

Maths 721 Notes

2020

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1 Lecture 1

In the first half of this course we will cover three main topics:

- representations;
- modules;
- characters.

We will further see that representations and modules are essentially the same, and that modules and characters are essentially the same; and hence all three are essentially the same.

From now on G is a group.

1.1 Representations

Definition 1.1. A **representation** of a group G over a field F is a group homomorphism from G to $\mathrm{GL}(n, F)$, where n is the **degree** of the representation.

Explicitly, a representation is a function $\rho : G \rightarrow \mathrm{GL}(n, F)$ such that for all $g, h \in G$;

- (i) $(gh)\rho = (g\rho)(h\rho)$,
- (ii) $1_G\rho = I_n$,
- (iii) $g^{-1}\rho = (g\rho)^{-1}$.

Note the use of the (incredibly shit) postfix function notation.

Example 1.2. Take D_4 , the Dihedral group of order 8. It has the following group presentations

$$\begin{aligned} D_4 &= \langle a, b \mid a^4 = 1, b^2 = 1, a^b = a^{-1} \rangle \\ &\cong \langle (1\ 2\ 3\ 4), (1\ 4)(2\ 3) \rangle, \end{aligned}$$

where $a^b = bab^{-1}$ is conjugation of a by b . By defining the matrix subgroup

$$H = \left\langle A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\rangle$$

and defining $\rho : D_4 \rightarrow \mathrm{GL}(n, F)$ where $F = \mathbb{R}, \mathbb{C}$, by $a \mapsto A$ and $b \mapsto B$, and $a^i b^j \mapsto A^i B^j$ for $0 \leq i \leq 3$, and $0 \leq j \leq 1$. Hence we have ρ is a representation of D_4 over F . ◀

Example 1.3. Take \mathbb{Q}_8 the Quaternion group of order 8, which has the following group presentations

$$\begin{aligned} \mathbb{Q}_8 &= \langle a, b \mid a^4 = 1, a^2 = b^2, a^b = a^{-1} \rangle \\ &\cong \langle \bar{a} = (1\ 6\ 2\ 5)(3\ 8\ 4\ 7), \bar{b} = (1\ 4\ 2\ 3)(5\ 7\ 6\ 8) \rangle \end{aligned}$$

Define

$$H = \left\langle A = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\rangle \subset \text{GL}(2, \mathbb{C}).$$

Then $\rho : \mathbb{Q}_8 \rightarrow \text{GL}(2, \mathbb{C})$ defined by $a^k b^\ell \mapsto A^k B^\ell$ is a group representation of \mathbb{Q}_8 over \mathbb{C} of degree 2. \blacktriangleleft

Definition 1.4. Let G be a group and define

$$\begin{aligned} \rho : G &\rightarrow \text{GL}(n, F) \\ g\rho &= I_n \end{aligned}$$

for all $g \in G$. Then ρ is a representation, called the **trivial representation** of arbitrary of degree.

It follows from the trivial representation that any group G has a representation of an arbitrary degree.

Let $\rho : G \rightarrow \text{GL}(n, F)$ be a group homomorphism, and take $T \in \text{GL}(n, F)$. Then

$$(T^{-1}AT)(T^{-1}BT) = T^{-1}(AB)T.$$

Thus, given ρ define σ such that

$$g\sigma = T^{-1}(g\rho)T$$

for all $g \in G$. As for all $g, h \in G$, one has

$$\begin{aligned} (gh)\sigma &= T^{-1}((gh)\rho)T \\ &= T^{-1}(g\rho)(h\rho)T \\ &= T^{-1}(g\rho)TT^{-1}(h\rho)T \\ &= (g\sigma)(h\sigma), \end{aligned}$$

and so σ is a group homomorphism; and hence a representation.

Definition 1.5. Define

$$\rho : G \rightarrow \text{GL}(m, F), \quad \sigma : G \rightarrow \text{GL}(n, F)$$

to both be representation of G over F . We say that ρ is **equivalent to** σ if $n = m$ and there exists $T \in \text{GL}(n, F)$ such that $g\sigma = T^{-1}(g\rho)T$.

Proposition 1.6. *Equivalence of representations is an equivalence relation.*

Proof. Reflexivity is clear by taking $T = I_n$. For symmetry, take T to be its inverse. For transitivity, if

$$g\sigma = T^{-1}(g\rho)T, \quad g\rho = S^{-1}(g\eta)S,$$

then

$$g\sigma = (ST)^{-1}(g\eta)(ST).$$

□

Definition 1.7. Define the **kernel** of the representation $\rho : G \rightarrow \text{GL}(n, F)$ as $\ker \rho = \{g \in G \mid g\rho = I_n\}$.

Proposition 1.8. *The kernel of a representation of G is a normal subgroup of G ; i.e. $\ker \rho \triangleleft G$.*

Proof. Suppose $g \in \ker \rho$ and $h \in G$ is arbitrary. Then

$$(hgh^{-1})\rho = (h\rho)(g\rho)(h^{-1}\rho) = (h\rho)I_n(h\rho)^{-1} = (h\rho)(h\rho)^{-1} = I_n,$$

and so $hgh^{-1} \in \ker \rho$. As $\ker \rho$ is closed under conjugation, it is a normal subgroup of G . \square

Definition 1.9. We say ρ is a **faithful** representation of G if $\ker \rho = \{1_G\}$.

Example 1.10. For the trivial representation $\rho : G \rightarrow \text{GL}(n, F)$ with $g \mapsto I_n$ for all $g \in G$, we have $\ker \rho = G$. Hence the representation is not faithful. \blacktriangleleft

Lemma 1.11. *Suppose G is a finite group, and ρ is a representation of G over F . Then ρ is faithful if, and only if, $\text{im } \rho \cong G$.*

Proof. Immediate from the first isomorphism theorem. \square

1.2 FG-Modules

Suppose G is a group, and $F = \mathbb{R}, \mathbb{C}$. Given $\rho : G \rightarrow \text{GL}(n, F)$, with $V = F^n$, let $v = (\lambda_1, \dots, \lambda_n) \in V$ for $\lambda_i \in F$ be a row vector. Moreover, note that $g\rho$ is an $n \times n$ matrix for all $g \in G$. Thus, we have $v \cdot (g\rho) \in V$, and satisfies the following properties:

- (i) $v \cdot ((gh)\rho) = v \cdot (g\rho)(h\rho)$;
- (ii) $v \cdot (1_G\rho) = v$;
- (iii) $(\lambda v) \cdot (g\rho) = \lambda(v \cdot (g\rho))$;
- (iv) $(u + v) \cdot (g\rho) = u \cdot (g\rho) + v \cdot (g\rho)$.

We often will omitted the \cdot in the operation, and write $v(a\rho)$ for $v \cdot (a\rho)$.

Example 1.12. Recall D_4 and its given presentation from a previous example. We have

$$a\rho = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad b\rho = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

If $v = (\lambda_1, \lambda_2)$, then we have

$$v(a\rho) = (-\lambda_2, \lambda_1), \quad v(b\rho) = (\lambda_1, -\lambda_2).$$

\blacktriangleleft

Definition 1.13. Let V a vector space over the field $F = \mathbb{R}, \mathbb{C}$, and let G be a group. We say V is a FG -**module** if a multiplication $v \cdot g$ for $v \in V$, and $g \in G$ is defined such that:

- (i) $v \cdot g \in V$;
- (ii) $v \cdot (gh) = (v \cdot g) \cdot h$;
- (iii) $v \cdot 1_G = v$;
- (iv) $(\lambda v) \cdot g = \lambda(v \cdot g)$;
- (v) $(u + v) \cdot g = u \cdot g + v \cdot g$.

This generalises the previous discussion from a matrix group to an arbitrary group.

Note that properties (i), (iv), and (v) imply that the map $v \mapsto v \cdot g$ is an endomorphism of V (a linear map from V to itself).

Definition 1.14. Suppose V is an FG -module and B is a basis for V . For $g \in G$, let $[g]_B$ denote the matrix of the endomorphism $v \mapsto v \cdot g$ of V relative to the basis B .

2 Lecture 2

Theorem 2.1. Let $\rho : G \rightarrow GL(n, F)$ be a representation of G over F .

- (I) If $V = F^n$ is an FG module and G acts on V by $v \cdot g = v(g\rho)$ there exists a basis B of V such that $g\rho = [g]_B$.
- (II) The map $g \mapsto [g]_B$ is a representation for G over F .

Proof. Choose the standard basis $B = [e_1, \dots, e_n]$.

Since V is an FG -module we have $v(gh) = (vg)h$ for all $g, h \in G$ and $v \in V$. Thus $[gh]_B = [g]_B[h]_B$ so the map is a homomorphism. We now check that $[g]_B$ is invertible for all $g \in G$. We know $v \cdot 1_G = (vg)g^{-1}$ so $I_n = [g]_B[g^{-1}]_B$ and thus $[g]_B$ has an inverse. \square

Example 2.2. Recall the representation of $G = D_4$ from a previous example. Define an FG -module $V = F^2$ with the action defined by taking vg to $v(g\rho)$.

$$\begin{aligned} v_1 &= (1, 0), & v_1 a &= v_2, & v_1 b &= v_1, \\ v_2 &= (0, 1), & v_1 a &= -v_1, & v_1 b &= -v_2. \end{aligned}$$

In this basis we recover our representation

$$a \mapsto [a]_B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad b \mapsto [b]_B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

◀

We now provide an equivalent basis-dependent definition for an FG -module.

Lemma 2.3. *Let V a vector space over the field $F = \mathbb{R}, \mathbb{C}$ with basis $B = [v_1, \dots, v_n]$, and let G be a group. If a multiplication $v \cdot g$ for $v \in B$, and $g \in G$ is defined such that:*

- (i) $v \cdot g \in V$;
- (ii) $v \cdot (gh) = (v \cdot g) \cdot h$;
- (iii) $v \cdot 1_G = v$;
- (iv) $(\sum_{i=1}^n \lambda_i v_i) \cdot g = \sum_{i=1}^n \lambda_i (v_i \cdot g)$ for all $\lambda_i \in F$;

then V is an FG -module.

Definition 2.4. The trivial module of a group over F is a one dimensional vector space V over F such that $vg = v$ for all $v \in V$ and $g \in G$.

Definition 2.5. An FG -module is faithful if 1_G is the only $g \in G$ such that $vg = v$ for all $v \in V$.

Theorem 2.6. *Let V be an FG -module with basis B and ρ a representation of group G over F defined by taking $g \mapsto [g]_B$.*

- (i) *If B' is another basis of V then the map $g \mapsto [g]_{B'}$ is a representation of G equivalent to ρ .*
- (ii) *If representation σ is equivalent to ρ then there exists basis B'' such that $\sigma(g) = [g]_{B''}$ for all $g \in G$.*

Proof. Taking T to be the change of basis matrices, the two representations are equivalent. \square

Example 2.7. Let $G = C_3 = \langle a \mid a^3 = 1 \rangle$ and representation $\rho : G \rightarrow GL(n, F)$ defined by

$$a \mapsto \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}.$$

We attempt to construct an FG -module with group action described by ρ . Take $V = F^2$ with basis $B = [v_1, v_2]$. Define the action of G on V by

$$v_1 a = v_2, \quad v_2 a = -v_1 - v_2.$$

Let us now choose alternate basis $B' = [u_1 = v_1, u_2 = v_1 + v_2]$. The action of G on this basis is described by

$$u_1 a = -u_1 + u_2, \quad u_2 a = -u_1.$$

This gives us a representation

$$a \mapsto [a]_{B'} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}.$$

To verify this construction we write our change of basis matrix as

$$T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

and verify that

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}.$$

◀

Definition 2.8. The permutation module of a group $G \leq S_n$ is an n -dimensional vector space V with basis $B = [v_1, \dots, v_n]$ and action by G defined by

$$v_i g = v_{ig}$$

for all $g \in G$ where ig is the image of i under $g \in S_n$.

It follows from Cayley's theorem that every group has a faithful FG -module.

Example 2.9. Take $G = S_4$ and pick $g = (1\ 2)$ and $h = (1\ 2\ 3\ 4)$. We have representations

$$[g]_B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad [h]_B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

◀

2.1 Module Reducibility

Definition 2.10. Let V be an FG -module. We call W a submodule of V if W is a vector subspace of V and W is closed under the action of G . We then write $W < V$.

Example 2.11. Let $G = C_3 = \langle (1\ 2\ 3) \rangle$ and V the permutation module of G with basis $B = [v_1, v_2, v_3]$. The subspace $W = \langle v_1 + v_2 + v_3 \rangle$ is a submodule but the subspace $U = \langle v_1 + v_2 \rangle$ is not.

For example, consider the action of $g = (1\ 2\ 3)$ on $v_1 + v_2 \in U$.

$$(v_1 + v_2)g = v_{1g} + v_{2g} = v_2 + v_3 \notin U$$

whereas G acts on W trivially.

◀

3 Lecture 3

3.1 Module and Representation Reducibility

For any module, it is clear that we have two trivial submodules: $0 < V$ and $V < V$. Where $0 = \{0\} \subset V$.

Definition 3.1. Let V be an FG -module. We say that V is irreducible if the only submodules of V are V and 0 . Otherwise V is reducible

In 2.11 we showed that the permutation module of C_3 is reducible.

Definition 3.2. Let $\rho : G \rightarrow GL(n, F)$ be a representation. We say that ρ is irreducible if the corresponding FG -module (as constructed in 2.1) is irreducible. Otherwise ρ is reducible.

If an FG -module, V is reducible, that is, $0 < W < V$, $0 \neq W \neq V$. Let B_W be a basis for W . If we extend B_W to B a basis of V , then we get the following representation of G :

$$g \mapsto [g]_B = \begin{bmatrix} X_g & 0 \\ Y_g & Z_g \end{bmatrix} \quad (3.1)$$

Where the matrices X_g, Y_g and Z_g are some block matrices and 0 is a block of zeros and X_g has the dimensions $m \times m$ and, in this case, $\dim(W) = m$.

Proposition 3.3. A representation $\rho : G \rightarrow GL(n, F)$ is reducible if and only if with respect to some basis, B , of F^n , $[g]_B$ has the form 3.1 for some $0 < m < \dim(V)$ for all $g \in G$. Then the maps $g \mapsto X_g$ and $g \mapsto Z_g$ are both representations of G .

Proof. Suppose we have a presentation, $\rho : G \rightarrow GL(n, F)$ and a basis B of $V = F^n$ such that $[g]_B$ has the form 3.1 for every $g \in G$. Then consider the subspace $0 \subset W \subset V$ spanned by the first m elements of B . It is clear that $v[g]_B \in W$ for all $v \in W$. Therefore the module induced by ρ is reducible, so ρ is reducible. Now, if we have a reducible representation, then the argument above this proposition shows that with respect to any basis extending B_W , the matrices $[g]_B$ have the required form.

Now, using elementary block matrix multiplication, we get the following for $g, h \in G$:

$$\rho(g)\rho(h) = [g]_B[h]_B = \begin{bmatrix} X_g X_h & 0 \\ Y_g X_h + Z_g Y_h & Z_g Z_h \end{bmatrix} = [gh]_B = \rho(gh)$$

Therefore $X_{gh} = X_g X_h$ and $Z_{gh} = Z_g Z_h$, so the maps $g \mapsto X_g$ and $g \mapsto Z_g$ are both representations of G . \square

Problem 1. Prove that the example representation of D_8 of degree 2 over \mathbb{R} or \mathbb{C} is irreducible.

3.2 Group Algebras

Recall that an algebra over a field F is a vector space over F equipped with a bilinear product $A \times A \rightarrow A$ that is compatible with scalar multiplication.

Definition 3.4. The group algebra over a finite group G over a field F is an algebra¹ of dimension $n = |G|$ over $F = \mathbb{R}$ or \mathbb{C} called FG , with basis $B = G = \{g_1, \dots, g_n\}$.

¹See Lemma 3.6

Where the algebra structure is given by the following for two arbitrary elements of FG , $u = \sum_{g \in G} \lambda_g g$, $v = \sum_{g \in G} \mu_g g$, $\lambda_g, \mu_g \in F$ and $\nu \in F$:

- (i) $u + v = \sum_{i=1}^n (\lambda_i + \mu_i) g_i$
- (ii) $\nu \cdot u = \sum_{i=1}^n (\nu \lambda_i) g_i$
- (iii) $u \cdot v = \sum_{(h,g) \in G \times G} \lambda_g \mu_h (gh)$

This is clearly a vector space.

Example 3.5. Consider $G = C_3 = \{e, a, a^2\} = \langle a | a^3 = e \rangle$ and $F = \mathbb{R}$ or \mathbb{C} . Then if we let $u = e - a + 2a^2$, $v = \frac{1}{2}e + 5a$, then:

$$u + v = \frac{3}{2}e + 4a + 2a^2, \quad \frac{1}{3}u = \frac{1}{3}e - \frac{1}{3}a + \frac{2}{3}a^2, \quad uv = \frac{21}{2}e + \frac{9}{2}a - 4a^2$$

◀

Lemma 3.6. Given a group algebra FG , $r, s, t \in FG$, $\lambda \in F$:

- (I) $rs \in FG$
- (II) $(rs)t = r(st)$
- (III) $1_G r = r 1_G = r$
- (IV) $(\lambda r)s = \lambda(rs)$
- (V) $(r + s)t = rt + st$
- (VI) $r(s + t) = rs + rt$
- (VII) $r0 = 0r = 0$

That is, FG is an associative algebra with unit

Proof. 1,3 and 7 are clear from the definition of FG , 4,5 and 6 follow from the distributive and associative laws of F and 2 follows from associativity in G . \square

3.3 The Regular FG -module, FG

Problem 2. $V = FG$ is an FG -module with the group action defined by $v \cdot g = vg$ for $v \in FG$, $g \in G \subset FG$.

Definition 3.7. For a finite group G and $F = \mathbb{R}$ or \mathbb{C} , the regular FG -module is FG . The associated module, $g \mapsto [g]_B$ is called the regular representation.

Lemma 3.8. FG is a faithful module for G over F

Proof. If $vg = v$ for all $v \in FG$, then specifically, $hg = h$ for all $h \in G$, so $g = 1_G$. \square

Example 3.9. For $C = C_3$, over the basis $B = G$, we get:

$$[e]_B = I_3, \quad [a]_B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad [a^2]_B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$



Now, if we have an FG -module, V , then FG acts on V in the following way:

$$v \cdot r = v \cdot \left(\sum_{g \in G} \mu_g g \right) = \sum_{g \in G} \mu_g (v \cdot g)$$

Lemma 3.10. For $u, v \in V$, $\lambda \in F$, $r, s \in FG$:

- (I) $vr \in FG$
- (II) $(vr)s = v(rs)$
- (III) $v1 = v$
- (IV) $(\lambda v)r = \lambda(vr) = v(\lambda r)$
- (V) $v(r + s) = vr + vs$
- (VI) $(u + v)r = ur + vr$
- (VII) $r0 = v0 = 0$

Proof. I, III and the first part of VII follow from V being an FG -module, the second equality of VII follows from scalar multiplication by 0 in V . The following calculation:

$$\begin{aligned} (\lambda v)r &= \sum_{g \in G} \mu_g ((\lambda v)g) \\ &= \sum_{g \in G} \mu_g (\lambda(vg)) \\ &= \sum_{g \in G} (\lambda \mu_g)(vg) = v(\lambda r) \\ &= \lambda \sum_{g \in G} \mu_g (vg) \\ &= \lambda(vg) \end{aligned}$$

proves IV. VI follows from the linearity of the action of G on V . V follows from distributivity of scalar multiplication in V . Finally, to prove II:

$$\begin{aligned}
v(rs) &= \sum_{(g,h) \in G \times G} (\mu_g \lambda_h(v(gh))) \\
&= \sum_{h \in G} \lambda_h \sum_{g \in G} \mu_g(gv)h \\
&= \sum_{h \in G} \lambda_h \left(\sum_{g \in G} \mu_g(gv) \right) h \\
&= \sum_{h \in G} \lambda_h(vr)h = (vr)s
\end{aligned}$$

□

4 Lecture 4

4.1 Homomorphisms

Definition 4.1. Let V and W be FG -modules. A *homomorphism* of FG -modules is a map $\sigma : V \rightarrow W$ which is a linear transformation and also satisfies $(vg)\sigma = (v\sigma)g$ for all $g \in G, v \in V$. The *kernel* and *image* are defined in the obvious way

Equivalently, it is a homomorphism of modules over the ring FG . Indeed:

Problem 3. Suppose $r \in FG$ is an element of the group algebra. Prove that $(vr)\sigma = (v\sigma)r$.

Lemma 4.2. Let $\sigma : V \rightarrow W$ be a homomorphism of FG -algebras. Then the kernel and image of σ are submodules

Proof. This is a matter of simple checking, which will be left to the reader. □

Example 4.3. Take $\sigma : V \rightarrow V$ to be $v \mapsto \lambda v$ for some $\lambda \in F^*$. Then $\ker \sigma = 0, \text{im } \sigma = V$. ◀

Example 4.4. Let $G = S_n$ and $V = \langle v_1, \dots, v_n \rangle$ be the permutation module for G over F , and let $W = \langle w \rangle$ be the trivial module. Now define $\sigma : V \rightarrow W$ by

$$\sum \lambda_i v_i \mapsto \sum \lambda_i w$$

Then $\ker \sigma = \{ \sum \lambda_i v_i \mid \sum \lambda_i = 0 \}$ and $\text{im } \sigma = W$. ◀

Definition 4.5. A homomorphism of FG -modules is an *isomorphism* if it is bijective

Remark 1. In class we originally said "if the homomorphism has trivial kernel". However, this is definitely not correct because inclusions are always homomorphisms, but obviously not isomorphisms. \blacklozenge

Lemma 4.6. *The inverse of an isomorphism is an isomorphism*

Proof. Once again, this is just an exercise in checking. The details will be left for the reader. \square

Some rather obvious invariants of FG -modules (under isomorphism) are dimension and irreducibility.

Lemma 4.7. *V and W are isomorphic if and only if there exists bases \mathcal{B}_1 of V and \mathcal{B}_2 of W such that*

$$[g]_{\mathcal{B}_1} = [g]_{\mathcal{B}_2}$$

for all g .

Proof. Suppose firstly that V and W are isomorphic, and let $\sigma : V \rightarrow W$ be one such isomorphism. Let $\mathcal{B}_1 = \{v_1, \dots, v_n\}$ be a basis for V . In particular, it is linearly independent, and it is easy to see that $\mathcal{B}_2 = \{v_1\sigma, \dots, v_n\sigma\}$ is also linearly independent. Since V and W are isomorphic, they have the same dimension, and thus \mathcal{B}_2 is a basis for W . Since $(vg)\sigma = (v\sigma)g$ for all g and v , the action of g on the basis vectors of both bases are the same, and thus we conclude $[g]_{\mathcal{B}_1} = [g]_{\mathcal{B}_2}$.

Conversely, suppose that the latter hypothesis is satisfied. Let $\{v_1, \dots, v_n\}$ be a basis for V and $\{w_1, \dots, w_n\}$ be a basis for W . We define a bijective linear map $\sigma : V \rightarrow W$ such that $v_i\sigma = w_i$ for each i . Now observe that for each i , we have $v_i g = \lambda_1 v_1 + \dots + \lambda_n v_n$ and $w_i g = \lambda_1 w_1 + \dots + \lambda_n w_n$, where $(\lambda_1, \dots, \lambda_n)$ is the i -th row of $[g]$. This means that

$$(v_i g)\sigma = (\lambda_1 v_1 + \dots + \lambda_n v_n)\sigma = \lambda_1 v_1\sigma + \dots + \lambda_n v_n\sigma = \lambda_1 w_1 + \dots + \lambda_n w_n = w_i g = (v_i\sigma)g$$

and thus σ is a homomorphism of FG -modules. Since it is bijective, it is an isomorphism. \square

Theorem 4.8. *Let V be an FG -module with basis \mathcal{B}_1 and W an FG -module with basis \mathcal{B}_2 . Then $W \cong V$ if and only if $g \mapsto [g]_{\mathcal{B}_1}$ and $g \mapsto [g]_{\mathcal{B}_2}$ are equivalent.*

Proof. This follows from the previous Lemma and the fact that two matrices are conjugate (A and B are conjugate if $A = P^{-1}BP$ for some P) if and only if the linear transformations they define differ by a change of basis (that is they define the same transformation but with respect to different bases) \square

Example 4.9. Let $G = C_3 = \{e, a, a^2\}$. Let V be the regular representation, that is the natural representation induced by the module $FG = \langle e, a, a^2 \rangle$. Write $B := \{e, a, a^2\}$ as a basis for FG . Then

$$[a]_B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Now let W be the permutation module where $a = (1, 2, 3)$ and C_3 is considered a subgroup of S_3 . Write $B' = \{v_1, v_2, v_3\}$ for the basis of W . Then

$$[a]_{B'} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Note that these two modules are isomorphic. ◀

Example 4.10. Let $G = D_4 = \langle a, b \mid a^4 = b^2 = 1, a^b = a^{-1} \rangle$. Now we can act on either F^4 or F^8 . On F^4 , we have the representation described in Example 1.2. On W , we have the regular representation. Clearly are not isomorphic. ◀

4.2 Sums

We now consider how modules behave with respect to direct sums. Let V be an FG -module and suppose $V = U \oplus W$, where U and W are submodules. Let $\mathcal{B}_1 = \{u_1, \dots, u_n\}$ be a basis for U and $\mathcal{B}_2 = \{w_1, \dots, w_m\}$ one for W , so that $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ is a basis for V . Then

$$[g]_{\mathcal{B}} = \begin{pmatrix} [g]_{\mathcal{B}_1} & 0 \\ 0 & [g]_{\mathcal{B}_2} \end{pmatrix}$$

Lemma 4.11. *Let V be an FG -module such that we have the decomposition*

$$V = \bigoplus_{i=1}^n U_i$$

Define the projection map $\pi_i : u_1 + u_2 + \dots + u_n \mapsto u_i$. Then

- (I) π_i is a homomorphism
- (II) $\pi_i \circ \pi_i = \pi_i$

Proof. Trivial □

Lemma 4.12. *Suppose we have a finite decomposition*

$$V = \sum U_i$$

where the U_i are irreducible. Then V is the direct sum of some subset of the U_i .

Proof. This follows from the fact that the intersection of two distinct irreducible modules is trivial (again, simple checking). \square

We will now present an important result

Theorem 4.13 (Maschke's Theorem). *Let G be a finite group, F a field of characteristic 0, V an FG -module and U a submodule. Then there exists some W such that $V = U \oplus W$.*

Proof. We first choose some W_1 such that $V = U \oplus W_1$ as vector spaces. Note that each $v \in V$ can be uniquely decomposed as $v = u + w$, where $u \in U, w \in W_1$. Now define the canonical projection $\sigma : V \rightarrow U$ where $v \mapsto u$. Clearly $\ker \sigma = W_1$ and $\text{im } \sigma = U$. However, we note that σ is NOT necessarily a homomorphism of FG -modules. We modify it as follows: Define $\varphi : V \rightarrow V$ by

$$v \mapsto \frac{1}{|G|} \sum_{g \in G} v g \sigma g^{-1}$$

We claim that φ IS a homomorphism. Indeed, suppose $x \in G, v \in V$. Then

$$\begin{aligned} (xv)\varphi &= \frac{1}{|G|} \sum_{g \in G} (vx) g \sigma g^{-1} \\ &= \frac{1}{|G|} \sum_{h \in G} v h \sigma h^{-1} x \\ &= \left(\frac{1}{|G|} \sum_{h \in G} v h \sigma h^{-1} \right) x = (v\varphi)x \end{aligned}$$

where the equality

$$\frac{1}{|G|} \sum_{g \in G} (vx) g \sigma g^{-1} = \frac{1}{|G|} \sum_{h \in G} v h \sigma h^{-1} x$$

follows from the change of variables $h = xg$. Clearly φ maps into U , and we now check it is a projection. Indeed, supposing $u \in U$ we have

$$\begin{aligned} (u)\varphi &= \frac{1}{|G|} \sum_{g \in G} u g \sigma g^{-1} \\ &= \frac{1}{|G|} \sum_{g \in G} u \sigma g g^{-1} \\ &= \frac{1}{|G|} \sum_{g \in G} u \\ &= u \end{aligned}$$

as desired.

Now clearly $U = \text{im } \varphi$ and we define $W := \ker \varphi$. Then for each $v \in V$, write $u := v\varphi \in U$ and $w := v - u \in W$ so that $v = u + w$. It only remains to check that this is unique. To see this, suppose $u' + w' = v = u + w$. Then

$$u' = \varphi(u') = \varphi(v) = \varphi(u) = u$$

which implies the result. \square

5 Lecture 5

We begin with consequences of Maschke's theorem.

Example 5.1. Let $G = S_3$, and $V = \langle v_1, v_2, v_3 \rangle$ is the permutation module. Let U be the submodule of V defined as $U = \langle v_1 + v_2 + v_3 \rangle < V$. Suppose $W_0 = \langle v_1, v_2 \rangle$, so that $V = U \oplus W_0$ as subspaces. Define a projection $\phi : V \rightarrow U$ by $v_1 \mapsto 0$, $v_2 \mapsto 0$, and $v_3 \mapsto v_1 + v_2 + v_3$. Further, define $\theta : V \rightarrow V$, as in proof of Maschke's theorem, so that

$$v\theta = \frac{1}{|G|} \sum_{g \in G} vg\phi g^{-1} = \frac{1}{6} \sum_{g \in S_3} vg\phi g^{-1}.$$

Consider the action of θ on the basis elements v_i , then a short computation shows that

$$v_i\theta = \frac{1}{3}(v_1 + v_2 + v_3), \quad i = 1, 2, 3.$$

Moreover, we have

$$\ker \theta = \{v \in V : v\theta = 0\} = \left\{ \sum_{i=1}^3 \lambda_i v_i : \sum_{i=1}^3 \lambda_i = 0 \right\}.$$

Hence $V = U \oplus \ker \theta$, is a direct summand of submodules. Moreover, if $\mathcal{B} = [v_1 + v_2 + v_3, v_1, v_2]$ is a basis for V , and $\mathcal{B}' = [v_1 + v_2 + v_3, v_1 - v_2, v_2 - v_3]$ is another, one has

$$[g]_{\mathcal{B}} = \begin{pmatrix} * & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix}, \quad [g]_{\mathcal{B}'} = \begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}.$$

◀

In fact, it follows from Maschke's theorem that if we choose a basis \mathcal{B} for V such that

$$[g]_{\mathcal{B}} = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix},$$

then there exists a basis \mathcal{B}' for V such that

$$[g]_{\mathcal{B}'} = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}.$$

Definition 5.2. Let V be a FG -module, V is said to be **completely reducible** if $V = U_1 \oplus \cdots \oplus U_r$ with each U_i an irreducible FG -module.

Theorem 5.3. Suppose G is a finite group, and $F = \mathbb{R}, \mathbb{C}$. Then every FG -module is completely reducible

Proof. Induction using Maschke's theorem. \square

Lemma 5.4. Suppose G is a finite group, $F = \mathbb{R}, \mathbb{C}$, and V a FG -module. If U is a FG -submodule, then there exists a surjective FG -homomorphism from V onto U .

Proof. By Maschke's theorem, there exists a complementary submodule W to U such that $V = U \oplus W$. Thus, defining $\pi : V \rightarrow U$ by $u + w \mapsto u$ gives the result. \square

Example 5.5. Take

$$G = \left\langle \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} \mid n \in \mathbb{Z} \right\rangle$$

and $V = \mathbb{C}^2$. Then V is not completely reducible. \blacktriangleleft

Example 5.6. Let $G = C_p = \langle a \mid a^p = 1 \rangle$ where p is prime, and take the representation

$$a^j \mapsto \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix}, \quad 0 \leq j \leq p-1$$

over the finite field $F = \mathbb{Z}_p$. If $V = \langle v_1, v_2 \rangle$ is the FG -module of the representation and $U = \langle v_2 \rangle$, then there does not exist a submodule $W < V$ such that $V = U \oplus W$. \blacktriangleleft

The previous two examples show that the assumptions of Maschke's theorem are required, and cannot be relaxed.

5.1 Schur's Lemma

Theorem 5.7 (Schur's lemma). Suppose V and W are irreducible $\mathbb{C}G$ -modules.

- (I) If $\theta : V \rightarrow W$ is a $\mathbb{C}G$ -homomorphism, then θ is either a $\mathbb{C}G$ -isomorphism or the zero homomorphism.
- (II) If $\theta : V \rightarrow V$ is a $\mathbb{C}G$ -isomorphism, then θ is scalar multiple of the identity endomorphism of V .

Proof. (I): Suppose that $v\theta \neq 0$ for some $v \in V$, then the image of θ is not trivial, $\text{im } \theta \neq \{0\}$. However, $\text{im } \theta$ is a submodule of W , and so the irreducibility of W forces $\text{im } \theta = W$. Likewise, as the kernel of θ is a submodule of V , but not all of V , we have $\ker \theta = \{0\}$ as V is irreducible. Therefore, θ is a bijective $\mathbb{C}G$ -homomorphism, and so it is a $\mathbb{C}G$ -isomorphism.

(II): Suppose θ is a $\mathbb{C}G$ -isomorphism. Then as \mathbb{C} is algebraically closed, θ has an eigenvalue $\lambda_v \in \mathbb{C}$ with $v\theta = \lambda_v v$ for some $v \in V$. Now, as $\ker(\theta - \lambda_v 1_V) \neq \{0\}$ is a submodule of V , V being irreducible implies that $\ker(\theta - \lambda_v 1_V) = V$. Therefore, $w(\theta - \lambda_v 1_V) = 0$ for all $w \in V$, and so $\theta = \lambda 1_V$ as required. \square

Further, we actually have a converse to the second statement of Schur's lemma.

Proposition 5.8. *Let V be a nontrivial $\mathbb{C}G$ -module, and suppose that every $\mathbb{C}G$ -homomorphism from V to V is a scalar multiple of the identity endomorphism of V . Then V is irreducible.*

Proof. Suppose V is reducible. Then there exists a nontrivial submodule U of V such that, by Maschke's theorem, $V = U \oplus W$ with W also a submodule of V . Defining $\pi : V \rightarrow V$ by $u + w \mapsto u$ gives a $\mathbb{C}G$ -homomorphism that is not a multiple of the identity endomorphism. A contradiction. \square

We now interpret Schur's lemma as representation statement.

Lemma 5.9. *Let $\rho : G \rightarrow \mathrm{GL}(n, \mathbb{C})$ be a representation. Then ρ is irreducible if, and only if, every $n \times n$ matrix A which satisfies $(g\rho)A = A(g\rho)$ for all $g \in G$, has the form $A = \lambda I_n$.*

Proof. Result follows from Schur's lemma and its partial converse. \square

Example 5.10. Suppose $G = C_3 = \langle a \mid a^3 = 1 \rangle$, and $\rho : G \rightarrow \mathrm{GL}(2, \mathbb{C})$ is the representation defined by

$$a\rho = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

Let

$$A = a\rho = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

Then it is clear that $A(g\rho) = (g\rho)A$ for all $g \in G$. As A is not a scalar multiple of the identity, the representation is reducible. \blacktriangleleft

Example 5.11. Suppose $G = D_5 = \langle a, b \mid a^5 = 1, b^2 = 1, a^b = a^{-1} \rangle$. Set $\omega = e^{2\pi i/5}$. Let $\rho : G \rightarrow \mathrm{GL}(2, \mathbb{C})$ be the representation defined by

$$a \mapsto \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

If

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

commutes with $a\rho$ and $b\rho$; one calculates that

$$A = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} = \alpha I_n,$$

and so the representation is irreducible. ◀

Lemma 5.12. *Let G be a finite abelian group. Then every irreducible $\mathbb{C}G$ -module has dimension 1.*

Proof. Choose $x \in G$, then $v(gx) = v(xg)$ for all $g \in G$, and so $v \mapsto vx$ is a $\mathbb{C}G$ -homomorphism. It is actually an isomorphism with inverse $v \mapsto vx^{-1}$. Hence by Schur this isomorphism is a scalar multiple of the identity, say $\lambda_x 1_V$. Thus $vx = \lambda_x 1_V$ for all $v \in V$ and the group action by G is just usual scalar multiplication. This means that every subspace is a submodule. However, V is irreducible and so has no non-trivial submodules; which forces $\dim V = 1$. ◻

6 Lecture 6

Continuing our discussion of the representations of abelian groups, we provide a stronger theorem in which we construct these 1-dimensional representations by mapping group elements to roots of unity. But first recall the Fundamental Theorem of Abelian Groups which states that any finite abelian group G is isomorphic to the direct sum of cyclic groups

$$G \simeq C_{n_1} \oplus \cdots \oplus C_{n_r}$$

where $n_i \mid n_{i+1}$ for all $1 \leq i \leq r-1$. Note that if $C_i = \langle g_i \rangle$ then we can write $G = \langle g_1, \dots, g_r \rangle$ and g_i has order n_i .

We define a homomorphism ρ from G to \mathbb{C} by taking $g_i \mapsto \lambda_i$ where λ_i is the n_i -th root of unity. This defines a representation and is specified by roots of unity $\lambda_1, \dots, \lambda_r$. Thus for the representation ρ defined by roots of unity $\lambda_1, \dots, \lambda_r$ we write $\rho = \rho_{\lambda_1, \dots, \lambda_r}$.

Theorem 6.1. *Suppose $G \simeq C_{n_1} \times \cdots \times C_{n_r}$ for cyclic groups C_{n_i} of order n_i . The representation $\rho_{\lambda_1, \dots, \lambda_r}$ of G is irreducible of degree 1. There are $|G|$ many such representations and every irreducible representation of G over \mathbb{C} is equivalent to one of these.*

Example 6.2. Take $G = \langle a \mid a^n = 1 \rangle$ and $\omega = e^{2\pi i/n}$. The irreducible representations of G are ρ_{ω^j} for $0 \leq j \leq n-1$ defined by

$$a^k \rho_{\omega^j} = \omega^{jk}, \quad 0 \leq k \leq n-1.$$

◀

6.1 Application of Schur's to $\mathbb{C}G$

Definition 6.3. If G is a finite group then we define the center of the group algebra $Z(\mathbb{C}G)$ by

$$Z(\mathbb{C}G) = \{z \in \mathbb{C}G \mid zr = rz \quad \forall r \in \mathbb{C}G\}.$$

We now state some simple properties of the center:

1. $Z(\mathbb{C}G)$ is a subspace of $\mathbb{C}G$.
2. If G is abelian then $Z(\mathbb{C}G) = \mathbb{C}Z(G)$.
3. If H is a normal subgroup of G then $\sum_{h \in H} h \in Z(\mathbb{C}G)$.

We provide a simple proof for the last statement.

Proof. Take $z = \sum_{h \in H} h$ and $g \in G$. We then have

$$z^g = \sum_{h \in H} h^g = z.$$

because H is normal and hence fixed by conjugation of elements in G . □

Example 6.4. Take $G = S_3 = \langle a = (1\ 2\ 3), b = (1\ 2) \rangle$. Then $\sum_{g \in G} g \in Z(\mathbb{C}G)$. ◀

Lemma 6.5. Let V be an irreducible $\mathbb{C}G$ -module and let $z \in Z(\mathbb{C}G)$. There exists $\lambda \in \mathbb{C}$ such that $vz = \lambda v$ for all $v \in V$.

Proof. For all $r \in \mathbb{C}G$ and $v \in V$ we know $vrz = vZR$. Thus the mapping $v \mapsto vz$ is a $\mathbb{C}G$ -homomorphism from V to V . The result then follows by Schur. □

Remark 2. Note that $\mathbb{C}G$ is a faithful module. ♦

Lemma 6.6. If there exists a faithful, irreducible $\mathbb{C}G$ -module then $Z(G)$ is cyclic.

Proof. Let V be an irreducible, faithful $\mathbb{C}G$ -module and take $z \in Z(G) \subset Z(\mathbb{C}G)$. There exists $\lambda_z \in \mathbb{C}$ such that $vz = \lambda_z v$ for all $v \in V$. But since V is faithful the mapping $z \mapsto \lambda_z$ is injective from $Z(G)$ to \mathbb{C}^\times and so $Z(G) \simeq \{\lambda_z \mid z \in Z(G)\}$ is a finite subgroup of \mathbb{C}^\times . All of which are cyclic. □

Example 6.7. \mathbb{Z}_4 has a faithful, irreducible representation by taking 1 to the 4-th root of unity. However $\mathbb{Z}_2 \times \mathbb{Z}_2$ has no faithful irreducible representation as its center is not cyclic. ◀

Remark 3. Note that the converse is not true in general. There exists groups with cyclic centers but no faithful, irreducible representations. See Frobenius Groups. ♦

Lemma 6.8. Suppose G finite and every irreducible $\mathbb{C}G$ -module has dimension 1. Then G is abelian.

Proof. Since $\mathbb{C}G$ is a $\mathbb{C}G$ -module and it is completely reducible, we can decompose it as follows

$$\mathbb{C}G = \bigoplus_{i=1}^{|G|} V_i$$

where each V_i is one-dimensional. For any $v_i \in V_i$ and $x, y \in G$, note that

$$v_i xy = \lambda_x v_i y = \lambda_y \lambda_x v_i = \lambda_x \lambda_y v_i = v_i yx$$

Since the v_i form a basis for $\mathbb{C}G$ this means $vxy = v yx$ for all $v \in \mathbb{C}G$ and $x, y \in G$. Since $\mathbb{C}G$ is faithful, the result follows. \square

6.2 The Group Algebra and Irreducible Modules

Lemma 6.9. *Let V, W be $\mathbb{C}G$ -modules and $\theta : V \rightarrow W$ is a $\mathbb{C}G$ -homomorphism. There exists a submodule $U < V$ such that $V = \ker \theta \oplus U$ and $U \simeq \text{im } \theta$*

Proof. We apply Maschke's theorem to $\ker \theta < V$ and obtain submodule $U < V$ such that $V = \ker \theta \oplus U$. Now define the map $\bar{\theta} : U \rightarrow \text{im } \theta$ by $u \mapsto u\theta$ which is a homomorphism because it is the restriction of θ to a subspace. The kernel of $\bar{\theta}$ is trivial as $\ker \bar{\theta} = \ker \theta \cap U = \{0\}$. And $\text{im } \theta = \text{im } \bar{\theta}$ because if $w \in \text{im } \theta$ then $w = v\theta$ for some $v \in V$ and $v = k + u$ for some $u \in U$ and $k \in \ker \theta$. We then have

$$w = v\theta = (u + k)\theta = u\theta = u\bar{\theta} \in \text{im } \bar{\theta}.$$

Finally, by the first isomorphism theorem we obtain

$$U \simeq U / \ker \bar{\theta} \simeq \text{im } \bar{\theta} = \text{im } \theta.$$

\square

7 Lecture 7

7.1 The Group Algebra and Irreducible Modules: Part II

Definition 7.1. For V , a $\mathbb{C}G$ -module, U an irreducible $\mathbb{C}G$ -module, U is called a composition factor for V if V has a $\mathbb{C}G$ -submodule isomorphic to U .

Lemma 7.2. *Let V be a $\mathbb{C}G$ -module, suppose we have a finite decomposition into into a direct sum of irreducible $\mathbb{C}G$ -modules:*

$$V = \bigoplus_{i=1}^n U_i$$

Then, if $0 < U < V$ is an irreducible, $\mathbb{C}G$ -submodule of V , then $U \simeq U_i$ for some $1 \leq i \leq n$.

Proof. Consider the maps $\pi_i : U \rightarrow U_i$, since each $\pi_i : V \rightarrow U_i$ is a $\mathbb{C}G$ -homomorphism, and since U is non-trivial, it is clear that not all the π_i are the zero map, let π_j be such a projection, then by Schur, $\pi_j : U \rightarrow U_j$ is a $\mathbb{C}G$ -isomorphism. \square

Theorem 7.3. *Let the following be a decomposition of the regular representation into irreducible $\mathbb{C}G$ -modules:*

$$\mathbb{C}G = \bigoplus_{i=1}^n U_i$$

Then every irreducible $\mathbb{C}G$ -module is isomorphic to U_i for some $1 \leq i \leq n$

Proof. Let W be an irreducible $\mathbb{C}G$ -module, let $0 \neq w \in W$, then $wG = \{wr : r \in \mathbb{C}G\}$ is a $\mathbb{C}G$ -submodule of W . However, W is irreducible, so $W = wG$. Define $\theta : \mathbb{C}G \rightarrow W$ by $r\theta = wr$ for $r \in \mathbb{C}G$, then θ is a linear $\mathbb{C}G$ -homomorphism by 3.10 and $\text{im } \theta = W$. So $\mathbb{C}G = \ker \theta \oplus U$, where $U \simeq \text{im } \theta \simeq W$ by 6.9. Now W is irreducible, so U is irreducible, so by 7.2, $U \simeq U_i \simeq W$ for some $1 \leq i \leq n$. \square

Now for a finite group G , we can characterise all irreducible $\mathbb{C}G$ -modules by decomposing $\mathbb{C}G$.

Example 7.4. Let $G = C_3 = \langle a | a^3 = 1 \rangle$, then let $\omega = e^{\frac{2\pi i}{3}}$ be a primitive 3rd root of unity and let the following be elements of $\mathbb{C}G$:

$$v_1 = 1 + a + a^2, \quad v_2 = 1 + \omega^2 a + \omega a^2, \quad v_3 = 1 + \omega a + \omega^2 a^2$$

We have $(v_1)a = v_1$, $(v_2)a = \omega + a + \omega^2 a^2 = \omega v_2$ and $(v_3)a = \omega^2 + a + \omega a = \omega^2 v_3$. Therefore, $V_i = \langle v_i \rangle < \mathbb{C}G$ for $i = 1, 2, 3$ are all irreducible submodules of $\mathbb{C}G$.

Furthermore, it is clear that $\{v_1, v_2, v_3\}$ forms a basis for $\mathbb{C}G$, hence $\mathbb{C}G = V_1 \oplus V_2 \oplus V_3$. \blacktriangleleft

Example 7.5. Let $G = D_6 = \langle a, b | a^3 = 1, a^b = a^{-1} \rangle$ be the dihedral group of order 6, and let $\omega = e^{2\pi i/3}$ be a third root of unity. Define $v_i \in \mathbb{C}G$ by $v_i a = \omega^i v_i$, then

$$\begin{aligned} v_0 &= 1 + a + a^2, \\ v_1 &= 1 + \omega^2 a + \omega a^2, \\ v_2 &= 1 + \omega a + \omega^2 a^2. \end{aligned}$$

Further define $u_i = bv_i$ for all $i = 0, 1, 2$. Then it follows that $\langle v_i \rangle$ and $\langle u_i \rangle$ are $\mathbb{C}\langle a \rangle$ -modules for all i . Further, a short computation shows that

$$\langle v_0, u_0 \rangle, \quad \langle v_1, u_2 \rangle, \quad \langle v_2, u_1 \rangle$$

are $\mathbb{C}\langle b \rangle$ -modules. Hence, we see that

$$\begin{aligned} U_1 &= \langle v_0 + u_0 \rangle, \\ U_2 &= \langle v_0 - u_0 \rangle, \\ U_3 &= \langle v_1, u_2 \rangle, \\ U_4 &= \langle v_2, u_1 \rangle, \end{aligned}$$

are irreducible $\mathbb{C}G$ -submodules. It is clear that $U_3 \simeq U_4$ via the map that sends $v_1 \mapsto u_1$ and $u_2 \mapsto v_2$. Finding the representations of each U_i we have

$$\begin{aligned}\rho_1 : a &\mapsto (1), & b &\mapsto (1), \\ \rho_2 : a &\mapsto (1), & b &\mapsto (-1), \\ \rho_3 : a &\mapsto \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}, & b &\mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.\end{aligned}$$

Hence U_1 , U_2 , and U_3 are non-isomorphic irreducible $\mathbb{C}G$ -submodules. Moreover, we have

$$\mathbb{C}G = \underbrace{U_1}_{\text{trivial}} \oplus U_2 \oplus \underbrace{U_3 \oplus U_4}_{\text{isomorphic}}.$$

◀

7.2 The Vector Space of $\mathbb{C}G$ -homomorphisms

Definition 7.6. Let V, W be $\mathbb{C}G$ -modules, then define $H = \text{Hom}_{\mathbb{C}G}(V, W) = \{\theta : V \rightarrow W : \theta \text{ is a } \mathbb{C}G \text{ homomorphism}\}$.

We can then define addition and scalar multiplication operations on this set for $\theta, \phi \in H$, $\lambda \in \mathbb{C}$ as follows:

$$v(\theta + \phi) := v\theta + v\phi, \quad v(\lambda\theta) = \lambda(v\theta)$$

Lemma 7.7. *The space H with the operations defined above is a vector space over \mathbb{C} .*

Proof. Clear from the definition, note that the zero homomorphism is: $v0 = 0_W$. □

Lemma 7.8. *For irreducible $\mathbb{C}G$ -modules, V, W ,*

$$\dim(H) = \begin{cases} 1 & V \simeq W \\ 0 & V \not\simeq W \end{cases}$$

Proof. If $V \not\simeq W$, then it follows that $\theta = 0$ for all $\theta \in H$ by Schur.

Now, suppose $V \simeq W$ and let $\theta : V \rightarrow W$ be an isomorphism. Now, let $\phi \in H$, we have that $\phi\theta^{-1} : V \rightarrow V$ is a homomorphism, so by Schur, $\phi\theta^{-1} = \lambda \text{Id}_V$, so for $v \in V$, $v\phi\theta^{-1} = \lambda v \implies v\phi\theta^{-1}\theta = v\phi = (\lambda v)\theta = \lambda(v\theta)$, therefore $\phi = \lambda\theta$, so $\dim H = 1$. □

Proposition 7.9. *Given V, V_1, V_2, W, W_1, W_2 $\mathbb{C}G$ -modules,*

1. $\dim \text{Hom}_{\mathbb{C}G}(V, W_1 \oplus W_2) = \dim \text{Hom}_{\mathbb{C}G}(V, W_1) + \dim \text{Hom}_{\mathbb{C}G}(V, W_2)$
2. $\dim \text{Hom}_{\mathbb{C}G}(V_1 \oplus V_2, W) = \dim \text{Hom}_{\mathbb{C}G}(V_1, W) + \dim \text{Hom}_{\mathbb{C}G}(V_2, W)$

Proof. I will prove the first statement and the conversion of the proof of the second statement is left as a simple exercise.

Firstly, we define the projection homomorphisms: $\pi_i : W_1 \oplus W_2 \rightarrow W_i$ defined by $(w_1 + w_2)\pi_i = w_i$. Now, if $\theta \in \text{Hom}_{\mathbb{C}G}(V, W_1 \oplus W_2)$, we have that $\theta\pi_i \in \text{Hom}_{\mathbb{C}G}(V, W_i)$.

Define a linear map $f : \text{Hom}_{\mathbb{C}G}(V, W_1 \oplus W_2) \rightarrow \text{Hom}_{\mathbb{C}G}(V, W_1) \oplus \text{Hom}_{\mathbb{C}G}(V, W_2)$ by $\theta \mapsto \theta\pi_1 \oplus \theta\pi_2$ then it is clear that the following map is an inverse of f :

$$v(\theta_1 \oplus \theta_2)f^{-1} = v\theta_1 + v\theta_2$$

therefore we have an isomorphism of vector spaces, $\text{Hom}_{\mathbb{C}G}(V, W_1 \oplus W_2) \simeq \text{Hom}_{\mathbb{C}G}(V, W_1) \oplus \text{Hom}_{\mathbb{C}G}(V, W_2)$, giving us $\dim \text{Hom}_{\mathbb{C}G}(V, W_1 \oplus W_2) = \dim \text{Hom}_{\mathbb{C}G}(V, W_1) \oplus \dim \text{Hom}_{\mathbb{C}G}(V, W_2) = \dim \text{Hom}_{\mathbb{C}G}(V_1, W) + \dim \text{Hom}_{\mathbb{C}G}(V_2, W)$ \square

Corollary 7.9.1. For $V_1, \dots, V_n, W_1, \dots, W_m$, if $V = \bigotimes_{i=1}^n V_i$ and $W = \bigotimes_{i=1}^m W_i$

$$\dim H = \sum_{i=1}^n \sum_{j=1}^m \dim \text{Hom}_{\mathbb{C}G}(V_i, W_j)$$

Corollary 7.9.2. Suppose

$$V = \bigoplus_{j=1}^n U_j$$

Then for any irreducible module W

$$\dim \text{Hom}(V, W) = \dim \text{Hom}(W, V) = |\{j \mid U_j \cong W\}|$$

Example 7.10. Recall that

$$\mathbb{C}D_3 = \bigoplus_{i=1}^4 U_i$$

where $U_3 \cong U_4$. Then

$$\dim \text{Hom}(\mathbb{C}D_3, U_3) = 2$$

◀

8 Lecture 8

8.1 The Dimension of $\text{Hom}(U, V)$

We continue the discussion last lecture about $\text{Hom}(U, V)$.

Lemma 8.1. Let V, W be two $\mathbb{C}G$ -modules such that $\text{Hom}(V, W)$ is nonzero. Then V and W share a composition factor.

Proof. Suppose we have a morphism $\theta : V \rightarrow W$. Then there exists some $v \in V$ such that $v\theta \neq 1$. Now v is contained in some irreducible submodule, say V_0 . Then $V_0\theta \cong V_0$. \square

Lemma 8.2. *Let U be a $\mathbb{C}G$ -module. Then*

$$\dim \text{Hom}(\mathbb{C}G, U) = \dim U$$

.

Proof. Fix a basis $\{u_1, \dots, u_n\}$ for U and define $\varphi_i : \mathbb{C}G \rightarrow U$ as $r \mapsto u_i r$. We claim the φ_i form a basis for $\text{Hom}(\mathbb{C}G, U)$. Indeed, let $\varphi \in \text{Hom}(\mathbb{C}G, U)$ and suppose

$$(1)\varphi = \sum_{i=1}^n \lambda_i u_i$$

Then for all $r \in \mathbb{C}G$ we have

$$(r)\varphi = (1)\varphi r = \left(\sum_{i=1}^n \lambda_i u_i\right)r = \sum_{i=1}^n \lambda_i u_i r = \sum_{i=1}^n \lambda_i (r)\varphi_i$$

where the first equality follows from Problem 3 \square

Theorem 8.3. *Suppose*

$$\mathbb{C}G = \bigoplus_{j=1}^n V_j$$

and U is an irreducible module. Then the number of j such that $V_j \cong U$ is exactly $\dim U$

Proof. Combine Lemma 8.2 and Corollary 7.9.2 \square

Example 8.4. Recall that

$$\mathbb{C}D_3 = \bigoplus_{i=1}^4 U_i$$

where $U_3 \cong U_4$ but $U_1 \not\cong U_2$. Then U_1 and U_2 occur once whereas U_3 occurs twice, consistently with the theorem. \blacktriangleleft

Theorem 8.5. *Let V_1, \dots, V_n denote a complete set of irreducible modules that are pairwise non-isomorphic. Then*

$$\sum_{i=1}^n (\dim V_i)^2 = |G|$$

Proof. Suppose

$$\mathbb{C}G = \bigoplus_{j=1}^N U_j$$

where for each V_i there are exactly $\dim V_i$ of the U_j isomorphic to V_i . Thus we have

$$|G| = \dim \mathbb{C}G = \sum_{i=1}^N \sum_{j=1}^{\dim V_i} \dim U_j = \sum_{i=1}^N \sum_{j=1}^{\dim V_i} \dim V_i = \sum_{i=1}^n (\dim V_i)^2$$

□

Observe that $\mathbb{C}G$ always has a trivial submodule, namely the module spanned by $\sum_{g \in G} g$.

Example 8.6. Note that $|D_3| = 6$ and $6 = 1^2 + 1^2 + 2^2$. This is the only way; indeed, if all irreducible submodules are of dimension 1, then D_3 would be abelian, which is obviously false. ◀

9 Lecture 9

9.1 Group Theoretic Diversion

Suppose G is a group. We define an equivalence relation on G called **conjugacy** by

$$x \sim y \iff y = x^g = g^{-1}xg, \quad \text{for some } g \in G.$$

The equivalence class

$$x^G = G^{-1}xG = \{g^{-1}xg \mid g \in G\},$$

is called the **conjugacy class** of x .

Lemma 9.1. *Every group is a union of conjugacy classes and distinct classes are disjoint.*

Proof. Every equivalence relation on a set corresponds to a partition of said set. □

Example 9.2. For any group G , $1^G = \{1\}$ is a conjugacy class in G . More generally, if $x \in Z(G)$ then $xg = gx$ for all $g \in G$; from which it follows that $x^G = \{x\}$. ◀

Example 9.3. Let $G = D_6$ the dihedral group of 6 elements, generated by the elements a, b . Then $a^G = \{a, a^2\}$, and $b^G = \{b, ab, a^2b\}$. Hence $D_6 = 1^G \amalg a^G \amalg b^G$. ◀

Example 9.4. If G is an abelian group, then for all $x \in G$, $x^G = \{x\}$. This follows from a previous example as G is abelian if, and only if, $G = Z(G)$. ◀

Lemma 9.5. *Suppose that $x, y \in G$ with $x \sim y$, then $x^n \sim y^n$ for all $n \in \mathbb{N}$. In particular, $|x| = |y|$.*

Proof. As $x \sim y$ there exists $g \in G$ such that $x = g^{-1}yg$. By induction, it follows that $x^n = g^{-1}y^n g$ which shows that $x^n \sim y^n$. To see that the orders are equal, note that $x^n = 1$ if, and only if $g^{-1}y^n g = 1$. Hence $y \in 1^G = \{1\}$ and so $y^n = 1$. \square

Suppose $x \in G$. Define the **centraliser** of x in G to be the set

$$C_G(x) = \{g \in G \mid xg = gx\} = \{g \in G \mid x^g = x\},$$

i.e. the set of $g \in G$ which fix x under conjugation. It is clear that $C_G(x) \leq G$

Theorem 9.6 (Orbit-stabiliser). *Suppose G is a finite group and $x \in G$. Then $|x^G| = |G : C_G(x)| = |G|/|C_G(x)|$, and in particular $|x^G| \mid |G|$.*

Proof. First we have the chain of equivalences:

$$\begin{aligned} g^{-1}xg = h^{-1}xh &\iff hg^{-1}x = xhg^{-1} \\ &\iff hg^{-1} \in C_G(x) \\ &\iff C_G(x)g = C_G(x)h. \end{aligned}$$

Hence let Λ denote the set of right cosets of $C_G(x)$ in G , and define the function

$$\begin{aligned} f : x^G &\rightarrow \Lambda \\ g^{-1}xg &\mapsto C_G(x). \end{aligned}$$

Then f is well-defined by the previous working. Moreover, the previous working also shows that f is injective, and it is clearly surjective. Thus $|x^G| = |G : C_G(x)|$. \square

Observe that

$$\begin{aligned} |x^G| = 1 &\iff g^{-1}xg = x \quad \forall g \in G \\ &\iff xg = gx \quad \forall g \in G \\ &\iff x \in Z(G). \end{aligned}$$

Theorem 9.7 (Class equation). *Let G be a finite group and suppose $G = \coprod_i x_i^G$. Then*

$$|G| = |Z(G)| + \sum_{x_i \notin Z(G)} |x_i^G|,$$

where $|x_i^G| = |G : C_G(x_i)|$ and both components divide $|G|$.

Proof. As G is a disjoint union of conjugacy classes, we have

$$|G| = \left| \coprod_i x_i^G \right| = \sum_i |x_i^G|.$$

Finally use the fact that $x \in Z(G)$ if, and only if, $|x^G| = 1$. The fact $|x_i^G| = |G : C_G(x_i)|$, and both components divide $|G|$ follow from the orbit-stabiliser theorem. \square

10 Lecture 10

10.1 Class Sums

Definition 10.1. Let C be a conjugacy class of group G . We define a class sum to be the sum of all elements in our conjugacy class denoted

$$\overline{C} = \sum_{g \in C} g.$$

We now note the importance of these sums in the following theorem.

Theorem 10.2. The class sums $\overline{C}_1, \dots, \overline{C}_l$ form a basis for $Z(\mathbb{C}G)$.

Proof. We first note that the class sums are closed under conjugation and therefore are elements of the center. Now suppose $\sum_{i=1}^l \lambda_i \overline{C}_i = 0$. Since conjugacy classes are pairwise disjoint we obtain $\lambda_i = 0$ for all $1 \leq i \leq l$ so the \overline{C}_i are linearly independent. We now need to show they span $Z(\mathbb{C}G)$. Pick some $r = \sum_{g \in G} \lambda_g g \in Z(\mathbb{C}G)$ and $h \in G$. Since r is central $r^h = r$ and therefore

$$\sum_{g \in G} \lambda_g g^h = \sum_{g \in G} \lambda_g g.$$

Since the $h \in G$ was arbitrary we see that if $x \sim y$ then $\lambda_x = \lambda_y$ so r can be written as a sum of class sums. \square

Note now that we immediately obtain an important result. The dimension of $Z(\mathbb{C}G)$ is exactly the number of conjugacy classes of G .

Example 10.3. Let $G = S_3$. The conjugacy classes of S_3 are given by

$$\{\varepsilon\}, \{(1\ 2), (2\ 3), (1\ 3)\}, \{(1\ 2\ 3), (1\ 3\ 2)\}.$$

Then $Z(\mathbb{C}G)$ has dimension 3.

$$Z(\mathbb{C}G) = \langle 1, (1\ 2) + (2\ 3) + (1\ 3), (1\ 2\ 3) + (1\ 3\ 2) \rangle.$$

◀

10.2 Characters

Definition 10.4. If $A = (a_{ij})$ is an $n \times n$ matrix, then the trace of A is the sum of the diagonal elements.

$$\text{tr } A = \sum_{i=1}^n a_{ii}$$

We now present some basic properties of the trace.

Lemma 10.5. *Let $A, B, T \in M_n(\mathbb{C})$ and T be invertible.*

- (I) $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$
- (II) $\text{tr}(AB) = \text{tr}(BA)$
- (III) $\text{tr}(T^{-1}AT) = \text{tr}(A)$

Proof. (I) and (II) follow from the fact that if $C = A + B$ then $c_{ii} = a_{ii} + b_{ii}$ and if $D = AB$ then $d_{ii} = a_{ii}b_{ii}$. (III) follows from an application of (II)

$$\text{tr}(T^{-1}AT) = \text{tr}(T^{-1}(AT)) = \text{tr}((AT)T^{-1}) = \text{tr}(A).$$

□

Now that we have the machinery to describe them we define a character.

Definition 10.6. Let V be a $\mathbb{C}G$ module with basis B . The character of V is the map $\chi : G \rightarrow \mathbb{C}$ defined by

$$\chi(g) = \text{tr}[g]_B.$$

Note that by property (III) of the trace that a character is independent of basis and so we uniquely associate a character to a $\mathbb{C}G$ module. In the following lemma we see that we can associate a character to a module up to isomorphism.

Lemma 10.7.

- (I) *Isomorphic $\mathbb{C}G$ modules have the same character.*
- (II) *If $x, y \in G$ then $x \sim y \implies \text{tr}[x] = \text{tr}[y]$.*

Proof. (I) If V and W are isomorphic $\mathbb{C}G$ -modules then there exists bases B_1, B_2 such that $[g]_{B_1} = [g]_{B_2}$ for all $g \in G$.

(II) If $x \sim y$ then $x = g^{-1}yg$ for some $g \in G$. Then $\text{tr}[x] = \text{tr}[g^{-1}yg] = \text{tr}[y]$. □

Example 10.8. Let $G = S_3 = \langle a = (1\ 2), b = (1\ 2\ 3) \rangle$ and let $V = \langle v_1, v_2, v_3 \rangle$ be the permutation module of G . We then have representations

$$[a] = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad [b] = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

and therefore character of the permutation module χ satisfies

$$\chi(a) = 1, \quad \chi(b) = 0.$$

Note that because characters are constant on conjugacy classes, to specify a character of a group we only need to define it on the group's classes. As a result here we know that χ will take any 2-cycle to 1 and any 3-cycle to 0. ◀

Definition 10.9. The dimension of a character χ is $\chi(1)$.

The character of a one-dimensional $\mathbb{C}G$ -module is called a linear character. By Schur's lemma for each $g \in G$ there exists some λ_g such that $vg = \lambda_g v$ for all $v \in V$. Thus a linear character will take $v \mapsto \lambda_g$.

Lemma 10.10. *Every linear character is a homomorphism from G to \mathbb{C}^* ; The multiplicative group of \mathbb{C} .*

Proof. Suppose χ is a linear character of G . Note that $\chi(e) = \lambda_e = 1$ and if $g \in G$ then $\chi(g) \neq 0$ as $\chi(g)\chi(g^{-1}) = \chi(e) = 1$. Now pick $g, h \in G$. Then

$$\chi(gh) = \lambda_{gh} = \lambda_g \lambda_h = \chi(g)\chi(h).$$

□

Note that the multiplicative properties of a linear character don't hold for all characters in general. That is for matrices A, B , $\text{tr}(AB) \neq \text{tr}(A)\text{tr}(B)$ in general.

We now summarise some properties of a character.

Lemma 10.11. *Let V be a $\mathbb{C}G$ -module with character χ and let $g \in G$ with $|g| = m$.*

- (I) $\chi(1) = \dim V$,
- (II) $\chi(g)$ is a sum of m -th roots of unity,
- (III) $\chi(g^{-1}) = \overline{\chi(g)}$,
- (IV) $\chi(g)$ is real if $g \sim g^{-1}$.

Proof. (I) The representation of 1 will be the identity matrix. The trace of which is the dimension of the representation.

(II) There is a basis of V in which the representation of g is a diagonal matrix of m -th roots of unity.

(III) Note by the previous part we can write $\chi(G) = \sum i = 1^n \omega_i$ where ω_i is an m -th root of unity. The representation of g^{-1} will be the same diagonal matrix but with the inverse of each root of unity so $\chi(g^{-1}) = \sum_{i=1}^n \omega_i^{-1}$. However the inverse of a root of unity is its conjugate and the sum on conjugates is the conjugate of the sum so

$$\chi(g^{-1}) = \sum_{i=1}^n \overline{\omega_i} = \overline{\chi(g)}.$$

(IV) This follows from (III). □

Definition 10.12. The character of the trivial module is called the trivial character. It sends all elements of G to 1.

11 Lecture 11

11.1 Restrictions on Characters

We start with a quick corollary to Lemma 10.11

Corollary 11.0.1. *Let V be a $\mathbb{C}G$ -module of dimension n with character χ and let $g \in G$, suppose $|g| = 2$, then $\chi(g) \in \mathbb{Z}$ and $\chi(g) \equiv \chi(1) \pmod{2}$*

Proof. By 10.11, we have that $\chi(g) = \sum_{i=1}^n \omega_i$ for each ω_i a 2th-root of unity, that is, $\omega_i = \pm 1 \equiv 1 \pmod{2}$, therefore it is clear that $\chi(g) \in \mathbb{Z}$. We can also calculate,

$$\chi(g) = \sum_{i=1}^n \omega_i \equiv \sum_{i=1}^n 1 \pmod{2} \equiv n \pmod{2} \equiv \chi(1) \pmod{2}$$

as $\chi(1) = n$ by 10.11. □

Note, that we can assign a character to a representation even more naturally as we assigned one to a module by $\chi = \text{tr} \circ \rho$

Theorem 11.1. *Let $\phi : G \rightarrow GL(n, \mathbb{C})$ be a representation of G and let χ be the representation of ϕ .*

(I) *For $g \in G$, $|\chi(g)| = \chi(1) \iff g\rho = \lambda I_n$ for some $\lambda \in \mathbb{C}$*

(II) $\ker \rho = \{g \in G : \chi(g) = \chi(1) = n\}$

Proof. For (I), we firstly prove the reverse implication. Let $g \in G$ and let $m = |g|$, then $I_n = 1\rho = g^m\rho = \lambda^m I_n \implies \lambda^m = 1$, that is, λ is an m -th root of unity. Since $\chi(g) = n\lambda$, we have $|\chi(g)| = n|\lambda| = n = \chi(1)$.

Now, instead suppose that $|\chi(g)| = \chi(1) = n$. By 10.11, we can write $\chi(g) = \sum_{i=1}^n \omega_i$ where ω_i are m -th roots of unity. Therefore $|\chi(g)| = |\sum_{i=1}^n \omega_i| = n = \sum_{i=1}^n |\omega_i|$, by strict convexity of \mathbb{C} with the absolute value, this is only true if each ω_i is a positive real multiple of every other ω_j , however, all have length 1, therefore all the ω_i are equal. So $\chi(g) = \omega_1 I_n$.

(II) Suppose $g \in \ker \rho$, then $\chi(g) = \text{tr}(I_n) = n = \chi(1)$. Now, instead, suppose that $\chi(g) = \chi(1) = n$, then by (I), $g\rho = \lambda I_n$ for some $\lambda \in \mathbb{C}$. Therefore $\chi(1) = \chi(g) = \lambda\chi(1)$, so $\lambda = 1$ and $g\rho = I_n \implies g \in \ker \rho$. □

Definition 11.2. Let χ be a character of G . We define the kernel of χ by:

$$\ker \chi = \{g \in G : \chi(g) = \chi(1)\}$$

If ρ is a representation of G and χ is its character, we get $\ker \chi = \ker \rho$. Therefore $\ker \chi \trianglelefteq G$. And ρ is faithful if and only if $\ker \chi = \{1\}$.

Example 11.3. $G = D_6$

	1	a	a^2	b	ab	a^2b
χ_1	1	1	1	1	1	1
χ_2	1	1	1	-1	-1	-1
χ_3	2	-1	-1	0	0	0

$$\ker \chi_1 = G, \quad \ker \chi_2 = \langle a \rangle, \quad \ker \chi_3 = 1$$

Therefore the only faithful character of the three is χ_3 . ◀

12 Lecture 12

12.1 Some Functional Analysis

Definition 12.1. Let V be a vector space over \mathbb{C} . An *inner product* on V is a function $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{C}$ satisfying the following for all $u, v, w \in V$, $\alpha \in \mathbb{C}$

- (I) $\langle \alpha u + v, w \rangle = \alpha \langle u, w \rangle + \langle v, w \rangle$
- (II) $\langle u, v \rangle = \overline{\langle v, u \rangle}$
- (III) $\langle u, u \rangle \geq 0$
- (IV) $\langle u, u \rangle = 0$ if and only if $u = 0$.

A character is a function (not necessarily a homomorphism) from a group G into \mathbb{C} . Denote the set of all functions $G \rightarrow \mathbb{C}$ by $\text{Hom}_{\text{Sets}}(G, \mathbb{C})$. Recall that $\text{Hom}_{\text{Sets}}(G, \mathbb{C})$ has a natural vector space structure. We define an inner product on it as follows: For $\varphi, \psi \in \text{Hom}_{\text{Sets}}(G, \mathbb{C})$ define

$$\langle \varphi, \psi \rangle := \frac{1}{|G|} \sum_{g \in G} \varphi(g) \overline{\psi(g)}$$

Problem 4. Check that the above function defines an inner product.

Lemma 12.2. Let G be a finite group and χ_1 and χ_2 two characters. Then

- (I) $\langle \chi_1, \chi_2 \rangle$ is real
- (II) If there are ℓ conjugacy classes with $\{g_i\}_{i=1}^{\ell}$ a representative of each class then

$$\langle \chi_1, \chi_2 \rangle = \sum_{i=1}^{\ell} \frac{\chi_1(g_i) \overline{\chi_2(g_i)}}{|C_G(g_i)|}$$

Proof. Recall that for any character χ and $g \in G$ we have $\chi(g^{-1}) = \overline{\chi(g)}$. Thus

$$\langle \chi_1, \chi_2 \rangle = \sum_{g \in G} \chi_1(g) \overline{\chi_2(g)} = \sum_{g \in G} \chi_2(g) \chi_1(g^{-1}) = \sum_{g \in G} \chi_1(g) \chi_2(g^{-1}) = \langle \chi_2, \chi_1 \rangle$$

and since $\langle \chi_2, \chi_1 \rangle = \overline{\langle \chi_1, \chi_2 \rangle}$ this proves (I).

Denote the conjugacy class of g_i by g_i^G . Note that

$$\sum_{g \in G} \chi_1(g) \chi_2(g^{-1}) = |g_i^G| \chi_1(g_i) \chi_2(g_i^{-1})$$

since conjugates have the same character. Thus

$$\begin{aligned} \langle \chi_1, \chi_2 \rangle &= \frac{1}{|G|} \sum_{g \in G} \chi_1(g) \chi_2(g^{-1}) \\ &= \frac{1}{|G|} \sum_{i=1}^{\ell} \sum_{g \in g_i^G} \chi_1(g) \chi_2(g^{-1}) \\ &= \frac{1}{|G|} \sum_{i=1}^{\ell} |g_i^G| \chi_1(g_i) \chi_2(g_i^{-1}) \\ &= \sum_{i=1}^{\ell} \frac{\chi_1(g_i) \overline{\chi_2(g_i)}}{|C_G(g_i)|} \end{aligned}$$

since $|G| = |C_G(g_i)| |g_i^G|$ by the Orbit-Stabiliser Theorem. □

12.2 Irreducible Characters form an Orthonormal Set

Recall that $\mathbb{C}G = U_1 \oplus \dots \oplus U_r$ where each U_i is irreducible. Find W_1 and W_2 that have no common composition factors such that $\mathbb{C}G = W_1 \oplus W_2$ (for example, take W_1 to be the unique trivial submodule, then apply Maschke). Then we can write $1 = e_1 + e_2$ where $e_i \in W_i$.

Lemma 12.3. *Suppose $w_i \in W_i$ for $i = 1, 2$. Then $w_i e_j = \delta_{ij} w_i$ where δ_{ij} is the Kronecker Delta. In particular $e_i e_j = \delta_{ij} e_i$.*

Proof. Observe that $\varphi : W_1 \rightarrow W_2$ given by $w \mapsto w_2 w$ is a homomorphism of $\mathbb{C}G$ -modules. But since W_1 and W_2 have no common components, $\text{Hom}(W_1, W_2) = 0$ and thus $w_2 e_1 = 0$. Similarly $w_1 e_2 = 0$. Now observe $w_1 e_1 = w_1 - w_1 e_2 = w_1$. □

Proposition 12.4. *Suppose χ is the character of W_1 . Then*

$$e_1 = \frac{1}{|G|} \sum_{g \in G} \chi(g^{-1}) g$$

Proof. Fix $x \in G$ and consider the endomorphism $\varphi : \mathbb{C}G \rightarrow \mathbb{C}G$ given by

$$w \mapsto w e_1 x^{-1}$$

We calculate the trace of φ in two ways. Firstly note that for $w_1 \in W_1, w_2 \in W_2$ we have

$$w_1\varphi = w_1e_1x^{-1} = w_1x^{-1}$$

and

$$\varphi w_2 = 0$$

Thus the trace of φ is simply $\chi(x^{-1})$.

On the other hand, suppose

$$e_1 = \sum_{g \in G} \lambda_g g$$

Then

$$w\varphi = w \sum_{g \in G} \lambda_g g x^{-1}$$

Now we consider the homomorphism $w \mapsto wgx^{-1}$ for some g . This is simply the regular action of gx^{-1} , and thus the trace is $|G|$ if $x = g$ and 0 otherwise. Thus the trace of φ is simply $\lambda_x |G|$.

Now putting it together, we have $\lambda_x |G| = \chi(x^{-1})$ for all x . Thus

$$e_1 = \sum_{x \in G} \lambda_x x = \frac{1}{|G|} \sum_{g \in G} \chi(g^{-1}) g$$

□

Corollary 12.4.1. *Let χ be the characteristic of W_1 . Then*

$$\langle \chi, \chi \rangle = \chi(1)$$

Proof. The coefficient of 1 in e_1^2 is

$$\frac{1}{|G|^2} \sum_{g \in G} \chi(g^{-1}) \chi(g) = \langle \chi, \chi \rangle$$

And the coefficient of 1 in e_1 is $\chi(1)$ But since $e_1 = e_1^2$ we have

$$\langle \chi, \chi \rangle = \chi(1)$$

as required. □

Theorem 12.5. *Let U, V be two non-isomorphic $\mathbb{C}G$ -modules with corresponding characters χ_U, χ_V . Then*

$$(I) \quad \langle \chi_U, \chi_U \rangle = 1$$

(II) $\langle \chi_U, \chi_V \rangle = 0$.

Proof. Let

$$\mathbb{C}G = \bigoplus_{i=1}^N U_i$$

be a decomposition of $\mathbb{C}G$. Write $m := \dim U$ and

$$X := \bigoplus_{U_i \cong U} U_i \cong \bigoplus_{i=1}^m U$$

and let W_1 denote the direct sum of the remaining U_i . Then $\mathbb{C}G = X \oplus W_1$ and X and W_1 share no composition factors. Writing χ_X as the character of X we note that $\chi_X = m\chi_U$ apply Corollary 12.4.1 to get

$$\langle \chi_X, \chi_X \rangle = \chi_X(1) = m\chi_U(1) = m^2$$

But since we also have

$$\langle \chi_X, \chi_X \rangle = \langle m\chi_U, m\chi_U \rangle = m^2 \langle \chi_U, \chi_U \rangle$$

and putting it together we get (I).

To prove (II), we write $n := \dim V$ and

$$Y := \left(\bigoplus_{U_i \cong V} U_i \right) \oplus X \cong \left(\bigoplus_{i=1}^n V \right) \oplus X$$

and let W_2 denote the remainder. Then $\mathbb{C}G = Y \oplus W_2$ and Y and W_2 share no common factors. Writing χ_Y as the character of Y we have that $\chi_Y = m\chi_U + n\chi_V$. Once again, applying Corollary 12.4.1 we have

$$\langle \chi_Y, \chi_Y \rangle = \chi_Y(1) = m\chi_U(1) + n\chi_V(1) = n^2 + m^2$$

But we also have

$$\langle \chi_Y, \chi_Y \rangle = \langle m\chi_U + n\chi_V, m\chi_U + n\chi_V \rangle = m^2 + 2mn\langle \chi_U, \chi_V \rangle + n^2$$

And thus $\langle \chi_U, \chi_V \rangle = 0$ □

Corollary 12.5.1. *Non-isomorphic representations have different characters.*

12.3 Some applications

An easy consequence of the above theorem is the following. Suppose V_1, \dots, V_r is a complete set of irreducible modules of $\mathbb{C}G$. Then if

$$W = (V_1 \oplus \dots \oplus V_1) \oplus \dots \oplus (V_r \oplus \dots \oplus V_r)$$

where there are d_i copies of each V_i then the character χ of W satisfies

$$\chi = \sum_{i=1}^r d_i \chi_i$$

where χ_i is the character of V_i , and thus

$$\langle \chi_i, \chi \rangle = \langle \chi, \chi_i \rangle = d_i$$

and

$$\langle \chi, \chi \rangle = \sum_{i=1}^r d_i^2$$