Definition 3.0.1. A (combinatorial) **design** consists a set V and a collection $\mathcal{B}$ of subsets of V. The elements of V are the **points** of the design, and the subsets in $\mathcal{B}$ are called **blocks**. An **automorphism** of a design is a permutation of V that preserves the blocks. More precisely, a permutation π of V is an automorphism of a design $(V, \mathcal{B})$ if $\mathcal{B}\pi = \mathcal{B}$, where $\mathcal{B}\pi = \{B\pi : B \in \mathcal{B}\}$ and $B\pi = \{x\pi : x \in B\}$. A design with v points is **cyclic** if it has an automorphism that permutes its points in a single cycle of length v.

$V = \{0, 1, 2, 3, 4, 5\}$

$\mathcal{B} = \{\{0,2,4\}, \{1,3,5\}, \{0,2\}, \{1,3\}, \{2,4\}, \{3,5\}, \{4,0\}, \{5,1\}\}$

$(V, \mathcal{B})$ cyclic. $(0\ 1\ 2\ 3\ 4\ 5)$ is an automorphism.

3.1 (v,k,λ)-designs.

Definition 3.1.1. Let v, k and λ be positive integers with $k < v$. A **(v, k, λ)-design** is a design with v points where every block has k elements, and where every pair of points occurs in exactly λ blocks.

$(v,k,1)$-design $\equiv S(2,k,v)$ Steiner system.

Number of blocks $b = \dfrac{\lambda v(v-1)}{k(k-1)}$

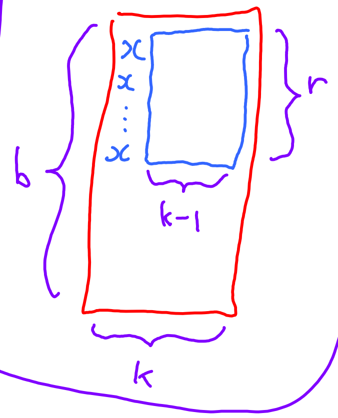
obvious necessary conditions

Each point is in $r = \dfrac{\lambda(v-1)}{k-1}$ blocks

replication number.

$vr = bk$

$\lambda(v-1) = r(k-1)$



Four special families of (v,k,λ)-designs:

- Projective planes $\longrightarrow$ $(n^2+n+1, n+1, 1)$-designs
- Affine planes $\longrightarrow$ $(n^2, n, 1)$-designs
- Hadamard designs
- Biplanes

Hadamard designs.

closely related to Hadamard matrices

H ← $m \times m$ matrix $1, -1$

$HH^T = mI$

Any two distinct rows (columns) are orthogonal.

$(4n-1, 2n-1, n-1)$-designs

$n \geq 2$

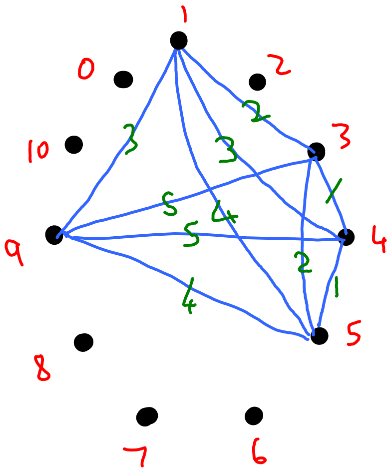
Unsolved problem: Does there exist a Hadamard design of order n for each $n \geq 2$?

The parameters for a Hadamard design of order n for $n = 2, 3, \ldots, 25$ are shown below.

$(7,3,1)$	$(11,5,2)$	$(15,7,3)$	$(19,9,4)$	$(23,11,5)$	$(27,13,6)$
$(31,15,7)$	$(35,17,8)$	$(39,19,9)$	$(43,21,10)$	$(47,23,11)$	$(51,25,12)$
$(55,27,13)$	$(59,29,14)$	$(63,31,15)$	$(67,33,16)$	$(71,35,17)$	$(75,37,18)$
$(79,39,19)$	$(83,41,20)$	$(87,43,21)$	$(91,45,22)$	$(95,47,23)$	$(99,49,24)$

$(11, 5, 2)$-design

1 3 4 5 9
2 4 5 6 10
⋮
0 2 3 4 8



$(15, 7, 3)$-design

points and hyperplanes of $PG(3,2)$

$\binom{4}{1}_2 = \frac{2^4-1}{2-1} = 15.$

$\frac{2^3-1}{2-1} = 7.$

$\binom{n-1}{d-1}_2 = \binom{2}{1}_2 = \frac{2^2-1}{2-1} = 3.$

Theorem 2.6.8. Let $n \geq 2$ and let q be a prime power. Each pair of distinct points of $\mathrm{PG}(n, q)$ occurs together in exactly $\binom{n-1}{d-1}_q$ d-dimensional subspaces of $\mathrm{PG}(n, q)$.

Theorem: If $q \equiv 3 \pmod 4$ is a prime power, then the orbit of the quadratic residues of $\mathbb{F}_q$ under $(\mathbb{F}_q, +)$ forms a Hadamard design of order $n = \frac{q+1}{4}$ (q points).

$\Rightarrow (11,5,2), (19,9,4), (23,11,5), (27,13,6), (31,15,7)$, etc

From 1985 until 2005 the smallest unresolved case was the existence of a Hadamard design of order 107, or $(427, 213, 106)$-design. Such a design was constructed by Kharaghani and Tayfeh-Rezaie in 2005, see [38]. The smallest unresolved case is now the existence of a Hadamard design of order 167, or $(667, 333, 166)$-design. Various other cases have been resolved in the last few years. For example, a Hadamard design of order 191, or $(763, 381, 190)$-design, was constructed by Doković in 2008 [21].

Biplanes: $\left(\binom{n+2}{2}+1,\ n+2,\ 2\right)$ - designs.

← biplane of order n

Any two distinct points are incident with exactly two lines.

Any two distinct lines intersect in exactly two points.

$n=2 \implies (7,4,2)$-design

3	5	6	7
4	6	7	1
5	7	1	2
6	1	2	3
7	2	3	4
1	3	4	5
2	4	5	6

1 2 4
2 3 5
⋮
7 1 3

(complement of Fano plane)

$n=3 \implies (11,5,2)$-design

Hadamard design of order 3

1 3 4 5 9
2 4 5 6 10
⋮
0 2 3 4 8

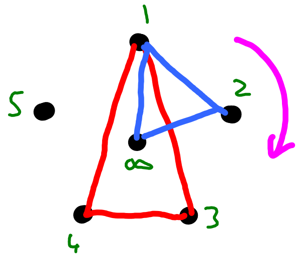
$n=4 \implies (16,6,2)$-design.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

The only known biplanes are of order $2, 3, 4, 7, 9$ and 11. There is no biplane of order $5, 6, 8, 10$. It is unknown whether there are any biplanes of order $n > 11$. □

Other than projective planes, affine planes, Hadamard designs, biplanes, there are many other (v,k,λ)-designs and families of (v,k,λ)-designs.

- $(6,3,2)$-design



∞ 1 2	1 3 4
∞ 2 3	2 4 5
∞ 3 4	3 5 1
∞ 4 5	4 1 2
∞ 5 1	5 2 3

- $(13,3,1)$-design

0 1 4	0 2 7
1 2 5	1 3 8
2 3 6	2 4 9
⋮	⋮
12 0 3	12 1 6

$(39,13,6)$-design
$(40,14,7)$-design
$(40,10,3)$-design
???

- Steiner triple systems, $(v,3,1)$-designs
  Exist iff $v \equiv 1,3 \pmod 6$ (Kirkman, 1847).

- $(v,4,1)$-designs exist iff $v \equiv 1,4 \pmod{12}$

- Two-fold triple systems, $(v,3,2)$-designs, exist iff $v \equiv 0,1 \pmod 3$

A $(v,3,1)$-design is called a Steiner triple system, and these were shown to exist if and only if $v \equiv 1,3 \pmod 6$ by Kirkman in 1847 [39]. By 1975, the existence problem for (v,k,λ)-designs was completely settled for $k \in \{3,4,5\}$, and also for $k = 6$ with $\lambda \geq 2$ [33]. For $k \in \{3,4,5\}$ and for each $\lambda \geq 1$, it is known that there exists a (v,k,λ)-design whenever the obvious necessary conditions are satisfied; except that there is no $(15,5,2)$-design. For $k = 6$ and for each $\lambda \geq 2$ the situation is similar: it is known that there exists a (v,k,λ)-design whenever the obvious necessary conditions are satisfied; except that there is no $(21,6,2)$-design.

For $k = 6$ and $\lambda = 1$, the existence problem is not yet completely settled. The most recent new results were obtained in 2007 [1]. There remain 29 unresolved values of v (ranging from $v = 51$ to $v = 801$) and four cases where the obvious necessary conditions are satisfied but no design exists ($v = 16, 21, 36, 46$). For values of $k > 6$ less in known, especially for $k \geq 10$. A comprehensive summary of results is given in [2]. For $k \leq \frac{v}{2}$ (see Theorem 3.1.11), the smallest, in terms of number of points, three designs whose existence is unknown are a $(39,13,6)$-design, a $(40,14,7)$-design and a $(40,10,3)$-design.

Symmetric (v,k,λ)-designs: $v=b$ $r=k$ $(vr=bk)$

- Projective planes $\equiv$ Symmetric $(v,k,1)$-designs
- Biplanes $\equiv$ Symmetric $(v,k,2)$-design
- Hadamard designs are also symmetric.
- Affine planes are not symmetric.

Theorem 3.1.8. In a symmetric (v, k, λ)-design, we have $\lambda(v-1) = k(k-1)$, each point occurs in exactly k blocks, and any two blocks intersect in exactly λ points.

v points — each block is incident with k points
v blocks — each point is incident with k blocks

each pair of points occurs together in λ blocks.
each pair of blocks have λ points in common.

The dual of a design is obtained by taking the blocks as points, and each point x of the original design defines a block B_x in the dual design, where the points in B_x are the blocks that contain x.

- The dual of a symmetric (v,k,λ)-design is another symmetric (v,k,λ)-design.
- The dual of a non-symmetric (v,k,λ)-design is not a (v,k,λ)-design. (For example, if a design has a pair of disjoint blocks, then its dual has a pair of points that occur together in no blocks.)

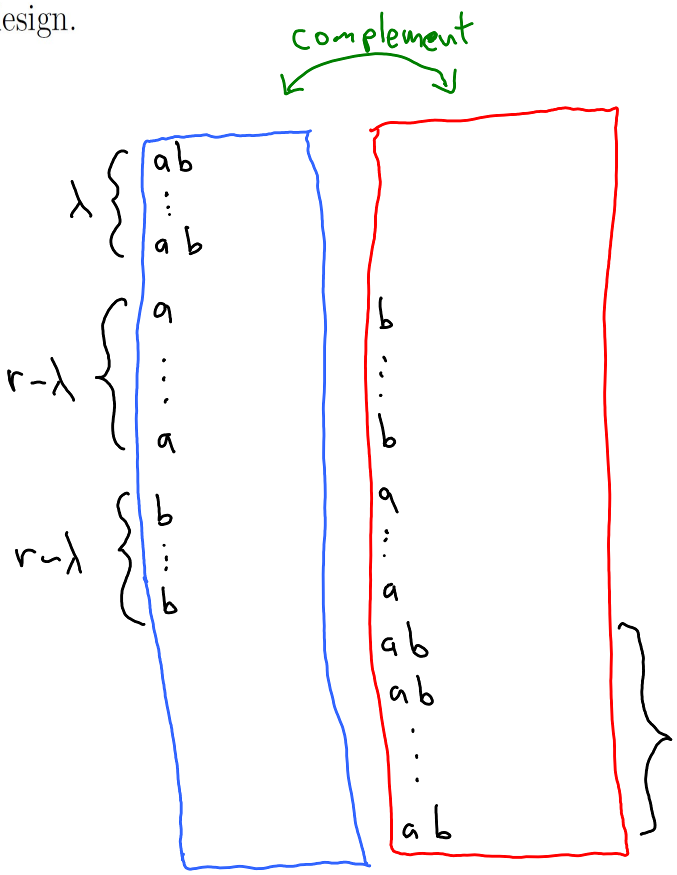
Theorem 3.1.9. (Wilson's Theorem, [60]) For all $k \geq 2$ and $\lambda \geq 1$ there exists a constant $C(k, \lambda)$ such that for all $v \geq C(k, \lambda)$, there exists a (v, k, λ)-design if and only if $k(k-1)$ divides $\lambda v(v-1)$ and $k - 1$ divides $\lambda(v-1)$.

Definition 3.1.10. The **complement** of a design $(V, \mathcal{B})$ is the design $(V, \mathcal{B}^c)$ where $\mathcal{B}^c = \{V \setminus B : B \in \mathcal{B}\}$. □

Theorem 3.1.11. If $k \leq v - 2$, then the complement of a (v, k, λ)-design is a $(v, v-k, b-2r+\lambda)$-design.

points $= v$.

block size $= v - k$.



$b-(r-\lambda)-(r-\lambda)-\lambda = b-2r+\lambda$