Stat3004 Assignment 3

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Q1

First we note that $N((a,b]) \sim Poi(2 \cdot (b-a))$.

a)

$$\begin{aligned}
\mathbb{P}(N_{t_1} = n_1, N_{t_2} = n_2) &= \mathbb{P}(N_2 = 7, N_{12} = 10) \\
&= \mathbb{P}(N_2 = 7, N_{12-2} = 10 - 7) \\
&= \mathbb{P}(N_2 = 7)\mathbb{P}(N_{10} = 3) \\
&= e^{-2\cdot 2}\frac{(2\cdot 2)^7}{7!} e^{-2\cdot 10}\frac{(2\cdot 10)^3}{3!} \\
&\approx 1.636291121 \times 10^{-7}
\end{aligned}$$

b)

$$\begin{aligned}
\mathbb{P}(N(1,t_1] = n_1, N(t_1 - 1, t_2] = n_2) &= \mathbb{P}(N(1,2] = 7, N(1,12] = 10) \\
&= \mathbb{P}(N(1,2] = 7, N(2,12] = 3) \\
&= \mathbb{P}(N(1,2] = 7)\mathbb{P}(N(2,12] = 3) \\
&= \frac{e^{-2}(2)^7}{7!}\frac{e^{-20}(20)^3}{3!} \\
&\approx 9.4458178806 \times 10^{-9}
\end{aligned}$$

c)

$$
\begin{aligned}
\mathbb{E}[N(1,t_1] \mid N(t_1-1,t_2]=n_2] &= \mathbb{E}[N(1,2] \mid N(1,12]=10] \\
&= \sum_{t\geq 0} t\cdot \mathbb{P}(N(1,2]=t|N(1,12]=10) \\
&= \sum_{t=0}^{10} t\cdot \frac{\mathbb{P}(N(1,2]=t, N(1,12]=10)}{\mathbb{P}(N(1,12]=10)} \\
&= \frac{1}{\mathbb{P}(N(1,12]=10)} \sum_{t=0}^{10} t\cdot \mathbb{P}(N(1,2]=t, N(2,12]=10-t) \\
&= \frac{1}{\mathbb{P}(N(1,12]=10)} \sum_{t=0}^{10} t\cdot \mathbb{P}(N(1,2]=t)\mathbb{P}(N(2,12]=10-t) \\
&= \frac{1}{\frac{e^{-22}(22)^{10}}{10!}} \sum_{t=0}^{10} t\cdot \frac{e^{-2}(2)^t}{t!}\frac{e^{-20}(20)^{10-t}}{(10-t)!} \\
&= \frac{1}{(22)^{10}} \sum_{t=0}^{10} \binom{10}{t} t\cdot (2)^t(20)^{10-t} \\
&= \frac{24145384355840}{22^{10}} \\
&= \frac{10}{11}
\end{aligned}
$$

d)

$$
\begin{aligned}
\mathbb{E}[N(t_1-1,t_2] \mid N(1,t_1]=n_1] &= \mathbb{E}[N(1,12] \mid N(1,2]=7] \\
&= \sum_{t\geq 0} t\cdot \mathbb{P}(N(1,12]=t|N(1,2]=7) \\
&= \sum_{t\geq 7} t\cdot \mathbb{P}(N(1,12]=t|N(1,2]=7) \\
&= \sum_{t\geq 7} t\cdot \frac{\mathbb{P}(N(1,12]=t,N(1,2]=7)}{\mathbb{P}(N(1,2]=7)} \\
&= \sum_{t\geq 7} t\cdot \frac{\mathbb{P}(N(2,12]=t-7)\mathbb{P}(N(1,2]=7)}{\mathbb{P}(N(1,2]=7)} \\
&= \sum_{t\geq 7} t\cdot \mathbb{P}(N(2,12]=t-7) \\
&= \sum_{t\geq 7} t\cdot \frac{e^{-20}20^{t-7}}{(t-7)!} \\
&= \sum_{x\geq 0} (x+7)\cdot \frac{e^{-20}20^{x}}{x!}, \quad \text{note } x=t-7 \\
&= \sum_{x\geq 0} x\cdot \frac{e^{-20}20^{x}}{x!} + 7\sum_{x\geq 0} \frac{e^{-20}20^{x}}{x!} \\
&= \mathbb{E}[x] + 7\cdot 1, \quad x\sim Poi(20) \\
&= 20+7 \\
&= 27
\end{aligned}
$$

Q2

First note that $N_{t+s} - N_t \sim Poi\left(\int_t^{t+s} \lambda(t)dt\right)$

$$
\begin{aligned}
\mathbb{P}(N_s = m | N_t = n) &= \frac{\mathbb{P}(N_s = m, N_t = n)}{\mathbb{P}(N_t = n)} \\
&= \frac{\mathbb{P}(N_s = m, N_t - N_s = n - m)}{\mathbb{P}(N_t = n)} \\
&= \frac{\mathbb{P}(N_s = m)\mathbb{P}(N_t - N_s = n - m)}{\mathbb{P}(N_t = n)} \\
&= \frac{\frac{\left(\int_0^s 1-e^{-t}dt\right)^m e^{-\int_0^s 1-e^{-t}dt}}{m!} \frac{\left(\int_s^t 1-e^{-t}dt\right)^{n-m} e^{-\int_s^t 1-e^{-t}dt}}{(n-m)!}}{\frac{\left(\int_0^t 1-e^{-t}dt\right)^n e^{-\int_0^t 1-e^{-t}dt}}{n!}} \\
&= \frac{n!(s-1+e^{-s})^m e^{1-s-e^{-s}}(t-s+e^{-t}-e^{-s})^{n-m} e^{t-s+e^{-t}-e^{-s}}}{m!(n-m)!(t-1+e^{-t})^n e^{1-t-e^{-t}}} \\
&= \frac{e^{2e^{-t}-2e^{-s}+2t-2s}}{(t-1+e^{-t})^n} \binom{n}{m} (s-1+e^{-s})^m (t-s+e^{-t}-e^{-s})^{n-m}
\end{aligned}
$$

Q3

a)

First we have that $N_A \sim Poi(\lambda|A|) = Poi(0.5 \cdot 0.25) = Poi(1/8)$.

$$
\begin{aligned}
\mathbb{P}(N_A \geq 1) &= 1 - \mathbb{P}(N_A = 0) \\
&= 1 - \frac{e^{-\frac{1}{8}}\left(\frac{1}{8}\right)^0}{0!} \\
&= 1 - e^{-\frac{1}{8}} \\
&\approx 0.1175
\end{aligned}
$$

b)

As each trip explores a new area we have that each trip is an independent Poisson process. This means if make n trips we have the Poisson process $\sum_{i=1}^n N_i \sim Poi(\sum_{i=1}^n \lambda|A|) = Poi(n\lambda|A|)$ (super position principle). So the expectation is $\mathbb{E}[\sum_{i=1}^n N_i] = n\lambda|A| = \frac{n}{8}$. Now we want to find the n that makes this greater or equal to 6, thus $n = 48$ ($\frac{48}{8} = 6$). Therefore we can expect to make 48 trips before we see 6 fat-tailed dunnarts.

c)

Assume the survey was also a spatial Poisson process with the same rate parameter. In the calculation we will see three seperate Poisson processes, $N_A \sim Poi(0.5 \cdot 0.25) = Poi(1/8)$, $N_B \sim Poi(0.5 \cdot 29750) = Poi(14785)$ and $N_A \sim Poi(0.5 \cdot (29750 - 0.25)) = Poi(14874.875)$. So we have that A is the area we search, B is the area that the survey searched and C is the area the survey searched minus the area we search.

$$\begin{aligned}
\mathbb{P}(N_A \geq 1 | N_B = 5 \cdot 10^5) &= 1 - \mathbb{P}(N_A = 0 | N_B = 5 \cdot 10^5) \\
&= 1 - \frac{\mathbb{P}(N_A = 0, N_B = 5 \cdot 10^5)}{\mathbb{P}(N_B = 5 \cdot 10^5)} \\
&= 1 - \frac{\mathbb{P}(N_A = 0)\mathbb{P}(N_C = 5 \cdot 10^5)}{\mathbb{P}(N_B = 5 \cdot 10^5)} \\
&= 1 - \frac{\frac{e^{-(1/8)(1/8)^0}}{0!} \frac{e^{-(14874.875)(14874.5)^{5 \cdot 10^5}}}{(5 \cdot 10^5)!}}{\frac{e^{-(14875)(14875)^{5 \cdot 10^5}}}{(5 \cdot 10^5)!}} \\
&= 1 - \frac{e^{-(1/8)} e^{-14874.875} (14874.875)^{5 \cdot 10^5}}{e^{-14875} (14875)^{5 \cdot 10^5}} \\
&= 1 - \left(\frac{14874.875}{14875} \right)^{5 \cdot 10^5} \\
&\approx 1 - 0.01497 \\
&\approx 0.98503
\end{aligned}$$

Q4

a)

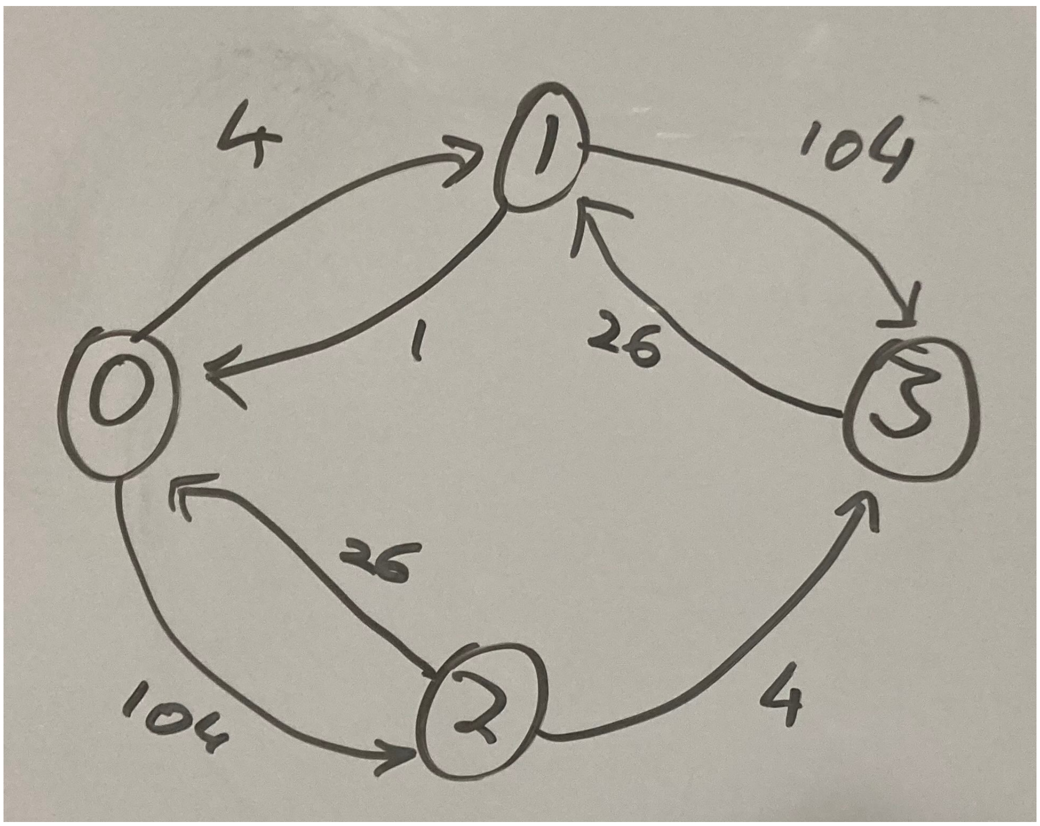
We have four states for this problem:

$$\begin{aligned}
0 &: \text{Both machines are functioning} \\
1 &: \text{The machine has failed, but the repair robot is functioning} \\
2 &: \text{The repair robot has failed, but the machine is functioning} \\
3 &: \text{Both machines have failed}
\end{aligned}$$

So $E = \{0, 1, 2, 3\}$. We know that both machines initially are working, so $\pi^{(0)} = (1, 0, 0, 0)^T$. Now noting the given lifetime and repair distributions and that we can't have an infitesimal change between state 1 and 2, 0 and 3, and, we can't go from state 3 to 2 as the repair robot needs to be repaired before the machine we get that the Q-matrix is:

$$Q = \begin{pmatrix} -108 & 4 & 104 & 0 \\ 1 & -105 & 0 & 104 \\ 26 & 0 & -30 & 4 \\ 0 & 26 & 0 & -26 \end{pmatrix}$$

b)



c)

We see that the chain is irreducible and positive reccurent, hence the limiting distribution π exists and is given by solving $\pi Q = \mathbf{0}$ such that $\pi \mathbf{1} = 1$.

$$0 = 104\pi_0 - 30\pi_2 \implies \pi_0 = \frac{15}{52}\pi_2 \quad (1)$$

$$0 = -108\pi_0 + \pi_1 + 26\pi_2 \implies \pi_1 = 108\pi_0 - 26\pi_2 = (\frac{3240}{104} - 26)\pi_2 = \frac{67}{13}\pi_2 \quad (2)$$

$$0 = 4\pi_0 - 105\pi_1 + 26\pi_3 \implies \pi_3 = \frac{105\pi_1 - 4\pi_0}{26} = \frac{270}{13}\pi_2 \quad (3)$$

$$1 = \pi_0 + \pi_1 + \pi_2 + \pi_3 \implies \frac{1}{\frac{15}{52} + \frac{67}{13} + 1 + \frac{270}{13}} = \pi_2 = \frac{52}{1415} \quad (4)$$

Subbing (4) into (1), (2) and (3) we get:

$$\pi = \left(\frac{3}{283}, \frac{268}{1415}, \frac{52}{1415}, \frac{216}{283}\right)^T$$

So the long run probability that both the machine and the machine repair robot are under repair is $\frac{216}{283}$.

d)

If the machine has failed and is under repair we are in state 1. Denote T_{xyt} the event of transitioning from state x to state y at time t. This is an event that is exponentially distributed with parameter given by q_{xy}. We then find:

$$\begin{aligned}
\mathbb{P}(T_{10t} < T_{13t}) &= \int_{s=0}^{\infty} \int_{t=0}^{s} 104 e^{-t} e^{-104s} dt ds \\
&= \int_{s=0}^{\infty} -104 e^{-104s}(e^{-s} - 1) ds \\
&= -104 \int_{s=0}^{\infty} e^{-105s} - e^{-104s} ds \\
&= -104 \left[-\frac{1}{105} e^{-105s} + \frac{1}{104} e^{-104s} \right]_{s=0}^{\infty} \\
&= \frac{1}{105}
\end{aligned}$$

Note here that $\lim_{x \to \infty} e^{-x} = 0$. Hence the probability that the machine repair robot fixes the machine before it itself fails is $\frac{1}{105}$.