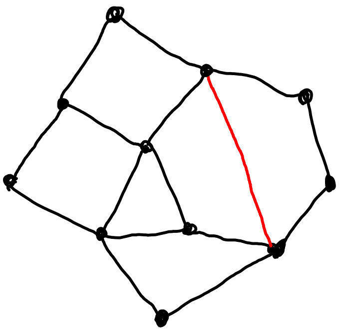
Polygons and Pick's Theorem:

Theorem 2.1.1. If a connected plane graph has n vertices, e edges and f faces, then



$$n - e + f = 2.$$

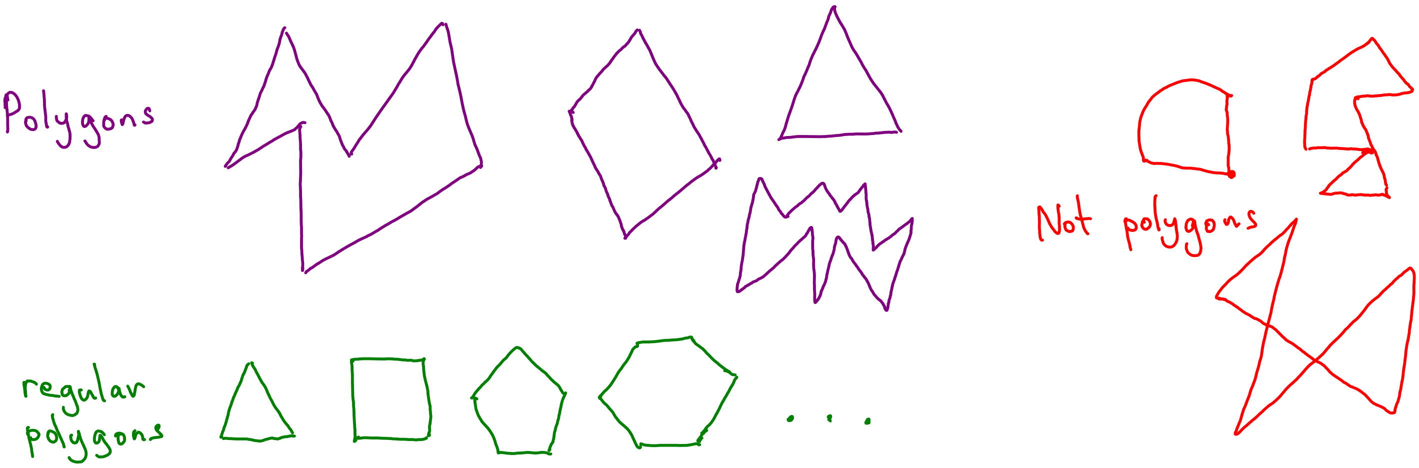
Euler's Formula.

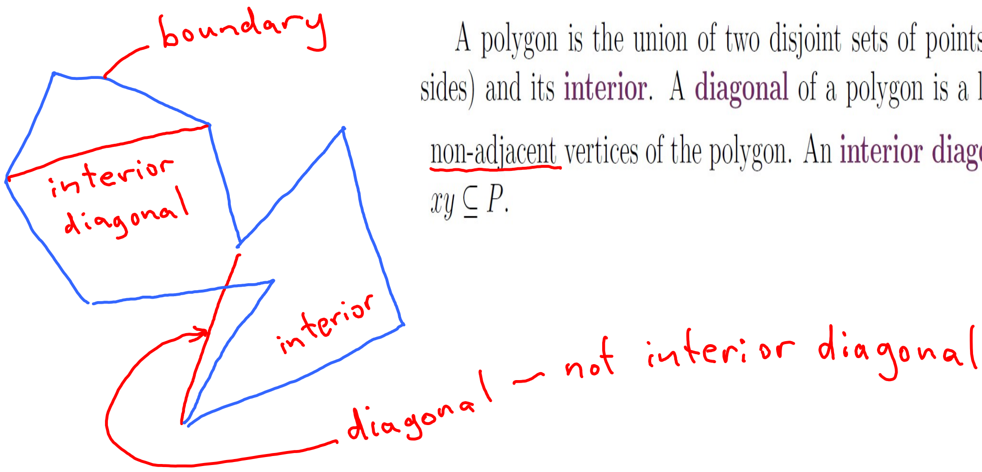
$n \longrightarrow n$

$e \longrightarrow e+1$

$f \longrightarrow f+1$

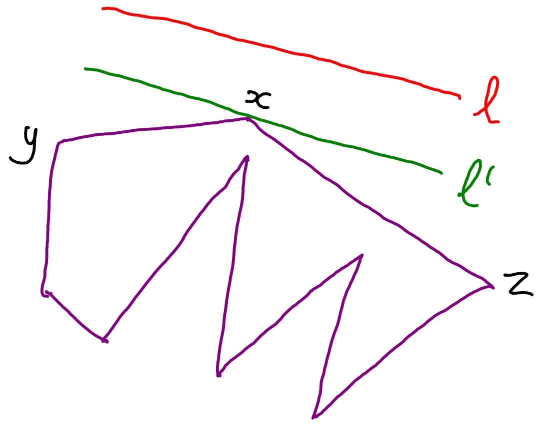
A **polygon** is a 2-dimensional region whose boundary is a simple closed curve which consists of straight line segments. These line segments are the polygon's **sides** or **edges**, and their endpoints are the polygon's **vertices**. A polygon with n sides, and hence also n vertices, is called an **n-gon**.





A polygon is the union of two disjoint sets of points: its **boundary** (which is the union of its sides) and its **interior**. A **diagonal** of a polygon is a line segment xy where x and y are distinct non-adjacent vertices of the polygon. An **interior diagonal** of polygon P is a diagonal xy such that $xy \subseteq P$.

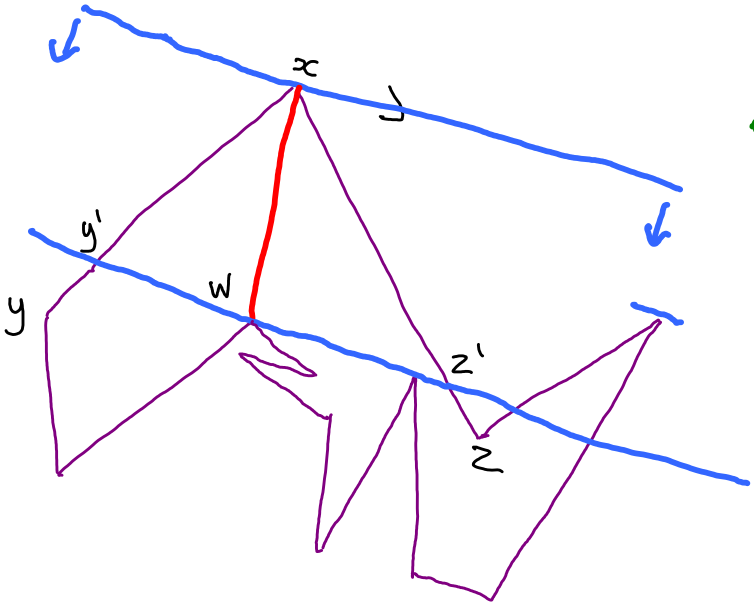
Theorem 2.1.2. For $n \geq 4$, every n-gon has an interior diagonal.



Let P be a polygon

$\exists \ell$, $\ell \cap P = \emptyset$, ℓ not parallel to sides, diagonals
(P finite)

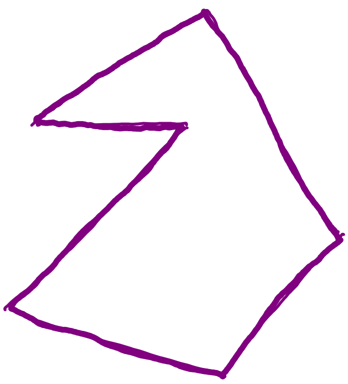
$\exists \ell'$, ℓ' parallel to ℓ, $\ell' \cap P$ is a vertex x.



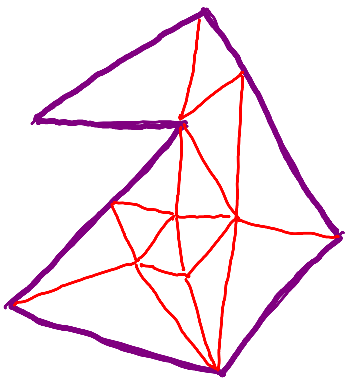
- if yz is interior diagonal ✓
- otherwise, take line through x parallel to yz, and move it towards yz until first vertex w in Δxyz is encountered.
- wx is interior diagonal.

A **triangulation** of a polygon P is a set $\mathcal{T}$ of triangles such that

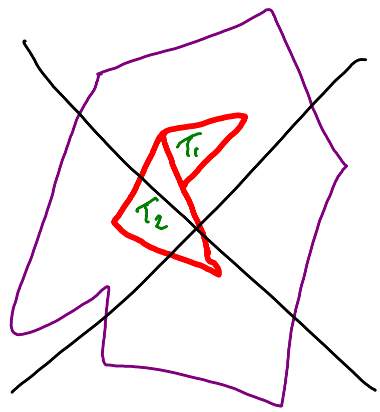
- $\bigcup_{T \in \mathcal{T}} T = P$,
- for all distinct $T_1, T_2 \in \mathcal{T}$, $T_1 \cap T_2$ is either empty, or is a side or vertex of both T_1 and T_2.



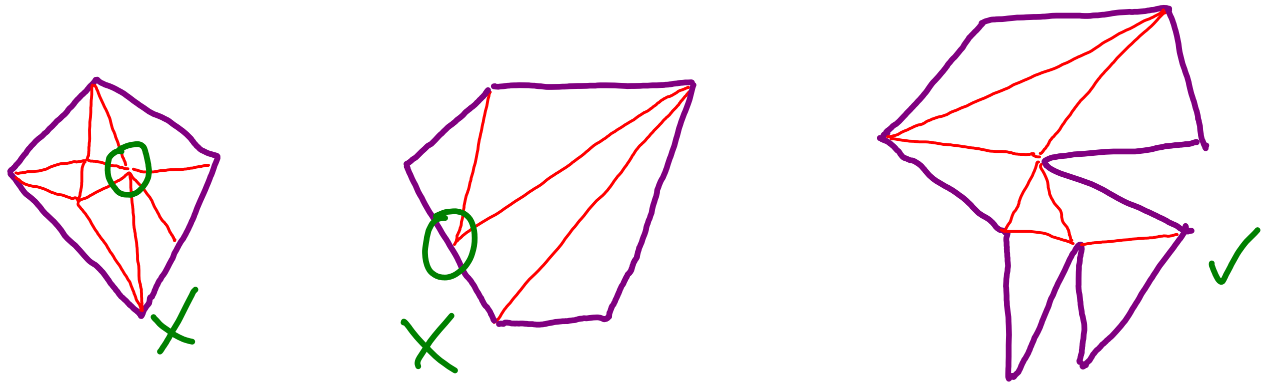
P



triangulation of P.

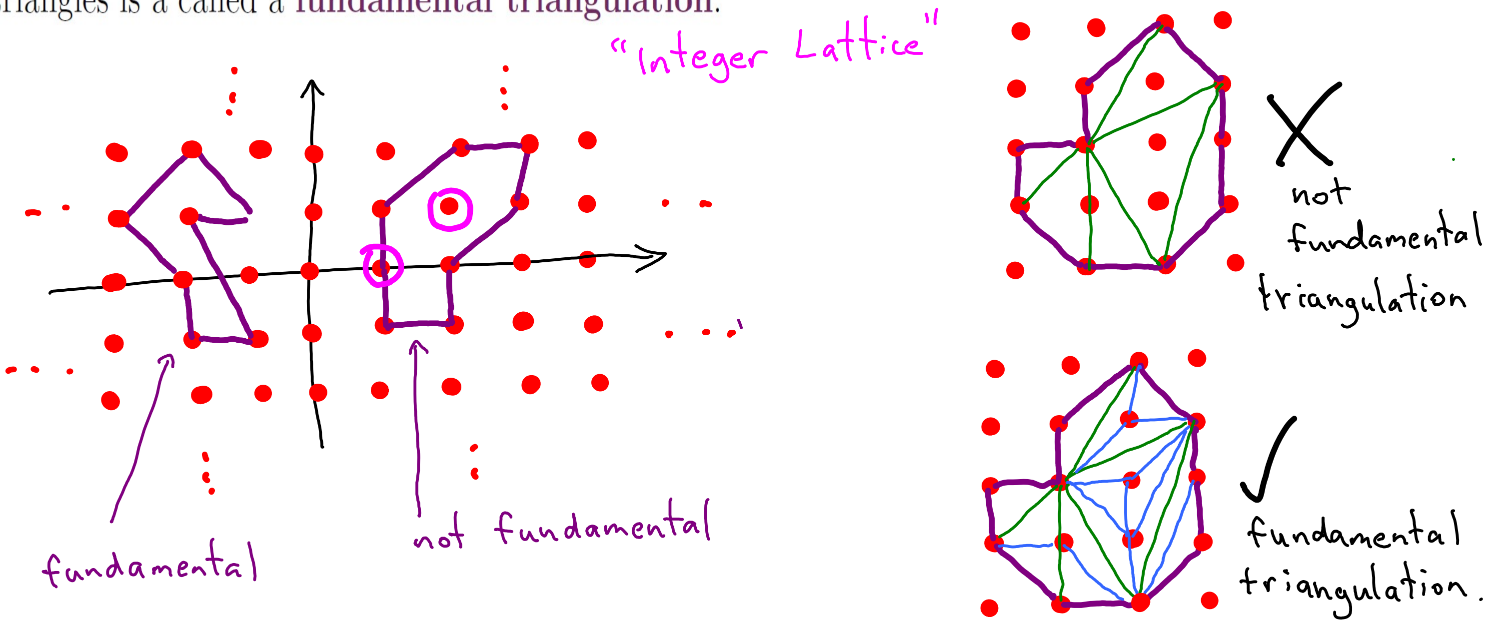


Theorem 2.1.4. Every polygon has a triangulation in which the vertices of the triangles are vertices of the polygon.



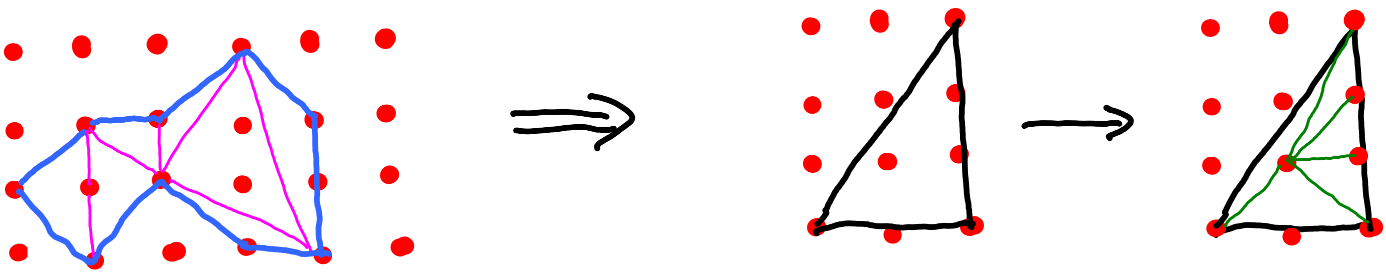
Proof: Keep choosing interior diagonals.

A point in the plane is a **lattice point** if its coordinates are integers. A polygon P is a **lattice polygon** if all of its vertices are lattice points. A lattice polygon containing no lattice points other than its vertices is called **fundamental**, and a triangulation consisting entirely of fundamental triangles is a called a **fundamental triangulation**.

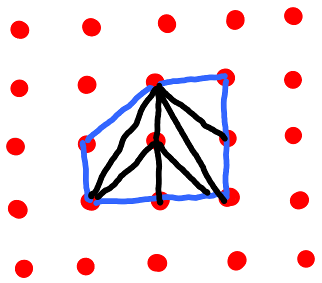


Theorem 2.1.5. Every lattice polygon has a fundamental triangulation.

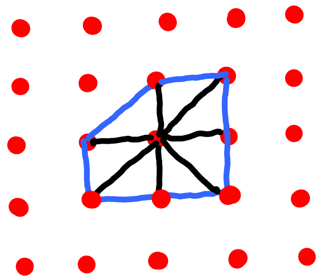
Proof: Thm 2.1.4 $\longrightarrow$ triangulation with lattice triangles.



Theorem 2.1.6. The number of triangles in a fundamental triangulation of a lattice polygon P is $b + 2i - 2$ where b is the number of lattice points on the boundary of P and i is the number of lattice points in the interior of P.



$7 + 2 - 2 = 7$ ✓



Planar graph

Proof:

$$\underbrace{\sum_{\text{faces}} \#\,\text{edges on boundary of face}}_{3x+b \;=\; 2e} \quad \text{(includes exterior face)}$$

$(x = \#\,\text{triangles})$

Euler's formula.

$$\left.\begin{aligned} v &= b+i \\ e &= \frac{3x+b}{2} \\ f &= x+1 \end{aligned}\right\} \Longrightarrow (b+i) - \frac{3x+b}{2} + (x+1) = 2$$

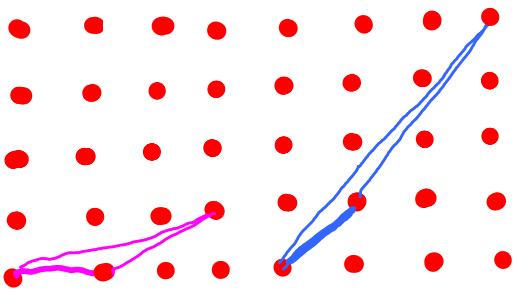
$$2b + 2i - 3x - b + 2x + 2 = 4$$

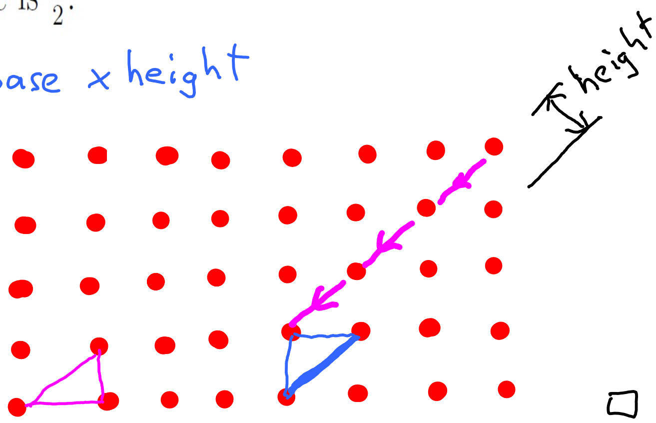
$$\underline{\underline{x = b + 2i - 2}}$$

□

Theorem 2.1.7. The area of a fundamental triangle is $\frac{1}{2}$.

$\text{Area} = \frac{1}{2}\,\text{base} \times \text{height}$





□

Theorem 2.1.8. (Pick's Theorem, 1899) If a lattice polygon has b lattice points on its boundary and i lattice points in its interior, then its area is $\frac{1}{2}b + i - 1$.

Proof Let P be a lattice polygon having b lattice points on its boundary and i lattice points in its interior. By Theorem 2.1.5, P has a fundamental triangulation, and by Theorem 2.1.6, the number of triangles is $b + 2i - 2$. Since each fundamental triangle has area $\frac{1}{2}$ (Theorem 2.1.7), the area of P is $\frac{1}{2}b + i - 1$. □