Section 1.

1.1 Basics of Group Theory

- assumed knowledge MATH 2301
- Blackboard has MATH2301 Notes

Consider a permutation $\pi : \{1,2,\dots,6\} \to \{1,2,\dots,6\}$ given by

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 3 & 2 & 6 & 5 & 1 \end{array} \quad \pi$$

fixed point

$(1\ 4\ 6)(2\ 3)(5)$

or

$(1\ 4\ 6)(2\ 3)$

Let S be a set. $\mathrm{Sym}(S)$ Symmetric group on S

all permutations of S.

composition of functions.

S_n denotes any group isomorphic to $\mathrm{Sym}(\{1,2,\dots,n\})$ $\mathrm{Sym}(n)$

Functions on left or right ???

The image of x under f $\to xf$, $\to f(x)$

Issue: $x(f \circ g) = (xf)g$ ... first f then g.

$(f \circ g)(x) = f(g(x))$ ... first g then f.

Example: $f = (1\ 2\ 3)$ $\quad g = (1\ 2\ 4)$

With functions on the right $\quad f \circ g = (1 2 3) \circ (1 2 4) = (1\ 4)(2\ 3)$

But with functions on the left $\quad f \circ g = (1\ 2\ 3) \circ (1\ 2\ 4) = (1\ 3)(2\ 4)$

Both "left" and "right" notation is used, but be consistent.

Alternating group A_n

- subgroup of S_n
- order $n!/2$
- consists of <u>even</u> permutations.

Note: Cycles of odd length are even permutations
Cycles of even length are odd permutations.

Exercise: Write out the elements of A_4 (using cycle representations)

Solution:

$()$	$(1 2 3)$	$(1\ 3\ 2)$
	$(1\ 2\ 4)$	$(1\ 4\ 2)$
$(12)(34)$		
$(13)(24)$	$(1\ 3\ 4)$	$(1\ 4\ 3)$
$(14)(2\ 3)$	$(2\ 3\ 4)$	$(2\ 4\ 3)$

1.2 Permutation Groups.

Defn: $G \leq \mathrm{Sym}(S)$

permutation group acting on S.
degree $|S|$.

Set of "points"

S_n, A_n are permutation groups of degree n.
(order $n!$, $\frac{n!}{2}$)

Defn: An action of G on S is a homomorphism

$$\phi : G \longrightarrow \mathrm{Sym}(S)$$

(each element of g has an associated permutation of S)

For $g \in G$ and $x \in S$, $g(x)$ denotes the image of x under $\phi(g) \in \mathrm{Sym}(S)$.

$$\phi(gh) = \phi(g)\,\phi(h) \quad \longleftarrow \text{homomorphism}$$

So $(gh)(x) = \big(\phi(gh)\big)(x) = \big(\phi(g)\,\phi(h)\big)(x) = \phi(g)\big(\phi(h)(x)\big)$

$$= \phi(g)\big(h(x)\big)$$

$$= g\big(h(x)\big). \qquad \square$$

The image $\mathrm{Im}\,\phi = \{\phi(g) : g \in G\}$ of an action $\phi : G \to \mathrm{Sym}(S)$ is a permutation group acting on S.

Example: $G = \mathbb{Z}_6$

Define an action ϕ of G on $S = \{1, 2, 3\}$ by

$$\phi(0) = \phi(3) = ()$$
$$\phi(1) = \phi(4) = (1\ 2\ 3)$$
$$\phi(2) = \phi(5) = (1\ 3\ 2)$$

It can be checked that $\phi : \mathbb{Z}_6 \to \mathrm{Sym}(\{1,2,3\})$ is indeed a homomorphism. For example

$3, 4 \in \mathbb{Z}_6$

$$\left.\begin{array}{l} \phi(3+4) = \phi(1) = (1\ 2\ 3). \\ \phi(3) \circ \phi(4) = () \circ (1\ 2\ 3) = (1\ 2\ 3). \end{array}\right\} \quad \phi(3+4) = \phi(3) \circ \phi(4).$$

$\mathrm{Im}(\phi) = \{(1), (1\,2\,3), (1\,3\,2)\} \leq \mathrm{Sym}(\{1,2,3\}) = \{(1), (1\,2\,3), (1\,3\,2), (1\,2), (1\,3), (2\,3)\}$

Defn: An action ϕ of G on S is faithful if $\underbrace{\ker \phi \text{ is } \{1\}}$.

(the mapping ϕ is one-to-one)

the only element of g that fixes all the points is the identity.

above example is not faithful: $\ker \phi = \{0, 3\}$.

Note: If ϕ is faithful, then by 1st isomorphism theorem,

$\mathrm{Im}\,\phi \cong G/\ker\phi \cong G$.

G is isomorphic to the permutation group $\mathrm{Im}\,\phi \leq \mathrm{Sym}(S)$.

Exercises up to 5.

1.1.29

Theorem [redacted] (First Isomorphism Theorem). Let G and H be groups and let $f : G \to H$ be a homomorphism. Then

(a) $\ker f \trianglelefteq G$;

(b) $\mathrm{Im}\, f \leq H$; and

(c) $G/\ker f \simeq \mathrm{Im}\, f$.

Orbit-Stabilizer Theorem:

Notation: G acts on S, $T \subseteq S$, $g(T) = \{g(x) : x \in T\}$.

(This is actually defining an action of G on $\mathcal{P}(S)$.)

Orbits: G acts on S.

Define equivalence relation $\sim$ on S by

$x \sim y$ iff $\exists\, g \in G$ such that $g(x) = y$.

Defn. Equivalence classes of $\sim$ are the orbits of G

form a partition of S.

$[x]$ is denoted $\mathcal{O}(x)$ the orbit of x under G

Exercise: (a) Show that $\sim$ is an equivalence relation

(b) Consider the natural induced action of $\mathbb{Z}_6$ on the 3-element subsets of $\{0,1,2,3,4,5\}$. Write out the orbits of this action.

Solution:

the identity of G.

(a) Reflexive: Let $x \in S$. Then $e \in G$ and $e(x) = x$ so $x \sim x$.

Symmetric: Suppose $x \sim y$. Then $\exists\, g \in G$ such that $g(x) = y$.

So $g^{-1} \in G$ and $g^{-1}(y) = g^{-1}(g(x)) = g^{-1}g\,(x) = e(x) = x$.

Thus we have $y \sim x$.

Transitive: Suppose $x \sim y$ and $y \sim z$.

$\longrightarrow \exists\, g, h \in G$ such that $g(x) = y$, $h(y) = z$.

$\longrightarrow hg \in G$ and $(hg)(x) = h(g(x)) = h(y) = z$.

$\longrightarrow x \sim z$.

(b)

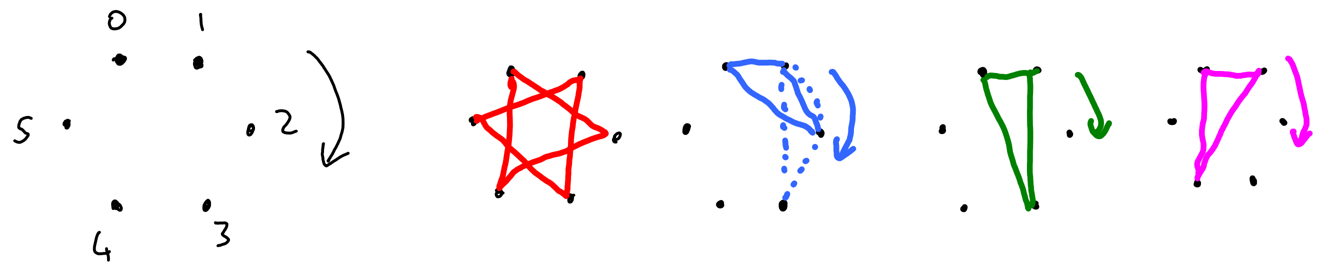
$\boxed{\binom{6}{3} = 20}$

0 3 6
1 2 4

0 1 2
1 2 3
2 3 4
3 4 5
4 5 0
5 0 1

0 1 3
1 2 4
2 3 5
3 4 0
4 5 1
5 0 2

0 1 4
1 2 5
2 3 0
3 4 1
4 5 2
5 0 3



Stabilizers: pointwise stabilizer, setwise stabilizer.

G acts on S, $T \subseteq S$.

* pointwise stabilizer of T:

$$G_T = \{g \in G : g(x) = x, \text{ for all } x \in T\}.$$

* setwise stabilizer of T:

$$G_{\{T\}} = \{g \in G : g(T) = T\}.$$

(If $T = \{x\}$, we write just G_x rather than $G_{\{x\}}$).

Thm: Stabilizers are subgroups.

Thm 1.2.7 G acts on S, $x \in S$. Let $\Theta(x) = \{\overset{x=x_1}{g_1(x)}, \overset{x_2}{g_2(x)}, \dots, \overset{x_t}{g_t(x)}\}$. $g_1 =$ identity.

Let $H = G_x$. Then the left cosets of H are

$$g_1H, g_2H, \dots, g_tH.$$

and $g_iH = \{g \in G : g(x) = g_i(x)\}$

① $g^* \in g_iH \longrightarrow g^* = g_ih \rightarrow g^*(x) = g_ih(x) = g_i(x)$

② $g^*(x) = g_i(x) \longrightarrow g_i^{-1}g^*(x) = x \longrightarrow g_i^{-1}g^* \in H \longrightarrow g^* \in g_iH.$

<u>Theorem 1.2.8</u> (Orbit-Stabilizer Theorem):- (follows from Thm 1.2.7).

G acts on finite set S. $x \in S$.

$$|G| = |G_x| \cdot |\Theta(x)|.$$

Exercise: Consider the induced action of A_4 on the set of pairs of elements of $\{1,2,3,4\}$.

* Determine the stabilizer $G_{\{1,2\}}$ of $\{1,2\}$.
* Write out the left cosets of $G_{\{1,2\}}$.
* Compare with Theorem 1.2.7.

Solution:

* $G_{\{1,2\}} = \{(), (1\,2)(3\,4)\}$.

* Left cosets

$()$ $(12)(34)$	$(1\,3)(2\,4)$ $(1\,4)(2\,3)$	$(1\,2\,3)$ $(1\,3\,4)$	$(1\,3\,2)$ $(2\,3\,4)$	$(1\,2\,4)$ $(1\,4\,3)$	$(2\,4\,3)$ $(1\,4\,2)$
$\{1,2\} \mapsto \{1,2\}$	$\{1,2\} \mapsto \{3,4\}$	$\{1,2\} \mapsto \{2,3\}$	$\{1,2\} \mapsto \{1,3\}$	$\{1,2\} \mapsto \{2,4\}$	$\{1,2\} \mapsto \{1,4\}$

eg. Last coset consists of
$(2\,4\,3) \circ () = (2\,4\,3)$ and
$(2\,4\,3) \circ (1\,2)(3\,4) = (1\,4\,2)$ $\square$

Definition: A permutation group with a single orbit is <u>transitive</u>.

G acting on S: $\forall x, y \in S$, $\exists g \in G$ such that $g(x) = y$.

Definition: A permutation group G acting on S is <u>regular</u> if for all $x, y \in S$, there exists a <u>unique</u> $g \in G$ such that $g(x) = y$. (regular $\rightarrow$ transitive)
(transitive $\not\rightarrow$ regular).

Theorem: Suppose G is transitive acting on finite set S. The following are equivalent:-

- G is regular
- $g \in G$ and $\exists x$ such that $g(x) = x \longrightarrow g = e$.
- $|G| = |S|$.

The only group element with a fixed point is the identity.

(don't confuse with faithful)

The only group element that fixes ALL the points is the identity.

Exercise 9. Prove this theorem.

Defn: Let G be a permutation group acting on S, and let $1 \le t \le |S|$. Then G is t-transitive if for any pairwise distinct $x_1, x_2, \ldots, x_t \in S$ and any pairwise distinct $y_1, y_2, \ldots, y_t \in S$, there exists $g \in G$ such that

$$g(x_1) = y_1, \; g(x_2) = y_2, \; \ldots, \; g(x_t) = y_t. \qquad \square$$

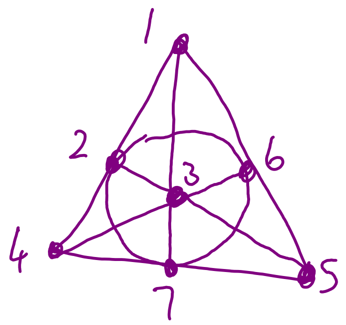
Note: 1-transitive $\equiv$ transitive

t-transitive $\longrightarrow$ s-transitive for $1 \le s \le t$

G is t-transitive iff G acts transitively on the set of t-tuples of distinct points.

Examples:

Fano Plane

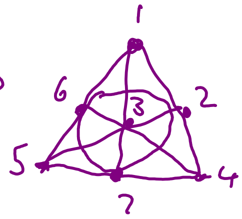


7 points
7 lines

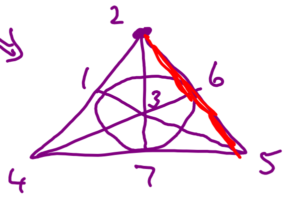
124
235
346
457
561
672
713

- each pair of points is on a unique line.
- each pair of lines intersects in a unique point.

Symmetries: Permutations of the point set that preserve the lines. Eg. $(2\,6)(4\,5) \Rightarrow$



Not $(1\,2) \Rightarrow$



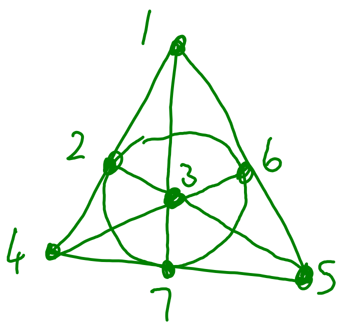
265 not a line.

G = Symmetry group of Fano Plane.

- permutation group acting on the points $1, 2, \ldots, 7$.
- subgroup of $\mathrm{Sym}(7)$.

Claim: G is 2-transitive but not 3-transitive.

$(1\ 2\ 3\ 4\ 5\ 6\ 7) \in G \longrightarrow G$ is transitive.



124
235
346
457
561 ✓
672
713

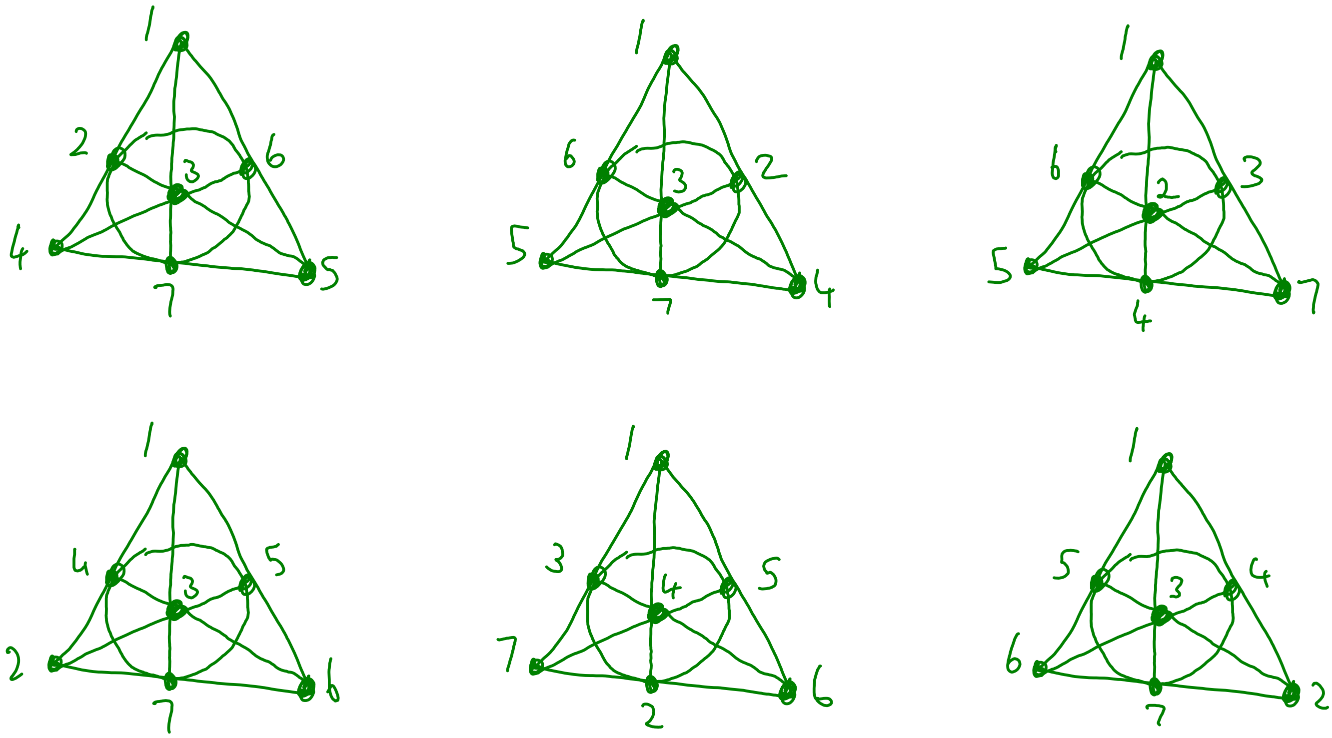
Will show that G_1 is transitive on $\{2,3,\ldots,7\}$.

Exercise: How does this prove G is 2-transitive?

Solution: We seek $g : (a,b) \longmapsto (c,d)$

We have $g_1 : (a,b) \longmapsto (1,x)$
$g_2 : (c,d) \longmapsto (1,y)$
$g_3 : (1,x) \longmapsto (1,y)$

Take $g = g_2^{-1} g_3 g_1$



$\square$

G is not 3-transitive... $\nexists g : (1,2,4) \longrightarrow (1,2,3)$.

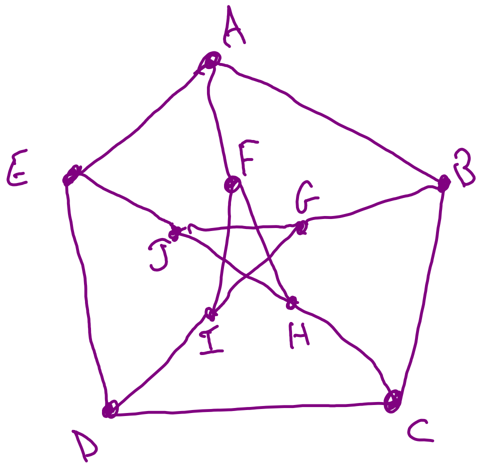
Fact $|G| = 168 = 7 \times 6 \times 4$ (# hours in a week)

$\uparrow$ $PSL(2,7)$ – Projective special linear group on $\mathbb{F}_7$

– 2nd smallest non-abelian simple group (behind A_5).

Symmetries of Petersen Graph

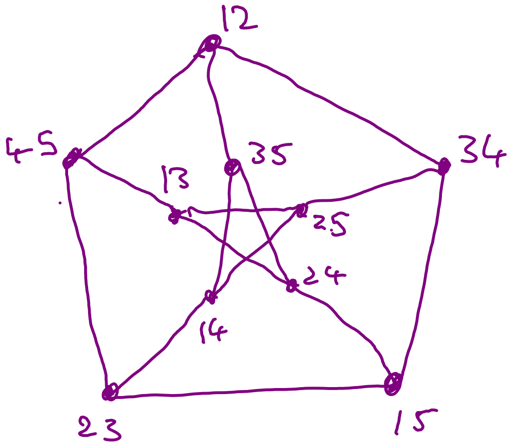
P



$G = \mathrm{Aut}(P)$

$(AB) \notin G$

$G \leq \mathrm{Sym}(\{A, B, \ldots, J\}) = S_{10}$



Label with $\binom{\{1,2,3,4,5\}}{2}$

$S_5 = \mathrm{Sym}(\{1,2,3,4,5\})$ acts on $V(P)$.

Example:

$\phi: \underset{\substack{\uparrow \\ \text{permutation}}}{(12)} \longmapsto (13\ \ 23)(14\ \ 24)(15\ \ 25)(12)(34)(35)(45)$

vertex label

Homomorphism ϕ from $\underset{S_5}{\mathrm{Sym}(\{1,2,3,4,5\})} \longrightarrow \underset{S_{10}}{\mathrm{Sym}\left(\binom{\{1,2,3,4,5\}}{2}\right)}$

$\mathrm{Im}\,\phi \leq G = \mathrm{Aut}(P)$ Faithful ???

The only $g \in S_5$ that fixes all vertices is $g = $ identity.

$\iff \ker \phi = \{\text{identity}\}$

1st Isomorphism Thm. $\mathrm{Im}\,\phi \cong S_5 / \ker\phi \cong S_5$.

So $\quad \mathrm{Im}\,\phi \cong S_5 \leq \mathrm{Aut}(P)$.

$\boxed{\text{Are there any other automorphism?}}$

NO

We show $\quad S_5 = \mathrm{Aut}(P)$.

Calculate order of $\mathrm{Aut}(P)$. $\longleftarrow$ Use Orb-Stab Thm.

$$\mathcal{O}(12) = V(P)$$

Orb Stab Thm $\quad |G| = 10 \cdot |G_{12}|$

Which automorphisms fix 12 ???

Orbit of 34 in G_{12} is ~~a subset of~~ $\{34, 35, 45\}$

$(3\ 4\ 5)$ does it

$$|\mathcal{O}_{G_{12}}(34)| = 3$$

$$|G_{12}| = 3 \cdot |G_{12_{34}}|$$

Notation

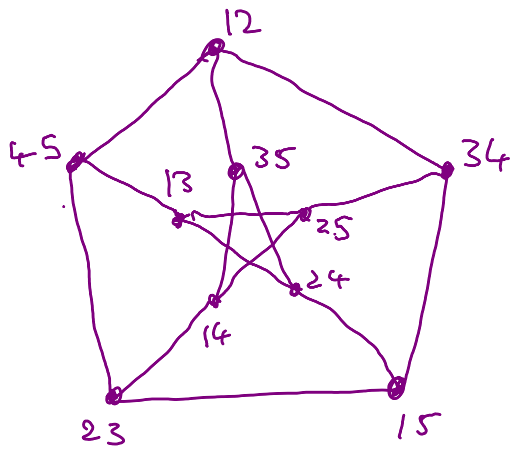
$G_{12_{34}} \qquad G_{12,34}$

So $\quad |G| = 10 \cdot 3 \cdot |G_{12,34}|$

Orbit of 15 in $G_{12,34}$

$$\mathcal{O}_{G_{12,34}}(15) = \{15, 25\}$$

$(1\ 2)$ this permutation does it.



So $\quad |G| = 10 \cdot 3 \cdot 2 \cdot |G_{12,34,15}|$

Orbit of 23 in $G_{12,34,15}$

$(3\,4) \in G_{12,34,15} \qquad 23 \longmapsto 24$

$\boxed{G \cong S_5}$

$$|G| = 10 \cdot 3 \cdot 2 \cdot 2 \cdot |G_{12,34,15,23}|$$

identity

$|G| = 120.$