Maths 721 Notes

2020

Contents

1 Lecture 1

In the first half of this course we will cover three main topics:

- representations;
- modules;
- characters.

We will further see that representations and modules are essentially the same, and that modules and characters are essentially the same; and hence all three are essentially the same.

From now on G is a group.

1.1 Representations

Definition 1.1. A **representation** of a group G over a field F is a group homomorphism from G to $\mathrm{GL}(n, F)$, where n is the **degree** of the representation.

Explicitly, a representation is a function $\rho : G \to \mathrm{GL}(n, F)$ such that for all $g, h \in G$;

(i) $(gh)\rho = (g\rho)(h\rho)$,

(ii) $1_G\rho = I_n$,

(iii) $g^{-1}\rho = (g\rho)^{-1}$.

Note the use of the (incredibly shit) postfix function notation.

Example 1.2. Take D_4, the Dihedral group of order 8. It has the following group presentations

$$\begin{aligned} D_4 &= \langle a, b \mid a^4 = 1, b^2 = 1, a^b = a^{-1} \rangle \\ &\cong \langle (1\ 2\ 3\ 4), (1\ 4)(2\ 3) \rangle, \end{aligned}$$

where $a^b = bab^{-1}$ is conjugation of a by b. By defining the matrix subgroup

$$H = \left\langle A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\rangle$$

and defining $\rho : D_4 \to \mathrm{GL}(n, F)$ where $F = \mathbb{R}, \mathbb{C}$, by $a \mapsto A$ and $b \mapsto B$, and $a^i b^j \mapsto A^i B^j$ for $0 \le i \le 3$, and $0 \le j \le 1$. Hence we have ρ is a representation of D_4 over F. ◀

Example 1.3. Take $\mathbb{Q}_8$ the Quaternion group of order 8, which has the following group presentations

$$\begin{aligned} \mathbb{Q}_8 &= \langle a, b \mid a^4 = 1, a^2 = b^2, a^b = a^{-1} \rangle \\ &\cong \langle \bar{a} = (1\ 6\ 2\ 5)(3\ 8\ 4\ 7), \bar{b} = (1\ 4\ 2\ 3)(5\ 7\ 6\ 8) \rangle \end{aligned}$$

Define

$$H = \left\langle A = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\rangle \subset \mathrm{GL}(2, \mathbb{C}).$$

Then $\rho : \mathbb{Q}_8 \to \mathrm{GL}(2, \mathbb{C})$ defined by $a^k b^\ell \mapsto A^k B^\ell$ is a group representation of $\mathbb{Q}_8$ over $\mathbb{C}$ of degree 2. ◀

Definition 1.4. Let G be a group and define

$$\begin{aligned} \rho : G &\to \mathrm{GL}(n, F) \\ g\rho &= I_n \end{aligned}$$

for all $g \in G$. Then ρ is a representation, called the **trivial representation** of arbitrary of degree.

It follows from the trivial representation that any group G has a representation of an arbitrary degree.

Let $\rho : G \to \mathrm{GL}(n, F)$ be a group homomorphism, and take $T \in \mathrm{GL}(n, F)$. Then

$$(T^{-1}AT)(T^{-1}BT) = T^{-1}(AB)T.$$

Thus, given ρ define σ such that

$$g\sigma = T^{-1}(g\rho)T$$

for all $g \in G$. As for all $g, h \in G$, one has

$$\begin{aligned} (gh)\sigma &= T^{-1}\big((gh)\rho\big)T \\ &= T^{-1}(g\rho)(h\rho)T \\ &= T^{-1}(g\rho)TT^{-1}(h\rho)T \\ &= (g\sigma)(h\sigma), \end{aligned}$$

and so σ is a group homomorphism; and hence a representation.

Definition 1.5. Define

$$\rho : G \to \mathrm{GL}(m, F), \qquad \sigma : G \to \mathrm{GL}(n, F)$$

to both be representation of G over F. We say that ρ **is equivalent to** σ if $n = m$ and there exists $T \in \mathrm{GL}(n, F)$ such that $g\sigma = T^{-1}(g\rho)T$.

Proposition 1.6. *Equivalence of representations is an equivalence relation.*

Proof. Reflexivity is clear by taking $T = I_n$. For symmetry, take T to be its inverse. For transitivity, if

$$g\sigma = T^{-1}(g\rho)T, \qquad g\rho = S^{-1}(g\eta)S,$$

then

$$g\sigma = (ST)^{-1}(g\eta)(ST).$$

□

Definition 1.7. Define the **kernel** of the representation $\rho : G \to \mathrm{GL}(n, F)$ as $\ker\rho = \{g \in G \mid g\rho = I_n\}$.

Proposition 1.8. *The kernel of a representation of G is a normal subgroup of G; i.e.* $\ker\rho \lhd G$.

Proof. Suppose $g \in \ker\rho$ and $h \in G$ is arbitrary. Then

$$(hgh^{-1})\rho = (h\rho)(g\rho)(h^{-1}\rho) = (h\rho)I_n(h\rho)^{-1} = (h\rho)(h\rho)^{-1} = I_n,$$

and so $hgh^{-1} \in \ker\rho$. As $\ker\rho$ is closed under conjugation, it is a normal subgroup of G. □

Definition 1.9. We say ρ is a **faithful** representation of G if $\ker\rho = \{1_G\}$.

Example 1.10. For the trivial representation $\rho : G \to \mathrm{GL}(n, F)$ with $g \mapsto I_n$ for all $g \in G$, we have $\ker\rho = G$. Hence the representation is not faithful. ◀

Lemma 1.11. *Suppose G is a finite group, and ρ is a representation of G over F. Then ρ is faithful if, and only if,* $\operatorname{im}\rho \cong G$.

Proof. Immediate from the first isomorphism theorem. □

1.2 FG-Modules

Suppose G is a group, and $F = \mathbb{R}, \mathbb{C}$. Given $\rho : G \to \mathrm{GL}(n, F)$, with $V = F^n$, let $v = (\lambda_1, \ldots, \lambda_n) \in V$ for $\lambda_i \in F$ be a row vector. Moreover, note that $g\rho$ is an $n \times n$ matrix for all $g \in G$. Thus, we have $v \cdot (g\rho) \in V$, and satisfies the following properties:

(i) $v \cdot \big((gh)\rho\big) = v \cdot (g\rho)(h\rho)$;

(ii) $v \cdot (1_G\rho) = v$;

(iii) $(\lambda v) \cdot (g\rho) = \lambda\big(v \cdot (g\rho)\big)$;

(iv) $(u + v) \cdot (g\rho) = u \cdot (g\rho) + v \cdot (g\rho)$.

We often will omitted the $\cdot$ in the operation, and write $v(a\rho)$ for $v \cdot (a\rho)$.

Example 1.12. Recall D_4 and its given presentation from a previous example. We have

$$a\rho = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad b\rho = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

If $v = (\lambda_1, \lambda_2)$, then we have

$$v(a\rho) = (-\lambda_2, \lambda_1), \qquad v(b\rho) = (\lambda_1, -\lambda_2).$$

◀

Definition 1.13. Let V a vector space over the field $F = \mathbb{R}, \mathbb{C}$, and let G be a group. We say V is a FG**-module** if a multiplication $v \cdot g$ for $v \in V$, and $g \in G$ is defined such that:

(i) $v \cdot g \in V$;

(ii) $v \cdot (gh) = (v \cdot g) \cdot h$;

(iii) $v \cdot 1_G = v$;

(iv) $(\lambda v) \cdot g = \lambda(v \cdot g)$;

(v) $(u + v) \cdot g = u \cdot g + v \cdot g$.

This generalises the previous discussion from a matrix group to an arbitrary group.

Note that properties (i), (iv), and (v) imply that the map $v \mapsto v \cdot g$ is an endomorphism of V (a linear map from V to itself).

Definition 1.14. Suppose V is an FG-module and B is a basis for V. For $g \in G$, let $[g]_B$ denote the matrix of the endomorphism $v \mapsto v \cdot g$ of V relative to the basis B.

2 Lecture 2

Theorem 2.1. *Let $\rho : G \to GL(n, F)$ be a representation of G over F.*

(I) *If $V = F^n$ is an FG module and G acts on V by $v \cdot g = v(g\rho)$ there exists a basis B of V such that $g\rho = [g]_B$.*

(II) *The map $g \mapsto [g]_B$ is a representation for G over F.*

Proof. Choose the standard basis $B = [e_1, \dots, e_n]$.

Since V is an FG-module we have $v(gh) = (vg)h$ for all $g, h \in G$ and $v \in V$. Thus $[gh]_B = [g]_B[h]_B$ so the map is a homomorphism. We now check that $[g]_B$ is invertable for all $g \in G$. We know $v \cdot 1_G = (vg)g^{-1}$ so $I_n = [g]_B[g^{-1}]_B$ and thus $[g]_B$ has an inverse. ☐

Example 2.2. Recall the representation of $G = D_4$ from a previous example. Define an FG-module $V = F^2$ with the action defined by taking vg to $v(g\rho)$.

$$\begin{aligned} v_1 &= (1, 0), \quad v_1 a = v_2, \quad v_1 b = v_1, \\ v_2 &= (0, 1), \quad v_1 a = -v_1, \quad v_1 b = -v_2. \end{aligned}$$

In this basis we recover our representation

$$a \mapsto [a]_B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad b \mapsto [b]_B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

◀

We now provide an equivalent basis-dependent definition for an FG-module.

Lemma 2.3. *Let V a vector space over the field $F = \mathbb{R}, \mathbb{C}$ with basis $B = [v_1, \ldots, v_n]$, and let G be a group. If a multiplication $v \cdot g$ for $v \in B$, and $g \in G$ is defined such that:*

(i) $v \cdot g \in V$;

(ii) $v \cdot (gh) = (v \cdot g) \cdot h$;

(iii) $v \cdot 1_G = v$;

(iv) $(\sum_{i=1}^{n} \lambda_i v_i) \cdot g = \sum_{i=1}^{n} \lambda_i (v_i \cdot g)$ *for all* $\lambda_i \in F$;

then V is an FG-module.

Definition 2.4. The trivial module of a group over F is a one dimensional vector space V over F such that $vg = v$ for all $v \in V$ and $g \in G$.

Definition 2.5. An FG-module is faithful if 1_G is the only $g \in G$ such that $vg = v$ for all $v \in V$.

Theorem 2.6. *Let V be an FG-module with basis B and ρ a representation of group G over F defined by taking $g \mapsto [g]_B$.*

(i) If B' is another basis of V then the map $g \mapsto [g]_{B'}$ is a representation of G equivalent to ρ.

(ii) If representation σ is equivalent to ρ then there exists basis B'' such that $\sigma(g) = [g]_{B''}$ for all $g \in G$.

Proof. Taking T to be the change of basis matrices, the two representations are equivalent. □

Example 2.7. Let $G = C_3 = \langle a \mid a^3 = 1 \rangle$ and representation $\rho : G \to GL(n, F)$ defined by

$$a \mapsto \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}.$$

We attempt to construct an FG-module with group action described by ρ. Take $V = F^2$ with basis $B = [v_1, v_2]$. Define the action of G on V by

$$v_1 a = v_2, \quad v_2 a = -v_1 - v_2.$$

Let us now choose alternate basis $B' = [u_1 = v_1, u_2 = v_1 + v_2]$. The action of G on this basis is described by

$$u_1 a = -u_1 + u_2, \quad u_2 a = -u_1.$$

This gives us a representation

$$a \mapsto [a]_{B'} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}.$$

To verify this construction we write our change of basis matrix as

$$T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

and verify that

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}.$$

◀

Definition 2.8. The permutation module of a group $G \leq S_n$ is an n-dimensional vector space V with basis $B = [v_1, \ldots, v_n]$ and action by G defined by

$$v_i g = v_{ig}$$

for all $g \in G$ where ig is the image of i under $g \in S_n$.

It follows from Caley's theorem that every group has a faithful FG-module.

Example 2.9. Take $G = S_4$ and pick $g = (1\ 2)$ and $h = (1\ 2\ 3\ 4)$. We have representations

$$[g]_B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad [h]_B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

◀

2.1 Module Reducibility

Definition 2.10. Let V be an FG-module. We call W a submodule of V if W is a vector subspace of V and W is closed under the action of G. We then write $W < V$.

Example 2.11. Let $G = C_3 = \langle (1\ 2\ 3) \rangle$ and V the permutation module of G with basis $B = [v_1, v_2, v_3]$. The subspace $W = \langle v_1 + v_2 + v_3 \rangle$ is a submodule but the subspace $U = \langle v_1 + v_2 \rangle$ is not.

For example, consider the action of $g = (1\ 2\ 3)$ on $v_1 + v_2 \in U$.

$$(v_1 + v_2)g = v_{1g} + v_{2g} = v_2 + v_3 \notin U$$

whereas G acts on W trivially. ◀

3 Lecture 3

3.1 Module and Representation Reducibility

For any module, it is clear that we have two trivial submodules: $0 < V$ and $V < V$. Where $0 = \{0\} \subset V$.

Definition 3.1. Let V be an FG-module. We say that V is irreducible if the only submodules of V are V and 0. Otherwise V is reducible

In 2.11 we showed that the permutation module of C_3 is reducible.

Definition 3.2. Let $\rho : G \to GL(n, F)$ be a representation. We say that ρ is irreducible if the corresponding FG-module (as constructed in 2.1) is irreducible. Otherwise ρ is reducible.

If an FG-module, V is reducible, that is, $0 < W < V$, $0 \neq W \neq V$. Let B_W be a basis for W. If we extend B_W to B a basis of V, then we get the following representation of G:

$$g \mapsto [g]_B = \begin{bmatrix} X_g & 0 \\ Y_g & Z_g \end{bmatrix} \tag{3.1}$$

Where the matrices X_g, Y_g and Z_g are some block matrices and 0 is a block of zeros and X_g has the dimensions $m \times m$ and, in this case, $\dim(W) = m$.

Proposition 3.3. *A representation $\rho : G \to GL(n, F)$ is reducible if and only if with respect to some basis, B, of F^n, $[g]_B$ has the form 3.1 for some $0 < m < \dim(V)$ for all $g \in G$. Then the maps $g \mapsto X_g$ and $g \mapsto Z_g$ are both representations of G.*

Proof. Suppose we have a presentation, $\rho : G \to GL(n, F)$ and a basis B of $V = F^n$ such that $[g]_B$ has the form 3.1 for every $g \in G$. Then consider the subspace $0 \subset W \subset V$ spanned by the first m elements of B. It is clear that $v[g]_B \in W$ for all $v \in W$. Therefore the module induced by ρ is reducible, so ρ is reducible. Now, if we have a reducible representation, then the argument above this proposition shows that with respect to any basis extending B_W, the matrices $[g]_B$ have the required form.
Now, using elementary block matrix multiplication, we get the following for $g, h \in G$:

$$\rho(g)\rho(h) = [g]_B[h]_B = \begin{bmatrix} X_g X_h & 0 \\ Y_g X_h + Z_g Y_h & Z_g Z_h \end{bmatrix} = [gh]_B = \rho(gh)$$

Therefore $X_{gh} = X_g X_h$ and $Z_{gh} = Z_g Z_h$, so the maps $g \mapsto X_g$ and $g \mapsto Z_g$ are both representations of G. ☐

Problem 1. Prove that the example representation of D_8 of degree 2 over $\mathbb{R}$ or $\mathbb{C}$ is irreducible.

3.2 Group Algebras

Recall that an algebra over a field F is a vector space over F equipped with a bilinear product $A \times A \to A$ that is compatible with scalar multiplication.

Definition 3.4. The group algebra over a finite group G over a field F is an algebra[1] of dimension $n = |G|$ over $F = \mathbb{R}$ or $\mathbb{C}$ called FG, with basis $B = G = \{g_1, \dots g_n\}$.

[1] See Lemma3.6

Where the algebra structure is given by the following for two arbitrary elements of FG, $u = \sum_{g\in G} \lambda_g g$, $v = \sum_{g\in G} \mu_g$, $\lambda_g, \mu_g \in F$ and $\nu \in F$:

(i) $u + v = \sum_{i=1}^{n} (\lambda_i + \mu_i) g_i$

(ii) $\nu \cdot u = \sum_{i=1}^{n} (\nu\lambda_i) g_i$

(iii) $u \cdot v = \sum_{(h,g)\in G\times G} \lambda_g \mu_h (gh)$

This is clearly a vector space.

Example 3.5. Consider $G = C_3 = \{e, a, a^2\} = \langle a | a^3 = e\rangle$ and $F = \mathbb{R}$ or $\mathbb{C}$. Then if we let $u = e - a + 2a^2$, $v = \frac{1}{2}e + 5a$, then:

$$u + v = \frac{3}{2}e + 4a + 2a^2, \quad \frac{1}{3}u = \frac{1}{3}e - \frac{1}{3}a + \frac{2}{3}a^2, \quad uv = \frac{21}{2}e + \frac{9}{2}a - 4a^2$$

◄

Lemma 3.6. *Given a group algebra FG, $r, s, t \in FG$, $\lambda \in F$:*

(I) $rs \in FG$

(II) $(rs)t = r(st)$

(III) $1_G r = r1_G = r$

(IV) $(\lambda r)s = \lambda(rs)$

(V) $(r + s)t = rt + st$

(VI) $r(s + t) = rs + rt$

(VII) $r0 = 0r = 0$

That is, FG is an associative algebra with unit

Proof. 1,3 and 7 are clear from the definition of FG, 4,5 and 6 follow from the distributive and associative laws of F and 2 follows from associativity in G. □

3.3 The Regular FG-module, FG

Problem 2. $V = FG$ is an FG-module with the group action defined by $v \cdot g = vg$ for $v \in FG$, $g \in G \subset FG$.

Definition 3.7. For a finite group G and $F = \mathbb{R}$ or $\mathbb{C}$, the regular FG-module is FG. The associated module, $g \mapsto [g]_B$ is called the regular representation.

Lemma 3.8. *FG is a faithful module for G over F*

Proof. If $vg = v$ for all $v \in FG$, then specifically, $hg = h$ for all $h \in G$, so $g = 1_G$. □

Example 3.9. For $C = C_3$, over the basis $B = G$, we get:

$$[e]_B = I_3, \quad [a]_B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad [a^2]_B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

◀

Now, if we have an FG-module, V, then FG acts on V in the following way:

$$v \cdot r = v \cdot \left(\sum_{g \in G} \mu_g g \right) = \sum_{g \in G} \mu_g (v \cdot g)$$

Lemma 3.10. *For $u, v \in V$, $\lambda \in F$, $r, s \in FG$:*

(I) $vr \in FG$

(II) $(vr)s = v(rs)$

(III) $v1 = v$

(IV) $(\lambda v)r = \lambda(vr) = v(\lambda r)$

(V) $(r + s)v = rv + sv$

(VI) $r(u + v) = ru + rv$

(VII) $r0 = v0 = 0$

Proof. 1,3 and the first part of 7 follow from V being an FG-module, the second equality of 7 follows from scalar multiplication by 0 in V. The following calculation:

$$\begin{aligned}
(\lambda v)r &= \sum_{g \in G} \mu_g ((\lambda v) g) \\
&= \sum_{g \in G} \mu_g (\lambda (vg)) \\
&= \sum_{g \in G} (\lambda \mu_g)(vg) = v(\lambda r) \\
&= \lambda \sum_{g \in G} \mu_g (vg) \\
&= \lambda (vg)
\end{aligned}$$

proves 4. 6 follows from the linearity of the action of G on V. 5 follows from distributivity of scalar multiplication in V. Finally, to prove 2:

$$
\begin{aligned}
v(rs) &= \sum_{(g,h)\in G\times G} (\mu_g \lambda_h (v(gh))) \\
&= \sum_{h\in G} \lambda_h \sum_{g\in G} \mu_g (gv) h \\
&= \sum_{h\in G} \lambda_h \left(\sum_{g\in G} \mu_g (gv) \right) h \\
&= \sum_{h\in G} \lambda_h (vr) h = (vr)s
\end{aligned}
$$

□

4 Lecture 4

4.1 Homomorphisms

Definition 4.1. Let V and W be FG-modules. A *homomorphism* of FG-modules is a map $\sigma : V \to W$ which is a linear transformation and also satisfies $(vg)\sigma = (v\sigma)g$ for all $g \in G, v \in V$. The *kernel* and *image* are defined in the obvious way

Equivalently, it is a homomorphism of modules over the ring FG. Indeed:

Problem 3. Suppose $r \in FG$ is an element of the group algebra. Prove that $(vr)\sigma = (v\sigma)r$.

Lemma 4.2. *Let $\sigma : V \to W$ be a homomorphism of FG-algebras. Then the kernel and image of σ are submodules*

Proof. This is a matter of simple checking, which will be left to the reader. □

Example 4.3. Take $\sigma : V \to V$ to be $v \mapsto \lambda v$ for some $\lambda \in F^*$. Then $\ker \sigma = 0, \operatorname{im} \sigma = V$. ◀

Example 4.4. Let $G = S_n$ and $V = \langle v_1, ..., v_n \rangle$ be the permutation module for G over F, and let $W = \langle w \rangle$ be the trivial module. Now define $\sigma : V \to W$ by

$$\sum \lambda_i v_i \mapsto \sum \lambda_i w$$

Then $\ker \sigma = \{\sum \lambda_i v_i \mid \sum \lambda_i = 0\}$ and $\operatorname{im} \sigma = W$. ◀

Definition 4.5. A homomorphism of FG-modules is an *isomorphism* if it is bijective

Remark 1. In class we originally said "if the homomorphism has trivial kernel". However, this is definitely not correct because inclusions are always homomorphisms, but obviously not isomorphisms. ♦

Lemma 4.6. *The inverse of an isomorphism is an isomorphism*

Proof. Once again, this is just an exercise in checking. The details will be left for the reader. □

Some rather obvious invariants of FG-modules (under isomorphism) are dimension and irreducibility.

Lemma 4.7. *V and W are isomorphic if and only if there exists bases $\mathcal{B}_1$ of V and $\mathcal{B}_2$ of W such that*

$$[g]_{\mathcal{B}_1} = [g]_{\mathcal{B}_2}$$

for all g.

Proof. Suppose firstly that V and W are isomorphic, and let $\sigma : V \to W$ be one such isomorphism. Let $\mathcal{B}_1 = \{v_1, ..., v_n\}$ be a basis for V. In particular, it is linearly independent, and it is easy to see that $\mathcal{B}_2 = \{v_1\sigma, ..., v_n\sigma\}$ is also linearly independent. Since V and W are isomorphic, they have the same dimension, and thus $\mathcal{B}_2$ is a basis for W. Since $(vg)\sigma = (v\sigma)g$ for all g and v, the action of g on the basis vectors of both bases are the same, and thus we conclude $[g]_{\mathcal{B}_1} = [g]_{\mathcal{B}_2}$.

Conversely, suppose that the latter hypothesis is satisfied. Let $\{v_1, ..., v_n\}$ be a basis for V and $\{w_1, ..., w_n\}$ be a basis for W. We define a bijective linear map $\sigma : V \to W$ such that $v_i\sigma = w_i$ for each i. Now observe that for each i, we have $v_ig = \lambda_1v_1 + ... + \lambda_nv_n$ and $w_ig = \lambda_1w_1 + ... + \lambda_nw_n$, where $(\lambda_1, ..., \lambda_n)$ is the i-th row of $[g]$. This means that

$$(v_ig)\sigma = (\lambda_1v_1 + ... + \lambda_nv_n)\sigma = \lambda_1v_1\sigma + ... + \lambda_nv_n\sigma = \lambda_1w_1 + ... + \lambda_nw_n = w_ig = (v_i\sigma)g$$

and thus σ is a homomorphism of FG-modules. Since it is bijective, it is an isomorphism. □

Theorem 4.8. *Let V be an FG-module with basis $\mathcal{B}_1$ and W an FG-module with basis $\mathcal{B}_2$. Then $W \cong V$ if and only if $g \mapsto [g]_{\mathcal{B}_1}$ and $g \mapsto [g]_{\mathcal{B}_2}$ are equivalent.*

Proof. This follows from the previous Lemma and the fact that two matrices are conjugate (A and B are conjugate if $A = P^{-1}BP$ for some P) if and only if the linear transformations they define differ by a change of basis (that is they define the same transformation but with respect to different bases) □

Example 4.9. Let $G = C_3 = \{e, a, a^2\}$. Let V be the regular representation, that is the natural representation induced by the module $FG = \langle e, a, a^2 \rangle$. Write $B := \{e, a, a^2\}$ as a basis for FG. Then

$$[a]_B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Now let W be the permutation module where $a = (1, 2, 3)$ and C_3 is considered a subgroup of S_3. Write $B' = \{v_1, v_2, v_3\}$ for the basis of W. Then

$$[a]_{B'} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Note that these two modules are isomorphic. ◀

Example 4.10. Let $G = D_4 = \langle a, b \mid a^4 = b^2 = 1, a^b = a^{-1} \rangle$. Now we can act on either F^4 or F^8. On F^4, we have the representation described in Example 1.2. On W, we have the regular representation. Clearly are not isomorphic. ◀

4.2 Sums

We now consider how modules behave with respect to direct sums. Let V be an FG-module and suppose $V = U \oplus W$, where U and W are submodules. Let $\mathcal{B}_1 = \{u_1, ..., u_n\}$ be a basis for U and $\mathcal{B}_2 = \{w_1, ..., w_m\}$ one for W, so that $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_\in$ is a basis for V. Then

$$[g]_\mathcal{B} = \begin{pmatrix} [g]_{\mathcal{B}_1} & 0 \\ 0 & [g]_{\mathcal{B}_2} \end{pmatrix}$$

Lemma 4.11. *Let V be an FG-module such that we have the decomposition*

$$V = \bigoplus_{i=1}^{n} U_i$$

Define the projection map $\pi_i : u_1 + u_2 + ... + u_n \mapsto u_i$. Then

(I) *π_i is a homomorphism*

(II) $\pi_i \circ \pi_i = \pi_i$

Proof. Trivial □

Lemma 4.12. *Suppose we have a finite decomposition*

$$V = \sum U_i$$

where the U_i are irreducible. Then V is the direct sum of the subset of the U_i.

Proof. This follows from the fact that the intersection of two distinct irreducible modules is trivial (again, simple checking). □

We will now present an important result

Theorem 4.13 (Maschke's Theorem)**.** *Let G be a finite group, F a field of characteristic 0, V an FG-module and U a submodule. Then there exists some W such that $V = U \oplus W$.*

Proof. We first choose some W_1 such that $V = U \oplus W_1$ as vector spaces. Note that each $v \in V$ can be uniquely decomposed as $v = u + w$, where $u \in U, w \in W_1$. Now define the canonical projection $\sigma : V \to U$ where $v \mapsto u$. Clearly $\ker \sigma = W_1$ and $\operatorname{im} \sigma = U$. However, we note that σ is NOT necessarily a homomorphism of FG-modules. We modify it as follows: Define $\varphi : V \to V$ by

$$v \mapsto \frac{1}{|G|} \sum_{g \in G} v g \sigma g^{-1}$$

We claim that φ IS a homomorphism. Indeed, suppose $x \in G, v \in V$. Then

$$\begin{aligned}
(xv)\varphi &= \frac{1}{|G|} \sum_{g \in G} (vx) g \sigma g^{-1} \\
&= \frac{1}{|G|} \sum_{h \in G} v h \sigma h^{-1} x \\
&= \left(\frac{1}{|G|} \sum_{h \in G} v h \sigma h^{-1} \right) x = (v\varphi)x
\end{aligned}$$

where the equality

$$\frac{1}{|G|} \sum_{g \in G} (vx) g \sigma g^{-1} = \frac{1}{|G|} \sum_{h \in G} v h \sigma h^{-1} x$$

follows from the change of variables $h = xg$. Clearly φ maps into U, and we now check it is a projection. Indeed, supposing $u \in U$ we have

$$\begin{aligned}
(u)\varphi &= \frac{1}{|G|} \sum_{g \in G} u g \sigma g^{-1} \\
&= \frac{1}{|G|} \sum_{g \in G} u \sigma g g^{-1} \\
&= \frac{1}{|G|} \sum_{g \in G} u \\
&= u
\end{aligned}$$

as desired.

Now clearly $U = \operatorname{im}\varphi$ and we define $W := \ker\varphi$. Then for each $v \in V$, write $u := v\varphi \in U$ and $w := v - u \in W$ so that $v = u + w$. It only remains to check that this is unique. To see this, suppose $u' + w' = v = u + w$. Then

$$u' = \varphi(u') = \varphi(v) = \varphi(u) = u$$

which implies the result. $\square$