1A $\mathbb{R}^n$ and $\mathbb{C}^n$

Problem 1

Show that $\alpha + \beta = \beta + \alpha$ for all $\alpha, \beta \in \mathbb{C}$.

Proof. Suppose $\alpha = a + bi$ and $\beta = c + di \in \mathbb{C}$ where $a, b, c, d \in \mathbb{R}$. Then

$$\alpha + \beta = a + bi + c + di = c + di + a + bi = \beta + \alpha \qquad \square.$$

Problem 2

Show that $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ for all $\alpha, \beta, \lambda \in \mathbb{C}$.

Proof. Suppose $\alpha = a+bi$ and $\beta = c+di$ and $\lambda = e+fi \in \mathbb{C}$ where $a, b, c, d, e, f \in \mathbb{R}$. Then

$$\begin{aligned}
(\alpha + \beta) + \lambda &= (a + bi + c + di) + e + fi \\
&= a + bi + c + di + e + fi \\
&= a + bi + (c + di + e + fi) = \alpha + (\beta + \lambda).
\end{aligned} \qquad \square$$

Problem 3

Show that $(\alpha\beta)\lambda = \alpha(\beta\lambda)$ for all $\alpha, \beta, \lambda \in \mathbb{C}$.

Proof. Suppose $\alpha = a+bi$ and $\beta = c+di$ and $\lambda = e+fi \in \mathbb{C}$ where $a, b, c, d, e, f \in \mathbb{R}$. Then

$$\begin{aligned}
(\alpha\beta)\lambda &= \big((a + bi)(c + di)\big)(e + fi) \\
&= (ac + adi + bci - bd)(e + fi) \\
&= ace + adei + bcei - bde + acfi - adf - bcf - bdfi \\
&= (a + bi)(ce + cfi + dei - df) \\
&= (a + bi)\big((c + di)(e + fi)\big) \\
&= \alpha(\beta\lambda).
\end{aligned} \qquad \square$$

Problem 4

Show that $\lambda\,(\alpha + \beta) = \lambda\alpha + \lambda\beta$ for all $\lambda, \alpha, \beta \in \mathbb{C}$.

Proof. Suppose $\alpha = a + bi$ and $\beta = c + di \in \mathbb{C}$ and let $a, b, c, d, \lambda \in \mathbb{R}$. Then

$$\begin{aligned}
\lambda\,(\alpha + \beta) &= \lambda(a + bi + c + di) \\
&= \lambda a + \lambda bi + \lambda c + \lambda di \\
&= \lambda(a + bi) + \lambda(c + di) = \lambda\alpha + \lambda\beta
\end{aligned} \qquad \square$$

Problem 5

Show that for every $\alpha \in \mathbb{C}$, there exists a unique $\beta \in \mathbb{C}$ such that $\alpha + \beta = 0$.

Proof. To show the existence, we let $\alpha = a + bi \in \mathbb{C}$ where $a, b \in \mathbb{R}$. Now, define $\beta = c + di \in \mathbb{C}$ such that $c = -a$ and $d = -b$. Then

$$\alpha + \beta = a + bi + -a - bi = 0$$

To show uniqueness, we suppose there exists $\tilde{\beta} \neq \beta$ such that $\alpha + \tilde{\beta} = 0$. But

$$\tilde{\beta} = \tilde{\beta} + \alpha + \beta = \alpha + \tilde{\beta} + \beta = \beta$$

gives us a contradiction. Result follows. □

Problem 6

Show that for every $\alpha \in \mathbb{C}$ with $\alpha \neq 0$, there exists a unique $\beta \in \mathbb{C}$ such that $\alpha\beta = 1$.

Proof. To show the existence, we let $\alpha = a + bi \neq 0 \in \mathbb{C}$ where $a, b \in \mathbb{R}$. Now, define $\beta = c + di \in \mathbb{C}$ such that $c = \frac{a}{a^2+b^2}$ and $d = \frac{b}{a^2+b^2}$. Then

$$\alpha\beta = (a + bi)(\frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i) = \frac{a^2}{a^2+b^2} + \frac{abi}{a^2+b^2} - \frac{abi}{a^2+b^2} + \frac{b^2}{a^2+b^2} = 1$$

To show uniqueness, we suppose there exists $\tilde{\beta} \neq \beta$ such that $\alpha\tilde{\beta} = 1$. But

$$\tilde{\beta} = \tilde{\beta}\alpha\beta = \alpha\tilde{\beta}\beta = \beta$$

gives us a contradiction. Result follows. □

Problem 7

Show that

$$\frac{-1 + \sqrt{3}i}{2}$$

is a cube root of 1 (meaning that its cube equals 1).

Proof.

$$\begin{aligned}
\Big(\frac{-1+\sqrt{3}i}{2}\Big)\Big(\frac{-1+\sqrt{3}i}{2}\Big)\Big(\frac{-1+\sqrt{3}i}{2}\Big) &= \Big(\frac{1-\sqrt{3}i-\sqrt{3}i-3}{4}\Big)\Big(\frac{-1+\sqrt{3}i}{2}\Big) \\
&= \Big(\frac{-2-2\sqrt{3}i}{4}\Big)\Big(\frac{-1+\sqrt{3}i}{2}\Big) \\
&= \Big(\frac{2-2\sqrt{3}i+2\sqrt{3}i+6}{8}\Big) = 1
\end{aligned}$$

□

Problem 8

Find two distinct square roots of i.

Proof. Similarly to question 7, to show a number is square root we want to square it and get the desired outcome. Here we want to find $\alpha \neq \beta$ such that $\alpha^2 = i$ and $\beta^2 = i$ where $\alpha, \beta \in \mathbb{C}$.

$$\begin{aligned}
(a+bi)^2 = i &\implies (a^2 - b^2) + 2abi = i \\
&\implies a = b \text{ and } 2ab = 1 \\
&\implies a^2 = \frac{1}{2} \\
&\implies \alpha = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i \quad \text{and} \quad \beta = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i.
\end{aligned}$$

□

Problem 9

Find $x \in \mathbb{R}^4$ such that

$$(4,\ -3,\ 1,\ 7) + 2x = (5,\ 9,\ -6,\ 8).$$

Proof. We get a system of following linear equations.

$$\begin{cases} 4 + 2x_1 = 5 \\ -3 + 2x_2 = 9 \\ 1 + 2x_3 = -6 \\ 7 + 2x_4 = 8 \end{cases} \quad \begin{aligned} &\implies \quad x_1 = \frac{1}{2}, \quad x_2 = 6, \quad x_3 = -\frac{7}{2}, \quad x_4 = \frac{1}{2} \\ &\implies \quad x = \left(\frac{1}{2}, 6, -\frac{7}{2}, \frac{1}{2}\right) \end{aligned}$$

□

Problem 10

Explain why there does not exist $\lambda \in \mathbb{C}$ such that

$$\lambda(2 - 3i,\ 5 + 4i,\ -6 + 7i) = (12 - 5i,\ 7 + 22i,\ -32 - 9i).$$

Proof. To show that such $\lambda \in \mathbb{C}$ does not exist we will try to solve the following system of linear equations.

$$\begin{cases} \lambda(2 - 3i) = 12 - 5i \implies \lambda = \dfrac{12 - 5i}{2 - 3i} \\ \lambda(5 + 4i) = 7 + 22i \\ \lambda(-6 + 7i) = -32 - 9i \implies \lambda = \dfrac{-32 - 9i}{-6 + 7i} \end{cases}$$

We see that λ in first and third row do not match, that is, we need a different scalar to get the desired outputs. Result follows. □

Problem 11

Show that $(x+y)+z = x+(y+z)$ for all $x, y, z \in \mathbb{F}^n$.

Proof. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ and $z = (z_1, \ldots, z_n)$ be $x, y, z \in \mathbb{F}^n$ where $x_i, y_i, z_i \in \mathbb{F}$ for every $i \in \{1, \ldots, n\}$. Then

$$\begin{aligned}
(x+y)+z &= \Big((x_1, \ldots, x_n) + (y_1, \ldots, y_n)\Big) + (z_1, \ldots, z_n) \\
&= (x_1 + y_1, \ldots, x_n + y_n) + (z_1, \ldots, z_n) \\
&= (x_1 + y_1 + z_1, \ldots, x_n + y_n + z_n) \\
&= (x_1, \ldots, x_n) + \Big((y_1 + z_1, \ldots, y_n + z_n)\Big) \\
&= (x_1, \ldots, x_n) + \Big((y_1, \ldots, y_n) + (z_1, \ldots, z_n)\Big) = x + (y+z)
\end{aligned}$$

□

Problem 12

Show that $(ab)x = a(bx)$ for all $x \in \mathbb{F}^n$ and all $a, b \in \mathbb{F}$.

Proof. Let $x = (x_1, \ldots, x_n) \in \mathbb{F}^n$ where $x_i \in \mathbb{F}$ for every $i \in \{1, \ldots, n\}$ and let $a, b \in \mathbb{F}$. Then

$$(ab)x = ab(x_1, \ldots, x_n) = (abx_1, \ldots, abx_n) = a(bx_1, \ldots, bx_n) = a(bx)$$

□

Problem 13

Show that $1x = x$ for all $x \in \mathbb{F}^n$.

Proof. Let $x = (x_1, \ldots, x_n) \in \mathbb{F}^n$ where $x_i \in \mathbb{F}$ for every $i \in \{1, \ldots, n\}$ and let $1 \in \mathbb{F}$. Then

$$1x = 1(x_1, \ldots, x_n) = (1x_1, \ldots, 1x_n) = (x_1, \ldots, x_n) = x$$

□

Problem 14

Show that $\lambda(x+y) = \lambda x + \lambda y$ for all $\lambda \in \mathbb{F}$ and all $x, y \in \mathbb{F}^n$.

Proof. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be $x, y \in \mathbb{F}^n$ where $x_i, y_i \in \mathbb{F}$ for every $i \in \{1, \ldots, n\}$ and let $\lambda \in \mathbb{F}$. Then

$$\begin{aligned}
\lambda(x+y) &= \lambda\Big((x_1, \ldots, x_n) + (y_1, \ldots, y_n)\Big) \\
&= \lambda(x_1 + y_1, \ldots, x_n + y_n) \\
&= \Big(\lambda(x_1 + y_1), \ldots, \lambda(x_n + y_n)\Big) \\
&= (\lambda x_1 + \lambda y_1, \ldots, \lambda x_n + \lambda y_n) \\
&= (\lambda x_1, \ldots, \lambda x_n) + (\lambda y_1, \ldots, \lambda y_n) \\
&= \lambda(x_1, \ldots, x_n) + \lambda(y_1, \ldots, y_n) = \lambda x + \lambda y.
\end{aligned}$$

□

Problem 15

Show that $(a+b)x = ax + bx$ for all $a, b \in \mathbb{F}$ and all $x \in \mathbb{F}^n$.

Proof. Let $x = (x_1, \ldots, x_n) \in \mathbb{F}^n$ where $x_i \in \mathbb{F}$ for every $i \in \{1, \ldots, n\}$ and let $a, b \in \mathbb{F}$. Then

$$\begin{aligned}(a+b)x &= (a+b)(x_1, \ldots, x_n) \\ &= \Big((a+b)x_1, \ldots, (a+b)x_n\Big) \\ &= \Big(ax_1 + bx_1, \ldots, ax_n + bx_n\Big) \\ &= a(x_1, \ldots, x_n) + b(x_1, \ldots, x_n) = ax + bx.\end{aligned}$$

□