Math3303 Assignment 4

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Q1

Consider the action of $GL_2(\mathbb{Z})$ on $\mathbb{Z}^2$. Determine all the orbits and stabilisers of this action.

We first note that for a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, it's inverse is $A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ and as the inverses must also be in the group the entries remain integers if and only if $\det(A) = ad - bc = \pm 1$. We also note that the orbit and stabiliser of the zero vector are:

$$\mathcal{O}(\mathbf{0}) = \{\mathbf{0}\}$$
$$\text{Stab}(\mathbf{0}) = GL_2(\mathbb{Z})$$

Now we want to show that the orbits in $\mathbb{Z}^2$ under the action of $GL_2(\mathbb{Z})$ are the vectors whose coordinates have a fixed greatest common divisor. Each orbit contains one vector of the form $\begin{pmatrix} m \\ 0 \end{pmatrix}$ for $m \geq 0$, and the stabiliser of $\begin{pmatrix} m \\ 0 \end{pmatrix}$ for $m > 0$ is $\left\{ \begin{pmatrix} 1 & x \\ 0 & y \end{pmatrix} \mid \ y = \pm 1, x \in \mathbb{Z} \right\} \subset GL_2(\mathbb{Z})$.

Proof. We first note that $\gcd(m, 0) = m$, so the fixed gcd in each orbit is going to be m. We also note that the stabiliser is trivially true as $\begin{pmatrix} 1 & x \\ 0 & y \end{pmatrix} \begin{pmatrix} m \\ 0 \end{pmatrix} = \begin{pmatrix} m \\ 0 \end{pmatrix}$. We also have that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} m \\ 0 \end{pmatrix} = \begin{pmatrix} ma \\ mc \end{pmatrix}$. We have that a and c are relatively prime as $ad - bc = \pm 1$, which means that $\gcd(ma, mc) = m$. Conversely, each vector of the form $\begin{pmatrix} g \\ h \end{pmatrix}$, where $\gcd(g, h) = m$ is in the orbit of $\begin{pmatrix} m \\ 0 \end{pmatrix}$. We can solve $gx + hy = m$ for some integers x and y so $\begin{pmatrix} \frac{g}{m} & -y \\ \frac{h}{m} & x \end{pmatrix}$ is in $GL_2(\mathbb{Z})$ and from the solution to $gx + hy = m$, we get that the determinant is $\frac{g}{m}x + \frac{h}{m}y = 1$.

Also note that these fractions are in the integers as $\gcd(g,h) = m$, so m is a divisor of both g and h. Finally $\begin{pmatrix} \frac{g}{m} & -y \\ \frac{h}{m} & x \end{pmatrix} \begin{pmatrix} m \\ 0 \end{pmatrix} = \begin{pmatrix} g \\ h \end{pmatrix}$. The stabiliser for all elements of the form $\begin{pmatrix} g \\ h \end{pmatrix}$ is $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$. $\square$

Q2

Let G be a finite group acting transitively on a set X satisfying $1 < |X| < \infty$. Show that there exists $g \in G$ which fixes no element of X.

Proof. As G is transitive (X has only one orbit) we get from the orbit stabiliser theorem, $|\text{Stab}(x)| = \frac{|G|}{|X|}$, $\forall x \in X$ (note that $|\text{Orb}(x)| = |X|$ by transistivity and the orbit stabiliser theorem states $|\text{Orb}(x)| = \frac{|G|}{|\text{stab}(x)|}$). Noting that every stabiliser contains the identity, we get:

$$\begin{aligned} \left| \bigcup_{x \in X} \text{Stab}(x) \right| &= \left| \bigcup_{x \in X} \{g \in G | gx = x\} \right| \\ &\leq 1 + |G| - |X| \\ &= 1 + |X||\text{Stab}(x)| - |X| \\ &= 1 + |X| \big(|\text{Stab}(x)| - 1| \big) \\ &= 1 + |X| \left(\frac{|G|}{|X|} - 1 \right) \\ &< |G| \end{aligned}$$

Hence $\bigcup_{x \in X} \text{Stab}(x) \neq G$ so $\exists g \in G$ s.t. $gx \neq x, \quad \forall x \in X$. We also note the last line holds as:

$$\begin{aligned} & 1 + |X| \left(\frac{|G|}{|X|} - 1 \right) < |G| \\ \iff & |G| - |X| < |G| - 1 \end{aligned}$$

Which holds as $1 < |X|$. $\square$