Exercises

1. The game "Conway's Soldiers" is modified so that the initial arrangement of soldiers has only n soldiers below a horizontal line (instead of having a soldier in every position below a horizontal line). Show that if it is possible to progress a soldier four rows forwards, then $n \geq 19$.

Basics of Group Theory

2. Consider the set $A = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\},\}$ of pairs of elements of $\{1,2,3,4\}$. Any permutation f of $\{1,2,3,4\}$ induces a permutation of the set of pairs of elements of $\{1,2,3,4\}$ in a natural way. Namely, $\{x,y\} \mapsto \{f(x), f(y)\}$. So, for example, the permutation $(1\ 2\ 4)$ of $\{1,2,3,4\}$ induces the permutation $(\{1,2\}\ \{2,4\}\ \{1,4\})(\{1,3\}\ \{2,3\}\ \{3,4\})$ of A. The following question generalises this and shows how to view it as a homomorphism.

   Let S be a finite non-empty set, let $0 \leq k \leq |S|$, and let $\binom{S}{k}$ denote the set of all k-element subsets of S. Consider the function $\theta : \mathrm{Sym}(S) \to \mathrm{Sym}(\binom{S}{k})$ given by $(\theta(f))(\{x_1, x_2, \ldots, x_k\}) = \{f(x_1), f(x_2), \ldots, f(x_k)\}$ for all $\{x_1, x_2, \ldots, x_k\} \in \binom{S}{k}$. Show that θ is a homomorphism. What is the kernel of θ?

3. Show that the function $f : \mathbb{Z}_{12} \to \mathbb{Z}_8$ given by $f([x]_{12}) = [x+x]_8$ for all $x \in \mathbb{Z}_{12}$ is well defined and is a homomorphism. Determine $\ker f$ and $\mathrm{Im}\, f$, and verify that the first isomorphism theorem holds.

4. List the elements of A_4.

5. (Challenge Question) A 1-factor in a graph is a 1-regular spanning subgraph, and a 1-factorisation of a k-regular graph is a set of k pairwise edge-disjoint 1-factors. A 1-factorisation of a graph is perfect is the union of any pair of distinct 1-factors is a Hamilton cycle. Show that if the complete bipartite graph $K_{n,n}$ has a perfect 1-factorisation, then $n = 2$ or n is odd. Hint: think of the 1-factors as permutations and make use of the alternating group A_n.

Permutation Groups

6. For which values n and t, $n \geq 2$ and $1 \leq t \leq n$, is the alternating group A_n t-transitive?

7. Let p be an odd prime and for each $a, b \in \mathbb{Z}_p$ define $\pi_{a,b} : \mathbb{Z}_p \to \mathbb{Z}_p$ by $\pi_{a,b}(x) = ax + b$ for all $x \in \mathbb{Z}_p$.

- Show that $G = \{\pi_{a,b} : a \in \mathbb{Z}_p \setminus \{0\}\}, b \in \mathbb{Z}_p\}$ is a permutation group acting on $\mathbb{Z}_p$.
- Show that G is 2-transitive.
- Show that $H = \{\pi_{a,b} : a \in \{1, 2, 4\} \setminus \{0\}\}, b \in \mathbb{Z}_7\}$ has a regular action on the set of unordered pairs of elements of $\mathbb{Z}_7$.

8. Show that a finite transitive permutation group of degree at least 2 has an element with no fixed points.

9. Show that if G is a transitive permutation group acting on a finite set S, then the following are equivalent.

   (a) G is regular.

   (b) The only element of G with any fixed points is the identity.

   (c) $|G| = |S|$.

10. Show that if G is a transitive abelian permutation group acting faithfully on a finite set S, then G is regular.

Some Exceptional Objects

11. If G is a group, then the conjugacy class of $x \in G$ is $\{g^{-1}xg : g \in G\}$. Any element of the conjugacy class of x is called a conjugate of x. Show that the conjugacy classes of G partition G. Show that automorphisms of a group map conjugacy classes to conjugacy classes.

Polygons and Pick's Theorem

12. What is the area of a fundamental n-gon? Explain briefly.

13. What is the maximum number of lattice points that can be covered by an m by n rectangle that has its vertices on lattice points? You should show that your answer is indeed a maximum.

14. Determine the maximum and minimum number of lattice points that can be covered by a 5 by 5 square whose vertices are lattice points? You should show that your answers are indeed the maximum and minimum.

15. Let $m \geq 2$ and $n \geq 2$ be integers, let V be the set of lattice points with x-coordinate in $\{0, 1, 2, \ldots, m\}$ and y-coordinate in $\{0, 1, 2, \ldots, n\}$, let E be the set of line segments of length 1 that join points of V, and let G be the graph with vertex set V and edge set E.

    Let P be a Hamilton path from $(0, 0)$ to (m, n) in the graph G, and let R be the rectangle with vertices $(0, 0)$, $(m, 0)$, $(0, n)$ and (m, n). The path P partitions R into regions. Let A be the set of regions which open to either the bottom side or to the right side of R ($(0, 0)(m, 0)$ is the bottom side of R, and $(m, 0)(m, n)$ is the right side of R).

    (a) Derive a formula in terms of m and n for the sum of the areas of the regions in A.

    (b) Substitute $m = 9$ and $n = 11$ into your formula and explain why the result is not an integer, even though the regions of A can be partitioned into unit squares.

16. The *Farey sequence* F_N consists of the ascending set containing all irreducible fractions $\frac{a}{b}$ where $0 \leq \frac{a}{b} \leq 1$ and $b \leq N$.

    Use Pick's Theorem to show that if $\frac{a}{b}$ and $\frac{c}{d}$ are successive terms of a Farey sequence, then $bc - ad = 1$. (Hint: Plot each fraction $\frac{a}{b}$ as the point (b, a).) Deduce that for three successive terms $\frac{a_1}{b_1}$, $\frac{a_2}{b_2}$ and $\frac{a_3}{b_3}$, we have $\frac{a_2}{b_2} = \frac{a_3+a_1}{b_3+b_1}$.

17. Let a, b and n be distinct positive integers with $\gcd(a, n) = \gcd(b, n) = 1$. Determine a formula for the number of lattice points in the interior of the triangle whose vertices are the $(0, 0)$, $(a, n - a)$ and $(b, n - b)$.

Sperner's Lemma

18. Outline the proof of Sperner's Lemma in n-dimensions.

Regular Polytopes

19. Prove that the full symmetry group of the dodecahedron is $A_5 \times \mathbb{Z}_2$.

20. For a d-polytope P, the Euler characteristic $\chi(P)$ is defined by

$$\chi(P) = n_0 - n_1 + n_2 - n_3 + \cdots + (-1)^{d-1} n_{d-1}$$

    where n_i is the number of i-faces. Prove that the Euler characteristic of the d-simplex is 2 if d is odd, and 0 if d is even.

Sphere Packing

21. Let A and B be two adjacent spheres of an optimal sphere packing in $\mathbb{R}^8$. How many spheres touch both A and B?

22. In $\mathbb{R}^3$, determine the maximum value of r such that 12 non-overlapping spheres of radius r can touch a sphere of radius 1.

23. In $\mathbb{R}^3$, determine the maximum value of r such that 20 non-overlapping spheres of radius r can touch a sphere of radius 1.

Projective and Affine Planes

24. Let $k \geq 3$, let P be a finite set of points and let L be a set of lines such that

    (a) $|P| = |L|$,

    (b) each line has k points, and

    (c) there is exactly one line through each pair of points from P.

    Show that (P, L) is a projective plane of order $k-1$.

25. In a projective plane, an *arc* is a set S of points such that no three distinct points of S are collinear.

    (a) Show that the maximum number of points in an arc of a projective plane of order 2 is 4.

    (b) Show that the maximum number of points in an arc of a projective plane of order 3 is 4.

    (c) Show that if S is an arc in a projective plane of order n, then $|S| \leq n+2$.

    (d) Show that if a projective plane of order n has an arc with $n+2$ points, then n is even.

    (e) Let $t \in S_6$ be a transposition, let T be the set of all transpositions of S_6, let θ be an automorphism of S_6, and suppose $t\theta$ is the composition of three disjoint transpositions. Show that $T\theta$ is the set of all permutations consisting of three disjoint transpositions.

    (f) Show that a projective plane of order 4 with an arc consisting of 6 points can be constructed as follows. Take the vertices and edges of K_6 as the points, with the vertices forming the arc. For each vertex v of K_6, there is a line whose points are the edges incident with v. For each edge uv of K_6, there is a line whose points are u, v and the edges ab, cd and ef where $(a\ b)(c\ d)(e\ f) = (u\ v)\theta$, and θ is as defined in part (e).

26. Use the construction which yields a projective plane of order n from an affine plane of order n to construct a projective plane of order 3 from an affine plane of order 3.

27. Prove that there are infinitely many values of n for which there is no projective plane of order n. You may use Dirichlet's Theorem (from 1837) which guarantees that if $\gcd(a, d) = 1$, then there are infinitely many primes congruent to $a \pmod{d}$.

Projective and Affine Geometries PG(n,q) and AG(n,q)

28. Construct an $S(2,4,16)$ system as the set of d-flats in AG(n,q) (for some suitable d, n and q).

29. Write down the parameters and blocks, with the blocks resolved into parallel classes, of the design $(X,\mathcal{B})$ arising from the 2-flats of AG$(3,2)$. Visualise this design by drawing a cube. What can we say about the design with point set $X\setminus\{(0,0,0)\}$ and block set $\{B\setminus\{(0,0,0)\} : (0,0,0)\in B, B\in\mathcal{B}\}$?

Singer's Theorem

30. Use Singer's theorem and the primitive polynomial x^4+x+1 over $\mathbb{F}_2$ to construct a block B such that the orbit of B under the action of $\mathbb{Z}_{15}$ is a $(15,7,3)$-design.

31. The polynomial x^3+x^2+x+y is a primitive polynomial over $\mathbb{F}_4$ where $\mathbb{F}_4$ consists of the residue classes of polynomials modulo $f(y)=y^2+y+1$ over $\mathbb{F}_2$. Use this fact to construct a line whose orbit under $\mathbb{Z}_{21}$ is a projective plane of order 4.

32. A primitive polynomial over $\mathbb{F}_3$ is x^3+2x+1. Construct a $(27,3,1)$-design $\mathcal{D}$ with point set $\{\infty\}\cup\mathbb{Z}_{26}$ such that $\mathbb{Z}_{26}\leq \mathrm{Aut}(\mathcal{D})$.

33. What is the order of the (full) automorphism group of the design $(V,\mathcal{B})$ where $V=\{1,2,\ldots,9\}$ and

$$\begin{aligned}\mathcal{B}=\{\ \ & \{1,2,3\},\ \{1,4,7\},\ \{1,5,9\},\ \{1,6,8\},\\ & \{4,5,6\},\ \{2,5,8\},\ \{2,6,7\},\ \{2,4,9\},\\ & \{7,8,9\},\ \{3,6,9\},\ \{3,4,8\},\ \{3,5,7\}\ \ \}.\end{aligned}$$

(v,k,λ)-designs

t-designs

34. Write out the blocks of a $1-(10,4,2)$-design.

Extensions and Contractions

35. A $3-(8,4,1)$-design can be constructed by taking the vertices of a (3-dimensional) cube as points and the following blocks. The four vertices on each face form a block. The four vertices on each pair of opposite edges form a block. The two sets of four pairwise nonadjacent vertices forms of block. Verify that this design is an extension of the $(7,3,1)$-design arising from $\mathrm{PG}(2,2)$.

36. Prove that $\mathbb{Z}_{11} \leq \mathrm{Aut}(\mathcal{D})$ where $\mathcal{D}$ is the $3-(22,6,1)$-design constructed in the notes.

Inversive Planes

37. Construct a $2-(9,3,1)$-design by letting $\mathrm{AGL}(1,9)$ act on $\mathbb{F}_3$.

38. Use the construction of a $2-(q^2,q,1)$-design via the action of $\mathrm{AGL}(1,q^2)$ on $\mathbb{F}_q$ to prove the affine analogue of Singer's Theorem (for the case of $(q^2,q,1)$-designs).

39. Using the construction of a $3-(q^2+1,q+1,1)$-design via the action of $\mathrm{PGL}(2,q^2)$ on the subset B of the projective line $\mathrm{PG}(1,q^2)$ given by $B = \{\langle\binom{x}{y}\rangle \in \mathrm{PG}(1,q^2) : x,y \in \mathbb{F}_q\}$, explicitly construct the points and three blocks of a $3-(10,4,1)$-design.

Steiner Systems

Baranyai's Theorem

40. Determine the minimum number of colours needed to colour the blocks of the complete design with 14 points and blocks of size 4 such that any two blocks of the same colour are disjoint.

41. Show that an r by s array A with entries from $\{1,2,\ldots,n\}$ can be completed to an n by n latin square (n by n array where each element of $\{1,2,\ldots,n\}$ occurs exactly once in each row and exactly once in each column) if and only if

    - each row of A contains s distinct symbols;
    - each column of A contains r distinct symbols; and

- each element of $\{1, 2, \ldots, n\}$ occurs at least $r + s - n$ times in A.

42. Use the method of the proof of Baranyai's Theorem to show that there is a resolvable $2-(v, 2, 1)$-design with a subdesign that is a resolvable $2-(u, 2, 1)$-design and having the property that each resolution class of the $2-(u, 2, 1)$-design is a subset of a resolution class of the $2-(v, 2, 1)$-design if and only if u and v are even and $v \geq 2u$.

43. Let u and v be positive integers with $v \geq u$, let C be a set of $v - 1$ colours, and suppose we have a proper edge colouring of the complete graph of order u with colour set C. Show that this proper edge colouring can be extended to a proper edge colouring of a complete graph of order v with colour set C if and only if v is even and each colour occurs on at least $u - \frac{v}{2}$ edges of the complete graph of order u. By "extended" it is meant that $v - u$ new vertices are added, then new edges are added to form a complete graph of order v, and then the newly added edges are coloured – the colours of the edges of the complete graph of order u remain unchanged.

    (Hint: See the proof of Baranyai's Theorem.)

Vertex-Transitive and s-Arc-Transitive Graphs

44. Determine the automorphism group of

    - the path with vertices $v_1, v_2, \ldots, v_n$ and edges $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n$;
    - the star with vertices $c, v_1, v_2, \ldots, v_{n-1}$ and edges $cv_1, cv_2, \ldots, cv_{n-1}$;
    - the cycle with vertices $v_1, v_2, \ldots, v_n$ and edges $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1$.

45. Show that the 2-transitive graphs (that is, graphs with 2-transitive automorphism groups) are precisely the complete graphs and the empty graphs (a graph is empty if it has no edges).

46. Show that the Heawood graph is a bipartite 3-regular graph of order 14 with girth 6 and determine the order of its automorphism group.

47. For integers $s \geq 1$ and $m \geq 3$, determine the number of s-arcs in an m-cycle. Investigate s-arc-transitivity of the m-cycle.

48. How many graphs with vertex set $\{1, 2, \ldots, 8\}$ are isomorphic to the graph of the cube?

49. Given any graph X, the line graph $L(X)$ of X is the graph with a vertex for each edge of X, and with two vertices of $L(X)$ adjacent if and only if their corresponding edges are adjacent. Let $n \geq 2$ be an integer. Show that $\mathrm{Aut}(K_n) \cong \mathrm{Aut}(L(K_n))$ if and only if $n \notin \{2, 4\}$.

50. Let X be a 3-regular graph and let X' be the graph obtained from X by replacing each vertex of X with a triangle. A more detailed description of the construction of X' from X is as follows. Let v be a vertex of X and let x, y and z be the neighbours of v. We delete the vertex v and the three edges vx, vy and vz from X, and we add three new vertices v_x, v_y and v_z, three new edges v_xv_y, v_yv_z and v_yv_z, and another three new edges xv_x, yv_y and zv_z. We then repeat this procedure for each vertex of X. Prove that X' is vertex-transitive if and only if X is arc-transitive.

51. Let P denote the Petersen graph.

    (a) Show that P is 3-arc-transitive, but not 4-arc-transitive.

    (b) Determine $\mathrm{Aut}(P)$.

    (c) Does P have an automorphism group (not necessarily the full automorphism group) that acts regularly on its 2-arcs?

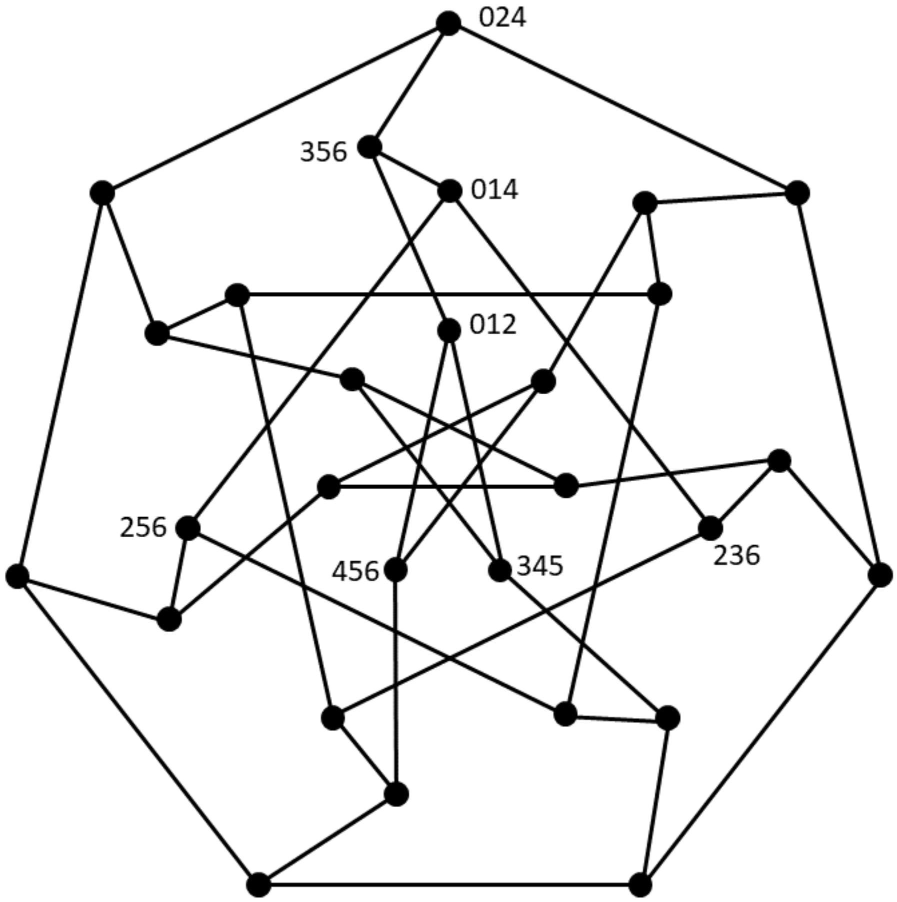
    (d) Determine the stabilizer of a

    - vertex
    - edge
    - arc

    (e) How many 5-cycles are there in P? Does $\mathrm{Aut}(P)$ act transitively on the 5-cycles? What is the stabilizer of a 5-cycle?

52. Recall that the Coxeter graph can be constructed from the Kneser graph $\mathcal{K}(7, 3)$ by deleting a set of 7 vertices whose corresponding 3-subsets form a Fano plane. It is easily proved (you do not need to include a proof of this, but you may like to verify it for yourself) that if $g : \mathbb{Z}_7 \to \mathbb{Z}_7$ is an automorphism of the deleted Fano plane, then the induced action of g (given by $g(\{a, b, c\}) = \{g(a), g(b), g(c)\}$) on the vertex set of the Coxeter graph is an automorphism of the Coxeter graph.

    (a) Shown below is a partial labelling of the vertices of the Coxeter graph. The 3-element subset $\{a, b, c\}$ is denoted by just abc. Complete this partial labelling so that the labels on each pair of adjacent vertices are disjoint 3-element subsets of $\mathbb{Z}_7$.



(b) Show that the Coxeter graph is 3-arc-transitive. Hint: Show that for $s \geq 1$, if a graph is $(s-1)$-arc-transitive, contains at least one s-arc, and the stabilizer of some $(s-1)$-arc $x_0, x_1, \ldots, x_{s-1}$ acts transitively on s-arcs of the form $x_0, x_1, \ldots, x_{s-1}, x$, then the graph is s-arc-transitive.

(c) Show that the Coxeter graph is not 4-arc-transitive.

53. Show that the Tutte-Coxeter graph is 5-arc-transitive, but not 6-arc-transitive.

54. Show that the Pappus Graph is 3-arc-transitive and has girth 6.

Cayley Graphs

55. Show that for any finite group G and any integer k satisfying $0 \leq k < |G|$, there exists a k-regular graph on which G acts regularly if and only if $k|G|$ is even.

56. Write Q_3 as a Cayley graph on each of the following groups, or show that it cannot be done.

    (a) $\mathbb{Z}_2 \times \mathbb{Z}_4$;

    (b) $\mathbb{Z}_8$;

    (c) D_4.

57. For each $k \geq 1$, determine the largest value of s such that Q_k is s-arc-transitive.

58. For those values of k where Q_k is 2-arc-transitive, determine the stabilizer of a 2-arc.

59. For each $k \geq 2$, show that $\mathrm{Aut}(Q_k)$ acts transitively on the 4-cycles of Q_k and determine the order of the stabilizer of a 4-cycle.

60. Show that for all $k \geq 1$, there is no automorphism of Q_k that fixes exactly one vertex.

61. Show that $S_4 \leq \mathrm{Aut}(Q_3)$.

62. Show that the Petersen graph is not a Cayley graph.

Kneser Graphs and the Erdos-Ko-Rado Theorem

63. Describe the Kneser graphs $\mathcal{K}(n, k)$ for $n \leq 2k$.

64. How many independent sets of cardinality $\binom{2k-1}{k-1}$ are there in $\mathcal{K}(2k, k)$?

65. For $n \geq 2k + 1$, determine the order of a vertex-stabilizer in $\mathcal{K}(n, k)$.

66. Use the fact that $\mathcal{K}(7, 3)$ is 3-arc-transitive to show that it has girth 6.

67. Determine all n and k such that $\mathcal{K}(n, k)$ is 2-arc-transitive.

68. Use the fact that the automorphism group of the Kneser graph $\mathcal{K}(6, 2)$ is S_6 to prove that any outer automorphism of S_6 maps transpositions to compositions of three disjoint transpositions.

Johnson Graphs

69. Which Johnson graphs are Kneser graphs? Explain fully with proofs.

Distance-Transitive Graphs

70. Show that $\mathcal{K}(2k+1, k)$ is distance-transitive for all $k \geq 1$.

Hoffman-Singleton Theorem

71. Prove that the Petersen graph is the unique 3-regular graph of diameter 2 on 10 vertices.

72. Prove that the Hoffman-Singleton graph is 7-regular and has girth 5 and diameter 2.