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Three topics:

- Methodologies (algorithms to perform approximations)
- Analysis (estimate errors and convergence properties)
- Implementation (Python versions of most topics)

Numerical Approximation: "something "close in value", but not the same, as a desired quantity"

Approximation is a matter of

- representation (floating point values VS real numbers)
- measure of the error (norms definition)

Elementary notions: $\boxed{\text{Vector space}}$: set of elements with two ops:

- $\oplus$ $X \times X \longrightarrow X$ "sum"
- $\odot$ $X \times \mathbb{R} \longrightarrow X$ "scale"

$u, v \in X$, $\alpha, \beta \in \mathbb{R}$

$w = \alpha u + \beta v \in X$

"with all the good properties" (see wikipedia!)

Two families:
- finite dimensional vector spaces e.g. $\mathbb{R}^n$
- infinite dimensional vector spaces e.g. $C^0([0,1])$

$\rightsquigarrow$ in a computer, only finite dimensional spaces

MOST of our problems are linked to the approximation of infinite dimensional spaces with finite dimensional ones

$\boxed{\text{(Semi) Norm}}$: function from $X \longrightarrow \mathbb{R}^+ \cup \{0\}$

- $\|a v\| = |a| \, \|v\|$
- $\|u + v\| \leq \|u\| + \|v\|$
- $(\|u\| = 0 \iff u = 0)$ $\Leftarrow$ only for norms.

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Notations : $X := \mathbb{R}^n$ vector space $u = \{u^i\}_{i=0}^{n-1} \in \mathbb{R}$

ℓ^p - norm : $\|u\|_{\ell^p}^p := \sum_{i=0}^{n-1} (u^i)^p$ $\ell^\infty := \max_{i=0}^{n-1} |u^i|$

$X :=$ functions on interval $[0,1]$

L^p - norm : $\|u\|_{L^p}^p := \int_0^1 |u|^p$ $L^\infty := \sup_{x \in I} |f(x)|$

Banach Space : Vectorspace + Norm

$\boxed{\text{INNER PRODUCT}}$: ~ bilinear form $(\cdot,\cdot) : X \times X \longrightarrow \mathbb{R}$

- commutative $(a,b) = (b,a)$
- positive $(a,a) \geq 0$

Hilbert Space : Banach, with norm $\|u\| := \sqrt{(u,u)}$

Schwartz : $|(u,v)| \leq \|u\| \, \|v\|$

Abstract Setting

1) Continuous problem $F(x,d) = 0$

"given $d \in D$ (Banach), find $x \in X$ (Banach) s.t. $F(x,d) = 0$

1) is $\boxed{\text{STABLE}}$ if $\boxed{\forall d, \exists! x}$ s.t. $F(x,d) = 0$, $\boxed{\text{cont. dep on } d}$

$\exists K$ s.t. $\forall \delta d \mid d + \delta d \in D$, $F(x + \delta x, d + \delta d) = 0$

$\|\delta x\| \leq K \|\delta d\|$ Absolute Stability

If $d \neq 0$, $x \neq 0$ $\frac{\|\delta x\|}{\|x\|} \leq K_{rel} \frac{\|\delta d\|}{\|d\|}$ Relative Stability

$G : D \longrightarrow X$ the resolvant :

$$\underline{F(G(d), d)} = 0 \quad \Rightarrow \quad \underline{x = G(d)}$$

Then Stability means $\|G(d+\delta d) - G(d)\| \leq k \|\delta d\|$

"Bounded Frechet Derivative of G"

— Only stable problems.

Discrete approximation: sequence of problems "approximating" F

$$F_n(x_n, d_n) = 0 \qquad d_n \in X_n, \quad x_n \in X_n$$

$\boxed{\text{Convergence :}}$ $\lim_{n \to \infty} \|x_n - x\|_X = 0$

when $\|d_n - d\|_D \longrightarrow 0$, and $F_n(x_n, d_n) = 0$

$$\|G_n(d_n) - G_n(d)\| \longrightarrow 0$$

$\boxed{\text{Consistency}}$ If $d_n = d$ $\forall n$, $x \in X_n \ \forall n$, Then pb is

consistent $\Leftrightarrow$ $\boxed{F_n(x, d) \longrightarrow 0}$

$$\|G_n(d) - G(d)\| \longrightarrow 0$$