Math3303 Assignment 1

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Q1

Let G $=\mathrm{GL}_n(\mathbb{R})$ be the group of $n\times n$ invertible matrices and N $=\mathrm{SL}_n(\mathbb{R})$ the subgroup of G consisting of those matrices which have determinant one. First we want to prove that N$\trianglelefteq$G.

Proof. By definition we have that, N$\trianglelefteq$G $\iff$ $\forall g\in$G and $n\in$N, $gng^{-1}\in$N. This means we require $\det(gng^{-1})$=1 $\forall$ g$\in$G and $n\in$N, as N is the group of invertible matrices whose determinant is 1. By calculation and properties of determinant ($\det(AB)=\det(A)\det(B)$, $\det(A^{-1})=\frac{1}{\det(A)}$, $\forall n\in$ N, $\det(n)=1$) we get:

$$\begin{aligned}\det(gng^{-1}) &= \det(g)\det(n)\det(g^{-1})\\ &= \frac{\det(g)}{\det(g)}\det(n)\\ &= 1\end{aligned}$$

$\square$

Now we want to prove G/N $\cong\mathbb{R}^*$.

Proof. First we define the homomorphism $\phi:\mathrm{GL}_n(\mathbb{R})\to\mathbb{R}^*$. This is a homomorphism because $\forall A,B\in\mathrm{GL}_n(\mathbb{R})$, $\phi(A\cdot B)=\det(A\cdot B)=\det(A)\cdot\det(B)\in\mathbb{R}^*$. As the identity element of $\mathbb{R}^*$ is 1 we have that:

$$\mathrm{Ker}\ \phi=\{A\in\mathrm{GL}_n(\mathbb{R}):\det(A)=1\}=\mathrm{SL}_n(\mathbb{R})$$

Now by the first isomorphism theorem we have:

$$\begin{aligned}\mathrm{GL}_n(\mathbb{R})/\mathrm{Ker}\ \phi &= \mathrm{GL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{R})\\ &= G/N\\ &\cong \phi(G)\end{aligned}$$

Now to get the result we require $\phi(G) \cong \mathbb{R}^*$ which occurs if the homorphism is surjective, i.e. $\forall a \in \mathbb{R}^*$, $\exists A \in$ G s.t. $\phi(A) = \det(A) = a$. To show this we consider the matrix whose top left value is a, the rest of the diagonal is 1 and every other entry is 0. The determinant of this matrix is clearly a, and so we have a surjective homomorphism, hence the result. $\square$

Q2

Q3