Regular Polytopes.

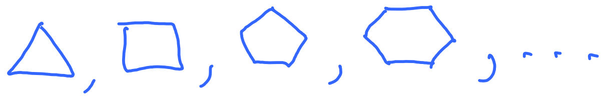
The term **polytope** has been defined in several different ways, only some of them equivalent. For our purposes, the following definition suffices. An **n-dimensional polytope**, or just **n-polytope** is a finite region of $\mathbb{R}^n$ that is enclosed by a finite number of hyperplanes (here, hyperplane means any translation of an $(n-1)$-dimensional subspace of $\mathbb{R}^n$).

2 dimensions: Polygons

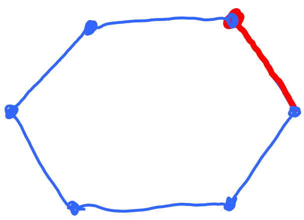
3 dimensions: Polyhedra.

Convex: $x, y \in P \rightarrow \overline{xy} \in P$

Convex regular polygons:



Symmetry group: D_n (dihedral group)

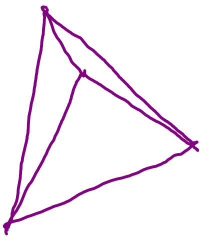


A <u>flag</u> of a polygon consists of an edge e and a vertex of e.

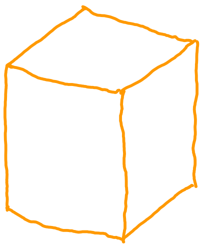
A polygon is <u>regular</u> if its symmetry group acts transitively on its flags.

Convex regular polyhedra (3-polytopes):

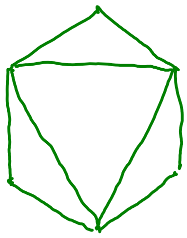
Polyhedra is the usual name for 3-polytopes. A polyhedron is bounded by polygons, which are called the **faces** of the poyhedron. A flag of polyhedron consists of a face f (which is a polygon), an edge e of f, and a vertex v of e. A polyhedron is **regular** if its symmetry group acts transitively on its flags. The convex regular 3-polytopes are precisely the **Platonic solids** – the **tetrahedron**, **cube**, **octahedron**, **icosahedron**, and **dodecahedron**.



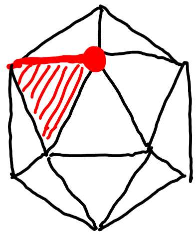
Tetrahedron



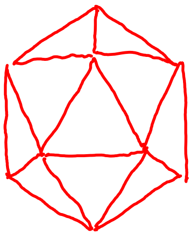
Cube



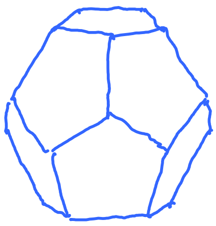
Octahedron



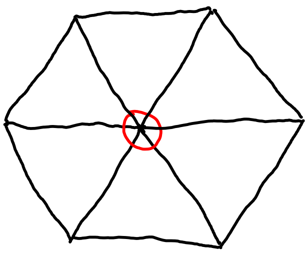
Platonic Solids

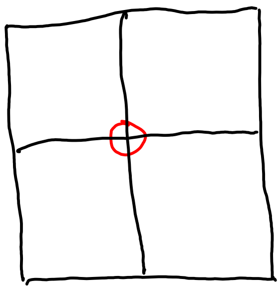


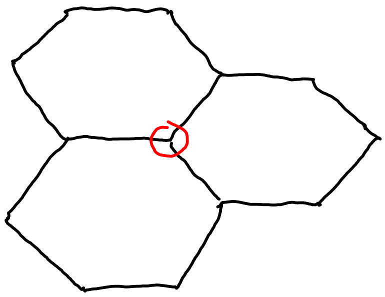
Icosahedron



Dodecahedron







The **dual** of an polyhedron is obtained by placing a vertex at the centre of each face, and joining two such vertices by an edge precisely when the corresponding faces share an edge. The octahedron is the dual of the cube, the dodecahedron is the dual of the icosahedron, and the tetrahedron is self-dual (the dual of itself). The dual of a polyhedron has the same symmetry group.

Symmetry groups:

The (full) symmetry group of a polyhedron includes both rotations and reflections. The group of rotations is a subgroup of the full symmetry group, which includes the reflections. The table below lists the rotational and full symmetry groups, and their orders in parentheses, for each the five Platonic solids.

Solid	Group of rotations	Full symmetry group
Tetrahedron	A_4 (12)	S_4 (24)
Cube	S_4 (24)	$S_4 \times \mathbb{Z}_2$ (48)
Octahedron	S_4 (24)	$S_4 \times \mathbb{Z}_2$ (48)
Icosahedron	A_5 (60)	$A_5 \times \mathbb{Z}_2$ (120)
Dodecahedron	A_5 (60)	$A_5 \times \mathbb{Z}_2$ (120)

Symmetry groups have regular action on flags (for regular polygons and polyhedra)

Unlike the other solids, the tetrahedron does not have a "reflection through the origin" symmetry.

For the other four, reflection through the origin

$$\text{Full symmetry group} = \text{Rotation Group} \times \mathbb{Z}_2$$

Reflection through the origin R_0 commutes with all other symmetries — if we write the automorphism group as a matrix group, then $R_0 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

(rotations have determinant $= 1$)

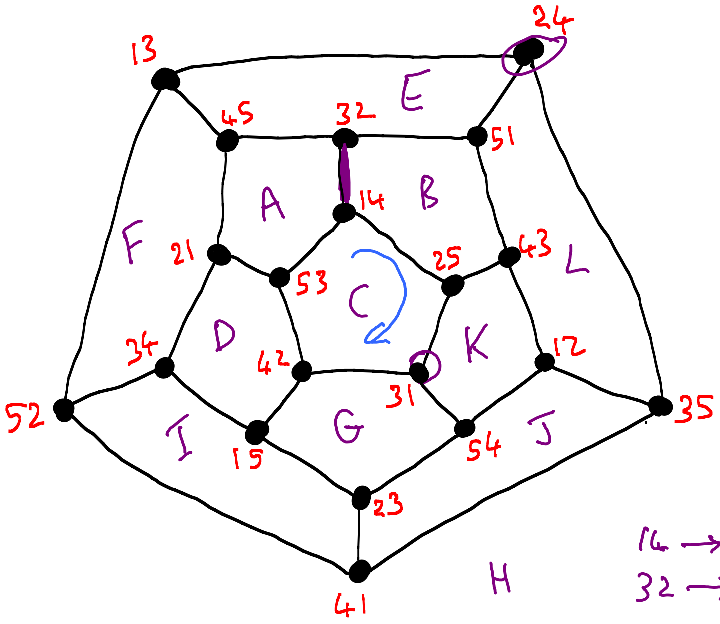
(In 2 dimensions, "reflection" through the origin is a rotation.)

(and commutes with all symmetries)

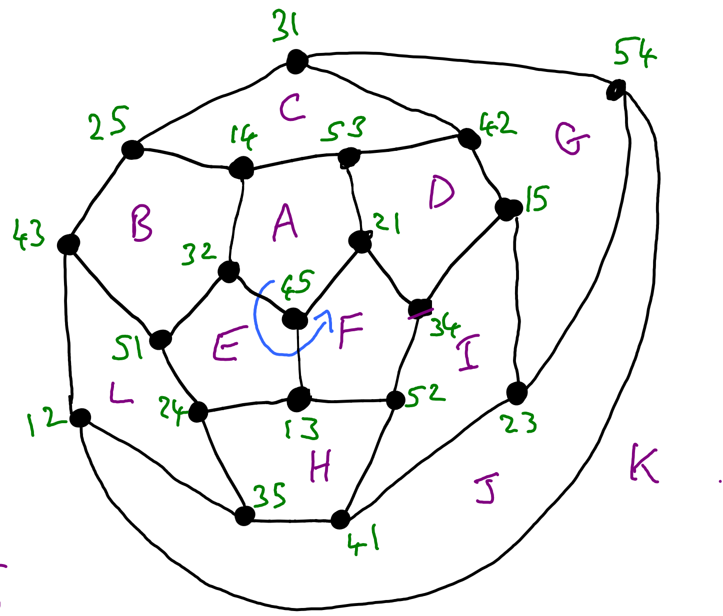
Reflections of a tetrahedron are single transpositions (so in S_4 but not A_4) and do not commute with rotations.

Reflections of polygons do not commute with rotations.

Symmetries of the dodecahedron $(\mathrm{Aut}(D) \cong A_5 \times \mathbb{Z}_2)$



$14 \to 24$
$32 \to 31$



$(1\ 2\ 3\ 4\ 5) \in \mathrm{Aut}(D)$

$(1\ 2\ 3) \in \mathrm{Aut}(D)$

$A_5 = \langle (1\ 2\ 3), (1\ 2\ 3\ 4\ 5) \rangle$

$\boxed{(12) \notin A_5,\ (12) \notin \mathrm{Aut}(D)}$

$\hookrightarrow A_5 \leq$ Rotation group

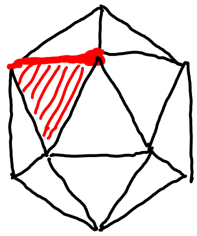
reflection through origin $\in \mathrm{Aut}(D)$

$ij \mapsto ji \notin A_5$

$ij \mapsto ji \in \mathrm{Aut}(D)$

Regular n-polytopes with $n \geqslant 4$.

For $n \geq 4$ (and also for $n = 2$ and $n = 3$), an n-polytope is bounded by $(n-1)$-polytopes, and these are the **$(n-1)$-faces** of the n-polytope. Each of the n-polytope's $(n-1)$-faces is bounded by $(n-2)$-polytopes, and these are the **$(n-2)$-faces** of the n-polytope. The pattern continues on down until we have the 2-faces (which are polygons) bounded by the 1-faces (which are the edges of the n-polytope), and finally the 1-faces (edges) bounded by the 0-faces (which are the vertices of the n-polytope). A **flag** of an n-polytope is a sequence $F_n, F_{n-1}, \ldots, F_1, F_0$ where F_n is the n-polytope itself, and F_i is an i-face of F_{i+1} for $i = 0, 1, \ldots, n-1$.

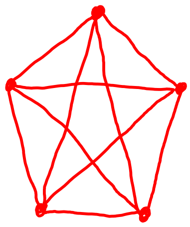


An n-polytope is **regular** if its symmetry group acts transitively on its flags. There are six convex regular 4-polytopes, each of these is called an n-cell, where the "n" refers to the number of 3-faces. For example, the 8-cell has 8 3-faces, each of which is a (3-dimensional) cube.

There are six convex regular 4-polytopes.

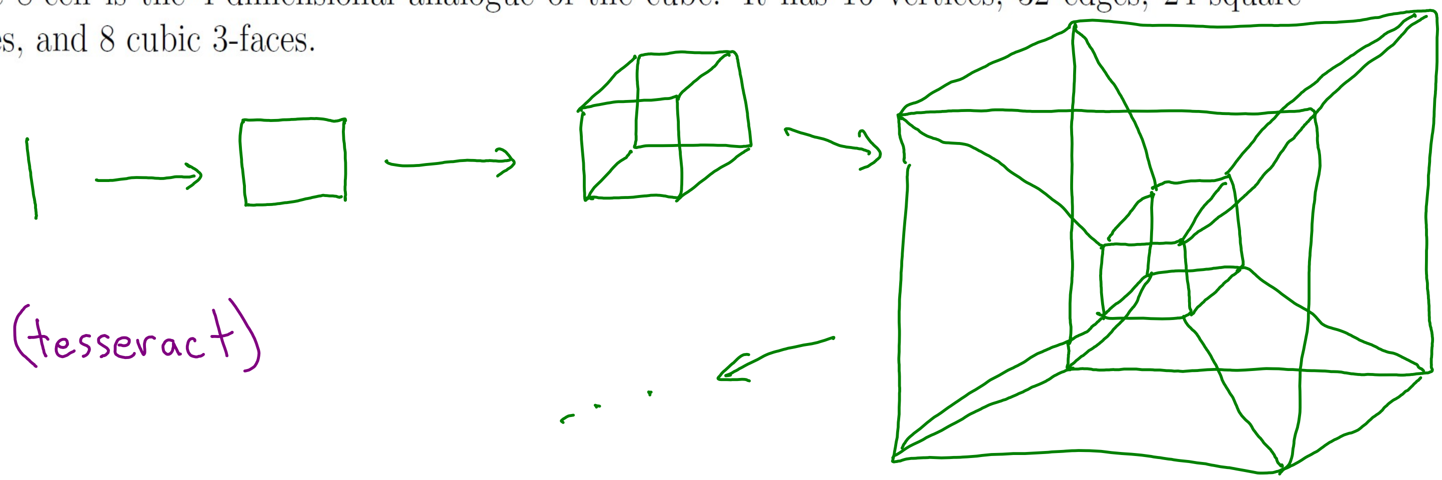
5-cell

The 5-cell is the 4-dimensional analogue of the tetrahedron. It has 5 vertices, 10 edges, 10 faces (2-faces) each of which is a regular 3-gon (equilateral triangle), and 5 tetrahedral 3-faces.



8-cell

The 8-cell is the 4-dimensional analogue of the cube. It has 16 vertices, 32 edges, 24 square faces, and 8 cubic 3-faces.



(tesseract)

16-cell

The 16-cell is the 4-dimensional analogue of the octahedron. It has 8 vertices, 24 edges, 32 triangular faces, and 16 tetrahedral 3-faces.

dual of the 8-cell (4-cube, tesseract)

24-cell

The 24-cell has no direct 3-dimensional analogue. It has 24 vertices, 96 edges, 96 triangular faces, and 24 octahedral 3-faces. (Self-dual)

600-cell

The 600-cell is the 4-dimensional analogue of the icosahedron. It has 120 vertices, 720 edges, 1200 triangular faces, and 600 tetrahedral 3-faces.

dual

120-cell

The 120-cell is the 4-dimensional analogue of the dodecahedron. It has 600 vertices, 1200 edges, 720 pentagonal faces, and 120 dodecahedral 3-faces.

Convex regular n-polytopes:

n=2 △, □, ⬠, ...

n=3 tetrahedron cube ⟷ octahedron

n=4 4-simplex 8-cell ⟷ 16-cell
(4-cube) (4-orthoplex)

n-simplex n-cube ⟷ n-orthoplex

icosahedron
⇕
dodecahedron

24 cell

120 cell ⇅
600 cell

Exceptional Convex regular polytopes.

Theorem 2.3.2. A 4-polytope has Euler characteristic 0 and so satisfies

$$v - e + f - c = 0$$

where v is the number of vertices, e is the number of edges, and f is the number of faces, and c is the number of 3-faces.