Math3303 Assignment 6

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May 2023

Q1

Let R be an integral domain such that $(a_0) \supseteq (a_1) \supseteq (a_2) \supseteq ...$ implies that $(a_n) = (a_{n+1}) = ...$ for n sufficiently large. Now we want to show that R is a field.

Proof. As R is an integral domain for R to also be a field we require that every element of R has a multiplicative inverse. We will show that every element has a multiplicative inverse via a proof by contradiction. Suppose a_0 is not a unit. We also have that $(a_n) = (a_{n+1})$ for n sufficiently large. This means that $a_n \in (a_{n+1})$, so we have that for some $r \in R$ (noting that we have commutivity and distributivity as R is an integral domain):

$$\begin{aligned} a_n &= ra_{n+1} \\ \iff a_n - ra_{n+1} &= 0 \\ \iff a_n - ra_n a_0 &= 0 \\ \iff a_n - a_n r a_0 &= 0 \\ \iff a_n(1 - ra_0) &= 0 \end{aligned}$$

As R is an integral domain this means that $1 = ra_0$, i.e. a_0 is a unit. Thus we have arrived at the contradiction. □

Note on Q2 and Q3

In Q2 and Q3 we will often consider the partial ordering of ideals of a commutative ring R with identity, where the ordering is given by $\subseteq$. It is not hard to see that the zero ideal is a subset of all other ideals (an ideal in a commutative ring requires $rx = xr \in I$, $\forall x \in I$ and $\forall r \in R$ and as $0 \in R$ we have that 0 must always be in an ideal). We also have by theorem 9.22 from Gregory Lee's abstract algebra that every maximal ideal in this ring is a prime ideal and we also note that each of these ideals is a subset of the ring itself. Thus we see that ideals with subset ordering form a finite partially ordered set with all elements bounded above by R and below by the zero ideal.

Q2

Let I be an ideal in a commutative unital ring R. Define

$$\hat{I} := \{r \in R | r^n \in I \text{ for some } n \in \mathbb{Z} > 0\}.$$

We first prove that for $S = \{r, r^2, ...\}$ and I, an ideal disjoint from S, i.e. $r^n \notin I$ for any n, there is a prime ideal that contains I and is disjoint from S.

Proof. First note that a prime ideal ($\mathcal{P}$) must satisfy 1) $\mathcal{P} \neq$R and 2) if $a, b \in$R and $ab \in \mathcal{P}$ then $a \in \mathcal{P}$ or $b \in \mathcal{P}$. In this proof we will show the equivalent condition of 2), $a \notin \mathcal{P}$ and $b \notin \mathcal{P}$ then $ab \notin \mathcal{P}$. As $r^n \notin I$ we must have $I \neq R$. Now from Gregory Lee's Abstract Algebra theorem 9.22 we have that every maximal ideal of R is also a prime ideal. This means that if we partially order the ideals from I with the order being given by $\subseteq$, $\exists \mathcal{P}$ such that $\mathcal{P} \geq I$. So we now have condition 1 as each chain from I is bounded above by a maximal prime ideal. If one of these maximal prime ideals does not intersect with S we are done, however this not guarenteed so we now consider the subset of ideals in the partial ordering that don't intersect with S. We know that this subset of the partial ordering is non-empty as $r^n \notin I$. From this we now consider an ideal Q that is maximal in a chain of this subset. As this subset is finite and non-empty we know that such a Q exists (Zorn's lemma). If we consider two ideals A and B that follow from Q in the partial ordering we must have $r^i \in A$ and $r^j \in B$ for some $i, j \in \mathbb{Z}_{>0}$. Now if we consider a third ideal C with $r^{i+j} \in C$ we must have $A \subseteq C$ and $B \subseteq C$. This now gives $r^i \notin Q$ and $r^j \notin Q$ then $r^{i+j} \notin Q$, thus condition 2) is satisfied and Q is a prime ideal containing I. □

We now want to show that $\hat{I}$ equals the intersection of all prime ideals of R which contain I.

Proof. First let $\mathcal{P}$ be some prime ideal containing I. We will prove this by proving two relations:

$$(1) \quad \hat{I} \subseteq \bigcap_{I \leq \mathcal{P}} \mathcal{P}$$

$$(2) \quad \hat{I} \supseteq \bigcap_{I \leq \mathcal{P}} \mathcal{P}$$

Where $\bigcap_{I \leq \mathcal{P}} \mathcal{P}$ is the intersect of all prime ideals containing I.

(1)
If $\mathcal{P}$ is some prime ideal containing I and we have some $r \in R$ such that $r^n \in I$, then as $I \leq \mathcal{P}$ we have $r^n \in \mathcal{P}$ and as $\mathcal{P}$ is a prime ideal we also must have $r \in \mathcal{P}$.

(2)
Now if we consider $r \notin \hat{I}$, then $r^n \notin I$ for any n, so $S = \{r, r^2, ...\}$ is a set

disjoint from I. From what we first proved we know that there is a prime ideal Q containing I with $r \notin Q$, thus we have $r \notin \bigcap_{I \leq \mathcal{P}} \mathcal{P}$.

These two relations immediantly imply that $\hat{I} = \bigcap_{I \leq \mathcal{P}} \mathcal{P}$. $\square$

Q3

a)

$V(I) = \emptyset$ if and only if I =R.

Proof. First consider the case $I = R$.
We then have $V(I) = V(R)$ and as there are no prime ideals that contain the entire ring (R is not a prime ideal as we require that a prime ideal isn't equal to R) we must have $V(I) = \emptyset$.

Now consider the case $I \neq R$.
We know from theorem 9.22 of Gregory Lee's Abstract Algebra that every maximal ideal in R is a prime ideal. Now partially ordering the ideals with the ordering given by $\subseteq$ we see that all ideals I such that $I \neq$R must be bounded above by a prime ideal (noting that an ideal can contain itself as ideal). Thus we always have $V(I) \neq \emptyset$. $\square$

b)

$V(I) \cup V(J) = V(IJ)$

Proof. We first note the partial ordering we have constructed and the fact from Gregory Lee's Abstract Algebra page 151 that by absorbtion property, $IJ \subseteq I \cap J$ and that if we have ideals A and B such that $A \subseteq B$ we must have $V(A) \supseteq V(B)$ as all prime ideals contained in B must also be contained in A due to the partial ordering.

(1) $V(IJ) \supseteq V(I) \cup V(J)$:
If we let $IJ \subseteq I \cap J = X$ we must have $X \subseteq I$, $X \subseteq J$ and $\mathbf{0} \subseteq X$ (This is from the partial ordering we constructed (the zero ideal must be contained in X)). This then gives $V(IJ) \supseteq V(X)$, $V(X) \supseteq V(I)$ and $V(X) \supseteq V(J)$. Now we see that we have (1), $V(IJ) \supseteq V(I) \cup V(J)$.

(2) $V(IJ) \subseteq V(I) \cup V(J)$:
Now if we take a prime ideal $\mathcal{P} \in V(IJ)$, we want to show $\mathcal{P} \in V(I)$ or $\mathcal{P} \in V(J)$. Suppose $IJ \subseteq \mathcal{P}$ and I is not contained in $\mathcal{P}$. We now show that for all $j \in J$, we have $j \in \mathcal{P}$. Fix $j \in J$ and $i \in I \backslash \mathcal{P}$ and note that $ij \in IJ$. Since $IJ \subseteq \mathcal{P}$, we have that $ij \in \mathcal{P}$ but $\mathcal{P}$ is prime, so we must have that either $i \in \mathcal{P}$ or $j \in \mathcal{P}$. Since $i \notin \mathcal{P}$ (This is because $i \in I \backslash \mathcal{P}$), we conclude that $j \in J$. This shows that $J \subseteq \mathcal{P}$. The argument is similar if we assume hat J is not contained

in $\mathcal{P}$. In that case we get that $I \subseteq \mathcal{P}$. So if we have a prime ideal $\mathcal{P} \in V(IJ)$, then $\mathcal{P} \in V(I)$ or $\mathcal{P} \in V(J)$. This then implies (2), $V(IJ) \subseteq V(I) \cup V(J)$.

From (1) and (2) we must have equality which is what we wanted to show. $\square$

c)

Let $\{I_\alpha\}$ be a set of ideals of R. Then $\cap_\alpha V(I_\alpha) = V(\sum_\alpha I_\alpha)$.

Proof. First note that for $I \in \{I_\alpha\}$ we have, $I \subseteq \sum_\alpha I_\alpha$.

(1)$\cap_\alpha V(I_\alpha) \supseteq V(\sum_\alpha I_\alpha)$
If we have a prime ideal $\mathcal{P}$ such that $\sum_\alpha I_\alpha \subseteq \mathcal{P}$, then for any $I \in \{I_\alpha\}$ we get $I \subseteq \sum_\alpha I_\alpha \subseteq \mathcal{P}$. We also have $\cap_\alpha V(I_\alpha)$, which is the set of prime ideals that contain all I_α. From above we see that $\mathcal{P}$ must contain all $I \in \{I_\alpha\}$ as all I are also contained in $\sum_\alpha I_\alpha$ which itself is contained in $\mathcal{P}$.

(2)$\cap_\alpha V(I_\alpha) \subseteq V(\sum_\alpha I_\alpha)$
Suppose we have a prime ideal $\mathcal{P} \in \cap_\alpha V(I_\alpha)$, i.e. $\mathcal{P}$ contains all $I \in \{I_\alpha\}$. We want to show that $\mathcal{P} \in V(\sum_\alpha I_\alpha)$. If we have $a \in I$ for some $I \in \{I_\alpha\}$ and $b \in$R such that $ab \in \mathcal{P}$ then we have either $a \in \mathcal{P}$ or $b \in \mathcal{P}$, by definition of prime ideal. We know that we must have $a \in \mathcal{P}$ from the premise. Now let $ab = i_\alpha b$. As ideals are closed under addition and we have this holding for every ideal in $\{I_\alpha\}$ we must also have $\sum_\alpha bi_\alpha = (\sum_\alpha i_\alpha)b \in \mathcal{P}$ and as $b \notin \mathcal{P}$ we must have $\sum_\alpha i_\alpha \in \mathcal{P}$ from definition of prime ideal. From this we must have $\mathcal{P} \in V(\sum_\alpha I_\alpha)$.

From (1) and (2) we get equality which is what we wanted to show. $\square$