

Stat3004 Assignment 3

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Q1

First we note that $N((a, b]) \sim \text{Poi}(2 \cdot (b - a))$.

a)

$$\begin{aligned}\mathbb{P}(N_{t_1} = n_1, N_{t_2} = n_2) &= \mathbb{P}(N_2 = 7, N_{12} = 10) \\ &= \mathbb{P}(N_2 = 7, N_{12-2} = 10 - 7) \\ &= \mathbb{P}(N_2 = 7) \mathbb{P}(N_{10} = 3) \\ &= e^{-2 \cdot 2} \frac{(2 \cdot 2)^7}{7!} e^{-2 \cdot 10} \frac{(2 \cdot 10)^3}{3!} \\ &\approx 1.636291121 \times 10^{-7}\end{aligned}$$

b)

$$\begin{aligned}\mathbb{P}(N(1, t_1] = n_1, N(t_1 - 1, t_2] = n_2) &= \mathbb{P}(N(1, 2] = 7, N(1, 12] = 10) \\ &= \mathbb{P}(N(1, 2] = 7, N(2, 12] = 3) \\ &= \mathbb{P}(N(1, 2] = 7) \mathbb{P}(N(2, 12] = 3) \\ &= \frac{e^{-2}(2)^7}{7!} \frac{e^{-20}(20)^3}{3!} \\ &\approx 9.4458178806 \times 10^{-9}\end{aligned}$$

c)

$$\begin{aligned}
\mathbb{E}[N(1, t_1) \mid N(t_1 - 1, t_2) = n_2] &= \mathbb{E}[N(1, 2) \mid N(1, 12) = 10] \\
&= \sum_{t \geq 0} t \cdot \mathbb{P}(N(1, 2) = t \mid N(1, 12) = 10) \\
&= \sum_{t=0}^{10} t \cdot \frac{\mathbb{P}(N(1, 2) = t, N(1, 12) = 10)}{\mathbb{P}(N(1, 12) = 10)} \\
&= \frac{1}{\mathbb{P}(N(1, 12) = 10)} \sum_{t=0}^{10} t \cdot \mathbb{P}(N(1, 2) = t, N(2, 12) = 10 - t) \\
&= \frac{1}{\mathbb{P}(N(1, 12) = 10)} \sum_{t=0}^{10} t \cdot \mathbb{P}(N(1, 2) = t) \mathbb{P}(N(2, 12) = 10 - t) \\
&= \frac{1}{\frac{e^{-22}(22)^{10}}{10!}} \sum_{t=0}^{10} t \cdot \frac{e^{-2}(2)^t}{t!} \frac{e^{-20}(20)^{10-t}}{(10-t)!} \\
&= \frac{1}{(22)^{10}} \sum_{t=0}^{10} \binom{10}{t} t \cdot (2)^t (20)^{10-t} \\
&= \frac{24145384355840}{22^{10}} \\
&= \frac{10}{11}
\end{aligned}$$

d)

$$\begin{aligned}
\mathbb{E}[N(t_1 - 1, t_2] \mid N(1, t_1] = n_1] &= \mathbb{E}[N(1, 12] \mid N(1, 2] = 7] \\
&= \sum_{t \geq 0} t \cdot \mathbb{P}(N(1, 12] = t \mid N(1, 2] = 7) \\
&= \sum_{t \geq 7} t \cdot \mathbb{P}(N(1, 12] = t \mid N(1, 2] = 7) \\
&= \sum_{t \geq 7} t \cdot \frac{\mathbb{P}(N(1, 12] = t, N(1, 2] = 7)}{\mathbb{P}(N(1, 2] = 7)} \\
&= \sum_{t \geq 7} t \cdot \frac{\mathbb{P}(N(2, 12] = t - 7) \mathbb{P}(N(1, 2] = 7)}{\mathbb{P}(N(1, 2] = 7)} \\
&= \sum_{t \geq 7} t \cdot \mathbb{P}(N(2, 12] = t - 7) \\
&= \sum_{t \geq 7} t \cdot \frac{e^{-20} 20^{t-7}}{(t-7)!} \\
&= \sum_{x \geq 0} (x+7) \cdot \frac{e^{-20} 20^x}{x!}, \quad \text{note } x = t - 7 \\
&= \sum_{x \geq 0} x \cdot \frac{e^{-20} 20^x}{x!} + 7 \sum_{x \geq 0} \frac{e^{-20} 20^x}{x!} \\
&= \mathbb{E}[x] + 7 \cdot 1, \quad x \sim Poi(20) \\
&= 20 + 7 \\
&= 27
\end{aligned}$$

Q2

First note that $N_{t+s} - N_t \sim Poi\left(\int_t^{t+s} \lambda(t)dt\right)$

$$\begin{aligned}
\mathbb{P}(N_s = m | N_t = n) &= \frac{\mathbb{P}(N_s = m, N_t = n)}{\mathbb{P}(N_t = n)} \\
&= \frac{\mathbb{P}(N_s = m, N_t - N_s = n - m)}{\mathbb{P}(N_t = n)} \\
&= \frac{\mathbb{P}(N_s = m) \mathbb{P}(N_t - N_s = n - m)}{\mathbb{P}(N_t = n)} \\
&= \frac{\frac{(\int_0^s 1 - e^{-t} dt)^m e^{-\int_0^s 1 - e^{-t} dt}}{m!} \frac{(\int_s^t 1 - e^{-t} dt)^{n-m} e^{-\int_s^t 1 - e^{-t} dt}}{(n-m)!}}{\frac{(\int_0^t 1 - e^{-t} dt)^n e^{-\int_0^t 1 - e^{-t} dt}}{n!}} \\
&= \frac{n!(s-1+e^{-s})^m e^{1-s-e^{-s}} (t-s+e^{-t}-e^{-s})^{n-m} e^{t-s+e^{-t}-e^{-s}}}{m!(n-m)!(t-1+e^{-t})^n e^{1-t-e^{-t}}} \\
&= \binom{n}{m} e^{2e^{-t}-2e^{-s}+2t-2s} \frac{(s-1+e^{-s})^m (t-s+e^{-t}-e^{-s})^{n-m}}{(t-1+e^{-t})^n}
\end{aligned}$$

Q3

a)

First we have that $N_A \sim Poi(\lambda|A|) = Poi(0.5 \cdot 0.25) = Poi(1/8)$.

$$\begin{aligned}
\mathbb{P}(N_A \geq 1) &= 1 - \mathbb{P}(N_A = 0) \\
&= 1 - \frac{e^{-\frac{1}{8}} \left(\frac{1}{8}\right)^0}{0!} \\
&= 1 - e^{-\frac{1}{8}} \\
&\approx 0.1175
\end{aligned}$$

b)

As each trip explores a new area we have that each trip is an independent Poisson process. This means if make n trips we have the Poisson process $\sum_{i=1}^n N_i \sim Poi(\sum_{i=1}^n \lambda|A|) = Poi(n\lambda|A|)$. So the expectation is $\mathbb{E}[\sum_{i=1}^n N_i] = n\lambda|A| = \frac{n}{8}$. Now we want to find the n that makes this greater or equal to 6, thus $n = 48$ ($\frac{48}{8} = 6$). Therefore we can expect to make 48 trips before we see 6 fat-tailed dunnarts.

c)

If we assume the survey was also a spatial Poisson process with the same rate parameter, let A be the event that there are a total of $5 \cdot 10^5$ fat-tailed dunnarts over the total 29750 km^2 . We then have:

$$\mathbb{P}(A) = \frac{e^{-0.5 \cdot 29750} (0.5 \cdot 29750)^{5 \cdot 10^5}}{(5 \cdot 10^5)!}$$

$$\begin{aligned} \mathbb{P}(N_A \geq 1|A) &= 1 - \mathbb{P}(N_A = 0|A) \\ &= 1 - \frac{\mathbb{P}(N_A = 0, A)}{\mathbb{P}(A)} \end{aligned}$$

Q4

a)

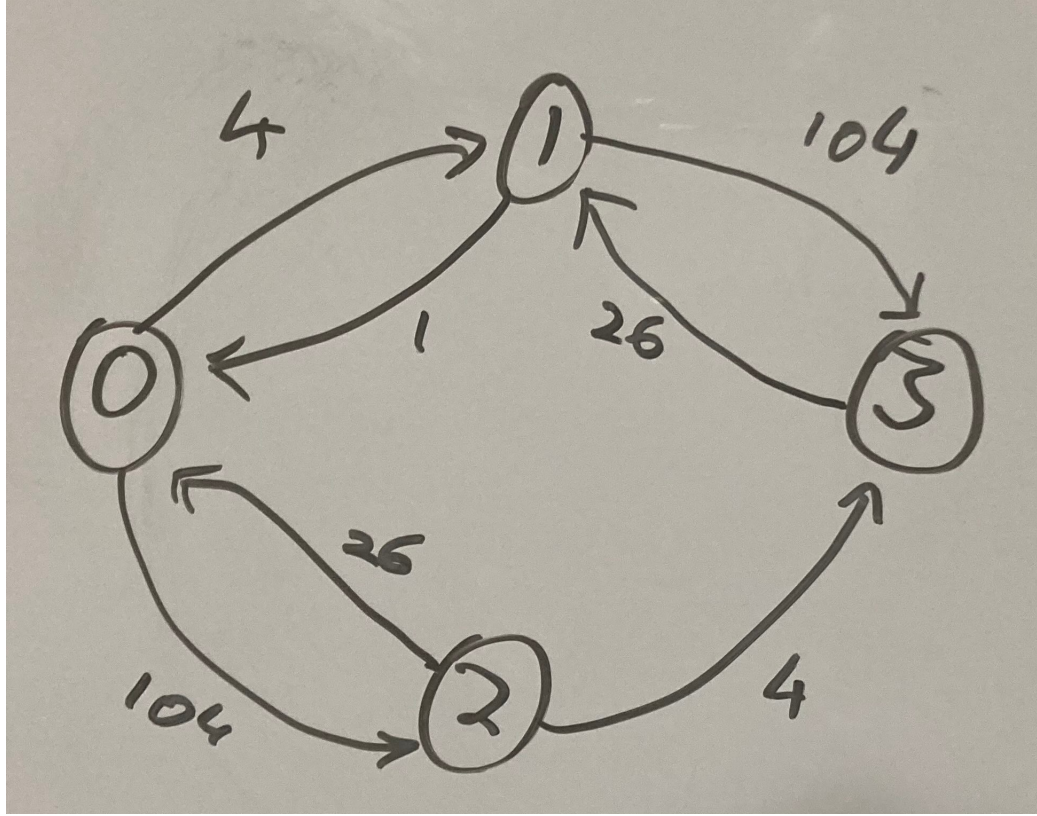
We have four states for this problem:

- 0 : Both machines are functioning
- 1 : The machine has failed, but the repair robot is functioning
- 2 : The repair robot has failed, but the machine is functioning
- 3 : Both machines have failed

So $E = \{0, 1, 2, 3\}$. We know that both machines initially are working, so $\pi^{(0)} = (1, 0, 0, 0)^T$. Now noting the given lifetime and repair distributions and that we can't have an infinitesimal change between state 1 and 2, 0 and 3, and, we can't go from state 3 to 2 as the repair robot needs to be repaired before the machine we get that the Q -matrix is:

$$Q = \begin{pmatrix} -108 & 4 & 104 & 0 \\ 1 & -105 & 0 & 104 \\ 26 & 0 & -30 & 4 \\ 0 & 26 & 0 & -26 \end{pmatrix}$$

b)



c)

We calculate the long run distribution π by solving $\pi Q = \mathbf{0}$ such that $\pi \mathbf{1} = 1$.

$$0 = 104\pi_0 - 30\pi_2 \implies \pi_0 = \frac{15}{52}\pi_2 \quad (1)$$

$$0 = -108\pi_0 + \pi_1 + 26\pi_2 \implies \pi_1 = 108\pi_0 - 26\pi_2 = \left(\frac{3240}{104} - 26\right)\pi_2 = \frac{67}{13}\pi_2 \quad (2)$$

$$0 = 4\pi_0 - 105\pi_1 + 26\pi_3 \implies \pi_3 = \frac{105\pi_1 - 4\pi_0}{26} = \frac{270}{13}\pi_2 \quad (3)$$

$$1 = \pi_0 + \pi_1 + \pi_2 + \pi_3 \implies \frac{1}{\frac{15}{52} + \frac{67}{13} + 1 + \frac{270}{13}} = \pi_2 = \frac{52}{1415} \quad (4)$$

Subbing (4) into (1), (2) and (3) we get:

$$\pi = \left(\frac{3}{283}, \frac{268}{1415}, \frac{52}{1415}, \frac{216}{283} \right)^T$$

So the long run probability that both the machine and the machine repair robot are under repair is $\frac{216}{283}$.

d)

To find this probability we first find the transition matrix of the embedded markov chain (D), where the entry $d_{ij} = \frac{q_{ij}}{-q_{ii}}$, where q_{ij} are entries from the Q -matrix.

$$D = \begin{pmatrix} 0 & \frac{4}{108} & \frac{104}{108} & 0 \\ \frac{1}{105} & 0 & 0 & \frac{104}{105} \\ \frac{26}{30} & 0 & 0 & \frac{4}{30} \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

If the machine has failed and is under repair we are in state 1 and the probability that it transitions to 0 and not 3 is from the matrix D, $\frac{1}{105}$.