Stat3004 Assignment 2

Dominic Scocchera

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Q1

a)

First we assume that the wallaby is on the integers and starts at 0. Let:

$$P_1 = \mathbb{P}(\text{the wallaby will ever reach x=1})$$
$$P_u = \mathbb{P}(\text{the wallaby will ever reach x=u})$$

By independence we have $(P_1)^u = P_u$. We also have:

$$\begin{aligned} P_1 &= p \cdot 1 + q \cdot P_2 = p + q \cdot (P_1)^2 \\ \iff 0 &= P_1^2 - \frac{1}{q}P_1 + \frac{p}{q} \\ \iff P_1 &= \frac{1 \pm \sqrt{1-4pq}}{2q} \\ &= \frac{1 \pm |p-q|}{2q} \\ &= \begin{cases} 1, & \text{if } p \geq q \\ \frac{p}{q}, & \text{if } p < q \end{cases} \end{aligned}$$

Hence:

$$P_u = \begin{cases} 1, & \text{if } p \geq q \\ (\frac{p}{q})^u, & \text{if } p < q \end{cases}$$

b)

Let N_{ou} denote the time it takes the wallaby to reach $x = u$ for the first time when starting at $x = 0$. We now have:

$$\mathbb{E}_u = \mathbb{E}(N_{0u}) = u \cdot \mathbb{E}_1$$

Conditioning yields:

$$\mathbb{E}_1 = 1 + p \cdot 0 + q \cdot \mathbb{E}_2 = 1 + 2q \cdot \mathbb{E}_1$$

Now we look at the cases $p < q$, p=q and $p > q$. For $p < q$ we have from a) that $\mathbb{P}(N_{01} = \infty) = 1 - P_1 > 0$, implying that $\mathbb{E}_1 = \infty$.

Now for $p = q = \frac{1}{2}$ we get:
$$\mathbb{E}_1 = 1 + \mathbb{E}_1$$
This is a contradiction so $\mathbb{E}_1 = \infty$.

Finally for $p > q$, we get:

$$\mathbb{E}_1 = \frac{1}{1 - 2q} = \frac{1}{p - q}$$

Hence:
$$\mathbb{E}_u = \begin{cases} \infty & \text{if } p \leq q \\ \frac{u}{p-q} & \text{if } p > q \end{cases}$$
And so the expected moves when $p = q = \frac{1}{2}$ is ∞.

e)

With almost identical working out to a) we can get:

$$P_d = \begin{cases} 1, & \text{if } p \leq q \\ (\frac{q}{p})^d, & \text{if } p > q \end{cases}$$

So the probability of ever reaching the seeds (P_s) is:

$$P_s = P_u \text{ or } P_d = \begin{cases} P_d = P_u = 1, & \text{if } p = q \\ P_u = (\frac{p}{q})^u, \quad P_d = 1, & \text{if } p < q \\ P_u = 1, \quad P_d = (\frac{q}{p})^d, & \text{if } p > q \end{cases}$$

So no matter what p and q are the wallaby is guaranteed to reach some seeds.

f)

With almost the exact same working out as in b) we can get:

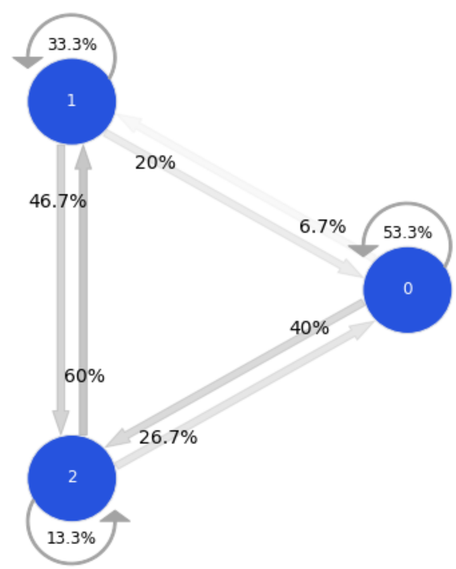
$$\mathbb{E}_d = \begin{cases} \infty & \text{if } p \geq q \\ \frac{|d|}{q-p} & \text{if } p < q \end{cases}$$

Now we get:

$$\mathbb{E}_s = \begin{cases} \mathbb{E}_u = \mathbb{E}_d = \infty & \text{if } p = q \\ \mathbb{E}_u = \frac{u}{p-q}, \ \ \mathbb{E}_d = \infty & \text{if } p > q \\ \mathbb{E}_u = \infty, \ \ \mathbb{E}_d = \frac{|d|}{q-p} & \text{if } p < q \end{cases}$$
$$= \begin{cases} \infty & \text{if } p = q \\ \frac{u}{p-q} & \text{if } p > q \\ \frac{|d|}{q-p} & \text{if } p < q \end{cases}$$

Q2

a)



b)

$$\begin{aligned}
\mathbb{P}(X_3 = 1|X_0 = 1) &= \mathbb{P}(X_3 = 1|X_0 = 1, X_1 = 0, X_2 = 0) + \mathbb{P}(X_3 = 1|X_0 = 1, X_1 = 0, X_2 = 1) \\
&+ \mathbb{P}(X_3 = 1|X_0 = 1, X_1 = 0, X_2 = 2) + \mathbb{P}(X_3 = 1|X_0 = 1, X_1 = 1, X_2 = 0) \\
&+ \mathbb{P}(X_3 = 1|X_0 = 1, X_1 = 1, X_2 = 1) + \mathbb{P}(X_3 = 1|X_0 = 1, X_1 = 1, X_2 = 2) \\
&+ \mathbb{P}(X_3 = 1|X_0 = 1, X_1 = 2, X_2 = 0) + \mathbb{P}(X_3 = 1|X_0 = 1, X_1 = 2, X_2 = 1) \\
&+ \mathbb{P}(X_3 = 1|X_0 = 1, X_1 = 2, X_2 = 2) \\
&= \frac{3}{15}\frac{8}{15}\frac{1}{15} + \frac{3}{15}\frac{1}{15}\frac{5}{15} + \frac{3}{15}\frac{6}{15}\frac{9}{15} + \frac{5}{15}\frac{3}{15}\frac{1}{15} + \frac{5}{15}\frac{5}{15}\frac{5}{15} + \frac{5}{15}\frac{7}{15}\frac{9}{15} \\
&+ \frac{7}{15}\frac{4}{15}\frac{1}{15} + \frac{7}{15}\frac{9}{15}\frac{5}{15} + \frac{7}{15}\frac{2}{15}\frac{9}{15} \\
&= \frac{1}{3}
\end{aligned}$$

c)

To find the limiting distribution we solve $\pi(P - I) = \mathbf{0}$.

```
import numpy as np
from scipy . linalg import null_space

P = (1/15)*np.array([[8, 1, 6], [3, 5, 7], [4, 9, 2]])
v = null_space(P - np.eye(3))
v = v/sum(v)
```

From this we find that $\pi = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)^T$.

Q3

a)

We want to show that for $x, y \in E$, if $x \to y$ but $r_{yx} = \mathbb{P}_y(\tau_y < \infty) < 1$ (i.e. $y \nrightarrow x$) then x is transient.

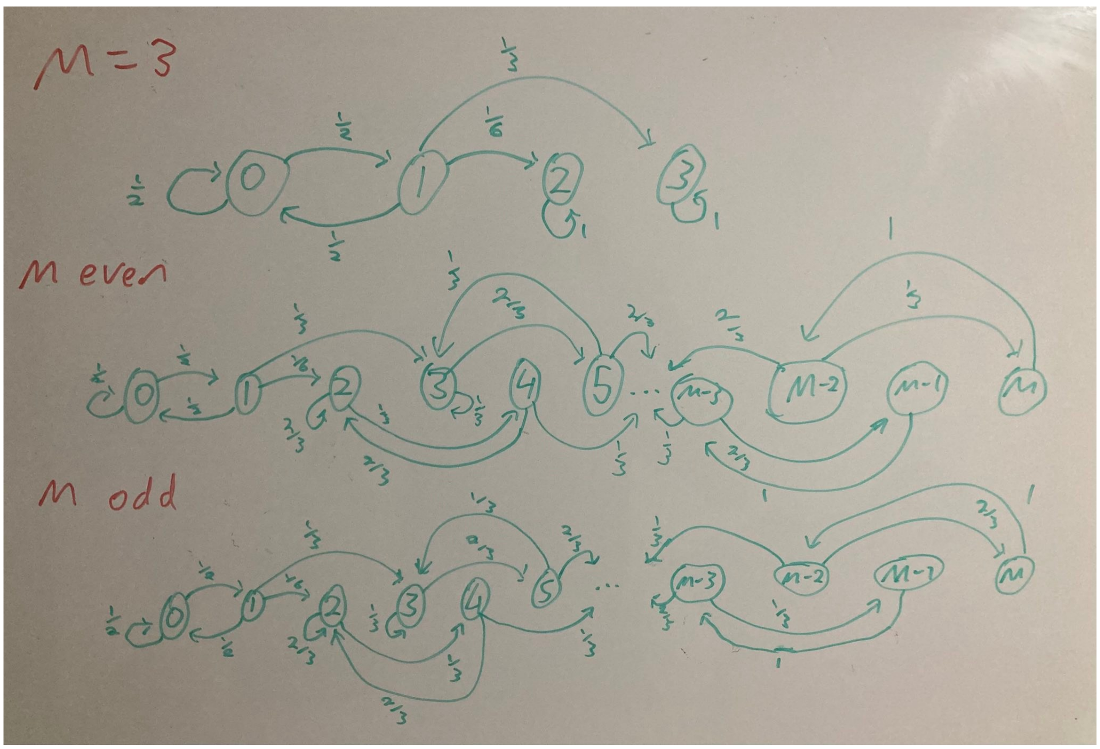
Proof. First consider the probability that we return to y after time n conditioned on whether or not $X_n = y$.

$$\begin{aligned}
\mathbb{P}(\text{return to x after time n}|X_0 = x) &= r_{xy}\mathbb{P}(\text{return to x after time n}|X_n = y, X_0 = x) \\
&+ (1 - r_{xy})\mathbb{P}(\text{return to x after time n}|X_n \neq y, X_0 = x) \\
&\leq 0 + (1 - r_{xy}) \\
&< 1
\end{aligned}$$

As $\mathbb{P}(\text{return to x after time n}|X_0 = x) < 1$ then x must be transient (if it were recurrent it would equal 1 as we would be able to return infinitely often). We also note here that $\mathbb{P}(\text{return to x after time n}|X_n = y, X_0 = x) = 0$ as we can't return to x from y and $\mathbb{P}(\text{return to x after time n}|X_n \neq y, X_0 = x) \leq 1$ as there exists at least one state namely y that makes it impossible to return to x. $\square$

Q4

a)



b)

There are three communicating classes, $A = \{0, 1\}$, $B = \{2i \mid 2 \leq 2i \leq M\}$ (even integers greater or equal to 2) and $C = \{2i + 1 \mid 3 \leq 2i + 1 \leq M\}$ (odd integers greater or equal to 3).

c)

We have three cases:
M=3:
A is transient. B is closed, recurrent and absorbing. C is closed, recurrent and

absorbing.
M=4:
A is transient. B is closed and recurrent. C is closed, recurrent and absorbing.
$M > 4$:
A is transient. B is closed and recurrent. C is closed and recurrent.

d)

First we note that as A is transient, so it is always aperiodic.

We have four cases:
M=3:
2 is absorbing so the period is 1 which makes it aperiodic by definition. Same for 3.
M=4:
3 is absorbing so the period is 1 which makes it aperiodic by definition. Starting at 2 we return to 2 after either 1 or 2 transitions, $\delta_x = \gcd(1, 2) = 1$, so 2 is aperiodic. $2 \leftrightarrow 4$ so 4 is also aperiodic.
$M = 5$:
Starting at 2 we return to 2 after either 1 or 2 transitions, $\delta_x = \gcd(1, 2) = 1$, so 2 is aperiodic. $2 \leftrightarrow 4$ so 4 is also aperiodic. Starting at 3 we return to 3 after either 1 or 2 transitions, $\delta_x = \gcd(1, 2) = 1$, so 3 is aperiodic. $3 \leftrightarrow 5$ so 5 is also aperiodic.
$M > 5$:
We note that $p_{22} = \frac{2}{3} > 0$ and $p_{33} = \frac{1}{3} > 0$. From lectures we know that this means 2 and 3 must be aperiodic. We also know that if $x \leftrightarrow y$ then x and y have the same period. So all states in B and C must be aperiodic.