EE3210 Signals and Systems

Semester A 2023-2024

Assignment 2

Due Date: 15 November 2023

1. Compute the Fourier transform of $x(t) = e^{-2|t-1|}$.

2. Find the frequency response $H(e^{j\omega})$ of a discrete-time linear time-invariant (LTI) system whose input $x[n]$ and output $y[n]$ satisfy the following difference equation:

$$y[n] - \frac{1}{2}y[n-1] = x[n] + 2x[n-1] + x[n-2]$$

3. Determine the difference equation that characterizes a discrete-time LTI system whose frequency response is

$$H(e^{j\omega}) = \frac{1 - \frac{1}{2}e^{-j\omega} + e^{-j3\omega}}{1 + \frac{1}{2}e^{-j\omega} + \frac{3}{4}e^{-j2\omega}}$$

4. Figure 1 shows a system which consists of an interconnection of two discrete-time LTI systems with impulse responses $h_1[n]$ and $h_2[n]$.

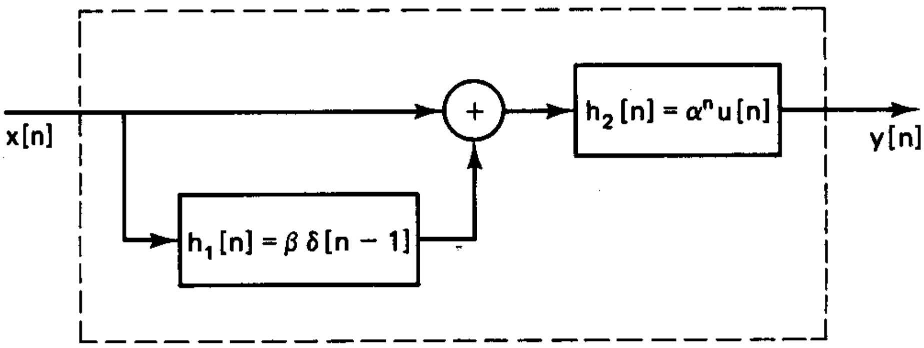

Figure 1

(a) Find the impulse response $h[n]$ of the overall system.
(b) Find the system transfer function $H(z)$ of the overall system, which is equal to $Y(z)/X(z)$ where $X(z)$ and $Y(z)$ are the z transforms of the input $x[n]$ and output $y[n]$, respectively.
(c) Write down the difference equation that relates $x[n]$ and $y[n]$.
(d) Is the system causal?
(e) Under what condition would the system be stable?

5. Given a discrete-time signal $x[n]$ which has the form of:

$$x[n] = \begin{cases} \alpha e^{j(\omega_o n+\phi)}, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

where α, ω_0 and ϕ are real numbers.

(a) Determine $X(e^{j\omega})$ which is the discrete-time Fourier transform of $x[n]$.
(b) Find the maximum value of $|X(e^{j\omega})|$. Determine the value of ω which maximizes $|X(e^{j\omega})|$.

6. Given a continuous-time signal $x(t)$:

$$x(t) = \sin\left(\frac{\pi}{2}t\right)$$

The signal is sampled with a sampling period $T=1$s to produce the discrete-time signal $x[n]$. Find $x[0]$, $x[1]$, $x[2]$, $x[3]$ and $x[4]$. Is $x[n]$ a periodic signal?

7. Determine the z transform of $x[n]$ which has the form of:

$$x[n] = \begin{cases} na^n, & 1 \le n \le N \\ 0, & \text{otherwise} \end{cases}$$

Specify the region of convergence (ROC).

8. Consider a discrete-time LTI system whose transfer function $H(z)$ is:

$$H(z) = \frac{z^{-2}}{(1-0.5z^{-1})(1-3z^{-1})}$$

(a) If the system is stable, determine the output $y[n]$ when the input is $x[n]=u[n]$.
(b) If the system is causal, determine the output $y[n]$ when $x[n]=\delta[n]$.

9. Use z transform and inverse z transform to compute the convolution of $x[n]=u[-n-1]$ and $h[n]=(0.5)^n u[n]$.

10. Watch the short video of the 2013 Shaw Prize winner for mathematics, Prof. David Donoho (start at 14:50): https://www.youtube.com/watch?v=5wv4grOMgIU

(a) Briefly describe a denoising system, which includes the system input, output and function, as well as the principle to achieve denoising. Use your own words in no more than 100 words.
(b) Suppose you are given the following observed continuous-time signal $x(t)$:

$$x(t) = \cos(100\pi t) + n(t)$$

where $n(t)$ is the unwanted noise. With the use of an appropriate transform you have learned in this course, briefly describe, in theory, how can you extract $\cos(100\pi t)$ from $x(t)$? Also, how can you achieve compression for $\cos(100\pi t)$?

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Solution for Assignment 2

1.
Re-expressing $x(t)$ as

$$x(t) = e^{-2|t-1|} = \begin{cases} e^{-2(t-1)}, & t > 1 \\ e^{2(t-1)}, & t < 1 \end{cases}$$

We then apply (5.1) to obtain:

$$\begin{aligned} X(j\Omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt \\ &= \int_{-\infty}^{1} e^{2(t-1)}e^{-j\Omega t}dt + \int_{1}^{\infty} e^{-2(t-1)}e^{-j\Omega t}dt \\ &= \frac{e^{-j\Omega}}{2-j\Omega} + \frac{e^{-j\Omega}}{2+j\Omega} \\ &= \frac{4e^{-j\Omega}}{4+\Omega^2} \end{aligned}$$

2.
Taking the DTFT on both sides of

$$y[n] - \frac{1}{2}y[n-1] = x[n] + 2[n-1] + x[n-2]$$

yields:

$$Y(e^{j\omega})\left(1 - 0.5e^{-j\omega}\right) = X(e^{j\omega})\left(1 + 2e^{-j\omega} + e^{-j2\omega}\right) \Rightarrow H(e^{j\omega}) = \frac{1 + 2e^{-j\omega} + e^{-j2\omega}}{1 - 0.5e^{-j\omega}}$$

3.
Let $X(e^{j\omega})$ and $Y(e^{j\omega})$ be the DTFTs of the system input $x[n]$ and output $y[n]$. We then have:

$$\begin{aligned} & H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1 - \frac{1}{2}e^{-j\omega} + e^{-j3\omega}}{1 + \frac{1}{2}e^{-j\omega} + \frac{3}{4}e^{-j2\omega}} \\ & \Rightarrow Y(e^{j\omega})\left(1 + \frac{1}{2}e^{-j\omega} + \frac{3}{4}e^{-j2\omega}\right) = X(e^{j\omega})\left(1 - \frac{1}{2}e^{-j\omega} + e^{-j3\omega}\right) \\ & \Rightarrow y[n] + \frac{1}{2}y[n-1] + \frac{3}{4}y[n-2] = x[n] - \frac{1}{2}x[n-1] + x[n-3] \end{aligned}$$

Alternatively, we can convert $X(e^{j\omega})$ and $Y(e^{j\omega})$ to $X(z)$ and $Y(z)$ and then apply inverse z transform.

4.(a)
From the figure, we have

$$\begin{aligned} y[n] &= (x[n] + x[n] \otimes h_1[n]) \otimes h_2[n] \\ &= (x[n] \otimes \delta[n] + x[n] \otimes h_1[n]) \otimes h_2[n] \\ &= x[n] \otimes ([\delta[n] + h_1[n]]) \otimes h_2[n] \end{aligned}$$

As a result, the overall impulse response $h[n]$ is:

$$h[n] \;=\; ([\delta[n] + h_1[n]]) \otimes h_2[n] = [\delta[n] + \beta\delta[n-1]] \otimes \alpha^n u[n] = \alpha^n u[n] + \beta\alpha^{n-1} u[n-1]$$

4.(b)
Taking the z transform of $h[n]$ yields

$$H(z) = \frac{1}{1-\alpha z^{-1}} + \frac{\beta z^{-1}}{1-\alpha z^{-1}}, \quad |z| > |\alpha|$$

4.(c)
Apply cross-multiplying and perform inverse z transform, we get:

$$\begin{aligned} &H(z) = \frac{Y(z)}{X(z)} = \frac{1+\beta z^{-1}}{1-\alpha z^{-1}} \\ &\Rightarrow Y(z)(1-\alpha z^{-1}) = X(z)(1+\beta z^{-1}) \\ &\Rightarrow y[n] - \alpha y[n-1] = x[n] + \beta x[n-1] \end{aligned}$$

4.(d)
As $h[n] = 0$ for $n < 0$, the system is causal.

4.(e)
The system is stable if the ROC of $H(z)$ includes the unit circle, i.e., $|\alpha| < 1$.

5.(a)

$$\begin{aligned} X(e^{j\omega}) &= \alpha e^{j\phi} \sum_{n=0}^{N-1} e^{j(\omega_0-\omega)n} \\ &= \alpha e^{j\phi} \frac{1-e^{j(\omega_0-\omega)N}}{1-e^{j(\omega_0-\omega)}} \\ &= \alpha e^{j(\phi-(\omega_0-\omega)(N-1)/2)} \frac{\sin(\frac{(\omega_0-\omega)N}{2})}{\sin(\frac{\omega_0-\omega}{2})} \end{aligned}$$

5.(b)
Using the result in Example 6.3, we have

$$\frac{\sin((\omega_0-\omega)N/2)}{\sin((\omega_0-\omega)/2)} = N \cdot \frac{\operatorname{sinc}((\omega_0-\omega)N/(2\pi))}{\operatorname{sinc}((\omega_0-\omega)/(2\pi))}$$

where its maximum appears at $\omega = \omega_0$, with a value of N. As a result, the maximum value of $|X(e^{j\omega})|$ is $|\alpha|N$. The value of ω which maximizes $|X(e^{j\omega})|$ is thus $\omega = \omega_0$.

6.
$x[0]=0,\ x[1]=1,\ x[2]=0,\ x[3]=-1$ and $x[4]=0$.

Yes. $x[n]$ is a periodic signal.

7.
The z transform is:

$$X(z)=\sum_{n=1}^{N} na^n z^{-n} = az^{-1}+2a^2z^{-2}+\cdots Na^N z^{-N}$$

Clearly, the ROC is $|z|>0$.

Considering $X(z)=X_1(z)+X_2(z)+\cdots+X_N(z)$ where

$$X_1(z)=az^{-1}+a^2z^{-2}+\cdots a^N z^{-N}=\frac{az^{-1}(1-(az^{-1})^N)}{1-az^{-1}}$$
$$X_2(z)=a^2z^{-2}+a^3z^{-3}+\cdots a^N z^{-N}=\frac{a^2z^{-2}(1-(az^{-1})^{N-1})}{1-az^{-1}}$$
……………………………….
$$X_N(z)=a^N z^{-N}=\frac{a^N z^{-N}(1-az^{-1})}{1-az^{-1}}$$

As a result, we have:

$$\begin{aligned}
X(z) &= \frac{az^{-1}(1-(az^{-1})^N)}{1-az^{-1}}+\frac{a^2z^{-2}(1-(az^{-1})^{N-1})}{1-az^{-1}}+\cdots+\frac{a^N z^{-N}(1-az^{-1})}{1-az^{-1}}\\
&= \frac{az^{-1}+a^2z^{-2}+\cdots+a^N z^{-N}-N(az^{-1})^{N+1}}{1-az^{-1}}\\
&= \frac{\frac{az^{-1}(1-(az^{-1})^N)}{1-az^{-1}}-N(az^{-1})^{N+1}}{1-az^{-1}}\\
&= \frac{az^{-1}-(N+1)(az^{-1})^{N+1}+N(az^{-1})^{N+2}}{(1-az^{-1})^2},\quad |z|>0
\end{aligned}$$

Alternatively, you can find $X(z)$ by first expressing $x[n]$ as:

$$x[n]=na^n(u[n-1]-u[n-N+1])$$

and then make use of Table 8.1 and time shifting property.

8.(a)
If the system is stable, then the ROC for $H(z)$ is $0.5<|z|<3$. On the other hand, for the unit step input, we have:

$$X(z)=\frac{1}{1-z^{-1}},\quad |z|>1$$

The z transform for $y[n]$ is $Y(z)=H(z)X(z)$. Using partial fraction expansion, we get:

$$Y(z) = H(z)X(z) = \frac{0.8}{1-0.5z^{-1}} + \frac{0.2}{1-3z^{-1}} - \frac{1}{1-z^{-1}}, \quad 1 < |z| < 3$$

Taking the inverse z transform, we get:

$$y[n] = (0.8)(0.5)^n u[n] - 0.2(3)^n u[-n-1] - u[n]$$

8.(b)
If the system is causal, then the ROC for $H(z)$ is $|z| > 3$. For $x[n] = \delta[n]$, $y[n] = h[n]$. Using partial fraction expansion, we get:

$$Y(z) = H(z) = \frac{-0.4z^{-1}}{1-0.5z^{-1}} + \frac{0.4z^{-1}}{1-3z^{-1}}, \quad |z| > 3$$

Taking the inverse z transform, we get:

$$y[n] = -(0.4)(0.5)^{n-1}u[n-1] + 0.4(3)^{n-1}u[n-1]$$

9.
The z transforms of $x[n] = u[-n-1]$ and $h[n] = (0.5)^n u[n]$ are:

$$X(z) = -\frac{1}{1-z^{-1}}, \quad |z| < 1 \qquad \text{and} \qquad H(z) = \frac{1}{1-0.5z^{-1}}, \quad |z| > 0.5$$

So we have:

$$Y(z) = -\frac{1}{1-z^{-1}} \cdot \frac{1}{1-0.5z^{-1}} = \frac{-2}{1-z^{-1}} + \frac{1}{1-0.5z^{-1}}, \quad 0.5 < |z| < 1$$

Taking the inverse z transform yields:

$$y[n] = 2u[-n-1] + (0.5)^n u[n]$$

10.(a)
In a denoising system, the input contains a signal-of-interest and additive noise, and the system attempts to extract the signal-of-interest while removing/suppressing the noise, thus it is expected that the output is a good approximation of the signal-of-interest. The principle is that, under a certain transform, the signal-of-interest is sparse, meaning that there are only a few non-zero entries and we only need to keep them and ignoring the rest.

10.(b)
In theory, we can transform the continuous-time signal to frequency domain using Fourier transform. The $\cos(100\pi t)$ corresponds to two impulses at -100π and 100π in the frequency domain. We keep only the two components at -100π and 100π, and assign the rest to zero, and convert this resultant frequency-domain signal to time domain using the inverse Fourier transform.

Compression is easily achieved. Instead of storing $\cos(100\pi t)$, we only need to store the amplitudes and locations of the two impulses in the frequency-domain signal.