Math3303 Assignment 1

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Febuary 2023

Q1

Let G = $\mathrm{GL}_n(\mathbb{R})$ be the group of $n \times n$ invertible matrices and N = $\mathrm{SL}_n(\mathbb{R})$ the subgroup of G consisting of those matrices which have determinant one. First we want to prove that N$\trianglelefteq$G.

Proof. By definition we have that, N$\trianglelefteq$G $\iff$ $\forall g \in$G and $n \in$N, $gng^{-1} \in$N. This means we require $\det(gng^{-1})$=1 $\forall$ g$\in$G and $n \in$N, as N is the group of invertible matrices whose determinant is 1. By calculation and properties of determinant ($\det(AB) = \det(A)\det(B)$, $\det(A^{-1}) = \frac{1}{\det(A)}$, $\forall n \in$ N, $\det(n) = 1$) we get:

$$\begin{aligned}\det(gng^{-1}) &= \det(g)\det(n)\det(g^{-1})\\ &= \frac{\det(g)}{\det(g)}\det(n)\\ &= 1\end{aligned}$$

□

Now we want to prove G/N $\cong \mathbb{R}^*$.

Proof. First we define the homomorphism $\phi : \mathrm{GL}_n(\mathbb{R}) \to \mathbb{R}^*$. This is a homomorphism because $\forall A, B \in \mathrm{GL}_n(\mathbb{R})$, $\phi(A \cdot B) = \det(A \cdot B) = \det(A) \cdot \det(B) \in \mathbb{R}^*$. As the identity element of $\mathbb{R}^*$ is 1 we have that:

$$\mathrm{Ker}\ \phi = \{A \in \mathrm{GL}_n(\mathbb{R}) : \det(A) = 1\} = \mathrm{SL}_n(\mathbb{R})$$

Now by the first isomorphism theorem we have:

$$\begin{aligned}\mathrm{GL}_n(\mathbb{R})/\mathrm{Ker}\ \phi &= \mathrm{GL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{R})\\ &= G/N\\ &\cong \phi(G)\end{aligned}$$

Now to get the result we require $\phi(G) \cong \mathbb{R}^*$ which occurs if the homorphism is surjective, i.e. $\forall a \in \mathbb{R}^*$, $\exists A \in$ G s.t. $\phi(A) = \det(A) = a$. To show this we consider the matrix whose top left value is a, the rest of the diagonal is 1 and every other entry is 0. The determinant of this matrix is clearly a, and so we have a surjective homomorphism, hence the result. $\square$

Q2

Let $G = \mathrm{SL}_2(\mathbb{Z})$ be the group of 2×2 matrices with integer coefficients and determinant equal to 1. We want to show that G is generated by:

$$S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Proof. We first note that $S^4 = I$ where I is the identity matrix and $T^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$. We first denote the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ generated by S and T as G. We now note that:

$$SA = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix}, \quad T^n A = \begin{pmatrix} a + nd & b + nd \\ c & d \end{pmatrix}$$

Where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Now choose A s.t. $A \in \mathrm{SL}_2(\mathbb{Z})$ and suppose $c \neq 0$. Now consider when $|a| \geq |c|$. If this is the case then we divide a by c yeilding $a = cp + q$ with $0 \leq q < |c|$. Applying the note from above we get that $T^{-q}A$ has $a - pc = q$ in its upper left corner. This is smaller in absolute value than the lower left entry c in $T^{-q}A$. As we saw above multiplying by S switches the top and bottom entries, with the top entries changing sign. Now we can apply the division algorithm in $\mathbb{Z}$ again if the lower left entry is nonzero in order to find another power of T to multiply by on the left so the lower left entry has smaller absolute value than before. Eventually multiplication of A on the left by enough copies of S and powers of T gives a matrix in $\mathrm{SL}_2(\mathbb{Z})$ with lower left entry 0. Such a matrix, m since it is integral with determinant 1, has the form $\begin{pmatrix} \pm 1 & m \\ 0 & \pm 1 \end{pmatrix}$ for some $m \in \mathbb{Z}$ and common m $-m$ signs on the diagonal. This matrix is either T or $-T$, so there is some $g \in G$ such that $gA = \pm T^n$ for some $n \in \mathbb{Z}$. Since $T^n \in G$ and $S^2 = -I_2$, we have $A = \pm g^{-1} T^n \in G$. $\square$

Q3

Let U denote the set of roots of unity in $\mathbb{C}^*$. That is,

$$U := \{x \in \mathbb{C} \mid x^n = 1, \text{ for some } n \in \mathbb{Z}_{\geq 0}\}$$

We want to show that $\mathbb{Q}/\mathbb{Z} \cong U$.

Proof. We begin by considering the homomorphism $\phi : x \to e^{2\pi i x},\ \ x \in \mathbb{Q}$. This is a homomorphism because if we consider $x, y \in \mathbb{Q}$ we get:

$$\begin{aligned}\phi(x+y) &= e^{2\pi i(x+y)} \\ &= e^{2\pi i x} e^{2\pi i y} \\ &= \phi(x) \cdot \phi(y)\end{aligned}$$

Now if we restrict x to the integers ($x \in \mathbb{Z}$) and apply Eulers identity and the fact that $\cos(2\pi x) = 1$ and $\sin(2\pi x) = 0\ \forall x \in \mathbb{Z}$ we get:

$$\begin{aligned}e^{2\pi i x} &= \cos(2\pi x) + i \sin(2\pi x) \\ &= 1 + i \cdot 0 \\ &= 1\end{aligned}$$

As 1 is the multiplicative identity in $\mathbb{C}^*$ we have that Ker $\phi = \mathbb{Z}$. Now by the first isomorphism theorem we have $\mathbb{Q}/\mathbb{Z} \cong \phi(\mathbb{Q})$. To get the result we must show that the homomorphism is surjective as that implies $\phi(\mathbb{Q}) \cong U$. This means that $\forall z \in U,\ \ \exists a \in \mathbb{Q}$ s.t. $\phi(a) = z$. We have that $\phi(a) = e^{2\pi i a} = \cos(2\pi a) + i \sin(2\pi a)$ which is exactly the roots of unity as for $a = n$ this is one. This means the homomorphism is surjective, and hence the result follows. □

Q4

Let G be the subgroup of $\mathrm{GL}_3(\mathbb{R})$ consisting of matrices of the form: $\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$

Determine the centre $Z(G)$ and prove that $G/Z(G) \cong \mathbb{R}^2$.

Proof. Consider $A, B \in G$, where $A = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & d & f \\ 0 & 1 & e \\ 0 & 0 & 1 \end{pmatrix}$. We define $Z(G) = \{A \in G | \forall B \in G, AB = BA\}$, i.e. we requrie:

$$\begin{aligned}AB - BA &= \begin{pmatrix} 1 & a+d & ea+c+f \\ 0 & 1 & b+e \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & a+d & bd+c+f \\ 0 & 1 & b+e \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & ea-bd \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \mathbf{0}\end{aligned}$$

So we require $ea - bd = 0$ and we claim that $e = d = 0$. If $d \neq 0$ then with $b \neq 0$ and $a = 0$ we have $AB - BA \neq 0$. Similar argument for e. Hence:

$$Z(G) = B = \begin{pmatrix} 1 & 0 & f \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad f \in \mathbb{R}$$

Now that we have $Z(G)$ we need to show $G/Z(G) \cong \mathbb{R}^2$. First consider the homomorphism $\phi : A \to \mathbb{R}^2$, where for $g \in G$, $\phi(g) = \begin{pmatrix} a \\ b \end{pmatrix}$. This is a homomorphism because if we consider $g, g' \in G$ we get (noting how the matrices multiply from above):

$$\begin{aligned} \phi(gg') &= \begin{pmatrix} g_1 + g_1' \\ g_2 + g_2' \end{pmatrix} \\ &= \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} + \begin{pmatrix} g_1' \\ g_2' \end{pmatrix} \\ &= \phi(g) + \phi(g') \end{aligned}$$

We also clearly have:

$$\ker\, \phi = \{(a, b) : a = b = 0\} = Z(G)$$

And as $g_1, g_2 \in \mathbb{R}$ we have $\phi(g) \in \mathbb{R}^2$ and hence:

$$\mathrm{Im}\ \phi = \mathbb{R}^2$$

This is also clearly a bijective mapping and hence by the first isomorphism theorem we have:

$$G/Z(G) \cong \mathbb{R}^2$$

$\square$