3A The Vector Space of Linear Maps

Problem 7

Show that every linear map from a one-dimensional vector space to itself is multiplication by some scalar. More precisely, prove that if $\dim V = 1$ and $T \in \mathcal{L}(V)$, then there exists $\lambda \in \mathbb{F}$ such that $Tv = \lambda v$ for all $v \in V$.

Proof. Since V is one dimensional, it has 1 basis. Let v_1 be basis of V. Now, $\forall v \in V$ can be expressed as $v = \alpha v_1$ for some $\alpha \in \mathbb{F}$.

Further, $T \in \mathcal{L}(V, V)$ and V is vector space, so $T(v_1) \in V$. Hence, we can write $T(v_1) = \lambda v_1$. Now combining:

$$T(v) = T(\alpha v_1) = \alpha T(v_1) = \alpha \lambda v_1 = \lambda v.$$

Result follows as desired $\square$

Problem 8

Give an example of a function $\varphi \colon \mathbb{R}^2 \to \mathbb{R}$ such that

$$\varphi(av) = a\varphi(v)$$

for all $a \in \mathbb{R}$ and all $v \in \mathbb{R}^2$, but φ is not linear.

Proof. For $v = (v_1, v_2) \in \mathbb{R}^2$, let a function $\varphi \colon \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$\varphi(v) = \begin{cases} \frac{{v_1}^2}{v_2} & \text{if } v_2 \neq 0, \\ 0 & \text{if } v_2 = 0. \end{cases}$$

Now we check that φ satisfies $\varphi(av) = a\varphi(v)$. Suppose $v_2 \neq 0$:

$$\varphi(\lambda v) = \varphi(\lambda v_1, \lambda v_2) = \frac{(\lambda v_1)^2}{\lambda v_2} = \lambda \frac{(v_1)^2}{v_2} = \lambda \varphi(v).$$

When $v_2 = 0$,

$$\varphi(\lambda v) = \varphi(\lambda v_1, 0) = 0 = \lambda 0 = \lambda \varphi(v).$$

To show that φ is not linear map we need to show that it is not additive, i.e. $\varphi(u + v) \neq \varphi(v) + \varphi(u)$. Let $v = (1, 1)$ and $u = (-1, 1)$. Then

$$\begin{aligned} &\varphi(v + u) = \varphi(1, 1) + \varphi(-1, 1) = \varphi(0, 1) = 0. \\ \text{But} \quad &\varphi(v) + \varphi(u) = \varphi(1, 1) + \varphi(-1, 1) = 2. \end{aligned}$$

We conclude that $\varphi(u + v) \neq \varphi(v) + \varphi(u)$ hence φ is not a linear map. $\square$

Problem 9

Give an example of a function $\varphi\colon \mathbb{C} \to \mathbb{C}$ such that

$$\varphi(w+z) = \varphi(w) + \varphi(z)$$

for all $w, z \in \mathbb{C}$, but φ is not linear. (Here $\mathbb{C}$ is considered as a complex vector space.)

Proof. For $z = a + ib \in \mathbb{C}$, let a function $\varphi\colon \mathbb{C} \to \mathbb{C}$ be defined as $\varphi(z) = \mathrm{Re}(z)$.

Now we check that φ satisfies $\varphi(w+z) = \varphi(w) + \varphi(z)$.

$\varphi(w+z) = \varphi((a_1+ib_1)+(a_2+ib_2)) = \varphi((a_1+a_2)+i(b_1+b_2)) = a_1+a_2 = \varphi(w)+\varphi(z)$.

So see φ is additive. Now we check if φ is homogeneous. Let $\lambda = i$ and $z = 1+i$.

$$\varphi(\lambda z) = \varphi(i(1+i)) = \varphi(-1+i) = -1$$
$$\text{But } \lambda\varphi(z) = i(\varphi(1+i)) = i$$

We see that φ is not homogeneous, hence we conclude φ is not a linear map. $\square$

Problem 10

Suppose U is a subspace of V with $U \neq V$. Suppose $S \in \mathcal{L}(U, W)$ and $S \neq 0$ (which means that $Su \neq 0$ for some $u \in U$). Define $T\colon V \to W$ by

$$Tv = \begin{cases} Sv & \text{if } v \in U, \\ 0 & \text{if } v \in V \text{ and } v \notin U. \end{cases}$$

Prove that T is not a linear map on V.

Proof. Let $u \in V$ be such that $S(u) \neq 0$ and let $v \in V$ be such that $v \notin U$. Then $u+v \notin U$ since otherwise $v = (v+u)-u \in U$ which would give us a contradiction.

Having shown that we have $T(v+u) = 0$ (by defn of T).

But $T(u) + T(v) = T(u) + 0 \neq 0$. Hence, $T(v+u) \neq T(u) + T(v)$ and we conclude T is not a linear map. $\square$

Problem 11

Let V be a finite-dimensional vector space, $U \subseteq V$ a subspace, and $S \in \mathcal{L}(U, W)$. Then there exists a linear map $T \in \mathcal{L}(V, W)$ such that

$$T(u) = S(u) \quad \text{for all } u \in U.$$

Proof. Let $u_1, ..., u_n$ be basis of U. We can extend this basis to the basis of V. That is $u_1, ..., u_n, v_1, ..., v_m$ is basis of V.

Now we define $T : V \to W$ such that $\sum_{i=1}^n a_i u_i + \sum_{i=1}^m b_i v_i \to \sum_{i=1}^n S(u_i)$. Clearly T and S agree on every $u \in U$. Now we need to show that T is linear map.

$$\begin{aligned}
T(\alpha u + \beta v) &= T(\alpha(\sum_{i=1}^n a_i u_i + \sum_{i=1}^m b_i v_i) + \beta(\sum_{i=1}^n c_i u_i + \sum_{i=1}^m d_i v_i)) \\
&= T(\sum_{i=1}^n (\alpha a_i u_i + \beta c_i u_i) + \sum_{i=1}^m (\alpha b_i v_i + \beta d_i v_i)) \\
&= S(\sum_{i=1}^n \alpha a_i u_i + \beta c_i u_i) \\
&= S(\sum_{i=1}^n \alpha a_i u_i) + S(\sum_{i=1}^n \beta c_i u_i) \\
&= \alpha S(\sum_{i=1}^n a_i u_i) + \beta S(\sum_{i=1}^n c_i u_i) \\
&= \alpha T(u) + \beta T(v).
\end{aligned}$$

We conclude T is a linear map and result follows as desired. $\square$