Math3303 Assignment 1

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Q1

a)

We want to show that $G^\vee$ is a group.

Proof. Associativity:
$\forall \phi, \varphi, \theta \in G^\vee$ we have $(\phi(g)\varphi(g))\theta(g) = \phi(g)(\varphi(g)\theta(g))$ because $\phi(g), \varphi(g), \theta(g) \in \mathbb{C}^\times$, which is a set where associativity holds ($(a+bi)(c+di) = ac+adi+bci-bd = (c+di)(a+bi)$) so $\mathbb{C}^\times$ is abelian).

Identity:
The identity is the identity homomorphism, $\phi(g) = 1$. This is a homomorphism as $\forall g, h \in G$ we have:

$$\begin{aligned} 1 &= \phi(g \cdot h) \\ &= \phi(g)\phi(h) \\ &= 1 \cdot 1 \\ &= 1 \end{aligned}$$

This is the identity as for some $\theta(g) \in \mathbb{C}^\times$ we have $\phi(g)\theta(g) = 1 \cdot (a+bi) = (a+bi) \cdot 1 = \theta(g)\phi(g)$.

Inverses: For inverses we have $(\phi)^{-1}(g) = (\phi(g))^{-1} = \phi(g^{-1})$, which holds as $\mathbb{C}^\times$ is abelian.

All group axioms hold so $G^\vee$ is a group. $\square$

b)

We want to show that $(\mathbb{Z}/n)^\vee \cong \mathbb{Z}/n$.

Proof. First we will show that homomorphisms map to the roots of unity. We have that 1 is the generator of $\mathbb{Z}/n$ and 0 is it's identity. As a homomorphism preserves identity we have for a homomorphism ϕ, $\phi(0) = 1$ and we let $a = \phi(1)$. So we have:

$$\phi(n) = \phi(n \cdot 1) = a^n = 1 \implies a = \exp\left(\frac{2\pi i k}{n}\right), \quad k \in \{0, ..., n-1\}$$

So there are n maps defined by:

$$\phi_k(1) = \exp\left(\frac{2\pi i k}{n}\right), \quad k \in \{0, ..., n-1\}$$

Noting this is what one maps to and by homomorphism $\phi(2) = \phi(1+1) = \phi(1)\phi(1)$ which can be extended until n-1 is reached (n maps back to identity). So the general homomorphisms for $g \in \mathbb{Z}/n$ are:

$$\phi_k(g) = \exp\left(\frac{2\pi i k}{n}\right)^g, \quad k \in \{0, ..., n-1\}$$

As there are n maps the order of $(\mathbb{Z}/n)^\vee$ is $(\mathbb{Z}/n)^\vee$ is n. $(\mathbb{Z}/n)^\vee$ is also abelian as $\phi_{k_1}(g_1)\phi_{k_2}(g_2) = \exp\left(\frac{2\pi i k_1}{n}\right)^{g_1} \exp\left(\frac{2\pi i k_2}{n}\right)^{g_2} = \exp\left(\frac{2\pi i k_2}{n}\right)^{g_2} \exp\left(\frac{2\pi i k_1}{n}\right)^{g_1} = \phi_{k_2}(g_2)\phi_{k_1}(g_1)$. We also have that $\mathbb{Z}/n$ is also of order n and is abelian, so by the finite theorem of abelian groups we have $(\mathbb{Z}/n)^\vee \cong \mathbb{Z}/n$. $\square$

c)

We want to show $(G \times H)^\vee \cong G^\vee \times H^\vee$.

Proof. Suppose $g \in G$ and $h \in H$. We define the map $\theta : (G \times H)^\vee \to G^\vee \times H^\vee$, where $\theta(\phi((g, h))) = (\phi(g), \phi(h))$. This is a homomorphism because:

$$\begin{aligned}
\theta(\phi_1((g_1, h_1)), \phi_2((g_2, h_2))) &= \theta(\phi_1((g_1, h_1))\phi_2((g_2, h_2))) \\
&= (\phi_1(g_1), \phi_1(h_1))(\phi_1(g_2), \phi_1(h_2)) \\
&= \theta(\phi_1((g_1, h_1))\theta(\phi_2((g_2, h_2))
\end{aligned}$$

We can also define the map $\alpha : G^\vee \times H^\vee \to (G \times H)^\vee$, where $\alpha((\phi(g), \phi(h)) = \phi((g, h)))$ which is a homomorphism because:

$$\begin{aligned}
\alpha((\phi_1(g_1), \phi_1(h_1)), (\phi_1(g_2), \phi_1(h_2))) &= \alpha((\phi_1(g_1)\phi_2(g_2), \phi_1(h_1)\phi_2(h_2))) \\
&= (\phi_1(g_1)\phi_2(g_2), \phi_1(h_1)\phi_2(h_2)) \\
&= \phi_1((g_1, h_1))\phi_2((g_2, h_2)) \\
&= \alpha((\phi_1(g_1), \phi_1(h_1)))\alpha((\phi_2(g_2), \phi_2(h_2)))
\end{aligned}$$

We also trivially see that trivially $\theta \circ \alpha = \mathbf{Id}$ and $\alpha \circ \theta = \mathbf{Id}$. Composed both ways they are the identity mapping, and hence $(G \times H)^\vee \cong G^\vee \times H^\vee$. We also note that this trivially extends to the direct product of n groups, and in this case the tuple is replaced with $(g_1, ..., g_n)$. $\square$

d)

We want to show that if G is a finite abelian group, then $G^\vee \cong G$.

Proof.

$$
\begin{aligned}
G^\vee &\cong (\mathbb{Z}_{p_1^{\alpha_1}} \times ... \times \mathbb{Z}_{p_n^{\alpha_n}})^\vee \quad (*\text{fundamental theorem of finite abelian groups}) \\
&\cong (\mathbb{Z}_{p_1^{\alpha_1}})^\vee \times ... \times (\mathbb{Z}_{p_n^{\alpha_n}})^\vee \quad (*\text{From c)}) \\
&\cong \mathbb{Z}_{p_1^{\alpha_1}} \times ... \times \mathbb{Z}_{p_n^{\alpha_n}} \quad (*\text{From b)}) \\
&\cong G \quad (*\text{fundamental theorem of finite abelian groups})
\end{aligned}
$$

□

Q2

a)

We want to show that the subgroup generated by A, $[G,G]$, is normal in G.

Proof. If $g \in G$ and $n \in [G,G] \leq G$, then we have that $gng^{-1}n^{-1} \in [G,G]$ and :

$$(gng^{-1}n^{-1})n = gng^{-1}$$

As $[G,G]$ is closed under products we have $gng^{-1} \in [G,G]$, hence by definition $[G,G]$ is normal in G. □

b)

We want to show that if G is a normal subgroup of M , then $[G,G]$ is also a normal subgroup of M.

Proof. Suppose $g,h \in G$ and $m \in M$. This means we have $mgm^{-1} \in G$ and $mhm^{-1} \in G$ because G is normal in M. As these are elements of G we have $mgm^{-1}mhm^{-1}(mgm^{-1})^{-1}(mhm^{-1})^{-1} \in [G,G]$. This gives:

$$
\begin{aligned}
mgm^{-1}mhm^{-1}(mgm^{-1})^{-1}(mhm^{-1})^{-1} &= mghm^{-1}mg^{-1}m^{-1}mh^{-1}m^{-1} \\
&= mghg^{-1}h^{-1}m^{-1} \\
&= mam^{-1}
\end{aligned}
$$

Here $a = ghg^{-1}h^{-1} \in [G,G]$. So a is a general element of $[G,G]$ and m a general element of M, so by definition of normality we have that $[G,G]$ is normal in M. □