Stat3004 Assignment 3

Dominic Scocchera

April 2023

Q1

First we note that $N((a, b]) \sim Poi(2 \cdot (b - a))$.

a)

$$\begin{aligned}
\mathbb{P}(N_{t_1} = n_1, N_{t_2} = n_2) &= \mathbb{P}(N_2 = 7, N_{12} = 10) \\
&= \mathbb{P}(N_2 = 7, N_{12-2} = 10 - 7) \\
&= \mathbb{P}(N_2 = 7)\mathbb{P}(N_{10} = 3) \\
&= e^{-2 \cdot 2} \frac{(2 \cdot 2)^7}{7!} e^{-2 \cdot 10} \frac{(2 \cdot 10)^3}{3!} \\
&\approx 1.636291121 \times 10^{-7}
\end{aligned}$$

b)

$$\begin{aligned}
\mathbb{P}(N(1, t_1] = n_1, N(t_1 - 1, t_2] = n_2) &= \mathbb{P}(N(1, 2] = 7, N(1, 12] = 10) \\
&= \mathbb{P}(N(1, 2] = 7, N(2, 12] = 3) \\
&= \mathbb{P}(N(1, 2] = 7)\mathbb{P}(N(2, 12] = 3) \\
&= \frac{e^{-2}(2)^7}{7!} \frac{e^{-20}(20)^3}{3!} \\
&\approx 9.4458178806 \times 10^{-9}
\end{aligned}$$

c)

$$
\begin{aligned}
\mathbb{E}[N(1,t_1] \mid N(t_1-1,t_2] = n_2] &= \mathbb{E}[N(1,2] \mid N(1,12] = 10] \\
&= \sum_{t\geq 0} t \cdot \mathbb{P}(N(1,2] = t | N(1,12] = 10) \\
&= \sum_{t=0}^{10} t \cdot \frac{\mathbb{P}(N(1,2] = t, N(1,12] = 10)}{\mathbb{P}(N(1,12] = 10)} \\
&= \frac{1}{\mathbb{P}(N(1,12] = 10)} \sum_{t=0}^{10} t \cdot \mathbb{P}(N(1,2] = t, N(2,12] = 10-t) \\
&= \frac{1}{\mathbb{P}(N(1,12] = 10)} \sum_{t=0}^{10} t \cdot \mathbb{P}(N(1,2] = t)\mathbb{P}(N(2,12] = 10-t) \\
&= \frac{1}{\frac{e^{-22}(22)^{10}}{10!}} \sum_{t=0}^{10} t \cdot \frac{e^{-2}(2)^t}{t!} \frac{e^{-20}(20)^{10-t}}{(10-t)!} \\
&= \frac{1}{(22)^{10}} \sum_{t=0}^{10} \binom{10}{t} t \cdot (2)^t (20)^{10-t} \\
&= \frac{24145384355840}{22^{10}} \\
&= \frac{10}{11}
\end{aligned}
$$

d)

$$
\begin{aligned}
\mathbb{E}[N(t_1-1,t_2] \mid N(1,t_1]=n_1] &= \mathbb{E}[N(1,12] \mid N(1,2]=7] \\
&= \sum_{t\geq 0} t\cdot \mathbb{P}(N(1,12]=t|N(1,2]=7) \\
&= \sum_{t\geq 7} t\cdot \mathbb{P}(N(1,12]=t|N(1,2]=7) \\
&= \sum_{t\geq 7} t\cdot \frac{\mathbb{P}(N(1,12]=t, N(1,2]=7)}{\mathbb{P}(N(1,2]=7)} \\
&= \sum_{t\geq 7} t\cdot \frac{\mathbb{P}(N(2,12]=t-7)\mathbb{P}(N(1,2]=7)}{\mathbb{P}(N(1,2]=7)} \\
&= \sum_{t\geq 7} t\cdot \mathbb{P}(N(2,12]=t-7) \\
&= \sum_{t\geq 7} t\cdot \frac{e^{-20}20^{t-7}}{(t-7)!} \\
&= \sum_{x\geq 0} (x+7)\cdot \frac{e^{-20}20^{x}}{x!}, \quad \text{note } x=t-7 \\
&= \sum_{x\geq 0} x\cdot \frac{e^{-20}20^{x}}{x!} + 7\sum_{x\geq 0}\frac{e^{-20}20^{x}}{x!} \\
&= \mathbb{E}[x] + 7\cdot 1, \quad x\sim Poi(20) \\
&= 20+7 \\
&= 27
\end{aligned}
$$

Q2

First note that $N_{t+s} - N_t \sim Poi\left(\int_t^{t+s} \lambda(t)dt\right)$

$$\begin{aligned}
\mathbb{P}(N_s = m | N_t = n) &= \frac{\mathbb{P}(N_s = m, N_t = n)}{\mathbb{P}(N_t = n)} \\
&= \frac{\mathbb{P}(N_s = m, N_{t \cap s^c} = n - m)}{\mathbb{P}(N_t = n)} \\
&= \frac{\mathbb{P}(N_s = m)\mathbb{P}(N_{t \cap s^c} = n - m)}{\mathbb{P}(N_t = n)} \\
&= \frac{\frac{\left(\int_0^s 1-e^{-t}dt\right)^m e^{-\int_0^s 1-e^{-t}dt}}{m!} \frac{\left(\int_s^t 1-e^{-t}dt\right)^{n-m} e^{-\int_s^t 1-e^{-t}dt}}{(n-m)!}}{\frac{\left(\int_0^t 1-e^{-t}dt\right)^n e^{-\int_0^t 1-e^{-t}dt}}{n!}} \\
&= \frac{n!(s-1+e^{-s})^m e^{1-s-e^{-s}} (t-s+e^{-t}-e^{-s})^{n-m} e^{t-s+e^{-t}-e^{-s}}}{m!(n-m)!(t-1+e^{-t})^n e^{1-t-e^{-t}}} \\
&= \binom{n}{m} e^{2e^{-t}-2e^{-s}+2t-2s} \frac{(s-1+e^{-s})^m (t-s+e^{-t}-e^{-s})^{n-m}}{(t-1+e^{-t})^n}
\end{aligned}$$

Q3

a)

b)

c)

Q4

a)

b)

c)

d)