1.3 Some Exceptional Objects.

Some classes of combinatorial/algebraic objects consist of a few infinite families plus a few sporadic/exceptional objects.

Examples:

① Convex regular polytopes:

- 2-dimensional regular polygons
- n-dimensional tetrahedron
- n-dimensional cube
- n-dimensional octahedron
- icosahedron, dodecahedron, 24-cell, 120-cell, 600-cell. } 5 exceptional objects.

② Automorphisms of S_n:

Aut(G): the set of all isomophisms $G \to G$ (automorphisms)

- for all $n \geq 3$, $\mathrm{Aut}(S_n) \cong S_n$;

  except that $\mathrm{Aut}(S_6) \cong S_6 \rtimes \mathbb{Z}_2$.

③ Finite Simple Groups:

- $\mathbb{Z}_p$ where p is prime.
- A_n $n \geq 5$.
- "Groups of Lie type" (infinite family, PSL(2,7))
- 26 sporadic groups

④ Unimodular Lattices: E_8 lattice, Leech lattice.

Automorphisms of S_n:

Let G be a group. $\forall g \in G$, define $\phi_g : G \to G$ by

$$a\phi_g = g^{-1} a g \quad \text{for all } a \in G \quad \text{(conjugation)}$$

ϕ_g is an automorphism: ("inner automorphism")

Homomorphism:

$$(ab)\phi_g = g^{-1}abg = g^{-1}agg^{-1}bg = a\phi_g \, b\phi_g$$

$\ker \phi_g = \{1\}$:

$$g^{-1}ag = 1 \longrightarrow ag = g \longrightarrow a = 1.$$

(1st Isomorphism Thm:
$\mathrm{Im}\, \phi_g \cong G/\ker \phi_g$
$\cong G$
ϕ_g is a bijection)

Conjugation in S_n:

Example: $(3\ 5)(1\ 2\ 3\ 4\ 5\ 6)(3\ 5) = (1\ 2\ 5\ 4\ 3\ 6)$

Let $g, \theta \in S_n$ and consider the image of the point xg under the permutation $g^{-1}\theta g$.

$$(xg)(g^{-1}\theta g) = x\theta g$$

So if $\quad \theta = (\ \ldots\ )(\ \ldots\ x\ \ x\theta\ \ldots\ )(\ \ldots\ )\ldots$

then $\quad g^{-1}\theta g = (\ \ldots\ )(\ \ldots\ xg\ \ x\theta g\ \ldots\ )(\ \ldots\ )\ldots$

The cycle representation of $g^{-1}\theta g$ is obtained by "applying" the permutation g to the cycle representation of θ.

Theorem: In S_n, conjugation preserves cycle structure.

Theorem: If G is a group, then
$$\pi : G \longrightarrow \mathrm{Aut}(G)$$
given by $g \longmapsto \phi_g$ is a homomorphism.

Proof: If $\theta \in G$, then

$$\theta(gh)\pi = \theta\phi_{gh} = (gh)^{-1}\theta gh = h^{-1}g^{-1}\theta gh = h^{-1}(\theta\phi_g)h$$
$$= \theta\phi_g\phi_h$$
$$= \theta(g\pi)(h\pi)$$

Theorem: $\mathrm{Aut}(S_2) = \{1\}$. For $n \geq 3$ $S_n \leq \mathrm{Aut}(S_n)$.

Proof: $\mathrm{Aut}(S_2) = \{1\}$ ✓ Let $n \geq 3$. $(S_2 = \{(), (1\ 2)\})$

Define $\pi : S_n \longrightarrow \mathrm{Aut}(S_n)$ by $g \mapsto \phi_g$

Consider $\ker \pi$: Let $g \in S_n$

$g \in \ker\pi$ iff ϕ_g is the identity of $\mathrm{Aut}(S_n)$
iff $\theta\phi_g = \theta$ for all $\theta \in S_n$
iff $g^{-1}\theta g = \theta$ for all $\theta \in S_n$
iff $\theta g = g\theta$ for all $\theta \in S_n$

$\ker \pi$ is the centre of S_n.

For $n \geq 3$, $Z(S_n) = \{1\}$: Let $g \in S_n \setminus \{1\}$. Then $\exists$ distinct points x, y such that $xg = y$. Let $z \notin \{x, y\}$ be a point $(n \geq 3)$

Then $g(y\,z)$ maps x to z
but $(y\,z)g$ maps x to y $\Bigg\} \Longrightarrow$ $g(y\,z) \neq (y\,z)g$
$g \notin Z(S_n)$.

$\Longrightarrow$ for $n \geq 3$, $\ker \Pi = \{1\}$.

1st Isomorphism Thm $\Longrightarrow$ • $\operatorname{Im} \Pi \leq \operatorname{Aut}(S_n)$

• $\operatorname{Im} \Pi \cong S_n / \ker \Pi \cong S_n$

$S_n \leq \operatorname{Aut}(S_n)$. $\square$

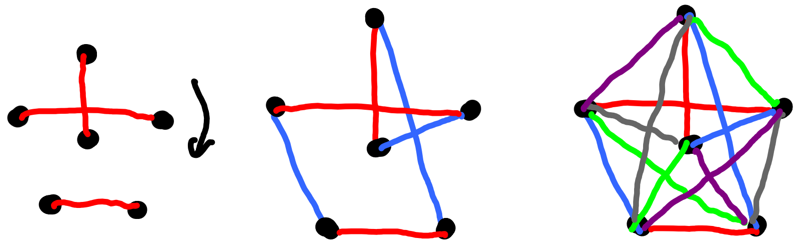
<u>Theorem</u>: $\operatorname{Aut}(S_2) = \{1\}$. For $n \geq 3$, $\operatorname{Aut}(S_n) \cong S_n$ except that $\operatorname{Aut}(S_6) \cong S_6 \rtimes \mathbb{Z}_2$ (semi-direct product)

$|\operatorname{Aut}(S_6)| = 6! \times 2 = 1440$.

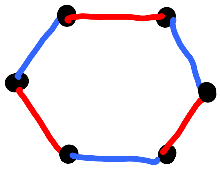
S_6 has "outer automorphisms".

Outer Automorphisms of S_6

1-factorisations of K_6



Any two 1-factors form a 6-cycle.



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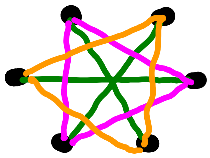
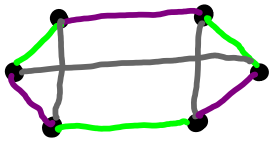
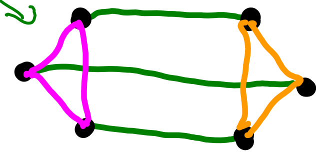


Any two edge-disjoint 1-factors of K_6 are in a unique 1-factorisation

unique 1-factorisation





$\cong$?

How many 1-factorisations of K_6???

Let x be the number. Count "occurrences of 6-cycles" in 1-factorisations.

pair of 1-factors

5 1-factors in each 1-factorisation $\longrightarrow \binom{5}{2} = 10$ pairs of 1-factors / 6-cycles in each 1-factorisation.

So $10x$ 6-cycles across all 1-factorisations.

This counts each 6-cycle exactly once.

There are $\dfrac{6!}{12} = 60$ 6-cycles in K_6 $\quad (\mathrm{Aut}(C_6) \cong D_6)$

So $\quad 10x = 60$ and $\underline{\underline{x = 6}}$.

Now count 1-factors of K_6. The number is $|\Theta(F)|$ where $F = \{12, 34, 56\}$ and $\Theta(F)$ denotes the orbit of F under S_6.

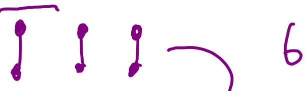
The order of the stabilizer of F in S_6 equals the number of ways of labelling to get $F = 6 \times 4 \times 2$.

So the number of 1-factors of K_6 is

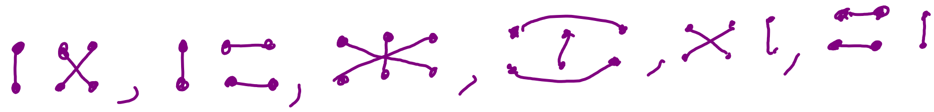
$$|\Theta(F)| = |S_6| / 6\times4\times2 = \frac{6!}{6\times4\times2} = 5\times3 = 15$$

The 6 1-factorisations of K_6 collectively contain $5 \times 6 = 30$ 1-factors.

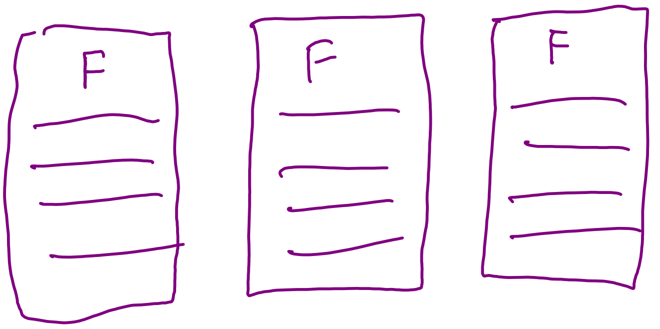
Each 1-factor must occur exactly twice because none can occur three times:–



6 1-factors share an edge with any given 1-factor.

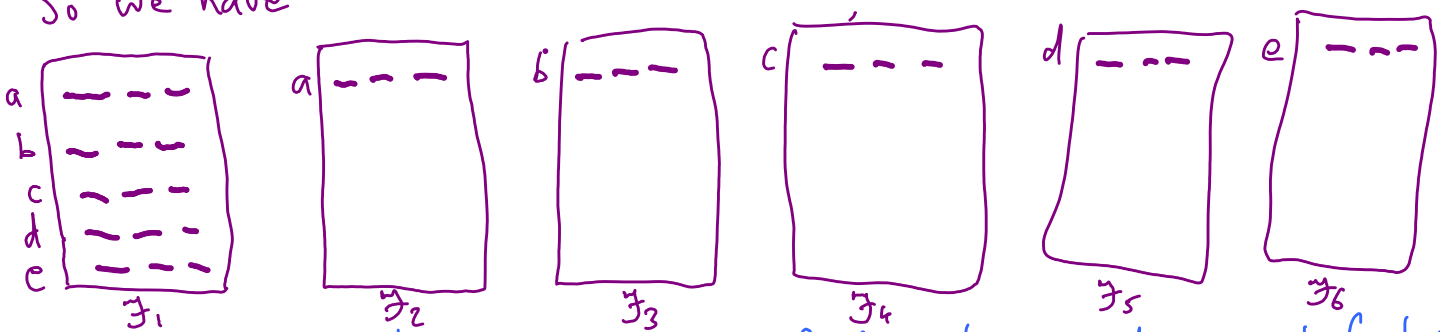


There are $15 - 6 - 1 = 8$ 1-factors that are edge-disjoint from any given 1-factor.

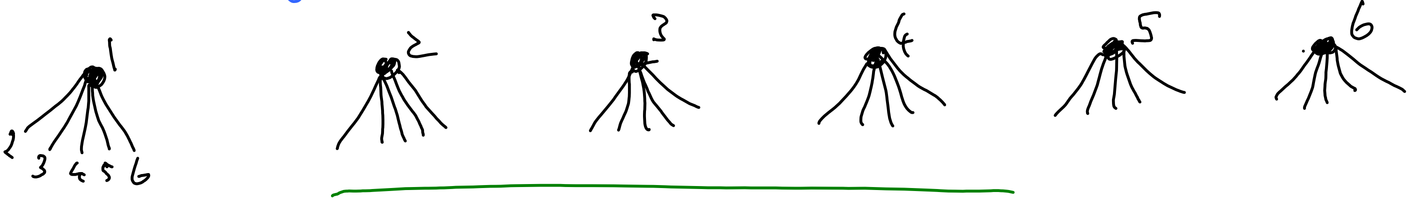


would require 12 distinct 1-factors that are disjoint from F

So we have

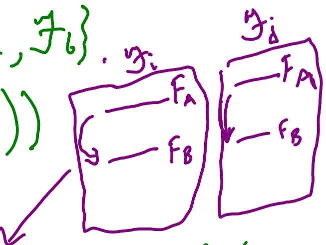


Notice the symmetry: vertices $\longleftrightarrow$ 1-factorisations; edges $\longleftrightarrow$ 1-factors



Now consider the induced action ϕ of S_6 on $\{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_6\}$.

(homomorphism $\phi : S_6 \longrightarrow \mathrm{Sym}(\{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_6\})$)



Suppose $\theta\phi$ fixes all 6 1-factorisations.

Then $\theta\phi$ fixes the individual 1-factors (because any two edge-disjoint 1-factors are in a unique 1-factorisation)

It follows that θ is the identity. ($\underset{\text{fixed}}{12\ 34\ 56}$, $\underset{\text{fixed}}{12\ 35\ 46}$ $\longrightarrow$ 12 $\overset{34}{\underset{56}{\leftrightarrows}}$ edges are fixed)

So $\ker \phi = \{1\}$ and ϕ is an isomorphism. ($\underset{\text{fixed}}{12}$, $\underset{\text{fixed}}{13}$ $\longrightarrow$ vertices are fixed)

Since $\mathrm{Sym}(\{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_6\}) \cong S_6$ we have an automorphism of S_6.

$\mathcal{F}_1$	$\mathcal{F}_2$	$\mathcal{F}_3$	$\mathcal{F}_4$	$\mathcal{F}_5$	$\mathcal{F}_6$
12 34 56	12 34 56	12 35 46	12 35 46	12 36 45	12 36 45
16 23 45	16 24 35	16 23 45	16 25 34	16 25 34	16 24 35
13 25 46	14 25 36	13 24 56	15 24 36	15 23 46	14 23 56
14 26 35	13 26 45	15 26 34	13 26 45	14 26 35	15 26 34
15 24 36	15 23 46	14 25 36	14 23 56	13 24 56	13 25 46

Examples of the induced action of S_6 on $\{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4, \mathcal{F}_5, \mathcal{F}_6\}$.

$(1\ 2) \longmapsto (\mathcal{F}_1\ \mathcal{F}_2)(\mathcal{F}_3\ \mathcal{F}_4)(\mathcal{F}_5\ \mathcal{F}_6)$ $\quad (1\ 2)(3\ 4)(5\ 6)$

$(1\ 3) \longmapsto (\mathcal{F}_1\ \mathcal{F}_6)(\mathcal{F}_2\ \mathcal{F}_4)(\mathcal{F}_3\ \mathcal{F}_5)$ $\quad (1\ 6)(2\ 4)(3\ 5)$

$(1\ 2\ 3) \longmapsto (\mathcal{F}_1\ \mathcal{F}_4\ \mathcal{F}_5)(\mathcal{F}_2\ \mathcal{F}_6\ \mathcal{F}_3)$ $\quad (1\ 4\ 5)(2\ 6\ 3)$

$(1\ 2\ 3\ 4\ 5) \longmapsto (\mathcal{F}_1\ \mathcal{F}_3\ \mathcal{F}_6\ \mathcal{F}_5\ \mathcal{F}_2)$ $\quad (1\ 3\ 6\ 5\ 2)$

Note $(1\ 2)(1\ 3) = (1\ 2\ 3)$ and $(1\ 2)(3\ 4)(5\ 6)(1\ 6)(2\ 4)(3\ 5)$
$= (1\ 4\ 5)(2\ 6\ 3)$

How do we know this automorphism of S_6 is not conjugation?

(It does not preserve cycle structure)

2-transitive action of S_5 on 6 points.

In S_6, the stabilizer of a point is S_5.

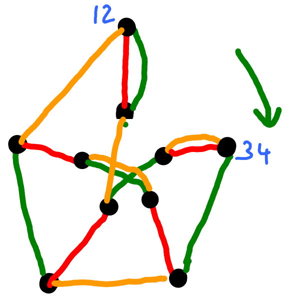
Conjugation permutes these S_5 subgroups amongst themselves.

The outer automorphism maps these S_5 subgroups to a different subgroup that is isomorphic to S_5 but acts on 6 points.

see above examples, elements from stabilizer of 6 move all 6 1-factorisations

2-transitively

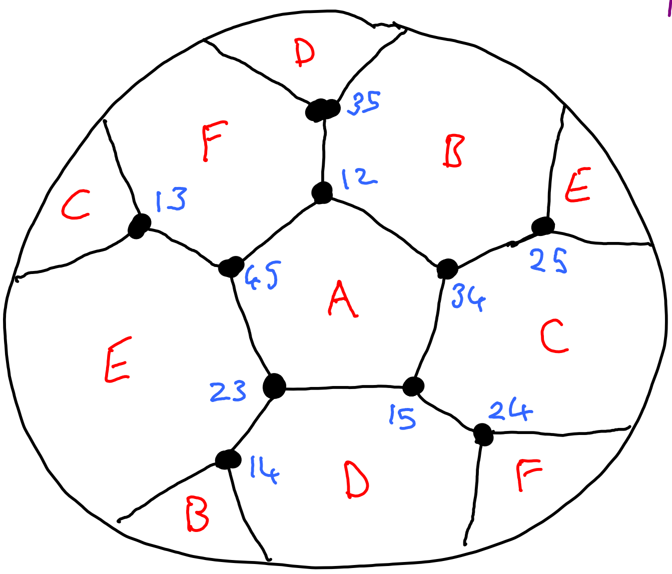
This 2-transitive action of S_5 on 6 points is given by the action of S_5 on the 6 1-factors of the Petersen graph.



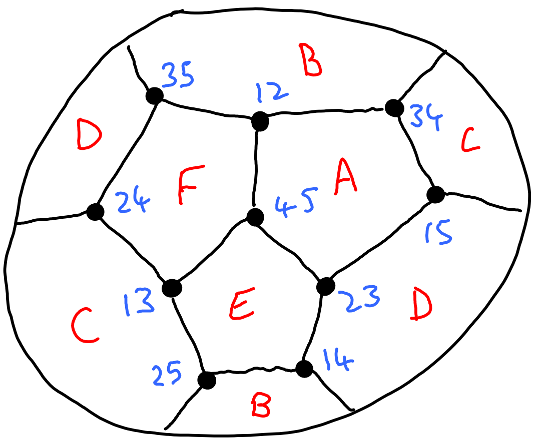
In fact A_5 acts 2-transitively on 6 points

A_5 acts 2-transitively on the 6 faces of the Petersen graph embedded on the projective plane.

$(1\,2\,3) \longmapsto (A\,E\,F)(B\,D\,C)$



$(1\,3\,5\,2\,4) \longmapsto (A)(B\,C\,D\,E\,F)$



The Mathieu groups.

Simple Groups:

Simple groups play a fundamental role in group theory. In the second half of the twentieth century (and with some small corrections/omissions made later), a program to classify all the finite simple groups was successfully undertaken. Up to isomorphism, the finite simple groups are

(a) $\mathbb{Z}_p$ where p is prime.

(b) A_n where $n \geq 5$.

(c) The so-called "groups of Lie type", which form an infinite family.

(d) 26 "sporadic groups". (exceptional objects)

Five of the ten smallest sporadic groups are the "Mathieu groups"

$$M_{11}, \qquad M_{12}, \qquad M_{22}, \qquad M_{23} \qquad \text{and} \qquad M_{24}$$

which have orders

$$7,920, \qquad 95,040, \qquad 443,520, \qquad 10,200,960 \qquad \text{and} \qquad 244,823,040$$

respectively. The largest sporadic group, the "Monster group", has order

$$808,017,424,794,512,875,886,459,904,961,710,757,005,754,368,000,000,000.$$

Multiply Transitive Groups:

The only t-transitive group actions with $t \geq 4$ are as follows. The proof of this fact uses the classification of finite simple groups.

- The symmetric group S_n is sharply n-transitive on n points.
- The alternating group A_n is sharply $(n-2)$-transitive on n points. (exercise)
- The Mathieu group M_{11} is sharply 4-transitive on 11 points.
- The Mathieu group M_{12} is sharply 5-transitive on 12 points.
- The Mathieu group M_{23} is 4-transitive on 23 points.
- The Mathieu group M_{24} is 5-transitive on 24 points.

Steiner Systems

$S(5,6,12)$ Automorphism group M_{12}

132 6-subsets of 12-set, each 5-subset occurs once.

$S(5,8,24)$ Automorphism group M_{24}

759 8-subsets of 24-set, each 5-subset occurs once.

$S(t,k,v)$ ← None known with $t>5$.

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Theorem 1.3.2. (Keevash, [36]) For all $t \geq 1$, $k \geq t$ and $\lambda \geq 1$, there is a constant $C(t,k,\lambda)$ such that for all $v \geq C(t,k,\lambda)$, there exists a $t-(v,k,\lambda)$-design if and only if $\binom{k-s}{t-s}$ divides $\lambda\binom{v-s}{t-s}$ for $0 \leq s \leq t$.