Math3303 Assignment 3

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Q1

a)

We want to show G^{ab} is abelian.

Proof. In the previous assignment we showed that $[G, G]$ is a normal subgroup, so $G/[G, G] = \{g[G, G] : g \in G\}$. Hence we now get, $(g_1[G, G])(g_2[G, G]) = (g_1g_2)[G, G] = ([g_1, g_2]g_2g_1)[G, G] = g_2g_1[G, G] = (g_2[G, G])(g_1[G, G])$ where $g_1, g_2 \in G$. $\square$

b)

First we want to prove the fundamental homomorphism theorem.

Theorem 1. *Let G, H be groups, $f : G \to H$ a homomorphism, and let N be a normal subgroup of G such that $N \subseteq ker\ f$. Then there exists a unique homomorphism $f' : G/N \to H$ so that $f' \circ \pi = f$, where π denotes the obvious homomorphism from G to G/N, $\pi(g) = gN$.*

Proof. We first show the uniqueness of the mapping. Let $f_1', f_2' : G/N \to H$ be functions such that $f_1' \circ \pi = f_2' \circ \pi$. For $y \in G/N$ there exists $x \in G$ such that $\pi(x) = y$, so we have $f_1'(y) = (f_1' \circ \pi)(x) = (f_2' \circ \pi)(x) = f_2'(y)$ for all $y \in G/N$, thus $f_1' = f_2'$. Now we define $f' : G/N \to H, \quad f'(gN) = f(g) \forall g \in G$. Now let $gN = kN$, or $k \in gN$. Since $N \subseteq \text{Ker } f$, $g^{-1}k \in N$ implies $g^{-1}k \in \text{Ker } f$, hence $f(g) = f(k)$. Clearly $f' \circ \pi = f$. $\square$

Now we want to show that if we have the homomorphisms $\pi : G \to G^{ab}$, $f : G \to A$, with A abelian then there exists a homomorphism $f' : G^{ab} \to A$ such that $f = f' \circ \pi$.

Proof. This is almost a direct consequence of the above theorem. As f is mapping to an abelian group A we have for $[a,b] \in [G,G]$:

$$\begin{aligned}
f([a,b]) &= f(aba^{-1}b^{-1}) \\
&= f(a)f(b)f(a^{-1})f(b^{-1}) \\
&= f(a)f(a^{-1})f(b)f(b^{-1}) \\
&= f(aa^{-1}bb^{-1}) \\
&= f(e) \\
&= e
\end{aligned}$$

So we have $[G,G] \subseteq \text{Ker } f$. Plugging our values into the fundamental homomorphism theorem we get the desired result (note here that $N = [G,G]$ and $H = A$). $\square$

c)

Now we want to prove $G^\vee \cong (G^{ab})^\vee$.

Proof. Plugging the groups into the result from b) we get:

$$\begin{tikzcd}
G \arrow[r, "\phi"] \arrow[d, "\pi"] & \mathbb{C}^\times \\
G^{ab} \arrow[ur, "\varphi"] &
\end{tikzcd}$$

We note that we can do this as $[G,G]$ is a normal subgroup and $\mathbb{C}^\times$ is an abelian group. We also note that from b) we get $\phi = \varphi \circ \pi$ and that the homomorphisms are $\pi(g) = g[G,G]$ and $\varphi(\pi(g)) = \phi(g)$ (From b)). Now consider $\theta : G^\vee \to (G^{ab})^\vee$, where $\theta(\phi(g)) = \varphi(\pi(g))$. This is a homomorphism as:

$$\begin{aligned}
\theta(\phi_1(g_1)\phi_2(g_2)) &= \varphi(\pi(g_1 g_2)) \\
&= \varphi(\pi(g_1)\pi(g_2)) \\
&= \varphi(\pi(g_1))\varphi(\pi(g_2)) \\
&= \theta(\phi_1(g_1))\theta(\phi_1(g_2))
\end{aligned}$$

Noting lines 2 and 3 are possible as ϕ and π are homomorphisms. It is also bijective as $\phi(g) = \varphi(\pi(g)), \forall g \in G$, hence the result $G^\vee \cong (G^{ab})^\vee$. $\square$

Q2

We want to prove that S_n can be generated by (12) and $(12...n)$.

Proof. First we have:

$$\begin{aligned}(12...n)(1,2)(12...n)^{-1} &= (2,3)\\(12...n)(2,3)(12...n)^{-1} &= (3,4)\\&\vdots\\(12...n)(n-2,n-1)(12...n)^{-1} &= (n-1,n)\end{aligned}$$

Hence $(i,i+1) \in \langle (1,2),(12...n)\rangle, \quad \text{for all } \ 1 \leq i \leq n-1$. Next we have:

$$\begin{aligned}(2,3)(1,2)(2,3)^{-1} &= (1,3)\\(3,4)(2,3)(3,4)^{-1} &= (1,4)\\&\vdots\\(n-1,n)(1,n-1)(n-1,n)^{-1} &= (1,n)\end{aligned}$$

From this we now have $(1,i) \in \langle (1,2),(12...n)\rangle$ for all $1 \leq i \leq n$. Now for any $1 \leq i < j \leq n$ we get the following:

$$(i,j) = (1,i)(1,j)(1,j)^{-1} \in \langle (1,2),(12...n)\rangle$$

Hence all transpositions are contained in $\langle (1,2),(12...n)\rangle$ and hence $\langle (1,2),(12...n)\rangle = S_n$ □

Q3

We want to show that $Z(S_n) = \{e\}$. We will first assume that $n \geq 3$ as this trivially holds for $n = 1$ and $n = 2$.

Proof. By definition of identity in a group ($\forall g \in G, \quad g \cdot e = e \cdot g$) we have that $e \in Z(S_n)$. Also by the definition of center we have $Z(S_n) = \{\tau \in S_n : \forall \sigma \in S_n : \tau\sigma = \sigma\tau\}$. Let $\pi, \rho \in S_n$ be permutations of $\{1,...,n\}$. Suppose we have $\pi \in S_n$ such that $\pi \neq e, \pi(i) = j, i \neq j$. Since $n \geq 3$, we can find $\rho \in S_n$ which interchanges j and k (where $k \neq i,j$) and fixes everything else. It follows that $\rho - 1$ does the same thing, and in particular both ρ and $\rho - 1$ fix i. So:

$$\begin{aligned}\rho\pi\rho^{-1}(i) &= \rho\pi(i)\\&= \rho(j)\\&= k\end{aligned}$$

So:

$$\rho\pi\rho^{-1}(i) = k \neq j = \pi(i)$$

Now we notice that if ρ and π were to commute, $\rho\pi\rho^{-1} = \pi$, but in S_n this isn't the case. So for any $\pi \in S_n$ we can always find a ρ such that $\rho\pi\rho^{-1} \neq \pi$. So no we have that no elements other than the identity of S_n commute with all elements of S_n. Hence the result, $Z(S_n) = \{e\}$. $\square$