MTHM006 MASTERS PROJECT REPORT

Exploring Feedback Effects in Multi-agent Models for Trading

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1 Introduction

In this project I aim to produce a model of the behaviour of a population of traders in a financial market. Specifically, I consider a market which initially is stable, and has a large and unchanging population, and then investigate the populations buying and selling trends in model simulations.

The methods involved in this start from modelling a single trader's thought process over time, and how this leads to a decision the trader makes when it comes to buying or selling the asset. Once the theory behind this is established, I apply this model to every member of the population, from which the feedback terms (determined from the population's behaviour, which simulate agent interaction) are introduced.

It is not feasibly possible to make a totally accurate model of a person's thought process, and even less so when attempting to do this for a large population (as no two people will think exactly the same). For this reason, I take a fairly generalised and simplistic approach when modelling the psychological behaviour of a single trader, which means the population size should be taken to be large enough for a generalised approach to be justified. A trader's thought process (which in my model dictates whether they buy or sell) is mainly affected by two things, the first being the arrival of news about the asset being traded, which is modelled as a Poisson process. The second is the actions of other traders in the same market, which is often referred to as the feedback/interaction effect. The interaction effect is supposed to imitate herd mentality in financial markets, where a trader's decision to buy or sell the asset is influenced by the population's buying and selling tendencies.

I investigate the trends we see in the general behaviour of the population, and how these trends change upon varying the parameters which dictate the population dynamics. Our area of interest lies in the buying and selling rates of the population, and when these rates remain stable and when they sharply diverge.

The methods used in the model in this report are partly similar to one made by Ahmet Omurtag and Lawrence Sirovich in a 2006 article [14], specifically the use of the Poisson distribution to model news arrival. Initial model simulations show similar results (although not identically the same, as I use different methods to emulate the interaction term) to those from that article, particularly those on instability. I aim to provide a better understanding of where instability occurs with different parameter values, and also introduce an "initial condition" simulation into the model to remove the transient behaviour of the order rates at the start of simulations.

I also compare my results with various literature on financial markets to see if the model's simulations reflect the findings from other studies.

2 Literature Review

Modelling financial markets has long been an area of interest in applied mathematics, as have multi-agent models (not just those modelling a population of traders). Multi-agent models are a method of analysing complex systems [13], whether that be the dynamics of infection rates in a susceptible-infected population of individuals [11]; the behaviour of cells in wound contraction [3]; or in the case of this project, modelling the buying and selling trends of a population of traders.

As previously stated, the first step to making the model is modelling the thought process of a trader in their decision to buy or sell the asset. The mathematics behind the psychology of the traders in the model made in this report is based around the theory discussed in a 1997 article by Adele Diederich [8]. In this article, an individuals thought process when it comes to making a decision is modelled by a stochastic process, say X_t. When X_t crosses a certain value, say a or b, then a decision is made, and X_t resets back to a value in between a and b [8]. A limit of modelling an individuals thought process this way is the individual can only take two different actions: the decision made when crossing a or the decision made when crossing b. This limitation is not an issue for the model made in this report, as in my model a trader can only do two things: buy or sell.

Similar to the work of Adele Diedrich, I use a discrete time stochastic process to model the decision process of trader, when it comes to deciding to buy or sell. A 2006 article by Ahmet Omurtag and Lawrence Sirovich [14] also took a similar approach to create a multi-agent trading model similar to the one made in this report. In their model, the stochastic jumps in a trader's thought process are caused by the same two main things that affect the psychology of traders in my model: arrival of news about the asset being traded and the actions of the other traders [14]. In Omurtag and Sirovich's model, the arrival of news about the asset being traded is assumed to be a Poisson process [14], and it is modelled the same way in my model. There is plenty of literature on the Poisson distribution which suggests that modelling news arrival as a Poisson process is justifiable [17].

In the both the Omurtag and Sirovich 2006 article and the Diedrich 1997 article, the stochastic jumps from one value to the next are quite small in magnitude but relatively high in frequency [14, 8]. To achieve a similar effect in my model simulations I take relatively high arrival rates for the news arrival (in the range 20-30) and take the amount by which each piece of news changes a trader's "thought process variable", to be comparatively smaller (in the range 0.05-0.1).

The other main factor that is considered when modelling a trader's thought process is the actions of others in the population. The influence of other traders on a given trader's psychology is a result of herd mentality, which is the tendency for people's behaviour or beliefs to conform to those of the group to which they belong. The influence of herd mentality can be seen in everyday life [1], for example we often make a decision on things like what food to eat and what TV shows to watch based on popularity and recommendations of others. Literature on economics and finance widely agrees that this behaviour occurs in financial markets (Banerjee 1992 [1], Bikhchandani 2000 [2], Dang 2016 [7] for example).

To include the effect of herd mentality in the model of this report, one must introduce terms that affect a trader's thought process and are dependent on the behaviour of other individuals. This causes the agents to interact. There is a lot of published literature on multi-agent models with agent interactions that are similar to that of herd mentality. A 1978 article by Mark Granovetter [10] introduced a threshold model, where a certain number of agents reaching a decision resulted in a given agent doing the same. A 1996 article by Knight et al. [12] produced a model of the interactions of cortical neurons.

There are many approaches to take when modelling agent interactions in the form of herd mentality. Models proposed by Ahmet Omurtag and Lawrence Sirovich [14] and Alain Corcos et al. [4] suggest taking agents to be "in contact" with an average number of other agents, who are then affected by the behaviour of that particular agent.

In the model in this report, I derive an agent interaction term with the goal of changing a trader's decision value to move towards the buying or selling decision making threshold value, depending on the recent buying and selling behaviour of all the agents in the model. An agent becomes more likely to buy the asset if the recent history of the population reflects more agents buying than selling (which can be seen as herd mentality [16]). I also construct the model in such a way so that the influence of the population's behaviour on each trader's psychology is different for each trader. The psychology and behaviour of participants in a financial market is discussed in more detail in a 1999 article by Robert J Shiller [15].

3 Derivation of the Multi-agent Trading Model

3.1 Model Overview

To start the model, I first derive equations to describe the thinking process of agents and how they lead to traders taking action (buying or selling). This involves a value (the thought process) crossing a boundary resulting in the trader taking action, similar to theory discussed by Adele Diederich (1997) [8].

Each trader in the population is assigned an "internal preference state", which shall be represented by the variable x. This variable represents how likely a trader is to buy or sell, where $x > 0$ represents a preference to buy, and $x < 0$ represents a preference to sell. There is also a reset condition on x, where if $x > 1$ or $x < -1$ then it resets to 0. In the event where x resets from $+1$ to 0 the trader buys, and when it resets from -1 to 0 they sell.

A trader's internal preference state is affected by three things: the arrival of external information, the actions of other traders in the same market and the traders own "forgetfulness". The evolution of x in time (t) can then be described by the following differential equation:

$$\frac{dx}{dt} = E(t) + I(t) - px \tag{1}$$

where $E(t)$ and $I(t)$ are functions which represent the effect of external information and other traders' actions respectively on a specific trader's internal preference state x at time t. The effect of a trader's forgetfulness is represented by the $-px$ term [14], where p is a small, positive constant ($p << 1$). The effect of this term is then to slightly decrease or increase a trader's internal preference state, depending on if the trader's current x value is positive or negative. The premise behind this term is that in the event of a trader receiving no external information or feedback from other traders, their internal preference state becomes slightly more "neutral", where they become slightly more likely to neither buy nor sell (in essence, the traders x value goes slightly more towards 0).

3.2 Exogenous Information Term

The arrival of news about the asset being traded is thought of as a Poisson process. Each piece of news, for simplicity, is thought of as either "good" or "bad", and the arrival of a piece of good news results in x being increased by a small amount υ^+, whilst the arrival of bad news means x decreases by υ^-. The arrival of news about the asset is assumed to be Poisson distributed, with arrival rates L^+ and L^- for good and bad news respectively.

The exogenous information term is then taken to be the total effect of the arrival of good and bad news on a traders x value taken over a short time period Δt. So at a time t, we take the the news arrivals between a time period with upper bound t and lower bound $t - \Delta t$ (in the case $t < \Delta t$ the lower bound is set to $t = 0$). Let S^+ and S^- denote the number of good and bad pieces of information about the asset. The $E(t)$ term can then be written as:

$$E(t) = \upsilon^+ [S^+]^t_{t-\Delta t} - \upsilon^- [S^-]^t_{t-\Delta t}$$

where $[S^+]^t_{t-\Delta t}$ is the number of pieces of good (or bad in the S^- case) news that arrive between times $t - \Delta t$ and t.

3.3 Interaction Term

A traders preference state is also affected by the buying and selling of other traders in the same market. If a trader, say trader 1, is thought of as having an "interaction" with another trader, say trader 2, then if trader 1 buys the asset, trader 2's x value increases by υ^+, and if trader 1 sells the asset, trader 2's x value decreases by υ^-.

Similar to the $E(t)$ term, the interaction term is taken to be the total effect other traders have on a trader's x value in a short time period Δt. Take F^+ to be the number of traders that buy the asset and also "interact" with a trader, and F^- to be the number of traders that sell the asset and "interact" with the same trader (and as with the $E(t)$ term, $[F^\pm]^b_a$ denotes this number between times a and b). The $I(t)$ term can then be written:

$$I(t) = \upsilon^+ [F^+]^t_{t-\Delta t} - \upsilon^- [F^-]^t_{t-\Delta t}$$

where $[F^\pm]^t_{t-\Delta t}$ is the number of good (or bad) feedbacks a trader receives in a small time period Δt.

3.4 Modelling a Single Trader's Internal State in Discrete Time

I now look to model a trader's internal state in discrete time. Take h to be the time step size (taking h to be small) so that t is discretely modelled by $t^{(n)} = nh$ $(n = 0, 1, 2, \ldots)$. I set $\Delta t = h$ so that the short time period over which the $E(t)$ and $I(t)$ terms are evaluated is simply the time from the previous time step $(t^{(n-1)})$ to the current one $(t^{(n)})$. The expressions for $E(t)$ and $I(t)$ can then be written in discrete time as follows:

$$E(t^{(n)}) = \upsilon^+ [S^+]_{t^{(n-1)}}^{t^{(n)}} - \upsilon^- [S^-]_{t^{(n-1)}}^{t^{(n)}} \tag{2}$$

$$I(t^{(n)}) = \upsilon^+ [F^+]_{t^{(n-1)}}^{t^{(n)}} - \upsilon^- [F^-]_{t^{(n-1)}}^{t^{(n)}} \tag{3}$$

I also apply a forward Euler approximation to the forgetfulness term (by taking $px \approx hpx$ in equation (1) over a small time interval of length h) so that equation (1) can be re-written in discrete time as:

$$\frac{dx(t^{(n)})}{dt} = E(t^{(n)}) + I(t^{(n)}) - hpx(t^{(n-1)})$$

(note that the x value is taken at time $t^{(n-1)}$ as opposed to $t^{(n)}$).

Now introducing the notation $x^{(n)} = x(t^{(n)})$, and using the initial condition $x^{(0)} = 0$ (for now) the evolution of a trader's internal state can be modelled in discrete time as a stochastic process:

$$x^{(n)} = x^{(n-1)} + \frac{dx^{(n)}}{dt} \tag{4}$$

Substituting equations (2) and (3) into the expression for $\frac{dx}{dt}$ in discrete time and factorising out υ^+ and υ^- gives:

$$\frac{dx^{(n)}}{dt} = \upsilon^+ \left([S^+]_{t^{(n-1)}}^{t^{(n)}} + [F^+]_{t^{(n-1)}}^{t^{(n)}} \right) - \upsilon^- \left([S^-]_{t^{(n-1)}}^{t^{(n)}} + [F^-]_{t^{(n-1)}}^{t^{(n)}} \right) - hpx^{(n-1)}$$

and substituting this expression for $\frac{dx^{(n)}}{dt}$ into equation (4) gives the iterative equation used for the simulations of a trader's internal state:

$$x^{(n)} = x^{(n-1)} - hpx^{(n-1)} + \upsilon^+ \left([S^+]_{t^{(n-1)}}^{t^{(n)}} + [F^+]_{t^{(n-1)}}^{t^{(n)}} \right) - \upsilon^- \left([S^-]_{t^{(n-1)}}^{t^{(n)}} + [F^-]_{t^{(n-1)}}^{t^{(n)}} \right) \tag{5}$$

The $[S^{\pm}]_{t^{(n-1)}}^{t^{(n)}}$ and $[F^{\pm}]_{t^{(n-1)}}^{t^{(n)}}$ terms are calculated as random numbers drawn from a specified distribution at each time step.

3.5 Simulating the Arrival of External Information

First consider the internal state of a single trader, who is not in contact with any other traders, which means the $I(t)$ term can be neglected from $\frac{dx}{dt}$, and the iterative equation for this particular trader simplifies to:

$$x^{(n)} = x^{(n-1)} - hpx^{(n-1)} + \upsilon^+ [S^+]_{t^{(n-1)}}^{t^{(n)}} - \upsilon^- [S^-]_{t^{(n-1)}}^{t^{(n)}} \tag{6}$$

As stated before, the arrival of news about the asset is assumed to be a Poisson process with arrival rate $L^{\pm}$. Thus, the variables S^+ and S^- are distributed as follows:

$$S^+ \sim \text{Poisson}(L^+)$$

$$S^- \sim \text{Poisson}(L^-)$$

Then given that the time between $t^{(n)}$ and $t^{(n-1)}$ is taken to be h, we have the following distributions for $[S^+]_{t^{(n-1)}}^{t^{(n)}}$ and $[S^-]_{t^{(n-1)}}^{t^{(n)}}$ [6]:

$$[S^+]_{t^{(n-1)}}^{t^{(n)}} \sim \text{Poisson}(hL^+)$$

$$[S^-]_{t^{(n-1)}}^{t^{(n)}} \sim \text{Poisson}(hL^-)$$

The values of $[S^{\pm}]_{t^{(n-1)}}^{t^{(n)}}$ are then taken to be random numbers drawn from these distributions.

Figure 1 illustrates a possible evolution of a trader's internal state over time period of $0 \leq t \leq 20$. The time step size is taken to be 0.01, and the plot shows the trader's x value at each time step, which is calculated via equation (6). In this particular simulation, the trader buys three times (marked by the blue points) and sells twice (marked in red).

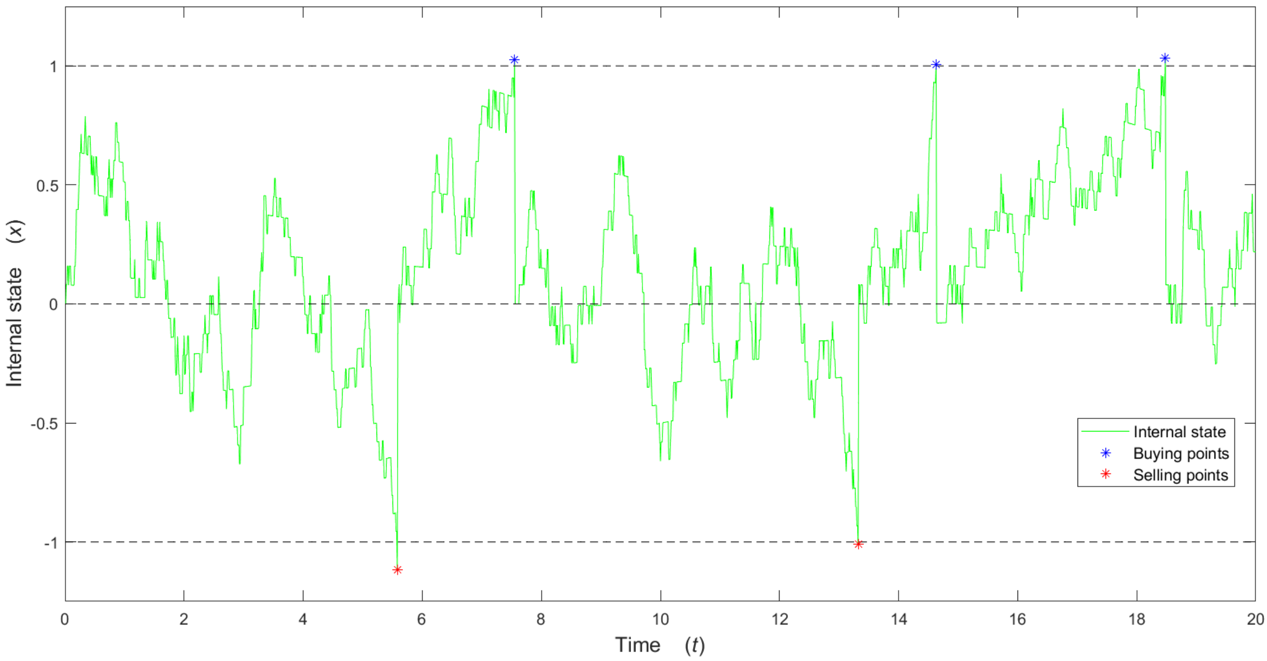

Figure 1: Plot of a trader's internal state with $L^+ = L^- = 22$, $\upsilon^+ = \upsilon^- = 0.08$ and influence from other traders is neglected. The horizontal dashed lines illustrate the reset boundary for x and the reset value (0).

3.6 Modelling the Internal States of a Population of Traders

I now look to model the internal states of an entire population of traders. Let N denote the size of the population. Each trader is indexed by the variable i ($i = 1, 2, \ldots, N$) and has internal state x_i. The exogenous information each trader receives about the asset at each time step is denoted by $[S_i^+]_{t^{(n-1)}}^{t^{(n)}}$ and $[S_i^-]_{t^{(n-1)}}^{t^{(n)}}$. N random numbers are taken from the distribution for $[S^+]_{t^{(n-1)}}^{t^{(n)}}$ and $[S^-]_{t^{(n-1)}}^{t^{(n)}}$ (giving $2N$ in total) at each time step, which gives the $[S_i^{\pm}]_{t^{(n-1)}}^{t^{(n)}}$ values for each trader. This means each trader receives their own personal stream of information arrival about the asset being traded. Similarly for the affect each trader receives from the action of other traders, we say that each trader receives their own unique positive and negative feedbacks from the population at each time step, denoted by $[F_i^+]_{t^{(n-1)}}^{t^{(n)}}$ and $[F_i^-]_{t^{(n-1)}}^{t^{(n)}}$.

Now take $\boldsymbol{X}^{(n)}$ to be a vector of length N, whose entries are the internal states $x_i^{(n)}$ of every trader at time $t^{(n)}$. Also take $(\boldsymbol{S}^+)^{(n)}$, $(\boldsymbol{S}^-)^{(n)}$, $(\boldsymbol{F}^+)^{(n)}$ and $(\boldsymbol{F}^-)^{(n)}$ to be vectors of length N, with entries $[S_i^+]_{t^{(n-1)}}^{t^{(n)}}$, $[S_i^-]_{t^{(n-1)}}^{t^{(n)}}$, $[F_i^+]_{t^{(n-1)}}^{t^{(n)}}$ and $[F_i^-]_{t^{(n-1)}}^{t^{(n)}}$ respectively. The evolution of the internal states of the entire population can then be modelled by:

$$\boldsymbol{X}^{(n)} = \boldsymbol{X}^{(n-1)} - hp\boldsymbol{X}^{(n-1)} + \upsilon^+ \left((\boldsymbol{S}^+)^{(n)} + (\boldsymbol{F}^+)^{(n)} \right) - \upsilon^- \left((\boldsymbol{S}^-)^{(n)} + (\boldsymbol{F}^-)^{(n)} \right) \tag{7}$$

with the initial condition (for now) that each $x_i^{(0)}$ value is zero (giving $\boldsymbol{X}^{(0)} = (0, 0, \ldots)'$).

3.7 Simulating the Interaction Term

To complete the model, I now define the feedback terms $[F_i^{\pm}]_{t^{(n-1)}}^{t^{(n)}}$. To do this I define some more variables. Take the variable g to be the average number of trader's that one specific trader is "in contact" with, meaning that when a trader buys (or sells), the expected number of traders whose x value increases (decreases) by υ^+ (υ^-) is g. Take y to be the number of traders in the population who buy, and $y^{(n)}$ to be the number that buy at time $t^{(n)}$. Similarly take z to be the number of sells, so that $z^{(n)}$ is the number of traders that sell at time $t^{(n)}$. Also take δt to be a small time period (although at least as big as Δt and h) and take $\overline{y}^{(n)}$ and $\overline{z}^{(n)}$ to be the average number of buys/sells from time $t^{(n)} - \delta t$ to $t^{(n)}$. As t is modelled discretely, I define d as $d = \lfloor \frac{\delta t}{h} \rfloor$ (where $\lfloor u \rfloor$ represents the floor function that returns an integer less than or equal to u), then take $\overline{y}^{(n)}$ and $\overline{z}^{(n)}$ to be the mean of each $y^{(n)}$ (or $z^{(n)}$) value at the previous d time steps. So $\overline{y}^{(n)}$ and $\overline{z}^{(n)}$ are calculated by:

$$\overline{y}^{(n)} = \frac{1}{d} \sum_{j=n-d}^{n} y^{(j)}, \qquad \overline{z}^{(n)} = \frac{1}{d} \sum_{j=n-d}^{n} z^{(j)}$$

(For $n < d$ we start the sum from $j = 0$)

Then $[F_i^{+}]_{t^{(n-1)}}^{t^{(n)}}$ and $[F_i^{-}]_{t^{(n-1)}}^{t^{(n)}}$ are taken to be random numbers, which have the following distributions:

$$[F_i^{+}]_{t^{(n-1)}}^{t^{(n)}} \sim \text{Poisson}\left(\frac{g\overline{y}^{(n-1)}}{N}\right)$$

$$[F_i^{-}]_{t^{(n-1)}}^{t^{(n)}} \sim \text{Poisson}\left(\frac{g\overline{z}^{(n-1)}}{N}\right)$$

The terms are defined this way for the following reasons. First consider a case where one trader in the population buys at time $t^{(0)}$ (and no others buy or sell at this time), and think of the corresponding $(\boldsymbol{F}^{\pm})^{(1)}$ vectors. From how g is defined, and from the description of trader interactions, in such a case one would expect g traders in the population to have their internal states increased by υ^+ from the $(\boldsymbol{F}^{+})^{(1)}$ vector, whilst the $(\boldsymbol{F}^{-})^{(1)}$ vector should have no effect.

First note that no sellers at time $t^{(0)}$ would give $\overline{z}^{(0)} = 0$, which results in the $[F_i^{-}]_{t^{(0)}}^{t^{(1)}}$ terms having the distribution $[F_i^{-}]_{t^{(0)}}^{t^{(1)}} \sim \text{Poisson}(0)$. Thus, $[F_i^{-}]_{t^{(0)}}^{t^{(1)}} = 0$ for each i, giving the required $(\boldsymbol{F}^{-})^{(1)}$ vector described above.

Now consider the $(\boldsymbol{F}^{+})^{(1)}$ vector. Take $(\boldsymbol{F}^{+}_{\text{Sum}})^{(1)}$ to be the sum of all the entries of $(\boldsymbol{F}^{+})^{(1)}$, so that $(\boldsymbol{F}^{+}_{\text{Sum}})^{(1)}$ is defined by:

$$(\boldsymbol{F}^{+}_{\text{Sum}})^{(1)} = \sum_{i=1}^{N} [F_i^{+}]_{t^{(0)}}^{t^{(1)}}$$

This sum can be thought of as the total number of positive feedbacks the population receives at time $t^{(1)}$, which we would expect to be g, given the definition of g and the description of the buying and selling of the population at time $t^{(0)}$. Consider the expectation of $(\boldsymbol{F}^{+}_{\text{Sum}})^{(1)}$:

$$\text{E}\left[(\boldsymbol{F}^{+}_{\text{Sum}})^{(1)}\right] = \text{E}\left[\sum_{i=1}^{N} [F_i^{+}]_{t^{(0)}}^{t^{(1)}}\right]$$

$$\Rightarrow \text{E}\left[(\boldsymbol{F}^{+}_{\text{Sum}})^{(1)}\right] = \text{E}\left[[F_1^{+}]_{t^{(0)}}^{t^{(1)}} + [F_2^{+}]_{t^{(0)}}^{t^{(1)}} + \ldots + [F_N^{+}]_{t^{(0)}}^{t^{(1)}}\right]$$

Then by linearity of expectation:

$$\Rightarrow \text{E}\left[(\boldsymbol{F}^{+}_{\text{Sum}})^{(1)}\right] = \text{E}\left[[F_1^{+}]_{t^{(0)}}^{t^{(1)}}\right] + \text{E}\left[[F_2^{+}]_{t^{(0)}}^{t^{(1)}}\right] + \ldots + \text{E}\left[[F_N^{+}]_{t^{(0)}}^{t^{(1)}}\right]$$

As one trader buys at time $t^{(0)}$ we have $y^{(0)} = 1$, which gives $\overline{y}^{(0)} = 1$, and so each $[F_i^{+}]_{t^{(0)}}^{t^{(1)}}$ term has distribution:

$$[F_i^{+}]_{t^{(0)}}^{t^{(1)}} \sim \text{Poisson}\left(\frac{g}{N}\right)$$

Then as the expectation of a Poisson distribution is equal to it's arrival rate [6], this gives:

$$\mathrm{E}\left[[F_i^+]_{t^{(0)}}^{t^{(1)}}\right] = \frac{g}{N}$$

Thus, the expectation of $(\boldsymbol{F}_{\text{Sum}}^+)^{(1)}$ is:

$$\mathrm{E}\left[(\boldsymbol{F}_{\text{Sum}}^+)^{(1)}\right] = \frac{g}{N} + \frac{g}{N} + \ldots + \frac{g}{N}$$

$$\Rightarrow \mathrm{E}\left[(\boldsymbol{F}_{\text{Sum}}^+)^{(1)}\right] = N(\frac{g}{N}) = g$$

So the vector $(\boldsymbol{F}^+)^{(1)}$ will give an expected number of g positive feedbacks to the population. There is a chance that a trader could receive more than 1 positive feedback as it is possible to have $[F_i^+]_{t^{(0)}}^{t^{(1)}} > 1$, but the chances of this are very small as we generally take $g << N$. Thus the vector $(\boldsymbol{F}^+)^{(1)}$ gives the desired effect on the population's x values, in the case of 1 trader buying at time $t^{(0)}$.

Now consider the case where multiple traders are buying at a later time $t^{(k-1)}$, so take $\overline{y}^{(k-1)} = a$ ($a > 1$). In this case one would expect there to be ag positive feedbacks distributed throughout the population due to the $(\boldsymbol{F}^+)^{(k)}$ vector. By considering the expectation of the sum of this vector as before, this is indeed the case as:

$$[F_i^+]_{t^{(k-1)}}^{t^{(k)}} \sim \text{Poisson}\left(\frac{ag}{N}\right)$$

$$\Rightarrow \mathrm{E}\left[(\boldsymbol{F}_{\text{Sum}}^+)^{(k)}\right] = \frac{ag}{N} + \frac{ag}{N} + \ldots + \frac{ag}{N}$$

$$\Rightarrow \mathrm{E}\left[(\boldsymbol{F}_{\text{Sum}}^+)^{(k)}\right] = ag$$

Note taking $\overline{z} > 1$ at a time $t^{(k-1)}$ gives the same result for the $(\boldsymbol{F}^-)^{(k)}$ vector. Indeed the interaction terms in equation (7) are better interpreted as the effect that the population's buying/selling behaviour, over the previous δt length of time, has on each trader's internal state.

The $[F_i^+]_{t^{(k-1)}}^{t^{(k)}}$ terms are taken as random numbers from a Poisson distribution, as opposed to just distributing $\overline{y}g/\overline{z}g$ positive/negative feedbacks throughout the population, so that there is randomness in how much the behaviour of the population affects each trader's internal state. It would be unrealistic to say that every trader is affected by the population's behaviour in the same way.

The y and z values are not used in the interaction terms as these numbers generally have a lot of variation, which in turn can cause large changes in the internal states of the population, leading to sharp divergences in the population behaviour and giving unstable results. This is why the momentary mean of y and z ($\overline{y}$ and $\overline{z}$) are taken, as it reduces the fluctuations in the interaction terms and gives more stable simulations. Note that the decision to use $\overline{y}$ and $\overline{z}$ means that process of obtaining the internal states is not a Markov process, as these values depend on the recent history of the buying and selling rates of the population, meaning the process is not memoryless [9].

4 Results from Model Simulations

Note that the forgetfulness parameter p, and the small time period δt used to calculate momentary means $\overline{y}$ and $\overline{z}$, are unchanged for every model simulation. They are set to:

$$p = 0.01, \qquad \delta t = 0.2$$

The population size N is assumed to always be 10^4, apart from section 4.4, where it is specified in the text and the figure captions.

4.1 Defining Buying and Selling Rates

The buying order rates and selling order rates per agent (r^+ and r^- respectively) are the quantities of interest when understanding the behaviour of the population. These rates are rather self-explanatory; they are simply the average rates at which a single agent in the population buys or sells the asset. The buying rate is calculated as the number of buys y over a period of time t, then dividing this number by the population size N gives the buying rate per agent. We make the decision to use the momentary mean of y ($\overline{y}$) over the time period in the calculation for r^+ to get more realistic, stable order rate estimates. We apply the same method with the momentary mean number of sells $\overline{z}$ to get the selling rates, which gives the following equations for r^+ and r^-:

$$r^+ = \frac{\overline{y}}{tN}, \qquad r^- = \frac{\overline{z}}{tN}$$

We take the order rates per agent in order to get results that are comparable between different population sizes.

The values we study from the simulations are the momentary values of the order rates, denoted by R^+ and R^-. These are defined as the order rates over a small time interval of size Δt, so the momentary order rate at time t is the order rate over the time period starting at $t - \Delta t$ and ending at t. We set $\Delta t = h$ as before so that the momentary values of $R^\pm$ are the order rates over the time from one time step to the next. Denoting the momentary values of $R^\pm$ at time $t^{(n)}$ by $(R^\pm)^{(n)}$ we get the following formulae for momentary order rates:

$$(R^+)^{(n)} = \frac{\overline{y}^{(n)}}{hN}, \qquad (R^-)^{(n)} = \frac{\overline{z}^{(n)}}{hN}$$

4.2 Preliminary Hypotheses and Simulations

The process of running simulations involves iterating equation (7). Take T to be the end time of the simulations, which are then run from $0 \leq t \leq T$. The total number of time steps from 0 to T is given by $\frac{T}{h}$ (we choose a time step size that divides 1, and note that there will be a total of $\frac{T}{h} + 1$ discrete time values in the interval for t). We then plot $(R^\pm)^{(n)}$ for each $t^{(n)}$ value, where $n = 0, 1, 2, \ldots, \frac{T}{h} - 1, \frac{T}{h}$.

Upon studying the terms in equation (7), we can make the assumption that sharp transitions in the order rates will be caused by the interaction terms. The terms in the vectors $(\boldsymbol{S}^\pm)^{(n)}$ have a constant distribution throughout the simulation which is why we wouldn't expect these to cause instability. Instead we would expect it to come from the terms that change for each iteration depending on the previous state of the population. The $hp\boldsymbol{X}^{n-1}$ term should not cause instability as this term actually keeps the system in a more stable state (shifts all x_i values slightly towards 0, keeping order rates lower).

Having deduced that the $(\boldsymbol{F}^\pm)^{(n)}$ vectors are the cause of the order rates becoming unstable, it is logical to make the prediction that increasing the parameter g will make order rates becoming unstable more likely. This is because increasing g increases the expected values of the entries of $(\boldsymbol{F}^+)^{(n)}$ and $(\boldsymbol{F}^-)^{(n)}$.

Upon running a few simulations with some different g values, we confirm these initial hypotheses to be true. Figures 2 and 3 show simulations for $0 \leq t \leq 15$, one with $g = 14$ and the other with $g = 18$. The blue line is the momentary buying rate per agent, and the red line is the momentary selling rate per agent.

The sharp transition in R^+ in figure 3 confirms the hypothesis about trader interactions being the cause of instability. Due to the system settling to a stable state for $t < 7.5$ before the buying rate explodes, we can deduce that the divergence of the order rates must be caused by terms in equation (7) that change depending on the previous state of the population (the feedback terms). The lack of a sharp divergence in order rates in figure 2 also illustrates the hypothesis about increasing g resulting in more unstable order rates.

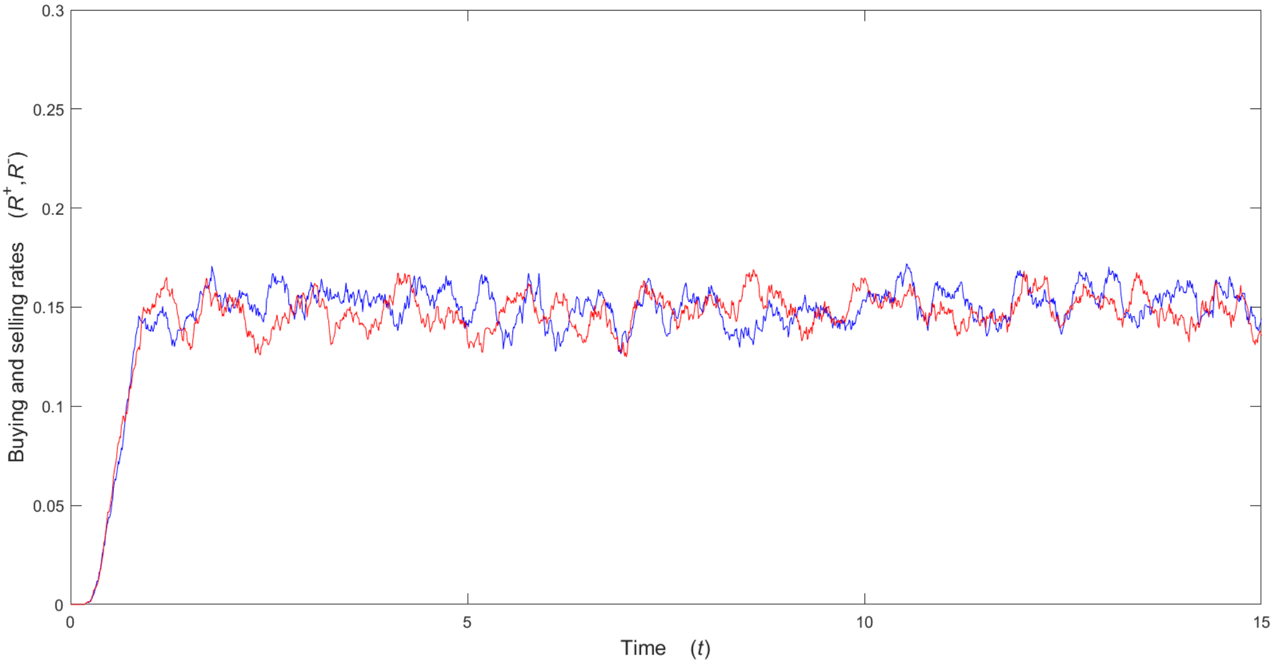

Figure 2: Plot of order rates with $g = 14$. Other parameters set as $L^+ = L^- = 30$, $\upsilon^+ = \upsilon^- = 0.075$ and $h = 0.01$.

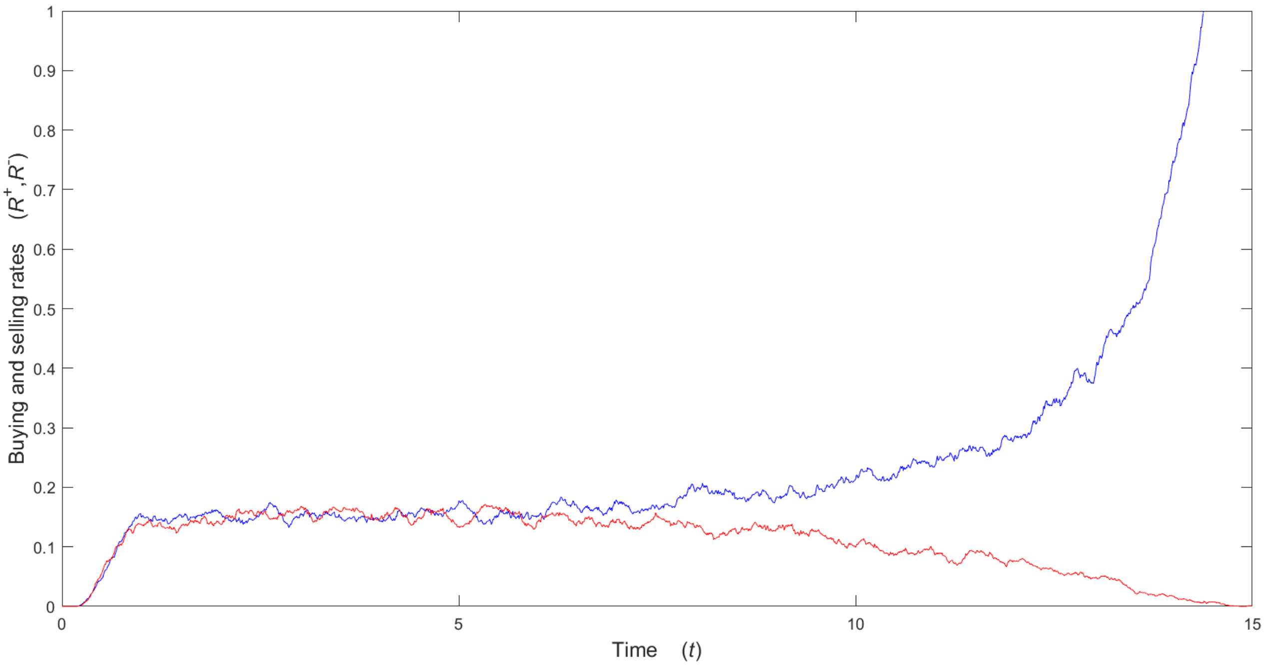

Figure 3: Plot of order rates with $g = 18$. Other parameters set as $L^+ = L^- = 30$, $\upsilon^+ = \upsilon^- = 0.075$ and $h = 0.01$.

4.3 Slight Model Adjustments

4.3.1 Adding Initial Condition Simulation

Recall that an aim stated at the start of this report is to investigate the trends we see from feedback effects in a market that is initially stable. It is clear to see in figures 2 and 3 that in the current model order rates are not initially stable, but rather undergo transient behaviour before reaching a value around which they fluctuate. I introduce an initial conditions to the model to remove this transient behaviour.

Figure 4 shows the distribution of the internal states of the population simulated in figure 2 at the end of the simulation in the form of a histogram. A population with internal states distributed like this is the point we wish to start simulations from when the initial conditions are introduced. This distribution of $x^{(0)}$ values looks very different to the one for the starting point of simulations in the current version of the model, where $x_i^{(0)} = 0$ for all i.

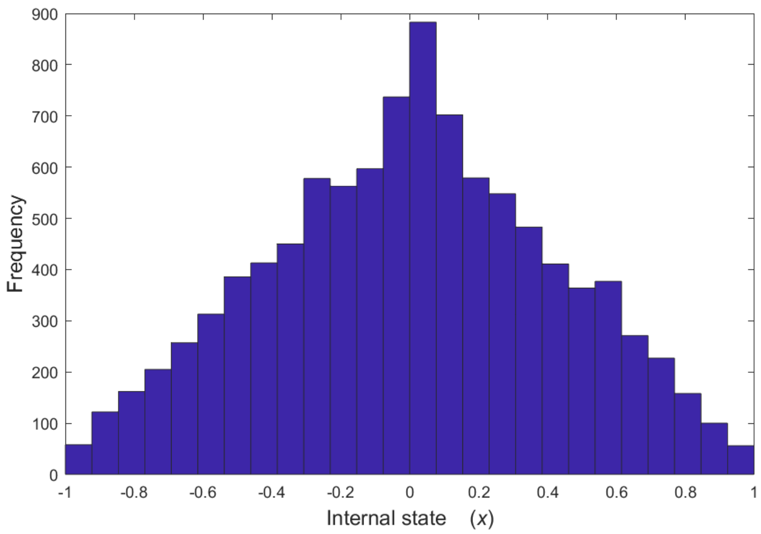

Figure 4: Histogram of internal states of entire population at the end of the simulation plotted in figure 2

One method of determining initial conditions could be to approximate the distribution of internal states, take the x_i values as random numbers from this distribution, then use the simulated order rates to calculate suitable y and z values and so forth, however I use a more efficient method.

My method is to run an "initial condition" simulation. Given that we know it is the interaction terms that cause instability, I set g to a low enough value such that the order rates don't diverge, and set $L^+ = L^-$, $v^+ = v^-$ to get similar buying and selling rates. I then use the endpoint of this initial condition simulation (which includes the $y^{(n)}$, $z^{(n)}$, $\overline{y}^{(n)}$ and $\overline{z}^{(n)}$ values at the end of the simulation, as well as the $x_i^{(n)}$ values) as the starting point of the actual simulation whose results are studied.

We set the start values of the initial condition simulation (x_i, (y), z etc.) to 0, as we have previously been doing. An end time T_{In} is chosen to be 5 for the initial condition simulation (figures 2 and 3 suggest transient behaviour passes much sooner but this can change depending on parameters so we take 5 to be safe). For the actual simulation, set:

$$x_i^{(0)} = x_i^{(\mathrm{In})}, \qquad y^{(0)} = y^{(\mathrm{In})}, \qquad z^{(0)} = z^{(\mathrm{In})}, \qquad \overline{y}^{(0)} = \overline{y}^{(\mathrm{In})}, \qquad \overline{z}^{(0)} = \overline{z}^{(\mathrm{In})}$$

where the "In" superscript denotes the value at the end of the initial condition simulation.

Figure 5 shows the significance of the initial condition simulation. Two simulations are ran with the exact same parameters, one with an initial condition simulation and one without. We notice that order rates remain stable for the simulation with an initial condition, but diverge in the simulation without.

Closer inspection of the no initial condition simulation (figure 6) reveals that due to the difference in good and bad news arrival rates, the buying rate approaches the stable value it reaches in the simulation with an initial condition faster than the selling rate. This causes the population to receive fewer negative feedbacks whilst receiving a significant amount more positive feedbacks, and as a result the buying rate starts to grow exponentially.

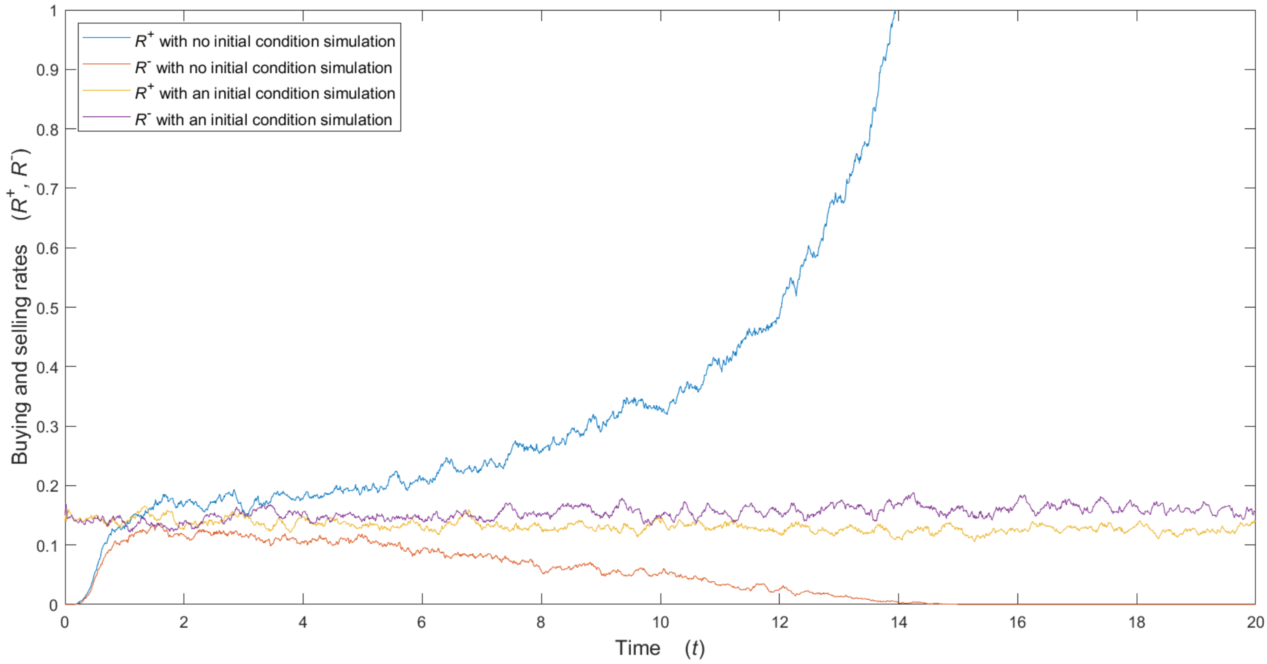

Figure 5: Plot of order rates from two simulations, one with an initial condition simulation and one without. Parameters in both simulations are the same, with $g = 14$, $\upsilon^+ = \upsilon^- = 0.08$, $L^+ = 23.2$, $L^- = 23$, $h = 0.01$, $T = 20$.

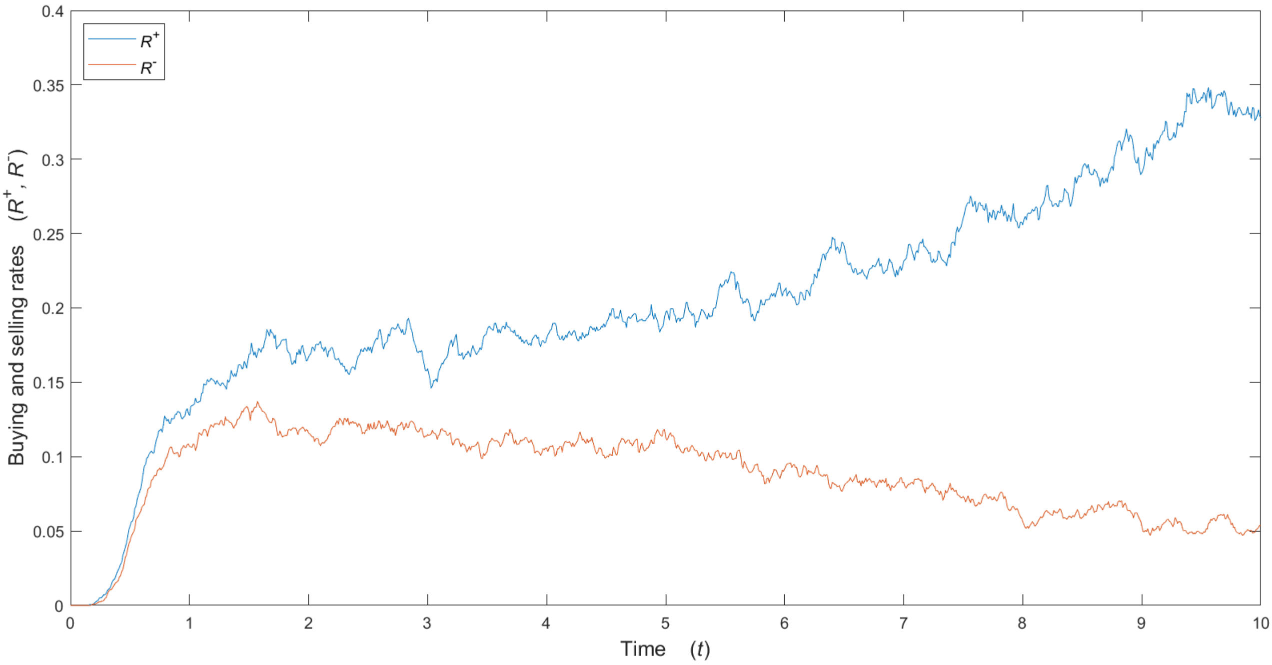

Figure 6: Zoomed in view of the no initial condition simulation from figure 5.

Whilst in many cases a simulation with no initial conditions would start to model the same behaviour as one with them after a period of time, the fact that they are different in some cases means it can make significant changes in results, especially in cases like figure 11 where 10^4 different simulations are ran.

4.3.2 Adjusting the Iterative Equation

I also make a small adjustment to the iterative equation used for simulations (equation (7)) by introducing parameters $\epsilon^{\pm}$. The value of ϵ^{+} (ϵ^{-}) becomes the amount by which a trader's internal state jumps up (down) as a result of receiving a positive (negative) feedback from the population. The values of $\epsilon^{\pm}$ are in the same range as $\upsilon^{\pm}$, and often equivalent. This adjustment is made in order to allow me to change the effect good and bad news has on the population, whilst keeping the influence of the population's behaviour on an agent the same (or change the population influence and keep external information impact the same).

The new iterative equation becomes:

$$\boldsymbol{X}^{(n)} = \boldsymbol{X}^{(n-1)} - hp\boldsymbol{X}^{(n-1)} + \upsilon^{+}(\boldsymbol{S}^{+})^{(n)} - \upsilon^{-}(\boldsymbol{S}^{-})^{(n)} + \epsilon^{+}(\boldsymbol{F}^{+})^{(n)} - \epsilon^{-}(\boldsymbol{F}^{-})^{(n)} \tag{8}$$

4.4 Testing the Model

Some tests we can run on the model to verify the mathematics behind it is to check the order rates for different time step and population sizes.

Provided the time step size is small enough, changing the time step size should not affect the order rate values, only the number of order rates number calculated over the simulation (a smaller time step means more iterations). Figure 7 shows the buying rates for five different h values ($h = 0.1, 0.05, 0.01, 0.005, 0.001$).

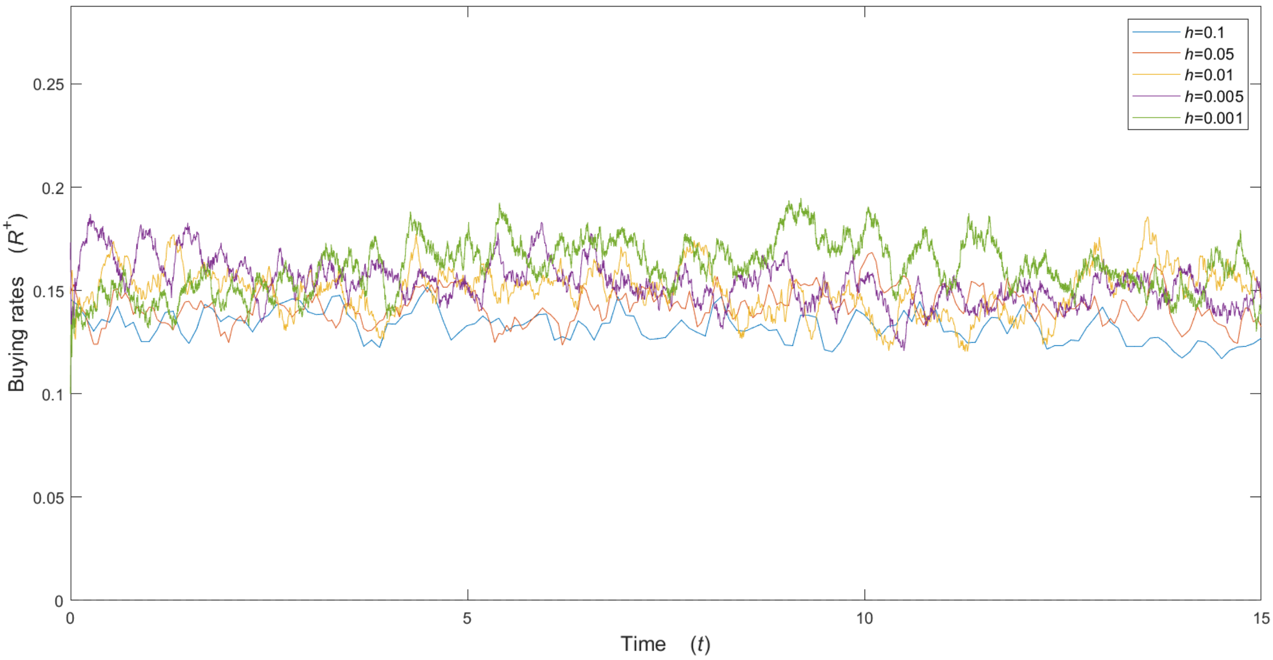

Figure 7: Plot of buying order rates for a range of different time step sizes. Other parameters are the same for each simulation with $g = 12$, $\upsilon^{+} = \upsilon^{-} = 0.077$, $\epsilon^{+} = \epsilon^{-} = 0.077$, $L^{+} = L^{-} = 28$, $T = 15$.

The results from figure 7 suggest that time step sizes of 0.1 and 0.05 are too large, as the mean values for each $(R^{+})^{(n)}$ over the entire simulation are 0.1328 and 0.1424 respectively. We get the desired result for the smaller time step sizes, with $h = 0.01$, $h = 0.005$ and $h = 0.001$ returning mean R^{+} values of 0.1539, 0.1510 and 0.1568 respectively. I make the decision to use $h = 0.01$ for the rest of the simulations as it will give faster simulations than smaller time step sizes, but is also small enough to get accurate results.

The population size should not change the average value of the order rates, but we would expect there to be smaller fluctuations around the average value. When comparing order rates, I take $L^{+} = L^{-}$, $\upsilon^{+} = \upsilon^{-}$, $\epsilon^{+} = \epsilon^{-}$ and g to be small enough so that order rates do not diverge. This means that R^+ and R^- should tend to the same value, meaning the expected value of $(R^+ - R^-)^{(n)}$ is 0.

Figure 8 shows the value of $(R^+ - R^-)^{(n)}$ for five different simulations, where all parameters are the same apart from the population size N. Each iteration can be thought of as a trial where the expected value is 0. More traders for each iteration means a higher number trials, and so by the law of large numbers, we would expect the value of $(R^+ - R^-)^{(n)}$ to be closer to 0 for larger population sizes [18], which is exactly the result we see in figure 8

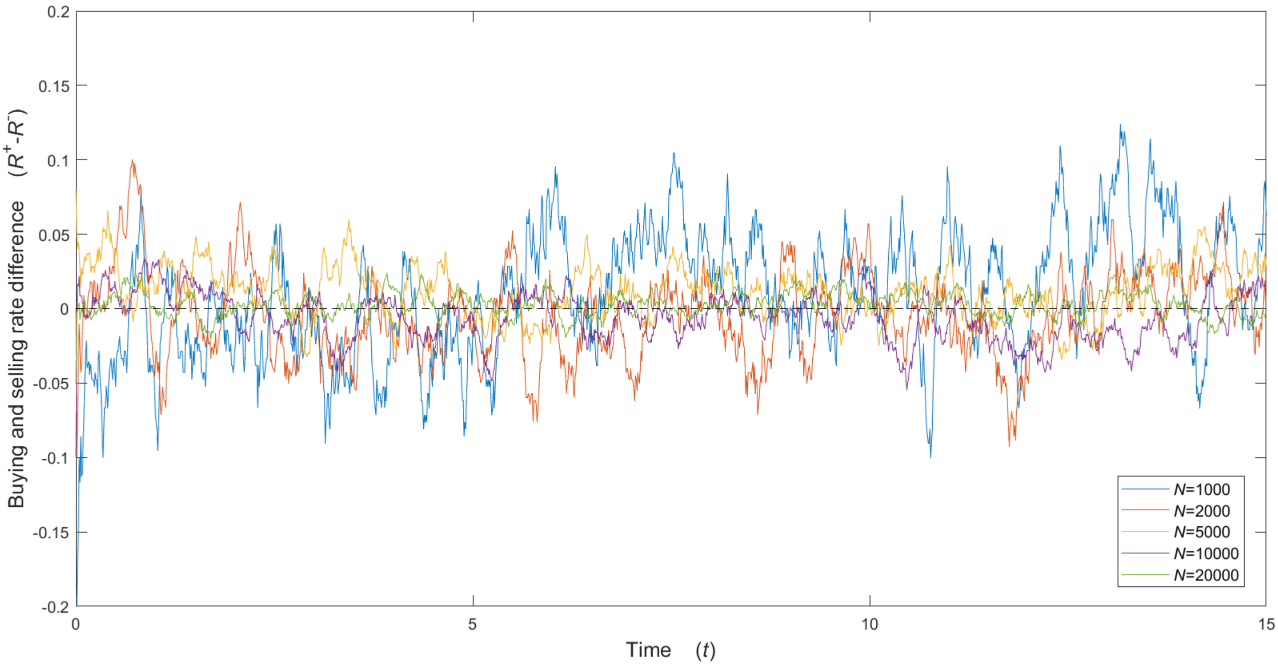

Figure 8: $\upsilon^+ = \upsilon^- = 0.077$, $\epsilon^+ = \epsilon^- = 0.077$, $L^+ = L^- = 28$, $h = 0.01$, $T = 15$. Horizontal dashed black line is plotted at 0 to show the expected value of $R^+ - R^-$.

4.5 Investigating the Effects of Changing Population Influence

Having previously established that the agent interaction terms are the cause of instability in order rates, I now investigate the results from model simulations whilst changing the interaction parameter g.

I initially start with simulations where $\epsilon^+ = \epsilon^-$, $\upsilon^+ = \upsilon^-$ and $L^+ = L^-$. I then run multiple simulations, increasing g in increments of 0.5, starting at $g = 12$. Order rates were found to not diverge until g was 16 or greater. Interestingly though, some simulations with $g \geq 16$ had their order rates remain stable, but no simulations with $g < 16$ ended up having order rate explosions.

Figure 9 shows order rates from multiple simulations with $g = 16$. The order rates behaved very differently in some cases, despite having the same parameters for each simulation.

I now change the variables υ^+ and υ^- (whilst keeping all others the same as in figure 9) to $\upsilon+ = 0.079$, $\upsilon^- = 0.077$, then run multiple simulations with different g values as before (in increments of 0.5 again).

For g values where order rates did not explode, the values of R^+ and R^- stabilise around two different values due to the difference in υ values (figure 10). The smallest value of g for which order rates grew exponentially was 13.5. Interestingly in these simulations there were no cases where the order rates remained stable for $g \geq 13.5$, and also no cases where they diverged for $g < 13.5$. Figure 10 shows two simulations, one with $g = 12$ and another with $g = 13.5$.

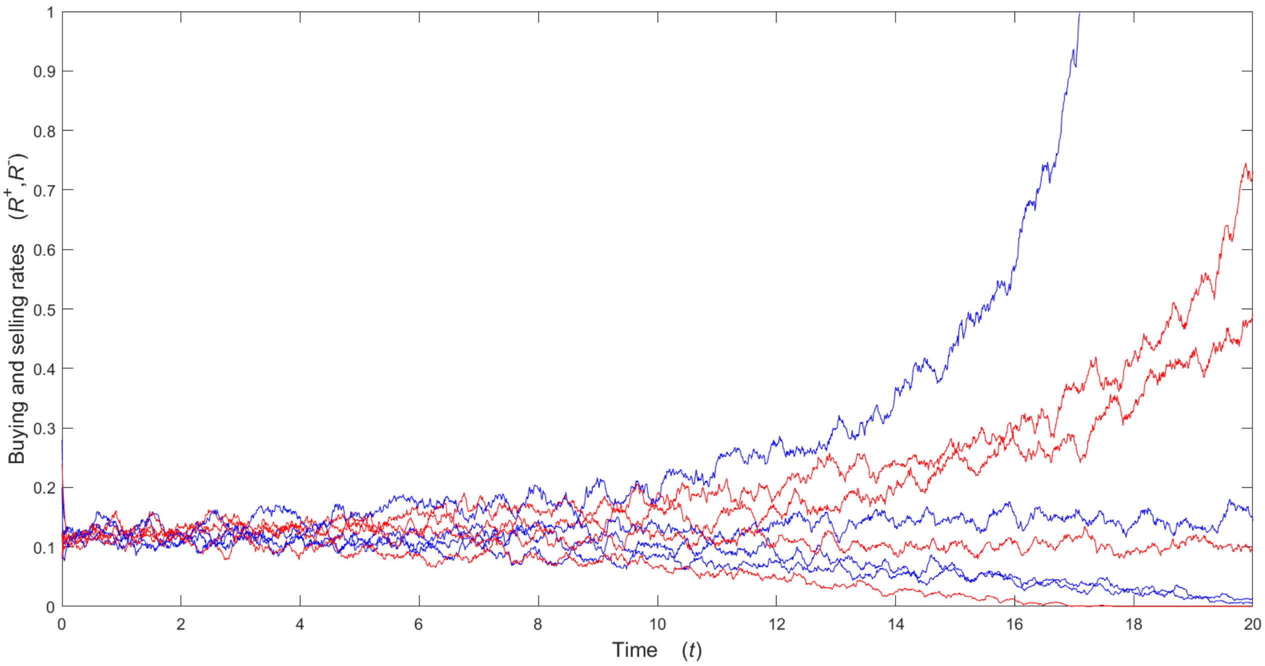

Figure 9: Plot of order rates from multiple simulations, all with parameters $g = 16$, $\upsilon^+ = \upsilon^- = 0.078$, $\epsilon^+ = \epsilon^- = 0.078$, $L^+ = L^- = 23$, $h = 0.01$, $T = 20$. The blue line represents the buying rate and the red line represents the selling rate.

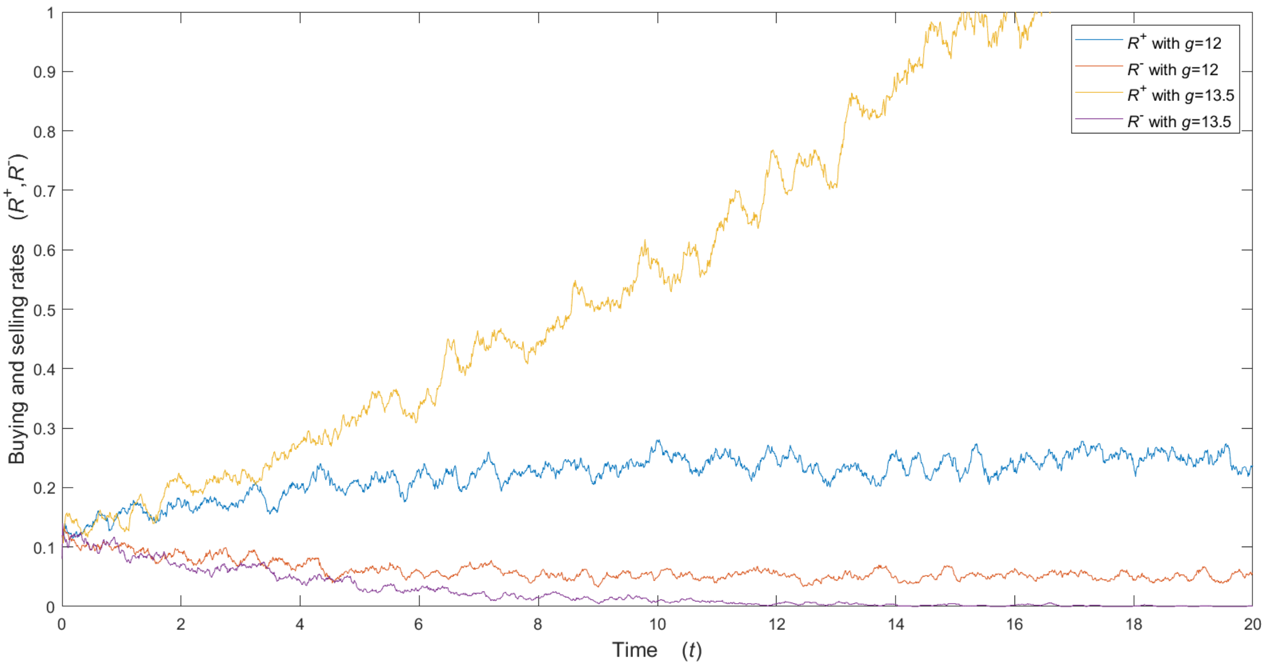

Figure 10: Plot of order rates from two different simulations, one with $g = 12$ and the other with $g = 16$. Other parameters are $\upsilon^+ = 0.079$, $\upsilon^- = 0.077$, $\epsilon^+ = \epsilon^- = 0.078$, $L^+ = L^- = 23$, $h = 0.01$, $T = 20$.

4.6 Investigating Where Instability Occurs

The simulations so far suggest that when increasing g there is a threshold value g^* where when $g \geq g^*$, it is possible for the order rates to diverge, whilst when $g < g^*$ the order rates always remain stable. The simulations also suggest that when there is a difference in the effect of "positive" external information and "negative" external information, the value of g^* decreases.

To investigate when the model becomes unstable, I vary the parameters υ^+ and υ^- to create a difference in the news effect, and also vary g. Define $\Delta\upsilon$ to be the difference between υ^+ and υ^- ($\Delta\upsilon = \upsilon^+ - \upsilon^-$). The $\upsilon^\pm$ values are centered around a base value υ_0, so the values of υ^+ and υ^- for a given $\Delta\upsilon$ value are given by:

$$\upsilon^+ = \upsilon_0 + \frac{1}{2}\Delta\upsilon$$

$$\upsilon^- = \upsilon_0 - \frac{1}{2}\Delta\upsilon$$

I run simulations over a range of $\Delta\upsilon$ and g values for simulation length $T = 20$, and record the difference in order rates at the end of the simulation $(|R^+ - R^-|)^{(T)}$ (given by the absolute value of the difference of the final $R^\pm$ values of the simulation). Before doing this it is important to set a value R^*, where when $(|R^+ - R^-|)^{(T)} > R^*$, I deem the model unstable. From simulation results I decide to take $R^* = 1$, as when one of the order rates exceeds the other by this much, one of the order rates is either constantly at 0 or very close to it, whilst the other order rate is greater than or equal to 1. It is certainly justifiable to say that the market being modelled has become destabilised when either the buying or selling rate remains this low, whilst the other rate remains high [2].

I set $\upsilon_0 = 0.08$ and run simulations for $0 \leq \Delta\upsilon \leq 0.007$ and $10 \leq g \leq 16$ and plot the value of $(|R^+ - R^-|)^{(T)}$ in the form of a heatmap in the $(g, \Delta\upsilon)$-plane (figure 11). I take 100 different linearly spaced g and $\Delta\upsilon$ values over the specified ranges, giving 10^4 simulations in total.

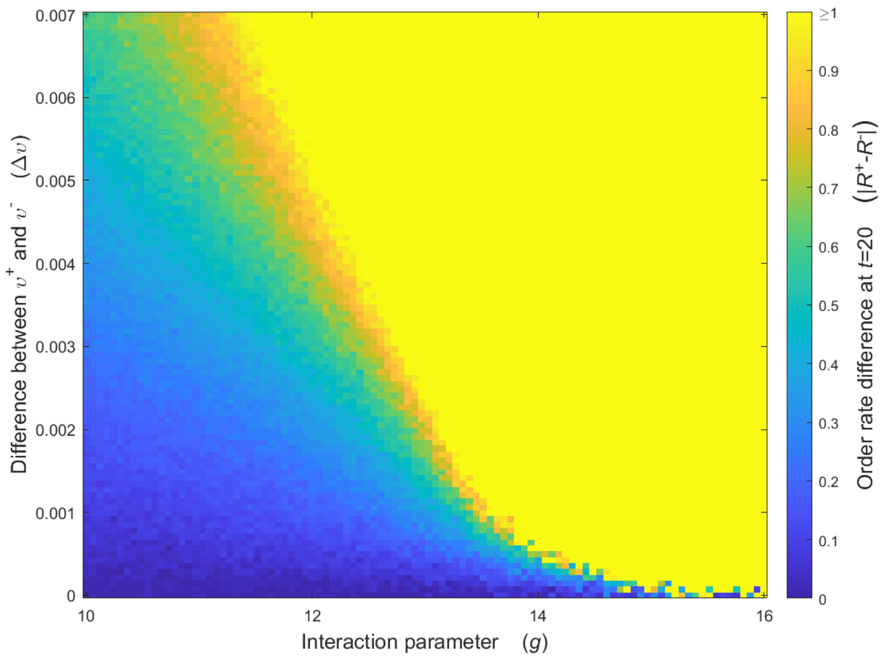

Figure 11: Heatmap of the difference in order rates after simulations in the $(g, \Delta\upsilon)$-plane. Other parameters are kept the same for each simulation at each point: $\epsilon^+ = \epsilon^- = 0.08$, $L^+ = L^- = 25$, $h = 0.01$, $T = 20$. Note that the bright yellow region indicates that $|R^+ - R^-| \geq 1$ (not equal to 1).

The results from figure 11 do clearly indicate that as $\Delta\upsilon$ increases, the minimum value of g at which the order rates become unstable (g^*) decreases. The dark yellow/orange region would suggest that for $g <\sim 13.25$ the relationship between $\Delta\upsilon$ and g^* is a negative, linear one. The results also reflect findings from the simulations in section 4.5, where for υ^+ and υ^- approximately equal, there is a region where both stability and instability in order rates can occur.

For larger values of $\Delta\upsilon$ the smoother colour gradient indicates that the value of g^* is harder to pinpoint, whereas for lower $\Delta\upsilon$ values there is a far more clearly defined boundary (assuming we take $g < 13.25$), either side of which stability and instability in the order rates occurs. This proposition is demonstrated in figure 12. I take two different g values fairly close together, and run two different simulations for the different g values at the same $\Delta\upsilon$ value. I set $\Delta\upsilon = 0.0008$ and set the two g values to be 13.5 and 13.8, meaning the simulations are above and below the stable-unstable boundary in the $(g, \Delta\upsilon)$-plane. My hypothesis from figure 11 would suggest that the order rates should sharply diverge for the simulation with $g = 13.8$ and remain stable for $g = 13.5$, which is seen in figure 12.

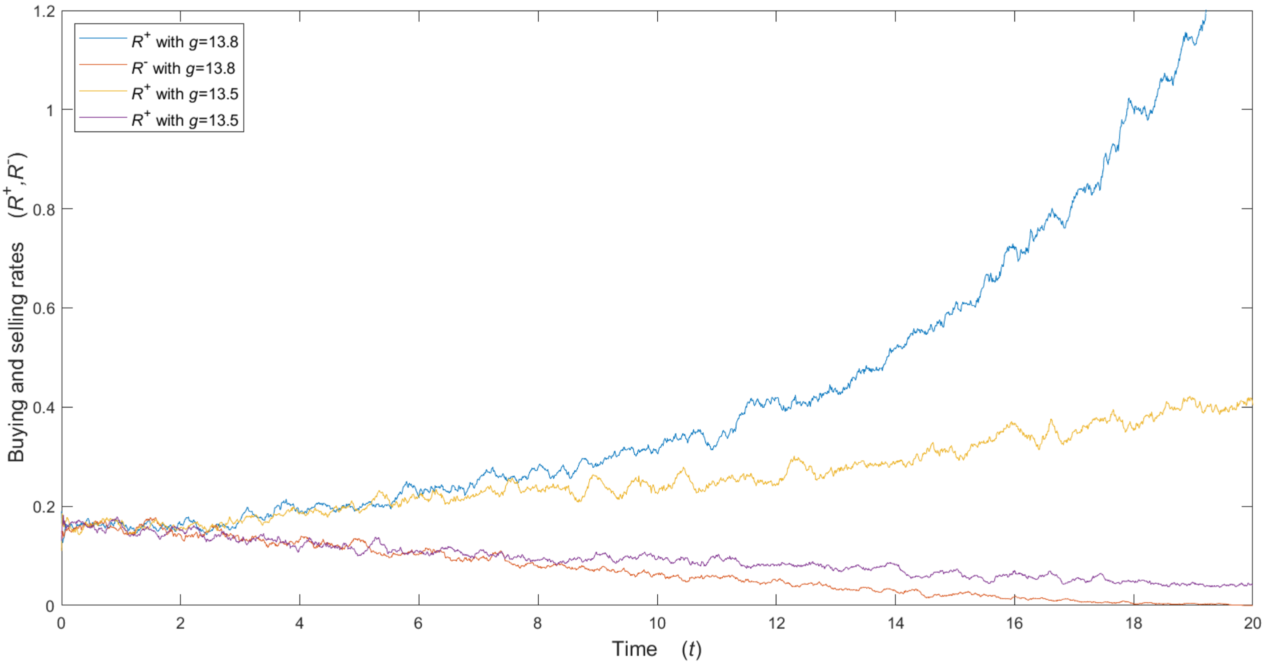

Figure 12: Plot of order rates from two different simulations with different g values. Other parameters are the same in both simulations; $\upsilon^+ = 0.0804$, $\upsilon^- = 0.0796$, $\epsilon^+ = \epsilon^- = 0.08$, $L^+ = L^- = 25$, $h = 0.01$, $T = 20$.

We also notice from the bottom right region of figure 11 that if g is sufficiently large then the population dynamics become unpredictable, no matter how close υ^+ and υ^- are to each other (although note when g takes even larger values outside the range shown in figure 11 we can be almost certain that the order rates will diverge). This is shown in figure 13 where I take $g = 15.5$, $\Delta\upsilon = 0$ and run three different simulations, all with the same parameters, and plot the order rate difference. The order rate differences all take different patterns, with two of them growing exponentially at different times and one of them remaining close to 0.

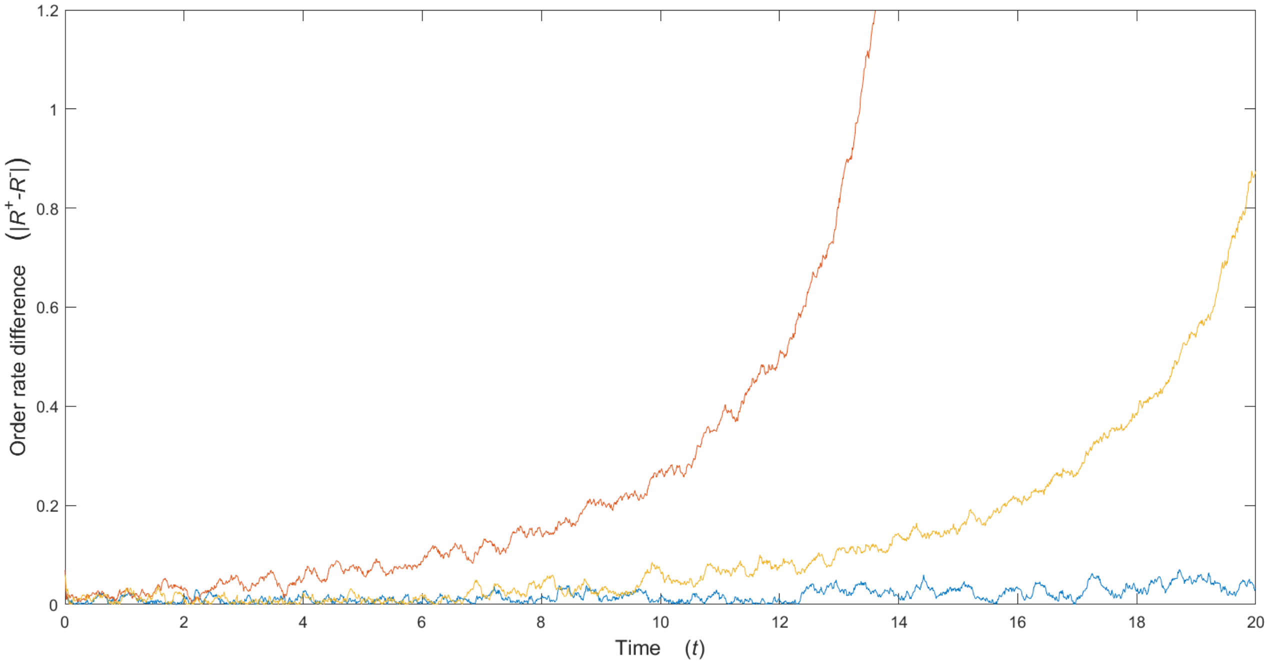

Figure 13: Plot of order rate difference for three simulations, all with $g = 15.5$, $\upsilon^+ = \upsilon^- = 0.08$, $\epsilon^+ = \epsilon^- = 0.08$, $L^+ = L^- = 25$, $h = 0.01$, $T = 20$.

5 Conclusions

5.1 General Conclusions From Model Simulations

The model can be deemed successful at producing realistic estimates of the behaviour of a large population of participants in a financial market, providing the parameters are set to suitable values. Some of the results are similar to conclusions reached by Ahmet Omurtag and Lawrence Sirovich (2006) [14], such as the order rate explosions caused by a larger amount of trader interactions. The trends of order rates in simulations with no initial condition (figures 2 and 3) are also comparable to results from this paper. The parameter plane plot (figure 11) helps provide further understanding of the model. Not only in understanding what values of g cause instability, but also in predicting the modelled financial market behaviour without having to run the simulations. I was able to simulate results similar to those shown in figure 10 from a model with different parameter values (plotted in figure 12) by reading off of figure 11, instead of doing trial and error. I was also able to remove the initial transient behaviour from simulations.

The model does successfully demonstrate how herd mentality by participants in financial markets can cause the markets to destabilise and become more fragile [2]. This is reflected by the results, as increasing g can be interpreted as "increasing herd mentality", and increasing g leads to more unstable order rates (figures 9, 11 and 13).

A generally known fact in economic literature is that large market moves do not occur when there is a small amount of external information being released about the asset [5]. This deduction can be reached from results from simulations of my model, by noticing that when $\upsilon^\pm$ values are closer together and smaller, a larger value of g is required to make the order rates diverge.

5.2 Current Model Restrictions and Potential for Model Improvements

An obvious model restriction is the invariable parameters. The population size is something that obviously can undergo significant change when individuals decide to enter or leave the market, particularly when order rates rise or drop, which could incentivise people to leave or enter the market [15].

The model's current underlying theory on an individual's psychology is very basic, and certainly not realistic on a small scale (hence N is always taken to be large). There are behavioural aspects that could be introduced to give a more realistic model of a trader's thought process. Behavioural concept's like regret and overconfidence could be implemented by changing the reset value, perhaps to a value closer to or further from the threshold just crossed for a decision to be made. Other aspects like gambling, over-reacting, under-reacting and anchoring, which is suggested to be present in financial markets [15], could also be explored, perhaps by setting $\epsilon^{\pm}$ (for under and over reacting) and $\upsilon^{\pm}$ (for gambling and banking) to different values for different traders.

When the order rates explode in the current model, the modelled order rates return values such as $R^{+} = 30$, essentially implausibly high values. In reality, order rates never get this high due to stabilising forces in the form of value investors [14]. Integrating this into the model would require setting parameters $\upsilon^{\pm}$ and $\epsilon^{\pm}$ to values that change depending on the value of order rates to bring the order rates back to more stable values. Successfully integrating this into the model would give more realistic estimates for $R^{\pm}$ and increase the range of parameter values for which the model would still provide useful results.

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