Further Complex Methods

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Books: "Complex Variables," M.J Ablowitz & A. Fokes (CUP)
"A Course in Modern Analysis," Whittaker & Watson

Any function of x, y can be written as a function of $z,\ \bar{z}$ for $z = x + iy$.

Functions of a complex variable are defined to be those functions of x and y that can be written entirely in terms of z only.

A function of a complex variable is continous if

$$\lim_{z \to z_0} f(z) = f(z_0) \quad \text{(as in real analysis)}$$

The derivative of a function of a complex variable is

$$f'(z) - \lim_{\delta z \to 0} \frac{f(z + \delta z) - f(z)}{\delta z}$$

For a function to be differentiable, the limit must be independent of the direction that the limit is taken.

If this is true, then the function is said to be differentiable at z. If $f'(z)$ exists, then $f(z)$ is continous (converse not true).

Write $f(z) = u(x, y) + iv(x, y)$ with u, v both real. Then

$$dz\, f'(z) = \lim_{\delta z = \delta x + i\delta y \to 0} (u(x + \delta x, y + \delta y) + iv(x + \delta x, y + \delta y) - u(x, y) - iv(x, y))$$

If $\delta y = 0$, $dz = dx$ and we get that

$$f'(z) = u_x + iv_x$$

Suppose now that $\delta x = 0$.

$$i\delta y\, f'(z) = u_y + iv_y$$
$$\implies f'(z) = v_y - iu_y$$
$$\implies v_y - iu_y = u_x + iv_x$$
$$\implies \left.\begin{aligned} v_y &= u_x \\ v_x &= -u_y \end{aligned}\right\} \text{The Cauchy-Riemann equations}$$

If the Cauchy-Riemann equations hold, the derivatives exist and are continous, then $f(z)$ is differentiable. If the Cauchy-Rieman equations hold then $u,\ v$ are harmonic.

$$u_{xx} = v_{xy} = -u_{yy} \implies u_{xx} + u_{yy} = 0$$

A similar equation holds with v.

Consider surfaces of $u = const$, $v = const$. These surfaces are orthogonal.

$$\nabla u = (u_x, u_y) \text{ - normal to } u = const$$
$$\nabla v = (v_x, v_y) \text{ - normal to } v = const$$

and so

$$\nabla u \cdot \nabla v = u_x v_x + u_y v_y = 0 \text{ from C-R}$$

Definition: Analytic function
$f(z)$ is analytic at z_0 if $f(z)$ is differentiable in some neighbourgood of z_0. $f(z)$ is analytic in a region if a similar condition applies.

Examples:

(i) e^z is analytic in the finite complex z-plane

(ii) $\bar{z}$ is analytic nowhere

(iii) $1/z^3$ is analytic everywhere except at $z = 0$

Definition: Entire functions
A function is entire if it is analytic in the finite complex plane

Examples:

(i) e^z , this only fails to be analytic at ∞

(ii) $\sin z$

(iii) z^2

Definition: Isolated singularity
A function is said to have an isolated singularity if it fails to be analytic at a point.

Example: $1/z^3$ has an isolated singularity at the origin.

Suppose that a function has an isolated singularity at $z = z_0$. Then it can be expanded as a Laurent series around z_0.

$$f(z) = \sum_{-\infty}^{\infty} c_n (z - z_0)^n$$

Note that this sum is over all positive and negative powers.

Suppose that $c_n = 0$ for all $n < -N$ where $N > 0$.

- If $c_n = 0 \; \forall \, n > 0$ then it is not singular.
- If $c_n = 0$ for all $n < -N$ for $N > 0$, then one has a pole of order N.

Example: $1/z^3$ has a pole of order 3 at $z = 0$.
The coefficient c_{-1} is special, it is the residue of the pole at z_0.

Definition: Removable singularities
Fake singularities where the building blocks of $f(z)$ have isolated singularities, but $f(z)$ does not.

Example:

$$f(z) = \frac{\sin z}{z} = \frac{1}{z}(z - \frac{z^3}{6} + \dots) = 1 - \frac{z^2}{6}$$

$f(z)$ has a removable singularity at $z = 0$.

Example:

$$f(z) = \frac{1}{z} - \frac{1}{z + z^2} = \frac{1}{1 + z}$$

so $f(z)$ has a removable singularity at the origin.

Definition: Essential Singularity
An essential singularity is where the order of the pole of an isolated singularity is infinite.

Example: $f(z) = e^{1/z}$, $z = 0$ is an isolated singularity, as a Laurent series

$$f(z) = \sum_{\infty}^{0} \frac{1}{(-n)!} z^n$$

Note that in this exaple, $f(z)$ is not even continous at $z = 0$, its value depends on how one approaches $z = 0$.

Definition: Meromorphic functions
These are functions of z that only have poles of any finite order in the finite complex plane.

Examples:

(a) $1/z^2$ has a pole of order 2 at the origin.

(b) $\cot z$ has poles of order 1 at $z = n\pi$, $n \in \mathbb{Z}$

Theorem: Cauchy's Theorem

$$\int_C f(z)\, dz = 2\pi i \left(\sum \text{ Residues of the poles enclosed by C}\right)$$

The integral is taken around C in the anti-clockwise direction, and $f(z)$ is meromorphic.

The complex plane is really a sphere, the Riemann sphere.
w is the perpendicular distance from the $x - y$ plane. The north pole corresponds to infinity, and all of infinity has become a point.

Definition: Stereographic projection
Construct a straight line staring at N, through P to meet the complex plane at C.

$$N = (0, 0, 2), \qquad P = (X, Y, \ )$$

Construct this line by saying that s is a parameter along the line such that $s = 0$ at the north pole and $s = 1$ at P.

$$\left.\begin{aligned} x &= X_s \\ y &= Y_s \end{aligned}\right\} \text{ What about w?}$$

$$w = 2 - (1 \pm \sqrt{1 - X^2 - Y^2})s$$

at C, $w = 0$

$$\implies s = \frac{2}{1 \pm \sqrt{1 - X^2 - Y^2}} \text{ at } C$$

Hence the coordinates of the point C are

$$x = \frac{2X}{1 \pm \sqrt{1 - X^2 - Y^2}} \ , \quad y = \frac{2Y}{1 \pm \sqrt{1 - X^2 - Y^2}}$$

Thus if X, Y both $\to 0$, then $x, y \to \infty$ with the choice of sign. All of infinity gets mapped to the north pole of the Riemann sphere. This motivates how to think about infinity.

$$z \mapsto 1/z = w \text{ maps infinity to the origin}$$

Suppose $f(z) = z$, then $f(w) = 1/w$. $f(w)$ has a simple pole of residue 1 at $w = 0 \implies f(z) = z$ has a simple pole of residue 1 at infinity.

This holds true for any function of a complex variable; to examine the behaviour of a funcion at ∞, send $z \mapsto 1/z = w$ and ask what happens at $w = 0$.

Example: $f(z) = e^z = e^{1/w}$ has an essential singularity at $w = 0$.

Theorem: Liouville's theorem
If f is analytic everywhere including ∞ then it must be a constant.

Multi-valued functions

For a real variable, the square root of a positive number has two forms $\pm\sqrt{x}$.
Now consider $z^{1/2}$ and decompose into modulus and argument.

$$z^{1/2} = \rho^{1/2} e^{i\theta/2}$$

As one moves around the circle,

$$\theta \mapsto \theta + 2\pi$$
$$z^{1/2} \mapsto \rho^{1/2} e^{i(\theta+2\pi)/2} = -\rho^{1/2} e^{i\theta/2}$$

So $f(z)$ changes sign. If one goes around the circle twice then $\theta \mapsto \theta + 4\pi$, and so $f(z)$ is invariant.

The effect of going around the circle is usually called the monodromy, and for the case $f(z) = z^{1/2}$ this is $(-1)^n$.

The monodromy always forms a group. So in this case, the monodromy group is just $\mathbb{Z}_2$,

A point where the monodromy is not 1 is called a branch point

For $f(z) = z^{1/2}$, the origin is a branch point. Around z_0 the function returns to its starting point. This holds for any $z_0 \neq 0$ in the finite complex plane.

Infinity is also a branch point.

$$z \mapsto 1/w\,, \quad w^{-\frac{1}{2}} \mapsto (\rho')^{-\frac{1}{2}} e^{-\frac{i\theta'}{2}}$$

and so when $\theta \mapsto \theta + 2\pi$, $w^{-1/2}$ changes sign. Therefore $z = \infty,\ w = 0$ is also a branch point.

There is always more than one branch point, so always look at infinity. Branch points represent a failure of analycity.