Dynamical Systems

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Introduction

A dynamical system is a set of equations describing the evolution of a system with respect to a time-like variable. Usually they are non-linear.
The possible states of the system define the state space/phase space.

Example: The logistic map

$$x_{n+1} = \mu x_n(1 - x_n)$$

For $0 \leq \mu \leq 4$ this describes evolution with respect to a discrete time n in a state space $[0, 1]$.

Example: The Lotka-Voltera equations

$$\begin{aligned} \dot{r} &= r(a - br - cs) \\ \dot{s} &= s(d - er - fs) \end{aligned}$$

Where a-f are positive constants. These describe continous evolution in a state space $(r, s) \in [0, \infty] \times [0, \infty]$ as a model for the population of two species competing for the same food supply.

Example: The non-linear Schrödinger equation

$$i\frac{\partial \Psi}{\partial t} = \nabla^2 \Psi + |\Psi|^2 \Psi$$

This describes evolution in an infinite dimensional statespace of possible wavefunctions.

Because the equations are non-linear, it is often impossible to find a complete set of closed form analytic solutions. Instead, we resort to a mixture of geometric and analytic arguments, and aim to say something about the generic long-term behaviour.

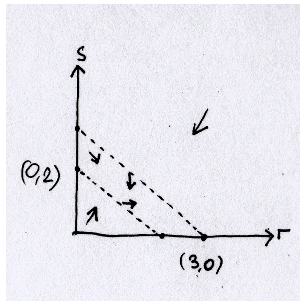
Example:

$$\begin{aligned} \dot{r} &= r(3 - r - s) \\ \dot{s} &= s(2 - r - s) \end{aligned}$$

Consider the regions where $\dot{r}$ and $\dot{s}$ are > 0, < 0, $= 0$.
If $r, s > 0$ then

$$\begin{aligned} r + s < 2 &\implies \dot{r}, \dot{s} > 0 \\ 2 < r + s < 3 &\implies \dot{r} > 0,\ \dot{s} < 0 \\ r + s > 3 &\implies \dot{r}, \dot{s} < 0 \end{aligned}$$

$\dot{r} = 0$ if $r = 0$ or $r + s = 3$.
$\dot{s} = 0$ if $s = 0$ or $r + s = 2$.
Therefore $\dot{r} = \dot{s} = 0$ at the fixed points $(0,0),\ (3,0),\ (0,2)$. This gives the phase portrait/diagram/plane[1]



The most important feature of the phase portrait is that all solutions with $r > 0$ tend to the stable fixed point $(3,0)$. The fixed points $(0,0)$ and $(0,2)$ are unstable. There are no periodic orbits.

Example:

$$\begin{aligned}\dot{r} &= r(3 - r - s)\\ \dot{s} &= s(2 - \mu r - s)\end{aligned}$$

In this case, a new fixed point $(\frac{1}{1-\mu}, \frac{2-3\mu}{1-\mu})$ appears in the state space at $\mu = \frac{2}{3}$ and for $\mu < \frac{2}{3}$ is the long term stable attractor.
A qualitative change in the solution structure is called a bifurcation.

Example:

$$\begin{aligned}\dot{x} &= -y + \epsilon x(\mu - x^2 - y^2)\\ \dot{y} &= \quad x + \epsilon y(\mu - x^2 - y^2)\end{aligned}$$

Use polar coordinates which are a more natural choice for this problem. In general:

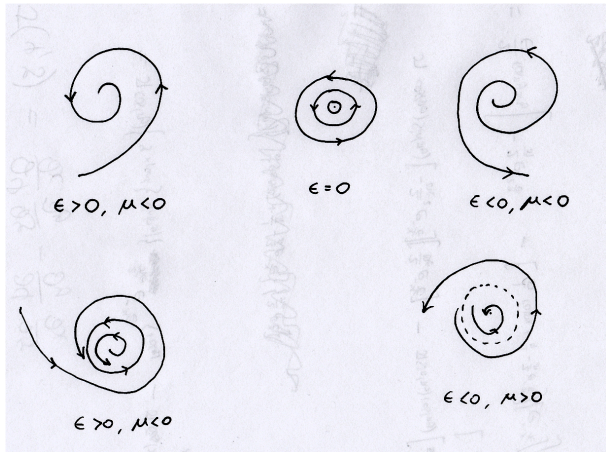
$$\boxed{\begin{aligned}\dot{r} &= \frac{x\dot{x} + y\dot{y}}{r}\\ \dot{\theta} &= \frac{x\dot{y} - y\dot{x}}{r^2}\end{aligned}}$$

Which are equations that will be referred to frequently. [2] In our example, they become

$$\begin{aligned}\dot{r} &= \epsilon r(\mu - r^2)\\ \dot{\theta} &= 1\end{aligned}$$

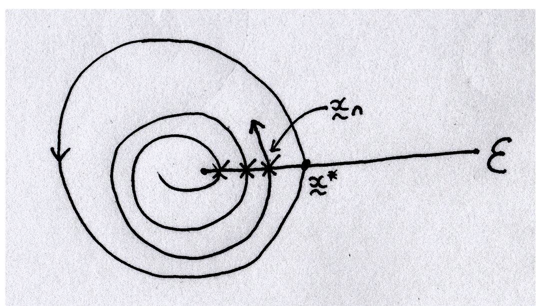
Consider $\dot{r}$ and $\dot{\theta}$:

[1] Phase portrait, phase diagram, phase plane will be used interchangably
[2] so learn them now!



The infinite set of periodic solutions for $\epsilon = 0$ is destroyed by any perturbation to $\epsilon \neq 0$. This is an example of structural instability. If $\mu > 0$ then just one limit cycle survives and is stable (unstable) for $\epsilon > 0$ ($\epsilon < 0$). The appearance of the limit cycle as $\mu \uparrow$ through 0 is another form of bifurcation.

Example: In 2D the points of successive interection x_n of a solution near a limit cycle with a line ε perpendicular to the cycle, move monotonically towards/away from the point of intersection x^* of ε with the limit cycle.



The point x^* is a stable/unstable fixed point of this Poincaré recurrence map.
In 3 or higher dimensions, or in 2D with time-dependent coefficients there is room for much more complicated behaviour including chaos.

1 Basic Definitions

We need some termonology.

1.1 Notation

We only consider ODEs of the form

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) \tag{*}$$

for $\mathbf{x}$ in a phase space/state space $E \subset \mathbb{R}^n$. The n first order ODEs form a dynamical system of order (dimension) n.

Since

$$\frac{\partial \mathbf{f}}{\partial t} = 0$$

we call the system autonomous.

A non-autonomous system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$ can be made autonomous by setting

$$\mathbf{y} = (\mathbf{x}, t) \quad \text{with} \quad \dot{\mathbf{y}} = (\mathbf{f}(\mathbf{y}), 1)$$

The nth order ODE

$$\frac{d^n x}{dt^n} = g\left(x, \frac{dx}{dt}, \ldots, \frac{d^{n-1}x}{dt^{n-1}}\right)$$

can be put in the form (*) by setting

$$\mathbf{y} = \left(x, \frac{dx}{dt}, \ldots, \frac{d^{n-1}x}{dt^{n-1}}\right)$$

with

$$\dot{\mathbf{y}} = (y_2, y_3, \ldots, g(\mathbf{y}))$$

Similarly we will consider maps in the form

$$\mathbf{x}_{n+1} = \mathbf{F}(\mathbf{x}_n)$$

1.2 Initial Value Problem

Consider the IVP:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$
$$\mathbf{x}(t_0) = \mathbf{x}_0$$

In Analysis II we showed that if $\mathbf{f}$ satisfies a Lipshitz condition then we are guaranteed that a solution exists in a neighbourhood of $\mathbf{x}_0$, t_0 and is unique.
(Lipshitz condition is that $\exists\, L$, a, s.t. $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x})| < L|\mathbf{x} - \mathbf{y}|\ , \quad \forall\, |\mathbf{x} - \mathbf{x}_0|,\ |\mathbf{y} - \mathbf{x}_0| < a.$)

Moreover, solutions $\mathbf{x}(t\,; \mathbf{x}')$ to

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$
$$\mathbf{x}(t_0) = \mathbf{x}'$$

exist, are unique and they depend continously on $\mathbf{x}'$, t.

Note we are not guaranteed existence for all times.

Example:

$$\dot{x} = x^2$$
$$x(0) = 1$$

This has solution

$$x(t) = \frac{1}{1-t}$$

and so $x \to \infty$ as $t \uparrow 1$.

If $|\mathbf{x}(t)| \to \infty$ as $t \to T$ $(< \infty)$ then we call this finite time blow up.

Example: Non-uniqueness when $\mathbf{f}$ is non-Lipshitz. If

$$\dot{x} = \begin{cases} \sqrt{x} & \text{for } x > 0 \\ 0 & \text{for } x \le 0 \end{cases} \quad \text{and } x(0) = 0$$

Then there is a family of solutions

$$\left.\begin{array}{ll} x = 0 & t < \tau \\ x = \frac{1}{4}(t-\tau)^2 & t > \tau \end{array}\right\} \text{ for any } \tau \geq 0$$

From now on we will assume that $\mathbf{f}$ is differentiable (and so Lipshitz).

1.3 Trajectories and Flows

Consider the autonomous system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ The solution $\mathbf{x}(t)$ to the IVP with $\mathbf{x}(0) = \mathbf{x}_0$ defines a trajectory. (orbit/integral curve)

The distance travelled along this curve clearly only depends on $t - t_0$, and we could consider lots of starting points $\mathbf{x}_0$.
This motivates the idea of a flow.

Definition: (Flow)
Given $\mathbf{f}$, the corresponding flow is defined to be a (the) function $\boldsymbol{\phi}_t(\mathbf{x})$ from $E \times \mathbb{R} \to E$ such that

$$\frac{\partial}{\partial t}\boldsymbol{\phi}_t(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{\phi}_t(\boldsymbol{x})), \qquad \boldsymbol{\phi}_0(\boldsymbol{x}) = \boldsymbol{x}$$

The solution to the IVP $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}),\ \boldsymbol{x}(t_0) = \boldsymbol{x}_0$, is just $\boldsymbol{x}(t) = \boldsymbol{\phi}_{t-t_0}(\boldsymbol{x}_0)$
Clearly

$$\boldsymbol{\phi}_{s+t}(\boldsymbol{x}) = \boldsymbol{\phi}_s(\boldsymbol{\phi}_t(\boldsymbol{x})) = \boldsymbol{\phi}_t(\boldsymbol{\phi}_s(\boldsymbol{x}))$$

Aside: We can establish another link with maps, by defining $\boldsymbol{x}_{n+1} = \boldsymbol{F}(\boldsymbol{x}_n) = \boldsymbol{\phi}_{\Delta t}(\boldsymbol{x}_n)$.
Clearly $\boldsymbol{\phi}_{n\Delta t}(\boldsymbol{x}) = \boldsymbol{F}(\boldsymbol{\phi}_{(n-1)\Delta t}(\boldsymbol{x})) = \boldsymbol{F}^n(\boldsymbol{x})$

1.4 Flows, Trajectories, Orbits, Invariant Sets, & Limiting Sets

Using the idea of a flow, we define the following:

Definition: Orbits/Trajectories
The orbit of $\boldsymbol{\phi}_t(\boldsymbol{x})$ through $\boldsymbol{x}_0$ is the set $\mathcal{O}(\boldsymbol{x}_0) = \{\boldsymbol{\phi}_t(\boldsymbol{x}_0) : -\infty < t < \infty\}$.
The forwards (backwards) orbit is

$$\mathcal{O}^{+\,(-)}(\boldsymbol{x}_0) = \{\boldsymbol{\phi}_t(\boldsymbol{x}_0) : t \geq 0 \ (t \leq 0)\}$$

Definition: Invariant sets
A set of points $\Lambda \subset E$ is invariant if $\boldsymbol{x} \in \Lambda \implies \mathcal{O}(\boldsymbol{x}) \subset \Lambda$
Clearly $\mathcal{O}(\boldsymbol{x}_0)$ is invariant, and so is any union of orbits.

Particular cases of interest are

Definition: Fixed points
$\boldsymbol{x}_0$ is a periodic point with period T if $\boldsymbol{\phi}_T(\boldsymbol{x}_0) = \boldsymbol{x}_0$ for some $T > 0$ and $\boldsymbol{\phi}_t(\boldsymbol{x}_0) \neq \boldsymbol{x}_0$ for $0 < t < T$.
The set $\{\boldsymbol{\phi}_t(\boldsymbol{x}_0) : 0 \leq t < T\}$ is the periodic orbit through $\boldsymbol{x}_0$.

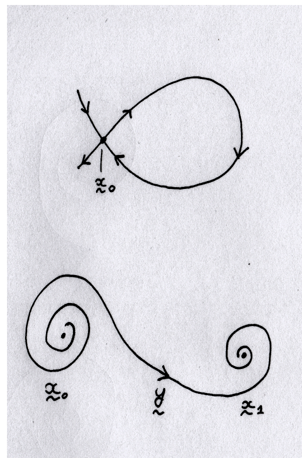
Definition: Limit Cycle
A limit cycle is an isolated periodic orbit, i.e. there are no other periodic orbits within a sufficiently

small neighbourhood.

Definition: Homoclinic and Heteroclinic orbits
If $\boldsymbol{x}_0$ is a fixed point and $\exists\ \boldsymbol{y} \neq \boldsymbol{x}_0$ such that $\boldsymbol{\phi}_t(\boldsymbol{y}) \to \boldsymbol{x}_0$ as $t \to \pm\infty$ then $\mathcal{O}(\boldsymbol{y})$ is a homoclinic orbit.

If $\boldsymbol{x}_0$, $\boldsymbol{x}_1$ are fixed points and $\exists\ \boldsymbol{y} \neq \boldsymbol{x}_0, \boldsymbol{x}_1$ with $\boldsymbol{\phi}_t(\boldsymbol{y}) \to \boldsymbol{x}_0$ as $t \to -\infty$, $\boldsymbol{\phi}_t(\boldsymbol{y}) \to \boldsymbol{x}_1$ as $t \to +\infty$ then $\mathcal{O}(\boldsymbol{y})$ is a heteroclinic orbit.

Example of a Heteroclinic and a Homoclinic orbit



If we are interested in the long-term behaviour of trajectories, it is not enough simply to think of

$$\lim_{t\to\infty} \boldsymbol{\phi}_t(\boldsymbol{x})$$

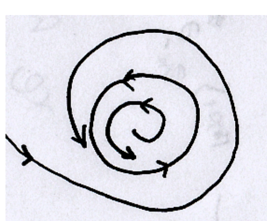
because the limit might not exist; for example a limit cycle.

Definition: Limit set
The ω-limit set of $\boldsymbol{x}$ is

$$\omega(\boldsymbol{x}) = \{\boldsymbol{y} : \exists \text{ a sequence } (t_n) \text{ with } \ \boldsymbol{\phi}_{t_n}(\boldsymbol{x}) \to \boldsymbol{y} \text{ and } t_n \to \infty\}$$

Similarly the α limit set is defined by sequences with $t_n \to -\infty$.

Example:



$$\begin{aligned} \dot{r} &= r(1-r^2) \\ \dot{\theta} &= 1 \end{aligned}$$

For $\boldsymbol{x}$ with $0 < |\boldsymbol{x}| < 1$ we have that

$$\omega(\boldsymbol{x}) = \{\boldsymbol{y} : |\boldsymbol{y}| = 1\} \qquad \alpha(\boldsymbol{x}) = \{\boldsymbol{0}\}$$

For $|\boldsymbol{x}| > 1$

$$\omega(\boldsymbol{x}) = \{\boldsymbol{y} : |\boldsymbol{y}| = 1\} \qquad \alpha(\boldsymbol{x}) = \{\}$$

In fact, the limit sets are always invariant sets since

- If $\mathcal{O}(\boldsymbol{x})$ is bounded then $\omega(\boldsymbol{x})$ is non-empty.
- If $\boldsymbol{\phi}_t(\boldsymbol{x}) \to \infty$ then $\omega(\boldsymbol{x}) = \{\}$.

Definition: (For maps)
Fixed points solve $\boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{x}$. Orbits, periodic points etc. are defined as above by replacing $\boldsymbol{\phi}_t(\boldsymbol{x})$ by $\boldsymbol{F}^n(\boldsymbol{x})$, etc.

A periodic orbit with period N is called an N-cycle.

1.5 Topological equivalence and structural stability of flows

This section is in a handout.

2 Flow near fixed points

The simplest features of a flow are the fixed points. Find them by first solving $\boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{0}$.

2.1 Linearisation

If $\boldsymbol{f}$ is sufficiently smooth and $\boldsymbol{x}_0$ is a fixed point, we expand in a Taylor series with $\boldsymbol{f}(\boldsymbol{x}_0) = \boldsymbol{0}$ to obtain

$$\dot{\boldsymbol{y}} = A\boldsymbol{y} + O(|\boldsymbol{y}|^2)$$

where $\boldsymbol{y} = \boldsymbol{x} - \boldsymbol{x}_0$ and

$$A_{ij} = \left.\frac{\partial f_i}{\partial x_j}\right|_{\boldsymbol{x}_0}$$

is the Jacobian matrix (Written $D\boldsymbol{f}$ in Glendinning).

The hope (see below when it is true) is that the flow $\boldsymbol{\phi}_t^{\boldsymbol{f}}$ is like the flow corresponding to the linearization $\dot{\boldsymbol{y}} = A\boldsymbol{y}$.

2.2 Classification of fixed points

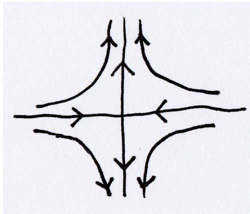
In 2D,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $D = ad - bc$, $T = a + d$ and the eigenvalues are

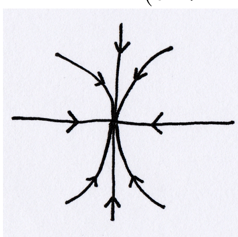
$$\lambda_{1,2} = \frac{1}{2}(T \pm \sqrt{T^2 - 4T})$$

(i) Saddle points ($D < 0$, real λ's with opposite signs)



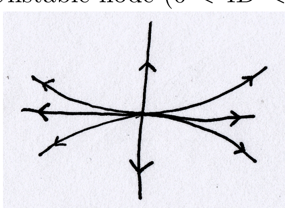
$$A = \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} \qquad \lambda_1 < 0 < \lambda_2$$

(ii) Stable nodes ($0 < 4D < T^2$, $T < 0$, real roots, both negative)



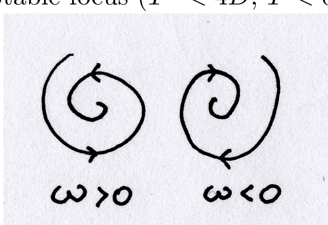
$$\lambda_1 < \lambda_2 < 0 \qquad \frac{y_2}{y_1} \propto e^{(\lambda_2 - \lambda_1)t} \to \infty \text{ as } t \to \infty$$

(iii) Unstable node ($0 < 4D < T^2$, $T > 0$, real roots, both positive)



$$\lambda_1 < \lambda_2 < 0 \qquad \frac{y_2}{y_1} \propto e^{(\lambda_2 - \lambda_1)t} \to 0 \text{ as } t \to -\infty$$

(iv) Stable focus ($T^2 < 4D$, $T < 0$, complex roots, $\lambda = \rho \pm i\omega$, $\rho < 0$)

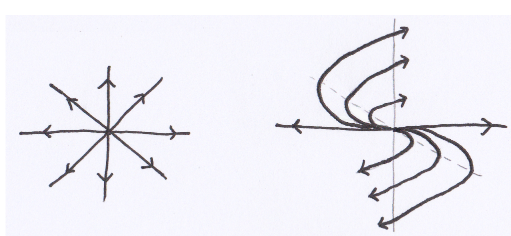


$$A = \begin{pmatrix} \rho & -\omega \\ \omega & \rho \end{pmatrix} \qquad \dot{r} = \rho r \qquad \dot{\theta} = \omega$$

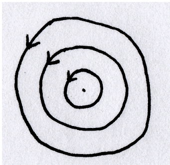
(v) Unstable focus ($T^2 < 4D$, $T > 0$, complex roots, $\rho > 0$)

(vi) Two degenerate cases with equal eigenvalues on the border $T^2 = 4D$ between nodes and foci. (e.g. $\lambda > 0$)

Stellar Node	Improper Node
$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$	$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$

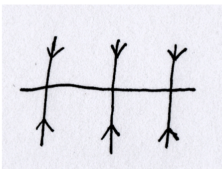


(vii) Centres ($T = 0$, $D > 0$, $\lambda = \pm i\omega$)

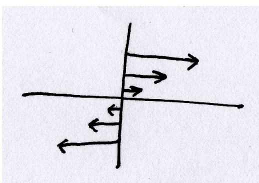


All trajectories are closed. This is on the border between stable foci and unstable foci.

(viii) Line of Fixed points. Specia case on the border $D = 0$ between saddles and nodes, where at least one λ is zero. For example,

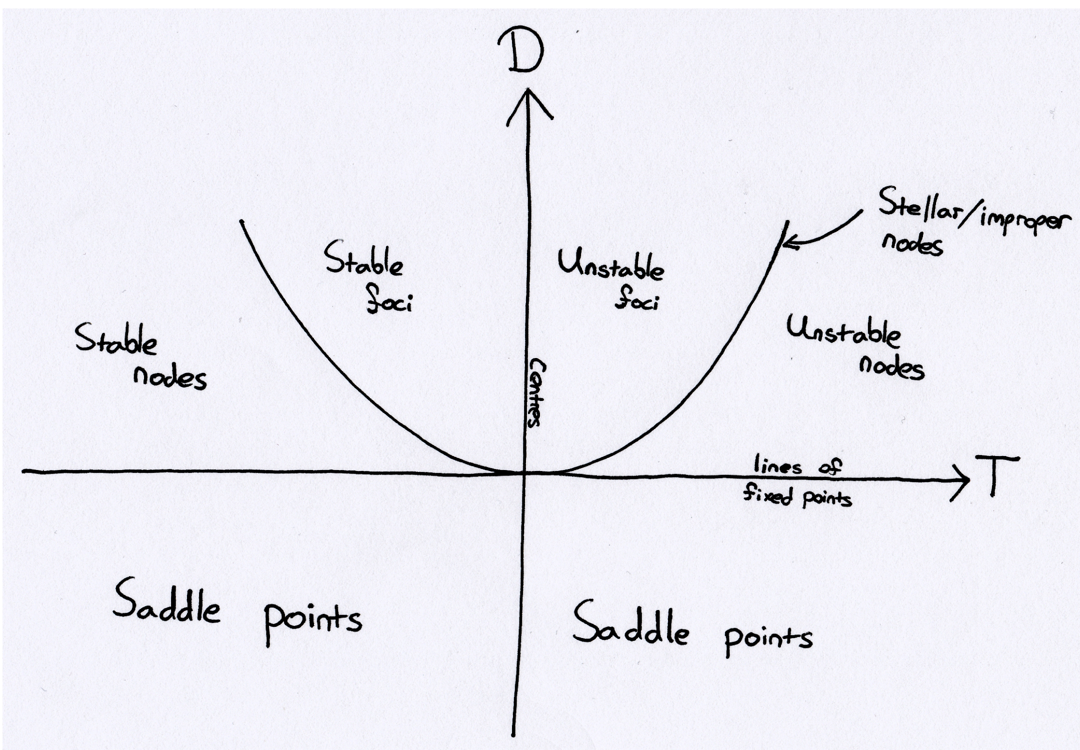


$$A = \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix} \quad (\lambda < 0)$$



$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

In summary



Notes:

(1) A Hamiltonian system is of the form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \partial H/\partial y \\ -\partial H/\partial x \end{pmatrix}$$

for some $H(x,y)$. Fixed points are always saddles or centers because

$$A = \begin{pmatrix} H_{xy} & H_{yy} \\ -H_{xx} & -H_{xy} \end{pmatrix} \implies T = 0$$

Also $\dot{H} = \dot{\boldsymbol{x}} \cdot \nabla H = 0$ so the trajectories are (parts of) the contours of H.

(2) The cannonical forms are obtained by using the eigenvectors $\boldsymbol{e}_i$ if $\lambda_i \in \mathbb{R}$ (possibly "generalised" for equal λ's and JNF, see Glendinning p 63) and $\mathrm{Re}(\boldsymbol{e}_1)$ and $\mathrm{Im}(\boldsymbol{e}_1)$ for $\lambda = \rho \pm i\omega$.

(3) It is not necessary to change basis to classify the fixed point since T, D, λ's are invariant under $A \mapsto P^{-1}AP$, but knowing the eigenvectors may help sketch the phase diagram for the case of a saddle point.

Pertubation to A (linear perturbations)
Cases (i)-(v) are robust - a small perturbation to A gives the same sort of eigenvalues and hence the same sort of fixed point.

Case (vi), if perturbed, may become a node or a focus, but these are topologically equivalent. The stability is not changed.

Cases (vii) and (viii) are fragile, - a small perturbation to (vii), ($\lambda = \pm i\omega$) may destroy the closed trajectories to give slow spiralling inwards or outwards. A small perturbation to (viii) ($\lambda = 0$) may destroy the line of fixed points to give a slow drift towards/away from the fixed point, and hence a saddle or node.

The fragility is because there are one or two eigenvalues on the border $\mathrm{Re}(\lambda) = 0$ between stability and instability.

Definition: Hyperbolic fixed point

A fixed point $\boldsymbol{x}_0$ of the system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ is a hyperbolic fixed point if none of the eigenvalues λ of the Jacobian $\left.\frac{\partial f_i}{\partial x_j}\right|_{\boldsymbol{x}=\boldsymbol{0}}$ satisfy $\mathrm{Re}(\lambda) = 0$, and is non-hyperbolic otherwise.

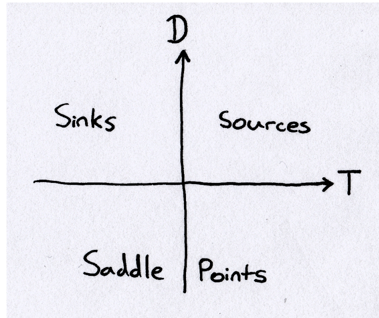
In n-dimensions ($n \geq 0$) we classify a hyperbolic fixed point as

(i) a sink if all eigenvalues have Re $(\lambda) < 0$.

(ii) a source if all eigenvalues have Re $(\lambda) > 0$.

(iii) a saddle point otherwise (some > 0, some < 0)

In 2D



Non hyperbolic fixed points are important in bifurcation theory.

2.3 The effects of non-linear terms

It is in fact true that the linearised system $\dot{\boldsymbol{y}} = A\boldsymbol{y}$ is essentially the same as the non linear system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ near a fixed point $\boldsymbol{x}_0$ provided

(i) The fixed point is hyperbolic

(ii) The non linear terms are $O(|\boldsymbol{x} - \boldsymbol{x}_0|^2)$

(If (i) holds then the systems are topologically equivalent, if (ii) also holds, then $\boldsymbol{h}(\boldsymbol{x})$ is a near identity map, so nodes are nodes and foci are foci.)

2.3.1 Stable and Unstable Manifold

First, we formalise the idea of stable and unstable directions in the linearised system.

Definition: The stable, unstable and center invariant subspaces of the linearisation of $\boldsymbol{f}$ at a fixed point are the local linear subspaces.

E^s, E^u, E^c spanned by the eigenvectors of the Jacobian, whose eigenvalues have real parts < 0, > 0, $= 0$ respectively. (Generalised eigenvectors for JNF)

For some types of fixed point, these spaces may be empty, e.g. for a hyperbolic fixed point, E^c is empty by definition.

Then we extend th linear picture into the non-linear domain for a hyperbolic fixed point.

Stable Manifold Theorem
If $\mathbf{0}$ is a hyperbolic fixed point of the system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$, with linear stable and unstable invariant subspaces E^s and E^u then in a sufficiently small neighbourhood of the origin, there exist local stable and unstable manifolds

$$W^s_{loc}(\mathbf{0}) = \{\boldsymbol{x} : \boldsymbol{\phi}_t(\boldsymbol{x}) \to \mathbf{0} \text{ as } \ t \to \infty\}$$
$$W^u_{loc}(\mathbf{0}) = \{\boldsymbol{x} : \boldsymbol{\phi}_t(\boldsymbol{x}) \to \mathbf{0} \text{ as } \ t \to -\infty\}$$

these have the same dimension as E^s and E^u and are tangent to them at $\mathbf{0}$.

Notes:

(1) For a saddle point in $\mathbb{R}^2$, this guarantees the existence of two specific trajectories (the separatrices) that approach and leave the saddle.

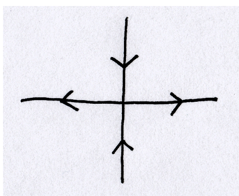
(2) For a sink this guarantees that all trajectories in some neighbourhood of the sink tend to it

(3) The local stable (unstable) manifolds can be extended to global stable (unstable) invariant manifolds W^s (W^u) by following the flow backwards (forwards).

The proof of this is not in the course - See Glendinning.

It is easy to calculate approximations to W^s and W^u for a saddle point in $\mathbb{R}^2$.

Wlog (Change of origin and basis) assume that the saddle is at $\boldsymbol{x} = \mathbf{0}$ and that E^s is $x = 0$ and E^u is $y = 0$.



Then W^s_{loc} becomes $x = S(y)$ with $S(0) = 0,\ S'(0) = 0$
W^u_{loc} is $y = U(x)$ with $U(0) = 0,\ U'(0) = 0$.

Since W^s and W^u are invariant,

$$\dot{x}|_{(S,y)} = \frac{dS}{dy}\,\dot{y}|_{(S,y)} \qquad\qquad \dot{y}|_{(x,U)} = \frac{dU}{dx}\,\dot{y}|_{(x,U)}$$

Example:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x - xy \\ -y + x^2 \end{pmatrix} \qquad\qquad A|_{\boldsymbol{x}=\mathbf{0}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Assume that

$$y = U(x) = a_2x^2 + a_3x^3 + a_4x^4 + \dots$$

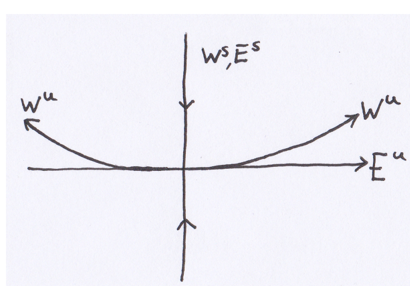
substitute into

$$\dot{y} = U'\dot{x} \implies -(a_2x^2 + a_3x^3 + a_4x^4 + \dots) + x^2 = (2a_2x + 3a_3x^2 + 4a_4x^3 + \dots)(x - a_2x^3 - \dots)$$

Equate coefficients:

$$\begin{aligned} a_2 &= \frac{1}{3} \\ a_3 &= 0 \\ a_4 &= \frac{2}{45} \quad \text{etc.} \end{aligned}$$

Locally for $|\boldsymbol{x}| \ll 1$



In higher dimensions, would write $\boldsymbol{y} = \boldsymbol{U}(\boldsymbol{x})$ and solve

$$\dot{y}_i = \frac{\partial U_i}{\partial x_i} \dot{x}_j$$

Where the x_j's span E^u and the y_i's span E^s.

2.3.2 Non-Linear terms in non-hyperbolic cases

There are many possibilities depending on the non-linear terms.

(a) $\lambda = \pm i\omega$ Linear system is a centre. Are the trajectories really closed?

Example:

$$\left.\begin{aligned} \dot{x} &= -y \pm x^3 \\ \dot{y} &= \quad x \pm y^3 \end{aligned}\right\} \implies \begin{aligned} \dot{r} &= \pm \tfrac{x^4+y^4}{r} \\ \dot{\theta} &= 1 \pm \tfrac{xy^3 - yx^3}{r^2} \end{aligned}$$

Now as $r \to 0$, $\dot{\theta} \to 1$. Thus the system is a stable focus if we choose '+' and an unstable focus if we choose '−'.

Example:

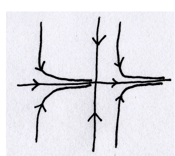
$$\begin{aligned} \dot{x} &= -y - 2yx^2 = \frac{\partial H}{\partial y} \\ \dot{y} &= \quad x + 2xy^2 = -\frac{\partial H}{\partial x} \end{aligned}$$

for $H = -\frac{x^2+y^2}{2} - x^2y^2$. The trajectories are contours of H, which are closed near the maximum at $\mathbf{0}$. So the fixed point really is a non-linear centre with nested periodic orbits.

(b) $\lambda_1 = 0$, $\lambda_2 \neq 0$. Which way does the system drift along $\boldsymbol{e}_1$?

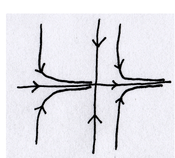
Example:

$$\begin{aligned} \dot{x} &= x^2 \\ \dot{y} &= -y \end{aligned}$$



Saddle-node

$$\begin{aligned} \dot{x} &= x^3 \\ \dot{y} &= -y \end{aligned}$$

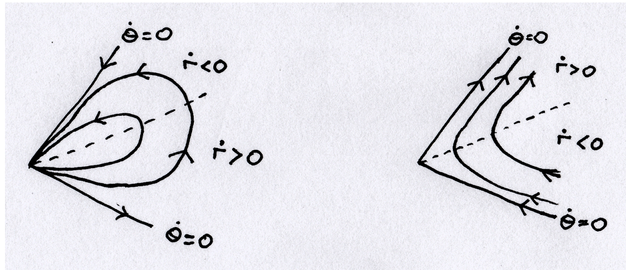


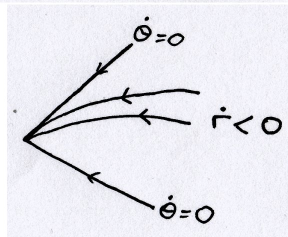
Non-linear saddle

(c) $\lambda_1 = \lambda_2 = 0$. Ad hoc methods needed.

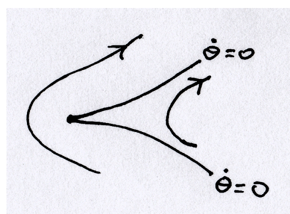
In some cases, it is useful to switch to polars and consider the sign of $\dot{r}$ and $\dot{\theta}$ as $r \to 0$.

Example: $\dot{\theta} > 0$





In nastier cases, the curves $\dot{\theta} = 0$ can form a cusp.



2.4 Sketching phase planes/portraits

Example:

$$\begin{aligned}\dot{x} &= x(1-y)\\ \dot{y} &= -y + x^2\end{aligned}$$

This has fixed points at $(0,0)$ and at $(\pm 1, 1)$.

$$(0,0) \qquad A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{saddle point}$$

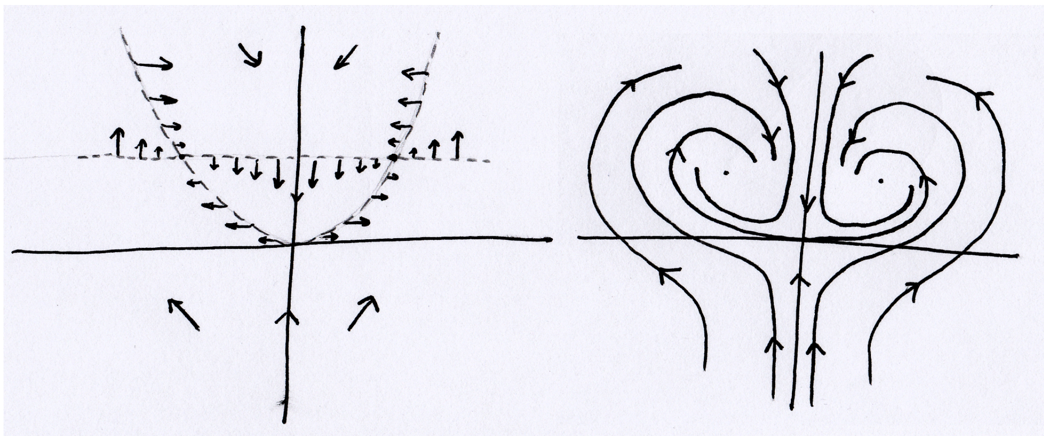
$$(\pm 1,1) \qquad A = \begin{pmatrix} 0 & \mp 1 \\ \pm 2 & -1 \end{pmatrix} \qquad \text{stable foci}$$

It is clear why $(0,0)$ is a saddle point. For $(\pm 1, 1)$ observe that $T = -1$, $D = 2$, $T^2 < 4D$, and therefore $(\pm 1, 1)$ are stable foci. Before sketching, observe that.

- $x = 0$ is a trajectory.
- $\dot{x} = 0$ on $y = 1$
- $\dot{y} = 0$ on $y = x^2$
- $\dot{y} \lesseqgtr 0$ if $y \gtreqless x^2$

- Symmetry in y axis; under $x \mapsto -x$, $\dot{x} \mapsto -\dot{x}$, and $\dot{y} \mapsto \dot{y}$
- Know that the unstable manifold is given by $y \approx \frac{1}{3}x^2 + \frac{2}{45}x^4$ near $x = 0$ (See earlier example)

Put all of this together to sketch the following:



In general, here are some steps to follow when sketching a phase plane.

1. Find the fixed points.
2. (a) Calculate the Jacobian to find the kind of eigenvalues and classify the fixed points.
   (b) If any are non-hyperbolic, consider non-linear terms, possible Hamiltonian structure, signs of $\dot{x}$, $\dot{y}$, $\dot{r}$ or $\dot{\theta}$ as $|\boldsymbol{x} - \boldsymbol{x}_0| \to 0$.
3. Calculate the eigenvectors for saddles.
4. Consider the sign of $\dot{x}$ and $\dot{y}$ - the curves where these are zero are called nullclines.
5. Construct the global picture by joining up the local behaviour near fixed points. (especially the separatrices of saddles) using the arrows from $\dot{x}$ and $\dot{y}$.
6. At some stage, decide whether there are periodic orbits.

For an additional example, see handout 2.4, "Example of Phase-Plane sketching."

3 Stability

3.1 Definitions

It is clear what it means for a hyperbolic node or focus to be stable, but we ned to be more careful with the stability of other kinds of fixed points or invariand sets, because there are at least two notions of stability.

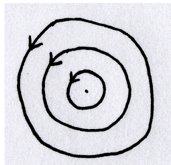
Definition: Lyapunov stability
A fixed point $\boldsymbol{x}_0$ of a flow $\boldsymbol{\phi}_t$ is Lyapunov stable if $\forall \epsilon > 0,\ \exists \delta > 0$ s.t. $|\boldsymbol{x} - \boldsymbol{x}_0| < \delta \implies |\boldsymbol{\phi}_t(\boldsymbol{x}) - \boldsymbol{x}_0| < \epsilon$ $\forall t > 0$. A motto is "Start near, stay near" stability.

Definition: Quasi-asymptotic stability (QAS)
A fixed point $\boldsymbol{x}_0$ of a flow $\boldsymbol{\phi}_t$ is quasi-asymptotically stable if $\exists \delta > 0$ s.t. $|\boldsymbol{x} - \boldsymbol{x}_0| < \delta \implies \boldsymbol{\phi}_t(\boldsymbol{x}) \to \boldsymbol{x}_0$ as $t \to \infty$.

These are not the same!

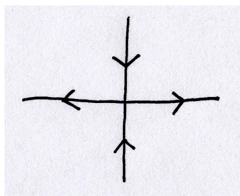
Example:

$$\begin{aligned} \dot{r} &= 0 \\ \dot{\theta} &= 1 \end{aligned}$$



Lyapunov stable, but not asyptotically stable.

$$\begin{aligned} \dot{r} &= r(1-r^2) \\ \dot{\theta} &= \sin^2(\theta/2) \end{aligned}$$



The point $r = 1$, $\theta = 0$ is QAS but not Lyapunov stable

Definition: Asymptotic stability (AS)
A fixed point $\boldsymbol{x}_0$ is asymptotically stable if it is both Lyapunov stable and quasi-asymptotically stable.

Example: A sink (Re $\lambda_i < 0\ \forall\, i$) is asymptotically stable. (Choose δ sufficiently small that the linear terms dominate) A saddle or source is obviously not Lyapunov stable, hence not AS.

For other kinds of invariant sets Λ e.g. limit cycles, define

$$N_\delta(\Lambda) = \{\boldsymbol{x} : \exists \boldsymbol{y} \in \Lambda \text{ with } \ |\boldsymbol{x} - \boldsymbol{y}| < \delta\}$$

and say

$$\boldsymbol{\phi}_t(\boldsymbol{x}) \to \Lambda$$

if

$$\min_{\boldsymbol{y}\in\Lambda} \{|\boldsymbol{\phi}_t(\boldsymbol{x}) - \boldsymbol{y}|\} \to 0 \quad \text{as} \quad t \to \infty$$

Then say Λ is Lyapunov stable if $\forall \epsilon > 0\ \exists \delta > 0$ s.t. $\boldsymbol{x} \in N_\delta(\Lambda) \implies \boldsymbol{\phi}_t(\boldsymbol{x}) \in N_\epsilon(\Lambda)\ \forall t > 0$.

Say Λ is QAS if $\exists \delta > 0$ s.t. $\boldsymbol{x} \in N_\delta(\Lambda) \implies \boldsymbol{\phi}_t(\boldsymbol{x}) \to \Lambda$ as $t \to \infty$.

Say Λ is AS if it is both LS and QAS.

3.2 Lyapunov functions

These allow us to say more about the stability of a fixed point which wlog we take to be $\boldsymbol{x} = 0$.

Definition: Lyapunov function
A continously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ is a Lyapunov function for $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ in a domain $\mathcal{D} \subset \mathbb{R}^n$ if

(i) $V(\boldsymbol{0}) = 0$ and $V(\boldsymbol{x}) > 0$ for $\boldsymbol{x} \in \mathcal{D}\backslash\{\boldsymbol{0}\}$. "*Positive semi-definite*".

(ii) $\dot{V}(\boldsymbol{x}) \equiv \boldsymbol{f} \cdot \nabla V \leq 0\ \forall \boldsymbol{x} \in \mathcal{D}$. "*Non-increasing*".

Note that (ii) means trajectories head "downhill" (or at least not uphill) and (i) means that $\mathbf{0}$ is the lowest point. These properties allow us to prove:

Theorem: Lyapunov's First Theorem
If a Lyapunov function exists, then $\mathbf{0}$ is Lyapunov stable.

Proof. Wlog we can assume ϵ is sufficiently small that $\{|\boldsymbol{x}| < \epsilon\} \subset \mathcal{D}$. Let

$$m = \inf\{V(\boldsymbol{x}) : |\boldsymbol{x}| = \epsilon\}$$

m is attained, and $m > 0$ since $\{|\boldsymbol{x}| = \epsilon\}$ is compact. Define

$$C_{m,\epsilon} = \{\boldsymbol{x} : V(\boldsymbol{x}) < m \text{ and } |\boldsymbol{x}| < \epsilon\}$$

Then $\boldsymbol{x} \in C_{m,\epsilon} \implies \boldsymbol{\phi}_t(\boldsymbol{x}) \in C_{m,\epsilon}\ \forall\, t > 0$ since V is non-increasing and $V \geq m$ on the boundary of $C_{m,\epsilon}$.

Finally, choose $\delta > 0$ such that $\{|\boldsymbol{x}| < \delta\} \subset C_{m,\epsilon}$. Then

$$|\boldsymbol{x}| < \delta \implies |\boldsymbol{\phi}_t(\boldsymbol{x})| < \epsilon \ \ \forall\, t > 0$$

□

A trajectory can head downhill and still not end up at $\mathbf{0}$. However, there is an important constraint.

Theorem: La Sale's Invariance Principle
If V is a Lyapunov function on a bounded domain $\mathcal{D}$ and $\mathcal{O}^+(\boldsymbol{x}) \subset \mathcal{D}$, then $\boldsymbol{\phi}_t(\boldsymbol{x}) \to$ an invariant subset of $\{\boldsymbol{x} : \dot{V}(\boldsymbol{x}) = 0\}$.

Proof.
$\mathcal{O}^+(\boldsymbol{x}) \subset \mathcal{D}$ so $V(\boldsymbol{\phi}_t(\boldsymbol{x}))$ is monotonically decreasing and bounded below by 0 $\implies V(\boldsymbol{\phi}_t(\boldsymbol{x})) \to \alpha$ for some $\alpha \geq 0$ as $t \to \infty$.

$\mathcal{D}$ is bounded $\implies \omega(\boldsymbol{x})$ is non empty.

Let $\boldsymbol{y} \in \omega(\boldsymbol{x})$ then $\exists\, \{t_n\}$ s.t. $\boldsymbol{\phi}_{t_n}(\boldsymbol{x}) \to \boldsymbol{y} \implies V(\boldsymbol{y}) = \alpha$ by continuity of V.

Therefore $V(\boldsymbol{\phi}_t(\boldsymbol{y})) = \alpha$ since $\boldsymbol{\phi}_t(\boldsymbol{y}) \in \omega(\boldsymbol{x})$.

Now $V(\boldsymbol{\phi}_t(\boldsymbol{y})) = \alpha \ \ \forall\, t \implies \dot{V}(\boldsymbol{y}) = 0$.

This holds for any $\boldsymbol{y} \in \omega(\boldsymbol{x})$ and thus

$$\omega(\boldsymbol{x}) \subset \{\dot{V} = 0\}$$

We know ω is invariant, and thus we are done. □

Corollary: If V is a Lyapunov function on a domain $\mathcal{D}$ containing $\mathbf{0}$, and the only invariant subset of $\{\boldsymbol{x} : \dot{V}(\boldsymbol{x}) = 0\} \cap \mathcal{D}$ is $\{\mathbf{0}\}$, then $\boldsymbol{x} = \mathbf{0}$ is asymptotically stable.

Proof.
Choose k such that

$$C_k \equiv \{\boldsymbol{x} : V(\boldsymbol{x}) < k\} \subset \mathcal{D}$$

and choose δ such that $\{\boldsymbol{x} : |\boldsymbol{x}| < \delta\} \subset C_k$.
Then $|\boldsymbol{x}| < \delta \implies V(\boldsymbol{x}) < k \implies V(\boldsymbol{\phi}_t(\boldsymbol{x})) < k \implies \ \boldsymbol{\phi}_t(\boldsymbol{x}) \subset \mathcal{D}$ □

Example: Let

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + o(|\boldsymbol{x}|)$$

and let λ_i be the distinct eigenvalues of A with corresponding eigenvectors $\boldsymbol{e}_i$.
Suppose that $\mathrm{Re}(\lambda_i) < 0$,and let v_i be any positive real constants. Then, writing

$$\boldsymbol{x} = \sum_{i=1}^{n} a_i \boldsymbol{e}_i, \qquad V(\boldsymbol{x}) = \sum_{i=1}^{n} v_i |a_i|^2$$

have that $V \geq 0$ ($= 0$ only at $\mathbf{0}$) and $\dot{V}$ in a sufficiently small neighbourhood of $\mathbf{0}$. (See example sheet)
Therefore have that $\boldsymbol{x} = \mathbf{0}$ is asymptotically stable.

The main use of Lyapunov functions, however, is to find information about the domain of stability.

Definition: Domain of stability
The domain of stability (basin of attraction) of an asymptotically stable invariant set Λ is

$$\{\boldsymbol{x} : \boldsymbol{\phi}_t(\boldsymbol{x}) \to \Lambda \text{ as } t \to \infty\}$$

i.e. $\boldsymbol{x}$ such that

$$\inf_{\boldsymbol{y} \in \Lambda} \{|\boldsymbol{\phi}_t(\boldsymbol{x}) - \boldsymbol{y}|\} \to 0 \quad \text{as } t \to \infty$$

If the domain of stability in $\mathbb{R}^n$ then Λ is globally stable.

Example: We can show that the non-hyperbolic fixed point of the system

$$\begin{aligned} \dot{x} &= y - xy^2 \\ \dot{y} &= x^3 \end{aligned}$$

is asymptotically stable.

Without the $-xy^2$ term, the system would be Hamiltonian with $H = \frac{1}{2}y^2 + \frac{1}{4}x^4$ and $\dot{H} = 0$.
With the $-xy^2$ term, $\dot{H} = -x^4y^2 \leq 0$.
Therefore H is a Lyapunov function and so $\boldsymbol{\phi}_t(\boldsymbol{x}) \to$ an invariant subset of $\{\dot{H} = 0\} = \{x = 0 \text{ or } y = 0\}$.
But the only invariant subset of $\{\dot{H} = 0\}$ is $\boldsymbol{x} = \mathbf{0}$. Hence by the corollary, $\boldsymbol{x} = \mathbf{0}$.

Example:

$$\left.\begin{aligned} \dot{x} &= \ y + \mu(\tfrac{1}{3}x^3 - x) \\ \dot{y} &= \ -x \end{aligned}\right\} \text{ equivalent to } \ddot{x} + \mu(1 - x^2)\dot{x} + x = 0$$

The "Time-reversed van der Pol oscillator". Observe that $\boldsymbol{x} = \mathbf{0}$ is a sink for $\mu > 0$.

Guess $V = x^2 + y^2$ (e.g. from case $\mu = 0$). This implies that $\dot{V} = 2\mu x^2(\frac{x^2}{3} - 1)$ so $\dot{V} \leq 0$ in $x^2 < 3$.
Let $C_3 = \{V < 3\}$ so $\boldsymbol{x} \in C_3 \implies \boldsymbol{\phi}_t(\boldsymbol{x}) \in C_3$.

La Salle's invariance principle $\implies$ $\boldsymbol{x} \in C_3$ then $\boldsymbol{\phi}_t(\boldsymbol{x})$ tends to an invariant subset of $\{\dot{V} = 0\}$.
But the only invariant subset of $\{x = 0\}$ is $\{x = y = 0\}$ $\implies$ $\mathbf{0}$ is AS and the domain of stability includes C_3.

General method

(1) Find a domain $\mathcal{D}$ and function V such that

   (i) $V \geq 0$ on $\mathcal{D}$ and $V = 0$ iff $\boldsymbol{x} = \mathbf{0}$ (easy)

   (ii) $\dot{V} \leq 0$ on $\mathcal{D}$ (harder, often restricts the size of $\mathcal{D}$)

(2) Choose k such that $C_k = \{V < k\} \subset \mathcal{D}$.

(3) If necessary, adjust k (or V), so that the only invariant subset of $\{\dot{V} = 0\} \cap C_k$ is $\boldsymbol{x} = \mathbf{0}$. Then La Salle implies that the origin is AS stable <u>and</u> the domain of stability includes C_k.

(4) Try different V to maximise C_k or take a union of them.

Example:

$$\left.\begin{aligned} \dot{x} &= -x + xy^2 \\ \dot{y} &= -y + yx^2 \end{aligned}\right\} \text{ Try } V = x^2 + by^2$$

$$\implies \dot{V} = -2(x^2 + b^2y^2) + 2(1 + b^2)x^2y^2$$

Clearly $\dot{V} \leq 0$ sufficiently close to $\mathbf{0}$.
Let $(x, by) = \sqrt{V}(\cos\phi, \sin\phi)$

$$\implies \dot{V} = -2V + 2V^2\left(\frac{1+b^2}{b^2}\right) \underbrace{\cos^2\phi\sin^2\phi}_{\leq 1/4}$$

$$\implies \dot{V} \leq 0 \text{ in } 0 \leq V \leq \frac{4b^2}{1+b^2}$$

The domain of stability includes the union over b of these regions.

For another example see handout 3.2, "Examples of Domains of Stability".

3.3 Bounding Functions

The idea of a function decreasing along trajectories.
If V is a function with bounded contours and $\dot{V} < \delta < 0$ in the region $V(\boldsymbol{x}) > M$, then all trajectories eventually enter and remain in the set $\{V(\boldsymbol{x}) < M\}$. Such a V is called a bounding function.

4 Existence and Stability of Periodic orbits in $\mathbb{R}^n$

There is a toolkit of standard techniques, particular to $\mathbb{R}^2$, to prove that there either is or isn't one or more periodic orbits.

4.1 Poincaré index test

Consider

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

Except at a fixed point ($\boldsymbol{f} = \mathbf{0}$), the trajectory through a point $\boldsymbol{x}$ makes a well defined angle $\psi(\boldsymbol{x})$ with the x-axis.

Definition: Poincaré index of a curve
If Γ is a simple closed curve, not necessarily a trajectory, that doesn't pass through any fixed points,

then moving around Γ once anticlockwise, $\psi = \tan^{-1} \frac{f_1}{f_2}$ changes continously and returns to its original value plus an integer multiple of 2π. This multiple is the Poincaré index $I(\Gamma)$ of Γ.

Properties:

1.
$$2\pi I(\Gamma) = \oint_\Gamma d\psi = \oint_\Gamma d\left(\tan^{-1} \frac{f_1}{f_2}\right) = \oint_\Gamma \frac{f_1 df_2 - f_2 df_1}{f_1^2 + f_2^2}$$

2. $I(\Gamma)$ is unchanged by deformations of Γ that do not cross a fixed point.
   (Proof: $I(\Gamma)$ is integrable and a continous function of $\boldsymbol{f}$ (from 1) provided $\boldsymbol{f} \neq \mathbf{0}$.
3. If Γ encloses no fixed points, then $I(\Gamma) = 0$.
4. The Poincaré index is additive, $I(\Gamma_1) = I(\Gamma_2) + I(\Gamma_3)$
5. If Γ is a closed trajectory, i.e. a periodic orbit, then $I(\Gamma) = 1$.
6. If time is reversed ($\boldsymbol{f} \to -\boldsymbol{f}$) then $I(\Gamma)$ is unaffected (from 1).

   Definition: Poincaré index of a fixed point
   The Poincaré index of an isolated fixed point is the Poincaré index of any simple closed curve enclosing this fixed point and no others. (Well defined by 2.)

7. $I(\Gamma)$ is the sum of the Poincaré indices of the fixed points it encloses.
8. The Poincaré index of a node, focus, or center is $+1$.
9. The Poincaré index of a saddle is -1.

Non-hyperbolic fixed points other than centers need to be done on a case by case basis.

Example: $\dot{x} = x^2, \quad \dot{y} = -y$
Index $= 0$

Important Corollaries
If there are any periodic orbits, then each must contain at least one fixed point. The fixed point(s) enclosed by a periodic orbit must have indicies summing to $+1$.

If the above is not possible then there are no periodic orbits. This is the Poincaré index test. (a negative test)

Example:

$$\dot{r} = r(3 - r - s)$$
$$\dot{s} = s(2 - r - s)$$

Any periodic orbit canot cross the trajectories $r = 0$ and $s = 0$. Therefore it cannot enclose any of the nodes. A periodic orbit cannot enclose only the saddle (index -1), hence there are no periodic orbits.

4.2 Dulac's Criterion

Another negative test.

Theorem: Dulac's criterion
If there is a continously differentiable funciton $\phi(x, y)$ such that $\nabla \cdot (\phi \boldsymbol{f}) \neq 0$ in a simply connected domain $\mathcal{D} \subset \mathbb{R}^2$ then there are no periodic orbits lying entirely within $\mathcal{D}$.

Proof.
Assume, wlog, $\nabla \cdot (\phi \boldsymbol{f}) > 0$, suppose that Γ is a periodic orbit lying entirely in $\mathcal{D}$, and enclosing an area A with outward normal $\boldsymbol{n}$.

$$\int_{\Gamma} \phi \boldsymbol{f} \cdot \boldsymbol{n} \, ds = 0 = \int_{A} \nabla \ \cdot (\phi \boldsymbol{f}) dA \ > 0$$

which is a contradiction. □

Often we just take $\phi = 1$, this is known as the divergence test.

Corollary: If $\nabla \cdot (\phi \boldsymbol{f}) \neq 0$ in a doubly connected domain $\mathcal{D}$, then there is at most one periodic orbit lying entirely in $\mathcal{D}$.

Proof. Apply the divergence theorem to the area between two hypothetical periodic orbits in $\mathcal{D}$. □

Example: The Lotka-Voltera equations

$$\begin{aligned} \dot{r} &= r(a - br - cs) \\ \dot{s} &= s(d - er - fs) \end{aligned}$$

Choose $\phi = \frac{1}{rs}$. Then $\nabla \cdot (\phi \boldsymbol{f}) = \ -\frac{b}{s} - \frac{f}{r} < 0$ in $r, s \ > 0$, for $b, f \ > 0$. Therefore there can be no periodic orbits entirely in $r, s \ > 0$.

Example: Damped pendulum with torque

$$\begin{aligned} \dot{\theta} &= p \\ \dot{p} &= F - kp - \sin\theta \end{aligned}$$

The argument is slightly complicated by the phase space having the topology of a cylinder.
Can work on the cylinder and use $\frac{\partial \dot{\theta}}{\partial \theta} + \ \frac{\partial \dot{p}}{\partial p} \ = -k < 0$ to deduce that there is at most one periodic orbit, which must encircle the cylinder.

Alternatively can let $r = e^p$ to map onto plane polars, and use $\phi = 1/r^2$,

$$\nabla \cdot (\phi \boldsymbol{f}) = \nabla \cdot (\phi(\dot{r}\boldsymbol{e}_r + r \ \dot{\theta}\boldsymbol{e}_\theta)) = -\frac{k}{r^2} < 0$$

4.3 Poincaré-Bendixson Theorem

A positive test!

Theorem: Poincaré-Bendixson
If the forward orbit $\mathcal{O}^+(\boldsymbol{x})$ of a point $\boldsymbol{x}$ remains in a compact (closed and bounded) set $K \subset \mathbb{R}^2$, and K contains no fixed points, then $\omega(\boldsymbol{x})$ is a periodic orbit.

Note:

1. From 3.1, K must be multiply connected with a fixed point in one of the holes.
2. Idea: The trajectory must go somewhere, and the conditions of the theorem mean it can't go to ∞ or a fixed point.

The full (sketch) proof is in handout 4.4.
Example:

$$\dot{x} = x - y - 2x(x^2 + y^2)$$
$$\dot{y} = x + y - y(x^2 + y^2)$$

The linear terms $\implies$ $\mathbf{0}$ is an unstable foci. The cubic terms look inward from $-\infty$.
Try for an annulus in polar coordinates

$$\dot{r} = \frac{1}{r}(x^2 + y^2)(1 - 2x^2 - y^2) = r(1 - r^2(1 + \cos^2\theta))$$

minimising/maximising over θ gives

$$r(1 - 2r^2) \leq \dot{r} \leq r(1 - r^2)$$

and therefore

$$\dot{r} > 0 \text{ in } r < \frac{1}{\sqrt{2}}$$
$$\dot{r} < 0 \text{ in } r > 1$$

So trajectories entering the annulus $C = \{\frac{1}{\sqrt{2}} \leq r \leq 1\}$ remain there.
No other fixed points?

$$\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2} = 1 + \frac{1}{2}r^2 \sin\ 2\theta \implies \dot{\theta} > 0 \text{ in } r < \sqrt{2}$$

C contains no fixed points. By Poincaré-Bendixson theorem, C contains a periodic orbit.

($\nabla \cdot \boldsymbol{f} = 2 - 5r^2 - 2x^2 < 0$ in C, therefore there is only one periodic orbit)

Example: Forced damped pendulum again

$$\dot{\theta} = p$$
$$\dot{p} = F - kp - \sin\theta$$

From bounding functions in §3.3, we know that all trajectories enter and remain in the compact set [3]

$$V \leq \frac{1}{2}\left(\frac{F}{k}\right)^2 + 2$$

If $F > 1$ then there are no fixed points $\implies$ there is a periodic orbit by Poincaré-Bendixson. (and only one by corollory to Dulac's criterion)

[3] I actually have no idea what V is! my notes don't contain any reference, and apparently I didn't notice at the time...

4.4 Nearly Hamiltonian Flows

This can be a negative or positive test.

Consider a system

$$\dot{x} = f_1(x,y) + \epsilon g_1(x,y)$$
$$\dot{y} = f_2(x,y) + \epsilon g_2(x,y)$$

Where

$$f_1 = \frac{\partial H}{\partial y}, \qquad f_2 = -\frac{\partial H}{\partial x}$$

If $\epsilon = 0$ then $\dot{H} = 0$ and closed contours of H are periodic orbits. If $\epsilon \neq 0$ then

$$\dot{H} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\ \dot{y} = \epsilon \dot{x} g_2 - \epsilon g_1 \dot{y} = \epsilon(g_2 f_1 - g_1 f_2)$$

For a periodic orbit Γ, we must have

$$\oint_\Gamma dH = 0 \implies \oint_\Gamma \epsilon(g_2 f_1 - g_1 f_2) dt = 0$$

Hence there are no periodic orbits that lie entirely in a region where $g_2 f_1 - g_1 f_2 > 0$ (or < 0).

If $0 < \epsilon \ll 1$ then $\dot{H} = O(\epsilon)$, therefore the trajectories of the peturbed system are very similar to the contours of H. So changes in H can be approximated by integrating along the trajectories $H = H_0$ of the unpeturbed Hamiltonian system.

$$\Delta H(H_0) = \epsilon \oint_{H=H_0} (g_2 f_1 - g_1 f_2) dt + O(\epsilon^2)\ = \epsilon \oint_{H=H_0} g_2 dx - g_1 dy + O(\epsilon^2)$$

Where the $O(\epsilon^2)$ terms arise as the trajectory is not exactly $H = H_0$.

Fixed points $\Delta H = 0$ of the energy return map $H_{n+1} = H_n + \Delta H(H_n)$ correspond to periodic orbits. Finding approximations to them by neglecting the $O(\epsilon^2)$ terms is called the energy balance equation.

Note that the $O(\epsilon^2)$ can make a double root of $\Delta H = 0$ disappear, but not a single root.

Example:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ -x + x^2 + \epsilon y(a-x) \end{pmatrix}$$

Settubg $\epsilon = 0$ yields a Hamiltonian system with $H = \frac{1}{2}(x^2 + y^2) - \frac{1}{3}x^3$.
If $\epsilon \neq 0$, the fixed points are still $(0,0)$ (node/focus) and $(1,0)$ (saddle). However, $\dot{H} = \epsilon y^2(a-x)$ so any periodic orbit must be in partly $x > a$ and partly in $x < a$. Any periodic orbit must enclose $(0,0)$ but not $(1,0)$ by Poincaré index.

$x_{\max}$ and $x_{\min}$ occur when $\dot{x} = 0 \implies y = 0$. Hence $x_{\min} < a$, $0 < x_{\max} < 1$.[4] Therefore there are no periodic orbits if $a > 1$.

If $0 < \epsilon \ll 1$ then

$$\Delta H \approx \oint_{H=H_0} \epsilon g_2 dx = \ 2\epsilon \int_{x_{\min}}^{x_{\max}} (2H_0 - x^2 + \frac{2}{3}x^3)^{\frac{1}{2}}\ (a-x) dx$$

[4] Consider the sign of $\dot{H}$!

(This integral gas to be integrated numerically except for the special case $H_0 = \frac{1}{6}$, when $()^{\frac{1}{2}} = (1-x)(1+\frac{2}{3}x)^{\frac{1}{2}}$ which corresponds to the homoclinic orbit.)

Setting $\Delta H(H_0, a) = 0$ gives a unique approximate periodic orbit $H = H^*(a)$ for $0 < a < \frac{1}{7}$ $(0 < H^* < \frac{1}{6})$.

The periodic orbit appears at $a = 0$ by a Hopf bifurcation and is destroyed at $a = \frac{1}{7}$ by a homoclinic bifurcation.

4.5 Stability of Periodic Orbits

Suppose $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ has a periodic orbit $\boldsymbol{x} = \boldsymbol{X}(t)$ with $\boldsymbol{X}(T) = \boldsymbol{X}(0) = \boldsymbol{X}_0$.
Consider a small perturbation $\boldsymbol{x} = \boldsymbol{X}(t) + \boldsymbol{\eta}(t)$. Linearising,

$$\dot{\boldsymbol{X}} + \dot{\boldsymbol{\eta}} = \boldsymbol{f}(\boldsymbol{X}) + (\boldsymbol{\eta}\cdot\nabla)\boldsymbol{f}(\boldsymbol{X}) + O(|\boldsymbol{\eta}|^2) \implies \dot{\boldsymbol{\eta}} = (\boldsymbol{\eta}\cdot\nabla)\boldsymbol{f}(\boldsymbol{X}) + O(|\boldsymbol{\eta}|^2)$$

and so

$$\dot{\boldsymbol{\eta}} = A\boldsymbol{\eta} \quad \text{where } A_{ij}(t) = \left.\frac{\partial f_i}{\partial x_j}\right|_{\boldsymbol{X}(t)}$$

This is a linear ODE.

$$\implies \boldsymbol{\eta}(t) = \Phi(t)\boldsymbol{\eta}(0)$$

Since $\boldsymbol{\eta}(0)$ is arbitrary

$$\begin{aligned} \dot{\Phi}_{ij} &= A_{ik}\Phi_{kj} \\ \Phi_{ij}(0) &= \delta_{ij} \end{aligned}$$

But $A(t)$ is periodic with period T, and so

$$\boldsymbol{\eta}(nT) = \Phi(T)\,\boldsymbol{\eta}\left((n-1)T\right) = \ldots (\Phi(T))^n\boldsymbol{\eta}(0)$$

Thus whether an arbitrary perturbation $\boldsymbol{\eta}(0)$ grows/decays after n times round the orbit, depends on the eigenvalues of $\Phi(T)$ compared to 1.

Definition: Floquet Multipliers
The Floquet multipliers of a periodic orbit are the eigenvalues λ_i of the matrix $\Phi(T)$ defined above.

One of the Floquet multipliers is always unity, since if $\boldsymbol{\eta}(0) = \boldsymbol{f}(\boldsymbol{x}_0)\delta t$ then $\dot{\boldsymbol{\eta}} = A\boldsymbol{\eta}$ has solution, $\boldsymbol{\eta}(t) = \boldsymbol{f}(\boldsymbol{X}(T))\delta t \implies \boldsymbol{\eta}(T) = \boldsymbol{f}(\boldsymbol{X}(T))\delta t = \boldsymbol{f}(\boldsymbol{X}_0)\delta t$

Hence

(i) A periodic orbit is asymptotically stable if all of the non-trivial Floquet multipliers satisfy $|\lambda| < 1$.

(ii) A periodic orbit is Lyapunov unstable if any satisfy $|\lambda| > 1$ (and so asymptotically unstable).

Definition: Hyperbolicity
A periodic orbit is hyperbolic if all of the non-trivial eigenvalues satisfy $|\lambda| \neq 1$ and is non-hyperbolic if at least one lies on the unit circle $|\lambda| = 1$.

Hyperbolic periodic orbits (like hyperbolic fixed points) are structurally stable i.e. the orbit is unaffected by small perturbations to $\boldsymbol{f}$.

Stability in $\mathbb{R}^2$
In $\mathbb{R}^2$ we only have one non trivial λ to worry about. (Which is real and positive -cf lemma for the Poincaré-Bendixson theorem[5])

$$\det(\Phi(T)) = \lambda_1\lambda_2, \qquad \lambda_1 = 1$$

Lemma:

$$\frac{d}{dt}\det\Phi = (\nabla\cdot\boldsymbol{f})\det\Phi$$

Proof. ($n = 2$ for simplicity)

$$\det\Phi = \epsilon_{ij}\Phi_{1i}\Phi_{2j} \implies \frac{d}{dt}\det\Phi = \epsilon_{ij}[\dot{\Phi}_{ij}\Phi_{2j} + \Phi_{1i}\dot{\Phi}_{2j}] = \epsilon_{ij}[A_{1k}\Phi_{ki}\Phi_{2j} + \Phi_{1i}(A_{2k}\Phi_{kj})]$$

Now $\epsilon_{ij}\Phi_{2i}\Phi_{1j} = 0$

$$\implies \frac{d}{dt}\det\Phi = A_{11}(\epsilon_{ij}\Phi_{1i}\Phi_{2j}) + A_{22}(\epsilon_{ij}\Phi_{1i}\Phi_{2j}) = (\nabla\cdot\boldsymbol{f})\det\Phi$$

□

Corollories:

1. In $\mathbb{R}^2$ the non trivial Floquet multiplier is

$$\lambda_2 = \exp\left[\oint_0^T \nabla\cdot\boldsymbol{f}|_{\boldsymbol{X}(t)}\,dt\right]$$

   Which is easy to calculate.

2. A periodic orbit is

$$\left.\begin{array}{l}\text{stable}\\ \text{unstable}\\ \text{non-hyperbolic}\end{array}\right\}\ \text{if}\ \oint_0^T \nabla\cdot\boldsymbol{f}\,dt\ \text{is}\ \left\{\begin{array}{l}<0\\ >0\\ =0\end{array}\right.$$

3. In higher dimensions $\oint_0^T \nabla\cdot\boldsymbol{f}\,dt\ < 0$ is necessary for stability, but not sufficient.

Example:

$$\begin{aligned}\dot{r} &= r(1-r^2)\\ \dot{\theta} &= \frac{1}{r}\end{aligned}$$

This has a periodic orbit $r = 1$, $\theta = t$, $T = 2\pi$. Make a linear perturbation to $r = 1+\delta$, $\theta = t+\epsilon$.

$$\implies \begin{array}{ll}\dot{\delta} & = -2\delta\\ \dot{\epsilon} & = -\delta\end{array} \implies \begin{array}{ll}\delta(t) & = \delta(0)e^{-2t}\\ \epsilon(t) & = \epsilon(0) + \delta(0)\frac{e^{-2t}-1}{2}\end{array}$$

Therefore

$$\Phi(T) = \begin{pmatrix} e^{-2T} & 0\\ \frac{1}{2}(e^{-2T}-1) & 1\end{pmatrix}$$

Which has eigenvalues

$$\begin{array}{ccc} e^{-4\pi} & \text{and} & 1\\ \text{stable} & & \text{trivial}\end{array}$$

[5]Not sure what this lemma is; here's my explanation. If $\lambda_1 = 1$ then λ_2 must be real. In addition it must be positive as Jordan curve lemma means peturbed trajectory cannot cross the periodic orbit.

More simply,

$$\nabla \cdot \boldsymbol{f} = \frac{1}{r}\frac{\partial}{\partial r}(r\dot{r}) + \frac{1}{r}\frac{\partial}{\partial \theta}(r\dot{\theta}) = 2 - 4r^2 = -2 \text{ at } r = 1$$

$$\implies \lambda = \exp\left[\int_0^{2\pi} (-2)dt\right] = \exp(-4\pi)$$

4.6 The Van der Pol Oscillator

$$\left.\begin{array}{rl}\dot{x} = & y + -\mu(\frac{1}{3}x^3 - x) \\ \dot{y} = & -x\end{array}\right\} \text{ equivalent to } \ddot{x} + \mu(x^2-1)\dot{x} + x = 0$$

(Lienard variables)
The only fixed point is $\boldsymbol{x} = \mathbf{0}$ and is unstable. (Unstable focus for $0 < \mu < 2$, unstable node for $\mu > 2$)

$$H = \frac{1}{2}(x^2+y^2) \implies \dot{H} = \mu x^2(1 - \frac{1}{3}x^2)$$

So if $0 < x^2 + y^2 < 3$ then $\dot{H} \geq 0$ and H increases monotonically along $\mathcal{O}^+(\boldsymbol{x})$ and eventually crosses $|\boldsymbol{x}|^2 = 3$ going outwards and cannot return.
(cf arguments in §3.3 and §3.4 for La Salle's invariance principle, bounding functions, and domain of stability of the time reversed problem $\mu \to -\mu$.)

So certainly "outwards" from $\mathbf{0}$, but is it "inwards" from ∞?
The question is complicated by the strip $|x| < \sqrt{3}$ where $\dot{H} \geq 0$ extends to ∞. Consider part trajectories of ABC as shown for different $y(A)$

$$H(B) - H(A) = \int_A^B \frac{\dot{H}}{\dot{x}}dx = \int_A^B \frac{\mu x^2(1-\frac{1}{3}x^2)}{y - \mu x(\frac{1}{3}x^2 - 1)}dx$$

Which is positive, but is a monotonically decreasing function of y.

$$H(C) - H(B) = \int_C^B \frac{\dot{H}}{\dot{y}}dy = \int_B^C \mu x(1 - \frac{1}{3}x^2)dy$$

is negative and increasingly so; as $y(B) - y(C)$ increases, there are larger values of $x(y)$.

Hence $H(C) - H(A)$ is a monotonic function of $y(A)$ with a unique root corresponding to a unique periodic orbit.

For $0 < \mu \ll 1$ we can use the energy balance method (§4.4) $H = \frac{1}{2}(x^2+y^2)$, the unperturbed orbits ($\mu = 0$) are $H = H_0 = const$, $x = \sqrt{2H_0}\cos t$.

$$\Delta H(H_0) = \oint_{r^2=2H_0} \mu x^2(1 - \frac{1}{3}x^2)dt + O(\mu^2) \approx \mu \int_0^{2\pi} (2H_0\cos^2 t - \frac{4H_0^2}{3}\cos^4 t)dt$$

$$2\pi\mu(H_0 - \frac{1}{2}H_0^2)$$

So the limit cycle is $H_0 = 2$ or $x^2 + y^2 = 4$.

Now consider $\mu \gg 1$. $\dot{x} = y - \mu x(\frac{1}{3}x^2 - 1)$ so $|\dot{x}| \gg 1$ except for $y \approx \mu x(\frac{1}{3}x^2 - 1)$.

Time taken: $A \to B$ or $C \to D \sim \frac{\Delta y}{\dot{y}} = O(\mu)$ which is large.

$B \to C$ or $D \to A \sim \frac{\Delta x}{x} = O(\frac{1}{\mu})$ which is small

The period is dominated by the time on the "Slow Manifold," $y = \mu x(\frac{1}{3}x^2 - 1)$ from $A \to B$ and $C \to D$. Hence

$$T \approx 2\int_A^B dt = 2\int_A^B \frac{1}{\dot{y}}\frac{dy}{dx}dx = 2\mu\int_{-2}^{-1}\frac{1-x^2}{x}dx = \mu(3 - 2\log 2)$$