

Stat3004 Assignment 1

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Q1

We want to show $\text{Var}(X) = \mathbb{E}[\text{Var}(X|Y)] + \text{Var}(\mathbb{E}[X|Y])$.

Proof.

$$\begin{aligned}\text{Var}(X) &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 \\ &= \mathbb{E}[\mathbb{E}[X^2|Y]] - \mathbb{E}[X]^2 \\ &= \mathbb{E}[\text{Var}(X|Y) + \mathbb{E}[X|Y]^2] - \mathbb{E}[X]^2 \\ &= \mathbb{E}[\text{Var}(X|Y) + \mathbb{E}[X|Y]^2 - \mathbb{E}[\mathbb{E}[X|Y]]^2] \\ &= \mathbb{E}[\text{Var}(X|Y)] + (\mathbb{E}[\mathbb{E}[X|Y]^2] - \mathbb{E}[\mathbb{E}[X|Y]]^2) \\ &= \mathbb{E}[\text{Var}(X|Y)] + \text{Var}(\mathbb{E}[X|Y])\end{aligned}$$

□

Q2

a)

Let X be a non-negative random variable with *p.d.f.*, f . Show $\mathbb{E}X = \int_0^\infty \mathbb{P}(X \geq x)dx$.

Proof.

$$\begin{aligned}
\mathbb{E}X &= \int_0^\infty xf(x)dx \\
&= - \int_0^\infty x(-f(x))dx \\
&= -x(1-F(x)) \Big|_0^\infty + \int_0^\infty 1-F(x)dx \quad (*\text{note, Integration by parts and } F(x) \text{ is the c.d.f}) \\
&= \int_0^\infty 1-F(x)dx \quad (*\text{note } \lim_{x \rightarrow \infty} 1-F(x) = 0 \quad \text{and} \quad -0(1-F(0)) = 0) \\
&= \int_0^\infty (1-\mathbb{P}(X \leq x))dx \\
&= \int_0^\infty \mathbb{P}(X \geq x)dx
\end{aligned}$$

□

b)

Show $\mathbb{E}[x^\alpha] = \int_0^\infty \alpha x^{\alpha-1} \mathbb{P}(X \geq x)dx$, where $\alpha > 0$.

Proof.

$$\begin{aligned}
\mathbb{E}[Y] &= \int_0^\infty \mathbb{P}(Y \geq y)dy \quad (*\text{note we have this from a) and } Y = X^\alpha) \\
&= \int_0^\infty \alpha x^{\alpha-1} \mathbb{P}(X^\alpha \geq x^\alpha)dx \quad (*\text{note the change of variable, } y = x^\alpha \iff dy = \alpha x^{\alpha-1}dx)
\end{aligned}$$

□

Q3

Suppose X_1, \dots, X_n are independent random variables with c.d.f's F_1, \dots, F_n respectively. Express the c.d.f of $M = \min(X_1, \dots, X_n)$ in terms of the $\{F_i\}$.

$$\begin{aligned}
F_M(x) &= \mathbb{P}(M \leq x) \\
&= \mathbb{P}(\min(X_1, \dots, X_n) \leq x) \\
&= 1 - \mathbb{P}(X_1 \geq x, \dots, X_n \geq x) \\
&= 1 - \mathbb{P}(X_1 \geq x) \dots \mathbb{P}(X_n \geq x) \\
&= 1 - (1 - F_1(x)) \dots (1 - F_n(x)) \\
&= 1 - \prod_{i=1}^n (1 - F_i(x))
\end{aligned}$$

Q4

a)

Determine $\mathcal{G}(z) = \mathbb{E}[z^X]$ for $z \in [0, 1]$.

$$\begin{aligned}
\mathcal{G}(z) &= \mathbb{E}[z^X] \\
&= z^0 \mathbb{P}(X = 0) + z^1 \mathbb{P}(X = 1) + z^2 \mathbb{P}(X = 2) \\
&= 1 - r - s + zr + z^2 s \\
&= 1 + r(z - 1) + s(z^2 - 1)
\end{aligned}$$

b)

We want to determine the mean and variance of S_n . We first determine the mean.

$$\begin{aligned}
\mathbb{E}[S_n] &= \mathbb{E} \left[\sum_{i=1}^{S_{n-1}} X_{i,n-1} \right] \\
&= \mathbb{E} \left[\mathbb{E} \left[\sum_{i=1}^{S_{n-1}} X_{i,n-1} \middle| S_{n-1} \right] \right] \\
&= \mathbb{E} \left[\sum_{i=1}^{S_{n-1}} \mathbb{E}[X_{i,n-1} | S_{n-1}] \right] \\
&= \mathbb{E} \left[\sum_{i=1}^{S_{n-1}} \mathbb{E}[X_{i,n-1}] \right] \\
&= \mathbb{E}[X_{i,n-1}] \mathbb{E}[S_{n-1}] \\
&= (0 \cdot (1-r-s) + 1 \cdot r + 2 \cdot s) \mathbb{E}[S_{n-1}] \\
&= (r+2s) \mathbb{E}[S_{n-1}]
\end{aligned}$$

We know $S_0 = 1 \implies S_j = \mathbb{E}[X]^j$, so we have:

$$\begin{aligned}
\mathbb{E}[S_n] &= (r+2s)(r+2s)^{n-1} \\
&= (r+2s)^n
\end{aligned}$$

Now we determine the variance. First noting that $\mu = \mathbb{E}[X] = r+2s$ and $\sigma^2 = \text{Var}(X) = \mathbb{E}[X^2] - \mu^2 = 0^2(1-r-s) + 1^2r + 2^2s - (r+2s)^2 = r+4s - (r+2s)^2$. And from lectures we have:

$$\begin{aligned}
\text{Var}(S_n) &= \begin{cases} \sigma^2 \mu^{n-1} \left(\frac{1-\mu^n}{1-\mu} \right), & \text{if } \mu \neq 1 \\ \sigma^2 n, & \text{if } \mu = 1 \end{cases} \\
&= \begin{cases} (r+4s - (r+2s)^2)(r+2s)^{n-1} \left(\frac{1-(r+2s)^n}{1-(r+2s)} \right), & \text{if } r+2s \neq 1 \\ (r+4s - (r+2s)^2)n, & \text{if } r+2s = 1 \end{cases}
\end{aligned}$$

c)

We know from lectures that the extinction probability is given by:

$$\eta = \mathcal{G}(\eta)$$

So we have:

$$\begin{aligned}\eta &= \mathcal{G}(\eta) \\ &= (1 - r - s) + r\eta + s\eta^2\end{aligned}$$

Rearranging we get:

$$0 = (1 - r - s) + (r - 1)\eta + s\eta^2$$

Solving for η we get:

$$\eta = \begin{cases} \frac{-(r-1) - \sqrt{(r-1)^2 - 4s(1-r-s)}}{2s} & \text{if } r + 2s > 1 \\ 1 & \text{if } r + 2s \leq 1 \\ 0 & \text{if } r + s = 1 \end{cases}$$

The 0 case is trivial as we have that $\mathbb{P}(X = 0) = 1 - r - s = 1 - (r + s) = 1 - 1 = 0$, so there all individuals are guaranteed to reproduce and hence extinction is impossible. The 1 case follows from lectures, we have that $\mu = r + 2s \leq 1$ and from above we know the variance is non-zero in this case. The last case follows from the solution to the above equation. We also only use the negative sign case as the positive sign case is ≥ 1 which violates the requirement derived in lectures (< 1).

$$\begin{aligned}\frac{-(r-1) + \sqrt{(r-1)^2 - 4s(1-r-s)}}{2s} &\geq 1 \\ \sqrt{(r-1)^2 - 4s(1-r-s)} &\geq 2s + (r-1) \\ (r-1)^2 - 4s(1-r-s) &\geq (2s + (r-1))^2 \\ r^2 - 2r + 1 - 4s + 4sr + 4s^2 &\geq 4s^2 + 4sr - 4s + r^2 - 2r + 1 \\ 0 &\geq 0\end{aligned}$$

Which always holds.

Q5

First we want to show $\mathcal{G}_n(z) = \mathcal{G}_{n-1}(\mathcal{G}(z))$.

Proof. First we will show that for $X = Y_1 + \dots + Y_N$ where Y_i is *i.i.d*, noting that X is a random sum of random variables so N is a random variable, that

$$\mathcal{G}_X(z) = \mathcal{G}_N(\mathcal{G}_{Y_1}(z)).$$

$$\begin{aligned}
\mathcal{G}_X(z) &= \mathbb{E}[z^X] \\
&= \sum_{x=0}^{\infty} \mathbb{P}(X = x) z^x \\
&= \sum_{n=0}^{\infty} \sum_{x=0}^{\infty} \mathbb{P}(X = x | N = n) \mathbb{P}(N = n) z^x \\
&= \sum_{n=0}^{\infty} \mathbb{P}(N = n) \sum_{x=0}^{\infty} \mathbb{P}(X = x | N = n) z^x \\
&= \sum_{n=0}^{\infty} \mathbb{P}(N = n) (\mathcal{G}_{Y_1}(z))^n \\
&= \mathcal{G}_N(\mathcal{G}_{Y_1}(z))
\end{aligned}$$

For a branching process we have $S_n = X_{1,n-1} + X_{2,n-1} + \dots + X_{N,n-1}$, where $X_{i,n-1}$ is the number of progeny produced by the i_{th} member of the previous generation. It is clear to see that this is the same situation as what was shown above (random sum of random variables) and thus plugging in our variables to what was shown above we immediately have our result:

$$\mathcal{G}_n(z) = \mathcal{G}_{n-1}(\mathcal{G}(z))$$

□

Now we want to show $\mathcal{G}_n(z) = \mathcal{G}(\mathcal{G}_{n-1}(z))$.

Proof. We can easily see from the above result that we can continue it to get $\mathcal{G}_{n-1}(z) = \mathcal{G}_{n-2}(\mathcal{G}(z))$. Continuing this until we reach $\mathcal{G}_2(z) = \mathcal{G}_1(\mathcal{G}(z))$. This holds as we have a random sum of random variables in each of these cases so this relation will hold regardless of the generation. We can see that $\mathcal{G}_n(z) = \mathcal{G}(\dots(\mathcal{G}(z))\dots)$ where this is n times. This is due to the recursive relation where for example $\mathcal{G}_3(z) = \mathcal{G}_2(\mathcal{G}(z)) = \mathcal{G}_1(\mathcal{G}(\mathcal{G}(z)))$, we note here that $\mathcal{G}_1(z) = \mathcal{G}(z)$. From this we see that all the terms in the middle of the brackets on the right hand side equal $\mathcal{G}_{n-1}(z)$, so we have our result:

$$\mathcal{G}_n(z) = \mathcal{G}(\mathcal{G}_{n-1}(z))$$

□