Stat3001 Assignment 1

Dominic Scocchera

March 2023

Q1

We have $x_1, ..., x_n \overset{iid}{\sim} N(\theta, \theta^2)$ so the likelihood function is:

$$
\begin{aligned}
L(\tilde{x};\theta) &= \prod_{i=1}^{n} f(\tilde{x};\theta) \\
&= \prod_{i=1}^{n} (2\pi)^{-\frac{1}{2}}(\theta^2)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}\frac{(x_i-\theta)^2}{\theta^2}\right) \\
&= (2\pi)^{-\frac{n}{2}}(\theta)^{-n} \exp\left(-\frac{1}{2}\sum_{i=1}^{n}\frac{(x_i-\theta)^2}{\theta^2}\right) \\
&= (2\pi)^{-\frac{n}{2}}(\theta)^{-n} \exp\left(-\frac{1}{2}\sum_{i=1}^{n}\frac{x_i^2-2x_i\theta+\theta^2}{\theta^2}\right) \\
&= (2\pi)^{-\frac{n}{2}}(\theta)^{-n} \exp\left(-\frac{1}{2\theta^2}\sum_{i=1}^{n}x_i^2+\frac{1}{\theta}\sum_{i=1}^{n}x_i-\frac{n}{2}\right) \\
&= \exp\left(-\frac{1}{2\theta^2}\sum_{i=1}^{n}x_i^2+\frac{1}{\theta}\sum_{i=1}^{n}x_i\right)(2\pi)^{-\frac{n}{2}}(\theta)^{-n}\exp\left(-\frac{n}{2}\right) \\
&= h_1(T(\tilde{x}))h_2(\tilde{x})
\end{aligned}
$$

Hence we have by the Fisher-Neyman factorisation theorem:

$$T(\tilde{x}) = \left(\sum_{i=1}^{n}x_i, \sum_{i=1}^{n}x_i^2\right)^T$$

Here $T(\tilde{x})$ is the sufficient statistic for θ.

We also have:

$$
\begin{aligned}
q &= \dim(T(\tilde{x})) = 2 \\
d &= \dim(\theta) = 1
\end{aligned}
$$

Noting that we can not factorise any further as it is a polynomial of different quadratic and linear terms so $q = 2$. As $q \neq d$ it does not belong to the regular exponential family.

Q2

We have $x_1, ..., x_n \overset{iid}{\sim} N(\theta, \theta)$ so the likelihood function is:

$$
\begin{aligned}
L(\tilde{x}; \theta) &= \prod_{i=1}^{n} f(\tilde{x}; \theta) \\
&= \prod_{i=1}^{n} (2\pi)^{-\frac{1}{2}} (\theta)^{-\frac{1}{2}} \exp\left(-\frac{1}{2} \frac{(x_i - \theta)^2}{\theta}\right) \\
&= (2\pi)^{-\frac{n}{2}} (\theta)^{-\frac{n}{2}} \exp\left(-\frac{1}{2} \sum_{i=1}^{n} \frac{(x_i - \theta)^2}{\theta}\right) \\
&= (2\pi)^{-\frac{n}{2}} (\theta)^{-\frac{n}{2}} \exp\left(-\frac{1}{2} \sum_{i=1}^{n} \frac{x_i^2 - 2x_i\theta + \theta^2}{\theta}\right) \\
&= (2\pi)^{-\frac{n}{2}} (\theta)^{-\frac{n}{2}} \exp\left(-\frac{1}{2\theta} \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} x_i + n\theta\right)
\end{aligned}
$$

Now taking the log likelihood we get:

$$
\log L(\tilde{x}; \theta) = -\frac{n}{2} (\log 2\pi + \log \theta) - \frac{1}{2\theta} \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} x_i + n\theta
$$

For the regular exponential family we require:

$$
L(\tilde{x}; \theta) = \frac{b(\tilde{x}) \exp(c(\theta)^T T(\tilde{x}))}{a(\theta)}
$$

And hence:

$$
\log L(\tilde{x}; \theta) = \log b(\tilde{x}) + c(\theta)^T T(\tilde{x}) - \log a(\theta)
$$

Here we have:

$$
\begin{aligned}
a(\theta) &= \frac{n}{2}(\log 2\pi + \log \theta + 2\theta) \\
b(\tilde{x}) &= \sum_{i=1}^{n} x_i \\
c(\theta) &= -\frac{1}{2\theta} \\
T(\tilde{x}) &= \sum_{i=1}^{n} x_i^2
\end{aligned}
$$

Here $T(\tilde{x})$ is the sufficient statistic for θ and from this we see that this distribution belongs to the regular exponential family.

Taking the derivative of the log likelihood equation and setting it to zero we get:

$$
\begin{aligned}
\frac{d}{d\theta} \log L(\tilde{x}; \theta) &= -\frac{n}{2\theta} + \frac{1}{2\theta^2} \sum_{i=1}^{n} x_i^2 + n \\
&= 0
\end{aligned}
$$

Multiplying through by $2\theta^2$ we get:

$$
0 = -n\theta + \sum_{i=1}^{n} x_i^2 + n\theta^2
$$

This is quadratic in θ so we get:

$$
\hat{\theta} = \frac{-1 + \sqrt{1 + 4\frac{1}{n}\sum_{i=1}^{n} x_i^2}}{2}
$$

As the likelihood function is from the regular exponential family we have:

$$\begin{aligned}
\mathbb{E}\left[\sum_{i=1}^{n} x_i^2\right] &= \sum_{i=1}^{n} \mathbb{E}\left[x_i^2\right] \\
&= \sum_{i=1}^{n} \mathrm{Var}(X_i) + \mathbb{E}\left[x_i\right]^2 \\
&= \sum_{i=1}^{n} \theta + \theta^2 \\
&= n(\theta + \theta^2) \\
&= \sum_{i=1}^{n} x_i^2
\end{aligned}$$

Rearranging we get:

$$0 = n\theta^2 + n\theta - \sum_{i=1}^{n} x_i^2$$

Which is quadratic in θ so we get:

$$\begin{aligned}
\hat{\theta} &= \frac{-n + \sqrt{n^2 + 2n \sum_{i=1}^{n} x_i^2}}{2n} \\
&= \frac{-1 + \sqrt{1 + 4\frac{1}{n} \sum_{i=1}^{n} x_i^2}}{2}
\end{aligned}$$

As the likelihood is from the regular exponential family the suffecient statistic is also complete which guarentees the estimator of θ is unique.