Further Complex Methods

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Books: "Complex Variables," M.J Ablowitz & A. Fokes (CUP)
"A Course in Modern Analysis," Whittaker & Watson

Introduction

Much of this section will be a recap of things learnt in the IB courses Complex Methods/Complex Analysis. In particular, the first three lectures seem to cover material familiar to anyone who understood IB Complex Methods.

Any function of x, y can be written as a function of z, $\bar{z}$ for $z = x + iy$. Functions of a complex variable are defined to be those functions of x and y that can be written entirely in terms of z only. A function of a complex variable is continous if

$$\lim_{z \to z_0} f(z) = f(z_0) \quad \text{(as in real analysis)}$$

The derivative of a function of a complex variable is

$$f'(z) - \lim_{\delta z \to 0} \frac{f(z + \delta z) - f(z)}{\delta z}$$

For a function to be differentiable, the limit must be independent of the direction that the limit is taken. If this is true, then the function is said to be differentiable at z. If $f'(z)$ exists, then $f(z)$ is continous (converse not true).

Cauchy Riemann equations

Write $f(z) = u(x, y) + iv(x, y)$ with u, v both real. Then

$$dz\, f'(z) = \lim_{\delta z = \delta x + i\delta y \to 0} (u(x + \delta x, y + \delta y) + iv(x + \delta x, y + \delta y) - u(x, y) - iv(x, y))$$

If $\delta y = 0$, $dz = dx$ and we get that

$$f'(z) = u_x + iv_x$$

Suppose now that $\delta x = 0$.

$$i\delta y\, f'(z) = u_y + iv_y$$

$$\implies f'(z) = v_y - iu_y$$

$$\implies v_y - iu_y = u_x + iv_x$$

$$\Longrightarrow \quad \left.\begin{aligned} v_y &= \quad u_x \\ v_x &= \ -u_y \end{aligned}\right\} \text{The Cauchy-Riemann equations}$$

If the Cauchy-Riemann equations (C-R) hold, the derivatives exist and are continous, then $f(z)$ is differentiable. If the C-R equations hold then $u,\ v$ are harmonic.

$$u_{xx} = v_{xy} = -u_{yy} \implies u_{xx} + u_{yy} = 0$$

A similar equation holds with v.

Consider surfaces of $u = const$, $v = const$. These surfaces are orthogonal.

$$\nabla u = (u_x, u_y) \text{ - normal to } u = const$$
$$\nabla v = (v_x, v_y) \text{ - normal to } v = const$$

and so

$$\nabla u \cdot \nabla v = u_x v_x + u_y v_y = 0 \text{ from C-R}$$

Analytic functions

Definition: Analytic function
$f(z)$ is analytic at z_0 if $f(z)$ is differentiable in some neighbourgood of z_0. $f(z)$ is analytic in a region if a similar condition applies.

Examples:

(i) e^z is analytic in the finite complex z-plane

(ii) $\bar{z}$ is analytic nowhere

(iii) $1/z^3$ is analytic everywhere except at $z = 0$

Definition: Entire functions
A function is entire if it is analytic in the finite complex plane

Examples:

(i) e^z , this only fails to be analytic at ∞

(ii) $\sin z$

(iii) z^2

Definition: Isolated singularity
A function is said to have an isolated singularity if it fails to be analytic at a point.

Example: $1/z^3$ has an isolated singularity at the origin.

Suppose that a function has an isolated singularity at $z = z_0$. Then it can be expanded as a Laurent series around z_0.

$$f(z) = \sum_{-\infty}^{\infty} c_n (z - z_0)^n$$

Note that this sum is over all positive and negative powers.

Suppose that $c_n = 0$ for all $n < -N$ where $N > 0$.

- If $c_n = 0 \; \forall \, n > 0$ then it is not singular.
- If $c_n = 0$ for all $n < -N$ for $N > 0$, then one has a pole of order N.

Example: $1/z^3$ has a pole of order 3 at $z = 0$.
The coefficient c_{-1} is special, it is the residue of the pole at z_0.

Definition: Removable singularities
Fake singularities where the building blocks of $f(z)$ have isolated singularities, but $f(z)$ does not.

Example:

$$f(z) = \frac{\sin z}{z} = \frac{1}{z}(z - \frac{z^3}{6} + \dots) = 1 - \frac{z^2}{6}$$

$f(z)$ has a removable singularity at $z = 0$.

Example:

$$f(z) = \frac{1}{z} - \frac{1}{z + z^2} = \frac{1}{1 + z}$$

so $f(z)$ has a removable singularity at the origin.

Definition: Essential Singularity
An essential singularity is where the order of the pole of an isolated singularity is infinite.

Example: $f(z) = e^{1/z}$, $z = 0$ is an isolated singularity, as a Laurent series

$$f(z) = \sum_{\infty}^{0} \frac{1}{(-n)!} z^n$$

Note that in this exaple, $f(z)$ is not even continous at $z = 0$, its value depends on how one approaches $z = 0$.

Definition: Meromorphic functions
These are functions of z that only have poles of any finite order in the finite complex plane.

Examples:

(a) $1/z^2$ has a pole of order 2 at the origin.

(b) $\cot z$ has poles of order 1 at $z = n\pi$, $n \in \mathbb{Z}$

Theorem: Cauchy's Theorem

$$\int_C f(z)\, dz = 2\pi i \left(\sum \text{ Residues of the poles enclosed by C} \right)$$

The integral is taken around C in the anti-clockwise direction, and $f(z)$ is meromorphic.

The Riemann Sphere

The complex plane is really a sphere, the Riemann sphere.
w is the perpendicular distance from the $x-y$ plane. The north pole corresponds to infinity, and all of infinity has become a point.

Definition: Stereographic projection
Construct a straight line staring at N, through P to meet the complex plane at C.

$$N = (0,0,2), \qquad P = (X, Y, \ )$$

Construct this line by saying that s is a parameter along the line such that $s = 0$ at the north pole and $s = 1$ at P.

$$\left.\begin{aligned} x &= X_s \\ y &= Y_s \end{aligned}\right\} \text{What about w?}$$

$$w = 2 - (1 \pm \sqrt{1 - X^2 - Y^2})s$$

at C, $w = 0$

$$\implies s = \frac{2}{1 \pm \sqrt{1 - X^2 - Y^2}} \text{ at } C$$

Hence the coordinates of the point C are

$$x = \frac{2X}{1 \pm \sqrt{1 - X^2 - Y^2}}\ , \quad y = \frac{2Y}{1 \pm \sqrt{1 - X^2 - Y^2}}$$

Thus if X, Y both $\to 0$, then $x, y \to \infty$ with the choice of sign. All of infinity gets mapped to the north pole of the Riemann sphere. This motivates how to think about infinity.

$$z \mapsto 1/z = w \text{ maps infinity to the origin}$$

Suppose $f(z) = z$, then $f(w) = 1/w$. $f(w)$ has a simple pole of residue 1 at $w = 0 \implies f(z) = z$ has a simple pole of residue 1 at infinity.

This holds true for any function of a complex variable; to examine the behaviour of a funcion at ∞, send $z \mapsto 1/z = w$ and ask what happens at $w = 0$.

Example: $f(z) = e^z = e^{1/w}$ has an essential singularity at $w = 0$.

Theorem: Liouville's theorem
If f is analytic everywhere including ∞ then it must be a constant.

Multi-valued functions

For a real variable, the square root of a positive number has two forms $\pm\sqrt{x}$.
Now consider $z^{1/2}$ and decompose into modulus and argument.

$$z^{1/2} = \rho^{1/2} e^{i\theta/2}$$

As one moves around the circle,

$$\theta \mapsto \theta + 2\pi$$
$$z^{1/2} \mapsto \rho^{1/2} e^{i(\theta+2\pi)/2} = -\rho^{1/2} e^{i\theta/2}$$

So $f(z)$ changes sign. If one goes around the circle twice then $\theta \mapsto \theta + 4\pi$, and so $f(z)$ is invariant.

The effect of going around the circle is usually called the monodromy, and for the case $f(z) = z^{1/2}$ this is $(-1)^n$.

The monodromy always forms a group. So in this case, the monodromy group is just $\mathbb{Z}_2$,

A point where the monodromy is not 1 is called a branch point

For $f(z) = z^{1/2}$, the origin is a branch point. Around z_0 the function returns to its starting point. This holds for any $z_0 \neq 0$ in the finite complex plane. Infinity is also a branch point.

$$z \mapsto 1/w\,, \quad w^{-\frac{1}{2}} \mapsto (\rho')^{-\frac{1}{2}} e^{-\frac{i\theta'}{2}}$$

and so when $\theta \mapsto \theta + 2\pi$, $w^{-1/2}$ changes sign. Therefore $z = \infty$, $w = 0$ is also a branch point. There is always more than one branch point, so always look at infinity. Branch points represent a failure of analycity.

Example: $f(z) - (z - z_0)^p$. If p is an integer, then $(z - z_0)^p$ is single valued. Consider instead $p = m/n$ for m, n integers.

$$z = z_0 + \rho e^{i\theta}$$

$$(z - z_0)^p = \rho^p e^{ip\theta} = \rho^{n/m} e^{im\theta/n}$$

Take $\theta \to \theta + 2\pi$. Then

$$(z - z_0)^p \to \rho^{m/n} e^{2\pi i m/n}$$

The change of the phase of the function is $e^{2\pi i m/n}$ for going round $z = z_0$ once anticlockwise.

Suppose one goes around $z = z_0$, s times, then the phase factor is $e^{2\pi i m s/n}$. Thus if $s = n$ one gets back to the original value, or indeed if s is any multiple of n times. The monodromy is therefore $e^{2\pi i m/n}, e^{4\pi i m/n}, \ldots$. Thus the monodromy group is the cyclic group of order n, i.e. $\mathbb{Z}_n$.

Suppose that p is not rational. Then one never gets back to the starting point. Monodromy for a single circle of z_0 is $e^{2\pi i p}$. The monodromy group is $\mathbb{Z}_\infty$.

Example: $f(z) = \operatorname{Log} z$. If $z = \rho e^{i\theta}$, then set

$$f(z) = \log \rho + i\theta$$

So going around a circle around the origin once has the effect that $\operatorname{Log} z \to \operatorname{Log} z + 2\pi i$. If one goes around the circle n times then $\operatorname{Log} z \to \operatorname{Log} z + 2n\pi i$. Thus there are an infinite number of possible values for $\operatorname{Log} z$. The monodromy is addition of $2\pi i$. The monodromy group is the integers under addition.

We can see that $z = 0$ is a branch point, but $z = \infty$ is also a branch point:

$$z \to 1/w, \quad \operatorname{Log} z \to \operatorname{Log} 1/w = -\operatorname{Log} w$$

Thus $w = 0$ is also a branch point, and so $z = \infty$ is a branch point.

Example: $f(z) = \sin^{-1} z$ is multi-valued, since $\sin^{-1} z$ is ambigous under the addition of $2\pi n$.

Branch Cuts

This is a method of making $f(z)$ simgle-valued in the complex z plane.

Example: $f(z) = z^{1/2}$. For $z = \rho e^{i\theta}$ this is

$$f(z) = \underbrace{\rho^{1/2}}_{\geq 0} e^{i\theta/2}$$

$f(z)$ is single-valued if θ lies in a range of 2π. If we restrict $0 < \theta < 2\pi$ then the function is single valued, but discontinous across the positive real axis. This is a failure of analycity. To get around this, exclude the positive real axis from the definition of the function.

If z is real, $z^{1/2}$ can still be defined either by

- taking the limit from the top half-plane
- taking the limit from the bottom half-plane

There is always a discontinuity across a branch cut. The branch cut extends all the way out to infinity since the discontinuity between $\theta = 0$ and $\theta = 2\pi$ is non vanishing for all ρ.

For a square root type branch cut, the discontinuity is always just a sign corresponding to the nature of the monodromy. However, this is not the only way to arrange a branch cut. Another possibility is for θ to run from $-\pi$ to $+\pi$. In fact, one could (peversely) pick any 2π interval for θ and it could be ρ dependent.

Example: $f(z) = (z-1)^{1/4}$. There is a branch point at $z = 1$. There are four possible values for $f(z)$. If one goes around a little circle enclosing $z = 1$ then

$$(z-1)^{1/4} \to e^{i\pi/2}(z-1)^{1/4}$$

There is a branch point at infinity, send $z \to 1/w$,

$$(z-1)^{1/4} \to \left(\frac{1}{w} - 1\right)^{1/4} = w^{-\frac{1}{4}}(1-w)^{\frac{1}{4}}$$

Hence a branch point at $w = 0$ and so at $z = \infty$.

Example: $I = \displaystyle\int_0^\infty \frac{x^{1/2}}{1+x^2}dx$ with x real. We can convert this to a contour integral taking $I = \displaystyle\int_0^\infty \frac{z^{1/2}}{1+z^2}dz$ where now $f(z) = \dfrac{z^{1/2}}{1+z^2}$ has Branch points at $0, \infty$ and simple poles at $\pm i$. We restrict the argument of z to run between 0 and 2π.

$$\int_C \frac{z^{1/2}}{1+z^2}dz = 2\pi i \left(\text{residues at } e^{i\pi/2} \text{ and } e^{-3i\pi/2}\right)$$

Along C_1, $z = xe^{0i}$ one just gets $I = \displaystyle\int_0^\infty \frac{x^{1/2}}{1+x^2}dx$
Along C_2, $z = Re^{i\theta}$ for R very large. Therefore $dz = iRe^{i\theta}d\theta$ and

$$\int_{C_2} f(z)dz = \int_0^{2\pi} \frac{R^{1/2}e^{i\theta/2}}{1+R^2e^{2i\theta}} iRe^{i\theta}d\theta = O(R^{-1/2}) \to 0 \text{ as } R \to \infty$$

Along C_3, $z = xe^{2\pi i}$ and so

$$\int_{C_3} f(z)dz = \int_\infty^0 \frac{x^{1/2}e^{\pi i}}{1+x^2e^{4\pi i}}dx = \int_0^\infty \frac{x^{1/2}}{1+x^2}dx = I$$

Along C_4, $z = \epsilon e^{i\theta}$ for θ very small. Therefore $dz = i\epsilon e^{i\theta}d\theta$ and

$$\int_{C_4} f(z)dz = \int_{2\pi}^0 \frac{\epsilon^{1/2}e^{i\theta/2}}{1+\epsilon^2e^{2i\theta}}i\epsilon e^{-i\theta}d\theta = O(\epsilon^{3/2}) \to 0 \text{ as } \epsilon \to 0$$

What are the residues at $z = e^{i\pi/2}$, $e^{3i\pi/2}$? Since these are simple poles, it is easiest to find

$$\lim_{z\to e^{i\pi/2}} \left(\frac{z-e^{i\pi/2}}{z^2+1}z^{1/2}\right) \to \lim_{z\to e^{i\pi/2}} \left(\frac{1}{2z}z^{1/2}\right) = \frac{1}{2}e^{-i\pi/4}$$

and at $z = e^{3i\pi/2}$ the residue is

$$\lim_{z\to e^{3i\pi/2}} \left(\frac{z-e^{3i\pi/2}}{z^2+1}z^{1/2}\right) \to \lim_{z\to e^{3i\pi/2}} \left(\frac{1}{2z}z^{1/2}\right) = \frac{1}{2}e^{-3i\pi/4}$$

$$\implies \int_C \frac{z^{1/2}}{1+z^2}dz = 2\pi i\left(\frac{1}{2}e^{-i\pi/4} + \frac{1}{2}e^{-3i\pi/4}\right) = 2\pi\sin\frac{\pi}{4} = \frac{\pi}{\sqrt{2}}$$

Example: Another example of a branch cut.

$$f(z) = \sqrt{(z-a)(z-b)}$$

with $a, b \in \mathbb{R}$, 0, $b < 0$ This has branch points at $z = a$, b. What aboutat infinity? As usual take $z = 1/w$

$$f(w) = \sqrt{\left(\frac{1}{w}-a\right)\left(\frac{1}{w}-b\right)} = \frac{1}{w}\sqrt{(1-aw)(1-bw)}$$

which is fine apart from a pole at ∞, which is not the same as a branch point.

$$f(z) = \sqrt{(z-a)(z-b)} = \sqrt{\rho_1\rho_2}e^{i(\theta_1+\theta_2)/2}$$

with say $0 \le \theta_1$, $\theta_2 < 2\pi$. In this case the cut is directly beween a and b. Making a different choice, $-\pi \le \theta_1 < \pi$. and $0 \le \theta_2 < \pi$.

This cut appears to end at $+\infty$ and $-\infty$ which is not a branch point. However the complex plane is really a sphere, infinity is really a point and the cut just happens to go through the point ∞.

Cauchy Principal Value of an integral

Sometimes it is possible to consruct the Cauchy Principal Value of an integral, defined to be

$$P\int_A^B f(x)dx = \lim_{\epsilon\to 0}\left[\int_A^{x_0-\epsilon} f(x)dx + \int_{x_0+\epsilon}^B f(x)dx\right]$$

Where the letter P before an integral indicates that it is the Principal Value (PV). Suppose that $f(x)$ has a singularity at x_0, then if the limit as $\epsilon \to 0$ exists, then it defines the principal value.

If the integral were convergent, then the principal value coincides with the original integral.

Examples:

$$P\int_{-1}^{2}\frac{dx}{x}=\lim_{\epsilon\to 0}\left[\int_{-1}^{-\epsilon}\frac{dx}{x}+\int_{\epsilon}^{2}\frac{dx}{x}\right]=\lim_{\epsilon\to 0}\left[\log|x|_{-1}^{-\epsilon}+\log|x|_{\epsilon}^{2}\right]=\log 2$$

Principal balues work nicely at poles. In the complex plane this turns out to be rather convenient. Suppose one integrates a function that has a pole on the real axis.

PV corresponds to the green contour.

This contour might be closed in the top half plane. Apply a small modification of Cauchy's theorem:

$$2\pi i\left(\sum \text{ Residues in } C\right)=\int_{\Gamma}f(z)dz+\int_{C}f(z)dz+\int_{C'}f(z)dz$$

Where $\int_C f(z)dz$ corresponds to the principal value. $C'=z_0+\epsilon e^{i\theta}$ where θ runs from π to 0. Near $z=z_0$,

$$f(z)=\sum_{n=-1}^{\infty}c_n(z-z_0)^n$$

So this has residue c_{-1} at $z=z_0$.

$$\int_{\pi}^{0}\sum_{n=-1}^{\infty}c_n(z-z_0)^n\,dz=\int_{\pi}^{0}\sum_{n=-1}^{\infty}c_n\epsilon^n e^{in\theta}(i\epsilon e^{i\theta}d\theta)=\int_{\pi}^{0}\sum_{n=-1}^{\infty}c_n\epsilon^{n+1}e^{i(n+1)\theta}id\theta$$

In the limit $\epsilon\to 0$ the only remaining term is $n=-1$. Therefore,

$$=\int_{\pi}^{0}c_{-1}id\theta=-i\pi c_{-1}$$

$$\implies \int_{c'}f(z)dz=-i\pi c_{-1}$$

This shows also that higher poles can give trouble. We will only look at the case of simple poles in the integrand.

Example: Calculate the PV of

$$P\int_{-\infty}^{\infty}\frac{e^{iax}}{x}dx$$

for a real and positive Apply our modified version of Cauchy's theorem to this contour.

$$2\pi i\left(\sum \text{ Residues in } C\right)=\int_{|z|=R,\,Im(z)>0}\frac{e^{iaz}}{z}dz+P\int_{-R}^{R}\frac{e^{iax}}{x}dx-i\pi c_{-1}$$

Where c_{-1} is the residue of $\frac{e^{iax}}{x}$ at $x=0$.

$$\implies 0=0+P\int_{-\infty}^{\infty}\frac{e^{iax}}{x}dx-i\pi$$

$$\implies P\int_{-\infty}^{\infty}\frac{e^{iax}}{x}dx=\pi i$$

Take the imaginary part of this expansion to get that

$$P\int_{-\infty}^{\infty}\frac{\sin ax}{x}dx=\pi$$

Since the principal part of a convergent integral is the same as the integral, one finds that

$$\int_{-\infty}^{\infty}\frac{\sin ax}{x}dx=\pi$$

and is indepenent of a for $a>0$.

Example:

$$I=\int_{-\infty}^{\infty}\frac{e^{px}-e^{qx}}{1-e^{x}}dx \qquad (0<p,q<1)$$

This is singular at $x=2\pi ni$ where the denominator zero. The singularity at zero is removable. Hence this integral is the same as its principal value

$$I=P\int_{-\infty}^{\infty}\frac{e^{px}-e^{qx}}{1-e^{x}}dx$$

we consider

$$I=\int_{-\infty}^{\infty}\frac{e^{pz}-e^{qz}}{1-e^{z}}dz$$

Choose a rectangular contour

$$0=\underbrace{P\int_{-R}^{R}\frac{e^{pz}-e^{qz}}{1-e^{z}}dz}_{C_1}+\underbrace{P\int_{R+2\pi i}^{-R+2\pi i}\frac{e^{pz}-e^{qz}}{1-e^{z}}dz}_{C_4}-\underbrace{i\pi\,(\text{Res at }0)}_{C_2}-\underbrace{i\pi\,(\text{Res at }2\pi i)}_{C_3}$$

Now we consider

$$P\int\frac{e^{pz}-e^{qz}}{1-e^{z}}=P\int\frac{e^{pz}}{1-e^{z}}dz-P\int\frac{e^{pz}}{1-e^{z}}dz$$

The residue at 0

$$\lim_{z\to 0}\frac{ze^{pz}}{1-e^{z}}=-1$$

The residue at $2\pi i$

$$\lim_{z\to 2\pi i}\frac{(x-2\pi i)e^{px}}{1-e^{x}}=-e^{2\pi ip}$$

$$\implies 0=\int_C\frac{e^{pz}}{1-e^{z}}dz=\underbrace{P\int_{-\infty}^{\infty}\frac{e^{px}}{1-e^{x}}dx}_{I_1}+P\int_{\infty}^{-\infty}\frac{e^{p(x+2\pi i)}}{1-e^{x+2\pi i}}dx-i\pi\,(\text{Res at }0)-i\pi\,(\text{Res at }2\pi i)$$

and so we can find I_1

$$0=I_1-I_1e^{2\pi ip}-i\pi(-1)-i\pi(-e^{2i\pi p})$$

$$\implies\frac{-i\pi(1+e^{2i\pi p})}{1-e^{2\pi ip}}=\pi\cot\pi p$$

therefore

$$I=\pi(\cot\pi p-\cot\pi q)$$

The Hilbert Transform

A Hilbert transform is somewhat like the Fourier transform. Take a function of a real variable s, $f(s)$.

The Hilbert transform is

$$H_f(t) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(s)}{t-s}$$

where t is another real variable.

This can be turned into a form that is easier to work with.

$$H_f(t) = \frac{1}{\pi} \lim_{\epsilon \to 0} \left[\int_{-\infty}^{t-\epsilon} \frac{f(s)}{t-s} ds + \int_{t+\epsilon}^{\infty} \frac{f(s)}{t-s} ds \right]$$

Shift the pole in s to the origin, $s - t = s'$

$$H_f(t) = -\frac{1}{\pi} \lim_{\epsilon \to 0} \left[\int_{-\infty}^{-\epsilon} \frac{f(s'+t)}{s'} ds' + \int_{\epsilon}^{\infty} \frac{f(s'+t)}{s'} ds' \right]$$

In the first integral take $s' \mapsto -s'$ to get that

$$H_f(t) = \frac{1}{\pi} \lim_{\epsilon \to 0} \left[\int_{\epsilon}^{\infty} \frac{f(t-s')}{s'} - \frac{f(t+s')}{s'} ds' \right]$$

Now lets find the inverse of the Hilbert transform. $\hat{H}_f(\omega)$ is the fourier transform of the Hilbert transform. Here we consider the Fourier transform to be

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt$$

Now

$$\hat{H}_f(\omega) = \frac{1}{\pi} \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \int_{\epsilon}^{\infty} (f(t-s) - f(t+s)) e^{i\omega t} ds\, dt$$

We will assume that interchanging the orders of integration is fine. Shift in the first integral $t - s \to t$, $t + s \to t$

$$\hat{H}_f(\omega) = \frac{1}{\pi} \lim_{\epsilon \to 0} \left[\int_{-\infty}^{\infty} \int_{\epsilon}^{\infty} \frac{1}{s} (\underbrace{f(t) e^{i\omega t}}_{\hat{f}(\omega)} e^{i\omega s} - \underbrace{f(t) e^{i\omega t}}_{\hat{f}(\omega)} e^{-i\omega s}) ds\, dt \right]$$

$$= \frac{1}{\pi} \lim_{\epsilon \to 0} \hat{f}(\omega) \int_{\epsilon}^{\infty} \frac{1}{s} (e^{i\omega s} - e^{-i\omega s}) ds = \frac{i}{\pi} \hat{f}(\omega) \int_{-\infty}^{\infty} \frac{\sin \omega s}{s} ds$$

Recall

$$P \int_{-\infty}^{\infty} \frac{\sin \omega s}{s} ds = \begin{cases} \pi & \omega > 0 \\ 0 & \omega = 0 \\ -i\pi & \omega < 0 \end{cases}$$

Therefore, for

$$\begin{aligned} \omega > 0 \quad \hat{H}_f(\omega) &= \quad i\hat{f}(\omega) \\ \omega < 0 \quad \hat{H}_f(\omega) &= -i\hat{f}(\omega) \\ \omega = 0 \quad \hat{H}_f(\omega) &= 0 \end{aligned}$$

So

$$\hat{H}_f \hat{H}_f(\omega) = -f(\omega) \quad (\omega \neq 0)$$

So we have that $H^{-1} = -H$. Use of the Hilbrt transform is to discover the Kramers-Kronig relations. Consider a function $f(s)$ of a complex variable s, analyic in the upper half plane, and dies off faster than $1/|s|$ in the upper half plane. Take t real.

$$\int_c \frac{f(s)}{t-s}$$

Integrating along the real axis gives

$$\underbrace{\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(s)}{t-s} ds}_{H_f(s)} + \frac{1}{\pi}(-i\pi(\text{Res at } s=t)) = 0$$

$$\implies \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(s)}{t-s} ds = -if(t)$$

Separate $f(s)$ into its real and imaginary parts. Real:

$$\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{u(x,0)}{t-x} dx = v(t,0)$$

Imaginary:

$$\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{v(x,0)}{t-x} dx = -u(t,0)$$

So if u is known on the real axis, then v can be found.