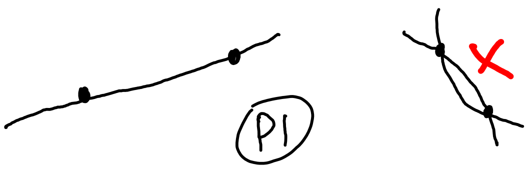
Projective and Affine Planes:

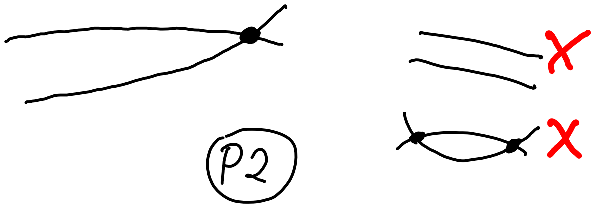
Definition 2.5.2. A **projective plane** is an incidence structure satisfying the following three axioms.

(P1) For any two distinct points x and y, there is a unique line incident with both x and y.

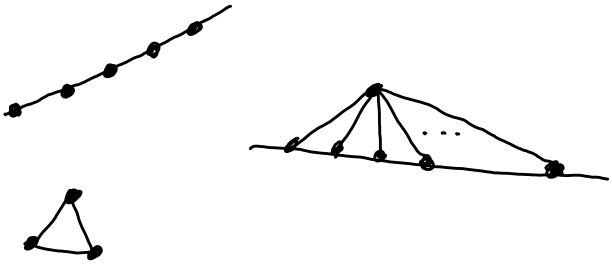
(P2) For any two distinct lines P and Q, there is a unique point incident with both P and Q.

(P3) There exist four points, no three of which are collinear.





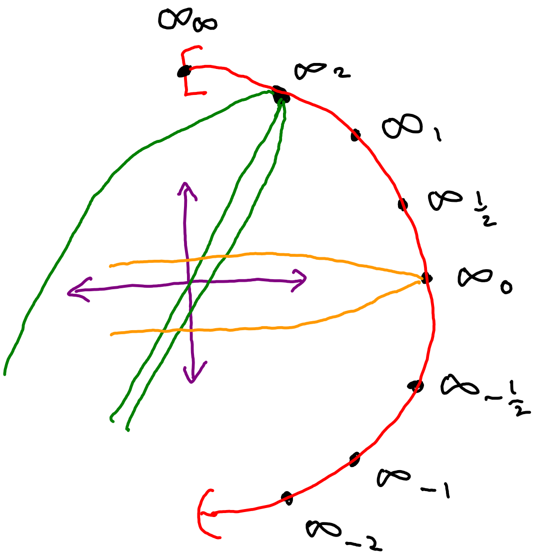
(P3) Rules out:



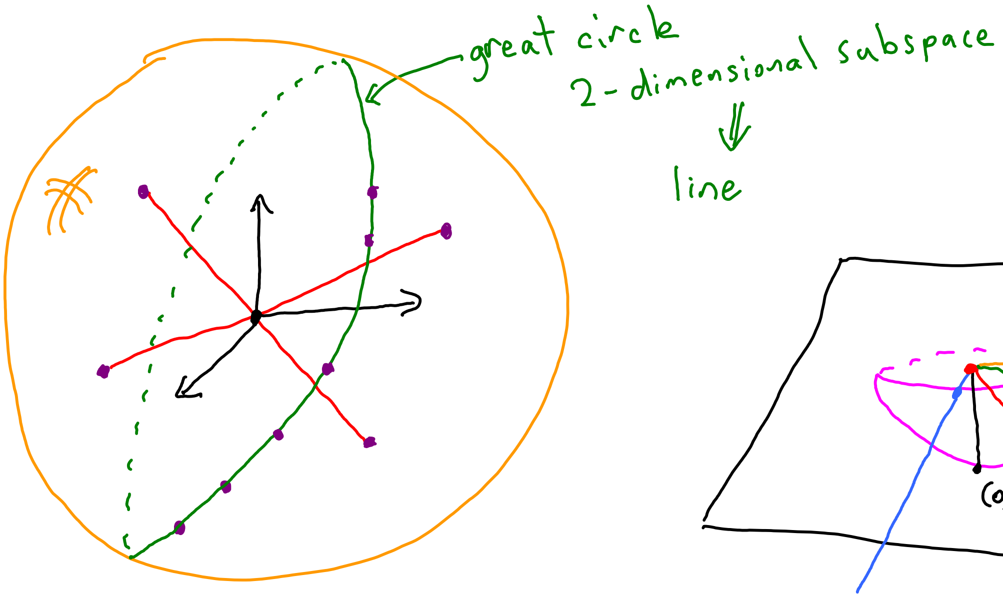
Example 2.5.3. The Euclidean plane $\mathbb{R}^2$ is not a projective plane because it has parallel lines, and so does not satisfy axiom P2. However a projective plane $\mathcal{P}$ can be constructed from $\mathbb{R}^2$ as follows.

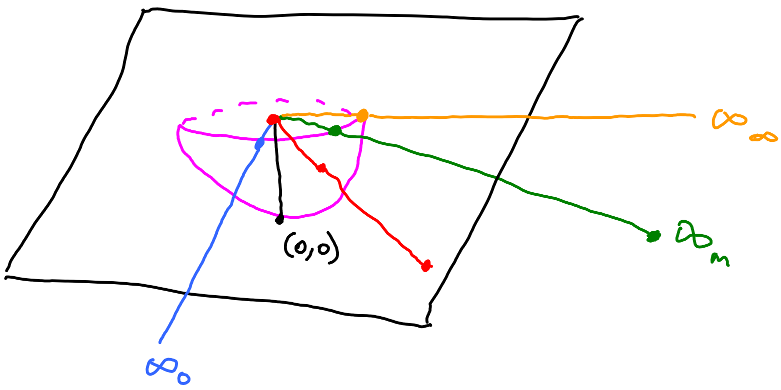
- Each point of $\mathbb{R}^2$ is a point in $\mathcal{P}$.
- For each $m \in \mathbb{R} \cup \{\infty\}$, there is a point ∞_m in $\mathcal{P}$ (these are the "points at infinity").
- For each $m \in \mathbb{R} \cup \{\infty\}$ and for each line L of gradient m in $\mathbb{R}^2$, the points of $L \cup \{\infty_m\}$ form a line of $\mathcal{P}$ (the parallel lines of gradient m "meet at infinity", namely at the point ∞_m).
- The line $\{\infty_m : m \in \mathbb{R} \cup \{\infty\}\}$ is in $\mathcal{P}$ (this is the "the line at infinity").

It is easy to verify that $\mathcal{P}$ is a projective plane – the **real projective plane**.



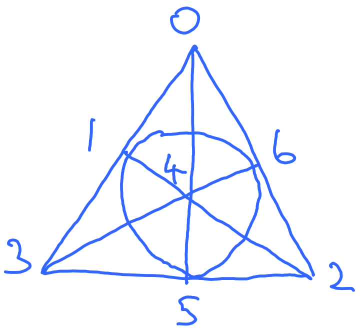
The real projective plane can also be constructed as follows. Let V be the vector space of dimension 3 over $\mathbb{R}$. The 1-dimensional subspaces of V are the points of $\mathcal{P}$, and the 2-dimensional subspaces of V are the lines $\mathcal{P}$. By this, we mean that for each 2-dimensional subspace U of V, there is a line in $\mathcal{P}$ whose points are the 1-dimensional subspaces of U. $\square$



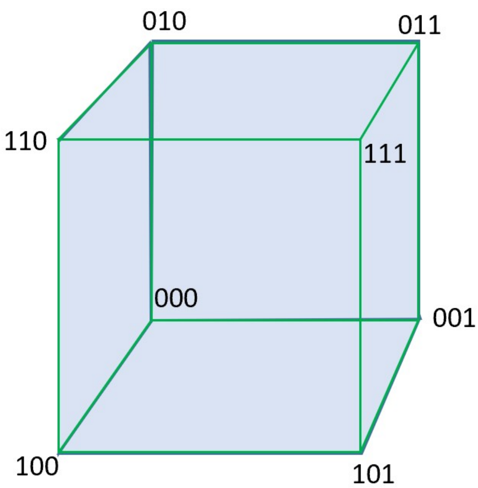


We now turn our attention to finite projective planes.

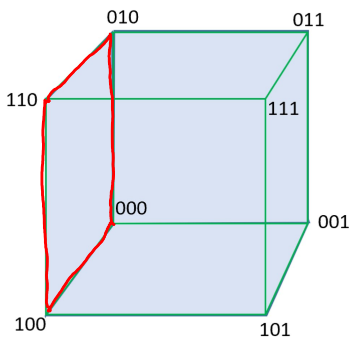
Example 2.5.4. Let $P = \mathbb{Z}_2^3 \setminus \{0\}$ and let $L = \{x, y, z \in P : x + y + z = 0\}$. Then (P, L) is a projective plane, called the **Fano plane**.



0	1	3
1	2	4
2	3	5
3	4	6
4	5	0
5	6	1
6	0	2



001	010	011
001	100	101
010	100	110
001	110	111
010	101	111
100	011	111
011	101	110



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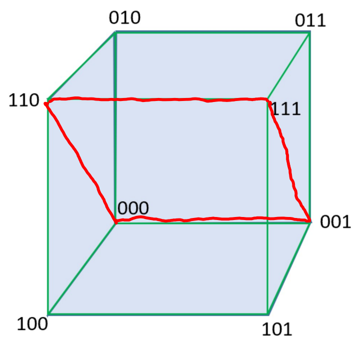

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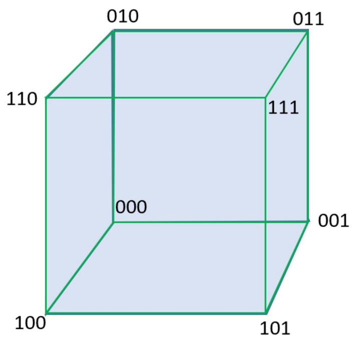
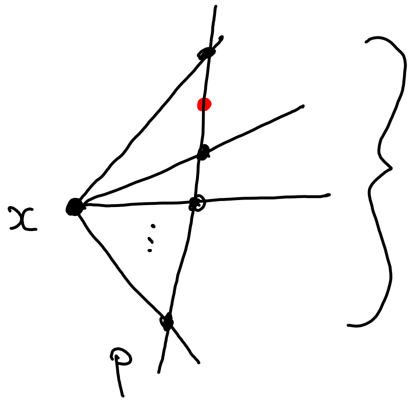


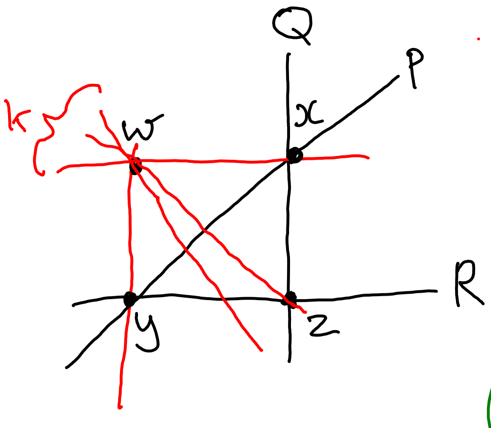
1

Theorem 2.5.5. If $\mathcal{P}$ is a finite projective plane, then there is a constant $n \geq 2$ such that there are exactly $n+1$ points on each line of $\mathcal{P}$, there are exactly $n+1$ lines through each point of $\mathcal{P}$, the number of points in $\mathcal{P}$ is n^2+n+1, and the number of lines in $\mathcal{P}$ is n^2+n+1.



suppose k lines through x

Line p not through x has k points.
(P1 and P2)



P3 $\Rightarrow$

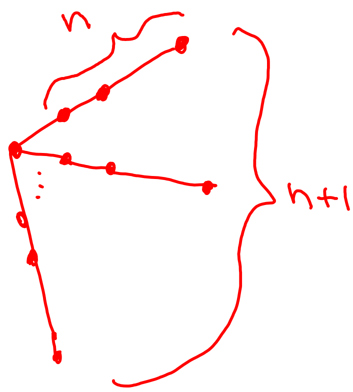
Suppose k lines through w.
wx, wy, wz $\rightarrow$ $k \geq 3$

Every line not through w, including P, Q, R, has k points.

Any point not on P has k lines.
Any point not on Q has k lines.
Any point not on R has k lines.

Every point has k lines.
Every line has k points.

Let $n = k-1$. ($k \geq 3 \rightarrow n \geq 2$).
Every point has $n+1$ lines. Every line has $n+1$ points.

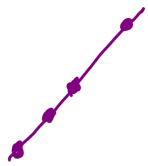


$\Rightarrow$ There are $n(n+1)+1 = n^2+n+1$ points.

$$\sum_{\text{points}} \text{number lines through point } x$$

$$= (n^2+n+1)(n+1)$$

This counts each line $n+1$ times



$\Rightarrow$ Number of lines $= \dfrac{(n^2+n+1)(n+1)}{n+1} = n^2+n+1.$

Projective plane of order n.

$S(2, n+1, n^2+n+1)$

- n^2+n+1 points
- n^2+n+1 lines
- $n+1$ points per line
- $n+1$ lines per point
- two points $\longrightarrow$ unique line
- two lines $\longrightarrow$ unique point.

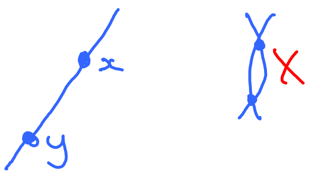
Definition 2.5.6. An **affine plane** is an incidence structure satisfying the following three axioms.

(A1) For any two distinct points x and y, there is a unique line incident with both x and y.

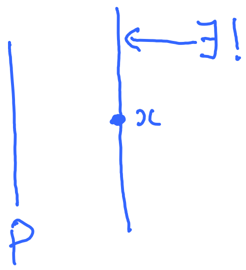
(A2) For any line P and any point x that is not on P, there is a unique line that is incident with x and incident with no point of P.

(A3) There exist four points, no three of which are collinear.

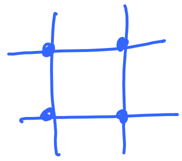
A1



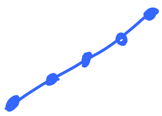
A2



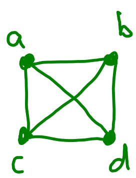
A3



A3 Rules out

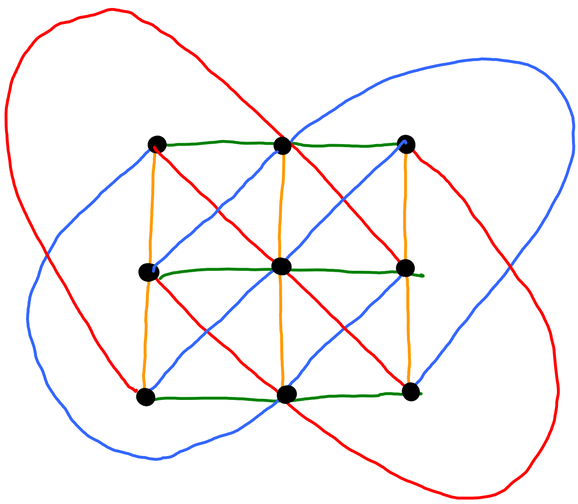


Examples: $\mathbb{R}^2$



ab, ac, ad

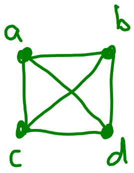
bc, bd, cd



Theorem 2.5.8. If $\mathcal{A}$ is a finite affine plane, then there is a constant $n \geq 2$ such that there are exactly n points on each line of $\mathcal{A}$, there are exactly $n+1$ lines through each point of $\mathcal{A}$, the number of points in $\mathcal{A}$ is n^2, and the number of lines in $\mathcal{A}$ is $n^2 + n$.

Affine plane of order n.

$S(2, n, n^2)$



ab, ac, ad

bc, bd, cd

Affine plane of order 2

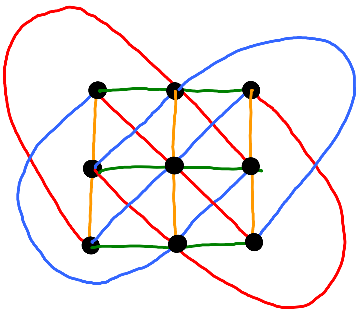
4 points $(2^2 = 4)$

6 lines $(2^2 + 2 = 6)$

3 lines per point $(2 + 1 = 3)$

2 points per line (2)

two points $\Rightarrow$ unique line



Affine plane of order 3

9 points $(3^2 = 9)$

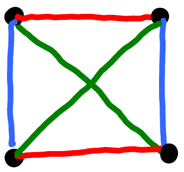
12 lines $(3^2 + 3 = 12)$

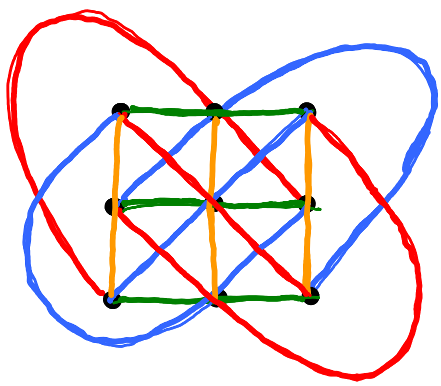
4 lines per point $(3 + 1 = 4)$

3 points per line (3)

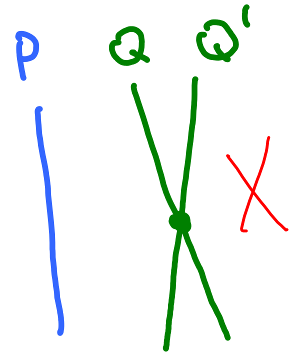
two points $\Rightarrow$ unique line

Theorem 2.5.9. The lines of an affine plane of order n can be partitioned into $n+1$ parallel classes, where each parallel class consists of a set of n pairwise parallel lines which collectively contain all the points.

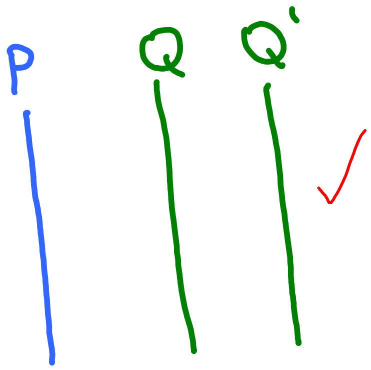


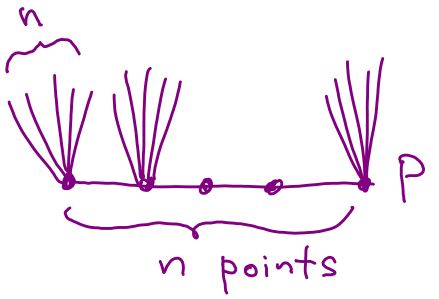


Proof Let P be a line and suppose there are distinct lines Q and Q' which are both parallel to P. By Axiom A2, Q and Q' must be parallel to each other. Now, since there are n points on P and $n+1$ lines through each point of P, there are n^2+1 lines, including P itself, which intersect P. This leaves $n-1$ lines which are parallel to P, and we have already noted that these are pairwise parallel. The result follows. $\square$



(A2) Given line P and point x not on P, there exists unique line through x and parallel to P.





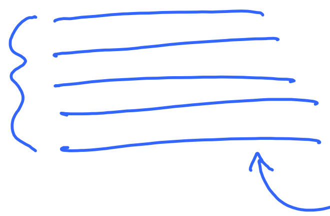
$\Longrightarrow$ n^2+1 lines incident with P. $\Longrightarrow n^2+n-(n^2+1)$

$= n-1$

lines parallel to P.

These are parallel to each other.

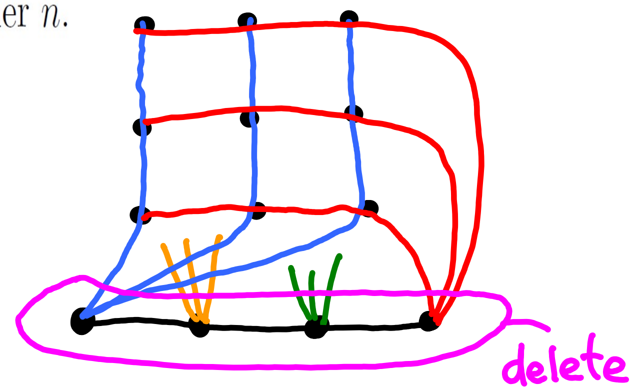
set of n parallel lines

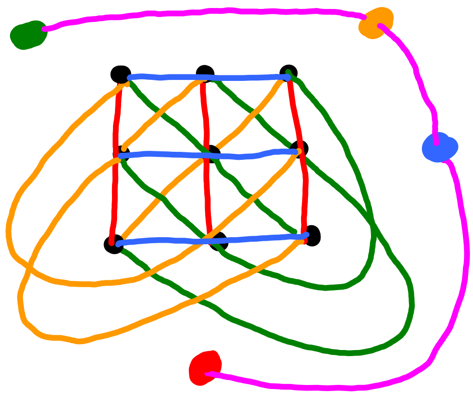


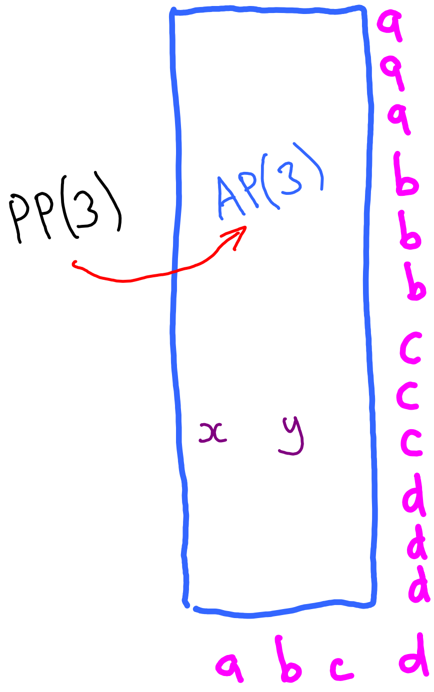
these contain all n^2 points.

Repeat until all n^2+n lines are grouped into $n+1$ sets of n parallel lines. $\square$

Theorem 2.5.10. There exists a projective plane of order n if and only if there exists an affine plane of order n.





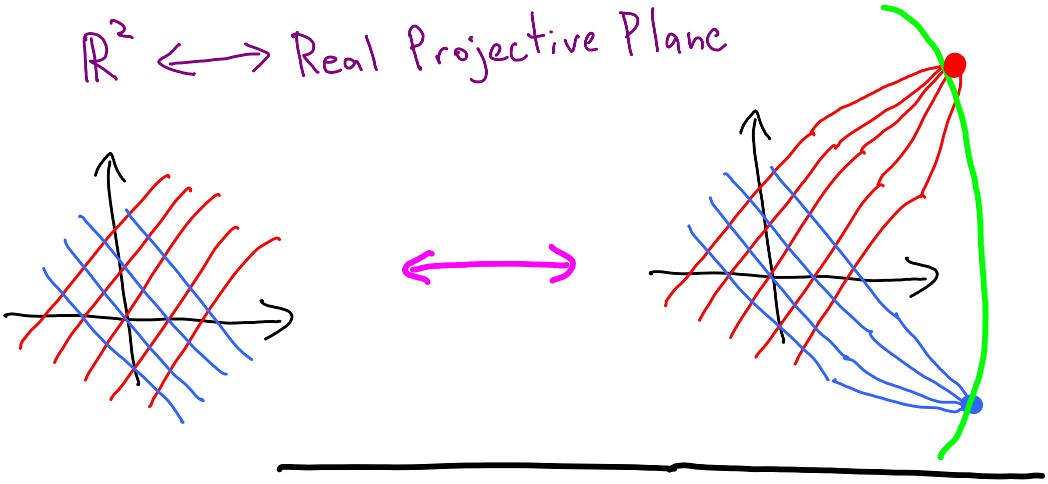


AP(3) $\longrightarrow$ PP(3)

1	2	3	10	1	4	7	11	1	5	9	12	1	6	8	13
4	5	6	10	2	5	8	11	2	6	7	12	2	4	9	13
7	8	9	10	3	6	9	11	3	4	8	12	3	5	7	13

10 11 12 13

$\mathbb{R}^2 \longleftrightarrow$ Real Projective Plane



Theorem 2.5.11. If q is a power of a prime, then there exist projective and affine planes of order q.

$\mathbb{F}_q \longrightarrow PP(q)$ and $AP(q)$

Bruck-Ryser-Chowla Theorem:

Theorem 2.5.12. If there exists a projective plane of order n with $n \equiv 1, 2 \pmod 4$ then for each prime $p \equiv 3 \pmod 4$ the largest α for which p^α divides n is even.

2	3	4	5	6	7	8	9	10	11	12	13	14	15
✓	✓	✓	✓	×	✓	✓	✓	×	✓	?	✓	×	?

16	17	18	19	20	21	22	23	24	25	26
✓	✓	?	✓	?	×	×	✓	?	✓	?

27	28	29	30	...
✓	?	✓	×	

n	$\# PP(n)$
2, 3, 4, 5, 7, 8	1
6, 14, 21, 22, ... 10	0
9	4
16	$\geqslant 22$
11, 13, 17, 19, ...	$\geqslant 1$
12, 15, 18, 20, ...	$\geqslant 0$