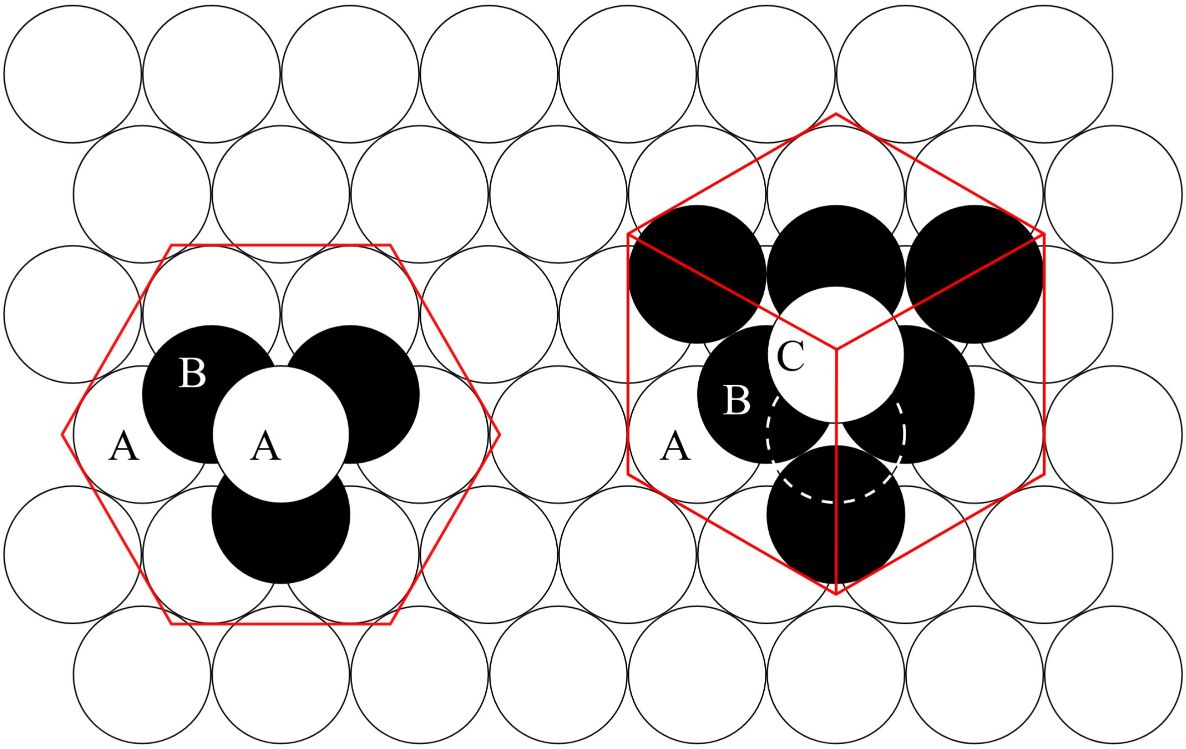
Sphere Packing

A **sphere packing** is an arrangement of spheres in some space such that the spheres do not overlap. A classical problem is to find the densest possible packing of non-overlapping equal-sized spheres in 3-dimensional Euclidean space. In a densest possible packing, the fraction of space filled by the spheres is $\frac{\pi}{\sqrt{18}}$, which is about 0.74. It has been known since ancient times how to pack spheres to achieve this density, but it was not proved to be the maximum until relatively recently. The proof was announced in 1998, but formal checking of the proof was not completed until 2014 [28].

Multiple ways of achieving the maximum density:



Other dimensions:

2-dimensions, $\frac{\pi\sqrt{3}}{6} \approx 90\%$

The sphere packing problem can be generalised in many ways, including to higher dimensions, with spheres of differing sizes, and to non-Euclidean spaces. In $\mathbb{R}^n$ with $n \geq 4$, the optimal sphere packing density is known only for $n = 8$ and $n = 24$, where the spheres are centred on the points of E_8 **lattice** and the **Leech lattice** respectively. It was only in 2017, that these packings were proved to be optimal [57, 15]. The Ukrainian mathematician Maryna Viazovska was awarded the Fields Medal in 2022, predominantly for her work on sphere packings.

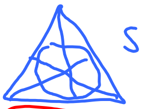
The E_8 lattice can be constructed from a Steiner system $S(3,4,8)$. The unique (up to isomorphism) $S(3,4,8)$ is given by

$S(t,k,v)$ points, blocksize, each t-subset of points occurs once.

$\frac{\binom{8}{3}}{\binom{4}{3}} = \frac{8\times7\times6}{4\times3\times2} = 14$

1	2	4	8	3	5	6	7
2	3	5	8	4	6	7	1
3	4	6	8	5	7	1	2
4	5	7	8	6	1	2	3
5	6	1	8	7	2	3	4
6	7	2	8	1	3	4	5
7	1	3	8	2	4	5	6

ab* ac* bc* a** b** c** ***



$S(2,3,7)$

a b c d e f g 8

abc8 defg

ade8 bcfg

afg8 bcde

(a) The complement of each block is a block.

(b) Any two blocks intersect in 0 or 2 points.

(c) The symmetric difference of any two distinct blocks is $\{1, 2\ldots, 8\}$ or is a block.

For each block B_j, $j = 1, 2, \ldots, 14$, of the $S(3,4,8)$, let v_j be the vector of $\mathbb{Z}^8$ having a 1 in coordinate i if $i \in B_j$, and having a 0 in coordinate i if $i \notin B_j$. Thus, $\{v_j : j = 1, 2, \ldots, 14\}$ is a set of 14 vectors in $\mathbb{Z}^8$, each having four coordinates that are 1 and four coordinates that are 0. These 14 vectors together with $(0, 0, \ldots, 0)$ and $(1, 1, \ldots, 1)$ form a 4-dimensional vector subspace V of $\mathbb{Z}_2^8$. This is the $(8,4,4)$ **extended Hamming code**. Closure, and the fact that any two distinct vectors of V differ in at least four coordinates, follows from property (c) mentioned above.

1	2	4	8	11010001	3	5	6	7	00101110
2	3	5	8	01101001	4	6	7	1	10010110
3	4	6	8	00110101	5	7	1	2	11001010
4	5	7	8	00011011	6	1	2	3	11100100
5	6	1	8	10001101	7	2	3	4	01110010
6	7	2	8	01000111	1	3	4	5	10111000
7	1	3	8	10100011	2	4	5	6	01011100
				00000000					11111111

(c) The symmetric difference of any two distinct blocks is $\{1, 2\ldots, 8\}$ or is a block.

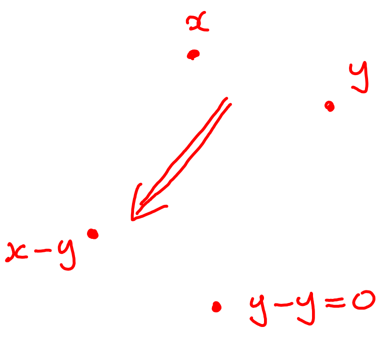
The E_8 lattice is

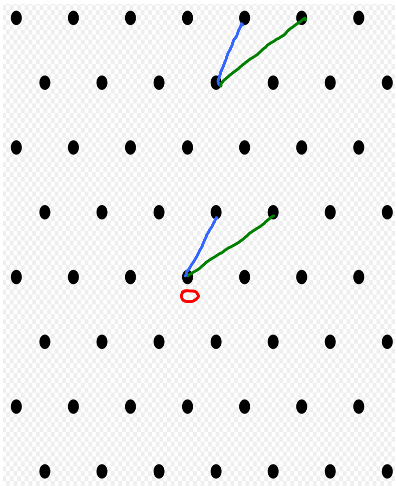
$$\{x \in \mathbb{Z}^8 : x \equiv v \pmod 2, v \in V\}$$

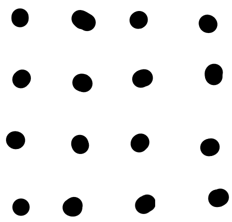
where $x \equiv v \pmod 2$ means that if $x = (x_1, \ldots, x_8)$ and $v = (v_1, \ldots, v_8)$, then $x_i \equiv v_i \pmod 2$ for $i = 1, \ldots, 8$. Notice that if $x, y \in E_8$, then $x + y \in E_8$, and from this it follows that E_8 is a subgroup of $\mathbb{Z}^8$.

(even + even = even, even + odd = odd, odd + odd = even
0 + 0 = 0, 0 + 1 = 1, 1 + 1 = 0)

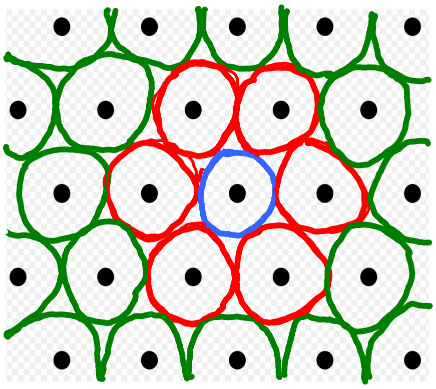
It can be seen that the minimum distance between distinct points of E_8 is 2. If points x and y are at distance d, then the points $x - y$ and 0 are also at distance d. So the minimum distance between distinct points of the lattice is equal to the minimum distance of a non-zero point from $(0, 0, \ldots, 0)$. This minimum is distance is 2, and is attained by points that have exactly four non-zero coordinates each equal to ± 1, and by points that have exactly one non-zero coordinate equal to ± 2.

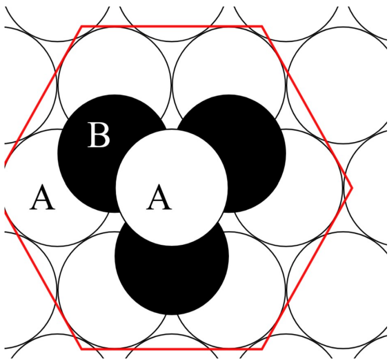






The number of points at distance 2 from any given point is equal to the number of points at distance 2 from $(0, 0, \ldots, 0)$. This number is $14 \cdot 16 + 2 \cdot 8 = 240$ (there are 14 blocks in $S(3, 4, 8)$ and for each block there are $2^4 = 16$ ways of assigning ± 1 to the coordinates corresponding to the four points of the block, and there are $2 * 8 = 16$ vectors having one non-zero coordinate equal to ± 2).





In an optimal sphere packing in 2 dimensions, each sphere touches 6 others.

In 3 dimensions, 12

In 8 dimensions, 240

The Leech lattice can be constructed from the blocks of the Steiner system $S(5,8,24)$ as follows, see [17]. Let $\{1,2,\dots,24\}$ be the point set of an $S(5,8,24)$ and let $\mathcal{B}$ be the set of blocks. For each block $B \in \mathcal{B}$ define v_B to be the vector in $\mathbb{Z}^{24}$ that has a 2 in coordinate i if $i \in B$, and a 0 in coordinate i if $i \notin B$. Let $w \in \mathbb{Z}^{24}$ be the vector $(1,1,\dots,1,-3)$. Then the integer linear combinations of the vectors v_B, $B \in \mathcal{B}$, and w give us the points of the Leech lattice. Notice that $\|w\| = \sqrt{32}$ and for each $B \in \mathcal{B}$ we have $\|v_B\| = \sqrt{32}$. Also,

$$\min\{\|v_B - w\| : B \in \mathcal{B}\} = \min\{\|v_B - v_{B'}\| : B, B' \in \mathcal{B}\} = \sqrt{32}.$$

Points $\{1,2,3,\dots,24\}$

$$\{1,2,3,5,6,7,10,11\} \Rightarrow \overbrace{2\,2\,2\,0\,2\,2\,2\,0\,0\,2\,2\,0\,0\cdots\cdots0}^{24}$$

B block $\qquad\qquad v_B$

$$w = (\underbrace{1,1,\dots,1,-3}_{24})$$

Blocks $B_1, B_2, \dots, B_{759}$

$$\Downarrow$$

$v_{B_1}, v_{B_2}, \dots, v_{B_{759}}, w$

$$\frac{\binom{24}{5}}{\binom{8}{5}} = \frac{24\cdot23\cdot22\cdot21\cdot20}{8\cdot7\cdot6\cdot5\cdot4} = 23\times11\times3 = 759$$

Leech Lattice

$$\{\lambda_0 w + \lambda_1 v_{B_1} + \lambda_2 v_{B_2} + \dots + \lambda_{759} v_{B_{759}} : \lambda_0, \lambda_1, \dots, \lambda_{759} \in \mathbb{Z}\}$$

Minimum length vectors:

$$\|v_B\|: \quad \underbrace{2\,2\cdots2}_{8}\ \underbrace{0\,0\cdots0}_{16} \Rightarrow \sqrt{8\times2^2} = \sqrt{32}$$

$$\|w\|: \quad 1\,1\ \cdots\cdots\ 1\ -3 \Rightarrow \sqrt{23+9} = \sqrt{32}$$

$$\min\|v_B - v_w\| = \min\|v_B - v_{B'}\| = \sqrt{32}$$

$$\begin{array}{rl} & 2\,2\,2\,2\,2\,2\,2\,2\,0\,0\cdots0\ \ 0 \\ - & 1\,1\,1\,1\,1\,1\,1\,1\,1\,1\cdots1\ -3 \\ \hline & 1\,1\cdots\cdots1\ -1\,-1\cdots-1\ \ 3 \end{array} \Rightarrow \sqrt{23+9}$$

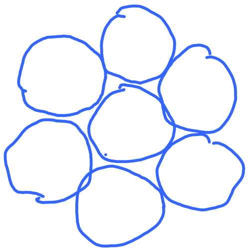
$$\begin{array}{l} 2\,2\,2\,2\,2\,2\,2\,2\,0\,0\,0\,0\,0\,0\,0\cdots0 \\ 2\,2\,2\,2\,0\,0\,0\,0\,2\,2\,2\,2\,0\,0\cdots0 \\ \hline 0\,0\,0\,0\,2\,2\,2\,2\,-2\,-2\,-2\,-2\,0\,0\cdots0 \end{array}$$

$$\Rightarrow \sqrt{8\times4}$$

In $\mathbb{R}^n$, how many non-overlapping unit spheres can touch a given unit sphere?

(kissing number problem).

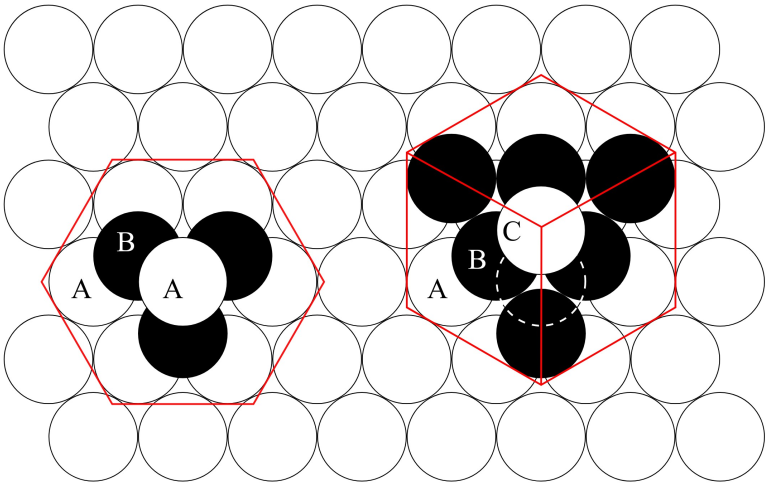
$\mathbb{R}^2$



6

$\mathbb{R}^3$

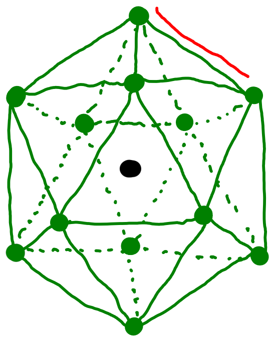
12



Proved maximum in 1953.

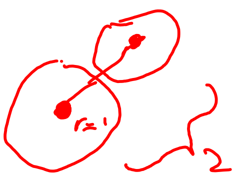
In $\mathbb{R}^2/\mathbb{R}^3$, is it possible to pack 6/12 larger spheres around a given sphere ???

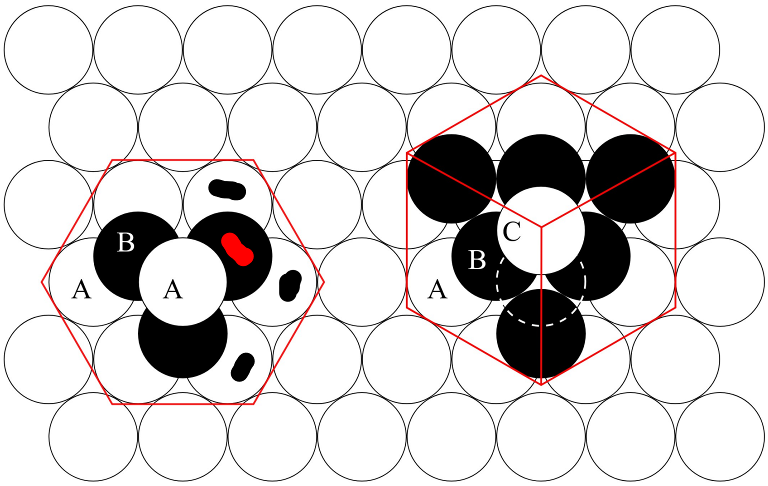
No in $\mathbb{R}^2$. Yes in $\mathbb{R}^3$.



Icosahedron

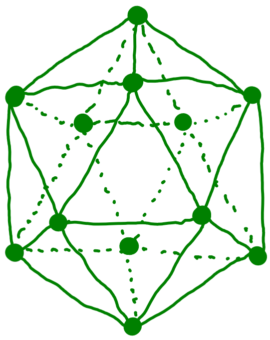
radius = 1

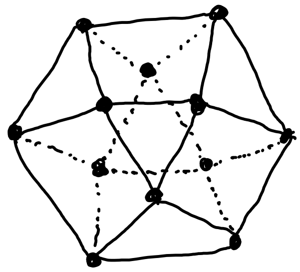




Among the 12 spheres touching a given sphere, each touches 4 others.

(does not extend to a sphere packing) →





cuboctahedron

$\mathbb{R}^2$	6	regular hexagon
$\mathbb{R}^3$	12	icosahedron, cuboctahedron, other arrangements

Kissing number problem in $\mathbb{R}^4, \mathbb{R}^5, \ldots$???

$\mathbb{R}^4$	24	24-cell	(2003)
$\mathbb{R}^8$	240	E_8 Lattice	
$\mathbb{R}^{24}$	196,560	Leech Lattice	

$n = 5, 6, 7, 9, \ldots, 23, 25, 26$ unsolved.