Math3303 Assignment 1

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Q1

Let G $=$ $\mathrm{GL}_n(\mathbb{R})$ be the group of $n \times n$ invertible matrices and N $=$ $\mathrm{SL}_n(\mathbb{R})$ the subgroup of G consisting of those matrices which have determinant one. First we want to prove that N$\trianglelefteq$G.

Proof. By definition we have that, N$\trianglelefteq$G $\iff$ $\forall g \in$G and $n \in$N, $gng^{-1} \in$N. This means we require $\det(gng^{-1})$=1 $\forall$ g$\in$G and $n \in$N, as N is the group of invertible matrices whose determinant is 1. By calculation and properties of determinant ($\det(AB) = \det(A)\det(B)$, $\det(A^{-1}) = \frac{1}{\det(A)}$, $\forall n \in \mathrm{N}, \;\; \det(n) = 1$) we get:

$$\begin{aligned}
\det(gng^{-1}) &= \det(g)\det(n)\det(g^{-1}) \\
&= \frac{\det(g)}{\det(g)}\det(n) \\
&= 1
\end{aligned}$$

$\square$

Now we want to prove G/N $\cong \mathbb{R}^*$.

Proof. First we define the homomorphism $\phi : \mathrm{GL}_n(\mathbb{R}) \to \mathbb{R}^*$. This is a homomorphism because $\forall A, B \in \mathrm{GL}_n(\mathbb{R})$, $\phi(A \cdot B) = \det(A \cdot B) = \det(A) \cdot \det(B) \in \mathbb{R}^*$. As the identity element of $\mathbb{R}^*$ is 1 we have that:

$$\mathrm{Ker}\ \phi = \{A \in \mathrm{GL}_n(\mathbb{R}) : \det(A) = 1\} = \mathrm{SL}_n(\mathbb{R})$$

Now by the first isomorphism theorem we have:

$$\begin{aligned}
\mathrm{GL}_n(\mathbb{R})/\mathrm{Ker}\ \phi &= \mathrm{GL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{R}) \\
&= G/N \\
&\cong \phi(G)
\end{aligned}$$

Now to get the result we require $\phi(G) \cong \mathbb{R}^*$ which occurs if the homorphism is surjective, i.e. $\forall a \in \mathbb{R}^*$, $\exists A \in$ G s.t. $\phi(A) = \det(A) = a$. To show this we consider the matrix whose top left value is a, the rest of the diagonal is 1 and every other entry is 0. The determinant of this matrix is clearly a, and so we have a surjective homomorphism, hence the result. $\square$

Q2

Let $G = \mathrm{SL}_2(\mathbb{Z})$ be the group of 2×2 matrices with integer coefficients and determinant equal to 1. We want to show that G is generated by:

$$S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Proof. We first note that $S^4 = I$ where I is the identity matrix and $T^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$. We first denote the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ generated by S and T as G. We now note that:

$$SA = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix}, \quad T^n A = \begin{pmatrix} a + nd & b + nd \\ c & d \end{pmatrix}$$

Where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Now choose A s.t. $A \in \mathrm{SL}_2(\mathbb{Z})$ and suppose $c \neq 0$. Now consider when $|a| \geq |c|$. If this is the case then we divide a by c yeilding $a = cp + q$ with $0 \leq q < |c|$. Applying the note from above we get that $T^{-q}A$ has $a - pc = q$ in its upper left corner. This is smaller in absolute value than the lower left entry c in $T^{-q}A$. As we saw above multiplying by S switches the top and bottom entries, with the top entries changing sign. Now we can apply division again if the lower left entry is nonzero in order to find another power of T to multiply by on the left so the lower left entry has smaller absolute value than before. If we keep multiplying A by copies of S and powers of T on the left we will get a matrix in $\mathrm{SL}_2(\mathbb{Z})$ which will have a lower left entry of 0. This matrix has the form $\begin{pmatrix} \pm 1 & k \\ 0 & \pm 1 \end{pmatrix}$ for some $k \in \mathbb{Z}$. Clearly this matrix has determinant 1. We also note that this matrix has common signs on the diagonal. We also see that this matrix is either T^k or $-T^k$, so there is some $g \in G$ such that $gA = \pm T^n$ for some $n \in \mathbb{Z}$. Since $T^n \in G$ and $S^2 = -I_2$, we have $A = \pm g^{-1} T^n \in G$. $\square$

Q3

Let U denote the set of roots of unity in $\mathbb{C}^*$. That is,

$$U := \{x \in \mathbb{C} \mid x^n = 1, \text{ for some } n \in \mathbb{Z}_{\geq 0}\}$$

We want to show that $\mathbb{Q}/\mathbb{Z} \cong U$.

Proof. We begin by considering the homomorphism $\phi : x \to e^{2\pi i x}, \;\; x \in \mathbb{Q}$. This is a homomorphism because if we consider $x, y \in \mathbb{Q}$ we get:

$$\begin{aligned}\phi(x+y) &= e^{2\pi i(x+y)} \\ &= e^{2\pi i x}e^{2\pi i y} \\ &= \phi(x)\cdot\phi(y)\end{aligned}$$

Now if we restrict x to the integers ($x \in \mathbb{Z}$) and apply Eulers identity and the fact that $\cos(2\pi x) = 1$ and $\sin(2\pi x) = 0 \; \forall x \in \mathbb{Z}$ we get:

$$\begin{aligned}e^{2\pi i x} &= \cos(2\pi x) + i\sin(2\pi x) \\ &= 1 + i\cdot 0 \\ &= 1\end{aligned}$$

As 1 is the multiplicative identity in $\mathbb{C}^*$ we have that Ker $\phi = \mathbb{Z}$. Now by the first isomorphism theorem we have $\mathbb{Q}/\mathbb{Z} \cong \phi(\mathbb{Q})$. To get the result we must show that the homomorphism is surjective as that implies $\phi(\mathbb{Q}) \cong U$. This means that $\forall z \in U, \;\; \exists a \in \mathbb{Q}$ s.t. $\phi(a) = z$. We have that $\phi(a) = e^{2\pi i a} = \cos(2\pi a) + i\sin(2\pi a)$ which is exactly the roots of unity as for $a = n$ this is one. This means the homomorphism is surjective, and hence the result follows. □

Q4

Let G be the subgroup of $\mathrm{GL}_3(\mathbb{R})$ consisting of matrices of the form: $\begin{pmatrix}1 & a & c\\ 0 & 1 & b\\ 0 & 0 & 1\end{pmatrix}$

Determine the centre $Z(G)$ and prove that $G/Z(G) \cong \mathbb{R}^2$.

Proof. Consider $A, B \in G$, where $A = \begin{pmatrix}1 & a & c\\ 0 & 1 & b\\ 0 & 0 & 1\end{pmatrix}$ and $B = \begin{pmatrix}1 & d & f\\ 0 & 1 & e\\ 0 & 0 & 1\end{pmatrix}$. We define $Z(G) = \{A \in G | \forall B \in G, AB = BA\}$, i.e. we requrie:

$$\begin{aligned}AB - BA &= \begin{pmatrix}1 & a+d & ea+c+f\\ 0 & 1 & b+e\\ 0 & 0 & 1\end{pmatrix} - \begin{pmatrix}1 & a+d & bd+c+f\\ 0 & 1 & b+e\\ 0 & 0 & 1\end{pmatrix} \\ &= \begin{pmatrix}0 & 0 & ea-bd\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \\ &= \mathbf{0}\end{aligned}$$

So we require $ea - bd = 0$ and we claim that $e = d = 0$. If $d \neq 0$ then with $b \neq 0$ and $a = 0$ we have $AB - BA \neq 0$. Similar argument for e. Hence:

$$Z(G) = B = \begin{pmatrix} 1 & 0 & f \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad f \in \mathbb{R}$$

Now that we have $Z(G)$ we need to show $G/Z(G) \cong \mathbb{R}^2$. First consider the homomorphism $\phi : A \to \mathbb{R}^2$, where for $g \in G$, $\phi(g) = \begin{pmatrix} a \\ b \end{pmatrix}$. This is a homomorphism because if we consider $g, g' \in G$ we get (noting how the matrices multiply from above):

$$\begin{aligned} \phi(gg') &= \begin{pmatrix} g_1 + g_1' \\ g_2 + g_2' \end{pmatrix} \\ &= \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} + \begin{pmatrix} g_1' \\ g_2' \end{pmatrix} \\ &= \phi(g) + \phi(g') \end{aligned}$$

We also clearly have:

$$\ker \phi = \{(a, b) : a = b = 0\} = Z(G)$$

And as $g_1, g_2 \in \mathbb{R}$ we have $\phi(g) \in \mathbb{R}^2$ and hence:

$$\text{Im } \phi = \mathbb{R}^2$$

This is also clearly a bijective mapping as we are taking an input from $\mathbb{R}^2$ (the two matrix entries) and mapping it to $\mathbb{R}^2$. Hence by the first isomorphism theorem we have:

$$G/Z(G) \cong \mathbb{R}^2$$

$\square$