

Dynamical Systems

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Introduction

A dynamical system is a set of equations describing the evolution of a system with respect to a time-like variable. Usually they are non-linear.

The possible states of the system define the state space/phase space.

Example: The logistic map

$$x_{n+1} = \mu x_n(1 - x_n)$$

For $0 \leq \mu \leq 4$ this describes evolution with respect to a discrete time n in a state space $[0, 1]$.

Example: The Lotka-Volterra equations

$$\begin{aligned}\dot{r} &= r(a - br - cs) \\ \dot{s} &= s(d - er - fs)\end{aligned}$$

Where $a-f$ are positive constants. These describe continuous evolution in a state space $(r, s) \in [0, \infty] \times [0, \infty]$ as a model for the population of two species competing for the same food supply.

Example: The non-linear Schrödinger equation

$$i \frac{\partial \Psi}{\partial t} = \nabla^2 \Psi + |\Psi|^2 \Psi$$

This describes evolution in an infinite dimensional statespace of possible wavefunctions.

Because the equations are non-linear, it is often impossible to find a complete set of closed form analytic solutions. Instead, we resort to a mixture of geometric and analytic arguments, and aim to say something about the generic long-term behaviour.

Example:

$$\begin{aligned}\dot{r} &= r(3 - r - s) \\ \dot{s} &= s(2 - r - s)\end{aligned}$$

Consider the regions where \dot{r} and \dot{s} are > 0 , < 0 , $= 0$.

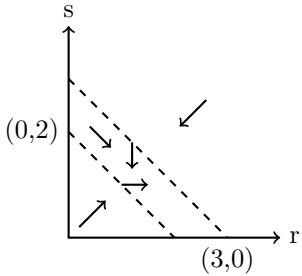
If $r, s > 0$ then

$$\begin{aligned} r + s < 2 &\implies \dot{r}, \dot{s} > 0 \\ 2 < r + s < 3 &\implies \dot{r} > 0, \dot{s} < 0 \\ r + s > 3 &\implies \dot{r}, \dot{s} < 0 \end{aligned}$$

$\dot{r} = 0$ if $r = 0$ or $r + s = 3$.

$\dot{s} = 0$ if $s = 0$ or $r + s = 2$.

Therefore $\dot{r} = \dot{s} = 0$ at the fixed points $(0, 0)$, $(3, 0)$, $(0, 2)$. This gives the phase portrait/diagram/plane¹



The most important feature of the phase portrait is that all solutions with $r > 0$ tend to the stable fixed point $(3, 0)$. The fixed points $(0, 0)$ and $(0, 2)$ are unstable. There are no periodic orbits.

Example:

$$\begin{aligned} \dot{r} &= r(3 - r - s) \\ \dot{s} &= s(2 - \mu r - s) \end{aligned}$$

In this case, a new fixed point $\left(\frac{1}{1-\mu}, \frac{2-3\mu}{1-\mu}\right)$ appears in the state space at $\mu = \frac{2}{3}$ and for $\mu < \frac{2}{3}$ is the long term stable attractor.

A qualitative change in the solution structure is called a bifurcation.

Example:

$$\begin{aligned} \dot{x} &= -y + \epsilon x(\mu - x^2 - y^2) \\ \dot{y} &= x + \epsilon y(\mu - x^2 - y^2) \end{aligned}$$

Use polar coordinates which are a more natural choice for this problem. In general:

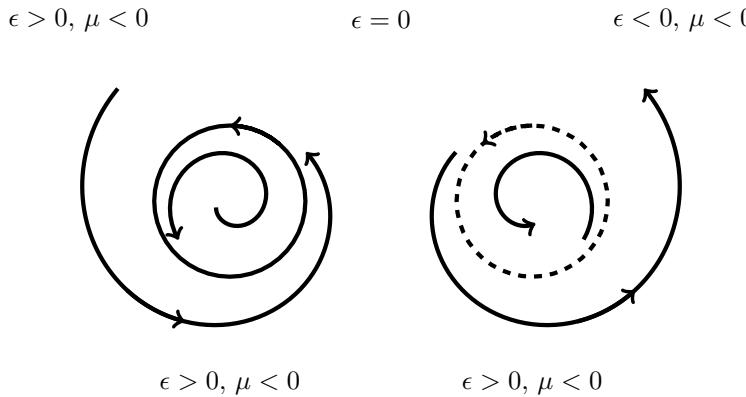
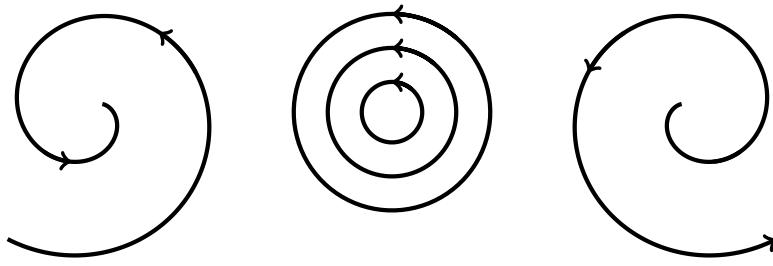
$$\boxed{\begin{aligned} \dot{r} &= \frac{x\dot{x} + y\dot{y}}{r} \\ \dot{\theta} &= \frac{x\dot{y} - y\dot{x}}{r^2} \end{aligned}}$$

Which are equations that will be referred to frequently.² In our example, they become

$$\begin{aligned} \dot{r} &= \epsilon r(\mu - r^2) \\ \dot{\theta} &= 1 \end{aligned}$$

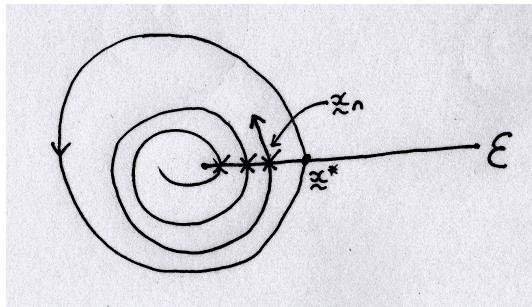
Consider \dot{r} and $\dot{\theta}$:

¹Phase portrait, phase diagram, phase plane will be used interchangably
²so learn them now!



The infinite set of periodic solutions for $\epsilon = 0$ is destroyed by any perturbation to $\epsilon \neq 0$. This is an example of structural instability. If $\mu > 0$ then just one limit cycle survives and is stable (unstable) for $\epsilon > 0$ ($\epsilon < 0$). The appearance of the limit cycle as $\mu \uparrow$ through 0 is another form of bifurcation.

Example: In 2D the points of successive intersection x_n of a solution near a limit cycle with a line ε perpendicular to the cycle, move monotonically towards/away from the point of intersection x^* of ε with the limit cycle.



The point x^* is a stable/unstable fixed point of this Poincaré recurrence map.

In 3 or higher dimensions, or in 2D with time-dependent coefficients there is room for much more complicated behaviour including chaos.

1 Basic Definitions

We need some terminology.

1.1 Notation

We only consider ODEs of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \quad (*)$$

for \mathbf{x} in a phase space/state space $E \subset \mathbb{R}^n$. The n first order ODEs form a dynamical system of order (dimension) n .

Since

$$\frac{\partial \mathbf{f}}{\partial t} = 0$$

we call the system autonomous.

A non-autonomous system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$ can be made autonomous by setting

$$\mathbf{y} = (\mathbf{x}, t) \quad \text{with} \quad \dot{\mathbf{y}} = (\mathbf{f}(\mathbf{y}), 1)$$

The n^{th} order ODE

$$\frac{d^n x}{dt^n} = g\left(x, \frac{dx}{dt}, \dots, \frac{d^{n-1}x}{dt^{n-1}}\right)$$

can be put in the form $(*)$ by setting

$$\mathbf{y} = \left(x, \frac{dx}{dt}, \dots, \frac{d^{n-1}x}{dt^{n-1}}\right)$$

with

$$\dot{\mathbf{y}} = (y_2, y_3, \dots, g(\mathbf{y}))$$

Similarly we will consider maps in the form

$$\mathbf{x}_{n+1} = \mathbf{F}(\mathbf{x}_n)$$

1.2 Initial Value Problem

Consider the IVP:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

$$\mathbf{x}(t_0) = \mathbf{x}_0$$

In Analysis II we showed that if \mathbf{f} satisfies a Lipschitz condition then we are guaranteed that a solution exists in a neighbourhood of \mathbf{x}_0, t_0 and is unique.

(Lipschitz condition is that $\exists L, a$, s.t. $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| < L|\mathbf{x} - \mathbf{y}|$, $\forall |\mathbf{x} - \mathbf{x}_0|, |\mathbf{y} - \mathbf{x}_0| < a$.)

Moreover, solutions $\mathbf{x}(t; \mathbf{x}')$ to

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

$$\mathbf{x}(t_0) = \mathbf{x}'$$

exist, are unique and they depend continuously on \mathbf{x}', t .

Note we are not guaranteed existence for all times.

Example:

$$\dot{x} = x^2$$

$$x(0) = 1$$

This has solution

$$x(t) = \frac{1}{1-t}$$

and so $x \rightarrow \infty$ as $t \uparrow 1$.

If $|\mathbf{x}(t)| \rightarrow \infty$ as $t \rightarrow T (< \infty)$ then we call this finite time blow up.

Example: Non-uniqueness when \mathbf{f} is non-Lipshitz. If

$$\dot{x} = \begin{cases} \sqrt{x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases} \quad \text{and } x(0) = 0$$

Then there is a family of solutions

$$\left. \begin{array}{ll} x = 0 & t < \tau \\ x = \frac{1}{4}(t - \tau)^2 & t > \tau \end{array} \right\} \text{ for any } \tau \geq 0$$

From now on we will assume that \mathbf{f} is differentiable (and so Lipshitz).

1.3 Trajectories and Flows

Consider the autonomous system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. The solution $\mathbf{x}(t)$ to the IVP with $\mathbf{x}(0) = \mathbf{x}_0$ defines a trajectory. (orbit/integral curve)

The distance travelled along this curve clearly only depends on $t - t_0$, and we could consider lots of starting points \mathbf{x}_0 .

This motivates the idea of a flow.

Definition: (Flow)

Given \mathbf{f} , the corresponding flow is defined to be a (the) function $\phi_t(\mathbf{x})$ from $E \times \mathbb{R} \rightarrow E$ such that

$$\frac{\partial}{\partial t} \phi_t(\mathbf{x}) = \mathbf{f}(\phi_t(\mathbf{x})), \quad \phi_0(\mathbf{x}) = \mathbf{x}$$

The solution to the IVP $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{x}(t_0) = \mathbf{x}_0$, is just $\mathbf{x}(t) = \phi_{t-t_0}(\mathbf{x}_0)$.
Clearly

$$\phi_{s+t}(\mathbf{x}) = \phi_s(\phi_t(\mathbf{x})) = \phi_t(\phi_s(\mathbf{x}))$$

Aside: We can establish another link with maps, by defining $\mathbf{x}_{n+1} = \mathbf{F}(\mathbf{x}_n) = \phi_{\Delta t}(\mathbf{x}_n)$.
Clearly $\phi_{n\Delta t}(\mathbf{x}) = \mathbf{F}(\phi_{(n-1)\Delta t}(\mathbf{x})) = \mathbf{F}^n(\mathbf{x})$

1.4 Flows, Trajectories, Orbits, Invariant Sets, & Limiting Sets

Using the idea of a flow, we define the following:

Definition: Orbit/Trajectories

The orbit of $\phi_t(\mathbf{x})$ through \mathbf{x}_0 is the set $\mathcal{O}(\mathbf{x}_0) = \{\phi_t(\mathbf{x}_0) : -\infty < t < \infty\}$.
The forwards (backwards) orbit is

$$\mathcal{O}^{+(-)}(\mathbf{x}_0) = \{\phi_t(\mathbf{x}_0) : t \geq 0 (t \leq 0)\}$$

Definition: Invariant sets

A set of points $\Lambda \subset E$ is invariant if $x \in \Lambda \implies \mathcal{O}(x) \subset \Lambda$
Clearly $\mathcal{O}(x_0)$ is invariant, and so is any union of orbits.

Particular cases of interest are

Definition: Fixed points

x_0 is a periodic point with period T if $\phi_T(x_0) = x_0$ for some $T > 0$ and $\phi_t(x_0) \neq x_0$ for $0 < t < T$.
The set $\{\phi_t(x_0) : 0 \leq t < T\}$ is the periodic orbit through x_0 .

Definition: Limit Cycle

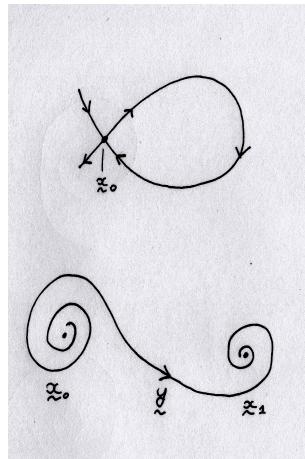
A limit cycle is an isolated periodic orbit, i.e. there are no other periodic orbits within a sufficiently small neighbourhood.

Definition: Homoclinic and Heteroclinic orbits

If x_0 is a fixed point and $\exists y \neq x_0$ such that $\phi_t(y) \rightarrow x_0$ as $t \rightarrow \pm\infty$ then $\mathcal{O}(y)$ is a homoclinic orbit.

If x_0, x_1 are fixed points and $\exists y \neq x_0, x_1$ with $\phi_t(y) \rightarrow x_0$ as $t \rightarrow -\infty$, $\phi_t(y) \rightarrow x_1$ as $t \rightarrow +\infty$ then $\mathcal{O}(y)$ is a heteroclinic orbit.

Example of a Heteroclinic and a Homoclinic orbit



If we are interested in the long-term behaviour of trajectories, it is not enough simply to think of

$$\lim_{t \rightarrow \infty} \phi_t(x)$$

because the limit might not exist; for example a limit cycle.

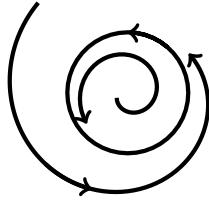
Definition: Limit set

The ω -limit set of x is

$$\omega(x) = \{y : \exists \text{ a sequence } (t_n) \text{ with } \phi_{t_n}(x) \rightarrow y \text{ and } t_n \rightarrow \infty\}$$

Similarly the α limit set is defined by sequences with $t_n \rightarrow -\infty$.

Example:



$$\begin{aligned}\dot{r} &= r(1 - r^2) \\ \dot{\theta} &= 1\end{aligned}$$

For \mathbf{x} with $0 < |\mathbf{x}| < 1$ we have that

$$\omega(\mathbf{x}) = \{\mathbf{y} : |\mathbf{y}| = 1\} \quad \alpha(\mathbf{x}) = \{\mathbf{0}\}$$

For $|\mathbf{x}| > 1$

$$\omega(\mathbf{x}) = \{\mathbf{y} : |\mathbf{y}| = 1\} \quad \alpha(\mathbf{x}) = \{\}$$

In fact, the limit sets are always invariant sets since

- If $\mathcal{O}(\mathbf{x})$ is bounded then $\omega(\mathbf{x})$ is non-empty.
- If $\phi_t(\mathbf{x}) \rightarrow \infty$ then $\omega(\mathbf{x}) = \{\}$.

Definition: (For maps)

Fixed points solve $\mathbf{F}(\mathbf{x}) = \mathbf{x}$. Orbits, periodic points etc. are defined as above by replacing $\phi_t(\mathbf{x})$ by $\mathbf{F}^n(\mathbf{x})$, etc.

A periodic orbit with period N is called an N-cycle.

1.5 Topological equivalence and structural stability of flows

This section is in a handout.

2 Flow near fixed points

The simplest features of a flow are the fixed points. Find them by first solving $\mathbf{F}(\mathbf{x}) = \mathbf{0}$.

2.1 Linearisation

If \mathbf{f} is sufficiently smooth and \mathbf{x}_0 is a fixed point, we expand in a Taylor series with $\mathbf{f}(\mathbf{x}_0) = \mathbf{0}$ to obtain

$$\dot{\mathbf{y}} = A\mathbf{y} + O(|\mathbf{y}|^2)$$

where $\mathbf{y} = \mathbf{x} - \mathbf{x}_0$ and

$$A_{ij} = \left. \frac{\partial f_i}{\partial x_j} \right|_{\mathbf{x}_0}$$

is the Jacobian matrix (Written $D\mathbf{f}$ in Glendinning).

The hope (see below when it is true) is that the flow ϕ_t^f is like the flow corresponding to the linearization $\dot{\mathbf{y}} = A\mathbf{y}$.

2.2 Classification of fixed points

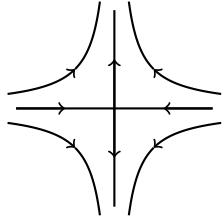
In 2D,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $D = ad - bc$, $T = a + d$ and the eigenvalues are

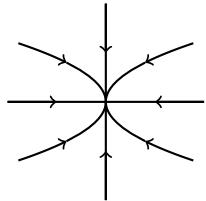
$$\lambda_{1,2} = \frac{1}{2}(T \pm \sqrt{T^2 - 4D})$$

- (i) Saddle points ($D < 0$, real λ 's with opposite signs)



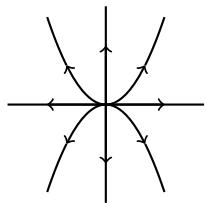
$$A = \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} \quad \lambda_1 < 0 < \lambda_2$$

- (ii) Stable nodes ($0 < 4D < T^2$, $T < 0$, real roots, both negative)



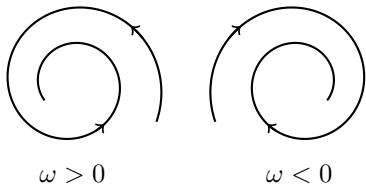
$$\lambda_1 < \lambda_2 < 0 \quad \frac{y_2}{y_1} \propto e^{(\lambda_2 - \lambda_1)t} \rightarrow \infty \text{ as } t \rightarrow \infty$$

- (iii) Unstable node ($0 < 4D < T^2$, $T > 0$, real roots, both positive)



$$0 < \lambda_1 < \lambda_2 \quad \frac{y_2}{y_1} \propto e^{(\lambda_2 - \lambda_1)t} \rightarrow 0 \text{ as } t \rightarrow -\infty$$

- (iv) Stable focus ($T^2 < 4D$, $T < 0$, complex roots, $\lambda = \rho \pm i\omega$, $\rho < 0$)

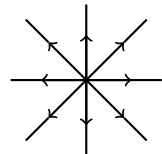


$$A = \begin{pmatrix} \rho & -\omega \\ \omega & \rho \end{pmatrix} \quad \dot{r} = \rho r \quad \dot{\theta} = \omega$$

- (v) Unstable focus ($T^2 < 4D$, $T > 0$, complex roots, $\rho > 0$)
- (vi) Two degenerate cases with equal eigenvalues on the border $T^2 = 4D$ between nodes and foci.
(e.g. $\lambda > 0$)

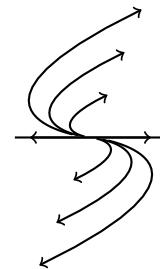
Stellar Node

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

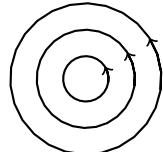


Improper Node

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

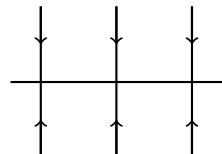


- (vii) Centres ($T = 0$, $D > 0$, $\lambda = \pm i\omega$)

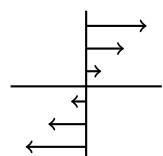


All trajectories are closed. This is on the border between stable foci and unstable foci.

- (viii) Line of Fixed points. Special case on the border $D = 0$ between saddles and nodes, where at least one λ is zero. For example,

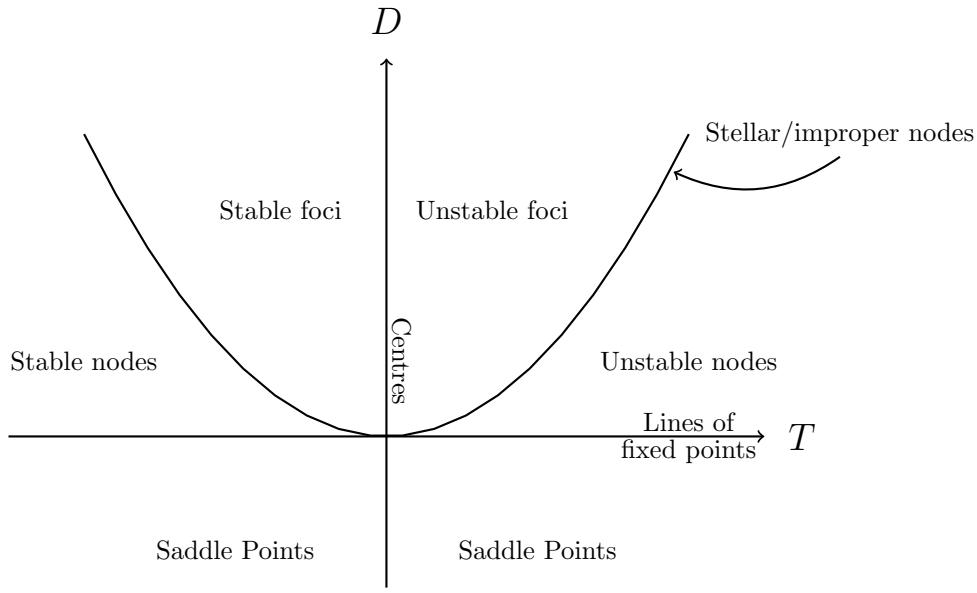


$$A = \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix} \quad (\lambda < 0)$$



$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

In summary



Notes:

- (1) A Hamiltonian system is of the form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \partial H / \partial y \\ -\partial H / \partial x \end{pmatrix}$$

for some $H(x, y)$. Fixed points are always saddles or centers because

$$A = \begin{pmatrix} H_{xy} & H_{yy} \\ -H_{xx} & -H_{xy} \end{pmatrix} \implies T = 0$$

Also $\dot{H} = \dot{\mathbf{x}} \cdot \nabla H = 0$ so the trajectories are (parts of) the contours of H .

- (2) The canonical forms are obtained by using the eigenvectors e_i if $\lambda_i \in \mathbb{R}$ (possibly “generalised” for equal λ 's and JNF, see Glendinning p 63) and $\text{Re}(e_1)$ and $\text{Im}(e_1)$ for $\lambda = \rho \pm i\omega$.
- (3) It is not necessary to change basis to classify the fixed point since T , D , λ 's are invariant under $A \mapsto P^{-1}AP$, but knowing the eigenvectors may help sketch the phase diagram for the case of a saddle point.

Perturbation to A (linear perturbations)

Cases (i)-(v) are robust - a small perturbation to A gives the same sort of eigenvalues and hence the same sort of fixed point.

Case (vi), if perturbed, may become a node or a focus, but these are topologically equivalent. The stability is not changed.

Cases (vii) and (viii) are fragile, - a small perturbation to (vii), ($\lambda = \pm i\omega$) may destroy the closed trajectories to give slow spiralling inwards or outwards. A small perturbation to (viii) ($\lambda = 0$) may destroy the line of fixed points to give a slow drift towards/away from the fixed point, and hence a saddle or node.

The fragility is because there are one or two eigenvalues on the border $\text{Re}(\lambda) = 0$ between stability and instability.

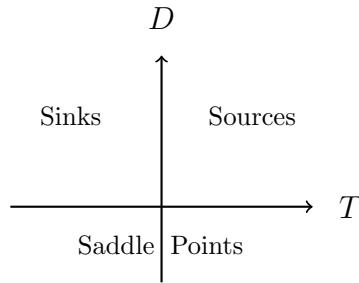
Definition: Hyperbolic fixed point

A fixed point \mathbf{x}_0 of the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is a hyperbolic fixed point if none of the eigenvalues λ of the Jacobian $\frac{\partial f_i}{\partial x_j} \Big|_{\mathbf{x}=0}$ satisfy $\text{Re}(\lambda) = 0$, and is non-hyperbolic otherwise.

In n-dimensions ($n \geq 0$) we classify a hyperbolic fixed point as

- (i) a sink if all eigenvalues have $\text{Re}(\lambda) < 0$.
- (ii) a source if all eigenvalues have $\text{Re}(\lambda) > 0$.
- (iii) a saddle point otherwise (some > 0 , some < 0)

In 2D



Non hyperbolic fixed points are important in bifurcation theory.

2.3 The effects of non-linear terms

It is in fact true that the linearised system $\dot{\mathbf{y}} = A\mathbf{y}$ is essentially the same as the non linear system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ near a fixed point \mathbf{x}_0 provided

- (i) The fixed point is hyperbolic
- (ii) The non linear terms are $O(|\mathbf{x} - \mathbf{x}_0|^2)$

(If (i) holds then the systems are topologically equivalent, if (ii) also holds, then $\mathbf{h}(\mathbf{x})$ is a near identity map, so nodes are nodes and foci are foci.)

2.3.1 Stable and Unstable Manifold

First, we formalise the idea of stable and unstable directions in the linearised system.

Definition: The stable, unstable and center invariant subspaces of the linearisation of \mathbf{f} at a fixed point are the local linear subspaces.

E^s , E^u , E^c spanned by the eigenvectors of the Jacobian, whose eigenvalues have real parts < 0 , > 0 , $= 0$ respectively. (Generalised eigenvectors for JNF)

For some types of fixed point, these spaces may be empty, e.g. for a hyperbolic fixed point, E^c is empty by definition.

Then we extend th linear picture into the non-linear domain for a hyperbolic fixed point.

Stable Manifold Theorem

If $\mathbf{0}$ is a hyperbolic fixed point of the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, with linear stable and unstable invariant subspaces E^s and E^u then in a sufficiently small neighbourhood of the origin, there exist local stable and unstable manifolds

$$W_{loc}^s(\mathbf{0}) = \{\mathbf{x} : \phi_t(\mathbf{x}) \rightarrow \mathbf{0} \text{ as } t \rightarrow \infty\}$$

$$W_{loc}^u(\mathbf{0}) = \{\mathbf{x} : \phi_t(\mathbf{x}) \rightarrow \mathbf{0} \text{ as } t \rightarrow -\infty\}$$

these have the same dimension as E^s and E^u and are tangent to them at $\mathbf{0}$.

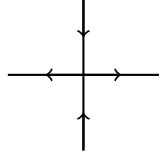
Notes:

- (1) For a saddle point in \mathbb{R}^2 , this guarantees the existence of two specific trajectories (the separatrices) that approach and leave the saddle.
- (2) For a sink this guarantees that all trajectories in some neighbourhood of the sink tend to it
- (3) The local stable (unstable) manifolds can be extended to global stable (unstable) invariant manifolds W^s (W^u) by following the flow backwards (forwards).

The proof of this is not in the course - See Glendinning.

It is easy to calculate approximations to W^s and W^u for a saddle point in \mathbb{R}^2 .

Wlog (Change of origin and basis) assume that the saddle is at $\mathbf{x} = \mathbf{0}$ and that E^s is $x = 0$ and E^u is $y = 0$.



Then W_{loc}^s becomes $x = S(y)$ with $S(0) = 0$, $S'(0) = 0$. W_{loc}^u is $y = U(x)$ with $U(0) = 0$, $U'(0) = 0$.

Since W^s and W^u are invariant,

$$\dot{x}|_{(S,y)} = \frac{dS}{dy} \dot{y}|_{(S,y)} \quad \dot{y}|_{(x,U)} = \frac{dU}{dx} \dot{x}|_{(x,U)}$$

Example:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x - xy \\ -y + x^2 \end{pmatrix} \quad A|_{\mathbf{x}=0} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Assume that

$$y = U(x) = a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots$$

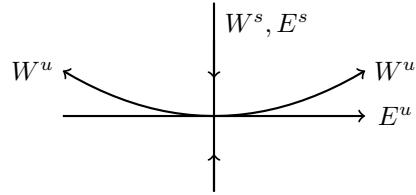
substitute into

$$\dot{y} = U' \dot{x} \implies -(a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots) + x^2 = (2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots)(x - a_2 x^3 - \dots)$$

Equate coefficients:

$$\begin{aligned} a_2 &= \frac{1}{3} \\ a_3 &= 0 \\ a_4 &= \frac{2}{45} \quad \text{etc.} \end{aligned}$$

Locally for $|\mathbf{x}| \ll 1$



In higher dimensions, would write $\mathbf{y} = \mathbf{U}(\mathbf{x})$ and solve

$$\dot{y}_i = \frac{\partial U_i}{\partial x_i} \dot{x}_j$$

Where the x_j 's span E^u and the y_i 's span E^s .

2.3.2 Non-Linear terms in non-hyperbolic cases

There are many possibilities depending on the non-linear terms.

- (a) $\lambda = \pm i\omega$ Linear system is a centre. Are the trajectories really closed?

Example:

$$\begin{aligned} \dot{x} &= -y \pm x^3 \\ \dot{y} &= x \pm y^3 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \dot{r} &= \pm \frac{x^4 + y^4}{r} \\ \dot{\theta} &= 1 \pm \frac{xy^3 - yx^3}{r^2} \end{aligned}$$

Now as $r \rightarrow 0$, $\dot{\theta} \rightarrow 1$. Thus the system is a stable focus if we choose '+' and an unstable focus if we choose '-'.

Example:

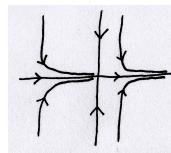
$$\begin{aligned} \dot{x} &= -y - 2yx^2 = -\frac{\partial H}{\partial y} \\ \dot{y} &= x + 2xy^2 = -\frac{\partial H}{\partial x} \end{aligned}$$

for $H = -\frac{x^2 + y^2}{2} - x^2y^2$. The trajectories are contours of H , which are closed near the maximum at $\mathbf{0}$. So the fixed point really is a non-linear centre with nested periodic orbits.

- (b) $\lambda_1 = 0, \lambda_2 \neq 0$. Which way does the system drift along e_1 ?

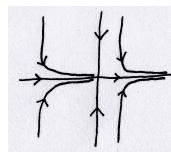
Example:

$$\begin{aligned} \dot{x} &= x^2 \\ \dot{y} &= -y \end{aligned}$$



Saddle-node

$$\begin{aligned} \dot{x} &= x^3 \\ \dot{y} &= -y \end{aligned}$$

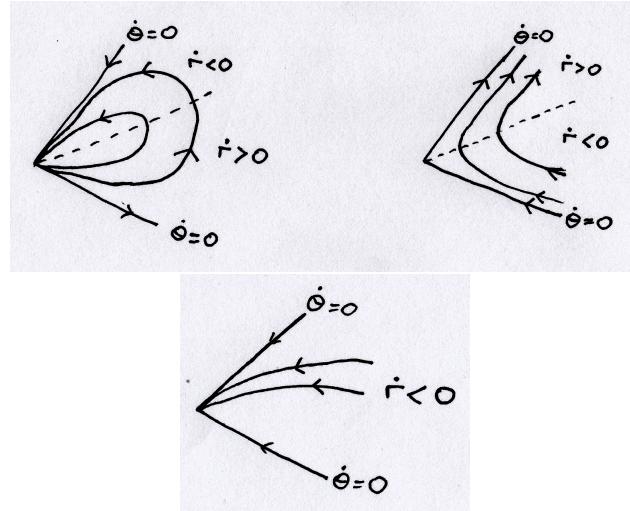


Non-linear saddle

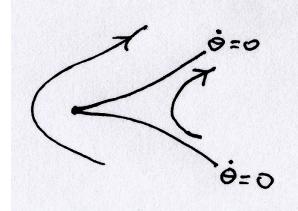
(c) $\lambda_1 = \lambda_2 = 0$. Ad hoc methods needed.

In some cases, it is useful to switch to polars and consider the sign of \dot{r} and $\dot{\theta}$ as $r \rightarrow 0$.

Example: $\dot{\theta} > 0$



In nastier cases, the curves $\dot{\theta} = 0$ can form a cusp.



2.4 Sketching phase planes/portraits

Example:

$$\begin{aligned}\dot{x} &= x(1-y) \\ \dot{y} &= -y + x^2\end{aligned}$$

This has fixed points at $(0, 0)$ and at $(\pm 1, 1)$.

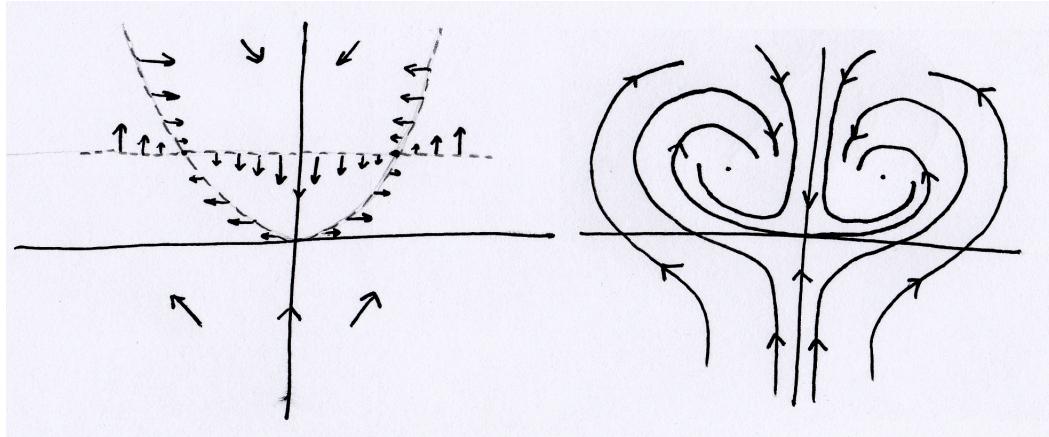
$$(0, 0) \quad A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{saddle point}$$

$$(\pm 1, 1) \quad A = \begin{pmatrix} 0 & \mp 1 \\ \pm 2 & -1 \end{pmatrix} \quad \text{stable foci}$$

It is clear why $(0, 0)$ is a saddle point. For $(\pm 1, 1)$ observe that $T = -1$, $D = 2$, $T^2 < 4D$, and therefore $(\pm 1, 1)$ are stable foci. Before sketching, observe that.

- $x = 0$ is a trajectory.
- $\dot{x} = 0$ on $y = 1$
- $\dot{y} = 0$ on $y = x^2$
- $\dot{y} \leq 0$ if $y \leq x^2$
- Symmetry in y axis; under $x \mapsto -x$, $\dot{x} \mapsto -\dot{x}$, and $\dot{y} \mapsto \dot{y}$
- Know that the unstable manifold is given by $y \approx \frac{1}{3}x^2 + \frac{2}{45}x^4$ near $x = 0$ (See earlier example)

Put all of this together to sketch the following:



In general, here are some steps to follow when sketching a phase plane.

1. Find the fixed points.
2. (a) Calculate the Jacobian to find the kind of eigenvalues and classify the fixed points.
(b) If any are non-hyperbolic, consider non-linear terms, possible Hamiltonian structure, signs of \dot{x} , \dot{y} , \dot{r} or $\dot{\theta}$ as $|\mathbf{x} - \mathbf{x}_0| \rightarrow 0$.
3. Calculate the eigenvectors for saddles.
4. Consider the sign of \dot{x} and \dot{y} - the curves where these are zero are called nullclines.
5. Construct the global picture by joining up the local behaviour near fixed points. (especially the separatrices of saddles) using the arrows from \dot{x} and \dot{y} .
6. At some stage, decide whether there are periodic orbits.

For an additional example, see handout 2.4, “Example of Phase-Plane sketching.”

3 Stability

3.1 Definitions

It is clear what it means for a hyperbolic node or focus to be stable, but we need to be more careful with the stability of other kinds of fixed points or invariant sets, because there are at least two notions of stability.

Definition: Lyapunov stability

A fixed point \mathbf{x}_0 of a flow ϕ_t is Lyapunov stable if $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $|\mathbf{x} - \mathbf{x}_0| < \delta \implies |\phi_t(\mathbf{x}) - \mathbf{x}_0| < \epsilon \forall t > 0$. A motto is “Start near, stay near” stability.

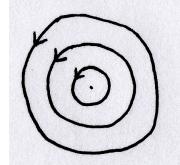
Definition: Quasi-asymptotic stability (QAS)

A fixed point \mathbf{x}_0 of a flow ϕ_t is quasi-asymptotically stable if $\exists \delta > 0$ s.t. $|\mathbf{x} - \mathbf{x}_0| < \delta \implies \phi_t(\mathbf{x}) \rightarrow \mathbf{x}_0$ as $t \rightarrow \infty$.

These are not the same!

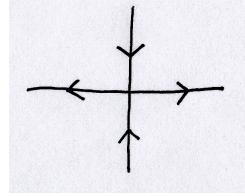
Example:

$$\begin{aligned}\dot{r} &= 0 \\ \dot{\theta} &= 1\end{aligned}$$



Lyapunov stable, but not asymptotically stable.

$$\begin{aligned}\dot{r} &= r(1 - r^2) \\ \dot{\theta} &= \sin^2(\theta/2)\end{aligned}$$



The point $r = 1, \theta = 0$ is QAS but not Lyapunov stable

Definition: Asymptotic stability (AS)

A fixed point \mathbf{x}_0 is asymptotically stable if it is both Lyapunov stable and quasi-asymptotically stable.

Example: A sink ($\text{Re } \lambda_i < 0 \ \forall i$) is asymptotically stable. (Choose δ sufficiently small that the linear terms dominate) A saddle or source is obviously not Lyapunov stable, hence not AS.

For other kinds of invariant sets Λ e.g. limit cycles, define

$$N_\delta(\Lambda) = \{\mathbf{x} : \exists \mathbf{y} \in \Lambda \text{ with } |\mathbf{x} - \mathbf{y}| < \delta\}$$

and say

$$\phi_t(\mathbf{x}) \rightarrow \Lambda$$

if

$$\min_{\mathbf{y} \in \Lambda} \{|\phi_t(\mathbf{x}) - \mathbf{y}|\} \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

Then say Λ is Lyapunov stable if $\forall \epsilon > 0 \ \exists \delta > 0$ s.t. $\mathbf{x} \in N_\delta(\Lambda) \implies \phi_t(\mathbf{x}) \in N_\epsilon(\Lambda) \ \forall t > 0$.

Say Λ is QAS if $\exists \delta > 0$ s.t. $\mathbf{x} \in N_\delta(\Lambda) \implies \phi_t(\mathbf{x}) \rightarrow \Lambda$ as $t \rightarrow \infty$.

Say Λ is AS if it is both LS and QAS.

3.2 Lyapunov functions

These allow us to say more about the stability of a fixed point which wlog we take to be $\mathbf{x} = 0$.

Definition: Lyapunov function

A continuously differentiable function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is a Lyapunov function for $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ in a domain $\mathcal{D} \subset \mathbb{R}^n$ if

- (i) $V(\mathbf{0}) = 0$ and $V(\mathbf{x}) > 0$ for $\mathbf{x} \in \mathcal{D} \setminus \{\mathbf{0}\}$. “Positive semi-definite”.
- (ii) $\dot{V}(\mathbf{x}) \equiv \mathbf{f} \cdot \nabla V \leq 0 \forall \mathbf{x} \in \mathcal{D}$. “Non-increasing”.

Note that (ii) means trajectories head “downhill” (or at least not uphill) and (i) means that $\mathbf{0}$ is the lowest point. These properties allow us to prove:

Theorem: Lyapunov’s First Theorem

If a Lyapunov function exists, then $\mathbf{0}$ is Lyapunov stable.

Proof. Wlog we can assume ϵ is sufficiently small that $\{|\mathbf{x}| < \epsilon\} \subset \mathcal{D}$. Let

$$m = \inf\{V(\mathbf{x}) : |\mathbf{x}| = \epsilon\}$$

m is attained, and $m > 0$ since $\{|\mathbf{x}| = \epsilon\}$ is compact. Define

$$C_{m,\epsilon} = \{\mathbf{x} : V(\mathbf{x}) < m \text{ and } |\mathbf{x}| < \epsilon\}$$

Then $\mathbf{x} \in C_{m,\epsilon} \implies \phi_t(\mathbf{x}) \in C_{m,\epsilon} \forall t > 0$ since V is non-increasing and $V \geq m$ on the boundary of $C_{m,\epsilon}$.

Finally, choose $\delta > 0$ such that $\{|\mathbf{x}| < \delta\} \subset C_{m,\epsilon}$. Then

$$|\mathbf{x}| < \delta \implies |\phi_t(\mathbf{x})| < \epsilon \quad \forall t > 0$$

□

A trajectory can head downhill and still not end up at $\mathbf{0}$. However, there is an important constraint.

Theorem: La Sale’s Invariance Principle

If V is a Lyapunov function on a bounded domain \mathcal{D} and $\mathcal{O}^+(\mathbf{x}) \subset \mathcal{D}$, then $\phi_t(\mathbf{x}) \rightarrow$ an invariant subset of $\{\mathbf{x} : \dot{V}(\mathbf{x}) = 0\}$.

Proof.

$\mathcal{O}^+(\mathbf{x}) \subset \mathcal{D}$ so $V(\phi_t(\mathbf{x}))$ is monotonically decreasing and bounded below by 0 $\implies V(\phi_t(\mathbf{x})) \rightarrow \alpha$ for some $\alpha \geq 0$ as $t \rightarrow \infty$.

\mathcal{D} is bounded $\implies \omega(\mathbf{x})$ is non empty.

Let $\mathbf{y} \in \omega(\mathbf{x})$ then $\exists \{t_n\}$ s.t. $\phi_{t_n}(\mathbf{x}) \rightarrow \mathbf{y} \implies V(\mathbf{y}) = \alpha$ by continuity of V .

Therefore $V(\phi_t(\mathbf{y})) = \alpha$ since $\phi_t(\mathbf{y}) \in \omega(\mathbf{x})$.

Now $V(\phi_t(\mathbf{y})) = \alpha \quad \forall t \implies \dot{V}(\mathbf{y}) = 0$.

This holds for any $\mathbf{y} \in \omega(\mathbf{x})$ and thus

$$\omega(\mathbf{x}) \subset \{\dot{V} = 0\}$$

We know ω is invariant, and thus we are done. □

Corollary: If V is a Lyapunov function on a domain \mathcal{D} containing $\mathbf{0}$, and the only invariant subset of $\{\mathbf{x} : \dot{V}(\mathbf{x}) = 0\} \cap \mathcal{D}$ is $\{\mathbf{0}\}$, then $\mathbf{x} = \mathbf{0}$ is asymptotically stable.

Proof.

Choose k such that

$$C_k \equiv \{\mathbf{x} : V(\mathbf{x}) < k\} \subset \mathcal{D}$$

and choose δ such that $\{\mathbf{x} : |\mathbf{x}| < \delta\} \subset C_k$.

Then $|\mathbf{x}| < \delta \implies V(\mathbf{x}) < k \implies V(\phi_t(\mathbf{x})) < k \implies \phi_t(\mathbf{x}) \subset \mathcal{D}$

□

Example: Let

$$\dot{\mathbf{x}} = A\mathbf{x} + o(|\mathbf{x}|)$$

and let λ_i be the distinct eigenvalues of A with corresponding eigenvectors \mathbf{e}_i .

Suppose that $\operatorname{Re}(\lambda_i) < 0$, and let v_i be any positive real constants. Then, writing

$$\mathbf{x} = \sum_{i=1}^n a_i \mathbf{e}_i, \quad V(\mathbf{x}) = \sum_{i=1}^n v_i |a_i|^2$$

have that $V \geq 0$ ($= 0$ only at $\mathbf{0}$) and \dot{V} in a sufficiently small neighbourhood of $\mathbf{0}$. (See example sheet)
Therefore have that $\mathbf{x} = \mathbf{0}$ is asymptotically stable.

The main use of Lyapunov functions, however, is to find information about the domain of stability.

Definition: Domain of stability

The domain of stability (basin of attraction) of an asymptotically stable invariant set Λ is

$$\{\mathbf{x} : \phi_t(\mathbf{x}) \rightarrow \Lambda \text{ as } t \rightarrow \infty\}$$

i.e. \mathbf{x} such that

$$\inf_{\mathbf{y} \in \Lambda} \{|\phi_t(\mathbf{x}) - \mathbf{y}|\} \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

If the domain of stability in \mathbb{R}^n then Λ is globally stable.

Example: We can show that the non-hyperbolic fixed point of the system

$$\begin{aligned}\dot{x} &= y - xy^2 \\ \dot{y} &= x^3\end{aligned}$$

is asymptotically stable.

Without the $-xy^2$ term, the system would be Hamiltonian with $H = \frac{1}{2}y^2 + \frac{1}{4}x^4$ and $\dot{H} = 0$.

With the $-xy^2$ term, $\dot{H} = -x^4y^2 \leq 0$.

Therefore H is a Lyapunov function and so $\phi_t(\mathbf{x}) \rightarrow$ an invariant subset of $\{\dot{H} = 0\} = \{x = 0 \text{ or } y = 0\}$.

But the only invariant subset of $\{\dot{H} = 0\}$ is $\mathbf{x} = \mathbf{0}$. Hence by the corollary, $\mathbf{x} = \mathbf{0}$.

Example:

$$\begin{aligned}\dot{x} &= y + \mu(\frac{1}{3}x^3 - x) \\ \dot{y} &= -x\end{aligned}\left.\right\} \text{ equivalent to } \ddot{x} + \mu(1 - x^2)\dot{x} + x = 0$$

The “Time-reversed van der Pol oscillator”. Observe that $\mathbf{x} = \mathbf{0}$ is a sink for $\mu > 0$.

Guess $V = x^2 + y^2$ (e.g. from case $\mu = 0$). This implies that $\dot{V} = 2\mu x^2(\frac{x^2}{3} - 1)$ so $\dot{V} \leq 0$ in

$x^2 < 3$.

Let $C_3 = \{V < 3\}$ so $\mathbf{x} \in C_3 \implies \phi_t(\mathbf{x}) \in C_3$.

La Salle's invariance principle $\implies \mathbf{x} \in C_3$ then $\phi_t(\mathbf{x})$ tends to an invariant subset of $\{\dot{V} = 0\}$. But the only invariant subset of $\{x = 0\}$ is $\{x = y = 0\} \implies \mathbf{0}$ is AS and the domain of stability includes C_3 .

General method

- (1) Find a domain \mathcal{D} and function V such that
 - (i) $V \geq 0$ on \mathcal{D} and $V = 0$ iff $\mathbf{x} = \mathbf{0}$ (easy)
 - (ii) $\dot{V} \leq 0$ on \mathcal{D} (harder, often restricts the size of \mathcal{D})
- (2) Choose k such that $C_k = \{V < k\} \subset \mathcal{D}$.
- (3) If necessary, adjust k (or V), so that the only invariant subset of $\{\dot{V} = 0\} \cap C_k$ is $\mathbf{x} = \mathbf{0}$. Then La Salle implies that the origin is AS stable and the domain of stability includes C_k .
- (4) Try different V to maximise C_k or take a union of them.

Example:

$$\begin{aligned} \dot{x} &= -x + xy^2 \\ \dot{y} &= -y + yx^2 \end{aligned} \left. \right\} \text{ Try } V = x^2 + by^2 \implies \dot{V} = -2(x^2 + b^2y^2) + 2(1 + b^2)x^2y^2$$

Clearly $\dot{V} \leq 0$ sufficiently close to $\mathbf{0}$.

Let $(x, by) = \sqrt{V}(\cos \phi, \sin \phi)$

$$\begin{aligned} \implies \dot{V} &= -2V + 2V^2 \left(\frac{1+b^2}{b^2} \right) \underbrace{\cos^2 \phi \sin^2 \phi}_{\leq 1/4} \\ \implies \dot{V} &\leq 0 \text{ in } 0 \leq V \leq \frac{4b^2}{1+b^2} \end{aligned}$$

The domain of stability includes the union over b of these regions.

For another example see handout 3.2, "Examples of Domains of Stability".

3.3 Bounding Functions

The idea of a function decreasing along trajectories.

If V is a function with bounded contours and $\dot{V} < \delta < 0$ in the region $V(\mathbf{x}) > M$, then all trajectories eventually enter and remain in the set $\{V(\mathbf{x}) < M\}$. Such a V is called a bounding function.

4 Existence and Stability of Periodic orbits in \mathbb{R}^n

There is a toolkit of standard techniques, particular to \mathbb{R}^2 , to prove that there either is or isn't one or more periodic orbits.

4.1 Poincaré index test

Consider

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

Except at a fixed point ($\mathbf{f} = \mathbf{0}$), the trajectory through a point \mathbf{x} makes a well defined angle $\psi(\mathbf{x})$ with the x -axis.

Definition: Poincaré index of a curve

If Γ is a simple closed curve, not necessarily a trajectory, that doesn't pass through any fixed points, then moving around Γ once anticlockwise, $\psi = \tan^{-1} \frac{f_1}{f_2}$ changes continuously and returns to its original value plus an integer multiple of 2π . This multiple is the Poincaré index $I(\Gamma)$ of Γ .

Properties:

1.

$$2\pi I(\Gamma) = \oint_{\Gamma} d\psi = \oint_{\Gamma} d\left(\tan^{-1} \frac{f_1}{f_2}\right) = \oint_{\Gamma} \frac{f_1 df_2 - f_2 df_1}{f_1^2 + f_2^2}$$

2. $I(\Gamma)$ is unchanged by deformations of Γ that do not cross a fixed point.

(Proof: $I(\Gamma)$ is integrable and a continuous function of \mathbf{f} (from 1) provided $\mathbf{f} \neq \mathbf{0}$.)

3. If Γ encloses no fixed points, then $I(\Gamma) = 0$.

4. The Poincaré index is additive, $I(\Gamma_1) = I(\Gamma_2) + I(\Gamma_3)$

5. If Γ is a closed trajectory, i.e. a periodic orbit, then $I(\Gamma) = 1$.

6. If time is reversed ($\mathbf{f} \rightarrow -\mathbf{f}$) then $I(\Gamma)$ is unaffected (from 1).

Definition: Poincaré index of a fixed point

The Poincaré index of an isolated fixed point is the Poincaré index of any simple closed curve enclosing this fixed point and no others. (Well defined by 2.)

7. $I(\Gamma)$ is the sum of the Poincaré indices of the fixed points it encloses.

8. The Poincaré index of a node, focus, or center is $+1$.

9. The Poincaré index of a saddle is -1 .

Non-hyperbolic fixed points other than centers need to be done on a case by case basis.

Example: $\dot{x} = x^2$, $\dot{y} = -y$

Index = 0

Important Corollaries

If there are any periodic orbits, then each must contain at least one fixed point. The fixed point(s) enclosed by a periodic orbit must have indices summing to $+1$.

If the above is not possible then there are no periodic orbits. This is the Poincaré index test. (a negative test)

Example:

$$\begin{aligned}\dot{r} &= r(3 - r - s) \\ \dot{s} &= s(2 - r - s)\end{aligned}$$

Any periodic orbit cannot cross the trajectories $r = 0$ and $s = 0$. Therefore it cannot enclose any of the nodes. A periodic orbit cannot enclose only the saddle (index -1), hence there are no periodic orbits.

4.2 Dulac's Criterion

Another negative test.

Theorem: Dulac's criterion

If there is a continuously differentiable function $\phi(x, y)$ such that $\nabla \cdot (\phi \mathbf{f}) \neq 0$ in a simply connected domain $\mathcal{D} \subset \mathbb{R}^2$ then there are no periodic orbits lying entirely within \mathcal{D} .

Proof.

Assume, wlog, $\nabla \cdot (\phi \mathbf{f}) > 0$, suppose that Γ is a periodic orbit lying entirely in \mathcal{D} , and enclosing an area A with outward normal \mathbf{n} .

$$\int_{\Gamma} \phi \mathbf{f} \cdot \mathbf{n} ds = 0 = \int_A \nabla \cdot (\phi \mathbf{f}) dA > 0$$

which is a contradiction. \square

Often we just take $\phi = 1$, this is known as the divergence test.

Corollary: If $\nabla \cdot (\phi \mathbf{f}) \neq 0$ in a doubly connected domain \mathcal{D} , then there is at most one periodic orbit lying entirely in \mathcal{D} .

Proof. Apply the divergence theorem to the area between two hypothetical periodic orbits in \mathcal{D} . \square

Example: The Lotka-Volterra equations

$$\begin{aligned}\dot{r} &= r(a - br - cs) \\ \dot{s} &= s(d - er - fs)\end{aligned}$$

Choose $\phi = \frac{1}{rs}$. Then $\nabla \cdot (\phi \mathbf{f}) = -\frac{b}{s} - \frac{f}{r} < 0$ in $r, s > 0$, for $b, f > 0$. Therefore there can be no periodic orbits entirely in $r, s > 0$.

Example: Damped pendulum with torque

$$\begin{aligned}\dot{\theta} &= p \\ \dot{p} &= F - kp - \sin \theta\end{aligned}$$

The argument is slightly complicated by the phase space having the topology of a cylinder.

Can work on the cylinder and use $\frac{\partial \dot{\theta}}{\partial \theta} + \frac{\partial \dot{p}}{\partial p} = -k < 0$ to deduce that there is at most one periodic orbit, which must encircle the cylinder.

Alternatively can let $r = e^p$ to map onto plane polars, and use $\phi = 1/r^2$,

$$\nabla \cdot (\phi \mathbf{f}) = \nabla \cdot (\phi(\dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta)) = -\frac{k}{r^2} < 0$$

4.3 Poincaré-Bendixson Theorem

A positive test!

Theorem: Poincaré-Bendixson

If the forward orbit $\mathcal{O}^+(\mathbf{x})$ of a point \mathbf{x} remains in a compact (closed and bounded) set $K \subset \mathbb{R}^2$, and K contains no fixed points, then $\omega(\mathbf{x})$ is a periodic orbit.

Note:

1. From 3.1, K must be multiply connected with a fixed point in one of the holes.
2. Idea: The trajectory must go somewhere, and the conditions of the theorem mean it can't go to ∞ or a fixed point.

The full (sketch) proof is in handout 4.4.

Example:

$$\begin{aligned}\dot{x} &= x - y - 2x(x^2 + y^2) \\ \dot{y} &= x + y - y(x^2 + y^2)\end{aligned}$$

The linear terms $\implies \mathbf{0}$ is an unstable foci. The cubic terms look inward from $-\infty$.

Try for an annulus in polar coordinates

$$\dot{r} = \frac{1}{r}(x^2 + y^2)(1 - 2x^2 - y^2) = r(1 - r^2(1 + \cos^2 \theta))$$

minimising/maximising over θ gives

$$r(1 - 2r^2) \leq \dot{r} \leq r(1 - r^2)$$

and therefore

$$\begin{aligned}\dot{r} > 0 \text{ in } r &< \frac{1}{\sqrt{2}} \\ \dot{r} < 0 \text{ in } r &> 1\end{aligned}$$

So trajectories entering the annulus $C = \{\frac{1}{\sqrt{2}} \leq r \leq 1\}$ remain there.

No other fixed points?

$$\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2} = 1 + \frac{r^2}{2} \sin 2\theta \implies \dot{\theta} > 0 \text{ in } r < \sqrt{2}$$

C contains no fixed points. By Poincaré-Bendixson theorem, C contains a periodic orbit.

($\nabla \cdot \mathbf{f} = 2 - 5r^2 - 2x^2 < 0$ in C , therefore there is only one periodic orbit)

Example: Forced damped pendulum again

$$\begin{aligned}\dot{\theta} &= p \\ \dot{p} &= F - kp - \sin \theta\end{aligned}$$

From bounding functions in §3.3, we know that all trajectories enter and remain in the compact set ³

$$V \leq \frac{1}{2} \left(\frac{F}{k} \right)^2 + 2$$

If $F > 1$ then there are no fixed points \implies there is a periodic orbit by Poincaré-Bendixson. (and only one by corollary to Dulac's criterion)

4.4 Nearly Hamiltonian Flows

This can be a negative or positive test.

Consider a system

$$\begin{aligned}\dot{x} &= f_1(x, y) + \epsilon g_1(x, y) \\ \dot{y} &= f_2(x, y) + \epsilon g_2(x, y)\end{aligned}$$

Where

$$f_1 = \frac{\partial H}{\partial y}, \quad f_2 = -\frac{\partial H}{\partial x}$$

If $\epsilon = 0$ then $\dot{H} = 0$ and closed contours of H are periodic orbits. If $\epsilon \neq 0$ then

$$\dot{H} = \frac{\partial H}{\partial x} \dot{x} + \frac{\partial H}{\partial y} \dot{y} = \epsilon \dot{x} g_2 - \epsilon g_1 \dot{y} = \epsilon(g_2 f_1 - g_1 f_2)$$

For a periodic orbit Γ , we must have

$$\oint_{\Gamma} dH = 0 \implies \oint_{\Gamma} \epsilon(g_2 f_1 - g_1 f_2) dt = 0$$

Hence there are no periodic orbits that lie entirely in a region where $g_2 f_1 - g_1 f_2 > 0$ (or < 0).

If $0 < \epsilon \ll 1$ then $\dot{H} = O(\epsilon)$, therefore the trajectories of the perturbed system are very similar to the contours of H . So changes in H can be approximated by integrating along the trajectories $H = H_0$ of the unperturbed Hamiltonian system.

$$\Delta H(H_0) = \epsilon \oint_{H=H_0} (g_2 f_1 - g_1 f_2) dt + O(\epsilon^2) = \epsilon \oint_{H=H_0} g_2 dx - g_1 dy + O(\epsilon^2)$$

Where the $O(\epsilon^2)$ terms arise as the trajectory is not exactly $H = H_0$.

Fixed points $\Delta H = 0$ of the energy return map $H_{n+1} = H_n + \Delta H(H_n)$ correspond to periodic orbits. Finding approximations to them by neglecting the $O(\epsilon^2)$ terms is called the energy balance equation.

Note that the $O(\epsilon^2)$ can make a double root of $\Delta H = 0$ disappear, but not a single root.

³I actually have no idea what V is! my notes don't contain any reference, and apparently I didn't notice at the time...

Example:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ -x + x^2 + \epsilon y(a-x) \end{pmatrix}$$

Settubg $\epsilon = 0$ yields a Hamiltonian system with $H = \frac{1}{2}(x^2 + y^2) - \frac{1}{3}x^3$.

If $\epsilon \neq 0$, the fixed points are still $(0, 0)$ (node/focus) and $(1, 0)$ (saddle). However, $\dot{H} = \epsilon y^2(a-x)$ so any periodic orbit must be in partly $x > a$ and partly in $x < a$. Any periodic orbit must enclose $(0, 0)$ but not $(1, 0)$ by Poincaré index.

x_{\max} and x_{\min} occur when $\dot{x} = 0 \implies y = 0$. Hence $x_{\min} < a, 0 < x_{\max} < 1$.⁴ Therefore there are no periodic orbits if $a > 1$.

If $0 < \epsilon \ll 1$ then

$$\Delta H \approx \oint_{H=H_0} \epsilon g_2 dx = 2\epsilon \int_{x_{\min}}^{x_{\max}} (2H_0 - x^2 + \frac{2}{3}x^3)^{\frac{1}{2}} (a-x) dx$$

(This integral gas to be integrated numerically except for the special case $H_0 = \frac{1}{6}$, when $(\cdot)^{\frac{1}{2}} = (1-x)(1+\frac{2}{3}x)^{\frac{1}{2}}$ which corresponds to the homoclinic orbit.)

Setting $\Delta H(H_0, a) = 0$ gives a unique approximate periodic orbit $H = H^*(a)$ for $0 < a < \frac{1}{7}$ ($0 < H^* < \frac{1}{6}$).

The periodic orbit appears at $a = 0$ by a Hopf bifurcation and is destroyed at $a = \frac{1}{7}$ by a homoclinic bifurcation.

4.5 Stability of Periodic Orbits

Suppose $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ has a periodic orbit $\mathbf{x} = \mathbf{X}(t)$ with $\mathbf{X}(T) = \mathbf{X}(0) = \mathbf{X}_0$. Consider a small perturbation $\mathbf{x} = \mathbf{X}(t) + \boldsymbol{\eta}(t)$. Linearising,

$$\dot{\mathbf{X}} + \dot{\boldsymbol{\eta}} = \mathbf{f}(\mathbf{X}) + (\boldsymbol{\eta} \cdot \nabla) \mathbf{f}(\mathbf{X}) + O(|\boldsymbol{\eta}|^2) \implies \dot{\boldsymbol{\eta}} = (\boldsymbol{\eta} \cdot \nabla) \mathbf{f}(\mathbf{X}) + O(|\boldsymbol{\eta}|^2)$$

and so

$$\dot{\boldsymbol{\eta}} = A\boldsymbol{\eta} \quad \text{where } A_{ij}(t) = \left. \frac{\partial f_i}{\partial x_j} \right|_{\mathbf{X}(t)}$$

This is a linear ODE.

$$\implies \boldsymbol{\eta}(t) = \Phi(t)\boldsymbol{\eta}(0)$$

Since $\boldsymbol{\eta}(0)$ is arbitrary

$$\begin{aligned} \dot{\Phi}_{ij} &= A_{ik}\Phi_{kj} \\ \Phi_{ij}(0) &= \delta_{ij} \end{aligned}$$

But $A(t)$ is periodic with period T , and so

$$\boldsymbol{\eta}(nT) = \Phi(T)\boldsymbol{\eta}((n-1)T) = \dots (\Phi(T))^n\boldsymbol{\eta}(0)$$

Thus whether an arbitrary perturbation $\boldsymbol{\eta}(0)$ grows/decays after n times round the orbit, depends on the eigenvalues of $\Phi(T)$ compared to 1.

⁴Consider the sign of \dot{H} !

Definition: Floquet Multipliers

The Floquet multipliers of a periodic orbit are the eigenvalues λ_i of the matrix $\Phi(T)$ defined above.

One of the Floquet multipliers is always unity, since if $\eta(0) = \mathbf{f}(\mathbf{x}_0)\delta t$ then $\dot{\eta} = A\eta$ has solution, $\eta(t) = \mathbf{f}(\mathbf{X}(t))\delta t \implies \eta(T) = \mathbf{f}(\mathbf{X}(T))\delta t = \mathbf{f}(\mathbf{X}_0)\delta t$

Hence

- (i) A periodic orbit is asymptotically stable if all of the non-trivial Floquet multipliers satisfy $|\lambda| < 1$.
- (ii) A periodic orbit is Lyapunov unstable if any satisfy $|\lambda| > 1$ (and so asymptotically unstable).

Definition: Hyperbolicity

A periodic orbit is hyperbolic if all of the non-trivial eigenvalues satisfy $|\lambda| \neq 1$ and is non-hyperbolic if at least one lies on the unit circle $|\lambda| = 1$.

Hyperbolic periodic orbits (like hyperbolic fixed points) are structurally stable i.e. the orbit is unaffected by small perturbations to \mathbf{f} .

Stability in \mathbb{R}^2

In \mathbb{R}^2 we only have one non trivial λ to worry about. (Which is real and positive -cf lemma for the Poincaré-Bendixson theorem⁵)

$$\det(\Phi(T)) = \lambda_1 \lambda_2, \quad \lambda_1 = 1$$

Lemma:

$$\frac{d}{dt} \det \Phi = (\nabla \cdot \mathbf{f}) \det \Phi$$

Proof. ($n = 2$ for simplicity)

$$\det \Phi = \epsilon_{ij} \Phi_{1i} \Phi_{2j} \implies \frac{d}{dt} \det \Phi = \epsilon_{ij} [\dot{\Phi}_{1i} \Phi_{2j} + \Phi_{1i} \dot{\Phi}_{2j}] = \epsilon_{ij} [A_{1k} \Phi_{ki} \Phi_{2j} + \Phi_{1i} (A_{2k} \Phi_{kj})]$$

Now $\epsilon_{ij} \Phi_{2i} \Phi_{1j} = 0$

$$\implies \frac{d}{dt} \det \Phi = A_{11} (\epsilon_{ij} \Phi_{1i} \Phi_{2j}) + A_{22} (\epsilon_{ij} \Phi_{1i} \Phi_{2j}) = (\nabla \cdot \mathbf{f}) \det \Phi$$

□

Corollaries:

1. In \mathbb{R}^2 the non trivial Floquet multiplier is

$$\lambda_2 = \exp \left[\oint_0^T \nabla \cdot \mathbf{f} |_{\mathbf{X}(t)} dt \right]$$

Which is easy to calculate.

2. A periodic orbit is

$$\begin{array}{c} \text{stable} \\ \text{unstable} \\ \text{non-hyperbolic} \end{array} \left\{ \begin{array}{l} \text{if } \oint_0^T \nabla \cdot \mathbf{f} dt \text{ is} \end{array} \right\} \begin{array}{l} < 0 \\ > 0 \\ = 0 \end{array}$$

⁵Not sure what this lemma is; here's my explanation. If $\lambda_1 = 1$ then λ_2 must be real. In addition it must be positive as Jordan curve lemma means perturbed trajectory cannot cross the periodic orbit.

3. In higher dimensions $\oint_0^T \nabla \cdot \mathbf{f} dt < 0$ is necessary for stability, but not sufficient.

Example:

$$\begin{aligned}\dot{r} &= r(1 - r^2) \\ \dot{\theta} &= \frac{1}{r}\end{aligned}$$

This has a periodic orbit $r = 1, \theta = t, T = 2\pi$. Make a linear perturbation to $r = 1 + \delta, \theta = t + \epsilon$.

$$\begin{aligned}\Rightarrow \frac{\dot{\delta}}{\dot{\epsilon}} &= -2\delta \quad \Rightarrow \frac{\delta(t)}{\epsilon(t)} = \delta(0)e^{-2t} \\ &= -\delta \quad \epsilon(t) = \epsilon(0) + \delta(0)\frac{e^{-2t}-1}{2}\end{aligned}$$

Therefore

$$\Phi(T) = \begin{pmatrix} e^{-2T} & 0 \\ \frac{1}{2}(e^{-2T} - 1) & 1 \end{pmatrix}$$

Which has eigenvalues

$$\begin{array}{ll} e^{-4\pi} & \text{and} \\ \text{stable} & \text{trivial} \end{array}$$

More simply,

$$\begin{aligned}\nabla \cdot \mathbf{f} &= \frac{1}{r} \frac{\partial}{\partial r}(r\dot{r}) + \frac{1}{r} \frac{\partial}{\partial \theta}(r\dot{\theta}) = 2 - 4r^2 = -2 \text{ at } r = 1 \\ \Rightarrow \lambda &= \exp \left[\int_0^{2\pi} (-2)dt \right] = \exp(-4\pi)\end{aligned}$$

4.6 The Van der Pol Oscillator

$$\begin{aligned}\dot{x} &= y + -\mu(\frac{1}{3}x^3 - x) \\ \dot{y} &= -x\end{aligned} \quad \left. \begin{array}{l} \text{equivalent to } \ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0 \\ \end{array} \right\}$$

(Lienard variables)

The only fixed point is $\mathbf{x} = \mathbf{0}$ and is unstable. (Unstable focus for $0 < \mu < 2$, unstable node for $\mu > 2$)

$$H = \frac{1}{2}(x^2 + y^2) \Rightarrow \dot{H} = \mu x^2(1 - \frac{1}{3}x^2)$$

So if $0 < x^2 + y^2 < 3$ then $\dot{H} \geq 0$ and H increases monotonically along $\mathcal{O}^+(\mathbf{x})$ and eventually crosses $|\mathbf{x}|^2 = 3$ going outwards and cannot return.

(cf arguments in §3.3 and §3.4 for La Salle's invariance principle, bounding functions, and domain of stability of the time reversed problem $\mu \rightarrow -\mu$.)

So certainly “outwards” from $\mathbf{0}$, but is it “inwards” from ∞ ?

The question is complicated by the strip $|x| < \sqrt{3}$ where $\dot{H} \geq 0$ extends to ∞ . Consider part trajectories of ABC as shown for different $y(A)$

$$H(B) - H(A) = \int_A^B \frac{\dot{H}}{\dot{x}} dx = \int_A^B \frac{\mu x^2(1 - \frac{1}{3}x^2)}{y - \mu x(\frac{1}{3}x^2 - 1)} dx$$

Which is positive, but is a monotonically decreasing function of y .

$$H(C) - H(B) = \int_C^B \frac{\dot{H}}{\dot{y}} dy = \int_B^C \mu x(1 - \frac{1}{3}x^2) dy$$

is negative and increasingly so; as $y(B) - y(C)$ increases, there are larger values of $x(y)$.

Hence $H(C) - H(A)$ is a monotonic function of $y(A)$ with a unique root corresponding to a unique periodic orbit.

For $0 < \mu \ll 1$ we can use the energy balance method (§4.4) $H = \frac{1}{2}(x^2 + y^2)$, the unperturbed orbits ($\mu = 0$) are $H = H_0 = \text{const}$, $x = \sqrt{2H_0} \cos t$.

$$\begin{aligned}\Delta H(H_0) &= \oint_{r^2=2H_0} \mu x^2 \left(1 - \frac{1}{3}x^2\right) dt + O(\mu^2) \approx \mu \int_0^{2\pi} \left(2H_0 \cos^2 t - \frac{4H_0^2}{3} \cos^4 t\right) dt \\ &\quad 2\pi\mu(H_0 - \frac{1}{2}H_0^2)\end{aligned}$$

So the limit cycle is $H_0 = 2$ or $x^2 + y^2 = 4$.

Now consider $\mu \gg 1$. $\dot{x} = y - \mu x(\frac{1}{3}x^2 - 1)$ so $|\dot{x}| \gg 1$ except for $y \approx \mu x(\frac{1}{3}x^2 - 1)$.

$A \rightarrow B$ or $C \rightarrow D \sim \frac{\Delta y}{\dot{y}} = O(\mu)$ which is large.
Time taken: $B \rightarrow C$ or $D \rightarrow A \sim \frac{\Delta x}{x} = O(\frac{1}{\mu})$ which is small

The period is dominated by the time on the “Slow Manifold,” $y = \mu x(\frac{1}{3}x^2 - 1)$ from $A \rightarrow B$ and $C \rightarrow D$. Hence

$$T \approx 2 \int_A^B dt = 2 \int_A^B \frac{1}{\dot{y}} \frac{dy}{dx} dx = 2\mu \int_{-2}^{-1} \frac{1-x^2}{x} dx = \mu(3 - 2 \log 2)$$

5 Bifurcation of flows

5.1 Introduction

Consider a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mu)$ where μ is a continuously varying parameter.

A bifurcation is a change in the topology of the flow as μ passes through a critical point μ_0 , called the bifurcation point.

Example:

$$\begin{cases} \dot{r} = r(\mu - r^2) \\ \dot{\theta} = 1 \end{cases} \quad \begin{array}{l} \text{If } \mu \leq 0 \text{ there are no periodic orbit} \\ \text{If } \mu > 0 \text{ there is one periodic orbit} \end{array}$$

Example:

$$\begin{cases} \dot{x} = x(y - \mu) \\ \dot{y} = -y + x^2 \end{cases} \quad \begin{array}{l} \text{If } \mu \leq 0 \text{ there is one fixed point } (0, 0) \\ \text{If } \mu > 0 \text{ there are three fixed points } (0, 0), (\pm\sqrt{\mu}, \mu) \end{array}$$

In both of these cases, there is a bifurcation at $\mu = 0$, when the fixed point $\mathbf{x} = \mathbf{0}$ is non-hyperbolic. ($\lambda = \mu \pm i$ and $\lambda = -\mu$)

Bifurcations can also occur when there are non-hyperbolic periodic orbits and when there are homo/heteroclinic orbits between saddles, because all of these features of the flow are structurally unstable.

Consider again

$$\begin{aligned}\dot{x} &= x(y - \mu) \\ \dot{y} &= -y + x^2\end{aligned}$$

If $|\mathbf{x}| \ll 1$ and $|\mu| \ll 1$ then expect that x varies slowly (since \dot{x} is quadratic) and y releases rapidly (like e^{-t}) onto $y = x^2$.

If $y \approx x^2$ (say y slaved to x) then $\dot{x} \approx x(x^2 - \mu)$ which gives the bifurcation diagram. a “subcritical pitchfork bifurcation”

Local bifurcations are analysed by generalising this simple example.

5.2 Center Manifold and Extended Centre Manifold

Theorem: Centre Manifold Theorem

If $\mathbf{x} = \mathbf{0}$ is a non-hyperbolic fixed point of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with linear stable, unstable, and centre subspaces E^s , E^u , and E^c then there exist stable, unstable and centre manifolds W^s , W^u and W^c that have the same dimensions as E^s , E^u , and E^c , are tangent to them at $\mathbf{0}$, and are invariant.

W^c is not necessarily unique.

Example:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix}$$

Where $\lambda > 0$ and f, g are $O(|\mathbf{x}|^2)$.

Linearly: E^s is $x = 0$, E^c is $y = 0$.

The centre manifold is $y = h(x)$ with $h(0) = h'(0) = 0$. A local approximation can be found as a power series $h(x) = a_2x^2 + a_3x^3 + \dots$ and by solving

$$\dot{y}(x) = \frac{dh}{dx} \dot{x}$$

For $|\mathbf{x}| \ll 1$, $y - h(x)$ tends to zero rapidly like $e^{-\lambda t}$ and there is then slow evolution on the centre manifold given by $\dot{x} = f(x, h(x))$ which is easier to analyse.

(The non-uniqueness of W^c is because many trajectories are exponentially close to $y = h(x)$. For example, if the slow evolution gave

$$x(t) = \frac{1}{\sqrt{t}}, \quad y = h(x) + Ae^{-\lambda/x^2}$$

then this is indistinguishable from $y = h(x)$ in a power series.)

The above analysis applies at the bifurcation point $\mu = 0$, but if now $f = f(x, y; \mu)$ etc. then the fixed point will not be at $x = 0$ if $\frac{\partial f}{\partial \mu} \neq 0$ or $y = 0$ if $\frac{\partial g}{\partial \mu} \neq 0$ and it won't be non-hyperbolic if $\frac{\partial^2 f}{\partial x \partial \mu} \neq 0$. We can get round this by means of a trick. We regard μ as a variable and add $\dot{\mu} = 0$ to the system (wlog assume $\mu_0 = 0$).

The extended system is

$$\begin{pmatrix} \dot{x} \\ \dot{\mu} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & \frac{\partial f}{\partial \mu}|_{\mu=0} & 0 \\ 0 & 0 & 0 \\ 0 & \frac{\partial g}{\partial \mu}|_{\mu=0} & -\lambda \end{pmatrix} \begin{pmatrix} x \\ \mu \\ y \end{pmatrix} + \begin{pmatrix} f - \mu \frac{\partial f}{\partial \mu}|_{\mu=0} \\ 0 \\ g - \mu \frac{\partial g}{\partial \mu}|_{\mu=0} \end{pmatrix}$$

The extended centre subspace is the plane $y = \frac{1}{\lambda} \frac{\partial g}{\partial \mu} \mu$ (with E^s the line $x = \mu = 0$) and the extended centre manifold is a surface $y = h(x, \mu)$ with $h = \frac{\partial h}{\partial x} = 0$, $\frac{\partial h}{\partial \mu} = \frac{1}{\lambda} \frac{\partial g}{\partial \mu}$ at $x = \mu = 0$.

We can find this by solving

$$\dot{y} = \frac{\partial h}{\partial x} \dot{x} + \underbrace{\frac{\partial h}{\partial \mu} \dot{\mu}}_{=0}$$

as a power series in x and μ . The evolution equation on the centre manifold is $\dot{x} = f(x, h(x, \mu); \mu)$

Example:

$$\dot{x} = \mu + x^2 + xy + y^2, \quad \dot{y} = 2\mu - y + x^2 + xy$$

At $\mu = 0$ this has a non-hyperbolic fixed point at $\mathbf{x} = \mathbf{0}$. The extended system is

$$\begin{pmatrix} \dot{x} \\ \dot{\mu} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ \mu \\ y \end{pmatrix} + \text{non-linear terms}$$

So the extended centre subspace is the plane (in \mathbb{R}^3), $y = 2\mu$. An extended centre manifold is the surface

$$y = 2\mu + a_{20}x^2 + a_{11}x\mu + a_{02}\mu^2 + O(3)$$

From $\dot{y} = \frac{dy}{dx} \dot{x} + 0$ get that

$$2\mu - \{2\mu + a_{20}x^2 + a_{11}x\mu + a_{02}\mu^2 + O(3)\} + x^2 + x\{2\mu + O(2)\} = \{2a_{20}x + a_{11}\mu + O(2)\}\{\mu + O(2)\}$$

Equating coefficients of x^2 , $x\mu$, $\mu^2 \implies a_{20} = 1$, $a_{11} = 0$, $a_{02} = 0$

Therefore the extended center manifold is

$$y = 2\mu + x^2 + O(3)$$

and the evolution equation on the centre manifold is

$$\dot{x} = \mu + x^2 + x(2\mu + O(2)) + 4\mu^2 + O(3)$$

For $|\mathbf{x}|, |\mu| \ll 1$, this gives the bifurcation diagram from $\dot{x} = \mu + x^2 + O(x^3, x\mu, \mu^2)$

Note:

1. Leading terms suggested that $\mu = O(x^2)$ rather than $\mu = O(x)$.
2. Method generalises to higher dimensions, $\mathbf{y} = \mathbf{h}(\mathbf{x}; \boldsymbol{\mu})$ where $(\mathbf{0}, \mathbf{y}) \in E^s$, $(\mathbf{x}, \mathbf{0}) \in E^c$. ($E^u = \phi$)

$$\dot{y}_i = \frac{\partial h_i}{\partial x_j} \dot{x}_j$$

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{h}(\mathbf{x}; \boldsymbol{\mu}), \boldsymbol{\mu})$$

Example: The Lorentz equations

$$\begin{aligned}\dot{x} &= \sigma(y - x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= -bz + xy\end{aligned}$$

These have a fixed point at $\mathbf{0}$ where

$$A = \begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix}$$

When $r = 1$ this is non hyperbolic;

$$\lambda = 0, \quad e = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \lambda = -(\sigma + 1), \quad e = \begin{pmatrix} \sigma \\ -1 \\ 0 \end{pmatrix}, \quad \lambda = -b, \quad e = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

So the centre subspace E^c is the line $x = y, z = 0$ (E^s is the plane $x + \sigma y = 0$).

Change variables to $\mu = r - 1, v = y - x$. Then the extended system becomes

$$\begin{pmatrix} \dot{x} \\ \dot{\mu} \\ \dot{v} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \sigma & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -(1 + \sigma) & 0 \\ 0 & 0 & 0 & -b \end{pmatrix} \begin{pmatrix} x \\ \mu \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ x(\mu - z) \\ x(x + v) \end{pmatrix}$$

The extended centre subspace is $v = z = 0$.

The extended centre manifold is $v = V(x, \mu), z = Z(x, \mu)$ where

$$V = \frac{\partial V}{\partial x} = \frac{\partial V}{\partial \mu} = Z = \frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial \mu} = 0 \text{ at } x = \mu = 0$$

Solve

$$\dot{V} = \frac{\partial V}{\partial x} \dot{x}, \quad \dot{Z} = \frac{\partial Z}{\partial x} \dot{x}$$

Find that

$$\begin{aligned}Z &= \frac{x^2}{b} + O(3), \quad V = \frac{1}{\sigma + 1} \left(\mu x - \frac{x^3}{b} - \frac{\sigma \mu^2 x}{1 + \sigma} + O(4) \right) \\ \implies \dot{x} &= \frac{\sigma}{\sigma + 1} \left(\mu x - \frac{x^3}{b} + O(3) \right)\end{aligned}$$

Key idea: If $E^u = \phi$ then all of the variables in E^s collapse onto the centre manifold, and all of the interesting dynamics are given by the slow evolution on the centre manifold.

5.3 Stationary Bifurcations ($\lambda = 0$)

If one λ passes through 0 at $\mu = 0$, and all the others satisfy $R(\lambda) < 0$ then the dynamics reduce to $\dot{x} = f(x, \mu), x \in \mathbb{R}$. A Taylor expansion gives

$$\dot{x} = \nu_0(\mu) + \nu_1(\mu)x + \frac{1}{2}\nu_2(\mu)x^2 + \dots$$

where $\nu_0(0) = \nu_1(0) = 0$, so that $x = 0$ is a non-hyperbolic fixed point at $\mu = 0$.

Normal forms

The simple stationary bifurcations in this course can all be reduced to one of the following normal forms by a suitable change of variables.

- (i) Saddle node bifurcation $\dot{x} = \mu - x^2$
Two fixed points $x = \pm\sqrt{\mu}$ for $\mu > 0$. In \mathbb{R}^2 with $\dot{y} = -\lambda y$
- (ii) Transcritical bifurcation $\dot{x} = x(\mu - x)$
Fixed points at $x = 0, \mu$ which cross and exchange stability at $\mu = 0$.
- (iii) Pitchfork bifurcation $\dot{x} = x(\mu - ax^2)$ with $a = \pm 1$ Fixed points at $x = 0$ (stable for $\mu < 0$) and $x = \pm\sqrt{\frac{\mu}{a}}$ if $\frac{\mu}{a} > 0$ (stable for $a = +1$)
There are two cases:
 - Supercritical : New solutions exist after the old solution becomes unstable
 - Subcritical : Other solutions exist before the old solution becomes unstable

In \mathbb{R}^2

| Supercritical | | Subcritical | |
|---------------|-----------|---------------------------|-----------|
| $\mu < 0$ | $\mu > 0$ | Saddle/Stable node/Saddle | $\mu < 0$ |

Classification and Reduction to Normal Form

The sort of bifurcation depends on which terms in

$$f = \nu_0(\mu) + \nu_1(\mu)x + \frac{1}{2}\nu_2(\mu)x^2 + \dots$$

are most important as $x, \mu \rightarrow 0$.

Since $\nu_0(0) = \nu_1(0) = 0$ we generally expect $\nu_0, \nu_1 = O(\mu)$ and $\nu_i = O(1)$ for $i \geq 2$.

- (i) If $\nu'_0(0) \neq 0$ and $\nu_2(0) \neq 0$ (the generic case) then

$$\dot{x} \approx \nu_0 + \nu_1 x + \frac{1}{2}\nu_2 x^2$$

Rescale time $T = -\frac{\nu_2 t}{2}$ and change origin $X = x + \frac{\nu_1}{\nu_2} t$ to obtain

$$X_T = \underbrace{\left[\left(\frac{\nu_1}{\nu_2} \right)^2 - \frac{2\nu_0}{\nu_2} \right]}_{\mu} - X^2 + O(3)$$

The normal form for a saddle-node bifurcation.

- (ii) If $\nu_2(0) \neq 0$, but $\nu_0(\mu) \equiv 0$ because $x = 0$ is always a fixed point of the system, then

$$\dot{x} \approx \nu_1 x + \frac{1}{2}\nu_2 x^2$$

so $T = -\frac{\nu_2 t}{2}$ gives

$$X_T = \left(-\frac{2\nu_1}{\nu_2} \right) x - x^2$$

The normal form for a transcritical bifurcation. (Can also get the transcritical case if $\nu_0 = O(\mu^2)$, e.g. $\dot{x} = (k_1\mu - x)(k_2\mu - x)$ with $k_1 \neq k_2$)

- (iii) If $\nu_0(\mu) \equiv \nu_2(\mu) \equiv 0$ because there is a symmetry $f(-x, \mu) = f(x, \mu)$ then rescaling gives the pitchfork normal form. (Pitchforks also occur when $\nu_0 = O(\mu^2)$ and $\nu_2 = O(\mu)$ e.g. $\dot{x} = (x - k_1\mu)(\mu + k_2\mu x - x^2)$)

The saddle node bifurcation is the generic case and is structurally stable. The transcritical and pitchfork bifurcations are structurally unstable because they depend on some coefficients being negligible.

Example: $\dot{x} = \epsilon + \mu x - x^2$ with $0 < |\epsilon| \ll 1$, ϵ fixed.

Example: $\dot{x} = \epsilon + \mu x - x^3$

Example: $\dot{x} = \mu x + \epsilon x^2 - x^3$

If more than one λ passes through zero at the same time, get more complicated bifurcations. (Takens-Bogdanov)

5.4 Oscillatory/Hopf Bifurcations ($\lambda = \pm i\omega$)

If a complex conjugate pair of eigenvalues crosses $\text{Re}(\lambda) = 0$ at $\mu = 0$, with all the others in $\text{Re}(\lambda) < 0$, then the dynamics on the centre manifold reduce by much grusome algebra to the following normal form.

$$\boxed{\begin{aligned}\dot{r} &= r(\mu - ar^2) + O(r^5) \\ \dot{\theta} &= \omega + \mu c - br^2 + O(r^4)\end{aligned}}$$

This is the Hopf bifurcation. The origin is a fixed point with eigenvalues $\lambda = \mu \pm i(\omega + \mu c)$ so we have an unstable focus for $\mu > 0$ and a stable focus for $\mu < 0$. There is a limit cycle if $\mu/a > 0$ with radius $\sqrt{\mu/a}$ and frequency $\omega + \mu(c - b/a)$. Two cases arise

The main point of the algebra is to find the sign of a . An alternative is to use the Poincaré-Bendixson theorem (for example) to show that there is a limit cycle for $\mu > 0$ if it is supercritical.

5.5 **Bifurcation of Periodic Orbits**

There are various interesting possibilities.

Examples:

1. Saddle node bifurcation on the periodic orbit.
2. One or more of the floquet multipliers cross $|\lambda| = 1$
 - $\lambda = -1$, Period doubling bifurcation
 - $\lambda = 1$, Torus bifurcation
3. Homoclinic bifurcation. When a periodic orbit collides with a saddle point (see §4.4).

6 Fixed Points and Bifurcations of Maps

6.1 Introduction

We've seen maps arise from our analysis of flows in various ways. However, maps are worth studying in their own right. In general $\mathbf{x}_{n+1} = \mathbf{F}(\mathbf{x}_n)$ or $\mathbf{F}(\mathbf{x}_n; \mu)$. The following simple maps are much discussed as illustrative examples.

1D maps

- (i) The logistic map, $x_{n+1} = \mu x_n(1 - x_n)$ (often restricted to $0 \leq x_n \leq 1$, $0 \leq \mu \leq 4$.)
This is an example of a unimodal (one humped) map.

- (ii) The tent map

$$x_{n+1} = \begin{cases} \mu x_n & \text{for } 0 \leq x_n \leq \frac{1}{2} \\ \mu(1 - x_n) & \text{for } \frac{1}{2} \leq x_n \leq 1 \end{cases}$$

for $1 < \mu \leq 2$.

Circle maps

Represent a circle by \mathbb{R}/\mathbb{Z} (Reals mod 1)

- (i) Rotation map, $x_{n+1} = x_n + \omega \pmod{1}$ This is periodic if $\omega \in \mathbb{Q}$. It is aperiodic and covers $[0, 1]$ if $\omega \notin \mathbb{Q}$.
- (ii) Bernoulli shift map/sawtooth map, $x_{n+1} = 2x_n \pmod{1}$. This is equivalent to writing x_n in binary, $x_n = 0.a_1a_2a_3\dots$ with $a_i \in \{0, 1\}$ and shifting left; $x_{n+1} = 0.a_2a_3a_4\dots$
 - This converges to the fixed point $x = 0$ if $x_0 = \frac{p}{2^n}$
 - This converges to a periodic orbit if $x_0 = p/q \in \mathbb{Q}$ and $q \neq 2^n$
 - This is aperiodic if $x_0 \notin \mathbb{Q}$

6.2 Fixed Points, Cycles, and Stability

Recall (§1.4) a map $\mathbf{x}_{n+1} = \mathbf{F}(\mathbf{x}_n)$ has

- A fixed point \mathbf{x}_0 if $\mathbf{x}_0 = \mathbf{F}(\mathbf{x}_0)$
- An N-cycle $(\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_{N-1})$ if $\mathbf{F}(\mathbf{x}_0) = \mathbf{x}_1, \mathbf{F}(\mathbf{x}_1) = \mathbf{x}_2, \dots, \mathbf{F}(\mathbf{x}_{N-1}) = \mathbf{x}_0 = \mathbf{F}^N(\mathbf{x}_0)$ and $\mathbf{x}_i \neq \mathbf{x}_0$ for $i = 1, \dots, N-1$
- An invariant set A if $\mathbf{F}(A) = A$
- Forward orbits $\mathcal{O}^+(\mathbf{x}) = \{\mathbf{x}, \mathbf{F}(\mathbf{x}), \mathbf{F}^2(\mathbf{x}), \dots\}$

Definitions of Lyapunov, quasi-asymptotic, and asymptotic stability of invariant sets follow in the obvious way. A stable invariant set is called an attractor. (usually QAS, AS)

The stability of a fixed point $\mathbf{x}_0 = \mathbf{F}(\mathbf{x})$ is determined by the eigenvalues of the Jacobian

$$A_{ij} = \left. \frac{\partial F_i}{\partial x_j} \right|_{\mathbf{x}_0}$$

Linearising and setting $\mathbf{y} = \mathbf{x} - \mathbf{x}_0$, gives

$$\mathbf{y}_{n+1} = A\mathbf{y}_n + O(|\mathbf{y}_n|^2)$$

Hence the fixed point is

- (i) Asymptotically stable if all of the eigenvalues of A satisfy $|\lambda| < 1$
- (ii) Lyapunov (\implies asymptotically) unstable if any eigenvalue satisfies $|\lambda| > 1$
- (iii) **Definition:** Non-hyperbolic if any eigenvalue is on the unit circle $|\lambda| = 1$

Each point on an N-cycle $(\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_{n-1})$ is a fixed point of \mathbf{F}^n . So we can use the same theory. Let $A_{ij}^{(k)} = \left. \frac{\partial F_i}{\partial x_j} \right|_{\mathbf{x}_k}$, then $\mathbf{y}_{k+1} = A^{(k)} \mathbf{y}_k$ and $\mathbf{y}_N = A^{(N-1)} A^{(N-2)} \cdots A^{(0)} \mathbf{y}_0$. Hence the stability is determined by the eigenvalues of the product $A^{(N-1)} A^{(N-2)} \cdots A^{(0)}$.

We get bifurcations as one or more eigenvalues cross $|\lambda| = 1$.

6.3 Local Bifurcations in 1D

$$x_{n+1} = F(x_n; \mu), \quad x \in \mathbb{R}$$

In 1D there is only one eigenvalue $\lambda = \left. \frac{dF}{dx} \right|_{x_0}$, which is real, and bifurcations occur when $\lambda = +1$ or -1 . Wlog (change of origin) we can assume that the bifurcation occurs at $x_0 = 0$ when $\mu = 0$ ($\implies F(0, 0) = 0$).

$$\left. \frac{dF}{dx} \right|_{x_0} = \pm 1$$

6.3.1 The case $\lambda = 1$

There are subcases which are directly analogous to those for 1D flows.

$$F = \nu_0(\mu) + \nu_1(\mu)x + \frac{1}{2}\nu_2(\mu)x^2 + \dots$$

Generically, ν_0 and ν_1 are $O(\mu)$ and $\nu_i = O(1)$ for $i \geq 2$ which gives a saddle node bifurcations. However, there are special cases (transcritical and pitchfork when $\nu_0 = O(\mu^2)$) Using, for variety, the partial derivatives of F at $(0, 0)$ to state the conditions,

- (i) Saddle-Node. $F_\mu \neq 0, F_{xx} \neq 0 \implies$

$$x_{n+1} = x_n + \mu F_\mu + \frac{1}{2}x^2 F_{xx} + O(\mu^{3/2})$$

where $x \sim \mu^{1/2}$ There is a fixed point at $x = \left(\frac{-2\mu F_\mu}{F_{xx}} \right)^{\frac{1}{2}}$ if $\frac{-2\mu F_\mu}{F_{xx}} > 0$.

The normal form is $x_{n+1} = x_n + \mu - x_n^2$

- (ii) Transcritical. $F_\mu = 0, F_{xx} \neq 0 \implies$

$$x_{n+1} = x_n + \frac{1}{2}x^2 F_{xx} + x_n \mu F_{x\mu} + \frac{1}{2}\mu^2 F_{\mu\mu} + O(\mu^3)$$

where $x \sim \mu$. The fixed points are at $x^2 F_{xx} + 2x\mu F_{x\mu} + \mu^2 F_{\mu\mu} = 0$ which are real if $F_{x\mu}^2 > F_{xx} F_{\mu\mu}$ and are $x = k_1\mu, k_2\mu$ for

$$k_{1,2} = \frac{-F_{x\mu} \pm \sqrt{F_{x\mu}^2 - F_{xx} F_{\mu\mu}}}{F_{xx}}$$

The normal form is $x_{n+1} = x_n + x_n(\mu - x_n)$

(iii) Pitchfork. $F_\mu = F_{xx} = 0 \implies$

$$x_{n+1} = x_n + x_n \mu F_{x\mu} + \frac{1}{6} x_n^3 F_{xxx} + O(\mu^2)$$

where $x \sim \mu^{1/2}$. Fixed points at $x = 0$ and $\left(\frac{-6\mu F_{x\mu}}{F_{xxx}}\right)^{1/2}$ when $\frac{-6\mu F_{x\mu}}{F_{xxx}} > 0$

There are two normal forms.

$$\begin{aligned} x_{n+1} &= x_n + x_n (\mu - x_n^2) && \text{(Supercritical)} \\ x_{n+1} &= x_n + x_n (\mu + x_n^2) && \text{(Subcritical)} \end{aligned}$$

6.3.2 Case $\lambda = -1$

This gives a period doubling bifurcations.

(Semi-gruesome algebra) With the bifurcation at $x = 0, \mu = 0$

$$x_{n+1} = -x_n + \alpha + \beta x_n + \gamma x_n^2 + \delta x_n^3 + O(4)$$

where α, β are $O(\mu)$ and γ, δ are $O(1)$. Keep the fixed point at the origin by setting $X = x - \alpha/2$.

$$X_{n+1} = -X_n + (\beta + \gamma\alpha)X_n + \gamma X_n^2 + \delta X_n^3 + O(4)$$

counting $\mu = O(X^2), \alpha = O(X^2)$ set $\beta + \gamma\alpha = -\tilde{\mu}$ and look for a 2-cycle.

$$\begin{aligned} X_{n+2} &= -(1 + \tilde{\mu})X_{n+1} + \gamma X_{n+1}^2 + \delta X_{n+1}^3 + h.o.t. \\ &= (1 + \tilde{\mu})[(1 + \tilde{\mu})X_{n+1} - \gamma X_{n+1}^2 - \delta X_{n+1}^3] + \gamma[(1 + \tilde{\mu})X_{n+1} - \gamma X_{n+1}^2]^2 - \delta X_n^3 + h.o.t. \\ &= (1 + 2\tilde{\mu})X_n - 2(\gamma^2 + \delta)X_n^3 + O(4) \end{aligned}$$

Which is a pitchfork bifurcation for F^2 . The fixed point $X = 0$ is just the original fixed point of F .

The fixed points $\pm \left(\frac{\tilde{\mu}}{\gamma^2 + \delta}\right)^{\frac{1}{2}}$ (where $\tilde{\mu}/(\gamma^2 + \delta) > 0$) form a 2-cycle.

6.4 The Logistic Map

$$x_{n+1} = \mu x_n (1 - x_n)$$

Fixed points satisfy $x = \mu x(1 - x) \implies x = 0, 1 - 1/\mu$

$$F'(x) = \mu(1 - 2x)$$

- At $x = 0, F' = \mu$ so stable for $-1 < \mu < 1$
- At $x = 1 - 1/\mu, F' = 2 - \mu$ so stable for $1 < \mu < 3$

Hence there is a transcritical bifurcation at $\mu_0 = 1$, and a period doubling bifurcation at $\mu = 3$ (Also at $\mu = -1$)

The bifurcating 2-cycle satisfies $F^2(x) = x \implies$

$$\begin{aligned} \mu[\mu x(1 - x)][1 - \mu x(1 - x)] &= x \\ \implies x(\mu^3 x^3 - 2\mu^3 x^2 + (\mu^2 - \mu^3)x + (1 - \mu)) &= 0 \end{aligned}$$

But we know that x and $x - (1 - 1/\mu)$ are factors from the fixed points of F .

$$\implies x[\mu x - (\mu - 1)][\mu^2 x^2 - \mu(\mu + 1)x\mu + 1] = 0$$

$$\implies x_1, x_2 = \frac{1}{2\mu}[(1 + \mu) \pm \sqrt{(\mu - 3)(\mu + 1)}]$$

So the 2-cycle exists in $\mu > 3$. The stability of 2-cycles is found from

$$\begin{aligned}(F^2)' &= F'(x_1)F'(x_2) \\ &= \mu^2(1 - 2x_1)(1 - 2x_2) \\ &= \mu^2(1 - 2(x_1 + x_2) + 4x_1x_2) \\ &= \mu^2 \left[1 - 2 \left(\frac{\mu + 1}{\mu} \right) + 4 \left(\frac{\mu + 1}{\mu^2} \right) \right] \\ &= 4 + 2\mu - \mu^2\end{aligned}$$

Where $-1 < 4 + 2\mu - \mu^2 < 1$ for $1 + \sqrt{6} > \mu > 3$. So the 2-cycle is stable in this interval and itself becomes unstable by a period doubling bifurcation at $\mu_2 = 1 + \sqrt{6}$ to a stable 4-cycle. ($1 + \sqrt{6} = 3.4495$) The fixed point $x = 0$ period doubles at $\mu = -1$, and the 2-cycle itself period doubles at $\mu = 1 - \sqrt{6}$.

Analysis has led us thus far:

Numerical calculations show that period doubling continues, with 2^k -cycles appearing at μ_k with $\mu_3 = 3.544, \mu_4 = 3.564, \dots, \mu_k \rightarrow \mu_\infty = 3.5700$

$$\frac{\mu_k - \mu_{k-1}}{\mu_{k+1} - \mu_k} \rightarrow 4.6692 \quad \text{Feigenbaum's Constant}$$

For $\mu > \mu_\infty = 3.5700$ get chaos.

7 Chaos

7.1 introduction

What do we mean by chaos in a dynamical system. Two main ideas:

- (a) Arbitrarily close orbits separate from each other
- (b) There is some sort of mixing between even the smallest sets.

Thus even the tiniest of errors in the initial conditions, or in numerical integration of the flow/iteration of the map will eventually grow to such an extent that it is impossible to predict the future evolution of the system, even though the system is entirely deterministic.

Sometimes only a subspace of a state space is chaotic. A stable invariant set on which there is chaos is called a strange attractor. We formalise ideas (a) and (b) as follows.

Let F be a continuous map $I \rightarrow I$ where I is a bounded interval in \mathbb{R} . Let $\Lambda \subseteq I$ be an F invariant set.

Definition: F has sensitive dependence on initial conditions (SDIC) if $\exists \delta > 0$ s.t. for any $x \in \Lambda$, and any $\epsilon > 0$, $\exists n > 0$ and $y \in \Lambda$ s.t. $|x - y| < \epsilon$ and $|F^n(x) - F^n(y)| > \delta$

Definition: F is topologically transitive (TT) on Λ if for all open sets U, V s.t. $U \cap V \neq \emptyset$ and $U \cap \Lambda \neq \emptyset \exists n > 0$ s.t. $F^n(U) \cap V \neq \emptyset$.

TT means that some points from any neighbourhood of Λ , however small, will eventually move into

any other neighbourhood of Λ . This implies that Λ cannot be decomposed into disjoint invariant sets.

Example: The rotation map $x_{n+1} = x_n + \omega \pmod{1}$ with $\omega \notin \mathbb{Q}$ is TT on $[0, 1]$ but not SDIC.

Example: The map

$$x_{n+1} = \begin{cases} 2x_n & |x| < \frac{1}{2} \\ 2(\text{sign}(x_n) - x_n) & \frac{1}{2} < |x| < 1 \end{cases}$$

Has SDIC on $[-1, 1]$.

Does not have TT on $[-1, 1]$ as there is no mixing of $x > 0$ and $x < 0$.

Definition: Chaos (according to Devaney) F is chaotic on Λ if

- (i) F has SDIC on Λ
- (ii) F has TT on Λ
- (iii) Periodic points are dense in Λ

The following definitions look quite different, but are actually almost the same, and are easier to work with.

Definition: Horseshoe property

F has a horseshoe if there exists an open interval J and disjoint open subintervals K_0, K_1 such that $F(K_0) = J$ and $F(K_1) = J$.

Definition: Chaos (according to Glendinning)

F is chaotic if F^n has has a horseshoe for some n .

Notes:

- (1) We shall show in the following sections that Glendinning's Chaos \implies Devaney's Chaos.
- (2) Glendinning's chaos \implies neigbouring orbits diverge exponentially. (though Devaney's chaos & SDIC only guarantees that some neigbouring orbits diverge, and possibly only algebraically.)
- (3) Glendinning's definition shows why "Stretching and Folding" is good for mixing.

7.2 The sawtooth map (Bernoulli shift map)

$$x_{n+1} = 2x_n \pmod{1}$$

Clearly F has a horseshoe with $J = (0, 1)$, $K_0 = (0, \frac{1}{2})$, $K_1 = (\frac{1}{2}, 1)$ \implies chaos (Glendinning)

- (a) If x and y differ first in the $(n+1)^{th}$ place in their binary expansion then $|F^r(x) - F^r(y)| = 2^r|x - y|$ for $r \leq n$. So let $0 < \delta < \frac{1}{2}$. Given x and $\epsilon > 0$, choose n such that $2^{-n-1} < \epsilon$ and y to differ from x in only the $(n+1)^{th}$ place. Then $|F^n(x) - F^n(y)| = \frac{1}{2} > \delta$.
- (b) Given any $u = 0.u_1u_2\dots$, and $v = 0.v_1v_2\dots$, define

$$u_N = 0.u_1u_2\dots u_Nv_1v_2v_3\dots$$

Then $F^N(u_N) = v$ and can make u_N arbitrarily close to u . Therefore the sawtooth map is TT on $[0, 1]$.

(c) Given any $w = 0.w_1w_2w_3\dots$ define

$$w_N = 0.w_1w_2\dots w_Nw_1w_2\dots w_Nw_1\dots$$

(written $0.\dot{w}_1\dots\dot{w}_N$) which is a periodic orbit arbitrarily close to w . Hence periodic orbits are dense in $[0, 1]$.

(a), (b) & (c) \implies the sawtooth map is chaotic (Devany).

7.3 Horseshoes, Symbolic Dynamics, and the Shift Map

We aim to show that a map with a horseshoe acts on a certain invariant subset Λ in essentially the same way as the sawtooth map acts on \mathbb{R}/\mathbb{Z} .

Suppose a continuous map F has a horseshoe on the open interval J and define the closed intervals $I = \bar{J}$, $I_0 = \bar{K}_0$, $I_1 = \bar{K}_1$. Suppose for simplicity that the graph of F has the following shape.

F is monotonic on I_0 and I_1 , $I_0 \cap I_1 = \emptyset$. If $x, F(x) \in I$ then $x \in I_0 \cup I_1$. Define

$$\Lambda = \{x \in I : F^n(x) \in I \ \forall n \geq 0\}$$

Clearly $x \in \Lambda \implies F(x) \in \Lambda$, and $x \in \Lambda \stackrel{IVT}{\implies} \exists y \in \Lambda$ s.t. $F(y) = x$. Therefore Λ is invariant, i.e. $F(\Lambda) = \Lambda$.

For each $x \in \Lambda$, $F^i(x) \in \Lambda \implies F^{i-1}(x) \in I_0 \cup I_1$. Define

$$x_i = \begin{cases} 0 & \text{if } F^{i-1}(x) \in I_0 \\ 1 & \text{if } F^{i-1}(x) \in I_1 \end{cases} \quad (\text{for } i \geq 1)$$

If x corresponds to the symbol sequence $x_1x_2x_3\dots$ then $F(x)$ corresponds to the sequence $x_2x_3x_4\dots$ i.e. F acts on Λ in the same way as the shift map acts on the space $\{0, 1\}^{\mathbb{N}}$ of ∞ sequences of 0's and 1's. This is not quite the same as the sawtooth map because 0111̄ and 1000̄ are distinct symbolic sequences, but the same as binary numbers. However, it differs only on a countable set $\{p/2^n\}$ and does not affect the constructions in §7.2.

I.e. Glendinning's chaos \iff horseshoe \implies like the sawtooth map \implies Devany's chaos.

To construct Λ note that for any closed interval $A \subseteq I$, $F^{-1}(A) \equiv \{x \in I : F(x) \in A\}$ comprises two closed intervals namely $F^{-1}(A) \cap I_0$ and $F^{-1}(A) \cap I_1$. Therefore $F^{-n}(I)$ is 2^n closed intervals and $\Lambda = \bigcap_{n=0}^{\infty} F^{-n}(I)$.

cf Cantor middle third set

7.4 Period 3 implies Chaos

Recall the intermediate value theorem (IVT):

If $f : [a, b] \rightarrow \mathbb{R}$ is continuous and $f(a) = c, f(b) = d$ then $\forall y \in [c, d] \exists x \in [a, b]$ s.t. $f(x) = y$, with the corollary that if f changes sign on $[a, b]$ then $\exists x_0$ s.t. $f(x_0) = 0$.

Theorem: Period 3 implies chaos

If a continuous map F on $I \subseteq \mathbb{R}$ has a 3-cycle then F^2 has a horseshoe and so F is chaotic (Glendinning).

Proof. Let $x_1 < x_2 < x_3$ be the elements of the 3-cycle and wlog assume $F(x_1) = x_2$, $F(x_2) = x_3$ and $F(x_3) = x_1$. (If $x_1 \rightarrow x_3 \rightarrow x_2$ just consider $-F(-x)$). Then

$$\begin{aligned} F(x_2) &= x_3 > x_2 \\ F(x_3) &= x_2 < x_3 \\ \implies \exists z \in (x_2, x_3) \text{ with } F(z) &= z \end{aligned}$$

Also

$$\begin{aligned} F(x_1) &= x_2 < z \\ F(x_2) &= x_3 > z \\ \implies \exists y \in (x_1, x_2) \text{ with } F(y) &= z \end{aligned}$$

Now $F^2(y) = F^2(z) = z$ and $F^2(x_2) = x_1 < y$. Therefore

$$\begin{aligned} \exists \text{ smallest } r \in (y, x_2) \text{ s.t. } F^2(r) &= y \\ \exists \text{ largest } s \in (x_2, z) \text{ s.t. } F^2(s) &= y \end{aligned}$$

Therefore F^2 has a horseshoe with $J = (y, z)$, $K_0 = (y, r)$, $K_1 = (s, z)$. \square

7.5 The existence of N-cycles

Since a 3-cycle in F implies that there is a horseshoe in F^2 , we know that F^2 has cycles of all orders (like the sawtooth/shift map). In fact we can show directly that F itself has cycles of all orders. We use the result from IA that if F is a continuous function, and $V \subseteq F(U)$ where U and V are closed intervals, then \exists a closed interval $K \subseteq U$, with $F(K) = V$.

Theorem:

If a continuous map F on $I \subseteq \mathbb{R}$ has a 3-cycle then it has an N-cycle $\forall N \geq 1$.

Proof. As before, wlog $x_1 < x_2 < x_3$ with $F(x_i) = x_{i+1}$.

$$N = 1 \quad x_1 = F(x_3) < x_2 < x_3 < F(x_2) \stackrel{IVT}{\implies} \exists \text{ a fixed point in } (x_2, x_3).$$

$$N \geq 2 \quad \text{Let } I_0 = [x_1, x_2], I_1 = [x_2, x_3]. \text{ by IVT, } F(I_0) \supseteq I_1, F(I_1) \supseteq I_0 \cup I_1$$

Let $J_N = I_1$, use the IA result repeatedly to construct $J_{N-1} \subseteq I_0, J_{N-2}, J_{N-3}, \dots, J_0 \subseteq I_1$ with $F(J_i) = J_{i+1}$ for $0 \leq i \leq N-1$

$$F^N(J_0) = I_1 \implies \exists a, b \in J_0 \text{ with } F^N(a) = x_2, F^N(b) = x_3. \text{ Now } J_0 \subseteq I_1 \implies$$

$$x_2 = F^N(a) \leq a, b \leq F^N(b) = x_3$$

Hence $\exists z \in [a, b]$ with $F^N(z) = z$

We are not done yet, we need to check that z has least period N . If $F^{N-1}(z) = x_2 \in I_0 \cap I_1$, then $F^N(z) = x_3$. But if

- $N = 2$, $F^2(x_3) = x_2 \neq x_3 \#$
- $N \neq 2$, then $F(x_3) = x_1 \notin I_1 \#$

so $J_1 \subseteq I_1$. Hence $F^{N-1}(z) \neq x_2$ and is in I_0 by construction.

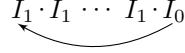
$$\implies F^{N-1}(z) \notin I_1$$

$\implies \mathcal{O}(z)$ leaves I_1 exactly once. Therefore this is an N cycle. \square

This kind of construction can be represented graphically. We write $F(I_0) \supseteq I_1$, $F(I_1) \supseteq I_0 \cup I_1$ as the directed graph.



On which the N-cycle follows the allowed closed path.

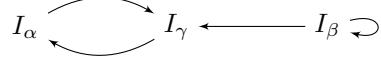


Example: If there is a 4-cycle with

$$x_1 = F(x_4) < x_3 = F(x_2) < x_2 = F(x_1) < x_4 = F(x_3)$$

Let $I_\alpha = [x_1, x_3]$, $I_\beta = [x_3, x_2]$, $I_\gamma = [x_2, x_4]$. Then

$$F(I_\alpha) \supseteq I_\gamma \quad F(I_\beta) \supseteq I_\beta \cup I_\gamma \quad F(I_\gamma) \supseteq I_\alpha$$



There is a fixed point in I_β , a 2-cycle oscillating between I_α and I_γ , but no obvious way to find a 3-cycle or 5-cycle etc. These sorts of arguments can be used to prove the remarkable result.

Theorem: Sharkovsky

If F is a continuous map $\mathbb{R} \rightarrow \mathbb{R}$, F has an N -cycle and $N \triangleright n$ with respect to the following ordering, then F has an n -cycle.

$$\begin{aligned} & 3 \triangleright 5 \triangleright 7 \triangleright 9 \triangleright \dots \\ & \triangleright 2 \cdot 3 \triangleright 2 \cdot 5 \triangleright 2 \cdot 7 \triangleright 2 \cdot 9 \triangleright \dots \\ & \triangleright 2^2 \cdot 3 \triangleright 2^2 \cdot 5 \triangleright 2^2 \cdot 7 \triangleright 2^2 \cdot 9 \triangleright \dots \\ & \vdots \\ & \triangleright 2^i \cdot 3 \triangleright 2^i \cdot 5 \triangleright 2^i \cdot 7 \triangleright \dots \\ & \vdots \\ & \triangleright 2^i \triangleright 2^{i-1} \triangleright \dots \triangleright 4 \triangleright 2 \triangleright 1 \end{aligned}$$

Proof: Not in course.

E.g.

- 3-cycles \implies cycles of all order (already proved)
- 4-cycles \implies 2 cycles & fixed point regardless of order x_1, \dots, x_4

7.6 The Tent Map

$$x_{n+1} = \begin{cases} \mu x_n & \text{for } 0 \leq x_n \leq \frac{1}{2} \\ \mu(1 - x_n) & \text{for } \frac{1}{2} \leq x_n \leq 1 \end{cases}$$

This only maps $[0, 1]$ into itself if $0 \leq \mu \leq 2$. There is always a fixed point at $x = 0$, which is stable for $\mu < 1$ (dull) and unstable for $\mu > 1$.

For $1 < \mu \leq 2$ there is another fixed point where $x_0 - \mu(1 - x_0) \implies x_0 = \frac{\mu}{1 + \mu} > \frac{1}{2}$ which is always unstable (because $F'(x_0) = -\mu < -1$).

The interesting range is $1 < \mu \leq 2$ for which neither fixed points is an attractor, and the orbits are bounded. We will show that F is chaotic throughout this range.

We proceed to establish a horseshoe for $\mu > 1$ in several steps:

1. Every orbit (apart from $x = 0$) eventually enters and remains in the interval

$$A = [F^2(1/2), F(1/2)] = [\mu(1 - \mu/2), \mu/2]$$

Proof. Note: since x_0 is unstable, orbits can only reach the neighbourhood of $x_0 = \frac{\mu}{1+\mu}$ if its preimage $x_{-1} = \frac{1}{1+\mu} \in A \implies \frac{1}{1+\mu} > \mu(1 - \frac{\mu}{2}) \implies \mu^2 > 2$ \square

2. Consider

$$F^2(x) = \begin{cases} \mu^2 x & 0 \leq x \leq 1/2\mu \\ \mu(1 - \mu x) & 1/2\mu \leq x \leq 1/2 \\ \mu(1 - \mu(1 - x)) & 1/2 \leq x \leq 1 - 1/2\mu \\ \mu^2(1 - x) & 1 - 1/2\mu \leq x \leq 1 \end{cases}$$

with $F(1/2\mu) = F(1 - 1/2\mu) = 1/2$.

Let x_{-2} be the preimage of x_{-1} in $x > 1/2$. $F^2(x_{-2}) = F(x_{-1}) = x_0$. Therefore

$$\mu(1 - x_{-2}) = \frac{1}{\mu + 1} \implies x_{-2} = 1 - \frac{1}{\mu(1 + \mu)}$$

3. Change of coordinates (renormalisation). F^2 acts on the intervals $J_L = [x_{-1}, x_0]$ and $J_R = [x_0, x_{-2}]$ like a tent map with parameter μ^2 . There are two cases:

- $\mu^2 \geq 2$.

$$F^2(1/2) = \mu(1 - \frac{\mu}{2}) < x_{-1} = \frac{1}{1 + \mu}$$

$$F(1/2) = \frac{\mu}{2} > x_{-2} = 1 - \frac{1}{\mu(1 + \mu)}$$

F^2 has horseshoes on both J_L and J_R . F is chaotic and the chaotic attractor is the whole of $A = [F^2(1/2), F(1/2)]$.

- $\mu^2 < 2$

F^2 maps both of J_L and J_R into themselves, and there are attracting subintervals

$$A_L = [F^2(1/2), F^4(1/2)] \subset J_L$$

$$A_R = [F^3(1/2), F(1/2)] \subset J_R$$

with $F(A_R) = A_L$, $F(A_L) \subseteq A_R$, $x_0 \notin A_L \cup A_R$

4. **Theorem:**

If $2^{\frac{1}{2n+1}} \leq \mu < 2^{\frac{1}{2n}}$ then $F^{2^{n+1}}$ has a horseshoe, F is chaotic and the chaotic attractor consists of 2^n intervals, which are permuted by F .

Proof. Apply arguments 1-3 inductively \square

7.7 The Logistic Map

$$x_{n+1} = \mu x_n (1 - x_n)$$

Find numerically that there exists a period doubling cascade in $3 < \mu < \mu_\infty = 3.5700\cdots$ with $\mu_\infty - \mu_k \sim A\delta^{-k}$ as $k \rightarrow \infty$ with $\delta = 4.6692016\cdots$. There is chaos for $\mu_\infty < \mu \leq 4$. In this range there are “windows” of stable N-cycles e.g. P3 in $1 + \sqrt{8} = 3.8284 < \mu < 3.8415$

There exists period-doubling cascades of the N-cycles, e.g. P3→P6→P12 etc. accumalating at $\mu_\infty^{(3)} = 3.8396$ with $\mu_\infty^{(3)} - \mu_k \sim B\delta^{-k}$ and the same δ . In fact, get the same δ for all maps with a quadratic maximum. Why?

(Note: “P3 \implies Chaos” so the chaotic sets are still there just not attracting.)