

# Asymptotic Methods

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January 22, 2016

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**Books:** Bender and Orszag, “*Advanced Mathematical methods for scientists and engineers*”, Chapters 3,6,10

More details can be found on the Moodle course site; login to Moodle at <https://www.vle.cam.ac.uk/login/index.php>, then self-enrol into the Asymptotic methods course.

## What we’ll learn in this course

### Examples:

1.  $I(\lambda) = \int_{-\infty}^{\infty} \exp[-\lambda \cosh u] du$

We expect that  $I(\lambda) \rightarrow 0$  as  $\lambda \rightarrow \infty$ . But how fast?

2.  $i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x)$  with  $\psi(x, t) \in \mathbb{C}$ ,  $V = V(x)$ .

Look for a solution  $\psi(x, t) = \exp\left[\frac{-iEt}{\hbar}\right] f(x) \implies \hbar^2 f'' = 2m(V(x) - E)f$

$\hbar$  is very small. So a natural problem is to try and understand  $\epsilon^2 \frac{d^2 y}{dx^2} = Q(x)y$  when  $\epsilon \ll 1$ .

The “semi-classical limit” or “geometric optics”.

3. Put  $\hbar = 1$ ,  $m = \frac{1}{2}$ ,  $V = 0$ ; specify  $\psi(x, 0) = \psi_0(x)$

Fourier transform  $\rightarrow \psi(x, t) = \frac{1}{(4\pi it)^{1/2}} \int_{\mathbb{R}} \exp\left[\frac{i|x-y|^2}{4t}\right] \psi_0(y) dy$ .

Question: Does  $\psi(x, t)$  really approach  $\psi_0(x)$  as  $t \rightarrow 0$ ?

## 1 Asymptotic expansions of functions

$$\sinh x = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

say  $\sinh x \sim x$  as  $x \rightarrow 0$ .

**Definition:**  $f \sim g$  as  $x \rightarrow x_0$  is  $|f(x) - g(x)| = o(g(x))$  as  $x \rightarrow x_0$ .

**Example:**

$$|\sinh x - x| = \left| \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \right| = O(x^3) = o(x)$$

( $F = O(G)$  as  $x \rightarrow x_0$  means  $\exists C > 0$  such that  $|F(x)| \leq C|G(x)|$  in some open interval  $I$ , with  $x_0 \in I$ )  
In fact, by remainder estimate for Taylor expansion

$$\left| \sinh x - \sum_0^N \frac{x^{2n+1}}{(2n+1)!} \right| = O(x^{2N+3}) = o(x^{2N+1}) \text{ as } x \rightarrow 0$$

We write  $\sinh x \sim \sum_0^\infty \frac{x^{2n+1}}{(2n+1)!}$

**Definition:** Asymptotic sequence and asymptotic expansion.

- (i)  $\{\phi_n\}_{n=0}^\infty$  is an asymptotic sequence (of functions) as  $x \rightarrow x_0$  if  $\phi_{n+1}(x) = o(\phi_n(x))$  as  $x \rightarrow x_0$ .
- (ii) A function  $f$  has asymptotic expansion w.r.t.  $\{\phi_n\}$  as  $x \rightarrow x_0$  written  $f \sim \sum_{n=0}^\infty a_n \phi_n$  if

$$\left| f(x) - \sum_{n=0}^N a_n \phi_n(x) \right| = o(\phi_N(x)) \text{ as } x \rightarrow x_0 \forall N$$

Notice the difference with Taylor expansion - an asymptotic expansion need not converge as  $N \rightarrow \infty$  for any  $x$ !

**Examples:**

- $\{\phi_n(x) = x^n\}$  as  $x \rightarrow 0$ , the most common sequence.
- $\{\phi_n(x) = x^{2n+1}\}$  as  $x \rightarrow 0$
- $\{\phi_n(x) = e^{-n/x}\}$  as  $x \rightarrow 0^+$  (i.e.  $x > 0$  and  $x \rightarrow 0$  on right)

**Warning:**  $\sin x \sim x - \frac{x^3}{3!} - \frac{x^5}{5!} + \dots$  as  $x \rightarrow 0$ .  
 $\sin x + e^{-1/x} \sim x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$  as  $x \rightarrow 0^+$ .

Why?

$$\left| \sin x + e^{-1/x} - \sum_0^N \frac{(-1)^n x^{2n+1}}{(2n+1)!} \right| = \left| \sum_{n=2N+1}^\infty \frac{(-1)^n x^{2n+1}}{(2n+1)!} + e^{-1/x} \right| = O(x^{2N+3}) = o(x^{2N+1})$$

Moral: information is lost in asymptotic expansions!

However, given  $f$  and asymptotic sequence, the  $a_j$ 's are unique, i.e.

$$\begin{aligned} a_0 &= \lim_{x \rightarrow x_0} \frac{f(x)}{\phi_0(x)} \\ a_1 &= \lim_{x \rightarrow x_0} \frac{f(x) - a_0 \phi_0(x)}{\phi_1(x)} \\ &\vdots \end{aligned}$$

**Question:** Is it possible that  $f(x) \sim 0$  as  $x \rightarrow 0$ ?

If  $|f(x) - 0| = o(0) = 0$  in some interval  $I$ , containing 0, then  $f \equiv 0$  on  $I$ .

**Example:** Consider  $E_1(x) = \int_x^\infty \frac{e^{-t}}{t} dt$  as  $x \rightarrow +\infty$ .

Consider the asymptotic sequence  $\phi_n(x) = 1/x^n$  as  $x \rightarrow +\infty$

$$E_1(x) = \int_x^\infty \frac{-d(e^{-t})}{t} = \left[ -\frac{e^{-t}}{t} \right]_x^\infty - \int_x^\infty \frac{e^{-t}}{t^2} dt = \frac{e^{-x}}{x} - \int_x^\infty \frac{e^{-t}}{t^2} dt$$

Claim:  $E_1(x) \sim e^{-x}/x$  as  $x \rightarrow +\infty$ .

$$\left| E_1(x) - \frac{e^{-x}}{x} \right| = \left| \int_x^\infty \frac{e^{-t}}{t^2} dt \right| \leq \frac{1}{x^2} \int_x^\infty e^{-t} dt = \frac{e^{-x}}{x^2} = o\left(\frac{e^{-x}}{x}\right)$$

Working out the full expansion of  $E_1$  with respect to  $\phi_n = e^{-x}/x^n$  gives that.

$$E_1(x) \sim \sum_{n=0}^{\infty} \frac{(-1)^n n! e^{-x}}{x^{n+1}}$$

What do we mean?

(i)  $\phi_n(x) = e^{-x}/x^{n+1}$  satisfies  $\phi_{n+1}(x) = o(\phi_n(x))$  as  $x \rightarrow +\infty$ . i.e. it forms an “asymptotic sequence.”

(ii) The notation “ $\sim$ ” (“asymptotic to”) means

$$\left| E_1(x) - \sum_{n=0}^N \frac{(-1)^n n! e^{-x}}{x^{n+1}} \right| = o\left(\frac{e^{-x}}{x^{N+1}}\right) \text{ as } x \rightarrow +\infty$$

This can be proved with integration by parts:

$$\begin{aligned} E_1(x) &= - \int_x^\infty \frac{1}{t} d(e^{-t}) = e^{-x}/x + \int_x^\infty \frac{1}{t^2} d(e^{-t}) \\ &= e^{-x}/x - e^{-x}/x^2 + 2 \int_x^\infty \frac{e^{-t}}{t^3} dt \\ &= e^{-x} \left[ \frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \cdots + \frac{(-1)^n n!}{x^{n+1}} \right] + \underbrace{(-1)^{n+1} (n+1)! \int_x^\infty \frac{e^{-t}}{t^{n+2}} dt}_{Rem_{n+1}(x)} \end{aligned}$$

Where

$$|Rem_{n+1}(x)| \leq \frac{(n+1)!}{x^{n+2}} \int_x^\infty e^{-t} dt = \frac{(n+1)! e^{-x}}{x^{n+2}} = o\left(\frac{e^{-x}}{x^{n+1}}\right) \text{ as } x \rightarrow +\infty$$

So it is an asymptotic expansion. Not convergent because  $\sum (-1)^n n! y^{n+1}$  has radius of convergence 0. (In fact for fixed  $y$  the terms become unbounded.)

Consider magnitudes of successive terms  $f_n(x) = \frac{(-1)^n n! e^{-x}}{x^{n+1}}$

$$\left| \frac{f_{n+1}(x)}{f_n(x)} \right| = \frac{n+1}{x}$$

Optimal truncation: Truncate the asymptotic expansion at the point  $n = N_x$ , such that the first term excluded is the smallest.

In our example, choose  $N_x = [x] - 1 = \sup\{j - 1 : j \leq x, j \in \mathbb{N}\}$

$$\left. \begin{aligned} |f_{N_x+1}(x)/f_{N_x}(x)| &= (N_x + 1)/x \leq 1 \\ |f_{N_x+2}(x)/f_{N_x+1}(x)| &= (N_x + 2)/x > 1 \end{aligned} \right\} \text{ so } f_{N_x+1} \text{ is the smallest term, later terms are larger}$$

So we write

$$E_1(x) = \sum_{n=0}^{N_x} \frac{(-1)^n n! e^{-x}}{x^{n+1}} + \text{Rem}_{N_x+1}(x)$$

$$|\text{Rem}_{n+1}(x)| \leq \frac{(N_x + 1)!}{x^{N_x+2}} e^{-x} = \frac{[x]! e^{-x}}{x^{[x]+1}} \leq \frac{2 \left(\frac{[x]}{e}\right)^{[x]} \sqrt{2\pi[x]} e^{-x}}{x^{[x]+1}} \leq \frac{2\sqrt{2\pi[x]}}{[x]} e^{-x} e^{-[x]}$$

Where we have used Stirling's formula.

$$\lim_{n \rightarrow \infty} \frac{n!}{(n/e)^n \sqrt{2\pi n}} \rightarrow 1 \text{ as } n \rightarrow \infty$$

The good new is the additional  $e^{-[x]}$  term. Optimal truncation (often) gives an exponentially small remainder.

**Examples:**

$$\sinh x = \frac{e^x - e^{-x}}{2} \sim e^{-x}/2 \text{ as } x \rightarrow +\infty$$

Works because  $e^{-x} = o(e^x)$  as  $x \rightarrow \infty$

$$\text{sech } x = \frac{2}{e^x + e^{-x}} = \frac{2}{e^x} (1 + e^{-2x})^{-1} = \frac{2}{e^x} (1 - e^{-2x} + e^{-4x} - \dots)$$

This gives an asymptotic expansion for the sequence  $\phi_n = e^{-nx}$  (Which is asymptotic since  $e^{-x} = o(e^x)$  as  $x \rightarrow +\infty$ )

Note:  $\sinh x \sim -e^{-x}/2$  as  $x \rightarrow -\infty$

Consider  $\sinh z$ , for  $z \in \mathbb{C}$

$$\sinh z = \frac{e^z - e^{-z}}{2} = \frac{e^{x+iy} - e^{-x-iy}}{2}$$

$$\begin{aligned} &\sim e^z/2 \text{ as } z \rightarrow \infty \text{ in sector } \left\{-\frac{\pi}{2} < \arg(z) < \frac{\pi}{2}\right\} \\ &\sim e^{-z}/2 \text{ as } z \rightarrow \infty \text{ in sector } \left\{\frac{\pi}{2} < \arg(z) < \frac{3\pi}{2}\right\} \end{aligned}$$

Conclusion: The asymptotic seems to change suddenly when going from sector to sector.

The lines separating the different sectors are Stokes Lines.

**Exercise:** Prove that the definition of asymptotics in a sector must satisfy that you do not approach Stokes lines too fast.