Dynamical Systems

Dominic Skinner

January 12, 2016

Introduction

A dynamical system is a set of equations describing the evolution of a system with respect to a time-like variable. Usually they are non-linear.
The possible states of the system define the state space/phase space.

Example: The logistic map

$$x_{n+1} = \mu x_n(1 - x_n)$$

For $0 \leq \mu \leq 4$ this describes evolution with respect to a discrete time n in a state space $[0, 1]$.

Example: The Lotka-Voltera equations

$$\begin{aligned} \dot{r} &= r(a - br - cs) \\ \dot{s} &= s(d - er - fs) \end{aligned}$$

Where a-f are positive constants. These describe continous evolution in a state space $(r, s) \in [0, \infty] \times [0, \infty]$ as a model for the population of two species competing for the same food supply.

Example: The non-linear Schrödinger equation

$$i\frac{\partial \Psi}{\partial t} = \nabla^2 \Psi + |\Psi|^2 \Psi$$

This describes evolution in an infinite dimensional statespace of possible wavefunctions.

Because the equations are non-linear, it is often impossible to find a complete set of closed form analytic solutions. Instead, we resort to a mixture of geometric and analytic arguments, and aim to say something about the generic long-term behaviour.

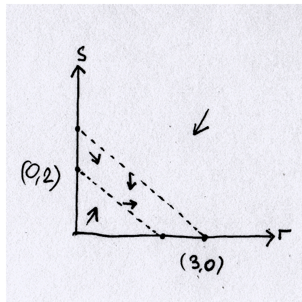
Example:

$$\begin{aligned} \dot{r} &= r(3 - r - s) \\ \dot{s} &= s(2 - r - s) \end{aligned}$$

Consider the regions where $\dot{r}$ and $\dot{s}$ are $> 0,\ < 0,\ = 0$.
If $r, s > 0$ then

$$\begin{aligned} r + s < 2 &\implies \dot{r}, \dot{s} > 0 \\ 2 < r + s < 3 &\implies \dot{r} > 0,\ \dot{s} < 0 \\ r + s > 3 &\implies \dot{r}, \dot{s} < 0 \end{aligned}$$

$\dot{r} = 0$ if $r = 0$ or $r + s = 3$.
$\dot{s} = 0$ if $s = 0$ or $r + s = 2$.
Therefore $\dot{r} = \dot{s} = 0$ at the fixed points $(0,0),\ (3,0),\ (0,2)$. This gives the phase portrait/diagram/plane[1]



The most important feature of the phase portrait is that all solutions with $r > 0$ tend to the stable fixed point $(3,0)$. The fixed points $(0,0)$ and $(0,2)$ are unstable. There are no periodic orbits.

Example:

$$\dot{r} = r(3 - r - s)$$
$$\dot{s} = s(2 - \mu r - s)$$

In this case, a new fixed point $(\frac{1}{1-\mu}, \frac{2-3\mu}{1-\mu})$ appears in the state space at $\mu = \frac{2}{3}$ and for $\mu < \frac{2}{3}$ is the long term stable attractor.
A qualitative change in the solution structure is called a bifurcation.

Example:

$$\dot{x} = -y + \epsilon x(\mu - x^2 - y^2)$$
$$\dot{y} = \quad x + \epsilon y(\mu - x^2 - y^2)$$

Use polar coordinates which are a more natural choice for this problem. In general:

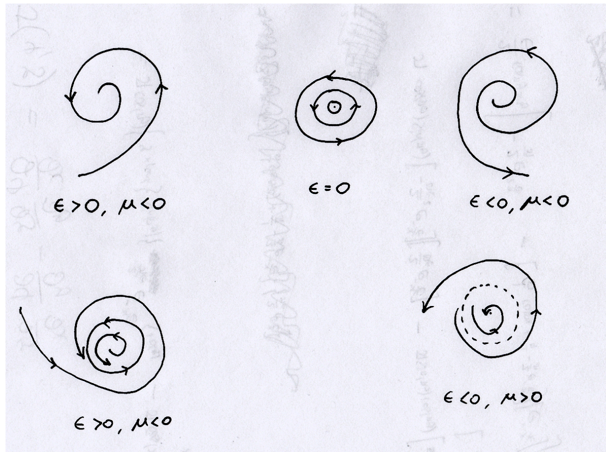
$$\boxed{\begin{aligned} \dot{r} &= \frac{x\dot{x} + y\dot{y}}{r} \\ \dot{\theta} &= \frac{x\dot{y} - y\dot{x}}{r^2} \end{aligned}}$$

Which are equations that will be referred to frequently. [2] In our example, they become

$$\dot{r} = \epsilon r(\mu - r^2)$$
$$\dot{\theta} = 1$$

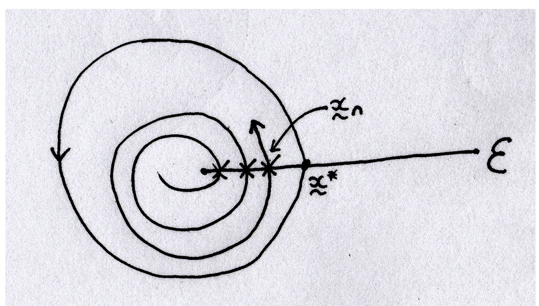
Consider $\dot{r}$ and $\dot{\theta}$:

[1]Phase portrait, phase diagram, phase plane will be used interchangably
[2]so learn them now!



The infinite set of periodic solutions for $\epsilon = 0$ is destroyed by any perturbation to $\epsilon \neq 0$. This is an example of structural instability. If $\mu > 0$ then just one limit cycle survives and is stable (unstable) for $\epsilon > 0$ ($\epsilon < 0$). The appearance of the limit cycle as $\mu \uparrow$ through 0 is another form of bifurcation.

Example: In 2D the points of successive interection x_n of a solution near a limit cycle with a line ε perpendicular to the cycle, move monotonically towards/away from the point of intersection x^* of ε with the limit cycle.



The point x^* is a stable/unstable fixed point of this Poincaré recurrence map.
In 3 or higher dimensions, or in 2D with time-dependent coefficients there is room for much more complicated behaviour including chaos.

1 Basic Definitions

We need some termonology.

1.1 Notation

We only consider ODEs of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \qquad (*)$$

for $\mathbf{x}$ in a phase space/state space $E \subset \mathbb{R}^n$. The n first order ODEs form a dynamical system of order (dimension) n.

Since

$$\frac{\partial \mathbf{f}}{\partial t} = 0$$

we call the system <u>autonomous</u>.

A non-autonomous system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$ can be made autonomous by setting

$$\mathbf{y} = (\mathbf{x}, t) \quad \text{with} \quad \dot{\mathbf{y}} = (\mathbf{f}(\mathbf{y}), 1)$$

The nth order ODE

$$\frac{d^n x}{dt^n} = g\left(x, \frac{dx}{dt}, \ldots, \frac{d^{n-1}x}{dt^{n-1}}\right)$$

can be put in the form (*) by setting

$$\mathbf{y} = \left(x, \frac{dx}{dt}, \ldots, \frac{d^{n-1}x}{dt^{n-1}}\right)$$

with

$$\dot{\mathbf{y}} = (y_2, y_3, \ldots, g(\mathbf{y}))$$

Similarly we will consider maps in the form

$$\mathbf{x}_{n+1} = \mathbf{F}(\mathbf{x}_n)$$

1.2 Initial Value Problem

Consider the IVP:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

$$\mathbf{x}(t_0) = \mathbf{x}_0$$

In Analysis II we showed that if $\mathbf{f}$ satisfies a Lipshitz condition then we are guaranteed that a solution exists in a neighbourhood of $\mathbf{x}_0$, t_0 and is unique.
(Lipshitz condition is that $\exists\, L$, a, s.t. $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x})| < L|\mathbf{x} - \mathbf{y}|$, $\quad \forall\, |\mathbf{x} - \mathbf{x}_0|,\ |\mathbf{y} - \mathbf{x}_0| < a$.)

Moreover, solutions $\mathbf{x}(t\,; \mathbf{x}')$ to

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

$$\mathbf{x}(t_0) = \mathbf{x}'$$

exist, are unique and they depend continously on $\mathbf{x}'$, t.

Note we are not guaranteed existence for all times.

Example:

$$\dot{x} = x^2$$

$$x(0) = 1$$

This has solution

$$x(t) = \frac{1}{1-t}$$

and so $x \to \infty$ as $t \uparrow 1$.

If $|\mathbf{x}(t)| \to \infty$ as $t \to T$ $(< \infty)$ then we call this finite time blow up.

Example: Non-uniqueness when $\mathbf{f}$ is non-Lipshitz. If

$$\dot{x} = \begin{cases} \sqrt{x} & \text{for } x > 0 \\ 0 & \text{for } x \le 0 \end{cases} \quad \text{and } x(0) = 0$$

Then there is a family of solutions

$$\left.\begin{array}{ll} x = 0 & t < \tau \\ x = \frac{1}{4}(t-\tau)^2 & t > \tau \end{array}\right\} \text{ for any } \tau \geq 0$$

From now on we will assume that $\mathbf{f}$ is differentiable (and so Lipshitz).

1.3 Trajectories and Flows

Consider the autonomous system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ The solution $\mathbf{x}(t)$ to the IVP with $\mathbf{x}(0) = \mathbf{x}_0$ defines a trajectory. (orbit/integral curve)

The distance travelled along this curve clearly only depends on $t - t_0$, and we could consider lots of starting points $\mathbf{x}_0$.
This motivates the idea of a flow.

Definition: (Flow)
Given $\mathbf{f}$, the corresponding flow is defined to be a (the) function $\boldsymbol{\phi}_t(\mathbf{x})$ from $E \times \mathbb{R} \to E$ such that

$$\frac{\partial}{\partial t}\boldsymbol{\phi}_t(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{\phi}_t(\boldsymbol{x})), \qquad \boldsymbol{\phi}_0(\boldsymbol{x}) = \boldsymbol{x}$$

The solution to the IVP $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}),\ \boldsymbol{x}(t_0) = \boldsymbol{x}_0$, is just $\boldsymbol{x}(t) = \boldsymbol{\phi}_{t-t_0}(\boldsymbol{x}_0)$
Clearly

$$\boldsymbol{\phi}_{s+t}(\boldsymbol{x}) = \boldsymbol{\phi}_s(\boldsymbol{\phi}_t(\boldsymbol{x})) = \boldsymbol{\phi}_t(\boldsymbol{\phi}_s(\boldsymbol{x}))$$

Aside: We can establish another link with maps, by defining $\boldsymbol{x}_{n+1} = \boldsymbol{F}(\boldsymbol{x}_n) = \boldsymbol{\phi}_{\Delta t}(\boldsymbol{x}_n)$.
Clearly $\boldsymbol{\phi}_{n\Delta t}(\boldsymbol{x}) = \boldsymbol{F}(\boldsymbol{\phi}_{(n-1)\Delta t}(\boldsymbol{x})) = \boldsymbol{F}^n(\boldsymbol{x})$

1.4 Flows, Trajectories, Orbits, Invariant Sets, & Limiting Sets

Using the idea of a flow, we define the following:

Definition: Orbits/Trajectories
The orbit of $\boldsymbol{\phi}_t(\boldsymbol{x})$ through $\boldsymbol{x}_0$ is the set $\mathcal{O}(\boldsymbol{x}_0) = \{\boldsymbol{\phi}_t(\boldsymbol{x}_0) : -\infty < t < \infty\}$.
The forwards (backwards) orbit is

$$\mathcal{O}^{+\,(-)}(\boldsymbol{x}_0) = \{\boldsymbol{\phi}_t(\boldsymbol{x}_0) : t \geq 0 \ (t \leq 0)\}$$

Definition: Invariant sets
A set of points $\Lambda \subset E$ is invariant if $\boldsymbol{x} \in \Lambda \implies \mathcal{O}(\boldsymbol{x}) \subset \Lambda$
Clearly $\mathcal{O}(\boldsymbol{x}_0)$ is invariant, and so is any union of orbits.

Particular cases of interest are

Definition: Fixed points
$\boldsymbol{x}_0$ is a periodic point with period T if $\boldsymbol{\phi}_T(\boldsymbol{x}_0) = \boldsymbol{x}_0$ for some $T > 0$ and $\boldsymbol{\phi}_t(\boldsymbol{x}_0) \neq \boldsymbol{x}_0$ for $0 < t < T$.
The set $\{\boldsymbol{\phi}_t(\boldsymbol{x}_0) : 0 \leq t < T\}$ is the periodic orbit through $\boldsymbol{x}_0$.

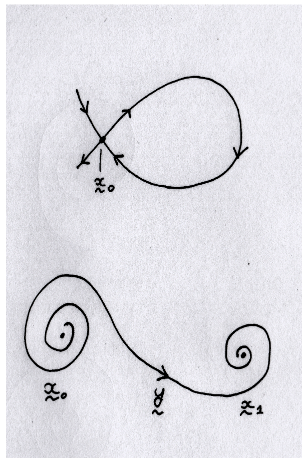
Definition: Limit Cycle
A limit cycle is an isolated periodic orbit, i.e. there are no other periodic orbits within a sufficiently

small neighbourhood.

Definition: Homoclinic and Heteroclinic orbits
If $\boldsymbol{x}_0$ is a fixed point and $\exists\ \boldsymbol{y} \neq \boldsymbol{x}_0$ such that $\boldsymbol{\phi}_t(\boldsymbol{y}) \to \boldsymbol{x}_0$ as $t \to \pm\infty$ then $\mathcal{O}(\boldsymbol{y})$ is a homoclinic orbit.

If $\boldsymbol{x}_0$, $\boldsymbol{x}_1$ are fixed points and $\exists\ \boldsymbol{y} \neq \boldsymbol{x}_0, \boldsymbol{x}_1$ with $\boldsymbol{\phi}_t(\boldsymbol{y}) \to \boldsymbol{x}_0$ as $t \to -\infty$, $\boldsymbol{\phi}_t(\boldsymbol{y}) \to \boldsymbol{x}_1$ as $t \to +\infty$ then $\mathcal{O}(\boldsymbol{y})$ is a heteroclinic orbit.

Example of a Heteroclinic and a Homoclinic orbit



If we are interested in the long-term behaviour of trajectories, it is not enough simply to think of

$$\lim_{t\to\infty} \boldsymbol{\phi}_t(\boldsymbol{x})$$

because the limit might not exist; for example a limit cycle.

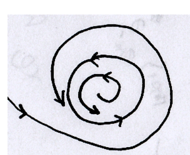
Definition: Limit set
The ω-limit set of $\boldsymbol{x}$ is

$$\omega(\boldsymbol{x}) = \{\boldsymbol{y} : \exists \text{ a sequence } (t_n) \text{ with } \ \boldsymbol{\phi}_{t_n}(\boldsymbol{x}) \to \boldsymbol{y} \text{ and } t_n \to \infty\}$$

Similarly the α limit set is defined by sequences with $t_n \to -\infty$.

Example:

$$\dot{r} = r(1 - r^2)$$
$$\dot{\theta} = 1$$



For $\boldsymbol{x}$ with $0 < |\boldsymbol{x}| < 1$ we have that

$$\omega(\boldsymbol{x}) = \{\boldsymbol{y} : |\boldsymbol{y}| = 1\} \qquad \alpha(\boldsymbol{x}) = \{\boldsymbol{0}\}$$

For $|\boldsymbol{x}| > 1$

$$\omega(\boldsymbol{x}) = \{\boldsymbol{y} : |\boldsymbol{y}| = 1\} \qquad \alpha(\boldsymbol{x}) = \{\}$$