Projective and Affine Geometries $PG(n,q)$ and $AG(n,q)$

Gaussian binomial coefficients

Definition 2.6.1. The ***q*-number** $[k]_q$ is defined by

$$[k]_q = \frac{1-q^k}{1-q} = 1 + q + q^2 + \cdots + q^{k-1},$$

the **q-factorial** $[k]_q!$ is defined by

$$[k]_q! = [k]_q[k-1]_q \cdots [1]_q,$$

$\boxed{k! = k(k-1)\dots 1}$

and the Gaussian binomial coefficient $\binom{n}{r}_q$ is defined by

$$\binom{n}{r}_q = \frac{[n]_q!}{[r]_q![n-r]_q!} = \frac{[n]_q[n-1]_q \cdots [n-r+1]_q}{[r]_q[r-1]_q \cdots [1]_q} = \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-r+1})}{(1-q^r)(1-q^{r-1})\cdots(1-q)}$$

for $r \leq n$ and $\binom{n}{r}_q = 0$ for $r > n$.

$$\boxed{\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\dots(n-r+1)}{r(r-1)\dots 1}}$$

Theorem 2.6.2. The number of r-dimensional subspaces of an n-dimensional vector space over a field of order q is given by the Gaussian binomial coefficient $\binom{n}{r}_q$.

Proof The number of ordered r-tuples of linearly independent vectors of an n-dimensional vector space over a field of order q is

$$(q^n-1)(q^n-q)\cdots(q^n-q^{r-1}).$$

$$r\left\{\begin{array}{lcl} q^n-1 & \to & x_1\ x_2\ \cdots\ x_n \leftarrow \text{anything but } 0 \\ q^n-q & \to & y_1\ y_2\ \cdots\ y_n \leftarrow \text{anything not in } \langle x\rangle \\ q^n-q^2 & \to & z_1\ z_2\ \cdots\ z_n \leftarrow \text{anything not in } \langle x,y\rangle. \\ & & \vdots \\ q^n-q^{r-1} & \to & w_1\ w_2\ \quad\ w_n \end{array}\right.$$

$\Longrightarrow$ each gives an r-dimensional subspace, but this counts each such subspace $(q^r-1)(q^r-q)\cdots(q^r-q^{r-1})$ times.

The number of r-tuples of linearly independent vectors of r-dimensional vector space $\longrightarrow (q^r-1)(q^r-q)\cdots(q^r-q^{r-1})$

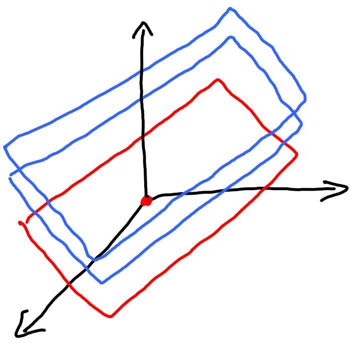
$$* \quad \frac{(q^n-1)(q^n-q)\cdots(q^n-q^{r-1})}{(q^r-1)(q^r-q)\cdots(q^r-q^{r-1})}$$

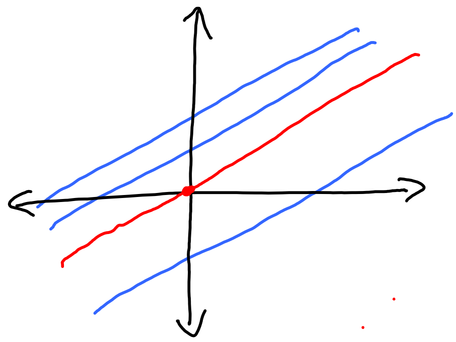
$$\begin{aligned}\frac{(q^n-1)(q^n-q)\cdots(q^n-q^{r-1})}{(q^r-1)(q^r-q)\cdots(q^r-q^{r-1})} &= \frac{(q^n-1)q(q^{n-1}-1)\cdots q^{r-1}(q^{n-r+1}-1)}{(q^r-1)q(q^{r-1}-1)\cdots q^{r-1}(q-1)} \\ &= \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-r+1})}{(1-q^r)(1-q^{r-1})\cdots(1-q)} \\ &= \binom{n}{r}_q.\end{aligned}$$

$\square$

AG(n, q)

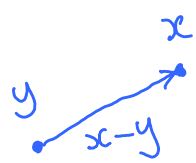
Definition 2.6.3. Let V be a vector space. The (additive) cosets in V of a d-dimensional subspace of V are called **d-flats**. The set of all d-flats, $0 \le d \le n-1$, of an n-dimensional vector space over a field of order q form the **n-dimensional affine geometry over $\mathbb{F}_q$**, which is denoted by AG(n, q). The 0-flats (vectors) are the **points** of AG(n, q), the 1-flats are the **lines**, and so on. The $(n-1)$-flats are the **hyperplanes** of AG(n, q). □





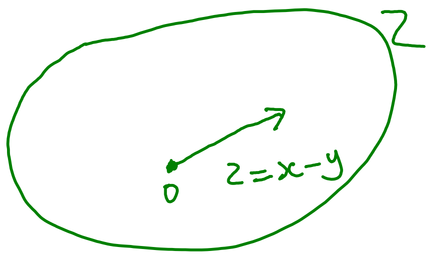
Theorem 2.6.4. Let q be a prime power, let $n \ge 2$, and let $1 \le d < n$. Each pair of distinct points of AG(n, q) occurs together in exactly $\binom{n-1}{d-1}_q$ d-flats.

Proof Let V be an n-dimensional vector space over $\mathbb{F}_q$ and let x and y be distinct vectors in V. So x and y are distinct points of AG(n, q). The number of d-flats containing both x and y is equal to the number of d-dimensional subspaces of V that contain the vector $z = x - y$. Let U be an $(n-1)$-dimensional subspace of V such that $z \notin U$. The d-dimensional subspaces of V containing z are exactly the subspaces spanned by $W \cup \{z\}$ where W is a $(d-1)$-dimensional subspace of U. Since the number of such W is $\binom{n-1}{d-1}_q$, each pair of points occurs in exactly $\binom{n-1}{d-1}_q$ d-flats. □



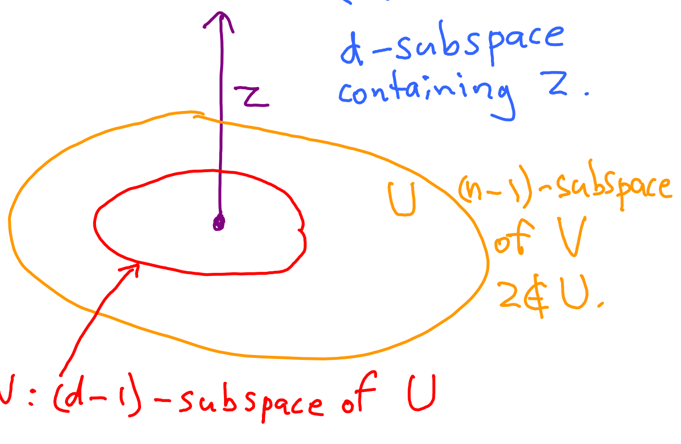
$x-y \in Z \Leftrightarrow x \in Z+y$

$0 \in Z \Leftrightarrow y \in Z+y$



$\boxed{\binom{n-1}{d-1}_q}$

$\langle W \cup z \rangle$: d-subspace containing z.

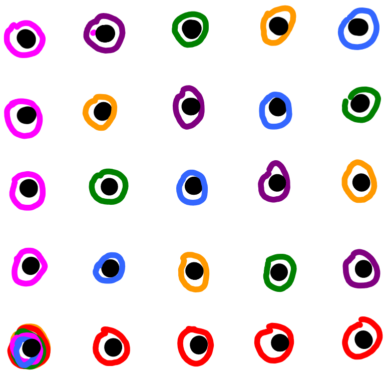


Corollary 2.6.5. The points and lines of AG$(2, q)$ form an affine plane of order q.

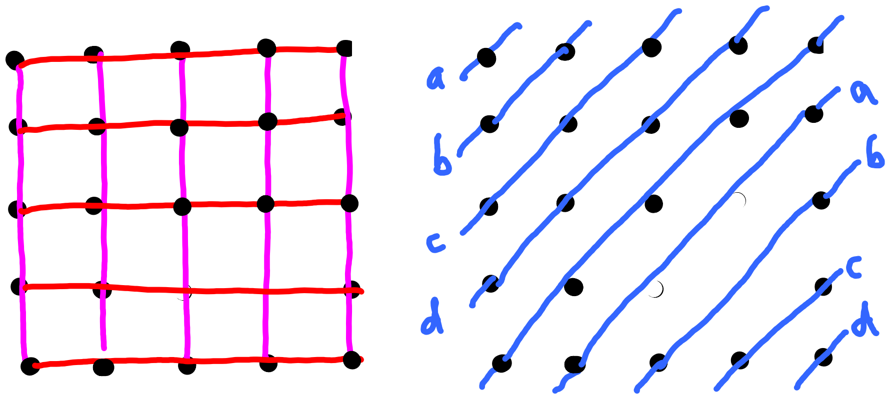
q^2 points, q points on each line,

each pair of points in $\binom{2-1}{1-1}_q = \binom{1}{0} = 1$ line.

AG(2,5)



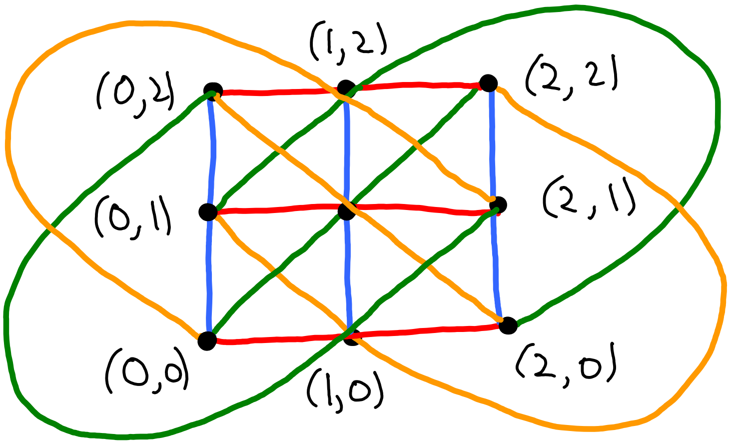
1-subspaces



1-flats

Example 2.6.6. The 1-flats in AG$(2,3)$, listed below, form an affine plane of order 3.

$\{(0,0),(0,1),(0,2)\}$	$\{(0,0),(1,0),(2,0)\}$	$\{(0,0),(1,1),(2,2)\}$	$\{(0,0),(1,2),(2,1)\}$
$\{(1,0),(1,1),(1,2)\}$	$\{(0,1),(1,1),(2,1)\}$	$\{(0,1),(1,2),(2,0)\}$	$\{(0,1),(1,0),(2,2)\}$
$\{(2,0),(2,1),(2,2)\}$	$\{(0,2),(1,2),(2,2)\}$	$\{(0,2),(1,0),(2,1)\}$	$\{(0,2),(1,1),(2,0)\}$



PG(n,q)

Definition 2.6.7. Let $n \geq 2$, let q be a prime power, and let V be the $(n+1)$-dimensional vector space over $\mathbb{F}_q$. The **projection** of any subspace U of V is defined to be $\{\langle x \rangle : x \in U\}$, where $\langle x \rangle$ denotes the 1-dimensional subspace of V generated by the vector x. The **projective geometry of dimension** n **over the field** $\mathbb{F}_q$, denoted $\mathrm{PG}(n,q)$, is the projection of V, and consists of the projections of all subspaces of V. The projections of the $(d+1)$-dimensional subspaces of V are the d**-dimensional subspaces** of $\mathrm{PG}(n,q)$. Thus, the projections of the 1-dimensional, 2-dimensional, and n-dimensional subspaces of V are the **points**, **lines** and **hyperplanes** respectively of $\mathrm{PG}(n,q)$.

Projection of U contains 1-subspaces of U as points.

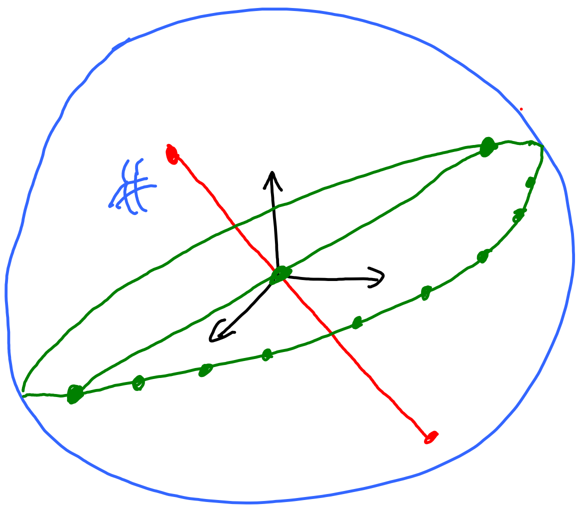
PG(n,q) is the projection of V.

d-subspaces of PG(n,q) are projections of (d+1)-subspaces of V.

lines of V $\longleftrightarrow$ points of PG(n,q)

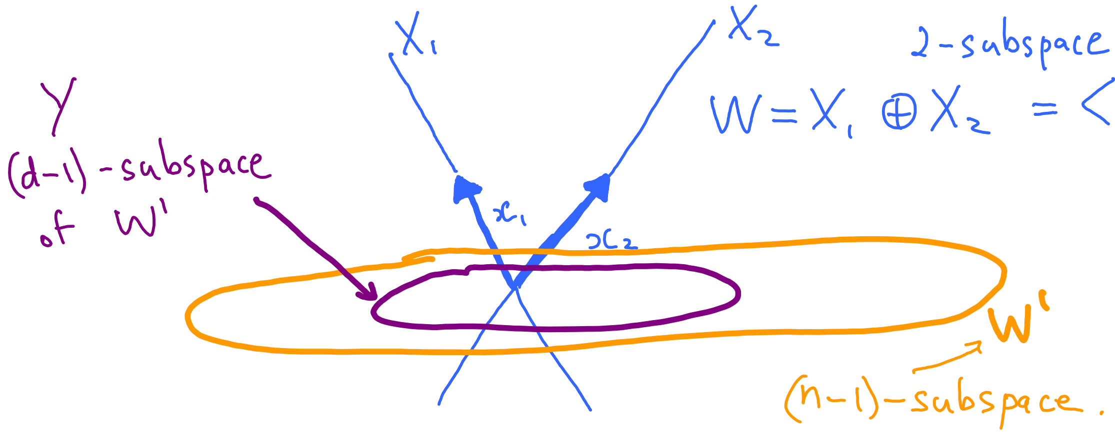
planes of V $\longleftrightarrow$ lines of PG(n,q)

n-subspaces/hyperplanes of V $\longleftrightarrow$ (n-1)-subspaces/hyperplanes of PG(n,q).



Theorem 2.6.8. Let $n \geq 2$ and let q be a prime power. Each pair of distinct points of $\mathrm{PG}(n,q)$ occurs together in exactly $\binom{n-1}{d-1}_q$ d-dimensional subspaces of $\mathrm{PG}(n,q)$.

Proof Let $\mathrm{PG}(n,q)$ be the projection of V. Let x_1 and x_2 be distinct points of $\mathrm{PG}(n,q)$, let X_1 and X_2 be their corresponding 1-dimensional subspaces of V, and let $W = X_1 \oplus X_2$ (the notation $\oplus$ is used for the direct sum; that is, the subspace spanned by two subspaces having trivial intersection). Let W' be an $(n-1)$-dimensional subspace of V such that $W \oplus W' = V$. The $(d+1)$-dimensional subspaces of V containing both X_1 and X_2 are precisely the spaces $W \oplus Y$ where Y is a $(d-1)$-dimensional subspace of W'. Since the number of such Y is $\binom{n-1}{d-1}_q$, there are exactly $\binom{n-1}{d-1}_q$ $(d+1)$-dimensional subspaces of V containing both X_1 and X_2. That is, there are exactly $\binom{n-1}{d-1}_q$ d-dimensional subspaces of $\mathrm{PG}(n,q)$ that contain x_1 and x_2. $\square$



2-subspace

$W = X_1 \oplus X_2 = \langle x_1, x_2 \rangle$.

$\boxed{W + W' = V}$

$Y + W$
$(d+1)$-subspace containing x_1, x_2

Corollary 2.6.9. The points and lines of $\mathrm{PG}(2,q)$ form a projective plane of order q.

$\binom{3}{1}_q = q^2+q+1$ points.

$\binom{2}{1}_q = q+1$ points on each line.

$\binom{3}{2}_q = q^2+q+1$ lines.

$\begin{matrix} n=2 \\ d=1 \end{matrix} \Rightarrow \binom{n-1}{d-1}_q = \binom{1}{0}_q = 1 \Longrightarrow$ each pair of points is in exactly one line.

It can be shown that if the construction given in the proof of Theorem 2.5.10 is applied to the affine plane arising from $\mathrm{AG}(2,q)$, then the resulting projective plane is the one arising from $\mathrm{PG}(2,q)$. Conversely, if we start with the projective plane arising from $\mathrm{PG}(2,q)$, delete a line and delete the points of that line from each of the remaining lines, then the resulting affine plane is the affine plane arising from $\mathrm{AG}(2,q)$.

Example 2.6.10. The points and hyperplanes of PG(2, 3) form a projective plane of order 3. Let V be the 3-dimensional vector space over $\mathbb{F}_3$. The points are the 1-dimensional subspaces of V and the lines are the 2-dimensional subspaces of V. We shall use

$$\frac{3^3-1}{3-1} = 13$$

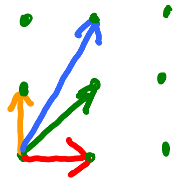
$$001, 010, 011, 012, 100, 101, 102, 110, 111, 112, 120, 121, 122$$

002 202 211

to denote the 1-dimensional subspaces of V, where xyz denotes the 1-dimensional subspace generated by the vector (x, y, z).

For each 1-dimensional subspace of V, there is a corresponding orthogonal 2-dimensional subspace of V. Moreover, a 1-dimensional subspace $x'y'z'$ is contained in the 2-dimensional subspace that is orthogonal to the 1-dimensional subspace xyz if and only if $xx' + yy' + zz' = 0$. Thus, the lines are as listed on the right below, with their corresponding orthogonal 1-dimensional subspaces listed on the left.

2-subspace of V has 4 1-subspaces



$\left(\frac{9-1}{3-1} = 4\right)$

001	010, 100, 110, 120
010	001, 100, 101, 102
011	012, 100, 112, 121
012	011, 100, 111, 122
100	001, 010, 011, 012
101	010, 102, 112, 122
102	010, 101, 111, 121
110	001, 120, 121, 122
111	012, 102, 111, 120
112	011, 101, 112, 120
120	001, 110, 111, 112
121	011, 102, 110, 121
122	012, 101, 110, 122

- lines of PG(2, 3).
- 2-subspaces of V orthogonal to given vector.