Math3303 Assignment 5

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Q1

We want to find all left, right and two-sided ideals of the ring of $n \times n$ complex matrices ($M_n(\mathbb{C})$). First we note that we have $1 \leq i \leq n$ and $1 \leq j \leq n$.

Left Ideals:
Let I_L be the set of left ideals, then we have:

1. $(I_L, +) \leq (M_n(\mathbb{C}), +)$
2. $\forall A \in M_n(\mathbb{C})$ and $B \in I_L, \;\; AB \in I_L$

So I_L is the set of matrices whose j_{th} column is all zeros. If we have $AB = C$ then the entry $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = 0$ as all $b_{kj} = 0$, and so the j_{th} column of C is all zeros and hence it is in I_L. It is also satisfies condition 1 as adding two matrices whose j_{th} columns are zero gives another matrix whose j_{th} column is zeros, hence it is closed. Also we note that the identity is just the zero matrix.

Right Ideals:
Let I_R be the set of right ideals, then we have:

1. $(I_R, +) \leq (M_n(\mathbb{C}), +)$
2. $\forall A \in M_n(\mathbb{C})$ and $B \in I_L, \;\; BA \in I_R$

So I_R is the set of matrices whose i_{th} row is all zeros. If we have $BA = C$ then the entry $c_{ij} = \sum_{k=1}^{n} b_{ik} a_{kj} = 0$ as all $b_{ik} = 0$, and so the i_{th} row of C is all zeros and hence it is in I_R. It is also satisfies condition 1 as adding two matrices whose i_{th} rows are zero gives another matrix whose i_{th} row is zeros, hence it is closed. Also we note that the identity is just the zero matrix.

Two Sided Ideals:
The two sided ideals are the set of matrices whose i_{th} row and j_{th} column are all zeros. It is clear to see this from above as such a matrix is a left sided ideal (j_{th} column is all zeros) and also a right sided ideal (i_{th} row is all zeros). As this set is both a left and right sided ideal it is a two sided ideal.

Q2

We want to prove that $\mathbb{R}[x]/(x^2+1)$ is isomorphic to $\mathbb{C}$. First we will prove three lemmas:

Lemma 1. *$x^2+1 \in \mathbb{R}[x]$ is irreducible.*

Proof. Suppose x^2+1 is reducible, then:

$$x^2+1 = (ax+b)(cx+d) = acx^2 + (ad+bc)x + bd$$

Hence we have that $ac = 1$, $bd = 1$ and $ad+bc = 0$. This yields:

$$\begin{aligned} ad &= -bc \\ \implies (ac)d &= -bc^2 \\ \implies d &= -bc^2 \\ \implies (bd) &= -(bc)^2 \\ \implies 1 &= -(bc)^2 \end{aligned}$$

Which isn't possible as this requires that $bc = i$ but this can only occur if b and/or c are complex numbers but they are real so x^2+1 is irreducible. $\square$

Definition 0.1 (Prinicipal Ideal Domain (PID))**.** A principal ideal domain is an integral domain R in which every ideal has the form:

$$\langle a \rangle = \{ar | r \in R\} = aR$$

for some a in R.

Lemma 2. *Let R be a PID, then every non-zero prime ideal is maximal.*

Proof. Let $P \subset R$ be a nonzero prime ideal. From the fact that R is a PID we get that $P = \langle p \rangle$ for some $p \in R$. Now suppose there is an ideal $I = \langle x \rangle$ such that $\langle p \rangle \subseteq \langle x \rangle \subseteq R$. From this we have that $p \in \langle x \rangle$ so that $p = kx$ for some $k \in R$. As $\langle p \rangle$ is a prime ideal, we get that either $x \in \langle p \rangle$ or $k \in \langle p \rangle$. If $x \in \langle p \rangle$, then $\langle x \rangle = \langle p \rangle$. If $k \in \langle p \rangle$, then $k = py \implies p = pyx \implies p(1-yx) = 0 \implies yx = 1 \implies x$ is a unit $\langle x \rangle = R$. Thus P is maximal. $\square$

Lemma 3. *$\mathbb{R}[x]$ is a principal ideal domain.*

Proof. Let I be an ideal of $\mathbb{R}[x]$. There are two cases to consider. In the first we have $I = \{0\}$, which means that $I = 0\mathbb{R}[x]$ and hence is a PID. In the second case $I \neq 0$. This means that there exists a nonzero polynomial $f \in I$ with minimal degree. Let $g \in I$. By the division algorithm there exists polynomials $q, r \in \mathbb{R}[x]$ such that:

$$g(x) = f(x)q(x) + r(x)$$

where $r(x) = 0$ or $0 \leq \deg r \leq \deg f$. We rewrite the equation above as:

$$r(x) = g(x) - f(x)q(x)$$

Since $f \in I$ and $q \in \mathbb{R}[x]$ we have by definition of I being an ideal that $fq \in I$. Since $g \in I$ we have that $(g - fq) \in I$. So $r \in I$. But by minimality of the degree of f in I we must have that $r(x) = 0$. Therefore:

$$g(x) = f(x)q(x)$$

Since $g \in I$ was arbitrary we have that $I = f(x)\mathbb{R}[x]$. So I is a principal ideal. Hence every ideal in $\mathbb{R}[x]$ is a principal ideal. So $\mathbb{R}[x]$ is a principal ideal domain. $\square$

Theorem 1. $\mathbb{R}[x]/(x^2 + 1) \cong \mathbb{C}$

Proof. From lemma 3 $\mathbb{R}[x]$ is a PID and thus from lemma 2 every prime ideal is maximal. From lemma 1 (x^2+1) is irreducible over $\mathbb{R}[x]$, and therefore (x^2+1) is a maximal ideal, so $\mathbb{R}[x]/(x^2 + 1)$ is a field. Now define the homomorphism $\phi : \mathbb{R}[x] \to \mathbb{C}$, such that $\phi(f) = f(i)$, this is the polynomial f evaluated at i). It is clear that $(x^2 + 1) \subseteq \ker \phi$, and equality follows since $(x^2 + 1)$ is maximal. From the 1st isomorphism theorem for rings, we have $\mathbb{R}[x]/(x^2 + 1) \cong \mathbb{C}$. $\square$

$\mathbb{R}[x]/(x^2 + x + 1)$ is isomorphic to $\mathbb{C}$ as $x^2 + x + 1$ is irreducible, so by the same argument as in the proof of theorem 1 except with the homomorphism being $\phi(f) = f\left(\frac{-1+\sqrt{3}i}{2}\right)$ we get that $\mathbb{R}[x]/(x^2 + x + 1) \cong \mathbb{C}$.