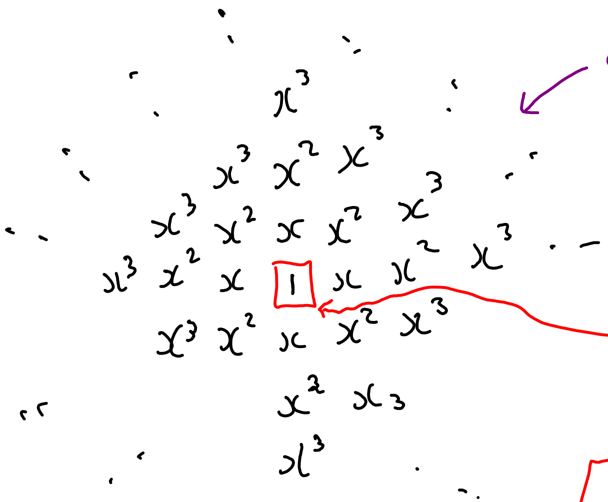
Conway's Soldiers:

John Conway — see Wikipedia

- Game of Life
- Classification of finite simple groups (Conway Groups) etc.
- Died of Covid 2020 (82 years old)

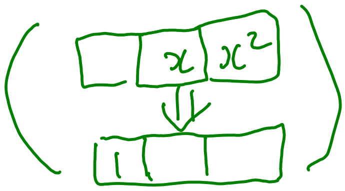
Link from Blackboard to Conways Soldiers.



each cell gets a "score"

"target cell"

We will choose x (positive) such that $1 = x + x^2$



Then also
$$x = x^2 + x^3$$
$$x^2 = x^3 + x^4$$
$$\vdots$$

$$\boxed{x^n - x^{n+1} - x^{n+2} = 0}$$

Quadratic formula

$$x^2 + x - 1 = 0$$
$$x = \frac{\pm\sqrt{5} - 1}{2}$$
$$x = \frac{\sqrt{5} - 1}{2}$$
$$\approx 0.618$$

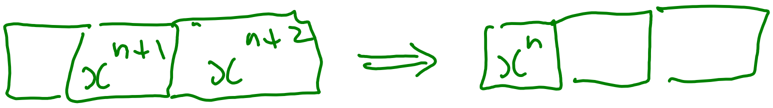
Aside: $x = \frac{1}{\varphi}$ ← golden ration

$$\varphi = \frac{1+\sqrt{5}}{2} = 1.618$$
$$x\varphi = \frac{(\sqrt{5}-1)}{2} \cdot \frac{(\sqrt{5}+1)}{2}$$
$$= \frac{5-1}{4} = 1$$

The "score" of an arrangement is the sum of the scores of the occupied cells.

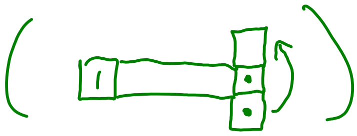
Claim: No move increases the score.

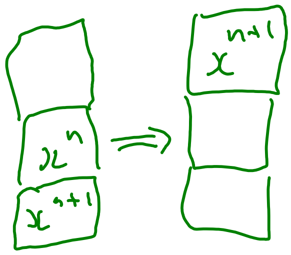
* jump closer to target cell



$$\Delta(\text{score}) = x^n - x^{n+1} - x^{n+2} = 0.$$

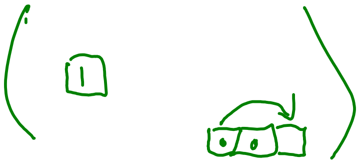
* jump to cell at same distance

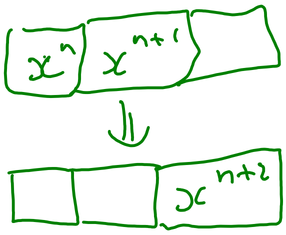




$$\Delta(\text{score}) = x^{n+1} - x^n - x^{n+1} = -x^n < 0.$$

* jump away from target cell





$$\begin{aligned}\Delta(\text{score}) &= x^{n+2} - x^{n+1} - x^n \\ &= x^n(x^2 - x - 1) \\ &= x^n(-2x) \\ &= -2x^{n+1} < 0.\end{aligned}$$

$$\boxed{\begin{aligned} &x^2 + x - 1 = 0 \\ \Rightarrow\ &x^2 - x - 1 = -2x \end{aligned}}$$

□

We show that the initial arrangement has score < 1

$\Longrightarrow$ getting to target cell is impossible.

We have

$$\begin{aligned} 1 &= x + x^2 \\ x &= x^2 + x^3 \\ &\vdots \\ x^n &= x^{n+1} + x^{n+2} \end{aligned} \quad\Longrightarrow\quad \begin{aligned} x^2 &= 1 - x \\ x^3 &= x - x^2 \\ x^4 &= x^2 - x^3 \\ &\vdots \end{aligned}$$

$$\underline{\overline{x^2 + x^3 + x^4 + \dots = 1}} \quad *$$

First calculate this arrangement:-

$$\dots\ x^3\ x^2\ \underset{x}{\boxed{1}}\ x^2\ x^3\ \dots$$

$\longleftarrow$ Sum of this row $= x + 2$ (by $*$)

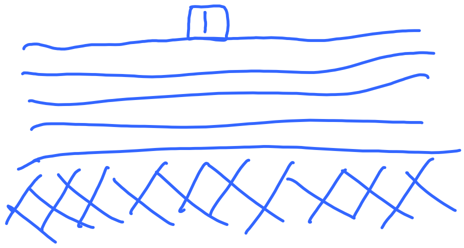
$\longleftarrow$ " " " row $= x(x+2)$

$\vdots$ $\quad x^2(x+2)$

$\vdots$

$$\text{Sum of all rows} = (x+2) + x(x+2) + x^2(x+2) + \dots$$
$$= (x+2)(1 + x + x^2 + x^3 + \dots)$$
$$= \underline{(x+2)^2}$$

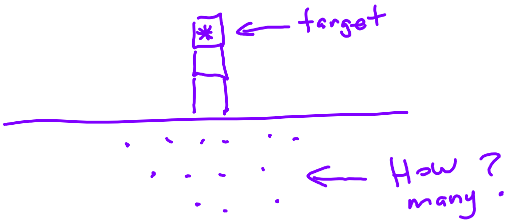
Now



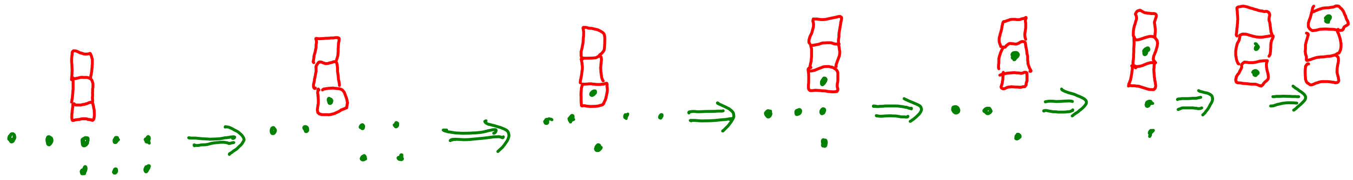
$$\text{Score} = x^4(x+2)^2$$
$$= (x^2)^2(x+2)^2$$
$$= (1-x)^2(x+2)^2$$
$$= ((1-x)(x+2))^2$$
$$= (x+2-x^2-2x)^2$$
$$= \underbrace{-x^2-x+1}_{0}+1 \qquad = 1$$

This shows the target square cannot be reached in a finite number of moves. (After a finite number of moves there are still infinitely many soldiers $\Rightarrow$ positive score)

Exercise: What is the minimum number of soldiers needed to advance 3 rows?



Solution:



It can be done with 8.

Now show at least 8 are needed. $\Rightarrow$

```
            21
            13
             8
1  2  3  5  3  2  1  1
1  1  2  3  2  1  1  1
   .  1  2  1  1  1
  :      1     .  .
```

No move increases the score

With 7 or fewer soldiers, maximum score is $5+3+3+3+2+2+2 = 20$.

$\boxed{\text{Exercises 1.}}$