Asymptotic Methods

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Contents

Books: Bender and Orszag, "*Advanced Mathematical methods for scientists and engineers*", Chapters 3,6,10

More details can be found on the Moodle course site; login to Moodle at https://www.vle.cam.ac.uk/login/index.php, then self-enrol into the Asymptotic methods course.

What we'll learn in this course

Examples:

1. $I(\lambda) = \int_{\infty}^{\infty} \exp[-\lambda \cosh u] du$

   We expect that $I(\lambda) \to 0$ as $\lambda \to \infty$. But how fast?

2. $i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V(x)$ with $\psi(x,t) \in \mathbb{C}$, $V = V(x)$.

   Look for a solution $\psi(x,t) = \exp\left[\frac{-iEt}{\hbar}\right] f(x) \implies \hbar^2 f'' = 2m(V(x) - E)f$

   $\hbar$ is very small. So a natural problem is to try and understand $\epsilon^2 \frac{d^2y}{dx^2} = Q(x)y$ when $\epsilon \ll 1$.

   The "semi-classical limit" or "geometric optics".

3. Put $\hbar = 1$, $m = \frac{1}{2}$, $V = 0$; specify $\psi(x,0) = \psi_0(x)$

   Fourier transform $\to \psi(x,t) = \frac{1}{(4\pi i t)^{1/2}} \int_{\mathbb{R}} \exp\left[\frac{i|x-y|^2}{4t}\right] \psi_0(y) dy$.

   Question: Does $\psi(x,t)$ really approach $\psi_0(x)$ as $t \to 0$?

1 Asymptotic expansions of functions

$$\sinh x = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

say $\sinh x \sim x$ as $x \to 0$.

Definition: $f \sim g$ as $x \to x_0$ is $|f(x) - g(x)| = o(g(x))$ as $x \to x_0$.

Example:

$$|\sinh x - x| = |\frac{x^3}{3!} + \frac{x^5}{5!} + \ldots| = O(x^3) = o(x)$$

($F = O(G)$ as $x \to x_0$ means $\exists\, C > 0$ such that $|F(x)| \le C|G(x)|$ in some open interval I, with $x_0 \in I$)
In fact, by remainder estimate for Taylor expansion

$$\left|\sinh x - \sum_0^N \frac{x^{2n+1}}{(2n+1)!}\right| = O(x^{2n+3}) = o(x^{2n+1}) \text{ as } x \to 0$$

We write $\sinh x \sim \sum_0^\infty \frac{x^{2n+1}}{(2n+1)!}$

Definition: Asymptotic sequence and asymptotic expansion.

(i) $\{\phi_n\}_{n=0}^\infty$ is an asymptotic sequence (of functions) as $x \to x_0$ if $\phi_{n+1}(x) = o(\phi_n(x))$ as $x \to x_0$.

(ii) A function f has asymptotic expansion w.r.t. $\{\phi_n\}$ as $x \to x_0$ written $f \sim \sum_{n=0}^\infty a_n \phi_n$ if

$$\left|f(x) - \sum_{n=0}^N a_n \phi_n(x)\right| = o(\phi_N(x)) \text{ as } x \to x_0 \forall N$$

Notice the difference with Taylor expansion - an asymptotic expansion need not converge as $N \to \infty$ for any x!

Examples:

- $\{\phi_n(x) = x^n\}$ as $x \to 0$, the most common sequence.
- $\{\phi_n(x) = x^{2n+1}\}$ as $x \to 0$
- $\{\phi_n(x) = e^{-n/x}\}$ as $x \to 0^+$ (i.e. $x > 0$ and $x \to 0$ on right)

Warning: $\sin x \sim x - \frac{x^3}{3!} - \frac{x^5}{5!} + \ldots$ as $x \to 0$.
$\sin x + e^{-1/x} \sim x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$ as $x \to 0^+$.

Why?

$$\left|\sin x + e^{-1/x} - \sum_0^N \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right| = \left|\sum_{n=2N+1}^\infty \frac{(-1)^n x^{2n+1}}{(2n+1)!} + e^{-1/x}\right| = O(x^{2N+3}) = o(x^{2N+1})$$

Moral: information is lost in asymptotic expansions!

However, given f and asymptotic sequence, the a_j's are unique, i.e.

$$a_0 = \lim_{x \to x_0} \frac{f(x)}{\phi_0(x)}$$
$$a_1 = \lim_{x \to x_0} \frac{f(x) - a_0\phi_0(x)}{\phi_1(x)}$$
$$\vdots$$

Question: Is it possible that $f(x) \sim 0$ as $x \to 0$?
If $|f(x) - 0| = o(0) = 0$ in some interval I, containing 0, then $f \equiv 0$ on I.

Example: Consider $E_1(x) = \displaystyle\int_x^\infty \frac{e^{-t}}{t} dt$ as $x \to +\infty$.

Consider the asymptotic sequence $\phi_n(x) = 1/x^n$ as $x \to +\infty$

$$E_1(x) = \int_x^\infty \frac{-d(e^{-t})}{t} = \left[-\frac{e^{-t}}{t}\right]_x^\infty - \int_x^\infty \frac{e^{-t}}{t^2} dt = \frac{e^{-x}}{x} - \int_x^\infty \frac{e^{-t}}{t^2} dt$$

Claim: $E_1(x) \sim e^{-x}/x$ as $x \to +\infty$.

$$\left| E_1(x) - \frac{e^{-x}}{x} \right| = \left| \int_x^\infty \frac{e^{-t}}{t^2} dt \right| \leq \frac{1}{x^2} \int_x^\infty e^{-t} dt = \frac{e^{-t}}{x^2} = o\left(\frac{e^{-x}}{x}\right)$$

Working out the full expansion of E_1 with respect to $\phi_n = e^{-x}/x^n$ gives that.

$$E_1(x) \sim \sum_{n=0}^\infty \frac{(-1)^n n! e^{-x}}{x^{n+1}}$$

<u>What do we mean?</u>

(i) $\phi_n(x) - e^{-x}/x^{n+1}$ satisfies $\phi_{n+1}(x) = o(\phi_n(x))$ as $x \to +\infty$. i.e. it forms an "asymptotic sequence."

(ii) The notation "$\sim$" ("asymptotic to") means

$$\left| E_1(x) - \sum_{n=0}^N \frac{(-1)^n n! e^{-x}}{x^{n+1}} \right| = o\left(\frac{e^{-x}}{x^{n+1}}\right) \text{ as } x \to +\infty$$

This can be proved with integration by parts:

$$\begin{aligned} E_1(x) &= -\int_x^\infty \frac{1}{t} d(e^{-t}) = e^{-x}/x + \int_x^\infty \frac{1}{t^2} d(e^{-t}) \\ &= e^{-x}/x - e^{-x}/x^2 + 2\int_x^\infty \frac{e^{-t}}{t^3} dt \\ &= e^{-x}\left[\frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \cdots + \frac{(-1)^n n!}{x^{n+1}}\right] + \underbrace{(-1)^{n+1}(n+1)! \int_x^\infty \frac{e^{-t}}{t^{n+2}} dt}_{Rem_{n+1}(x)} \end{aligned}$$

Where

$$|Rem_{n+1}(x)| \leq \frac{(n+1)!}{x^{n+2}} \int_x^\infty e^{-t} dt = \frac{(n+1)! e^{-x}}{x^{n+2}} = o\left(\frac{e^{-x}}{x^{n+1}}\right) \text{ as } x \to +\infty$$

So it is an asymptotic expansion. Not convergent because $\sum (-1)^n \, n! \, y^{n+1}$ has radius of convergence 0. (In fact for fixed y the terms become unbounded.)

Consider magnitudes of successive terms $f_n(x) = \dfrac{(-1)^n \, n! \, e^{-x}}{x^{n+1}}$

$$\left| \frac{f_{n+1}(x)}{f_n(x)} \right| = \frac{n+1}{x}$$

Optimal truncation: Truncate the asymptotic expansion at the point $n = N_x$, such that the first term excluded is the smallest.

In our example, choose $N_x = [x] - 1 = \sup\{j - 1 : j \leq x,\, j \in \mathbb{N}\}$

$$\left.\begin{aligned} |f_{N_x+1}(x)/f_{N_x}(x)| &= (N_x+1)/x \quad \leq 1 \\ |f_{N_x+2}(x)/f_{N_x+1}(x)| &= (N_x+2)/x \quad > 1 \end{aligned}\right\} \text{ so } f_{N_x+1} \text{ is the smallest term, later terms are larger}$$

So we write

$$E_1(x) = \sum^{N_x} \frac{(-1)^n n! e^{-x}}{x^{n+1}} + Rem_{N_x+1}(x)$$

$$|Rem_{n+1}(x)| \leq \frac{(N_x+1)!}{x^{N_x+2}} e^{-x} = \frac{[x]! e^{-x}}{x^{[x]+1}} \leq \frac{2\left(\frac{[x]}{e}\right)^{[x]} \sqrt{2\pi[x]} e^{-x}}{x^{[x]+1}} \leq \frac{2\sqrt{2\pi[x]}}{[x]} e^{-x} e^{-[x]}$$

Where we have used Stirling's formula.

$$\lim_{n\to\infty} \frac{n!}{(n/e)^n \sqrt{2\pi n}} \to 1 \text{ as } n \to \infty$$

The good new is the additional $e^{-[x]}$ term. Optimal truncation (often) gives an exponentially small remainder.

Examples:

$$\sinh x = \frac{e^x - e^{-x}}{2} \sim e^{-x}/2 \text{ as } x \to +\infty$$

Works because $e^{-x} = o(e^x)$ as $x \to \infty$

$$\operatorname{sech} x = \frac{2}{e^x + e^{-x}} = \frac{2}{e^x}(1 + e^{-2x})^{-1} = \frac{2}{e^x}(1 - e^{-2x} + e^{-4x} - \dots)$$

This gives an asymptotic expansion for the sequence $\phi_n = e^{-nx}$ (Which is asymptotic since $e^{-x} = o(e^x)$ as $x \to +\infty$)

Note: $\sinh x \sim -e^{-x}/2$ as $x \to -\infty$

Consider $\sinh z$, for $z \in \mathbb{C}$

$$\sinh z = \frac{e^z - e^{-z}}{2} = \frac{e^{x+iy} - e^{-x-iy}}{2}$$

$$\begin{aligned} &\sim e^z/2 \quad \text{as } z \to \infty \text{ in sector } \{-\frac{\pi}{2} < arg(z) < \frac{\pi}{2}\} \\ &\sim e^{-z}/2 \quad \text{as } z \to \infty \text{ in sector } \{\frac{\pi}{2} < arg(z) < \frac{3\pi}{2}\} \end{aligned}$$

Conclusion: The asymptotic seems to change suddenly when going from sector to sector.

The lines separating the different sectors are Stokes Lines.

Excercise: Prove that the definition of asymptotics in a sector must satisfy that you do not approach Stokes lines too fast.

Terminology:

$$\left.\begin{array}{rc} e^z/2 & \text{dominant} \\ -e^z/2 & \text{subdominant or recessive} \end{array}\right\} \text{ for Arg } z \in (-\frac{\pi}{2}, \frac{\pi}{2})$$

On Stokes lines, neither of these terms is dominant. This means that the asymptotic relation holds if "$z \to \infty$ *but not approaching Stokes lines.*"

Why? Consider $z_n = 1/n + in^2$ has Re $z_n > 0$, and $|z_n| \to \infty$.

$$\sinh z_n = \frac{1}{2}\left(e^{1/n+in^2} - e^{-1/n-in^2}\right)$$

where

$$\sinh z_n \sim \frac{1}{2}e^{1/n+in^2}$$

means that

$$\left|\sinh z_n - \frac{1}{2}e^{1/n+in^2}\right| = \frac{1}{2}\left|e^{-1/n-in^2}\right| = \frac{1}{2}e^{-1/n}$$

but $e^{-1/n} \neq o\left(e^{1/n}\right)$ as $n \to \infty$. So we must consider $z \to \infty$ with Arg $z \in [-\frac{\pi}{2}+\epsilon, \frac{\pi}{2}-\epsilon]$for some $\epsilon > 0$

Definition:

$$f(z) \sim a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots \qquad (z \to \infty,\ Arg\, z \in (\alpha, \beta))$$

means given $N \in \mathbb{N}$, $\epsilon > 0$ sufficiently small

$$\left|f(z) - \sum_{i=0}^{N} \frac{a_i}{z^i}\right| = o(z^{-N})$$

as $z \to \infty$ Arg $z \in [\alpha + \epsilon, \beta - \epsilon]$ In this case we write

$$f(z) = \sum_{i=0}^{N} \frac{a_i}{z^i} \qquad (z \to \infty;\ Arg\, z \in (\alpha, \beta))$$

Exercise: Write out corresponding definition for

$$f(z) = \sum_{i=0}^{N} a_i (z - z_0)^i \qquad (z \to z_0;\ Arg\,(z - z_0) \in (\alpha, \beta))$$

Example:

$$\sinh \frac{1}{z} \sim \frac{1}{2}e^{1/z} \qquad (z \to 0;\ Arg\, z \in (-\frac{\pi}{2}, \frac{\pi}{2}))$$

but

$$\sinh \frac{1}{z} \sim -\frac{1}{2}e^{-1/z} \qquad (z \to 0;\ Arg\, z \in (\frac{\pi}{2}, \frac{3\pi}{2}))$$

(fails on Stokes line)

Consider complex analytic (holomorphic) functions:

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

if f is holomorphic near z_0 and then

$$f(z) \sim \sum_{n=0}^{\infty} c_n (z - z_0)^n \qquad (z \to z_0;\ Arg\, z \in [0, 2\pi])$$

Fact: If z_0 is an isolated singularity, i.e. f is holomorphic in $\{z : 0 < |z - z_0| < r\}$ for some $r > 0$ and

$$f(z) \sim \sum_{n=0}^{\infty} c_n (z - z_0)^n \qquad (z \to z_0;\ Arg\, z \in [0, 2\pi])$$

then z_0 is a removable singularity, and $\sum a_n (z - z_0)^n$ converges to f in some neighbourhood of z_0.

Why? At isolated singularity z_0 either

- removable $\implies |f(z)|$ bounded as $z \to z_0$
- Pole $f(z) = \frac{a_{-N}}{(z-z_0)^N} + \frac{a_{-N+1}}{(z-z_0)^{N-1}} + \dots$
- essential singularity $f(z) = \sum_{-\infty}^{\infty} a_j (z - z_0)^j$ where some $a_{-n} \neq 0$ for arbitrarly large n

The last two have $|f(z)| \to \infty$ as $z \to z_0$ on some sequence.

Notice that if $f(z) \sim \sum_0^{\infty} a_n (z - z_0)^n$ holds for all Arg z then $|f(z) - a_0| = o(1)$ as $z \to z_0$ $\forall$ Arg$(z - z_0)$. I.e. $|f(z)|$ is bounded as $|z - z_0| \to 0$.

Therefore z_0 is a removable singularity, i.e. f is analytic at z_0. So $f(z) = \sum a_n (z - z_0)^n$ by uniqueness of asymptotic expansions. So we will often end up considering asymptotic expansions at essential singularities.

Differential Equations

$\sinh x$ solves $\dfrac{d^2y}{dx^2} = y$, with $y(0) = 0$, $y'(0) = 1$. $x = 0$ is an "ordinary point".

Recall:

$$y'' + C_1(x) y' + C_0(x) y = 0$$

$x = 0$ is an ordinary point if

$$C_j(x) = \sum_0^{\infty} c_{j_n} x^n, \quad j = 0, 1$$

(and these are convergent)
$x = 0$ is a regular singular point if

$$C_1(x) = \frac{1}{x} P_1(x), \quad C_0(x) = \frac{1}{x^2} P_0(x)$$

where $P_j = \sum_0^{\infty} P_{j_n} x^n, \quad j = 0, 1$ (and these are convergent).
(This is also written $x^2 y'' + x P_1(x) y' + P_0(x) y = 0$)

Let's see heuristically why y can be singular at a regular point but not an ordinary point. Assume

$$y = b_0 x^a + b_1 x^{a+1} + \dots$$

with $a < 0$.

$$\begin{aligned} y' &= ab_0 x^{a-1} + (a+1)b_1 x^a + \dots \\ y'' &= a(a-1)b_0 x^{a-2} + (a+1)ab_1 x^{a-1} + \dots \end{aligned}$$

There is no possibility to balance the "worst terms" x^{a-2} at an ordinary point. At a regular singular point we can hope to balance the worst terms because

$$x^2 x^{a-2} = x^a$$

Consider the Airy equation

$$\frac{d^2y}{dx^2} = xy \tag{A}$$

$x = 0$ is an ordinary point. The power series solution at an ordinary point is

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

This can be solved to get the relations

$$\begin{aligned} n(n-1)a_n &= 0 & n &= 0, 1, 2 \\ n(n-1)a_n &= a_{n-3} & n &= 3, 4, \dots \end{aligned}$$

Therefore we have two linearly independent solutions

$$y_0 = a_0 \sum_{n=0}^{\infty} \frac{x^{3n}}{9^n n! \Gamma(n + \frac{2}{3})}; \quad y_1 = a_1 \sum_{n=0}^{\infty} \frac{x^{3n+1}}{9^n n! \Gamma(n + \frac{4}{3})}$$

(Can be easily seen that $a_n = \dfrac{a_{n-3}}{n(n-1)} \implies a_{3n} = \dfrac{a_{3n-3}}{3n(3n-1)} = \cdots = \dfrac{a_0}{3^n n! 3^n (n - \frac{1}{3})(n - \frac{4}{3}) \cdots \frac{2}{3}}$)
The radius of convergence is ∞. (In general the radius of convergence of a power series solution at an ordinary point is the distance to the closest singularity.)

Question: Is $x = +\infty$ a singular point for (A)?
Also recall solutions of $y'' = y$, $y_0 = \cosh x$, $y_1 = \sinh x$. to analyse the point at infinity, let $z = 1/x$.

$$\begin{aligned} y(x) &= w(1/x) = w(z) \\ \implies y'(x) &= -\frac{1}{x^2} w'(1/x) \\ y''(x) &= \frac{1}{x^4} w''(1/x) + \frac{2}{x^3} w'(1/x) \end{aligned}$$

$$\left.\begin{aligned} y''(x) = y(x) &\iff z^4 w'' + 2z^3 w' = w \\ y''(x) = xy(x) &\iff z^4 w'' + 2z^3 w' = \tfrac{1}{z} w \end{aligned}\right\} \text{So an irregular singular point}$$

Often get essential singularities.

Behaviour of solutions as $x \to \infty$

Think of (A) as like $y'' = ay$ (which has solutions $\exp(\pm a^{1/2} x)$) but with a increasing with x. So maybe the solution looks like a speeded up exponential, $\exp[S(x)]$, where

$$S(x) \sim c x^{\frac{3}{2}} + \dots \tag{G}$$

$$y'' = xy \iff S'' + (S')^2 = x$$

If (G) holds, and also

$$S' \sim \frac{3c}{2}x^{\frac{1}{2}} + \dots$$
$$S'' \sim \frac{3c}{4}x^{-\frac{1}{2}} + \dots$$

Dominant balance:

$$(S_0')^2 = x \implies S_0' = \pm x^{1/2} \implies S_0 = \pm\frac{2}{3}x^{3/2} + c$$

Now substitute S_0 into the discarded S'' term and generate a new improved solution S_{NI} by solving $(S_{NI}')^2 = x - S_0'' = x \mp \frac{1}{2}x^{-1/2}$

$$\begin{aligned} S_{NI}' &= \pm x^{1/2}\left(1 \mp \frac{1}{2}x^{-3/2}\right)^{1/2} \\ &= \pm x^{1/2} - \frac{1}{4}x^{-1} + O(x^{-5/2}) \end{aligned}$$

Therefore

$$S_{NI}(x) = \pm\frac{2}{3}x^{3/2} - \frac{1}{4}\log x + O(x^{-3/2})$$

This suggest solutions of (A) look like

$$y_+(x) \sim x^{-1/4}\exp\left[+\frac{2}{3}x^{2/3}\right]$$
$$y_-(x) \sim x^{-1/4}\exp\left[-\frac{2}{3}x^{2/3}\right]$$

Note:

- Make informed guess
- throw away what you hope is small and solve the resulting equation ($(S_0')^2 = x$)
- Check for self consistency ($S_{NI} = \pm\frac{2}{3}x^{3/2} - \frac{1}{4}\log x + \dots$)

Bessel equation

$$x^2y'' + xy' - (x^2 + \nu^2)y = 0$$

Revise power series for this, look at operations on asyptotic expansions § 3.8 Bender-Orsag and Q1 on Ex Sheet 1

We can add, multiply, take reciprocals and integrate asymtotic expansions. But Cannot differentiate in general.
For example

$$f(x) \sim x^{-1/2} + x^{-3/2}\sin x^{x^3}$$