Dynamical Systems

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Introduction

A dynamical system is a set of equations describing the evolution of a system with respect to a time-like variable. Usually they are non-linear.
The possible states of the system define the state space/phase space.

Example: The logistic map

$$x_{n+1} = \mu x_n(1 - x_n)$$

For $0 \leq \mu \leq 4$ this describes evolution with respect to a discrete time n in a state space $[0, 1]$.

Example: The Lotka-Voltera equations

$$\begin{aligned}
\dot{r} &= r(a - br - cs) \\
\dot{s} &= s(d - er - fs)
\end{aligned}$$

Where a-f are positive constants. These describe continous evolution in a state space $(r, s) \in [0, \infty] \times [0, \infty]$ as a model for the population of two species competing for the same food supply.

Example: The non-linear Schrödinger equation

$$i\frac{\partial \Psi}{\partial t} = \nabla^2 \Psi + |\Psi|^2 \Psi$$

This describes evolution in an infinite dimensional statespace of possible wavefunctions.

Because the equations are non-linear, it is often impossible to find a complete set of closed form analytic solutions. Instead, we resort to a mixture of geometric and analytic arguments, and aim to say something about the generic long-term behaviour.

Example:

$$\begin{aligned}
\dot{r} &= r(3 - r - s) \\
\dot{s} &= s(2 - r - s)
\end{aligned}$$

Consider the regions where $\dot{r}$ and $\dot{s}$ are $> 0,\ < 0,\ = 0$.
If $r, s > 0$ then

$$\begin{aligned}
r + s < 2 &\implies \dot{r}, \dot{s} > 0 \\
2 < r + s < 3 &\implies \dot{r} > 0,\ \dot{s} < 0 \\
r + s > 3 &\implies \dot{r}, \dot{s} < 0
\end{aligned}$$

$\dot{r} = 0$ if $r = 0$ or $r + s = 3$.
$\dot{s} = 0$ if $s = 0$ or $r + s = 2$.
Therefore $\dot{r} = \dot{s} = 0$ at the fixed points $(0,0),\ (3,0),\ (0,2)$. This gives the phase portrait/diagram/plane[1]
The most important feature of the phase portrait is that all solutions with $r > 0$ tend to the stable fixed point $(3,0)$. The fixed points $(0,0)$ and $(0,2)$ are unstable. There are no periodic orbits.

Example:

$$\begin{aligned}\dot{r} &= r(3 - r - s)\\ \dot{s} &= s(2 - \mu r - s)\end{aligned}$$

In this case, a new fixed point $(\frac{1}{1-\mu}, \frac{2-3\mu}{1-\mu})$ appears in the state space at $\mu = \frac{2}{3}$ and for $\mu < \frac{2}{3}$ is the long term stable attractor.
A qualitative change in the solution structure is called a bifurcation.

Example:

$$\begin{aligned}\dot{x} &= -y + \epsilon x(\mu - x^2 - y^2)\\ \dot{y} &= \quad x + \epsilon y(\mu - x^2 - y^2)\end{aligned}$$

Use polar coordinates which are a more natural choice for this problem. In general:

$$\boxed{\begin{aligned}\dot{r} &= \frac{x\dot{x} + y\dot{y}}{r}\\ \dot{\theta} &= \frac{x\dot{y} - y\dot{x}}{r^2}\end{aligned}}$$

Which are equations that will be referred to frequently. [2] In our example, they become

$$\begin{aligned}\dot{r} &= \epsilon r(\mu - r^2)\\ \dot{\theta} &= 1\end{aligned}$$

Consider $\dot{r}$ and $\dot{\theta}$:
The infinite set of periodic solutions for $\epsilon = 0$ is destroyed by any perturbation to $\epsilon \neq 0$. This is an example of structural instability. If $\mu > 0$ then just one limit cycle survives and is stable (unstable) for $\epsilon > 0$ ($\epsilon < 0$). The appearance of the limit cycle as $\mu \uparrow$ through 0 is another form of bifurcation.

Example: In 2D the points of successive interection x_n of a solution near a limit cycle with a line ε perpendicular to the cycle, move monotonically towards/away from the point of intersection x^* of ε with the limit cycle.
The point x^* is a stable/unstable fixed point of this Poincaré recurrence map.
In 3 or higher dimensions, or in 2D with time-dependent coefficients there is room for much more complicated behaviour including <u>chaos</u>.

1 Basic Definitions

We need some termonology.

[1] Phase portrait, phase diagram, phase plane will be used interchangably
[2] so learn them now!

1.1 Notation

We only consider ODEs of the form

$$\dot{\mathbf{x}} = \mathbf{f(x)} \tag{*}$$

for $\mathbf{x}$ in a phase space/state space $E \subset \mathbb{R}^n$. The n first order ODEs form a dynamical system of order (dimension) n.

Since

$$\frac{\partial \mathbf{f}}{\partial t} = 0$$

we call the system autonomous.

A non-autonomous system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$ can be made autonomous by setting

$$\mathbf{y} = (\mathbf{x}, t) \quad \text{with} \quad \dot{\mathbf{y}} = (\mathbf{f}(\mathbf{y}), 1)$$

The nth order ODE

$$\frac{d^n x}{dt^n} = g\left(x, \frac{dx}{dt}, \ldots, \frac{d^{n-1}x}{dt^{n-1}}\right)$$

can be put in the form (*) by setting

$$\mathbf{y} = \left(x, \frac{dx}{dt}, \ldots, \frac{d^{n-1}x}{dt^{n-1}}\right)$$

with

$$\dot{\mathbf{y}} = (y_2, y_3, \ldots, g(\mathbf{y}))$$

Similarly we will consider maps in the form

$$\mathbf{x}_{n+1} = \mathbf{F}(\mathbf{x}_n)$$