

Further Complex Methods

Course given by Prof. M.Perry
L^AT_EX by Dominic Skinner
Dom-Skinner@github.com

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Books: “Complex Variables,” M.J Ablowitz & A. Fokes (CUP)
“A Course in Modern Analysis,” Whittaker & Watson

Introduction

Much of this section will be a recap of things learnt in the IB courses Complex Methods/Complex Analysis. In particular, the first three lectures seem to cover material familiar to anyone who understood IB Complex Methods.

Any function of x, y can be written as a function of z, \bar{z} for $z = x + iy$. Functions of a complex variable are defined to be those functions of x and y that can be written entirely in terms of z only. A function of a complex variable is continuous if

$$\lim_{z \rightarrow z_0} f(z) = f(z_0) \quad (\text{as in real analysis})$$

The derivative of a function of a complex variable is

$$f'(z) = \lim_{\delta z \rightarrow 0} \frac{f(z + \delta z) - f(z)}{\delta z}$$

For a function to be differentiable, the limit must be independent of the direction that the limit is taken. If this is true, then the function is said to be differentiable at z . If $f'(z)$ exists, then $f(z)$ is continuous (converse not true).

Cauchy Riemann equations

Write $f(z) = u(x, y) + iv(x, y)$ with u, v both real. Then

$$dz f'(z) = \lim_{\delta z = \delta x + i\delta y \rightarrow 0} (u(x + \delta x, y + \delta y) + iv(x + \delta x, y + \delta y) - u(x, y) - iv(x, y))$$

If $\delta y = 0$, $dz = dx$ and we get that

$$f'(z) = u_x + iv_x$$

Suppose now that $\delta x = 0$.

$$\begin{aligned} i\delta y f'(z) &= u_y + iv_y \\ \implies f'(z) &= v_y - iu_y \\ \implies v_y - iu_y &= u_x + iv_x \end{aligned}$$

$$\Rightarrow \left. \begin{aligned} v_y &= u_x \\ v_x &= -u_y \end{aligned} \right\} \text{The Cauchy-Riemann equations}$$

If the Cauchy-Riemann equations (C-R) hold, the derivatives exist and are continuous, then $f(z)$ is differentiable. If the C-R equations hold then u, v are harmonic.

$$u_{xx} = v_{xy} = -u_{yy} \implies u_{xx} + u_{yy} = 0$$

A similar equation holds with v .

Consider surfaces of $u = \text{const}$, $v = \text{const}$. These surfaces are orthogonal.

$$\nabla u = (u_x, u_y) \text{ - normal to } u = \text{const}$$

$$\nabla v = (v_x, v_y) \text{ - normal to } v = \text{const}$$

and so

$$\nabla u \cdot \nabla v = u_x v_x + u_y v_y = 0 \text{ from C-R}$$

Analytic functions

Definition: Analytic function

$f(z)$ is analytic at z_0 if $f(z)$ is differentiable in some neighbourhood of z_0 . $f(z)$ is analytic in a region if a similar condition applies.

Examples:

- (i) e^z is analytic in the finite complex z -plane
- (ii) \bar{z} is analytic nowhere
- (iii) $1/z^3$ is analytic everywhere except at $z = 0$

Definition: Entire functions

A function is entire if it is analytic in the finite complex plane

Examples:

- (i) e^z , this only fails to be analytic at ∞
- (ii) $\sin z$
- (iii) z^2

Definition: Isolated singularity

A function is said to have an isolated singularity if it fails to be analytic at a point.

Example: $1/z^3$ has an isolated singularity at the origin.

Suppose that a function has an isolated singularity at $z = z_0$. Then it can be expanded as a Laurent series around z_0 .

$$f(z) = \sum_{-\infty}^{\infty} c_n (z - z_0)^n$$

Note that this sum is over all positive and negative powers.

Suppose that $c_n = 0$ for all $n < -N$ where $N > 0$.

- If $c_n = 0 \forall n > 0$ then it is not singular.
- If $c_n = 0$ for all $n < -N$ for $N > 0$, then one has a pole of order N .

Example: $1/z^3$ has a pole of order 3 at $z = 0$.

The coefficient c_{-1} is special, it is the residue of the pole at z_0 .

Definition: Removable singularities

Fake singularities where the building blocks of $f(z)$ have isolated singularities, but $f(z)$ does not.

Example:

$$f(z) = \frac{\sin z}{z} = \frac{1}{z} \left(z - \frac{z^3}{6} + \dots \right) = 1 - \frac{z^2}{6}$$

$f(z)$ has a removable singularity at $z = 0$.

Example:

$$f(z) = \frac{1}{z} - \frac{1}{z + z^2} = \frac{1}{1 + z}$$

so $f(z)$ has a removable singularity at the origin.

Definition: Essential Singularity

An essential singularity is where the order of the pole of an isolated singularity is infinite.

Example: $f(z) = e^{1/z}$, $z = 0$ is an isolated singularity, as a Laurent series

$$f(z) = \sum_{-\infty}^0 \frac{1}{(-n)!} z^n$$

Note that in this example, $f(z)$ is not even continuous at $z = 0$, its value depends on how one approaches $z = 0$.

Definition: Meromorphic functions

These are functions of z that only have poles of any finite order in the finite complex plane.

Examples:

- $1/z^2$ has a pole of order 2 at the origin.
- $\cot z$ has poles of order 1 at $z = n\pi$, $n \in \mathbb{Z}$

Theorem: Cauchy's Theorem

$$\int_C f(z) dz = 2\pi i \left(\sum \text{Residues of the poles enclosed by } C \right)$$

The integral is taken around C in the anti-clockwise direction, and $f(z)$ is meromorphic.

The Riemann Sphere

The complex plane is really a sphere, the Riemann sphere.

w is the perpendicular distance from the $x - y$ plane. The north pole corresponds to infinity, and all of infinity has become a point.

Definition: Stereographic projection

Construct a straight line starting at N , through P to meet the complex plane at C .

$$N = (0, 0, 2), \quad P = (X, Y,)$$

Construct this line by saying that s is a parameter along the line such that $s = 0$ at the north pole and $s = 1$ at P .

$$\left. \begin{array}{l} x = X_s \\ y = Y_s \end{array} \right\} \text{What about } w?$$

$$w = 2 - (1 \pm \sqrt{1 - X^2 - Y^2})s$$

at C , $w = 0$

$$\implies s = \frac{2}{1 \pm \sqrt{1 - X^2 - Y^2}} \text{ at } C$$

Hence the coordinates of the point C are

$$x = \frac{2X}{1 \pm \sqrt{1 - X^2 - Y^2}}, \quad y = \frac{2Y}{1 \pm \sqrt{1 - X^2 - Y^2}}$$

Thus if X, Y both $\rightarrow 0$, then $x, y \rightarrow \infty$ with the choice of sign. All of infinity gets mapped to the north pole of the Riemann sphere. This motivates how to think about infinity.

$$z \mapsto 1/z = w \text{ maps infinity to the origin}$$

Suppose $f(z) = z$, then $f(w) = 1/w$. $f(w)$ has a simple pole of residue 1 at $w = 0 \implies f(z) = z$ has a simple pole of residue 1 at infinity.

This holds true for any function of a complex variable; to examine the behaviour of a function at ∞ , send $z \mapsto 1/z = w$ and ask what happens at $w = 0$.

Example: $f(z) = e^z = e^{1/w}$ has an essential singularity at $w = 0$.

Theorem: Liouville's theorem

If f is analytic everywhere including ∞ then it must be a constant.

Multi-valued functions

For a real variable, the square root of a positive number has two forms $\pm\sqrt{x}$.

Now consider $z^{1/2}$ and decompose into modulus and argument.

$$z^{1/2} = \rho^{1/2} e^{i\theta/2}$$

As one moves around the circle,

$$\begin{aligned} \theta &\mapsto \theta + 2\pi \\ z^{1/2} &\mapsto \rho^{1/2} e^{i(\theta+2\pi)/2} = -\rho^{1/2} e^{i\theta/2} \end{aligned}$$

So $f(z)$ changes sign. If one goes around the circle twice then $\theta \mapsto \theta + 4\pi$, and so $f(z)$ is invariant.

The effect of going around the circle is usually called the monodromy, and for the case $f(z) = z^{1/2}$ this is $(-1)^n$.

The monodromy always forms a group. So in this case, the monodromy group is just \mathbb{Z}_2 ,

A point where the monodromy is not 1 is called a branch point

For $f(z) = z^{1/2}$, the origin is a branch point. Around z_0 the function returns to its starting point. This holds for any $z_0 \neq 0$ in the finite complex plane. Infinity is also a branch point.

$$z \mapsto 1/w, \quad w^{-1/2} \mapsto (\rho')^{-1/2} e^{-i\theta'/2}$$

and so when $\theta \mapsto \theta + 2\pi$, $w^{-1/2}$ changes sign. Therefore $z = \infty$, $w = 0$ is also a branch point. There is always more than one branch point, so always look at infinity. Branch points represent a failure of analyticity.

Example: $f(z) = (z - z_0)^p$. If p is an integer, then $(z - z_0)^p$ is single valued. Consider instead $p = m/n$ for m, n integers.

$$\begin{aligned} z &= z_0 + \rho e^{i\theta} \\ (z - z_0)^p &= \rho^p e^{ip\theta} = \rho^{n/m} e^{im\theta/n} \end{aligned}$$

Take $\theta \rightarrow \theta + 2\pi$. Then

$$(z - z_0)^p \rightarrow \rho^{m/n} e^{2\pi im/n}$$

The change of the phase of the function is $e^{2\pi im/n}$ for going round $z = z_0$ once anticlockwise.

Suppose one goes around $z = z_0$, s times, then the phase factor is $e^{2\pi ims/n}$. Thus if $s = n$ one gets back to the original value, or indeed if s is any multiple of n times. The monodromy is therefore $e^{2\pi im/n}, e^{4\pi im/n}, \dots$. Thus the monodromy group is the cyclic group of order n , i.e. \mathbb{Z}_n .

Suppose that p is not rational. Then one never gets back to the starting point. Monodromy for a single circle of z_0 is $e^{2\pi ip}$. The monodromy group is \mathbb{Z}_∞ .

Example: $f(z) = \text{Log } z$. If $z = \rho e^{i\theta}$, then set

$$f(z) = \log \rho + i\theta$$

So going around a circle around the origin once has the effect that $\text{Log } z \rightarrow \text{Log } z + 2\pi i$. If one goes around the circle n times then $\text{Log } z \rightarrow \text{Log } z + 2n\pi i$. Thus there are an infinite number of possible values for $\text{Log } z$. The monodromy is addition of $2\pi i$. The monodromy group is the integers under addition.

We can see that $z = 0$ is a branch point, but $z = \infty$ is also a branch point:

$$z \rightarrow 1/w, \quad \text{Log } z \rightarrow \text{Log } 1/w = -\text{Log } w$$

Thus $w = 0$ is also a branch point, and so $z = \infty$ is a branch point.

Example: $f(z) = \sin^{-1} z$ is multi-valued, since $\sin^{-1} z$ is ambiguous under the addition of $2\pi n$.

Branch Cuts

This is a method of making $f(z)$ single-valued in the complex z plane.

Example: $f(z) = z^{1/2}$. For $z = \rho e^{i\theta}$ this is

$$f(z) = \underbrace{\rho^{1/2}}_{\geq 0} e^{i\theta/2}$$

$f(z)$ is single-valued if θ lies in a range of 2π . If we restrict $0 < \theta < 2\pi$ then the function is single valued, but discontinuous across the positive real axis. This is a failure of analyticity. To get around this, exclude the positive real axis from the definition of the function.

If z is real, $z^{1/2}$ can still be defined either by

- taking the limit from the top half-plane
- taking the limit from the bottom half-plane

There is always a discontinuity across a branch cut. The branch cut extends all the way out to infinity since the discontinuity between $\theta = 0$ and $\theta = 2\pi$ is non vanishing for all ρ .

For a square root type branch cut, the discontinuity is always just a sign corresponding to the nature of the monodromy. However, this is not the only way to arrange a branch cut. Another possibility is for θ to run from $-\pi$ to $+\pi$. In fact, one could (peversely) pick any 2π interval for θ and it could be ρ dependent.

Example: $f(z) = (z - 1)^{1/4}$. There is a branch point at $z = 1$. There are four possible values for $f(z)$. If one goes around a little circle enclosing $z = 1$ then

$$(z - 1)^{1/4} \rightarrow e^{i\pi/2} (z - 1)^{1/4}$$

There is a branch point at infinity, send $z \rightarrow 1/w$,

$$(z - 1)^{1/4} \rightarrow \left(\frac{1}{w} - 1 \right)^{1/4} = w^{-1/4} (1 - w)^{1/4}$$

Hence a branch point at $w = 0$ and so at $z = \infty$.

Example: $I = \int_0^\infty \frac{x^{1/2}}{1+x^2} dx$ with x real. We can convert this to a contour integral taking $I = \int_0^\infty \frac{z^{1/2}}{1+z^2} dz$ where now $f(z) = \frac{z^{1/2}}{1+z^2}$ has Branch points at $0, \infty$ and simple poles at $\pm i$. We restrict the argument of z to run between 0 and 2π .

$$\int_C \frac{z^{1/2}}{1+z^2} dz = 2\pi i \left(\text{residues at } e^{i\pi/2} \text{ and } e^{-3i\pi/2} \right)$$

Along C_1 , $z = xe^{0i}$ one just gets $I = \int_0^\infty \frac{x^{1/2}}{1+x^2} dx$

Along C_2 , $z = Re^{i\theta}$ for R very large. Therefore $dz = iRe^{i\theta} d\theta$ and

$$\int_{C_2} f(z) dz = \int_0^{2\pi} \frac{R^{1/2} e^{i\theta/2}}{1 + R^2 e^{2i\theta}} iRe^{i\theta} d\theta = O(R^{-1/2}) \rightarrow 0 \text{ as } R \rightarrow \infty$$

Along C_3 , $z = xe^{2\pi i}$ and so

$$\int_{C_3} f(z)dz = \int_{\infty}^0 \frac{x^{1/2}e^{\pi i}}{1+x^2e^{4\pi i}}dx = \int_0^{\infty} \frac{x^{1/2}}{1+x^2}dx = I$$

Along C_4 , $z = \epsilon e^{i\theta}$ for θ very small. Therefore $dz = i\epsilon e^{i\theta}d\theta$ and

$$\int_{C_4} f(z)dz = \int_{2\pi}^0 \frac{\epsilon^{1/2}e^{i\theta/2}}{1+\epsilon^2e^{2i\theta}} i\epsilon e^{-i\theta}d\theta = O(\epsilon^{3/2}) \rightarrow 0 \text{ as } \epsilon \rightarrow 0$$

What are the residues at $z = e^{i\pi/2}$, $e^{3i\pi/2}$? Since these are simple poles, it is easiest to find

$$\lim_{z \rightarrow e^{i\pi/2}} \left(\frac{z - e^{i\pi/2}}{z^2 + 1} z^{1/2} \right) \rightarrow \lim_{z \rightarrow e^{i\pi/2}} \left(\frac{1}{2z} z^{1/2} \right) = \frac{1}{2} e^{-i\pi/4}$$

and at $z = e^{3i\pi/2}$ the residue is

$$\begin{aligned} \lim_{z \rightarrow e^{3i\pi/2}} \left(\frac{z - e^{3i\pi/2}}{z^2 + 1} z^{1/2} \right) &\rightarrow \lim_{z \rightarrow e^{3i\pi/2}} \left(\frac{1}{2z} z^{1/2} \right) = \frac{1}{2} e^{-3i\pi/4} \\ \Rightarrow \int_C \frac{z^{1/2}}{1+z^2} dz &= 2\pi i \left(\frac{1}{2} e^{-i\pi/4} + \frac{1}{2} e^{-3i\pi/4} \right) = 2\pi \sin \frac{\pi}{4} = \frac{\pi}{\sqrt{2}} \end{aligned}$$

Example: Another example of a branch cut.

$$f(z) = \sqrt{(z-a)(z-b)}$$

with $a, b \in \mathbb{R}$, $0, b < 0$ This has branch points at $z = a, b$. What about at infinity? As usual take $z = 1/w$

$$f(w) = \sqrt{\left(\frac{1}{w} - a\right)\left(\frac{1}{w} - b\right)} = \frac{1}{w} \sqrt{(1-aw)(1-bw)}$$

which is fine apart from a pole at ∞ , which is not the same as a branch point.

$$f(z) = \sqrt{(z-a)(z-b)} = \sqrt{\rho_1 \rho_2} e^{i(\theta_1 + \theta_2)/2}$$

with say $0 \leq \theta_1, \theta_2 < 2\pi$. In this case the cut is directly between a and b . Making a different choice, $-\pi \leq \theta_1 < \pi$. and $0 \leq \theta_2 < \pi$.

This cut appears to end at $+\infty$ and $-\infty$ which is not a branch point. However the complex plane is really a sphere, infinity is really a point and the cut just happens to go through the point ∞ .