Asymptotic Methods

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Books: Bender and Orszag, "*Advanced Mathematical methods for scientists and engineers*", Chapters 3,6,10
More details can be found on the Moodle course site; self-enrol into the Asymptotic methods course.

What we'll learn in this course

Examples:

1. $I(\lambda) = \int_{\infty}^{\infty} \exp[-\lambda \cosh u] du$
   We expect that $I(\lambda) \to 0$ as $\lambda \to \infty$. But how fast?

2. $i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x)$ with $\psi(x,t) \in \mathbb{C}$, $V = V(x)$.
   Look for a solution $\psi(x,t) = \exp\left[\frac{-iEt}{\hbar}\right] f(x) \implies \hbar^2 f'' = 2m(V(x) - E)f$
   $\hbar$ is very small. So a natural problem is to try and understand $\epsilon^2 \frac{d^2 y}{dx^2} = Q(x)y$ when $\epsilon \ll 1$.
   The "semi-classical limit" or "geometric optics".

3. Put $\hbar = 1$, $m = \frac{1}{2}$, $V = 0$; specify $\psi(x,0) = \psi_0(x)$
   Fourier transform $\to \psi(x,t) = \frac{1}{(4\pi i t)^{1/2}} \int_{\mathbb{R}} \exp\left[\frac{i|x-y|^2}{4t}\right] \psi_0(y) dy$.
   Question: Does $\psi(x,t)$ really approach $\psi_0(x)$ as $t \to 0$?

1 Asymptotic expansions of functions

$$\sinh x = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

say $\sinh x \sim x$ as $x \to 0$.

Definition: $f \sim g$ as $x \to x_0$ is $|f(x) - g(x)| = o(g(x))$ as $x \to x_0$.

Example: $|\sinh x - x| = |\frac{x^3}{3!} + \frac{x^5}{5!} + \dots| = O(x^3) = o(x)$
($F = O(G)$ as $x \to x_0$ means $\exists C > 0$ such that $|F(x)| \leq C|G(x)|$ in some open interval I, with $x_0 \in I$)
In fact, by remainder estimate for Taylor expansion

$$\left|\sinh x - \sum_0^N \frac{x^{2n+1}}{(2n+1)!}\right| = O(x^{2n+3}) = o(x^{2n+1}) \text{ as } x \to 0$$

We write $\sinh x \sim \sum_0^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

Definition: Asymptotic sequence and asymptotic expansion.

(i) $\{\phi_n\}_{n=0}^{\infty}$ is an asymptotic sequence (of functions) as $x \to x_0$ if $\phi_{n+1}(x) = o(\phi_n(x))$ as $x \to x_0$.

(ii) A function f has asymptotic expansion w.r.t. $\{\phi_n\}$ as $x \to x_0$ written $f \sim \sum_{n=0}^{\infty} a_n \phi_n$ if

$$\left| f(x) - \sum_{n=0}^{N} a_n \phi_n(x) \right| = o(\phi_N(x)) \text{ as } x \to x_0 \forall N$$

Notice the difference with Taylor expansion - an asymptotic expansion need not converge as $N \to \infty$ for any x!

Examples:

- $\{\phi_n(x) = x^n\}$ as $x \to 0$, the most common sequence.
- $\{\phi_n(x) = x^{2n+1}\}$ as $x \to 0$
- $\{\phi_n(x) = e^{-n/x}\}$ as $x \to 0^+$ (i.e. $x > 0$ and $x \to 0$ on right)

Warning: $\sin x \sim x - \frac{x^3}{3!} - \frac{x^5}{5!} + \dots$ as $x \to 0$.
$\sin x + e^{-1/x} \sim x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$ as $x \to 0^+$.

Why?

$$\left| \sin x + e^{-1/x} - \sum_{0}^{N} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \right| = \left| \sum_{n=2N+1}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} + e^{-1/x} \right| = O(x^{2N+3}) = o(x^{2N+1})$$

Moral: information is lost in asymptotic expansions!

However, given f and asymptotic sequence, the a_j's are unique, i.e.

$$\begin{aligned} a_0 &= \lim_{x \to x_0} \frac{f(x)}{\phi_0(x)} \\ a_1 &= \lim_{x \to x_0} \frac{f(x) - a_0\phi_0(x)}{\phi_1(x)} \\ &\vdots \end{aligned}$$

Question: Is it possible that $f(x) \sim 0$ as $x \to 0$?
If $|f(x) - 0| = o(0) = 0$ in some interval I, containing 0, then $f \equiv 0$ on I.

Example: Consider $Ei(x) = \int_x^{\infty} \frac{e^{-t}}{t} dt$ as $x \to +\infty$.

Consider the asymptotic sequence $\phi_n(x) = 1/x^n$ as $x \to +\infty$

$$Ei(x) = \int_x^{\infty} \frac{-d(e^{-t})}{t} = \left[-\frac{e^{-t}}{t} \right]_x^{\infty} - \int_x^{\infty} \frac{e^{-t}}{t^2} dt = \frac{e^{-x}}{x} - \int_x^{\infty} \frac{e^{-t}}{t^2} dt$$

Claim: $Ei(x) \sim e^{-x}/x$ as $x \to +\infty$.

$$\left| Ei(x) - \frac{e^{-x}}{x} \right| = \left| \int_x^{\infty} \frac{e^{-t}}{t^2} dt \right| \le \frac{1}{x^2} \int_x^{\infty} e^{-t} dt = \frac{e^{-t}}{x^2} = o\left(\frac{e^{-x}}{x} \right)$$

Working out the full expansion of Ei with respect to $\phi_n = e^{-x}/x^n$ gives that.

$$Ei(x) \sim \sum_{n=0}^{\infty} \frac{(-1)^n n! e^{-x}}{x^{n+1}}$$

<u>What do we mean?</u>

(i) $\phi_n(x) - e^{-x}/x^{n+1}$ satisfies $\phi_{n+1}(x) = o(\phi_n(x))$ as $x \to +\infty$. i.e. it forms an "asymptotic sequence."

(ii) The notation "$\sim$" ("asymptotic to") means

$$\left| Ei(x) - \sum_{n=0}^{N} \frac{(-1)^n n! e^{-x}}{x^{n+1}} \right| = o\left(\frac{e^{-x}}{x^{n+1}}\right) \text{ as } x \to +\infty$$

This can be proved with integration by parts:

$$\begin{aligned} Ei(x) &= -\int_x^\infty \frac{1}{t} d(e^{-t}) = e^{-x}/x + \int_x^\infty \frac{1}{t^2} d(e^{-t}) \\ &= e^{-x}/x - e^{-x}/x^2 + 2\int_x^\infty \frac{e^{-t}}{t^3} dt \\ &= e^{-x}\left[\frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \cdots + \frac{(-1)^n n!}{x^{n+1}}\right] + \underbrace{(-1)^{n+1}(n+1)! \int_x^\infty \frac{e^{-t}}{t^{n+2}} dt}_{Rem_{n+1}(x)} \end{aligned}$$

Where

$$|Rem_{n+1}(x)| \le \frac{(n+1)!}{x^{n+2}} \int_x^\infty e^{-t} dt = \frac{(n+1)! e^{-x}}{x^{n+2}} = o\left(\frac{e^{-x}}{x^{n+1}}\right) \text{ as } x \to +\infty$$

So it is an asymptotic expansion. Not convergent because $\sum (-1)^n n! y^{n+1}$ has radius of convergence 0. (In fact for fixed y the terms become unbounded.)

Consider magnitudes of successive terms $f_n(x) = (-1)^n n! e^{-x}/x^{n+1}$

$$\left|\frac{f_{n+1}(x)}{f_n(x)}\right| = \frac{n+1}{x}$$

<u>Optimal truncation:</u> Truncate the asymptotic expansion at the point $n = N_x$, such that the first term excluded is the smallest.

In our example, choose $N_x = [x] - 1 = \sup\{j - 1 : j \le x, \, j \in \mathbb{N}\}$

$$\left.\begin{aligned} |f_{N_x+1}(x)/f_{N_x}(x)| &= (N_x+1)/x \quad \le 1 \\ |f_{N_x+2}(x)/f_{N_x+1}(x)| &= (N_x+2)/x \quad > 1 \end{aligned}\right\} \text{ so } f_{N_x+1} \text{ is the smallest term, later terms are larger}$$

So we write

$$Ei(x) = \sum^{N_x} \frac{(-1)^n n! e^{-x}}{x^{n+1}} + Rem_{N_x+1}(x)$$

$$|Rem_{n+1}(x)| \le \frac{(N_x+1)!}{x^{N_x+2}} e^{-x} = \frac{[x]! e^{-x}}{x^{[x]+1}} \le \frac{2\left(\frac{[x]}{e}\right)^{[x]} \sqrt{2\pi[x]} e^{-x}}{x^{[x]+1}} \le \frac{2\sqrt{2\pi[x]}}{[x]} e^{-x} e^{-[x]}$$

Where we have used Stirling's formula.

$$\lim_{n\to\infty} \frac{n!}{(n/e)^n\sqrt{2\pi n}} \to 1 \text{ as } n \to \infty$$

The good new is the additional $e^{-[x]}$ term. Optimal truncation (often) gives an exponentially small remainder.

Examples:

$$\sinh x = \frac{e^x - e^{-x}}{2} \sim e^{-x}/2 \text{ as } x \to +\infty$$

Works because $e^{-x} = o(e^x)$ as $x \to \infty$

$$\operatorname{sech} x = \frac{2}{e^x + e^{-x}} = \frac{2}{e^x}(1 + e^{-2x})^{-1} = \frac{2}{e^x}(1 - e^{-2x} + e^{-4x} - \dots)$$

This gives an asymptotic expansion for the sequence $\phi_n = e^{-nx}$ (Which is asymptotic since $e^{-x} = o(e^x)$ as $x \to +\infty$)

Note: $\sinh x \sim -e^{-x}/2$ as $x \to -\infty$

Consider $\sinh z$, for $z \in \mathbb{C}$

$$\sinh z = \frac{e^z - e^{-z}}{2} = \frac{e^{x+iy} - e^{-x-iy}}{2}$$

$$\sim e^z/2 \quad \text{as } z \to \infty \text{ in sector } \{-\frac{\pi}{2} < arg(z) < \frac{\pi}{2}\}$$
$$\sim e^{-z}/2 \quad \text{as } z \to \infty \text{ in sector } \{\frac{\pi}{2} < arg(z) < \frac{3\pi}{2}\}$$

<u>Conclusion:</u> The asymptotic seems to change suddenly when going from sector to sector.

The lines separating the different sectors are Stokes Lines.

Excercise: Prove that the definition of asymptotics in a sector must satisfy that you do not approach Stokes lines too fast.