

# Math3303 Assignment 1

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## Q1

Let  $G = GL_n(\mathbb{R})$  be the group of  $n \times n$  invertible matrices and  $N = SL_n(\mathbb{R})$  the subgroup of  $G$  consisting of those matrices which have determinant one. First we want to prove that  $N \trianglelefteq G$ .

*Proof.* By definition we have that,  $N \trianglelefteq G \iff \forall g \in G \text{ and } n \in N, gng^{-1} \in N$ . This means we require  $\det(gng^{-1}) = 1 \forall g \in G \text{ and } n \in N$ , as  $N$  is the group of invertible matrices whose determinant is 1. By calculation and properties of determinant ( $\det(AB) = \det(A)\det(B)$ ,  $\det(A^{-1}) = \frac{1}{\det(A)}$ ,  $\forall n \in N, \det(n) = 1$ ) we get:

$$\begin{aligned}\det(gng^{-1}) &= \det(g)\det(n)\det(g^{-1}) \\ &= \frac{\det(g)}{\det(g)}\det(n) \\ &= 1\end{aligned}$$

□

Now we want to prove  $G/N \cong \mathbb{R}^*$ .

*Proof.* First we define the homomorphism  $\phi : GL_n(\mathbb{R}) \rightarrow \mathbb{R}^*$ . This is a homomorphism because  $\forall A, B \in GL_n(\mathbb{R}), \phi(A \cdot B) = \det(A \cdot B) = \det(A) \cdot \det(B) \in \mathbb{R}^*$ . As the identity element of  $\mathbb{R}^*$  is 1 we have that:

$$\text{Ker } \phi = \{A \in GL_n(\mathbb{R}) : \det(A) = 1\} = SL_n(\mathbb{R})$$

Now by the first isomorphism theorem we have:

$$\begin{aligned}GL_n(\mathbb{R})/\text{Ker } \phi &= GL_n(\mathbb{R})/SL_n(\mathbb{R}) \\ &= G/N \\ &\cong \phi(G)\end{aligned}$$

Now to get the result we require  $\phi(G) \cong \mathbb{R}^*$  which occurs if the homomorphism is surjective, i.e.  $\forall a \in \mathbb{R}^*, \exists A \in G$  s.t.  $\phi(A) = \det(A) = a$ . To show this we consider the matrix whose top left value is  $a$ , the rest of the diagonal is 1 and every other entry is 0. The determinant of this matrix is clearly  $a$ , and so we have a surjective homomorphism, hence the result.  $\square$

## Q2

## Q3

Let  $U$  denote the set of roots of unity in  $\mathbb{C}^*$ . That is,

$$U := \{x \in \mathbb{C} \mid x^n = 1, \text{ for some } n \in \mathbb{Z}_{\geq 0}\}$$

We want to show that  $\mathbb{Q}/\mathbb{Z} \cong U$ .

*Proof.* We begin by considering the homomorphism  $\phi : x \rightarrow e^{2\pi i x}$ ,  $x \in \mathbb{Q}$ . This is a homomorphism because if we consider  $x, y \in \mathbb{Q}$  we get:

$$\begin{aligned} \phi(x+y) &= e^{2\pi i(x+y)} \\ &= e^{2\pi i x} e^{2\pi i y} \\ &= \phi(x) \cdot \phi(y) \end{aligned}$$

Now if we restrict  $x$  to the integers ( $x \in \mathbb{Z}$ ) and apply Eulers identity and the fact that  $\cos(2\pi x) = 1$  and  $\sin(2\pi x) = 0 \forall x \in \mathbb{Z}$  we get:

$$\begin{aligned} e^{2\pi i x} &= \cos(2\pi x) + i \sin(2\pi x) \\ &= 1 + i \cdot 0 \\ &= 1 \end{aligned}$$

As 1 is the multiplicative identity in  $\mathbb{C}^*$  we have that  $\text{Ker } \phi = \mathbb{Z}$ . Now by the first isomorphism theorem we have  $\mathbb{Q}/\mathbb{Z} \cong \phi(\mathbb{Q})$ . To get the result we must show that the homomorphism is surjective as that implies  $\phi(\mathbb{Q}) \cong U$ . This means that  $\forall z \in U, \exists a \in \mathbb{Q}$  s.t.  $\phi(a) = z$ . We have that  $\phi(a) = e^{2\pi i a} = \cos(2\pi a) + i \sin(2\pi a)$  which is exactly the roots of unity as for  $a = n$  this is one. This means the homomorphism is surjective, and hence the result follows.  $\square$