Stat3004 Assignment 4

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Q1

a)

For a $\sigma-$algebra we require it to contain Ω, to be closed under complements and closed under union. We know that $\mathcal{F}_3$ must contain $\emptyset$, Ω, A_1, A_2 and A_3. But we see that just this set is not closed under complements or unions (noting that A_i is a partition of Ω), so to do this we add $A_1 \cup A_2$, $A_2 \cup A_3$ and $A_1 \cup A_3$. Hence $\mathcal{F}_3 = \{\emptyset, \Omega, A_1, A_2, A_3, A_1 \cup A_2, A_2 \cup A_3, A_1 \cup A_3\}$, which contains 8 elements.

b)

In $\mathcal{F}_n$ we will have $\emptyset$, Ω, A_1, ... ,A_n and the unions of all except one A_i (there are n choose n-1 ways to select this union). Hence:

$$\begin{aligned}
|\mathcal{F}_n| &= 1 + 1 + n + \binom{n}{n-1} \\
&= 2 + n + n \\
&= 2(n+1)
\end{aligned}$$

Q2

Let $\Omega = \{a, b, c, d\}$, $\mathcal{F} = \{\emptyset, \Omega, \{a, b\}, \{c, d\}\}$ and $\mathcal{G} = \{\emptyset, \Omega, \{a, c\}, \{b, d\}\}$. We then have $\mathcal{H} = \mathcal{F} \cup \mathcal{G} = \{\emptyset, \Omega, \{a, b\}, \{c, d\}, \{a, c\}, \{b, d\}\}$. But then we have $\{a, b\} \cup \{a, c\} = \{a, b, c\}$ which isn't in $\mathcal{H}$. As $\mathcal{H}$ isn't closed under union it isn't a $\sigma-$algebra. Hence the union of two $\sigma-$algebras is not in general also a $\sigma-$algebra.

Q3

a)

First we require it to contain $\emptyset$, Ω, A and B. If the intersect of A and B is empty we can close it under complement and union by adding $\Omega\backslash A$, $\Omega\backslash B$, $A \cup B$ and $\Omega\backslash(A \cup B)$. If the intersect isn't empty we must further add $(\Omega\backslash A) \cup (B \cap A)$, $(\Omega\backslash B) \cup (A \cap B)$, $A\backslash B$ and $B\backslash A$. So in general we get the smallest $\sigma-$algebra is:

$$\mathcal{F} = \{\emptyset, \Omega, A, B, \Omega\backslash A, \Omega\backslash B, A \cup B, \Omega\backslash(A \cup B), (\Omega\backslash A) \cup (B \cap A), (\Omega\backslash B) \cup (A \cap B)\}$$

b)

Note that due to independence we have $B \cap A = \emptyset$.

$$\begin{aligned}
\mathbb{P}(\emptyset) &= 0 \\
\mathbb{P}(\Omega) &= 1 \\
\mathbb{P}(A) &= 0.4 \\
\mathbb{P}(B) &= 0.5 \\
\mathbb{P}(A \cup B) &= \mathbb{P}(A) + \mathbb{P}(B) = 0.5 + 0.4 = 0.9 \\
\mathbb{P}(\Omega\backslash(A \cup B)) &= \mathbb{P}(\Omega) - \mathbb{P}(A \cup B) = 1 - 0.9 = 0.1 \\
\mathbb{P}(\Omega\backslash A) &= \mathbb{P}(\Omega) - \mathbb{P}(A) = 1 - 0.4 = 0.6 \\
\mathbb{P}(\Omega\backslash B) &= \mathbb{P}(\Omega) - \mathbb{P}(B) = 1 - 0.5 = 0.5 \\
\mathbb{P}((\Omega\backslash A) \cup (B \cap A)) &= \mathbb{P}(\Omega\backslash A) = 0.6 \\
\mathbb{P}((\Omega\backslash B) \cup (A \cap B)) &= \mathbb{P}(\Omega\backslash B) = 0.5
\end{aligned}$$

Q4

Let $a, b \in \mathbb{R}$ such that $a < b$. As we have a borel $\sigma-$algebra everything is closed under compliment and countable union. We first know that we have intervals of the form $(-\infty, a]$, so taking the compliment we also have intervals of the form (a, ∞). Now $((-\infty, a] \cup (b, \infty))^c = (a, b]$. We also have $(a, b) = \bigcup_{n=1}^{\infty} \left(a, b - \frac{1}{n}\right]$. Also $(-\infty, b) = (-\infty, a] \cup (a, b)$ and complimenting $(-\infty, b)$ we also get $[b, \infty)$. Now we get $((-\infty, a) \cup [b, \infty))^c = [a, b)$, which is the first interval we wanted to show was contained in the borel $\sigma-$algebra. Now we finally get $((-\infty, a) \cup (b, \infty))^c = [a, b]$, which is the other interval we wanted to show was contained in the borel $\sigma-$algebra.

Q5

Using only the three Kolmogorov axioms of a probability measure we want to show that if A and B are events satisfying $A \subseteq B$ then $\mathbb{P}(A) \leq \mathbb{P}(B)$.

Proof. The third axiom states that a countable set of disjoint events $E_1, E_2, ...$ satisfies $\mathbb{P}\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(E_i)$. Letting $E_1 = A$, $E_2 = B \backslash A$ and $E_i = \emptyset$ where $i \geq 3$ we get $E_1 \cup E_2 \cup E_3 \cup ... = B$. Now applying the third axiom we get:

$$\mathbb{P}(A) + \mathbb{P}(B \backslash A) + \sum_{i=3}^{\infty} \mathbb{P}(\emptyset) = \mathbb{P}(B)$$

By the first axiom we must have $\mathbb{P}(E_j) \geq 0$ where $j \geq 0$ and as the left hand side of the equation above is a series of non-negative numbers converging to $\mathbb{P}(B)$ we can see that $\mathbb{P}(A) \leq \mathbb{P}(B)$ with equality if and only if $B \backslash A = \emptyset$, i.e. $B = A$. □

Q6

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $A_1, A_2, ...$ be an increasing sequence of events; that is, $A_1 \subseteq A_2 \subseteq$. Using only the Kolmogorov axioms, we want to prove that $\mathbb{P}$ is continuous from below:

$$\lim_{n \to \infty} \mathbb{P}(A_n) = \mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right)$$

Proof. First we define $B_1 = A_1$ and $B_n = A_n \backslash \left(\bigcup_{i=1}^{n-1} A_i\right)$ where $n \geq 2$. We can see that B_1 and each B_n are disjoint and hence we can apply the third axiom:

$$\begin{aligned}
\mathbb{P}\left(\bigcup_{n=1}^{\infty} B_n\right) &= \sum_{i=1}^{\infty} \mathbb{P}(B_i) \\
&= \lim_{n \to \infty} \sum_{i=1}^{n} \mathbb{P}(B_i) \\
&= \lim_{n \to \infty} \sum_{i=1}^{n} \mathbb{P}\left(A_i \backslash \left(\bigcup_{j=1}^{n-1} A_j\right)\right) \\
&= \lim_{n \to \infty} \mathbb{P}(A_n)
\end{aligned}$$

We also have that $\mathbb{P}\left(\bigcup_{n=1}^{\infty} B_n\right) = \mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right)$ as each A_{n-1} is contained in A_n we get that excluding each previous A_i and then unioning each B_i is the same as just unioning each A_i. Hence the result. □

Q8

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ the measurable space of reals and its borel $\sigma-$algebra, we also have the function $\mu_X(B) = \mathbb{P}(X^{-1}(B))$ for all

$B \in \mathcal{B}(\mathbb{R})$. We want to show that $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu_X)$ is a probability space.

Proof. As we have a sample space $(\mathbb{R})$ and what we know to be a $\sigma-$algebra $(\mathcal{B}(\mathbb{R}))$ we will just show that the properties of a probability measure hold for μ_X. A random variable is a function $X : \Omega \to \mathbb{R}$ such that the preimage of any set $B \in \mathcal{B}(\mathbb{R})$ is measurable in $\mathcal{F}$. This means we have $X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\} \in \mathcal{F}$. We also then have:

$$\begin{aligned}\mu_X(B) &= \mathbb{P}(X^{-1}(B)) \\ &= \mathbb{P}(\{\omega \in \Omega : X(\omega) \in B\})\end{aligned}$$

So we see that this is just the probability measure on $\mathcal{F}$ and hence it must also be a probability measure on $\mathcal{B}(\mathbb{R})$, i.e. we have:

$$\mu_X(\mathbb{R}) = \mathbb{P}(X^{-1}(\mathbb{R})) = \mathbb{P}(\Omega) = 1$$

and $\mu_X : \mathcal{B}(\mathbb{R}) \to \mathcal{F} \to [0, 1]$. As it maps to $\mathcal{F}$ we also get countable additivity and hence we see that it is indeed a probability measure and so $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu_X)$ is a probability space. $\square$