

# Dynamical Systems

Course given by Prof. J.Lister  
L<sup>A</sup>T<sub>E</sub>X by Dominic Skinner  
Dom-Skinner@github.com

January 16, 2016

## Introduction

A dynamical system is a set of equations describing the evolution of a system with respect to a time-like variable. Usually they are non-linear.

The possible states of the system define the state space/phase space.

**Example:** The logistic map

$$x_{n+1} = \mu x_n(1 - x_n)$$

For  $0 \leq \mu \leq 4$  this describes evolution with respect to a discrete time  $n$  in a state space  $[0, 1]$ .

**Example:** The Lotka-Volterra equations

$$\begin{aligned}\dot{r} &= r(a - br - cs) \\ \dot{s} &= s(d - er - fs)\end{aligned}$$

Where  $a-f$  are positive constants. These describe continuous evolution in a state space  $(r, s) \in [0, \infty] \times [0, \infty]$  as a model for the population of two species competing for the same food supply.

**Example:** The non-linear Schrödinger equation

$$i \frac{\partial \Psi}{\partial t} = \nabla^2 \Psi + |\Psi|^2 \Psi$$

This describes evolution in an infinite dimensional statespace of possible wavefunctions.

Because the equations are non-linear, it is often impossible to find a complete set of closed form analytic solutions. Instead, we resort to a mixture of geometric and analytic arguments, and aim to say something about the generic long-term behaviour.

**Example:**

$$\begin{aligned}\dot{r} &= r(3 - r - s) \\ \dot{s} &= s(2 - r - s)\end{aligned}$$

Consider the regions where  $\dot{r}$  and  $\dot{s}$  are  $> 0$ ,  $< 0$ ,  $= 0$ .

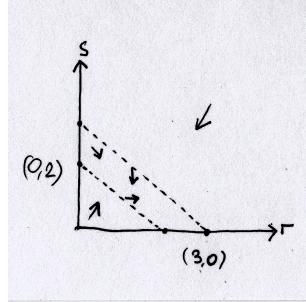
If  $r, s > 0$  then

$$\begin{aligned}r + s < 2 &\implies \dot{r}, \dot{s} > 0 \\ 2 < r + s < 3 &\implies \dot{r} > 0, \dot{s} < 0 \\ r + s > 3 &\implies \dot{r}, \dot{s} < 0\end{aligned}$$

$\dot{r} = 0$  if  $r = 0$  or  $r + s = 3$ .

$\dot{s} = 0$  if  $s = 0$  or  $r + s = 2$ .

Therefore  $\dot{r} = \dot{s} = 0$  at the fixed points  $(0, 0)$ ,  $(3, 0)$ ,  $(0, 2)$ . This gives the phase portrait/diagram/plane<sup>1</sup>



The most important feature of the phase portrait is that all solutions with  $r > 0$  tend to the stable fixed point  $(3, 0)$ . The fixed points  $(0, 0)$  and  $(0, 2)$  are unstable. There are no periodic orbits.

**Example:**

$$\begin{aligned}\dot{r} &= r(3 - r - s) \\ \dot{s} &= s(2 - \mu r - s)\end{aligned}$$

In this case, a new fixed point  $(\frac{1}{1-\mu}, \frac{2-3\mu}{1-\mu})$  appears in the state space at  $\mu = \frac{2}{3}$  and for  $\mu < \frac{2}{3}$  is the long term stable attractor.

A qualitative change in the solution structure is called a bifurcation.

**Example:**

$$\begin{aligned}\dot{x} &= -y + \epsilon x(\mu - x^2 - y^2) \\ \dot{y} &= x + \epsilon y(\mu - x^2 - y^2)\end{aligned}$$

Use polar coordinates which are a more natural choice for this problem. In general:

$$\boxed{\begin{aligned}\dot{r} &= \frac{x\dot{x} + y\dot{y}}{r} \\ \dot{\theta} &= \frac{x\dot{y} - y\dot{x}}{r^2}\end{aligned}}$$

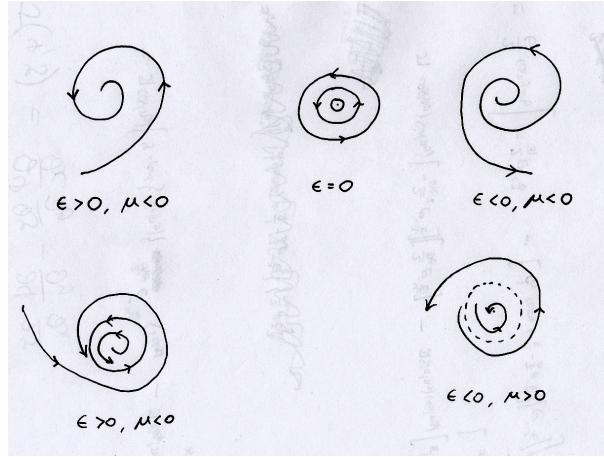
Which are equations that will be referred to frequently.<sup>2</sup> In our example, they become

$$\begin{aligned}\dot{r} &= \epsilon r(\mu - r^2) \\ \dot{\theta} &= 1\end{aligned}$$

Consider  $\dot{r}$  and  $\dot{\theta}$ :

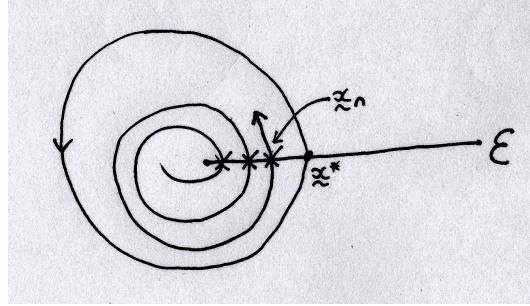
---

<sup>1</sup>Phase portrait, phase diagram, phase plane will be used interchangably  
<sup>2</sup>so learn them now!



The infinite set of periodic solutions for  $\epsilon = 0$  is destroyed by any perturbation to  $\epsilon \neq 0$ . This is an example of structural instability. If  $\mu > 0$  then just one limit cycle survives and is stable (unstable) for  $\epsilon > 0$  ( $\epsilon < 0$ ). The appearance of the limit cycle as  $\mu \uparrow$  through 0 is another form of bifurcation.

**Example:** In 2D the points of successive intersection  $x_n$  of a solution near a limit cycle with a line  $\varepsilon$  perpendicular to the cycle, move monotonically towards/away from the point of intersection  $x^*$  of  $\varepsilon$  with the limit cycle.



The point  $x^*$  is a stable/unstable fixed point of this Poincaré recurrence map. In 3 or higher dimensions, or in 2D with time-dependent coefficients there is room for much more complicated behaviour including chaos.

## 1 Basic Definitions

We need some terminology.

### 1.1 Notation

We only consider ODEs of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \quad (*)$$

for  $\mathbf{x}$  in a phase space/state space  $E \subset \mathbb{R}^n$ . The  $n$  first order ODEs form a dynamical system of order (dimension)  $n$ .

Since

$$\frac{\partial \mathbf{f}}{\partial t} = 0$$

we call the system autonomous.

A non-autonomous system  $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$  can be made autonomous by setting

$$\mathbf{y} = (\mathbf{x}, t) \quad \text{with} \quad \dot{\mathbf{y}} = (\mathbf{f}(\mathbf{y}), 1)$$

The  $n^{th}$  order ODE

$$\frac{d^n x}{dt^n} = g\left(x, \frac{dx}{dt}, \dots, \frac{d^{n-1}x}{dt^{n-1}}\right)$$

can be put in the form (\*) by setting

$$\mathbf{y} = \left(x, \frac{dx}{dt}, \dots, \frac{d^{n-1}x}{dt^{n-1}}\right)$$

with

$$\dot{\mathbf{y}} = (y_2, y_3, \dots, g(\mathbf{y}))$$

Similarly we will consider maps in the form

$$\mathbf{x}_{n+1} = \mathbf{F}(\mathbf{x}_n)$$

## 1.2 Initial Value Problem

Consider the IVP:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

$$\mathbf{x}(t_0) = \mathbf{x}_0$$

In Analysis II we showed that if  $\mathbf{f}$  satisfies a Lipschitz condition then we are guaranteed that a solution exists in a neighbourhood of  $\mathbf{x}_0$ ,  $t_0$  and is unique.

(Lipschitz condition is that  $\exists L, a$ , s.t.  $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| < L|\mathbf{x} - \mathbf{y}|$ ,  $\forall |\mathbf{x} - \mathbf{x}_0|, |\mathbf{y} - \mathbf{x}_0| < a$ .)

Moreover, solutions  $\mathbf{x}(t; \mathbf{x}')$  to

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

$$\mathbf{x}(t_0) = \mathbf{x}'$$

exist, are unique and they depend continuously on  $\mathbf{x}'$ ,  $t$ .

Note we are not guaranteed existence for all times.

**Example:**

$$\dot{x} = x^2$$

$$x(0) = 1$$

This has solution

$$x(t) = \frac{1}{1-t}$$

and so  $x \rightarrow \infty$  as  $t \uparrow 1$ .

If  $|\mathbf{x}(t)| \rightarrow \infty$  as  $t \rightarrow T (< \infty)$  then we call this finite time blow up.

**Example:** Non-uniqueness when  $\mathbf{f}$  is non-Lipschitz. If

$$\dot{x} = \begin{cases} \sqrt{x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases} \quad \text{and } x(0) = 0$$

Then there is a family of solutions

$$\left. \begin{array}{ll} x = 0 & t < \tau \\ x = \frac{1}{4}(t - \tau)^2 & t > \tau \end{array} \right\} \text{ for any } \tau \geq 0$$

From now on we will assume that  $\mathbf{f}$  is differentiable (and so Lipschitz).

### 1.3 Trajectories and Flows

Consider the autonomous system  $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ . The solution  $\mathbf{x}(t)$  to the IVP with  $\mathbf{x}(0) = \mathbf{x}_0$  defines a trajectory. (orbit/integral curve)

The distance travelled along this curve clearly only depends on  $t - t_0$ , and we could consider lots of starting points  $\mathbf{x}_0$ .

This motivates the idea of a flow.

**Definition:** (Flow)

Given  $\mathbf{f}$ , the corresponding flow is defined to be a (the) function  $\phi_t(\mathbf{x})$  from  $E \times \mathbb{R} \rightarrow E$  such that

$$\frac{\partial}{\partial t} \phi_t(\mathbf{x}) = \mathbf{f}(\phi_t(\mathbf{x})), \quad \phi_0(\mathbf{x}) = \mathbf{x}$$

The solution to the IVP  $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ ,  $\mathbf{x}(t_0) = \mathbf{x}_0$ , is just  $\mathbf{x}(t) = \phi_{t-t_0}(\mathbf{x}_0)$

Clearly

$$\phi_{s+t}(\mathbf{x}) = \phi_s(\phi_t(\mathbf{x})) = \phi_t(\phi_s(\mathbf{x}))$$

Aside: We can establish another link with maps, by defining  $\mathbf{x}_{n+1} = \mathbf{F}(\mathbf{x}_n) = \phi_{\Delta t}(\mathbf{x}_n)$ .  
 Clearly  $\phi_{n\Delta t}(\mathbf{x}) = \mathbf{F}(\phi_{(n-1)\Delta t}(\mathbf{x})) = \mathbf{F}^n(\mathbf{x})$

### 1.4 Flows, Trajectories, Orbits, Invariant Sets, & Limiting Sets

Using the idea of a flow, we define the following:

**Definition:** Orbit/Trajectories

The orbit of  $\phi_t(\mathbf{x})$  through  $\mathbf{x}_0$  is the set  $\mathcal{O}(\mathbf{x}_0) = \{\phi_t(\mathbf{x}_0) : -\infty < t < \infty\}$ .

The forwards (backwards) orbit is

$$\mathcal{O}^{+(-)}(\mathbf{x}_0) = \{\phi_t(\mathbf{x}_0) : t \geq 0 (t \leq 0)\}$$

**Definition:** Invariant sets

A set of points  $\Lambda \subset E$  is invariant if  $\mathbf{x} \in \Lambda \implies \mathcal{O}(\mathbf{x}) \subset \Lambda$

Clearly  $\mathcal{O}(\mathbf{x}_0)$  is invariant, and so is any union of orbits.

Particular cases of interest are

**Definition:** Fixed points

$\mathbf{x}_0$  is a periodic point with period  $T$  if  $\phi_T(\mathbf{x}_0) = \mathbf{x}_0$  for some  $T > 0$  and  $\phi_t(\mathbf{x}_0) \neq \mathbf{x}_0$  for  $0 < t < T$ .  
 The set  $\{\phi_t(\mathbf{x}_0) : 0 \leq t < T\}$  is the periodic orbit through  $\mathbf{x}_0$ .

**Definition:** Limit Cycle

A limit cycle is an isolated periodic orbit, i.e. there are no other periodic orbits within a sufficiently

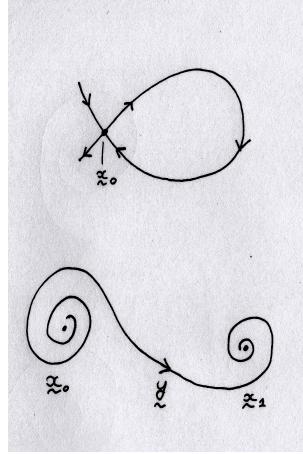
small neighbourhood.

**Definition:** Homoclinic and Heteroclinic orbits

If  $x_0$  is a fixed point and  $\exists \mathbf{y} \neq x_0$  such that  $\phi_t(\mathbf{y}) \rightarrow x_0$  as  $t \rightarrow \pm\infty$  then  $\mathcal{O}(\mathbf{y})$  is a homoclinic orbit.

If  $x_0, x_1$  are fixed points and  $\exists \mathbf{y} \neq x_0, x_1$  with  $\phi_t(\mathbf{y}) \rightarrow x_0$  as  $t \rightarrow -\infty$ ,  $\phi_t(\mathbf{y}) \rightarrow x_1$  as  $t \rightarrow +\infty$  then  $\mathcal{O}(\mathbf{y})$  is a heteroclinic orbit.

Example of a Heteroclinic and a Homoclinic orbit



If we are interested in the long-term behaviour of trajectories, it is not enough simply to think of

$$\lim_{t \rightarrow \infty} \phi_t(\mathbf{x})$$

because the limit might not exist; for example a limit cycle.

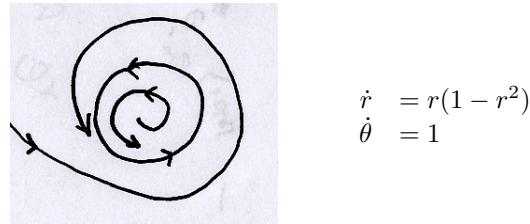
**Definition:** Limit set

The  $\omega$ -limit set of  $\mathbf{x}$  is

$$\omega(\mathbf{x}) = \{\mathbf{y} : \exists \text{ a sequence } (t_n) \text{ with } \phi_{t_n}(\mathbf{x}) \rightarrow \mathbf{y} \text{ and } t_n \rightarrow \infty\}$$

Similarly the  $\alpha$  limit set is defined by sequences with  $t_n \rightarrow -\infty$ .

**Example:**



For  $\mathbf{x}$  with  $0 < |\mathbf{x}| < 1$  we have that

$$\omega(\mathbf{x}) = \{\mathbf{y} : |\mathbf{y}| = 1\} \quad \alpha(\mathbf{x}) = \{\mathbf{0}\}$$

For  $|\mathbf{x}| > 1$

$$\omega(\mathbf{x}) = \{\mathbf{y} : |\mathbf{y}| = 1\} \quad \alpha(\mathbf{x}) = \{\}$$

In fact, the limit sets are always invariant sets since

- If  $\mathcal{O}(\mathbf{x})$  is bounded then  $\omega(\mathbf{x})$  is non-empty.
- If  $\phi_t(\mathbf{x}) \rightarrow \infty$  then  $\omega(\mathbf{x}) = \{\}$ .

**Definition:** (For maps)

Fixed points solve  $\mathbf{F}(\mathbf{x}) = \mathbf{x}$ . Orbits, periodic points etc. are defined as above by replacing  $\phi_t(\mathbf{x})$  by  $\mathbf{F}^n(\mathbf{x})$ , etc.

A periodic orbit with period N is called an N-cycle.

## 1.5 Topological equivalence and structural stability of flows

This section is in a handout.

## 2 Flow near fixed points

The simplest features of a flow are the fixed points. Find them by first solving  $\mathbf{F}(\mathbf{x}) = \mathbf{0}$ .

### 2.1 Linearisation

If  $\mathbf{f}$  is sufficiently smooth and  $\mathbf{x}_0$  is a fixed point, we expand in a Taylor series with  $\mathbf{f}(\mathbf{x}_0) = \mathbf{0}$  to obtain

$$\dot{\mathbf{y}} = A\mathbf{y} + O(|\mathbf{y}|^2)$$

where  $\mathbf{y} = \mathbf{x} - \mathbf{x}_0$  and

$$A_{ij} = \left. \frac{\partial f_i}{\partial x_j} \right|_{\mathbf{x}_0}$$

is the Jacobian matrix (Written  $D\mathbf{f}$  in Glendinning).

The hope (see below when it is true) is that the flow  $\phi_t^f$  is like the flow corresponding to the linearization  $\dot{\mathbf{y}} = A\mathbf{y}$ .

### 2.2 Classification of fixed points

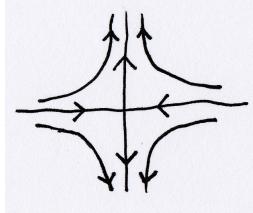
In 2D,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with  $D = ad - bc$ ,  $T = a + d$  and the eigenvalues are

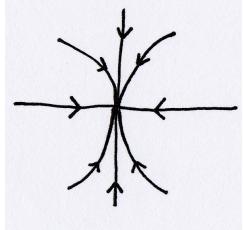
$$\lambda_{1,2} = \frac{1}{2}(T \pm \sqrt{T^2 - 4D})$$

(i) Saddle points ( $D < 0$ , real  $\lambda$ 's with opposite signs)



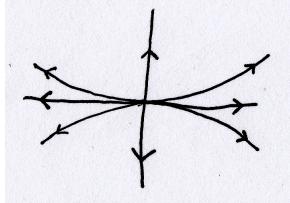
$$A = \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} \quad \lambda_1 < 0 < \lambda_2$$

(ii) Stable nodes ( $0 < 4D < T^2$ ,  $T < 0$ , real roots, both negative)



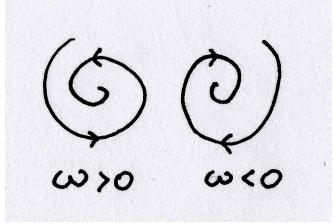
$$\lambda_1 < \lambda_2 < 0 \quad \frac{y_2}{y_1} \propto e^{(\lambda_2 - \lambda_1)t} \rightarrow \infty \text{ as } t \rightarrow \infty$$

(iii) Unstable node ( $0 < 4D < T^2$ ,  $T > 0$ , real roots, both positive)



$$\lambda_1 < \lambda_2 < 0 \quad \frac{y_2}{y_1} \propto e^{(\lambda_2 - \lambda_1)t} \rightarrow 0 \text{ as } t \rightarrow -\infty$$

(iv) Stable focus ( $T^2 < 4D$ ,  $T < 0$ , complex roots,  $\lambda = \rho \pm i\omega$ ,  $\rho < 0$ )



$$A = \begin{pmatrix} \rho & -\omega \\ \omega & \rho \end{pmatrix} \quad \dot{r} = \rho r \quad \dot{\theta} = \omega$$

(v) Unstable focus ( $T^2 < 4D$ ,  $T > 0$ , complex roots,  $\rho > 0$ )

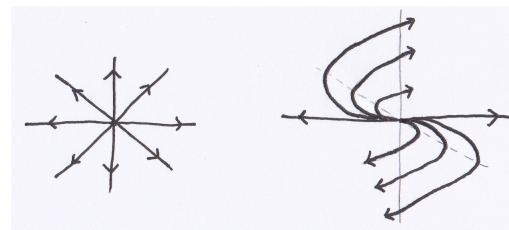
(vi) Two degenerate cases with equal eigenvalues on the border  $T^2 = 4D$  between nodes and foci.  
(e.g.  $\lambda > 0$ )

Stellar Node

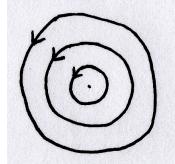
$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

Improper Node

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

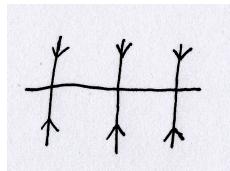


(vii) Centres ( $T = 0, D > 0, \lambda = \pm i\omega$ )

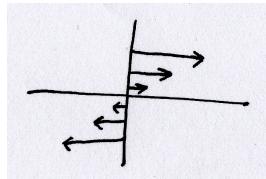


All trajectories are closed. This is on the border between stable foci and unstable foci.

(viii) Line of Fixed points. Special case on the border  $D = 0$  between saddles and nodes, where at least one  $\lambda$  is zero. For example,

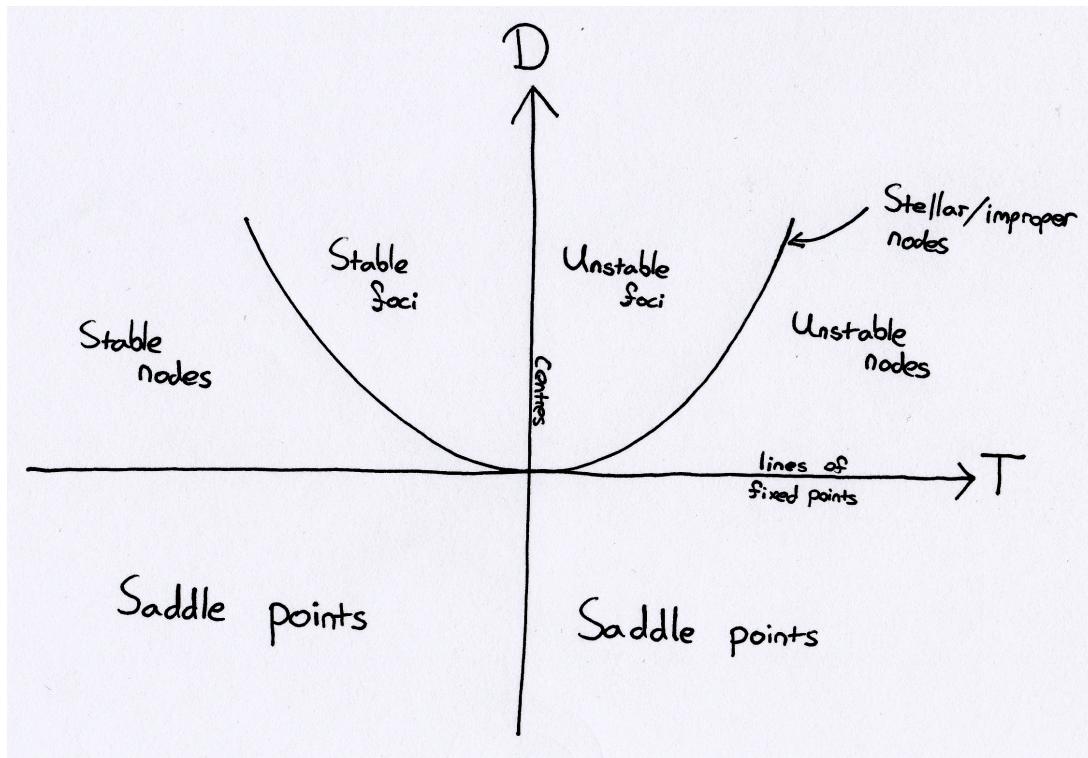


$$A = \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix} \quad (\lambda < 0)$$



$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

In summary



Notes:

(1) A Hamiltonian system is of the form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \partial H / \partial y \\ -\partial H / \partial x \end{pmatrix}$$

for some  $H(x, y)$ . Fixed points are always saddles or centers because

$$A = \begin{pmatrix} H_{xy} & H_{yy} \\ -H_{xx} & -H_{xy} \end{pmatrix} \implies T = 0$$

Also  $\dot{H} = \dot{\mathbf{x}} \cdot \nabla H = 0$  so the trajectories are (parts of) the contours of  $H$ .

- (2) The canonical forms are obtained by using the eigenvectors  $e_i$  if  $\lambda_i \in \mathbb{R}$  (possibly “generalised” for equal  $\lambda$ 's and JNF, see Glendinning p 63) and  $\text{Re}(e_1)$  and  $\text{Im}(e_1)$  for  $\lambda = \rho \pm i\omega$ .
- (3) It is not necessary to change basis to classify the fixed point since  $T$ ,  $D$ ,  $\lambda$ 's are invariant under  $A \mapsto P^{-1}AP$ , but knowing the eigenvectors may help sketch the phase diagram for the case of a saddle point.

#### Perturbation to A (linear perturbations)

Cases (i)-(v) are robust - a small perturbation to A gives the same sort of eigenvalues and hence the same sort of fixed point.

Case (vi), if perturbed, may become a node or a focus, but these are topologically equivalent. The stability is not changed.

Cases (vii) and (viii) are fragile, - a small perturbation to (vii), ( $\lambda = \pm i\omega$ ) may destroy the closed trajectories to give slow spiralling inwards or outwards. A small perturbation to (viii) ( $\lambda = 0$ ) may destroy the line of fixed points to give a slow drift towards/away from the fixed point, and hence a saddle or node.

The fragility is because there are one or two eigenvalues on the border  $\text{Re}(\lambda) = 0$  between stability and instability.

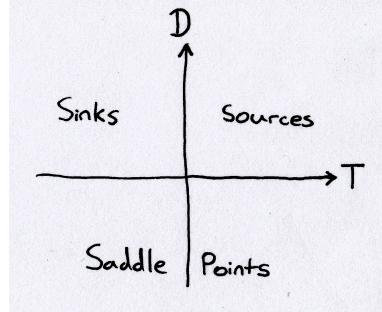
**Definition:** Hyperbolic fixed point

A fixed point  $\mathbf{x}_0$  of the system  $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$  is a hyperbolic fixed point if none of the eigenvalues  $\lambda$  of the Jacobian  $\left. \frac{\partial f_i}{\partial x_j} \right|_{\mathbf{x}=0}$  satisfy  $\text{Re}(\lambda) = 0$ , and is non-hyperbolic otherwise.

In n-dimensions ( $n \geq 0$ ) we classify a hyperbolic fixed point as

- (i) a sink if all eigenvalues have  $\text{Re}(\lambda) < 0$ .
- (ii) a source if all eigenvalues have  $\text{Re}(\lambda) > 0$ .
- (iii) a saddle point otherwise (some  $> 0$ , some  $< 0$ )

In 2D



Non hyperbolic fixed points are important in bifurcation theory.

### 2.3 The effects of non-linear terms

It is in fact true that the linearised system  $\dot{\mathbf{y}} = A\mathbf{y}$  is essentially the same as the non linear system  $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$  near a fixed point  $\mathbf{x}_0$  provided

- (i) The fixed point is hyperbolic
- (ii) The non linear terms are  $O(|\mathbf{x} - \mathbf{x}_0|^2)$

(If (i) holds then the systems are topologically equivalent, if (ii) also holds, then  $\mathbf{h}(\mathbf{x})$  is a near identity map, so nodes are nodes and foci are foci.)

#### 2.3.1 Stable and Unstable Manifold

First, we formalise the idea of stable and unstable directions in the linearised system.

**Definition:** The stable, unstable and center invariant subspaces of the linearisation of  $\mathbf{f}$  at a fixed point are the local linear subspaces.

$E^s$ ,  $E^u$ ,  $E^c$  spanned by the eigenvectors of the Jacobian, whose eigenvalues have real parts  $< 0$ ,  $> 0$ ,  $= 0$  respectively. (Generalised eigenvectors for JNF)

For some types of fixed point, these spaces may be empty, e.g. for a hyperbolic fixed point,  $E^c$  is empty by definition.

Then we extend the linear picture into the non-linear domain for a hyperbolic fixed point.

#### Stable Manifold Theorem

If  $\mathbf{0}$  is a hyperbolic fixed point of the system  $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ , with linear stable and unstable invariant subspaces  $E^s$  and  $E^u$  then in a sufficiently small neighbourhood of the origin, there exist local stable and unstable manifolds

$$\begin{aligned} W_{loc}^s(\mathbf{0}) &= \{\mathbf{x} : \phi_t(\mathbf{x}) \rightarrow \mathbf{0} \text{ as } t \rightarrow \infty\} \\ W_{loc}^u(\mathbf{0}) &= \{\mathbf{x} : \phi_t(\mathbf{x}) \rightarrow \mathbf{0} \text{ as } t \rightarrow -\infty\} \end{aligned}$$

these have the same dimension as  $E^s$  and  $E^u$  and are tangent to them at  $\mathbf{0}$ .

Notes:

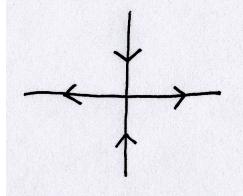
- (1) For a saddle point in  $\mathbb{R}^2$ , this guarantees the existence of two specific trajectories (the separatrices) that approach and leave the saddle.

- (2) For a sink this guarantees that all trajectories in some neighbourhood of the sink tend to it
- (3) The local stable (unstable) manifolds can be extended to global stable (unstable) invariant manifolds  $W^s$  ( $W^u$ ) by following the flow backwards (forwards).

The proof of this is not in the course - See Glendinning.

It is easy to calculate approximations to  $W^s$  and  $W^u$  for a saddle point in  $\mathbb{R}^2$ .

Wlog (Change of origin and basis) assume that the saddle is at  $\mathbf{x} = \mathbf{0}$  and that  $E^s$  is  $x = 0$  and  $E^u$  is  $y = 0$ .



Then  $W_{loc}^s$  becomes  $x = S(y)$  with  $S(0) = 0$ ,  $S'(0) = 0$   
 $W_{loc}^u$  is  $y = U(x)$  with  $U(0) = 0$ ,  $U'(0) = 0$ .

Since  $W^s$  and  $W^u$  are invariant,

$$\dot{x}|_{(S,y)} = \frac{dS}{dy} \dot{y}|_{(S,y)} \quad \dot{y}|_{(x,U)} = \frac{dU}{dx} \dot{x}|_{(x,U)}$$

**Example:**

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x - xy \\ -y + x^2 \end{pmatrix} \quad A|_{\mathbf{x}=\mathbf{0}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Assume that

$$y = U(x) = a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots$$

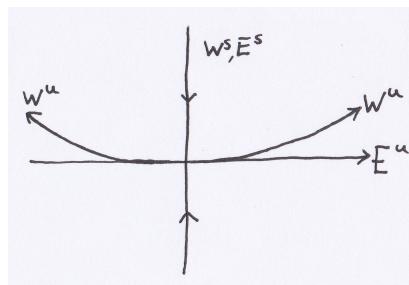
substitute into

$$\dot{y} = U' \dot{x} \implies -(a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots) + x^2 = (2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots)(x - a_2 x^3 - \dots)$$

Equate coefficients:

$$\begin{aligned} a_2 &= \frac{1}{3} \\ a_3 &= 0 \\ a_4 &= \frac{2}{45} \quad \text{etc.} \end{aligned}$$

Locally for  $|\mathbf{x}| \ll 1$



In higher dimensions, would write  $\mathbf{y} = \mathbf{U}(\mathbf{x})$  and solve

$$\dot{y}_i = \frac{\partial U_i}{\partial x_j} \dot{x}_j$$

Where the  $x_j$ 's span  $E^u$  and the  $y_i$ 's span  $E^s$ .

### 2.3.2 Non-Linear terms in non-hyperbolic cases

There are many possibilities depending on the non-linear terms.

- (a)  $\lambda = \pm i\omega$  Linear system is a centre. Are the trajectories really closed?

**Example:**

$$\begin{aligned} \dot{x} &= -y \pm x^3 \\ \dot{y} &= x \pm y^3 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \dot{r} &= \pm \frac{x^4 + y^4}{r} \\ \dot{\theta} &= 1 \pm \frac{xy^3 - yx^3}{r^2} \end{aligned}$$

Now as  $r \rightarrow 0$ ,  $\dot{\theta} \rightarrow 1$ . Thus the system is a stable focus if we choose '+' and an unstable focus if we choose '-'.

**Example:**

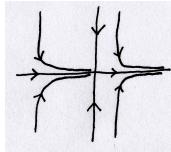
$$\begin{aligned} \dot{x} &= -y - 2yx^2 = -\frac{\partial H}{\partial y} \\ \dot{y} &= x + 2xy^2 = -\frac{\partial H}{\partial x} \end{aligned}$$

for  $H = -\frac{x^2+y^2}{2} - x^2y^2$ . The trajectories are contours of  $H$ , which are closed near the maximum at  $\mathbf{0}$ . So the fixed point really is a non-linear centre with nested periodic orbits.

- (b)  $\lambda_1 = 0, \lambda_2 \neq 0$ . Which way does the system drift along  $e_1$ ?

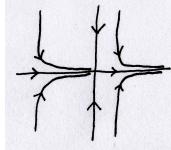
**Example:**

$$\begin{aligned} \dot{x} &= x^2 \\ \dot{y} &= -y \end{aligned}$$



Saddle-node

$$\begin{aligned} \dot{x} &= x^3 \\ \dot{y} &= -y \end{aligned}$$

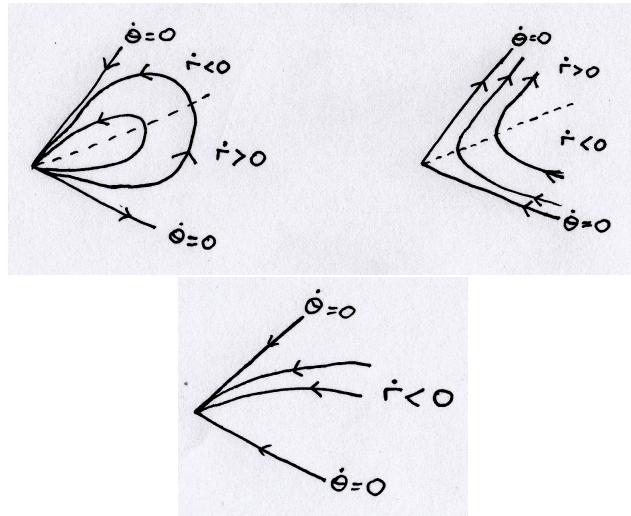


Non-linear saddle

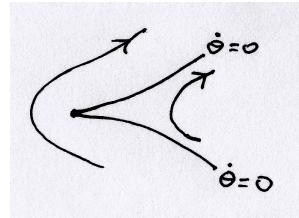
- (c)  $\lambda_1 = \lambda_2 = 0$ . Ad hoc methods needed.

In some cases, it is useful to switch to polars and consider the sign of  $\dot{r}$  and  $\dot{\theta}$  as  $r \rightarrow 0$ .

**Example:**  $\dot{\theta} > 0$



In nastier cases, the curves  $\dot{\theta} = 0$  can form a cusp.



## 2.4 Sketching phase planes/portraits

**Example:**

$$\begin{aligned}\dot{x} &= x(1-y) \\ \dot{y} &= -y + x^2\end{aligned}$$

This has fixed points at  $(0,0)$  and at  $(\pm 1, 1)$ .

$$(0,0) \quad A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{saddle point}$$

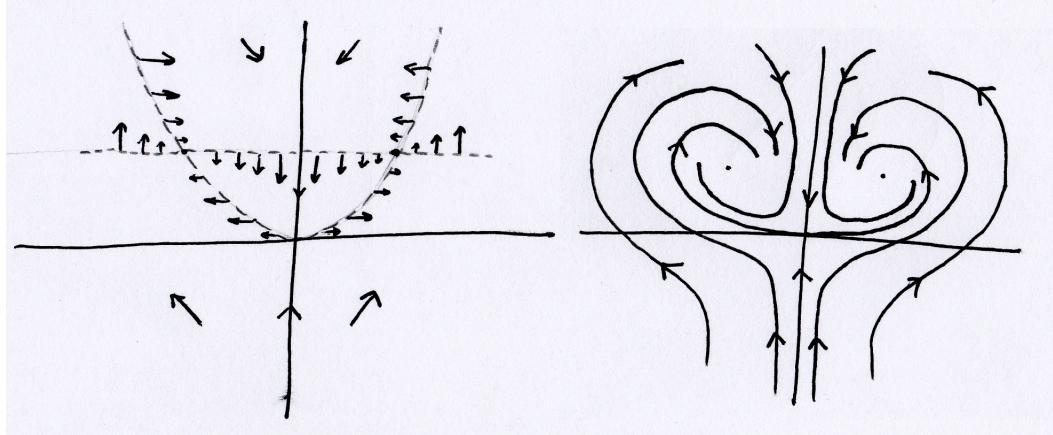
$$(\pm 1, 1) \quad A = \begin{pmatrix} 0 & \mp 1 \\ \pm 2 & -1 \end{pmatrix} \quad \text{stable foci}$$

It is clear why  $(0,0)$  is a saddle point. For  $(\pm 1, 1)$  observe that  $T = -1$ ,  $D = 2$ ,  $T^2 < 4D$ , and therefore  $(\pm 1, 1)$  are stable foci. Before sketching, observe that.

- $x = 0$  is a trajectory.
- $\dot{x} = 0$  on  $y = 1$
- $\dot{y} = 0$  on  $y = x^2$
- $\dot{y} \leq 0$  if  $y \leq x^2$

- Symmetry in y axis; under  $x \mapsto -x$ ,  $\dot{x} \mapsto -\dot{x}$ , and  $\dot{y} \mapsto \dot{y}$
- Know that the unstable manifold is given by  $y \approx \frac{1}{3}x^2 + \frac{2}{45}x^4$  near  $x = 0$  (See earlier example)

Put all of this together to sketch the following:



In general, here are some steps to follow when sketching a phase plane.

1. Find the fixed points.
2. (a) Calculate the Jacobian to find the kind of eigenvalues and classify the fixed points.  
(b) If any are non-hyperbolic, consider non-linear terms, possible Hamiltonian structure, signs of  $\dot{x}$ ,  $\dot{y}$ ,  $\dot{r}$  or  $\dot{\theta}$  as  $|\mathbf{x} - \mathbf{x}_0| \rightarrow 0$ .
3. Calculate the eigenvectors for saddles.
4. Consider the sign of  $\dot{x}$  and  $\dot{y}$  - the curves where these are zero are called nullclines.
5. Construct the global picture by joining up the local behaviour near fixed points. (especially the separatrices of saddles) using the arrows from  $\dot{x}$  and  $\dot{y}$ .
6. At some stage, decide whether there are periodic orbits.

### 3 Stability

#### 3.1 Definitions

It is clear what it means for a hyperbolic node or focus to be stable, but we ned to be more careful with the stability of other kinds of fixed points or invariand sets, because there are at least two notions of stability.

**Definition:** Lyapunov stability

A fixed point  $\mathbf{x}_0$  of a flow  $\phi_t$  is Lyapunov stable if  $\forall \epsilon > 0$ ,  $\exists \delta > 0$  s.t.  $|\mathbf{x} - \mathbf{x}_0| < \delta \implies |\phi_t(\mathbf{x}) - \mathbf{x}_0| < \epsilon \forall t > 0$ . A motto is “Start near, stay near” stability.

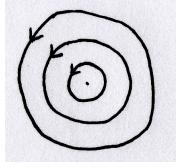
**Definition:** Quasi-asymptotic stability (QAS)

A fixed point  $\mathbf{x}_0$  of a flow  $\phi_t$  is quasi-asymptotically stable if  $\exists \delta > 0$  s.t.  $|\mathbf{x} - \mathbf{x}_0| < \delta \implies \phi_t(\mathbf{x}) \rightarrow \mathbf{x}_0$  as  $t \rightarrow \infty$ .

These are not the same!

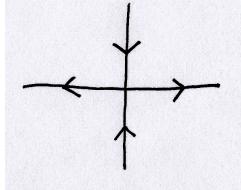
**Example:**

$$\begin{aligned}\dot{r} &= 0 \\ \dot{\theta} &= 1\end{aligned}$$



Lyapunov stable, but not asymptotically stable.

$$\begin{aligned}\dot{r} &= r(1 - r^2) \\ \dot{\theta} &= \sin^2(\theta/2)\end{aligned}$$



The point  $r = 1, \theta = 0$  is QAS but not Lyapunov stable

**Definition:** Asymptotic stability (AS)

A fixed point  $\mathbf{x}_0$  is asymptotically stable if it is both Lyapunov stable and quasi-asymptotically stable.

**Example:** A sink ( $\text{Re } \lambda_i < 0 \ \forall i$ ) is asymptotically stable. (Choose  $\delta$  sufficiently small that the linear terms dominate) A saddle or source is obviously not Lyapunov stable, hence not AS.

For other kinds of invariant sets  $\Lambda$  e.g. limit cycles, define

$$N_\delta(\Lambda) = \{\mathbf{x} : \exists \mathbf{y} \in \Lambda \text{ with } |\mathbf{x} - \mathbf{y}| < \delta\}$$

and say

$$\phi_t(\mathbf{x}) \rightarrow \Lambda$$

if

$$\min_{\mathbf{y} \in \Lambda} \{|\phi_t(\mathbf{x}) - \mathbf{y}|\} \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

Then say  $\Lambda$  is Lyapunov stable if  $\forall \epsilon > 0 \ \exists \delta > 0$  s.t.  $\mathbf{x} \in N_\delta(\Lambda) \implies \phi_t(\mathbf{x}) \in N_\epsilon(\Lambda) \ \forall t > 0$ .

Say  $\Lambda$  is QAS if  $\exists \delta > 0$  s.t.  $\mathbf{x} \in N_\delta(\Lambda) \implies \phi_t(\mathbf{x}) \rightarrow \Lambda$  as  $t \rightarrow \infty$ .

Say  $\Lambda$  is AS if it is both LS and QAS.

### 3.2 Lyapunov functions

These allow us to say more about the stability of a fixed point which wlog we take to be  $\mathbf{x} = 0$ .

**Definition:** Lyapunov function

A continuously differentiable function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  is a Lyapunov function for  $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$  in a domain  $\mathcal{D} \subset \mathbb{R}^n$  if

- (i)  $V(\mathbf{0}) = 0$  and  $V(\mathbf{x}) > 0$  for  $\mathbf{x} \in \mathcal{D} \setminus \{\mathbf{0}\}$ . “Positive semi-definite”.
- (ii)  $\dot{V}(\mathbf{x}) \equiv \mathbf{f} \cdot \nabla V \leq 0 \ \forall \mathbf{x} \in \mathcal{D}$ . “Non-increasing”.

Note that (ii) means trajectories head “downhill” (or at least not uphill) and (i) means that  $\mathbf{0}$  is the lowest point. These properties allow us to prove:

**Theorem:** Lyapunov's First Theorem

If a Lyapunov function exists, then  $\mathbf{0}$  is Lyapunov stable.

*Proof.* Wlog we can assume  $\epsilon$  is sufficiently small that  $\{|\mathbf{x}| < \epsilon\} \subset \mathcal{D}$ . Let

$$m = \inf\{V(\mathbf{x}) : |\mathbf{x}| = \epsilon\}$$

$m$  is attained, and  $m > 0$  since  $\{|\mathbf{x}| = \epsilon\}$  is compact. Define

$$C_{m,\epsilon} = \{\mathbf{x} : V(\mathbf{x}) < m \text{ and } |\mathbf{x}| < \epsilon\}$$

Then  $\mathbf{x} \in C_{m,\epsilon} \implies \phi_t(\mathbf{x}) \in C_{m,\epsilon} \forall t > 0$  since  $V$  is non-increasing and  $V \geq m$  on the boundary of  $C_{m,\epsilon}$ .

Finally, choose  $\delta > 0$  such that  $\{|\mathbf{x}| < \delta\} \subset C_{m,\epsilon}$ . Then

$$|\mathbf{x}| < \delta \implies |\phi_t(\mathbf{x})| < \epsilon \quad \forall t > 0$$

□

A trajectory can head downhill and still not end up at  $\mathbf{0}$ . However, there is an important constraint.

**Theorem:** La Sale's Invariance Principle

If  $V$  is a Lyapunov function on a bounded domain  $\mathcal{D}$  and  $\mathcal{O}^+(\mathbf{x}) \subset \mathcal{D}$ , then  $\phi_t(\mathbf{x}) \rightarrow$  an invariant subset of  $\{\mathbf{x} : \dot{V}(\mathbf{x}) = 0\}$ .

*Proof.*

$\mathcal{O}^+(\mathbf{x}) \subset \mathcal{D}$  so  $V(\phi_t(\mathbf{x}))$  is monotonically decreasing and bounded below by 0  $\implies V(\phi_t(\mathbf{x})) \rightarrow \alpha$  for some  $\alpha \geq 0$  as  $t \rightarrow \infty$ .

$\mathcal{D}$  is bounded  $\implies \omega(\mathbf{x})$  is non empty.

Let  $\mathbf{y} \in \omega(\mathbf{x})$  then  $\exists \{t_n\}$  s.t.  $\phi_{t_n}(\mathbf{x}) \rightarrow \mathbf{y} \implies V(\mathbf{y}) = \alpha$  by continuity of  $V$ .

Therefore  $V(\phi_t(\mathbf{y})) = \alpha$  since  $\phi_t(\mathbf{y}) \in \omega(\mathbf{x})$ .

Now  $V(\phi_t(\mathbf{y})) = \alpha \quad \forall t \implies \dot{V}(\mathbf{y}) = 0$ .

This holds for any  $\mathbf{y} \in \omega(\mathbf{x})$  and thus

$$\omega(\mathbf{x}) \subset \{\dot{V} = 0\}$$

We know  $\omega$  is invariant, and thus we are done. □

**Corollary:** If  $V$  is a Lyapunov function on a domain  $\mathcal{D}$  containing  $\mathbf{0}$ , and the only invariant subset of  $\{\mathbf{x} : \dot{V}(\mathbf{x}) = 0\} \cap \mathcal{D}$  is  $\{\mathbf{0}\}$ , then  $\mathbf{x} = \mathbf{0}$  is asymptotically stable.

*Proof.*

Choose  $k$  such that

$$C_k \equiv \{\mathbf{x} : V(\mathbf{x}) < k\} \subset \mathcal{D}$$

and choose  $\delta$  such that  $\{\mathbf{x} : |\mathbf{x}| < \delta\} \subset C_k$ .

Then  $|\mathbf{x}| < \delta \implies V(\mathbf{x}) < k \implies V(\phi_t(\mathbf{x})) < k \implies \phi_t(\mathbf{x}) \subset \mathcal{D}$  □

**Example:** Let

$$\dot{\mathbf{x}} = A\mathbf{x} + o(|\mathbf{x}|)$$

and let  $\lambda_i$  be the distinct eigenvalues of  $A$  with corresponding eigenvectors  $\mathbf{e}_i$ . Suppose that  $\operatorname{Re}(\lambda_i) < 0$ , and let  $v_i$  be any positive real constants. Then, writing

$$\mathbf{x} = \sum_{i=1}^n a_i \mathbf{e}_i, \quad V(\mathbf{x}) = \sum_{i=1}^n v_i |a_i|^2$$

have that  $V \geq 0$  ( $= 0$  only at  $\mathbf{0}$ ) and  $\dot{V}$  in a sufficiently small neighbourhood of  $\mathbf{0}$ . (See example sheet) Therefore have that  $\mathbf{x} = \mathbf{0}$  is asymptotically stable.

The main use of Lyapunov functions, however, is to find information about the domain of stability.

**Definition:** Domain of stability

The domain of stability (basin of attraction) of an asymptotically stable invariant set  $\Lambda$  is

$$\{\mathbf{x} : \phi_t(\mathbf{x}) \rightarrow \Lambda \text{ as } t \rightarrow \infty\}$$

i.e.  $\mathbf{x}$  such that

$$\inf_{\mathbf{y} \in \Lambda} \{|\phi_t(\mathbf{x}) - \mathbf{y}|\} \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

If the domain of stability in  $\mathbb{R}^n$  then  $\Lambda$  is globally stable.

**Example:** We can show that the non-hyperbolic fixed point of the system

$$\begin{aligned}\dot{x} &= y - xy^2 \\ \dot{y} &= x^3\end{aligned}$$

is asymptotically stable.

Without the  $-xy^2$  term, the system would be Hamiltonian with  $H = \frac{1}{2}y^2 + \frac{1}{4}x^4$  and  $\dot{H} = 0$ .

With the  $-xy^2$  term,  $\dot{H} = -x^4y^2 \leq 0$ .

Therefore  $H$  is a Lyapunov function and so  $\phi_t(\mathbf{x}) \rightarrow$  an invariant subset of  $\{\dot{H} = 0\} = \{x = 0 \text{ or } y = 0\}$ .

But the only invariant subset of  $\{\dot{H} = 0\}$  is  $\mathbf{x} = \mathbf{0}$ . Hence by the corollary,  $\mathbf{x} = \mathbf{0}$ .

**Example:**

$$\begin{aligned}\dot{x} &= y + \mu(\frac{1}{3}x^3 - x) \\ \dot{y} &= -x\end{aligned}\left.\right\} \text{ equivalent to } \ddot{x} + \mu(1 - x^2)\dot{x} + x = 0$$

The “Time-reversed van der Pol oscillator”