

# Stat3001 Assignment 1

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## Q1

We have  $x_1, \dots, x_n \stackrel{iid}{\sim} N(\theta, \theta^2)$  so the likelihood function is:

$$\begin{aligned} L(\tilde{x}; \theta) &= \prod_{i=1}^n f(\tilde{x}_i; \theta) \\ &= \prod_{i=1}^n (2\pi)^{-\frac{1}{2}} (\theta^2)^{-\frac{1}{2}} \exp\left(-\frac{1}{2} \frac{(x_i - \theta)^2}{\theta^2}\right) \\ &= (2\pi)^{-\frac{n}{2}} (\theta)^{-n} \exp\left(-\frac{1}{2} \sum_{i=1}^n \frac{(x_i - \theta)^2}{\theta^2}\right) \\ &= (2\pi)^{-\frac{n}{2}} (\theta)^{-n} \exp\left(-\frac{1}{2} \sum_{i=1}^n \frac{x_i^2 - 2x_i\theta + \theta^2}{\theta^2}\right) \\ &= (2\pi)^{-\frac{n}{2}} (\theta)^{-n} \exp\left(-\frac{1}{2\theta^2} \sum_{i=1}^n x_i^2 + \frac{1}{\theta} \sum_{i=1}^n x_i - \frac{n}{2}\right) \end{aligned}$$

Now taking the log likelihood we get:

$$\log L(\tilde{x}; \theta) = -\frac{n}{2} (\log 2\pi + \log \theta) - \frac{1}{2\theta^2} \sum_{i=1}^n x_i^2 + \frac{1}{\theta} \sum_{i=1}^n x_i - \frac{n}{2}$$

For the regular exponential family we require:

$$L(\tilde{x}; \theta) = \frac{b(\tilde{x}) \exp(c(\theta)^T T(\tilde{x}))}{a(\theta)}$$

And hence:

$$\log L(\tilde{x}; \theta) = \log b(\tilde{x}) + c(\theta)^T T(\tilde{x}) - \log a(\theta)$$

Here we have:

$$\begin{aligned}a(\theta) &= \frac{n}{2}(\log 2\pi + \log \theta + 1) \\b(\tilde{x}) &= 1 \\c(\theta) &= \left(\frac{1}{\theta}, -\frac{1}{2\theta^2}\right)^T \\T(\tilde{x}) &= \left(\sum_{i=1}^n x_i, \sum_{i=1}^n x_i^2\right)^T\end{aligned}$$

Here  $T(\tilde{x})$  is the sufficient statistic for  $\theta$  and from this we see that this distribution belongs to the regular exponential family.

## Q2

We have  $x_1, \dots, x_n \stackrel{iid}{\sim} N(\theta, \theta)$  so the likelihood function is:

$$\begin{aligned}L(\tilde{x}; \theta) &= \prod_{i=1}^n f(\tilde{x}; \theta) \\&= \prod_{i=1}^n (2\pi)^{-\frac{1}{2}} (\theta)^{-\frac{1}{2}} \exp\left(-\frac{1}{2} \frac{(x_i - \theta)^2}{\theta}\right) \\&= (2\pi)^{-\frac{n}{2}} (\theta)^{-\frac{n}{2}} \exp\left(-\frac{1}{2} \sum_{i=1}^n \frac{(x_i - \theta)^2}{\theta}\right) \\&= (2\pi)^{-\frac{n}{2}} (\theta)^{-\frac{n}{2}} \exp\left(-\frac{1}{2} \sum_{i=1}^n \frac{x_i^2 - 2x_i\theta + \theta^2}{\theta}\right) \\&= (2\pi)^{-\frac{n}{2}} (\theta)^{-\frac{n}{2}} \exp\left(-\frac{1}{2\theta} \sum_{i=1}^n x_i^2 + \sum_{i=1}^n x_i + n\theta\right)\end{aligned}$$

Now taking the log likelihood we get:

$$\log L(\tilde{x}; \theta) = -\frac{n}{2} (\log 2\pi + \log \theta) - \frac{1}{2\theta} \sum_{i=1}^n x_i^2 + \sum_{i=1}^n x_i + n\theta$$

For the regular exponential family we require:

$$L(\tilde{x}; \theta) = \frac{b(\tilde{x}) \exp(c(\theta)^T T(\tilde{x}))}{a(\theta)}$$

And hence:

$$\log L(\tilde{x}; \theta) = \log b(\tilde{x}) + c(\theta)^T T(\tilde{x}) - \log a(\theta)$$

Here we have:

$$\begin{aligned} a(\theta) &= \frac{n}{2}(\log 2\pi + \log \theta + 2\theta) \\ b(\tilde{x}) &= \sum_{i=1}^n x_i \\ c(\theta) &= -\frac{1}{2\theta} \\ T(\tilde{x}) &= \sum_{i=1}^n x_i^2 \end{aligned}$$

Here  $T(\tilde{x})$  is the sufficient statistic for  $\theta$  and from this we see that this distribution belongs to the regular exponential family.

Taking the derivative of the log likelihood equation and setting it to zero we get:

$$\begin{aligned} \frac{d}{d\theta} \log L(\tilde{x}; \theta) &= -\frac{n}{2\theta} + \frac{1}{2\theta^2} \sum_{i=1}^n x_i^2 + n \\ &= 0 \end{aligned}$$

Multiplying through by  $2\theta^2$  we get:

$$0 = -n\theta + \sum_{i=1}^n x_i^2 + n\theta^2$$

This is quadratic in  $\theta$  so we get:

$$\hat{\theta} = \frac{-1 + \sqrt{1 + 4\frac{1}{n} \sum_{i=1}^n x_i^2}}{2}$$

As the likelihood function is from the regular exponential family we have:

$$\begin{aligned}
\mathbb{E} \left[ \sum_{i=1}^n x_i^2 \right] &= \sum_{i=1}^n \mathbb{E} [x_i^2] \\
&= \sum_{i=1}^n \text{Var}(X_i) + \mathbb{E} [x_i]^2 \\
&= \sum_{i=1}^n \theta + \theta^2 \\
&= n(\theta + \theta^2) \\
&= \sum_{i=1}^n x_i^2
\end{aligned}$$

Rearranging we get:

$$0 = n\theta^2 + n\theta - \sum_{i=1}^n x_i^2$$

Which is quadratic in  $\theta$  so we get:

$$\begin{aligned}
\hat{\theta} &= \frac{-1 + \sqrt{n^2 + 2n \sum_{i=1}^n x_i^2}}{2n} \\
&= \frac{-1 + \sqrt{1 + 4 \frac{1}{n} \sum_{i=1}^n x_i^2}}{2}
\end{aligned}$$

As the likelihood is from the regular exponential family the sufficient statistic is also complete which guarantees an estimator of  $\theta$  is unique.