

Maths 721 Notes

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1 Lecture 1

In the first half of this course we will cover three main topics:

- representations;
- modules;
- characters.

We will further see that representations and modules are essentially the same, and that modules and characters are essentially the same; and hence all three are essentially the same.

From now on G is a group.

1.1 Representations

Definition 1.1. A **representation** of a group G over a field F is a group homomorphism from G to $\mathrm{GL}(n, F)$, where n is the **degree** of the representation.

Explicitly, a representation is a function $\rho : G \rightarrow \mathrm{GL}(n, F)$ such that for all $g, h \in G$;

- (i) $(gh)\rho = (g\rho)(h\rho)$,
- (ii) $1_G\rho = I_n$,
- (iii) $g^{-1}\rho = (g\rho)^{-1}$.

Note the use of the (incredibly shit) postfix function notation.

Example 1.2. Take D_4 , the Dihedral group of order 8. It has the following group presentations

$$\begin{aligned} D_4 &= \langle a, b \mid a^4 = 1, b^2 = 1, a^b = a^{-1} \rangle \\ &\cong \langle (1\ 2\ 3\ 4), (1\ 4)(2\ 3) \rangle, \end{aligned}$$

where $a^b = bab^{-1}$ is conjugation of a by b . By defining the matrix subgroup

$$H = \left\langle A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\rangle$$

and defining $\rho : D_4 \rightarrow \mathrm{GL}(n, F)$ where $F = \mathbb{R}, \mathbb{C}$, by $a \mapsto A$ and $b \mapsto B$, and $a^i b^j \mapsto A^i B^j$ for $0 \leq i \leq 3$, and $0 \leq j \leq 1$. Hence we have ρ is a representation of D_4 over F . ◀

Example 1.3. Take \mathbb{Q}_8 the Quaternion group of order 8, which has the following group presentations

$$\begin{aligned} \mathbb{Q}_8 &= \langle a, b \mid a^4 = 1, a^2 = b^2, a^b = a^{-1} \rangle \\ &\cong \langle \bar{a} = (1\ 6\ 2\ 5)(3\ 8\ 4\ 7), \bar{b} = (1\ 4\ 2\ 3)(5\ 7\ 6\ 8) \rangle \end{aligned}$$

Define

$$H = \left\langle A = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\rangle \subset \text{GL}(2, \mathbb{C}).$$

Then $\rho : \mathbb{Q}_8 \rightarrow \text{GL}(2, \mathbb{C})$ defined by $a^k b^\ell \mapsto A^k B^\ell$ is a group representation of \mathbb{Q}_8 over \mathbb{C} of degree 2. ◀

Definition 1.4. Let G be a group and define

$$\begin{aligned} \rho : G &\rightarrow \text{GL}(n, F) \\ g\rho &= I_n \end{aligned}$$

for all $g \in G$. Then ρ is a representation, called the **trivial representation** of arbitrary of degree.

It follows from the trivial representation that any group G has a representation of an arbitrary degree.

Let $\rho : G \rightarrow \text{GL}(n, F)$ be a group homomorphism, and take $T \in \text{GL}(n, F)$. Then

$$(T^{-1}AT)(T^{-1}BT) = T^{-1}(AB)T.$$

Thus, given ρ define σ such that

$$g\sigma = T^{-1}(g\rho)T$$

for all $g \in G$. As for all $g, h \in G$, one has

$$\begin{aligned} (gh)\sigma &= T^{-1}((gh)\rho)T \\ &= T^{-1}(g\rho)(h\rho)T \\ &= T^{-1}(g\rho)TT^{-1}(h\rho)T \\ &= (g\sigma)(h\sigma), \end{aligned}$$

and so σ is a group homomorphism; and hence a representation.

Definition 1.5. Define

$$\rho : G \rightarrow \text{GL}(m, F), \quad \sigma : G \rightarrow \text{GL}(n, F)$$

to both be representation of G over F . We say that ρ is **equivalent to** σ if $n = m$ and there exists $T \in \text{GL}(n, F)$ such that $g\sigma = T^{-1}(g\rho)T$.

Proposition 1.6. *Equivalence of representations is an equivalence relation.*

Proof. Reflexivity is clear by taking $T = I_n$. For symmetry, take T to be its inverse. For transitivity, if

$$g\sigma = T^{-1}(g\rho)T, \quad g\rho = S^{-1}(g\eta)S,$$

then

$$g\sigma = (ST)^{-1}(g\eta)(ST).$$

◻

Definition 1.7. Define the **kernel** of the representation $\rho : G \rightarrow \text{GL}(n, F)$ as $\ker \rho = \{g \in G \mid g\rho = I_n\}$.

Proposition 1.8. The kernel of a representation of G is a normal subgroup of G ; i.e. $\ker \rho \triangleleft G$.

Proof. Suppose $g \in \ker \rho$ and $h \in G$ is arbitrary. Then

$$(hgh^{-1})\rho = (h\rho)(g\rho)(h^{-1}\rho) = (h\rho)I_n(h\rho)^{-1} = (h\rho)(h\rho)^{-1} = I_n,$$

and so $hgh^{-1} \in \ker \rho$. As $\ker \rho$ is closed under conjugation, it is a normal subgroup of G . \square

Definition 1.9. We say ρ is a **faithful** representation of G if $\ker \rho = \{1_G\}$.

Example 1.10. For the trivial representation $\rho : G \rightarrow \text{GL}(n, F)$ with $g \mapsto I_n$ for all $g \in G$, we have $\ker \rho = G$. Hence the representation is not faithful. \blacktriangleleft

Lemma 1.11. Suppose G is a finite group, and ρ is a representation of G over F . Then ρ is faithful if, and only if, $\text{im } \rho \cong G$.

Proof. Immediate from the first isomorphism theorem. \square

1.2 FG-Modules

Suppose G is a group, and $F = \mathbb{R}, \mathbb{C}$. Given $\rho : G \rightarrow \text{GL}(n, F)$, with $V = F^n$, let $v = (\lambda_1, \dots, \lambda_n) \in V$ for $\lambda_i \in F$ be a row vector. Moreover, note that $g\rho$ is an $n \times n$ matrix for all $g \in G$. Thus, we have $v \cdot (g\rho) \in V$, and satisfies the following properties:

- (i) $v \cdot ((gh)\rho) = v \cdot (g\rho)(h\rho)$;
- (ii) $v \cdot (1_G\rho) = v$;
- (iii) $(\lambda v) \cdot (g\rho) = \lambda(v \cdot (g\rho))$;
- (iv) $(u + v) \cdot (g\rho) = u \cdot (g\rho) + v \cdot (g\rho)$.

We often will omit the \cdot in the operation, and write $v(a\rho)$ for $v \cdot (a\rho)$.

Example 1.12. Recall D_4 and its given presentation from a previous example. We have

$$a\rho = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad b\rho = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

If $v = (\lambda_1, \lambda_2)$, then we have

$$v(a\rho) = (-\lambda_2, \lambda_1), \quad v(b\rho) = (\lambda_1, -\lambda_2).$$

Definition 1.13. Let V a vector space over the field $F = \mathbb{R}, \mathbb{C}$, and let G be a group. We say V is a **FG-module** if a multiplication $v \cdot g$ for $v \in V$, and $g \in G$ is defined such that: \blacktriangleleft

- (i) $v \cdot g \in V$;
- (ii) $v \cdot (gh) = (v \cdot g) \cdot h$;
- (iii) $v \cdot 1_G = v$;
- (iv) $(\lambda v) \cdot g = \lambda(v \cdot g)$;
- (v) $(u + v) \cdot g = u \cdot g + v \cdot g$.

This generalises the previous discussion from a matrix group to an arbitrary group.

Note that properties (i), (iv), and (v) imply that the map $v \mapsto v \cdot g$ is an endomorphism of V (a linear map from V to itself).

Definition 1.14. Suppose V is an FG -module and B is a basis for V . For $g \in G$, let $[g]_B$ denote the matrix of the endomorphism $v \mapsto v \cdot g$ of V relative to the basis B .

2 Lecture 2

Theorem 2.1. Let $\rho : G \rightarrow GL(n, F)$ be a representation of G over F .

- (I) If $V = F^n$ is an FG module and G acts on V by $v \cdot g = v(g\rho)$ there exists a basis B of V such that $g\rho = [g]_B$.
- (II) The map $g \mapsto [g]_B$ is a representation for G over F .

Proof. Choose the standard basis $B = [e_1, \dots, e_n]$.

Since V is an FG -module we have $v(gh) = (vg)h$ for all $g, h \in G$ and $v \in V$. Thus $[gh]_B = [g]_B[h]_B$ so the map is a homomorphism. We now check that $[g]_B$ is invertible for all $g \in G$. We know $v \cdot 1_G = (vg)g^{-1}$ so $I_n = [g]_B[g^{-1}]_B$ and thus $[g]_B$ has an inverse. \square

Example 2.2. Recall the representation of $G = D_4$ from a previous example. Define an FG -module $V = F^2$ with the action defined by taking vg to $v(g\rho)$.

$$\begin{aligned} v_1 &= (1, 0), & v_1 a &= v_2, & v_1 b &= v_1, \\ v_2 &= (0, 1), & v_1 a &= -v_1, & v_1 b &= -v_2. \end{aligned}$$

In this basis we recover our representation

$$a \mapsto [a]_B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad b \mapsto [b]_B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$



We now provide an equivalent basis-dependent definition for an FG -module.

Lemma 2.3. Let V a vector space over the field $F = \mathbb{R}, \mathbb{C}$ with basis $B = [v_1, \dots, v_n]$, and let G be a group. If a multiplication $v \cdot g$ for $v \in B$, and $g \in G$ is defined such that:

- (i) $v \cdot g \in V$;
- (ii) $v \cdot (gh) = (v \cdot g) \cdot h$;
- (iii) $v \cdot 1_G = v$;
- (iv) $(\sum_{i=1}^n \lambda_i v_i) \cdot g = \sum_{i=1}^n \lambda_i (v_i \cdot g)$ for all $\lambda_i \in F$;

then V is an FG -module.

Definition 2.4. The trivial module of a group over F is a one dimensional vector space V over F such that $vg = v$ for all $v \in V$ and $g \in G$.

Definition 2.5. An FG -module is faithful if 1_G is the only $g \in G$ such that $vg = v$ for all $v \in V$.

Theorem 2.6. Let V be an FG -module with basis B and ρ a representation of group G over F defined by taking $g \mapsto [g]_B$.

- (i) If B' is another basis of V then the map $g \mapsto [g]_{B'}$ is a representation of G equivalent to ρ .
- (ii) If representation σ is equivalent to ρ then there exists basis B'' such that $\sigma(g) = [g]_{B''}$ for all $g \in G$.

Proof. Taking T to be the change of basis matrices, the two representations are equivalent. \square

Example 2.7. Let $G = C_3 = \langle a \mid a^3 = 1 \rangle$ and representation $\rho : G \rightarrow GL(n, F)$ defined by

$$a \mapsto \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}.$$

We attempt to construct an FG -module with group action described by ρ . Take $V = F^2$ with basis $B = [v_1, v_2]$. Define the action of G on V by

$$v_1 a = v_2, \quad v_2 a = -v_1 - v_2.$$

Let us now choose alternate basis $B' = [u_1 = v_1, u_2 = v_1 + v_2]$. The action of G on this basis is described by

$$u_1 a = -u_1 + u_2, \quad u_2 a = -u_1.$$

This gives us a representation

$$a \mapsto [a]_{B'} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}.$$

To verify this construction we write our change of basis matrix as

$$T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

and verify that

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}.$$

Definition 2.8. The permutation module of a group $G \leq S_n$ is an n -dimensional vector space V with basis $B = [v_1, \dots, v_n]$ and action by G defined by

$$v_i g = v_{ig}$$

for all $g \in G$ where ig is the image of i under $g \in S_n$.

It follows from Caley's theorem that every group has a faithful FG -module.

Example 2.9. Take $G = S_4$ and pick $g = (1\ 2)$ and $h = (1\ 2\ 3\ 4)$. We have representations

$$[g]_B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad [h]_B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

2.1 Module Reducibility

Definition 2.10. Let V be an FG -module. We call W a submodule of V if W is a vector subspace of V and W is closed under the action of G . We then write $W < V$.

Example 2.11. Let $G = C_3 = \langle (1\ 2\ 3) \rangle$ and V the permutation module of G with basis $B = [v_1, v_2, v_3]$. The subspace $W = \langle v_1 + v_2 + v_3 \rangle$ is a submodule but the subspace $U = \langle v_1 + v_2 \rangle$ is not.

For example, consider the action of $g = (1\ 2\ 3)$ on $v_1 + v_2 \in U$.

$$(v_1 + v_2)g = v_{1g} + v_{2g} = v_2 + v_3 \notin U$$

whereas G acts on W trivially.

3 Lecture 3

3.1 Module and Representation Reducibility

For any module, it is clear that we have two trivial submodules: $0 < V$ and $V < V$. Where $0 = \{0\} \subset V$.

Definition 3.1. Let V be an FG -module. We say that V is irreducible if the only submodules of V are V and 0 . Otherwise V is reducible

In 2.11 we showed that the permutation module of C_3 is reducible.

Definition 3.2. Let $\rho : G \rightarrow GL(n, F)$ be a representation. We say that ρ is irreducible if the corresponding FG -module (as constructed in 2.1) is irreducible. Otherwise ρ is reducible.

If an FG -module, V is reducible, that is, $0 < W < V$, $0 \neq W \neq V$. Let B_W be a basis for W . If we extend B_W to B a basis of V , then we get the following representation of G :

$$g \mapsto [g]_B = \begin{bmatrix} X_g & 0 \\ Y_g & Z_g \end{bmatrix} \quad (3.1)$$

Where the matrices X_g, Y_g and Z_g are some block matrices and 0 is a block of zeros and X_g has the dimensions $m \times m$ and, in this case, $\dim(W) = m$.

Proposition 3.3. A representation $\rho : G \rightarrow GL(n, F)$ is reducible if and only if with respect to some basis, B , of F^n , $[g]_B$ has the form 3.1 for some $0 < m < \dim(V)$ for all $g \in G$. Then the maps $g \mapsto X_g$ and $g \mapsto Z_g$ are both representations of G .

Proof. Suppose we have a presentation, $\rho : G \rightarrow GL(n, F)$ and a basis B of $V = F^n$ such that $[g]_B$ has the form 3.1 for every $g \in G$. Then consider the subspace $0 \subset W \subset V$ spanned by the first m elements of B . It is clear that $v[g]_B \in W$ for all $v \in W$. Therefore the module induced by ρ is reducible, so ρ is reducible. Now, if we have a reducible representation, then the argument above this proposition shows that with respect to any basis extending B_W , the matrices $[g]_B$ have the required form.

Now, using elementary block matrix multiplication, we get the following for $g, h \in G$:

$$\rho(g)\rho(h) = [g]_B[h]_B = \begin{bmatrix} X_g X_h & 0 \\ Y_g X_h + Z_g Y_h & Z_g Z_h \end{bmatrix} = [gh]_B = \rho(gh)$$

Therefore $X_{gh} = X_g X_h$ and $Z_{gh} = Z_g Z_h$, so the maps $g \mapsto X_g$ and $g \mapsto Z_g$ are both representations of G . \square

Problem 1. Prove that the example representation of D_8 of degree 2 over \mathbb{R} or \mathbb{C} is irreducible.

3.2 Group Algebras

Recall that an algebra over a field F is a vector space over F equipped with a bilinear product $A \times A \rightarrow A$ that is compatible with scalar multiplication.

Definition 3.4. The group algebra over a finite group G over a field F is an algebra¹ of dimension $n = |G|$ over $F = \mathbb{R}$ or \mathbb{C} called FG , with basis $B = G = \{g_1, \dots, g_n\}$. Where the algebra structure is given by the following for two arbitrary elements of FG , $u = \sum_{g \in G} \lambda_g g$, $v = \sum_{g \in G} \mu_g g$, $\lambda_g, \mu_g \in F$ and $\nu \in F$:

$$(i) \quad u + v = \sum_{i=1}^n (\lambda_i + \mu_i) g_i$$

¹See Lemma 3.6

$$(ii) \quad \nu \cdot u = \sum_{i=1}^n (\nu \lambda_i) g_i$$

$$(iii) \quad u \cdot v = \sum_{(h,g) \in G \times G} \lambda_g \mu_h(gh)$$

This is clearly a vector space.

Example 3.5. Consider $G = C_3 = \{e, a, a^2\} = \langle a | a^3 = e \rangle$ and $F = \mathbb{R}$ or \mathbb{C} . Then if we let $u = e - a + 2a^2$, $v = \frac{1}{2}e + 5a$, then:

$$u + v = \frac{3}{2}e + 4a + 2a^2, \quad \frac{1}{3}u = \frac{1}{3}e - \frac{1}{3}a + \frac{2}{3}a^2, \quad uv = \frac{21}{2}e + \frac{9}{2}a - 4a^2$$

Lemma 3.6. Given a group algebra FG , $r, s, t \in FG$, $\lambda \in F$:

- (I) $rs \in FG$
- (II) $(rs)t = r(st)$
- (III) $1_G r = r 1_G = r$
- (IV) $(\lambda r)s = \lambda(rs)$
- (V) $(r + s)t = rt + st$
- (VI) $r(s + t) = rs + rt$
- (VII) $r0 = 0r = 0$

That is, FG is an associative algebra with unit

Proof. 1,3 and 7 are clear from the definition of FG , 4,5 and 6 follow from the distributive and associative laws of F and 2 follows from associativity in G . \square

3.3 The Regular FG -module, FG

Problem 2. $V = FG$ is an FG -module with the group action defined by $v \cdot g = vg$ for $v \in FG$, $g \in G \subset FG$.

Definition 3.7. For a finite group G and $F = \mathbb{R}$ or \mathbb{C} , the regular FG -module is FG . The associated module, $g \mapsto [g]_B$ is called the regular representation.

Lemma 3.8. FG is a faithful module for G over F

Proof. If $vg = v$ for all $v \in FG$, then specifically, $hg = h$ for all $h \in G$, so $g = 1_G$. \square

Example 3.9. For $C = C_3$, over the basis $B = G$, we get:

$$[e]_B = I_3, \quad [a]_B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad [a^2]_B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Now, if we have an FG -module, V , then FG acts on V in the following way:

$$v \cdot r = v \cdot \left(\sum_{g \in G} \mu_g g \right) = \sum_{g \in G} \mu_g (v \cdot g)$$

Lemma 3.10. For $u, v \in V$, $\lambda \in F$, $r, s \in FG$:

- (I) $vr \in FG$
- (II) $(vr)s = v(rs)$
- (III) $v1 = v$
- (IV) $(\lambda v)r = \lambda(vr) = v(\lambda r)$
- (V) $v(r + s) = vr + vs$
- (VI) $(u + v)r = ur + vr$
- (VII) $r0 = v0 = 0$

Proof. I, III and the first part of VII follow from V being an FG -module, the second equality of VII follows from scalar multiplication by 0 in V . The following calculation:

$$\begin{aligned} (\lambda v)r &= \sum_{g \in G} \mu_g ((\lambda v)g) \\ &= \sum_{g \in G} \mu_g (\lambda(vg)) \\ &= \sum_{g \in G} (\lambda \mu_g)(vg) = v(\lambda r) \\ &= \lambda \sum_{g \in G} \mu_g (vg) \\ &= \lambda(vg) \end{aligned}$$

proves IV. VI follows from the linearity of the action of G on V . V follows from distributivity of scalar multiplication in V . Finally, to prove II:

$$\begin{aligned} v(rs) &= \sum_{(g,h) \in G \times G} (\mu_g \lambda_h (v(gh))) \\ &= \sum_{h \in G} \lambda_h \sum_{g \in G} \mu_g (gv)h \\ &= \sum_{h \in G} \lambda_h \left(\sum_{g \in G} \mu_g (gv) \right) h \\ &= \sum_{h \in G} \lambda_h (vr)h = (vr)s \end{aligned}$$

□

4 Lecture 4

4.1 Homomorphisms

Definition 4.1. Let V and W be FG -modules. A *homomorphism* of FG -modules is a map $\sigma : V \rightarrow W$ which is a linear transformation and also satisfies $(vg)\sigma = (v\sigma)g$ for all $g \in G, v \in V$. The *kernel* and *image* are defined in the obvious way

Equivalently, it is a homomorphism of modules over the ring FG . Indeed:

Problem 3. Suppose $r \in FG$ is an element of the group algebra. Prove that $(vr)\sigma = (v\sigma)r$.

Lemma 4.2. Let $\sigma : V \rightarrow W$ be a homomorphism of FG -algebras. Then the kernel and image of σ are submodules

Proof. This is a matter of simple checking, which will be left to the reader. □

Example 4.3. Take $\sigma : V \rightarrow V$ to be $v \mapsto \lambda v$ for some $\lambda \in F^*$. Then $\ker \sigma = 0, \text{im } \sigma = V$. ◀

Example 4.4. Let $G = S_n$ and $V = \langle v_1, \dots, v_n \rangle$ be the permutation module for G over F , and let $W = \langle w \rangle$ be the trivial module. Now define $\sigma : V \rightarrow W$ by

$$\sum \lambda_i v_i \mapsto \sum \lambda_i w$$

Then $\ker \sigma = \{ \sum \lambda_i v_i \mid \sum \lambda_i = 0 \}$ and $\text{im } \sigma = W$. ◀

Definition 4.5. A homomorphism of FG -modules is an *isomorphism* if it is bijective

Remark 1. In class we originally said "if the homomorphism has trivial kernel". However, this is definitely not correct because inclusions are always homomorphisms, but obviously not isomorphisms. ♦

Lemma 4.6. The inverse of an isomorphism is an isomorphism

Proof. Once again, this is just an exercise in checking. The details will be left for the reader. □

Some rather obvious invariants of FG -modules (under isomorphism) are dimension and irreducibility.

Lemma 4.7. V and W are isomorphic if and only if there exists bases \mathcal{B}_1 of V and \mathcal{B}_2 of W such that

$$[g]_{\mathcal{B}_1} = [g]_{\mathcal{B}_2}$$

for all g .

Proof. Suppose firstly that V and W are isomorphic, and let $\sigma : V \rightarrow W$ be one such isomorphism. Let $\mathcal{B}_1 = \{v_1, \dots, v_n\}$ be a basis for V . In particular, it is linearly independent, and it is easy to see that $\mathcal{B}_2 = \{v_1\sigma, \dots, v_n\sigma\}$ is also linearly independent. Since V and W are isomorphic, they have the same dimension, and thus \mathcal{B}_2 is a basis for W . Since $(vg)\sigma = (v\sigma)g$ for all g and v , the action of g on the basis vectors of both bases are the same, and thus we conclude $[g]_{\mathcal{B}_1} = [g]_{\mathcal{B}_2}$.

Conversely, suppose that the latter hypothesis is satisfied. Let $\{v_1, \dots, v_n\}$ be a basis for V and $\{w_1, \dots, w_n\}$ be a basis for W . We define a bijective linear map $\sigma : V \rightarrow W$ such that $v_i\sigma = w_i$ for each i . Now observe that for each i , we have $v_i g = \lambda_1 v_1 + \dots + \lambda_n v_n$ and $w_i g = \lambda_1 w_1 + \dots + \lambda_n w_n$, where $(\lambda_1, \dots, \lambda_n)$ is the i -th row of $[g]$. This means that

$$(v_i g)\sigma = (\lambda_1 v_1 + \dots + \lambda_n v_n)\sigma = \lambda_1 v_1\sigma + \dots + \lambda_n v_n\sigma = \lambda_1 w_1 + \dots + \lambda_n w_n = w_i g = (v_i\sigma)g$$

and thus σ is a homomorphism of FG -modules. Since it is bijective, it is an isomorphism. \square

Theorem 4.8. *Let V be an FG -module with basis \mathcal{B}_1 and W an FG -module with basis \mathcal{B}_2 . Then $W \cong V$ if and only if $g \mapsto [g]_{\mathcal{B}_1}$ and $g \mapsto [g]_{\mathcal{B}_2}$ are equivalent.*

Proof. This follows from the previous Lemma and the fact that two matrices are conjugate (A and B are conjugate if $A = P^{-1}BP$ for some P) if and only if the linear transformations they define differ by a change of basis (that is they define the same transformation but with respect to different bases) \square

Example 4.9. Let $G = C_3 = \{e, a, a^2\}$. Let V be the regular representation, that is the natural representation induced by the module $FG = \langle e, a, a^2 \rangle$. Write $B := \{e, a, a^2\}$ as a basis for FG . Then

$$[a]_B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Now let W be the permutation module where $a = (1, 2, 3)$ and C_3 is considered a subgroup of S_3 . Write $B' = \{v_1, v_2, v_3\}$ for the basis of W . Then

$$[a]_{B'} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Note that these two modules are isomorphic. \blacktriangleleft

Example 4.10. Let $G = D_4 = \langle a, b \mid a^4 = b^2 = 1, a^b = a^{-1} \rangle$. Now we can act on either F^4 or F^8 . On F^4 , we have the representation described in Example 1.2. On W , we have the regular representation. Clearly are not isomorphic. \blacktriangleleft

4.2 Sums

We now consider how modules behave with respect to direct sums. Let V be an FG -module and suppose $V = U \oplus W$, where U and W are submodules. Let $\mathcal{B}_1 = \{u_1, \dots, u_n\}$ be a basis for U and $\mathcal{B}_2 = \{w_1, \dots, w_m\}$ one for W , so that $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ is a basis for V . Then

$$[g]_{\mathcal{B}} = \begin{pmatrix} [g]_{\mathcal{B}_1} & 0 \\ 0 & [g]_{\mathcal{B}_2} \end{pmatrix}$$

Lemma 4.11. *Let V be an FG -module such that we have the decomposition*

$$V = \bigoplus_{i=1}^n U_i$$

Define the projection map $\pi_i : u_1 + u_2 + \dots + u_n \mapsto u_i$. Then

- (I) π_i is a homomorphism
- (II) $\pi_i \circ \pi_i = \pi_i$

Proof. Trivial □

Lemma 4.12. *Suppose we have a finite decomposition*

$$V = \sum U_i$$

where the U_i are irreducible. Then V is the direct sum of some subset of the U_i .

Proof. This follows from the fact that the intersection of two distinct irreducible modules is trivial (again, simple checking). □

We will now present an important result

Theorem 4.13 (Maschke's Theorem). *Let G be a finite group, F a field of characteristic 0, V an FG -module and U a submodule. Then there exists some W such that $V = U \oplus W$.*

Proof. We first choose some W_1 such that $V = U \oplus W_1$ as vector spaces. Note that each $v \in V$ can be uniquely decomposed as $v = u + w$, where $u \in U, w \in W_1$. Now define the canonical projection $\sigma : V \rightarrow U$ where $v \mapsto u$. Clearly $\ker \sigma = W_1$ and $\text{im } \sigma = U$. However, we note that σ is NOT necessarily a homomorphism of FG -modules. We modify it as follows: Define $\varphi : V \rightarrow V$ by

$$v \mapsto \frac{1}{|G|} \sum_{g \in G} vg\sigma g^{-1}$$

We claim that φ is a homomorphism. Indeed, suppose $x \in G, v \in V$. Then

$$\begin{aligned}(xv)\varphi &= \frac{1}{|G|} \sum_{g \in G} (vx)g\sigma g^{-1} \\ &= \frac{1}{|G|} \sum_{h \in G} vh\sigma h^{-1}x \\ &= \left(\frac{1}{|G|} \sum_{h \in G} vh\sigma h^{-1} \right) x = (v\varphi)x\end{aligned}$$

where the equality

$$\frac{1}{|G|} \sum_{g \in G} (vx)g\sigma g^{-1} = \frac{1}{|G|} \sum_{h \in G} vh\sigma h^{-1}x$$

follows from the change of variables $h = xg$. Clearly φ maps into U , and we now check it is a projection. Indeed, supposing $u \in U$ we have

$$\begin{aligned}(u)\varphi &= \frac{1}{|G|} \sum_{g \in G} ug\sigma g^{-1} \\ &= \frac{1}{|G|} \sum_{g \in G} u\sigma gg^{-1} \\ &= \frac{1}{|G|} \sum_{g \in G} u \\ &= u\end{aligned}$$

as desired.

Now clearly $U = \text{im } \varphi$ and we define $W := \ker \varphi$. Then for each $v \in V$, write $u := v\varphi \in U$ and $w := v - u \in W$ so that $v = u + w$. It only remains to check that this is unique. To see this, suppose $u' + w' = v = u + w$. Then

$$u' = \varphi(u') = \varphi(v) = \varphi(u) = u$$

which implies the result. □

5 Lecture 5

We begin with consequences of Maschke's theorem.

Example 5.1. Let $G = S_3$, and $V = \langle v_1, v_2, v_3 \rangle$ is the permutation module. Let U be the submodule of V defined as $U = \langle v_1 + v_2 + v_3 \rangle < V$. Suppose $W_0 = \langle v_1, v_2 \rangle$, so that

$V = U \oplus W_0$ as subspaces. Define a projection $\phi : V \rightarrow U$ by $v_1 \mapsto 0$, $v_2 \mapsto 0$, and $v_3 \mapsto v_1 + v_2 + v_3$. Further, define $\theta : V \rightarrow V$, as in proof of Maschke's theorem, so that

$$v\theta = \frac{1}{|G|} \sum_{g \in G} vg\theta g^{-1} = \frac{1}{6} \sum_{g \in S_3} vg\theta g^{-1}.$$

Consider the action of θ on the basis elements v_i , then a short computation shows that

$$v_i\theta = \frac{1}{3}(v_1 + v_2 + v_3), \quad i = 1, 2, 3.$$

Moreover, we have

$$\ker \theta = \{v \in V : v\theta = 0\} = \left\{ \sum_{i=1}^3 \lambda_i v_i : \sum_{i=1}^3 \lambda_i = 0 \right\}.$$

Hence $V = U \oplus \ker \theta$, is a direct summand of submodules. Moreover, if $\mathcal{B} = [v_1 + v_2 + v_3, v_1, v_2]$ is a basis for V , and $\mathcal{B}' = [v_1 + v_2 + v_3, v_1 - v_2, v_2 - v_3]$ is another, one has

$$[g]_{\mathcal{B}} = \begin{pmatrix} * & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix}, \quad [g]_{\mathcal{B}'} = \begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}.$$

◀

In fact, it follows from Maschke's theorem that if we choose a basis \mathcal{B} for V such that

$$[g]_{\mathcal{B}} = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix},$$

then there exists a basis \mathcal{B}' for V such that

$$[g]_{\mathcal{B}'} = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}.$$

Definition 5.2. Let V be a FG -module, V is said to be **completely reducible** if $V = U_1 \oplus \cdots \oplus U_r$ with each U_i an irreducible FG -module.

Theorem 5.3. Suppose G is a finite group, and $F = \mathbb{R}, \mathbb{C}$. Then every FG -module is completely reducible

Proof. Induction using Maschke's theorem. □

Lemma 5.4. Suppose G is a finite group, $F = \mathbb{R}, \mathbb{C}$, and V a FG -module. If U is a FG -submodule, then there exists a surjective FG -homomorphism from V onto U .

Proof. By Maschke's theorem, there exists a complementary submodule W to U such that $V = U \oplus W$. Thus, defining $\pi : V \rightarrow U$ by $u + w \mapsto u$ gives the result. □

Example 5.5. Take

$$G = \left\langle \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} \mid n \in \mathbb{Z} \right\rangle$$

and $V = \mathbb{C}^2$. Then V is not completely reducible. \blacktriangleleft

Example 5.6. Let $G = C_p = \langle a \mid a^p = 1 \rangle$ where p is prime, and take the representation

$$a^j \mapsto \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix}, \quad 0 \leq j \leq p-1$$

over the finite field $F = \mathbb{Z}_p$. If $V = \langle v_1, v_2 \rangle$ is the FG -module of the representation and $U = \langle v_2 \rangle$, then there does not exist a submodule $W < V$ such that $V = U \oplus W$. \blacktriangleleft

The previous two examples show that the assumptions of Maschke's theorem are required, and cannot be relaxed.

5.1 Schur's Lemma

Theorem 5.7 (Schur's lemma). *Suppose V and W are irreducible $\mathbb{C}G$ -modules.*

- (I) *If $\theta : V \rightarrow W$ is a $\mathbb{C}G$ -homomorphism, then θ is either a $\mathbb{C}G$ -isomorphism or the zero homomorphism.*
- (II) *If $\theta : V \rightarrow V$ is a $\mathbb{C}G$ -isomorphism, then θ is scalar multiple of the identity endomorphism of V .*

Proof. (I): Suppose that $v\theta \neq 0$ for some $v \in V$, then the image of θ is not trivial, $\text{im } \theta \neq \{0\}$. However, $\text{im } \theta$ is a submodule of W , and so the irreducibility of W forces $\text{im } \theta = W$. Likewise, as the kernel of θ is a submodule of V , but not all of V , we have $\ker \theta = \{0\}$ as V is irreducible. Therefore, θ is a bijective $\mathbb{C}G$ -homomorphism, and so it is a $\mathbb{C}G$ -isomorphism.

(II): Suppose θ is a $\mathbb{C}G$ -isomorphism. Then as \mathbb{C} is algebraically closed, θ has an eigenvalue $\lambda_v \in \mathbb{C}$ with $v\theta = \lambda_v v$ for some $v \in V$. Now, as $\ker(\theta - \lambda_v 1_V) \neq \{0\}$ is a submodule of V , V being irreducible implies that $\ker(\theta - \lambda_v 1_V) = V$. Therefore, $w(\theta - \lambda_v 1_V) = 0$ for all $w \in V$, and so $\theta = \lambda 1_V$ as required. \square

Further, we actually have a converse to the second statement of Schur's lemma.

Proposition 5.8. *Let V be a nontrivial $\mathbb{C}G$ -module, and suppose that every $\mathbb{C}G$ -homomorphism from V to V is a scalar multiple of the identity endomorphism of V . Then V is irreducible.*

Proof. Suppose V is reducible. Then there exists a nontrivial submodule U of V such that, by Maschke's theorem, $V = U \oplus W$ with W also a submodule of V . Defining $\pi : V \rightarrow V$ by $u + w \mapsto u$ gives a $\mathbb{C}G$ -homomorphism that is not a multiple of the identity endomorphism. A contradiction. \square

We now interpret Schur's lemma as representation statement.

Lemma 5.9. *Let $\rho : G \rightarrow \mathrm{GL}(n, \mathbb{C})$ be a representation. Then ρ is irreducible if, and only if, every $n \times n$ matrix A which satisfies $(g\rho)A = A(g\rho)$ for all $g \in G$, has the form $A = \lambda I_n$.*

Proof. Result follows from Schur's lemma and its partial converse. \square

Example 5.10. Suppose $G = C_3 = \langle a \mid a^3 = 1 \rangle$, and $\rho : G \rightarrow \mathrm{GL}(2, \mathbb{C})$ is the representation defined by

$$a\rho = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

Let

$$A = a\rho = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

Then it is clear that $A(g\rho) = (g\rho)A$ for all $g \in G$. As A is not a scalar multiple of the identity, the representation is reducible. \blacktriangleleft

Example 5.11. Suppose $G = D_5 = \langle a, b \mid a^5 = 1, b^2 = 1, a^b = a^{-1} \rangle$. Set $\omega = e^{2\pi i/5}$. Let $\rho : G \rightarrow \mathrm{GL}(2, \mathbb{C})$ be the representation defined by

$$a \mapsto \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

If

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

commutes with $a\rho$ and $b\rho$; one calculates that

$$A = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} = \alpha I_2,$$

and so the representation is irreducible. \blacktriangleleft

Lemma 5.12. *Let G be a finite abelian group. Then every irreducible $\mathbb{C}G$ -module has dimension 1.*

Proof. Choose $x \in G$, then $v(gx) = v(xg)$ for all $g \in G$, and so $v \mapsto vx$ is a $\mathbb{C}G$ -homomorphism. It is actually an isomorphism with inverse $v \mapsto vx^{-1}$. Hence by Schur this isomorphism is a scalar multiple of the identity, say $\lambda_x 1_V$. Thus $vx = \lambda_x 1_V$ for all $v \in V$ and the group action by G is just usual scalar multiplication. This means that every subspace is a submodule. However, V is irreducible and so has no non-trivial submodules; which forces $\dim V = 1$. \square

6 Lecture 6

Continuing our discussion of the representations of abelian groups, we provide a stronger theorem in which we construct these 1-dimensional representations by mapping group elements to roots of unity. But first recall the Fundamental Theorem of Abelian Groups which states that any finite abelian group G is isomorphic to the direct sum of cyclic groups

$$G \simeq C_{n_1} \oplus \cdots \oplus C_{n_r}$$

where $n_i \mid n_{i+1}$ for all $1 \leq i \leq r-1$. Note that if $C_i = \langle g_i \rangle$ then we can write $G = \langle g_1, \dots, g_r \rangle$ and g_i has order n_i .

We define a homomorphism ρ from G to \mathbb{C} by taking $g_i \mapsto \lambda_i$ where λ_i is the n_i -th root of unity. This defines a representation and is specified by roots of unity $\lambda_1, \dots, \lambda_r$. Thus for the representation ρ defined by roots of unity $\lambda_1, \dots, \lambda_r$ we write $\rho = \rho_{\lambda_1, \dots, \lambda_r}$.

Theorem 6.1. *Suppose $G \simeq C_{n_1} \times \cdots \times C_{n_r}$ for cyclic groups C_{n_i} of order n_i . The representation $\rho_{\lambda_1, \dots, \lambda_r}$ of G is irreducible of degree 1. There are $|G|$ many such representations and every irreducible representation of G over \mathbb{C} is equivalent to one of these.*

Example 6.2. Take $G = \langle a \mid a^n = 1 \rangle$ and $\omega = e^{2\pi i/n}$. The irreducible representations of G are ρ_{ω^j} for $0 \leq j \leq n-1$ defined by

$$a^k \rho_{\omega^j} = \omega^{jk}, \quad 0 \leq k \leq n-1.$$

◀

6.1 Application of Schur's to $\mathbb{C}G$

Definition 6.3. If G is a finite group then we define the center of the group algebra $Z(\mathbb{C}G)$ by

$$Z(\mathbb{C}G) = \{z \in \mathbb{C}G \mid zr = rz \quad \forall r \in \mathbb{C}G\}.$$

We now state some simple properties of the center:

1. $Z(\mathbb{C}G)$ is a subspace of $\mathbb{C}G$.
2. If G is abelian then $Z(\mathbb{C}G) = \mathbb{C}Z(G)$.
3. If H is a normal subgroup of G then $\sum_{h \in H} h \in Z(\mathbb{C}G)$.

We provide a simple proof for the last statement.

Proof. Take $z = \sum_{h \in H} h$ and $g \in G$. We then have

$$z^g = \sum_{h \in H} h^g = z.$$

because H is normal and hence fixed by conjugation of elements in G . □

Example 6.4. Take $G = S_3 = \langle a = (1\ 2\ 3), b = (1\ 2) \rangle$. Then $\sum_{g \in G} g \in Z(\mathbb{C}G)$. ◀

Lemma 6.5. Let V be an irreducible $\mathbb{C}G$ -module and let $z \in Z(\mathbb{C}G)$. There exists $\lambda \in \mathbb{C}$ such that $vz = \lambda v$ for all $v \in V$.

Proof. For all $r \in \mathbb{C}G$ and $v \in V$ we know $vrz = v zr$. Thus the mapping $v \mapsto vz$ is a $\mathbb{C}G$ -homomorphism from V to V . The result then follows by Schur. ◻

Remark 2. Note that $\mathbb{C}G$ is a faithful module. ♦

Lemma 6.6. If there exists a faithful, irreducible $\mathbb{C}G$ -module then $Z(G)$ is cyclic.

Proof. Let V be an irreducible, faithful $\mathbb{C}G$ -module and take $z \in Z(G) \subset Z(\mathbb{C}G)$. There exists $\lambda_z \in \mathbb{C}$ such that $vz = \lambda_z v$ for all $v \in V$. But since V is faithful the mapping $z \mapsto \lambda_z$ is injective from $Z(G)$ to \mathbb{C}^\times and so $Z(G) \simeq \{\lambda_z \mid z \in Z(G)\}$ is a finite subgroup of \mathbb{C}^\times . All of which are cyclic. ◻

Example 6.7. \mathbb{Z}_4 has a faithful, irreducible representation by taking 1 to the 4-th root of unity. However $\mathbb{Z}_2 \times \mathbb{Z}_2$ has no faithful irreducible representation as its center is not cyclic. ◀

Remark 3. Note that the converse is not true in general. There exists groups with cyclic centers but no faithful, irreducible representations. See Frobenius Groups. ♦

Lemma 6.8. Suppose G finite and every irreducible $\mathbb{C}G$ -module has dimension 1. Then G is abelian.

6.2 The Group Algebra and Irreducible Modules

Lemma 6.9. Let V, W be $\mathbb{C}G$ -modules and $\theta : V \rightarrow W$ is a $\mathbb{C}G$ -homomorphism. There exists a submodule $U < V$ such that $V = \ker \theta \oplus U$ and $U \simeq \text{im } \theta$

Proof. We apply Maschke's theorem to $\ker \theta < V$ and obtain submodule $U < V$ such that $V = \ker \theta \oplus U$. Now define the map $\bar{\theta} : U \rightarrow \text{im } \theta$ by $u \mapsto u\theta$ which is a homomorphism because it is the restriction of θ to a subspace. The kernel of $\bar{\theta}$ is trivial as $\ker \bar{\theta} = \ker \theta \cap U = \{0\}$. And $\text{im } \theta = \text{im } \bar{\theta}$ because if $w \in \text{im } \theta$ then $w = v\theta$ for some $v \in V$ and $v = k + u$ for some $u \in U$ and $k \in \ker \theta$. We then have

$$w = v\theta = (u + k)\theta = u\theta = u\bar{\theta} \in \text{im } \bar{\theta}.$$

Finally, by the first isomorphism theorem we obtain

$$U \simeq U / \ker \bar{\theta} \simeq \text{im } \bar{\theta} = \text{im } \theta.$$

◻

7 Lecture 7

7.1 The Group Algebra and Irreducible Modules: Part II

Definition 7.1. For V , a $\mathbb{C}G$ -module, U an irreducible $\mathbb{C}G$ -module, U is called a composition factor for V if V has a $\mathbb{C}G$ -submodule isomorphic to U .

Lemma 7.2. Let V be a $\mathbb{C}G$ -module, suppose we have a finite decomposition into a direct sum of irreducible $\mathbb{C}G$ -modules:

$$V = \bigoplus_{i=1}^n U_i$$

Then, if $0 < U < V$ is an irreducible, $\mathbb{C}G$ -submodule of V , then $U \simeq U_i$ for some $1 \leq i \leq n$.

Proof. Consider the maps $\pi_i : U \rightarrow U_i$, since each $\pi_i : V \rightarrow U_i$ is a $\mathbb{C}G$ -homomorphism, and since U is non-trivial, it is clear that not all the π_i are the zero map, let π_j be such a projection, then by Schur, $\pi_j : U \rightarrow U_j$ is a $\mathbb{C}G$ -isomorphism. \square

Theorem 7.3. Let the following be a decomposition of the regular representation into irreducible $\mathbb{C}G$ -modules:

$$\mathbb{C}G = \bigoplus_{i=1}^n U_i$$

Then every irreducible $\mathbb{C}G$ -module is isomorphic to U_i for some $1 \leq i \leq n$

Proof. Let W be an irreducible $\mathbb{C}G$ -module, let $0 \neq w \in W$, then $wG = \{wr : r \in \mathbb{C}G\}$ is a $\mathbb{C}G$ -submodule of W . However, W is irreducible, so $W = wG$. Define $\theta : \mathbb{C}G \rightarrow W$ by $r\theta = wr$ for $r \in \mathbb{C}G$, then θ is a linear $\mathbb{C}G$ -homomorphism by 3.10 and $\text{im } \theta = W$. So $\mathbb{C}G = \ker \theta \oplus U$, where $U \simeq \text{im } \theta \simeq W$ by 6.9. Now W is irreducible, so U is irreducible, so by 7.2, $U \simeq U_i \simeq W$ for some $1 \leq i \leq n$. \square

Now for a finite group G , we can characterise all irreducible $\mathbb{C}G$ -modules by decomposing $\mathbb{C}G$.

Example 7.4. Let $G = C_3 = \langle a \mid a^3 = 1 \rangle$, then let $\omega = e^{\frac{2\pi i}{3}}$ be a primitive 3rd root of unity and let the following be elements of $\mathbb{C}G$:

$$v_1 = 1 + a + a^2, \quad v_2 = 1 + \omega^2 a + \omega a^2, \quad v_3 = 1 + \omega a + \omega^2 a^2$$

We have $(v_1)a = v_1$, $(v_2)a = \omega + a + \omega^2 a^2 = \omega v_2$ and $(v_3)a = \omega^2 + a + \omega a = \omega^2 v_3$. Therefore, $V_i = \langle v_i \rangle < \mathbb{C}G$ for $i = 1, 2, 3$ are all irreducible submodules of $\mathbb{C}G$.

Furthermore, it is clear that $\{v_1, v_2, v_3\}$ forms a basis for $\mathbb{C}G$, hence $\mathbb{C}G = V_1 \oplus V_2 \oplus V_3$. \blacktriangleleft

Example 7.5. Let $G = D_6 = \langle a, b \mid a^3 = 1, a^b = a^{-1} \rangle$. \blacktriangleleft

7.2 The Vector Space of $\mathbb{C}G$ -homomorphisms

Definition 7.6. Let V, W be $\mathbb{C}G$ -modules, then define $H = \text{Hom}_{\mathbb{C}G}(V, W) = \{\theta : V \rightarrow W : \theta \text{ is a } \mathbb{C}G \text{ homomorphism}\}$.

We can then define addition and scalar multiplication operations on this set for $\theta, \phi \in H$, $\lambda \in \mathbb{C}$ as follows:

$$v(\theta + \phi) := v\theta + v\phi, \quad v(\lambda\theta) = \lambda(v\theta)$$

Lemma 7.7. *The space H with the operations defined above is a vector space over \mathbb{C} .*

Proof. Clear from the definition, note that the zero homomorphism is: $v0 = 0_W$. \square

Lemma 7.8. *For irreducible $\mathbb{C}G$ -modules, V, W ,*

$$\dim(H) = \begin{cases} 1 & V \simeq W \\ 0 & V \not\simeq W \end{cases}$$

Proof. If $V \not\simeq W$, then it follows that $\theta = 0$ for all $\theta \in H$ by Schur.

Now, suppose $V \simeq W$ and let $\theta : V \rightarrow W$ be an isomorphism. Now, let $\phi \in H$, we have that $\phi\theta^{-1} : V \rightarrow V$ is a homomorphism, so by Schur, $\phi\theta^{-1} = \lambda \text{Id}_V$, so for $v \in V$, $v\phi\theta^{-1} = \lambda v \implies v\phi\theta^{-1}\theta = v\phi = (\lambda v)\theta = \lambda(v\theta)$, therefore $\phi = \lambda\theta$, so $\dim H = 1$. \square

Proposition 7.9. *Given V, V_1, V_2, W, W_1, W_2 $\mathbb{C}G$ -modules,*

1. $\dim \text{Hom}_{\mathbb{C}G}(V, W_1 \oplus W_2) = \dim \text{Hom}_{\mathbb{C}G}(V, W_1) + \dim \text{Hom}_{\mathbb{C}G}(V, W_2)$
2. $\dim \text{Hom}_{\mathbb{C}G}(V_1 \oplus V_2, W) = \dim \text{Hom}_{\mathbb{C}G}(V_1, W) + \dim \text{Hom}_{\mathbb{C}G}(V_2, W)$

Proof. I will prove the first statement and the conversion of the proof of the second statement is left as a simple exercise.

Firstly, we define the projection homomorphisms: $\pi_i : W_1 \oplus W_2 \rightarrow W_i$ defined by $(w_1 + w_2)\pi_i = w_i$. Now, if $\theta \in \text{Hom}_{\mathbb{C}G}(V, W_1 \oplus W_2)$, we have that $\theta\pi_i \in \text{Hom}_{\mathbb{C}G}(V, W_i)$.

Define a linear map $f : \text{Hom}_{\mathbb{C}G}(V, W_1 \oplus W_2) \rightarrow \text{Hom}_{\mathbb{C}G}(V, W_1) \oplus \text{Hom}_{\mathbb{C}G}(V, W_2)$ by $\theta \mapsto \theta\pi_1 \oplus \theta\pi_2$ then it is clear that the following map is an inverse of f :

$$v(\theta_1 \oplus \theta_2)f^{-1} = v\theta_1 + v\theta_2$$

therefore we have an isomorphism of vector spaces, $\text{Hom}_{\mathbb{C}G}(V, W_1 \oplus W_2) \simeq \text{Hom}_{\mathbb{C}G}(V, W_1) \oplus \text{Hom}_{\mathbb{C}G}(V, W_2)$, giving us $\dim \text{Hom}_{\mathbb{C}G}(V, W_1 \oplus W_2) = \dim \text{Hom}_{\mathbb{C}G}(V, W_1) + \dim \text{Hom}_{\mathbb{C}G}(V, W_2) = \dim \text{Hom}_{\mathbb{C}G}(V_1, W) + \dim \text{Hom}_{\mathbb{C}G}(V_2, W)$ \square

Corollary 7.9.1. *For $V_1, \dots, V_n, W_1, \dots, W_m$, if $V = \bigotimes_{i=1}^n V_i$ and $W = \bigotimes_{i=1}^m W_i$*

$$\dim H = \sum_{i=1}^n \sum_{j=1}^m \dim \text{Hom}_{\mathbb{C}G}(V_i, W_j)$$