\omega t} d\omega + \frac{1}{2\pi} \int_{-\infty}^{+\infty} 3[\delta(\omega - 2\pi) + \delta(\omega + 2\pi)] e^{j\omega t} d\omega \\
&= \frac{1}{\pi} \left(e^{jt} - e^{-jt}\right) + \frac{3}{2\pi} \left(e^{j2\pi t} + e^{-j2\pi t}\right) \\
&= \frac{3}{\pi} \cos(2\pi t) + j \frac{2}{\pi} \sin t.
\end{aligned}
$$