On the impedance boundary condition

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(Received September 1981)

The use of the impedance boundary condition (IBC) provides a useful means of treating the problem of scattering by a material object and rough surfaces. A systematic development of the subject is presented. Alternative forms of the IBC are given and their range of validity is discussed. A special form is used in order to simplify the analysis of diffraction by a metallic sheet and its advantages are illustrated.

Key words: impedance boundary condition, diffraction, mathematical model

Introduction

The problem of scattering of waves by a material object placed in homogeneous media has many practical applications but its exact solution is known in only a few cases. The problem is complicated due to the need to find the field expansion both inside and outside the scatterer prior to the application of the boundary conditions. In the integral equation approach to the problem, in general volume integrals rather than surface integrals are encountered. This puts some constraints on the practical dimensions of the scatterer which can be handled by application of the numerical techniques.

When an electromagnetic field is incident on the object, considerable simplication results in the analysis by using the impedance boundary conditions (IBC). When these boundary conditions are applicable, they avoid the need to calculate the fields within the body. Hence, for the integral equation approach only surface integrals are involved.

The fundamental work on the topic is due to Rytov[1] and Leontovich[2] who used an asymptotic series representation of the field in order to derive the IBC. A simplified analysis for large refractive index was used by Senior[3] to show the exact electromagnetic boundary conditions can be approximated in order to yield the IBC. Mitzner[4] used the integral equation approach to develop and extend the Leontovich boundary condition.

In view of the importance of the subject, this communication is concerned with the different possible representations of the IBC and their range of validity. The case of scattering by a metallic sheet is considered and a specially convenient form of the boundary condition is given. This leads to a considerable simplification of the analysis.

Exact vectorial impedance identity

Aboul-Atta and Boerner[5] give an exact vectorial impedance relation at the scattering surface. The exact description of the harmonic electromagnetic field solution at the surface is shown, in general, to require two impedances to relate the tangential electric and magnetic field. For a nontrivial magnetic field solution for which the surface electric current $\boldsymbol{J} \neq 0$ over the entire surface, they choose the two linearly independent tangential vectors $\boldsymbol{J}$ and $\boldsymbol{J}^* \times \hat{n}$, where * denotes the complex conjugate, and $\hat{n}$ is an outward unit vector normal to the surface, as basis functions. These two functions are orthogonal and of equal norm, hence they are linearly independent. Consequently, the tangential electric field $\boldsymbol{M} = \boldsymbol{E} \times \hat{n}$, where $\boldsymbol{M}$ usually termed the magnetic current, can be represented by:[5]

$$\hat{n} \times \boldsymbol{M} = \zeta \boldsymbol{J} + \xi^* \boldsymbol{J}^* \times \hat{n} \tag{1}$$

where:

$$\zeta = -\hat{n} \cdot (\boldsymbol{E} \times \boldsymbol{H}^*)/\|\boldsymbol{J}\|^2$$
$$\xi^* = -\boldsymbol{M} \cdot \boldsymbol{J}/\|\boldsymbol{J}^* \times \hat{n}\|^2 \tag{2}$$

and the norm $\|\alpha\|$ of α is defned by:

$$\|\alpha\|^2 = \alpha \cdot \alpha^* \tag{3}$$

Equation (1) is called the vectorial impedance boundary condition (VIBC).

By vector product of $\hat{n}$ with equation (1) and the resubstitution of $\boldsymbol{J}^*$, one obtains the corresponding admittance identity:

$$\boldsymbol{J} \times \hat{n} = [\zeta^* \boldsymbol{M} + \xi^* \hat{n} \times \boldsymbol{M}^*]/[|\zeta|^2 + |\xi|^2] \tag{4}$$

In order to estimate the relative order of the magnitudes of ζ and ξ and consequently evaluate their significance, we use equation (4) to write:

$$\boldsymbol{M} = \frac{|\zeta|^2 + |\xi|^2}{\zeta^*}\left[\boldsymbol{J} \times \hat{n} - \frac{\xi^*}{|\zeta|^2 + |\xi|^2}\,\hat{n} \times \boldsymbol{M}^*\right] \tag{5}$$

Multiplying both sides scalarly with $\boldsymbol{J}$, we obtain:

$$\zeta^*/\xi^* = (\boldsymbol{M}^* \times \hat{n}) \cdot \boldsymbol{J}/(\boldsymbol{M} \cdot \boldsymbol{J}) \equiv \boldsymbol{M}^* \cdot (\hat{n} \times \boldsymbol{J})/(\boldsymbol{M} \cdot \boldsymbol{J}) \tag{6}$$

The Leontovich boundary condition

The subset of the VIBC given by equations (1) and (5) for $\xi^* = 0$ may be called the normal impedance boundary condition. The function ζ may then, under certain restrictions, be related to the physical properties of the scatterer.[2]

0307-904X/82/050405-03/$03.00

In such a case, equation (3) may be written in the form:

$$M = Z_c J \times \hat{n} \tag{7}$$

which is frequently referred to as the Leontovich boundary condition. The surface impedance Z_c is equal to the wave impedance in the conductor. This can be expressed in terms of the constitutive parameters of the conductor material as:

$$Z_c = \sqrt{\mu^*/\epsilon^*} = \sqrt{(\mu + i\tau/\omega)/(\epsilon + i\sigma/\omega)} \tag{8}$$

where μ^* and ϵ^* are the general permeability and permittivity of the conductor's material and a time dependence $\exp(-i\omega t)$ is assumed. It follows from equation (7) or (3) that:

$$J = Z_c^{-1} \hat{n} \times M \tag{9}$$

Equations (7) and (9) show that the Leontovich boundary condition requires the existence of nonzero magnetic as well as electric currents, but with the two currents trivially related.

When the surface impedance is represented by a tensor surface impedance $\tilde{Z}$, the boundary condition may then be written in the form:

$$M = -J \times (\tilde{Z} \times \hat{n}) \tag{10}$$

The Leontovich boundary condition is valid under the conditions that:[4]

(i) The absolute value of the complex refractive index (and of its imaginary part) of the material of the scatterer relative to free space must be large compared to unity.
(ii) The radii of curvature of the surface of the body must be large compared to both the penetration depth of the inward travelling field and the inverse of the absolute value of the propagation constant inside the body.

Condition (ii) may be clarified by considering the case when ϵ and τ are negligible and $|Z_c|$ is less than the free space impedance Z_0. Then, the most significant parameter characterizing the propagation in the conductor is the skin depth:

$$\delta = \sqrt{2/(\omega\mu\sigma)} \tag{11}$$

in terms of which, the wave impedance is given by:

$$Z_c \cong (1 - i)\, \omega\mu\delta/2 \tag{12}$$

One of the requirements for the validity of the Leontovich boundary condition is that the radii curvature of the body by large compared to δ. Let u and v be the principal curvature coordinates so oriented that:

$$\hat{v} \times \hat{u} = \hat{n} \tag{13}$$

where $\hat{u}$ and $\hat{v}$ are the associated unit vectors. A modification to treat smaller radii of curvature has been given by Mitzner[4] based on the integral equation approach. The modified boundary condition is:

$$M_u = -(1 - p) Z_c J_v \quad M_v = (1 + p) Z_v J_u \tag{14}$$

where $p = (1 + i)(\kappa_v - \kappa_u)\delta/4$, κ_v and κ_u are the principal curvatures, defined as negative when $\hat{n}$ points toward the centre of curvature. Then κ is positive whenever the conducting body is convex.

The Leontovich condition is valid to within errors of $0(\delta^2 k_0^2)$, $0(\delta^2/h^2)$ and $0(\delta\kappa)$ where k_0 is the wave number of the medium outside the conductor, h is the distance to the nearest significant source and κ is the larger of $|\kappa_u|$ and $|\kappa_v|$. On the othe hand, the curvature dependent boundary condition given by equation (14) is valid to within errors of $0(\delta^2 k_0^2)$, $0(\delta^2/h^2)$ and $(\delta^2\kappa^2)$.

A much more accurate relation between the surface magnetic and electric currents is also available.[4] However, this relation is more complicated since it includes surface integrals of these currents.

Application to the diffraction by a metallic half-plane at oblique incidence

The impedance boundary condition generally couples the tangential components of the electric and magnetic fields. This consequently complicates the analysis. The search for situations under which the condition reduces to a decoupled set of the components is justified in view of the expected simplicity of the analysis. In particular, we consider the case of the diffraction of an obliquely incident wave by a half plane.

The problem was treated by Senior[6] in terms of four functions which represent the surface electric and magnetic currents. The direct application of the IBC then led to four coupled Wiener-Hopf integral equations. Williams[7] showed that it is possible to deduce the solution for the three-dimensional incident field from the solution for the two-dimensional field. He presented the field in terms of axial electric and magnetic Hertz vectors. He then showed that certain linear combinations of the derivatives of these auxiliary vectors are directly related to the solutions of the corresponding two-dimensional problems.

Alternatively, the field may be expressed in terms of two of its components which satisfy simple decoupled boundary conditions. To illustrate this statement, let the metallic sheet occupy $y = 0, x \geqslant 0$ and z along the edge. If the upper surface of the sheet is denoted by S_1 and the lower by S_2, then the IBC to be applied are:

$$E_x = Z_c H_z \quad E_z = -Z_c H_x \text{ on } S_1 \tag{15a}$$

$$E_x = -Z_c H_z \quad E_z = Z_c H_x \text{ on } S_2 \tag{15b}$$

Since the sheet is uniform in the z-direction, the field dependence on z, which is assumed to be $\exp(-ik_0 z \cos\beta)$ where β is the angle of incidence with respect to the z-direction, is preserved. Invoking this condition and using Maxwell's equations in Cartesian coordinates, we find that:

$$\begin{aligned} \partial Ey/\partial y &= -i\omega\epsilon_0 Z_c E_y \\ \partial Hy/\partial y &= -i\omega\mu_0 Z_c^{-1} H_y \text{ on } S_1 \end{aligned} \tag{16a}$$

$$\begin{aligned} \partial Ey/\partial y &= i\omega\epsilon_0 Z_c E_y \\ \partial Hy/\partial y &= i\omega\mu_0 Z_c^{-1} H_y \text{ on } S_2 \end{aligned} \tag{16b}$$

Thus, the original boundary value problem is reduced to two independent boundary value problems for Ey and Hy. These latter problems are of a type whose solution is known. It may be of interest to note that the auxiliary functions f_1 and f_2 introduced by Williams[7] are actually Ey and Hy, respectively. An advantage in recognizing that these functions are field components is the direct application of the physical conditions on these components and the derivation of other field components from them.

Conclusion

The impedance boundary condition is a useful approximation which has a direct effect on simplifying the anaysis of scattering by material objects and reducing the computational efforts when a numerical approach is used. The errors in the approximations of the actual boundary conditions were discussed in this work. The accuracy was previously investigated by Alexopoulos and Tadler[8] in comparison with the exact solution. For a circular cylinder, the error was found to be largest at resonance.

As pointed out earlier, the nonezero surface impedance Z_c accounts for the finite conductivity of the scatterer. On the other hand, the IBC can be applied when the surface is perfectly conducting but rough. In this case, Z_c is related to the roughness characteristics.[9] The surface impedance may also be used to account for the presence of highly absorbing coating layers,[10] or for an overdense plasma.

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