

# Math3303 Assignment 5

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## Q1

Let  $R$  be an integral domain such that  $(a_0) \supseteq (a_1) \supseteq (a_2) \supseteq \dots$  implies that  $(a_n) = (a_{n+1}) = \dots$  for  $n$  sufficiently large. Now we want to show that  $R$  is a field.

*Proof.* As  $R$  is an integral domain for  $R$  to also be a field we require that every element of  $R$  has a multiplicative inverse. We will show that every element has a multiplicative inverse via a proof by contradiction. Suppose  $a_0$  is not a unit. We also have that  $(a_n) = (a_{n+1})$  for  $n$  sufficiently large. This means that  $a_n \in (a_{n+1})$ , so we have that for some  $r \in R$  (noting that we have commutativity and distributivity as  $R$  is an integral domain):

$$\begin{aligned} a_n &= ra_{n+1} \\ \iff a_n - ra_{n+1} &= 0 \\ \iff a_n - ra_n a_0 &= 0 \\ \iff a_n - a_n r a_0 &= 0 \\ \iff a_n(1 - ra_0) &= 0 \end{aligned}$$

As  $R$  is an integral domain this means that  $1 = ra_0$ , i.e.  $a_0$  is a unit. Thus we have arrived at the contradiction.  $\square$

## Note on Q2 and Q3

In Q2 and Q3 we will often consider the partial ordering of ideals of a commutative ring  $R$  with identity, where the ordering is given by  $\subseteq$ . It is not hard to see that the zero ideal is a subset of all other ideals (an ideal in a commutative ring requires  $rx = xr \in I$ ,  $\forall x \in I$  and  $\forall r \in R$  and as  $0 \in R$  we have that  $0$  must always be in an ideal). We also have by theorem 9.22 from Gregory Lee's abstract algebra that every maximal ideal in this ring is a prime ideal and we also note that each of these ideals is a subset of the ring itself. Thus we see that ideals with subset ordering form a finite partially ordered set with all elements bounded above by  $R$  and below by the zero ideal.

## Q2

Let  $I$  be an ideal in a commutative unital ring  $R$ . Define

$$\hat{I} := \{r \in R \mid r^n \in I \text{ for some } n \in \mathbb{Z} > 0\}.$$

We first prove that for  $S = \{r, r^2, \dots\}$  and  $I$ , an ideal disjoint from  $S$ , i.e.  $r^n \notin I$  for any  $n$ , there is a prime ideal that contains  $I$  and is disjoint from  $S$ .

*Proof.* First note that a prime ideal ( $\mathcal{P}$ ) must satisfy 1)  $\mathcal{P} \neq R$  and 2) if  $a, b \in R$  and  $ab \in \mathcal{P}$  then  $a \in \mathcal{P}$  or  $b \in \mathcal{P}$ . In this proof we will show the equivalent condition of 2),  $a \notin \mathcal{P}$  and  $b \notin \mathcal{P}$  then  $ab \notin \mathcal{P}$ . As  $r^n \notin I$  we must have  $I \neq R$ . Now from Gregory Lee's Abstract Algebra theorem 9.22 we have that every maximal ideal of  $R$  is also a prime ideal. This means that if we partially order the ideals from  $I$  with the order being given by  $\subseteq$ ,  $\exists \mathcal{P}$  such that  $\mathcal{P} \geq I$ . So we now have condition 1 as each chain from  $I$  is bounded above by a maximal prime ideal. If one of these maximal prime ideals does not intersect with  $S$  we are done, however this not guaranteed so we now consider the subset of ideals in the partial ordering that don't intersect with  $S$ . We know that this subset of the partial ordering is non-empty as  $r^n \notin I$ . From this we now consider an ideal  $Q$  that is maximal in a chain of this subset. As this subset is finite and non-empty we know that such a  $Q$  exists. If we consider two ideals  $A$  and  $B$  that follow from  $Q$  in the partial ordering we must have  $r^i \in A$  and  $r^j \in B$  for some  $i, j \in \mathbb{Z}_{>0}$ . Now if we consider a third ideal  $C$  with  $r^{i+j} \in C$  we must have  $A \subseteq C$  and  $B \subseteq C$ . This now gives  $r^i \notin Q$  and  $r^j \notin Q$  then  $r^{i+j} \notin Q$ , thus condition 2) is satisfied and  $Q$  is a prime ideal containing  $I$ .  $\square$

We now want to show that  $\hat{I}$  equals the intersection of all prime ideals of  $R$  which contain  $I$ .

*Proof.* First let  $\mathcal{P}$  be some prime ideal containing  $I$ . We will prove this by proving two relations:

$$(1) \hat{I} \subseteq \bigcap_{I \leq \mathcal{P}} \mathcal{P}$$

$$(2) \hat{I} \supseteq \bigcap_{I \leq \mathcal{P}} \mathcal{P}$$

Where  $\bigcap_{I \leq \mathcal{P}} \mathcal{P}$  is the intersect of all prime ideals containing  $I$ .

(1)

If  $\mathcal{P}$  is some prime ideal containing  $I$  and we have some  $r \in R$  such that  $r^n \in I$ , then as  $I \leq \mathcal{P}$  we have  $r^n \in \mathcal{P}$  and as  $\mathcal{P}$  is a prime ideal we also must have  $r \in \mathcal{P}$ .

(2)

Now if we consider  $r \notin \hat{I}$ , then  $r^n \notin I$  for any  $n$ , so  $S = \{r, r^2, \dots\}$  is a set

disjoint from  $I$ . From what we first proved we know that there is a prime ideal  $Q$  containing  $I$  with  $r \notin Q$ , thus we have  $r \notin \bigcap_{I \subseteq \mathcal{P}} \mathcal{P}$ .

These two relations immediately imply that  $\hat{I} = \bigcap_{I \subseteq \mathcal{P}} \mathcal{P}$ . □

### Q3

a)

$V(I) = \emptyset$  if and only if  $I = R$ .

*Proof.* First consider the case  $I = R$ .

We then have  $V(I) = V(R)$  and as there are no prime ideals that contain the entire ring ( $R$  is not a prime ideal as we require that a prime ideal isn't equal to  $R$ ) we must have  $V(I) = \emptyset$ .

Now consider the case  $I \neq R$ .

We know from theorem 9.22 of Gregory Lee's Abstract Algebra that every maximal ideal in  $R$  is a prime ideal. Now partially ordering the ideals with the ordering given by  $\subseteq$  we see that all ideals  $I$  such that  $I \neq R$  must be bounded above by a prime ideal (noting that an ideal can contain itself as ideal). Thus we always have  $V(I) \neq \emptyset$ . □

b)

$V(I) \cup V(J) = V(IJ)$

*Proof.* We first note the partial ordering we have constructed and the fact from Gregory Lee's Abstract Algebra page 151 that by absorption property,  $IJ \subseteq I \cap J$  and that if we have ideals  $A$  and  $B$  such that  $A \subseteq B$  we must have  $V(A) \supseteq V(B)$  as all prime ideals contained in  $B$  must also be contained in  $A$  due to the partial ordering.

(1)  $V(IJ) \supseteq V(I) \cup V(J)$ :

If we let  $IJ \subseteq I \cap J = X$  we must have  $X \subseteq I$ ,  $X \subseteq J$  and  $\mathbf{0} \subseteq X$  (This is from the partial ordering we constructed (the zero ideal must be contained in  $X$ )). This then gives  $V(IJ) \supseteq V(X)$ ,  $V(X) \supseteq V(I)$  and  $V(X) \supseteq V(J)$ . Now we see that we have (1),  $V(IJ) \supseteq V(I) \cup V(J)$ .

(2)  $V(IJ) \subseteq V(I) \cup V(J)$ :

Now if we take a prime ideal  $\mathcal{P} \in V(IJ)$ , we want to show  $\mathcal{P} \in V(I)$  or  $\mathcal{P} \in V(J)$ . Suppose  $IJ \subseteq \mathcal{P}$  and  $I$  is not contained in  $\mathcal{P}$ . We now show that for all  $j \in J$ , we have  $j \in \mathcal{P}$ . Fix  $j \in J$  and  $i \in I \setminus \mathcal{P}$  and note that  $ij \in IJ$ . Since  $IJ \subseteq \mathcal{P}$ , we have that  $ij \in \mathcal{P}$  but  $\mathcal{P}$  is prime, so we must have that either  $i \in \mathcal{P}$  or  $j \in \mathcal{P}$ . Since  $i \notin \mathcal{P}$  (This is because  $i \in I \setminus \mathcal{P}$ ), we conclude that  $j \in \mathcal{P}$ . This shows that  $J \subseteq \mathcal{P}$ . The argument is similar if we assume that  $I$  is not contained

in  $\mathcal{P}$ . In that case we get that  $I \subseteq \mathcal{P}$ . So if we have a prime ideal  $\mathcal{P} \in V(IJ)$ , then  $\mathcal{P} \in V(I)$  or  $\mathcal{P} \in V(J)$ . This then implies (2),  $V(IJ) \subseteq V(I) \cup V(J)$ .

From (1) and (2) we must have equality which is what we wanted to show.  $\square$

**c)**

Let  $\{I_\alpha\}$  be a set of ideals of  $R$ . Then  $\cap_\alpha V(I_\alpha) = V(\sum_\alpha I_\alpha)$ .

*Proof.*  $\square$