Best Approximation in Hilbert Spaces

17/11/2015 (1)

Let H be Hilbert space (Banach + scalar product $(\cdot,\cdot)$
with norm $:= \|u\|^2 := (u,u)$)

Example: $L^2([0,1]) :\Rightarrow \quad (a,b) := \int_0^1 a \cdot b \, ds \qquad \|a\| := \sqrt{\int_0^1 |a|^2 ds}$

Theorem p is B.A. of f in H (from $V \subset H$)

$$\Longleftrightarrow$$

$$(1) \quad (p,v) = (f,v) \qquad \forall\, v \in V$$

Proof $\Rightarrow$ B.A $\Rightarrow$ (1) $\qquad$ if p is B.A, then

$$\|f-p\|^2 = E(f)^2 := \inf_{q \in V} \|f-q\|^2$$

$$(*) \quad \|f-p\|^2 \leq \|f-p+tq\|^2 \qquad \forall t>0,\ \forall q \in V$$

consider that

$$\|a+b\|^2 - \|a-b\|^2 = 4(a,b)$$

then

$$\|f-p+\tfrac{t}{2}q + \tfrac{t}{2}q\|^2 - \|\overbrace{f-p+\tfrac{t}{2}q - \tfrac{t}{2}q}^{\|f-p\| \ (*)}\|^2$$

$$= 4\left(f-p+\tfrac{t}{2}q, \tfrac{t}{2}q\right) \geq 0 \quad \text{for } (*)$$

$$2t(f-p,q) + t^2\|q\|^2 \geq 0$$

$$(f-p,q) \geq -\frac{t}{2}\|q\|^2$$

same with $-q \Rightarrow \quad \equiv (f,q) = (p,q) \quad \forall q$

$$\boxed{-\frac{t}{2}\|q\|^2 \leq (f-p,q) \leq \frac{t}{2}\|q\|^2} \qquad \forall t>0,\ \forall q \in V$$

Proof ① $\Rightarrow$ B.A.

$$\|f-q\|^2 = \|f-q-p+p\|^2 = \|f-p\|^2 + \|p-q\|^2 + 2(\cancel{f}-p, p-q)$$

but $2(f-p, p-q) = 0$ $(p-q \in V)$

$$\|f-q\|^2 = \|f-p\|^2 + \|p-q\|^2 \quad \forall q \in V$$

$$\Rightarrow \quad \|f-p\| \leq \|f-q\| \quad \forall q \in V$$

Now consider $L^2(0,1)$, $(a,b) := \int_0^1 ab$

and take $V = \mathrm{span}\{v_i\}_{i=1}^n$

B.A. in L^2: find $p \in V$, s.t.

$$(p, v_i) = (f, v_i) \qquad i = 1, \dots, n$$

$$p \in V \Rightarrow p = p^J v_J \Rightarrow (p, v_i) = (p^J v_J, v_i)$$

$$(v_J, v_i)\, p^J = (f, v_i) \qquad i = 1 \dots n$$

$$\Leftrightarrow$$

$$Mp = F \qquad M_{iJ} := (v_J, v_i) \qquad F_i := (f, v_i)$$

Move from interpolating to integrating

Example $V^n := \text{span}\{x^i\}_{i=0}^{n-1}$ $\phi_i = x^i$? Take

$$M_{ij} := \int_0^1 x^j x^i = \left.\frac{x^{j+i+1}}{j+i+1}\right|_0^1 = \boxed{\frac{1}{j+i+1}}$$ Hilbert Matrix (H)

Hilbert Matrix is invertible BUT very ill conditioned.

$Mx = b$ $\qquad M(x+\delta x) = (b+\delta b)$

$K := \dfrac{\|\delta x\|/\|x\|}{\|\delta b\|/\|b\|}$ $\qquad \|Mx\| = \|b\|$

$\|\delta x\| = \|M^{-1}\delta b\|$

$\|b\| \leq \|M\| \, \|x\|$

$\|\delta x\| \leq \|M^{-1}\| \, \|\delta b\|$

$$K \leq \frac{\|M^{-1}\| \cancel{\|\delta b\|} / \cancel{\|x\|}}{\cancel{\|\delta b\|} / \|M\| \cancel{\|x\|}} = \|M\| \, \|M^{-1}\|$$

~~det H~~ ~~do~~ $K(H) \sim O\left((1+\sqrt{2})^{4n}/\sqrt{n}\right)$

Ideally $M_{ij} := \delta_{ij}$ identity (orthonormal basis)

GRAM SMIDTH: $V^n = \text{span}\{\phi_i\}$

$\phi_0 = 1/(b-a)$

$k_{i+1} = [x\phi_i - (x\phi_i, \phi_i)\phi_i]$ $\qquad \phi_{i+1} = \dfrac{k_{i+1}}{\|k_{i+1}\|}$

Is this ok? NO!

$$\|k_{i+1}\|^2 = \|x\phi_i\|^2 + (x\phi_i,\phi_i)^2\|\phi_i\|^2 - 2(x\phi_i,\phi_i)^2$$
$$= \|x\phi_i\|^2 - (x\phi_i,\phi_i)^2$$ the substraction can be very small

…w do we proceed for integration?

Take $p \in \mathbb{P}^n$, make sure you integrate p exactly, then use $\mathcal{L}^n f$

$$I^n(f) := \int_I \mathcal{L}^n f \qquad I(f) := \int_I f$$

Then we have $\displaystyle \left| I^n(f) - I(f) \right| \leq \int_I \left| f - \mathcal{L}^n f \right| \leq (b-a)\, e^n(f)$

Interpolating quadrature formulas:

given X of quadrature points $\{q_i\}$, define

$$I^n(f) := \int_I \mathcal{L}^n f = \int_I \ell_i^n(x)\, f(q_i)\, dx = f(q_i)\, \omega_i$$

where $\displaystyle \omega_i := \int_I \ell_i^n(x)\, dx$

More generally: ω_i : weights, q_i : quadrature points | define a generic quadrature formula I^n

If I^n is derived from $\mathcal{L}^n$, then it is interpolatory

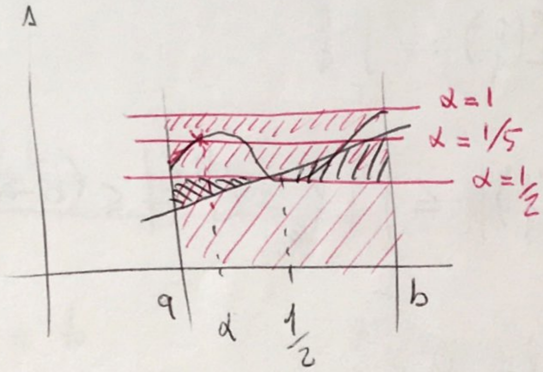
DEF: Degree of accuracy: q : $\boxed{\max_{n \in \mathbb{R}^n} \text{ such that } I_n p - \int_I p = 0 \text{ when } p \in \mathbb{P}^n}$

Theo: If I^n is interpolatory, then it has degree at least n

$(n+1)$ points

can the degree (q) be higher? Example:

mid point formula: $\underline{I^n(f) := f\left(\left(\frac{b+a}{2}\right)\right)(b-a)}$ (q_0, w_0)



↳ particular case of L, $q_0 = \alpha \in (a,b)$

If $\alpha = \frac{1}{2}$, then also linear functions are integrated exactly!

For symmetry the integral is exact also for linears.

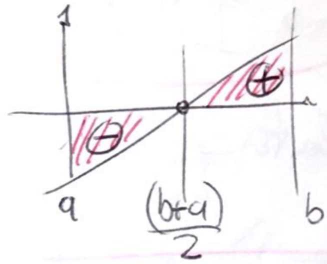
$$f(x) = f(c) + f'(c)(x-c) + f''(\eta(x,c))\frac{(x-c)^2}{2} \qquad c = \left(\frac{b+a}{2}\right)$$

$$E(f) := I^n(f) - \int_a^b f = \cancel{(b-a)f(c)} - \cancel{(b-a)f(c)}$$

$$\cancel{-\int f'(c)(x-c)} - \int f''(\eta(x,c))\frac{(x-c)^2}{2}\,dx$$

0

line passing throug $(c,0)$ with inclination $f'(c)$
~~$c(f(c))$~~



$$|E(f)| = |f''(\eta)|\left|\frac{(b-a)^3}{24}\right|$$

$\equiv 0 \iff f'' = 0$ linears

Altri: TRAPEZIO – <u>NEWTON COTES</u> – CHEBISHEV

$$E_n(f) \simeq \underline{h^{n+3} f^{(n+2)}(\xi)} \qquad n \text{ ~~pari~~ even}$$

$$\underline{h^{n+2} f^{(n+1)}(\xi)} \qquad n \text{ ~~dispa~~ odd}$$

FORMULE COMPOSITE

i) same order of accuracy

ii) one order less of infinitesimal in the error

q: order of accuracy or precision (exactness for polynomials of order q)

r: order of convergence (for composite formulas)

→ Composite formulas with low precision → more robust

("MULTIGRID" or Richardson Extrapolation methods)*

dopo

Can I raise the degree of accuracy, keeping n constant? Up to what value?

For example $p = (\omega(x))^2$ $\quad \omega = \prod_{i=0}^{n} (x - q_i)$

$p \in \mathbb{P}^{2n+2}$ $\quad (\omega \in \mathbb{P}^{n+1})$

$I_n p = 0 \quad \int_I p > 0$ NOT $2n+2$

Whats the maximum? $2n+1$ that is, $2M-1$ whe M = quadrature points. How?

Theo let $f \in \mathbb{P}^{n+m}$, $m \leq n+1$

$$I^n f = I f \iff \int_a^b \omega(x) \cdot p = 0 \quad \forall p \in \mathbb{P}^{m-1}$$

$$\omega = \prod_{i=0}^{n} (x - q_i)$$

① ②

$\forall \; p \in \mathbb{P}^{m+n}$ can be written as

$$p = \omega(x)\,\pi(x) + q(x)$$

where $\omega \in \mathbb{P}^{n+1}$, $\pi(x) \in \mathbb{P}^{m-1}$ and $q(x) \in \mathbb{P}^{m-1}$

Then $$\int_I p = \int_I \omega(x)\,\pi(x) + \int_I q(x)$$

$I_n(q) = I(q) \quad \forall \; m \leq n+1 \quad (q \in \mathbb{P}^n)$

From theo

$I_n(\omega\pi) = 0$ because $\omega = 0$ on q_i

Scanned by CamScanner

Quadrature Rules: methods of undetermined coefficients

Given a quadrature rule $I(f) := \sum_{i=0}^{n} f(x_i)\, w_i$

we want to determine w_i and x_i such that the degree of accuracy is as high as possible / desired.

We impose it to be exact for polynomials of order q. We have $2(n+1)$ unknown ($n+1$ points x_i, $n+1$ weights w_i).

We can impose it on all monomials of order $q = 2n+1$.

$$\sum_{i=0}^{n} (x_i)^j\, w_i = \int x^j \qquad \text{for } j = 0, \dots, 2n+1$$

$2n+2$ conditions on $2n+2$ unknowns.

$\Rightarrow$ NON linear system of equations. If you can solve it, you win!

How do we estimate Errors?

Call $I(f) := \sum_i f(q_i)\, w_i$ and $E(f) := \int f - I(f)$

the error.

Simple cases: Use Taylor expansion: MID POINT RULE

$$If := f\left(\tfrac{1}{2}\right) \cdot 1 = f\left(\tfrac{b+a}{2}\right) \cdot 1$$

If f is C^2

$$f(x) = f\left(\tfrac{1}{2}\right) + f'\left(\tfrac{1}{2}\right)\left(x - \tfrac{1}{2}\right) + f''(\xi)\left(x - \tfrac{1}{2}\right)^2$$

If we integrate, we obtain:

$$\int_a^b f(x) = (b-a) f(\tfrac{1}{2}) + \int_a^b f'(\tfrac{a+b}{2}) (\cancel{x} - \tfrac{b+a}{2})\, dx \quad 0$$

$$+ \int_a^b f''(\eta(x)) (x - \tfrac{b+a}{2})^2$$

This term decides all

$$I(f) := (b-a) f(\tfrac{1}{2})$$

$$\Rightarrow E(f) = \int_a^b f''(\eta(x)) \left(x - \frac{b+a}{2}\right)^2$$

$$\leq \|f''\|_\infty \int_a^b \left(x - \frac{b+a}{2}\right)^2$$

$$= \|f''\|_\infty \left[\frac{\left(x - \frac{b+a}{2}\right)^3}{3}\Bigg|_b - \frac{\left(x - \frac{b+a}{2}\right)^3}{3}\Bigg|_a \right]$$

$$= \|f''\|_\infty \left[\frac{(b-a)^3}{24} + \frac{(b-a)^3}{24} \right]$$

$$= \|f''\|_\infty \boxed{\frac{(b-a)^3}{12}}$$

Can we generalize this?

Yes

...ANO KERNEL : Given a quad. formula of degree d

Let $0 \leq k \leq d$ an integer, and let $f \in C^{k+1}([a,b])$

Taylor's Theorem

$$f(x) = p(x) + \frac{1}{k!}\int_a^x f^{(k+1)}(t)(x-t)^k \, dt \qquad \circledast$$

with $p(x) \in \mathbb{P}^k$ the Taylor Expansion to order k of f around a (i.e. : $p(x) = \sum_{i=0}^{k} \frac{f^{(i)}(a)}{i!}(x-a)^i$)

if we define x_+^n $\nearrow x^n$ for $x>0$, $\searrow 0$ for $x<0$

The remainder can be written as:

$$r(x) = \frac{1}{k!}\int_a^b f^{(k+1)}(x-t)_+^k \, dt$$

But $f = p + r$, and $I(f) = I(p+r)$

$$\Rightarrow E(f) := \int f - I(f) = I(r)$$

And $I(r) := \int_a^b f^{(k+1)}(t) \, K(t) \, dt$

where $K(t)$ is called the Peano Kernel $:= \boxed{E_x\left[(x-t)_+^k\right]}$

The Peano Kernel is the error we make in integrating -

$g(x) = (x-t)_+^k$ for a given t -

An explicit expression of the peano Kernel is:

$$\int_a^b (x-t)_+^k dx - I\left((x-t)_+^k\right)$$

$$\int_a^b (x-t)_+^k dx = \left.\frac{(x-t)_+^{k+1}}{k+1}\right|_{x=b} - \left.\frac{(x-t)_+^{k+1}}{k+1}\right|_{x=a}$$

but since $t \in [a,b] \Rightarrow \boxed{\frac{(b-t)^{k+1}}{k+1}}$ it does not depend on a