S&P 500: Portfolio Optimisation via Factor Analysis

Introduction: Modern Portfolio Theory

Modern Portfolio Theory (firt proposed by economist **Harry Markowitz**) draws on the logical idea that, given the choice between two portfolios with the same expected return, you would prefer the portfolio with the smallest variance. In this context, the variance of a portfolio is a direct metric for the portfolio's "risk".

Definition: A *Portfolio* of financial assets is defined as a set of weights $\boldsymbol{w} = (w_1, \ldots, w_p)$ such that $\boldsymbol{w}^T \mathbf{1}_p \leq 1$, where $\mathbf{1}_p$ is the p-length vector of 1's. This definition allows for *short-selling* (where an investor hold a negative quantity of a financial asset). In some cases the additional constraint that $w_i \geq 0$, $\forall i = 1 \ldots p$ which prevents short selling.

The idea of MPT boils down to a constrianed optimisation problem, taking the form

$$\max_{\boldsymbol{w}} \boldsymbol{w}^{\boldsymbol{T}} \boldsymbol{\mu}$$

$$\text{s.t. } \boldsymbol{w}^{\boldsymbol{T}} \boldsymbol{\Sigma} \boldsymbol{w} \leq \sigma_0^2, \quad \boldsymbol{w}^T \mathbf{1}_p \leq 1$$

Where $\boldsymbol{\mu}, \boldsymbol{\Sigma}$ are the mean and covariance of the returns respectively. That is we look for a portfolio that delivers the maximum expected return such that the variance of the portfolio does not exceed σ_0^2. The choice of σ_0^2 quantifies an investors aversion to risk.

In this form (i.e. no constraints on short-selling) there is a well known analytical solution to this optimization problem *see Proposition 2.1* (Bai, Liu, and Wong 2009):

$$\text{If } \sigma_0 B \leq \sqrt{A}, \text{ then } R = \sigma_0 \sqrt{A} \text{ and } \boldsymbol{w} = \frac{\sigma_0}{\sqrt{A}} \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}$$

$$\text{If } \sigma_0 B > \sqrt{A}, \text{ then } R = \frac{B}{C} + D(A - \frac{B^2}{C}) \text{ and } \boldsymbol{w} = \frac{1}{C} \boldsymbol{\Sigma}^{-1} \mathbf{1}_p + D(\boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} - \frac{B}{C} \boldsymbol{\Sigma}^{-1} \mathbf{1}_p)$$

Where R denotes the expected return of the portfolio, $A = \boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}$, $B = \mathbf{1}_p^T \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}$, $C = \mathbf{1}_p^T \boldsymbol{\Sigma}^{-1} \mathbf{1}_p$ and $D = \sqrt{\frac{C\sigma_0^2 - 1}{AC - B^2}}$.

Factor Analysis for Portfolio Optimisation

References

Bai, Zhidong, Huixia Liu, and Wing-Keung Wong. 2009. "ENHANCEMENT of the Applicability of Markowitz'S Portfolio Optimization by Utilizing Random Matrix Theory." *Mathematical Finance* 19 (4): 639–67. https://doi.org/10.1111/j.1467-9965.2009.00383.x.