

# Maths 721 Notes

2020

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# 1 Lecture 1

In the first half of this course we will cover three main topics:

- representations;
- modules;
- characters.

We will further see that representations and modules are essentially the same, and that modules and characters are essentially the same; and hence all three are essentially the same.

From now on  $G$  is a group.

## 1.1 Representations

**Definition 1.1.** A **representation** of a group  $G$  over a field  $F$  is a group homomorphism from  $G$  to  $\mathrm{GL}(n, F)$ , where  $n$  is the **degree** of the representation.

Explicitly, a representation is a function  $\rho : G \rightarrow \mathrm{GL}(n, F)$  such that for all  $g, h \in G$ ;

- (i)  $(gh)\rho = (g\rho)(h\rho)$ ,
- (ii)  $1_G\rho = I_n$ ,
- (iii)  $g^{-1}\rho = (g\rho)^{-1}$ .

Note the use of the (incredibly shit) postfix function notation.

**Example 1.2.** Take  $D_4$ , the Dihedral group of order 8. It has the following group presentations

$$\begin{aligned} D_4 &= \langle a, b \mid a^4 = 1, b^2 = 1, a^b = a^{-1} \rangle \\ &\cong \langle (1\ 2\ 3\ 4), (1\ 4)(2\ 3) \rangle, \end{aligned}$$

where  $a^b = bab^{-1}$  is conjugation of  $a$  by  $b$ . By defining the matrix subgroup

$$H = \left\langle A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\rangle$$

and defining  $\rho : D_4 \rightarrow \mathrm{GL}(n, F)$  where  $F = \mathbb{R}, \mathbb{C}$ , by  $a \mapsto A$  and  $b \mapsto B$ , and  $a^i b^j \mapsto A^i B^j$  for  $0 \leq i \leq 3$ , and  $0 \leq j \leq 1$ . Hence we have  $\rho$  is a representation of  $D_4$  over  $F$ . ◀

**Example 1.3.** Take  $\mathbb{Q}_8$  the Quaternion group of order 8, which has the following group presentations

$$\begin{aligned} \mathbb{Q}_8 &= \langle a, b \mid a^4 = 1, a^2 = b^2, a^b = a^{-1} \rangle \\ &\cong \langle \bar{a} = (1\ 6\ 2\ 5)(3\ 8\ 4\ 7), \bar{b} = (1\ 4\ 2\ 3)(5\ 7\ 6\ 8) \rangle \end{aligned}$$

Define

$$H = \left\langle A = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\rangle \subset \text{GL}(2, \mathbb{C}).$$

Then  $\rho : \mathbb{Q}_8 \rightarrow \text{GL}(2, \mathbb{C})$  defined by  $a^k b^\ell \mapsto A^k B^\ell$  is a group representation of  $\mathbb{Q}_8$  over  $\mathbb{C}$  of degree 2.  $\blacktriangleleft$

**Definition 1.4.** Let  $G$  be a group and define

$$\begin{aligned} \rho : G &\rightarrow \text{GL}(n, F) \\ g\rho &= I_n \end{aligned}$$

for all  $g \in G$ . Then  $\rho$  is a representation, called the **trivial representation** of arbitrary of degree.

It follows from the trivial representation that any group  $G$  has a representation of an arbitrary degree.

Let  $\rho : G \rightarrow \text{GL}(n, F)$  be a group homomorphism, and take  $T \in \text{GL}(n, F)$ . Then

$$(T^{-1}AT)(T^{-1}BT) = T^{-1}(AB)T.$$

Thus, given  $\rho$  define  $\sigma$  such that

$$g\sigma = T^{-1}(g\rho)T$$

for all  $g \in G$ . As for all  $g, h \in G$ , one has

$$\begin{aligned} (gh)\sigma &= T^{-1}((gh)\rho)T \\ &= T^{-1}(g\rho)(h\rho)T \\ &= T^{-1}(g\rho)TT^{-1}(h\rho)T \\ &= (g\sigma)(h\sigma), \end{aligned}$$

and so  $\sigma$  is a group homomorphism; and hence a representation.

**Definition 1.5.** Define

$$\rho : G \rightarrow \text{GL}(m, F), \quad \sigma : G \rightarrow \text{GL}(n, F)$$

to both be representation of  $G$  over  $F$ . We say that  $\rho$  is **equivalent to**  $\sigma$  if  $n = m$  and there exists  $T \in \text{GL}(n, F)$  such that  $g\sigma = T^{-1}(g\rho)T$ .

**Proposition 1.6.** *Equivalence of representations is an equivalence relation.*

*Proof.* Reflexivity is clear by taking  $T = I_n$ . For symmetry, take  $T$  to be its inverse. For transitivity, if

$$g\sigma = T^{-1}(g\rho)T, \quad g\rho = S^{-1}(g\eta)S,$$

then

$$g\sigma = (ST)^{-1}(g\eta)(ST).$$

□

**Definition 1.7.** Define the **kernel** of the representation  $\rho : G \rightarrow \text{GL}(n, F)$  as  $\ker \rho = \{g \in G \mid g\rho = I_n\}$ .

**Proposition 1.8.** *The kernel of a representation of  $G$  is a normal subgroup of  $G$ ; i.e.  $\ker \rho \triangleleft G$ .*

*Proof.* Suppose  $g \in \ker \rho$  and  $h \in G$  is arbitrary. Then

$$(hgh^{-1})\rho = (h\rho)(g\rho)(h^{-1}\rho) = (h\rho)I_n(h\rho)^{-1} = (h\rho)(h\rho)^{-1} = I_n,$$

and so  $hgh^{-1} \in \ker \rho$ . As  $\ker \rho$  is closed under conjugation, it is a normal subgroup of  $G$ .  $\square$

**Definition 1.9.** We say  $\rho$  is a **faithful** representation of  $G$  if  $\ker \rho = \{1_G\}$ .

**Example 1.10.** For the trivial representation  $\rho : G \rightarrow \text{GL}(n, F)$  with  $g \mapsto I_n$  for all  $g \in G$ , we have  $\ker \rho = G$ . Hence the representation is not faithful.  $\blacktriangleleft$

**Lemma 1.11.** *Suppose  $G$  is a finite group, and  $\rho$  is a representation of  $G$  over  $F$ . Then  $\rho$  is faithful if, and only if,  $\text{im } \rho \cong G$ .*

*Proof.* Immediate from the first isomorphism theorem.  $\square$

## 1.2 FG-Modules

Suppose  $G$  is a group, and  $F = \mathbb{R}, \mathbb{C}$ . Given  $\rho : G \rightarrow \text{GL}(n, F)$ , with  $V = F^n$ , let  $v = (\lambda_1, \dots, \lambda_n) \in V$  for  $\lambda_i \in F$  be a row vector. Moreover, note that  $g\rho$  is an  $n \times n$  matrix for all  $g \in G$ . Thus, we have  $v \cdot (g\rho) \in V$ , and satisfies the following properties:

- (i)  $v \cdot ((gh)\rho) = v \cdot (g\rho)(h\rho)$ ;
- (ii)  $v \cdot (1_G\rho) = v$ ;
- (iii)  $(\lambda v) \cdot (g\rho) = \lambda(v \cdot (g\rho))$ ;
- (iv)  $(u + v) \cdot (g\rho) = u \cdot (g\rho) + v \cdot (g\rho)$ .

We often will omit the  $\cdot$  in the operation, and write  $v(a\rho)$  for  $v \cdot (a\rho)$ .

**Example 1.12.** Recall  $D_4$  and its given presentation from a previous example. We have

$$a\rho = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad b\rho = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

If  $v = (\lambda_1, \lambda_2)$ , then we have

$$v(a\rho) = (-\lambda_2, \lambda_1), \quad v(b\rho) = (\lambda_1, -\lambda_2).$$

$\blacktriangleleft$

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**Definition 1.13.** Let  $V$  a vector space over the field  $F = \mathbb{R}, \mathbb{C}$ , and let  $G$  be a group. We say  $V$  is a  $FG$ -**module** if a multiplication  $v \cdot g$  for  $v \in V$ , and  $g \in G$  is defined such that:

- (i)  $v \cdot g \in V$ ;
- (ii)  $v \cdot (gh) = (v \cdot g) \cdot h$ ;
- (iii)  $v \cdot 1_G = v$ ;
- (iv)  $(\lambda v) \cdot g = \lambda(v \cdot g)$ ;
- (v)  $(u + v) \cdot g = u \cdot g + v \cdot g$ .

This generalises the previous discussion from a matrix group to an arbitrary group.

Note that properties (i), (iv), and (v) imply that the map  $v \mapsto v \cdot g$  is an endomorphism of  $V$  (a linear map from  $V$  to itself).

**Definition 1.14.** Suppose  $V$  is an  $FG$ -module and  $B$  is a basis for  $V$ . For  $g \in G$ , let  $[g]_B$  denote the matrix of the endomorphism  $v \mapsto v \cdot g$  of  $V$  relative to the basis  $B$ .

## 2 Lecture 2

**Theorem 2.1.** Let  $\rho : G \rightarrow GL(n, F)$  be a representation of  $G$  over  $F$ .

- (I) If  $V = F^n$  is an  $FG$  module and  $G$  acts on  $V$  by  $v \cdot g = v(g\rho)$  there exists a basis  $B$  of  $V$  such that  $g\rho = [g]_B$ .
- (II) The map  $g \mapsto [g]_B$  is a representation for  $G$  over  $F$ .

*Proof.* Choose the standard basis  $B = [e_1, \dots, e_n]$ .

Since  $V$  is an  $FG$ -module we have  $v(gh) = (vg)h$  for all  $g, h \in G$  and  $v \in V$ . Thus  $[gh]_B = [g]_B[h]_B$  so the map is a homomorphism. We now check that  $[g]_B$  is invertible for all  $g \in G$ . We know  $v \cdot 1_G = (vg)g^{-1}$  so  $I_n = [g]_B[g^{-1}]_B$  and thus  $[g]_B$  has an inverse.  $\square$

**Example 2.2.** Recall the representation of  $G = D_4$  from a previous example. Define an  $FG$ -module  $V = F^2$  with the action defined by taking  $vg$  to  $v(g\rho)$ .

$$\begin{aligned} v_1 &= (1, 0), & v_1 a &= v_2, & v_1 b &= v_1, \\ v_2 &= (0, 1), & v_1 a &= -v_1, & v_1 b &= -v_2. \end{aligned}$$

In this basis we recover our representation

$$a \mapsto [a]_B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad b \mapsto [b]_B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

◀

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We now provide an equivalent basis-dependent definition for an  $FG$ -module.

**Lemma 2.3.** *Let  $V$  a vector space over the field  $F = \mathbb{R}, \mathbb{C}$  with basis  $B = [v_1, \dots, v_n]$ , and let  $G$  be a group. If a multiplication  $v \cdot g$  for  $v \in B$ , and  $g \in G$  is defined such that:*

- (i)  $v \cdot g \in V$ ;
- (ii)  $v \cdot (gh) = (v \cdot g) \cdot h$ ;
- (iii)  $v \cdot 1_G = v$ ;
- (iv)  $(\sum_{i=1}^n \lambda_i v_i) \cdot g = \sum_{i=1}^n \lambda_i (v_i \cdot g)$  for all  $\lambda_i \in F$ ;

*then  $V$  is an  $FG$ -module.*

**Definition 2.4.** The trivial module of a group over  $F$  is a one dimensional vector space  $V$  over  $F$  such that  $vg = v$  for all  $v \in V$  and  $g \in G$ .

**Definition 2.5.** An  $FG$ -module is faithful if  $1_G$  is the only  $g \in G$  such that  $vg = v$  for all  $v \in V$ .

**Theorem 2.6.** *Let  $V$  be an  $FG$ -module with basis  $B$  and  $\rho$  a representation of group  $G$  over  $F$  defined by taking  $g \mapsto [g]_B$ .*

- (i) *If  $B'$  is another basis of  $V$  then the map  $g \mapsto [g]_{B'}$  is a representation of  $G$  equivalent to  $\rho$ .*
- (ii) *If representation  $\sigma$  is equivalent to  $\rho$  then there exists basis  $B''$  such that  $\sigma(g) = [g]_{B''}$  for all  $g \in G$ .*

*Proof.* Taking  $T$  to be the change of basis matrices, the two representations are equivalent.  $\square$

**Example 2.7.** Let  $G = C_3 = \langle a \mid a^3 = 1 \rangle$  and representation  $\rho : G \rightarrow GL(n, F)$  defined by

$$a \mapsto \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}.$$

We attempt to construct an  $FG$ -module with group action described by  $\rho$ . Take  $V = F^2$  with basis  $B = [v_1, v_2]$ . Define the action of  $G$  on  $V$  by

$$v_1 a = v_2, \quad v_2 a = -v_1 - v_2.$$

Let us now choose alternate basis  $B' = [u_1 = v_1, u_2 = v_1 + v_2]$ . The action of  $G$  on this basis is described by

$$u_1 a = -u_1 + u_2, \quad u_2 a = -u_1.$$

This gives us a representation

$$a \mapsto [a]_{B'} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}.$$

To verify this construction we write our change of basis matrix as

$$T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

and verify that

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}.$$

**Definition 2.8.** The permutation module of a group  $G \leq S_n$  is an  $n$ -dimensional vector space  $V$  with basis  $B = [v_1, \dots, v_n]$  and action by  $G$  defined by

$$v_i g = v_{ig}$$

for all  $g \in G$  where  $ig$  is the image of  $i$  under  $g \in S_n$ .

It follows from Cayley's theorem that every group has a faithful  $FG$ -module.

**Example 2.9.** Take  $G = S_4$  and pick  $g = (1\ 2)$  and  $h = (1\ 2\ 3\ 4)$ . We have representations

$$[g]_B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad [h]_B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

## 2.1 Module Reducibility

**Definition 2.10.** Let  $V$  be an  $FG$ -module. We call  $W$  a submodule of  $V$  if  $W$  is a vector subspace of  $V$  and  $W$  is closed under the action of  $G$ . We then write  $W < V$ .

**Example 2.11.** Let  $G = C_3 = \langle (1\ 2\ 3) \rangle$  and  $V$  the permutation module of  $G$  with basis  $B = [v_1, v_2, v_3]$ . The subspace  $W = \langle v_1 + v_2 + v_3 \rangle$  is a submodule but the subspace  $U = \langle v_1 + v_2 \rangle$  is not.

For example, consider the action of  $g = (1\ 2\ 3)$  on  $v_1 + v_2 \in U$ .

$$(v_1 + v_2)g = v_{1g} + v_{2g} = v_2 + v_3 \notin U$$

whereas  $G$  acts on  $W$  trivially.

## 3 Lecture 3

### 3.1 Module and Representation Reducibility

For any module, it is clear that we have two trivial submodules:  $0 < V$  and  $V < V$ . Where  $0 = \{0\} \subset V$ .

**Definition 3.1.** Let  $V$  be an  $FG$ -module. We say that  $V$  is irreducible if the only submodules of  $V$  are  $V$  and  $0$ . Otherwise  $V$  is reducible

In 2.11 we showed that the permutation module of  $C_3$  is reducible.

**Definition 3.2.** Let  $\rho : G \rightarrow GL(n, F)$  be a representation. We say that  $\rho$  is irreducible if the corresponding  $FG$ -module (as constructed in 2.1) is irreducible. Otherwise  $\rho$  is reducible.

If an  $FG$ -module,  $V$  is reducible, that is,  $0 < W < V$ ,  $0 \neq W \neq V$ . Let  $B_W$  be a basis for  $W$ . If we extend  $B_W$  to  $B$  a basis of  $V$ , then we get the following representation of  $G$ :

$$g \mapsto [g]_B = \begin{bmatrix} X_g & 0 \\ Y_g & Z_g \end{bmatrix} \quad (3.1)$$

Where the matrices  $X_g, Y_g$  and  $Z_g$  are some block matrices and  $0$  is a block of zeros and  $X_g$  has the dimensions  $m \times m$  and, in this case,  $\dim(W) = m$ .

**Proposition 3.3.** A representation  $\rho : G \rightarrow GL(n, F)$  is reducible if and only if with respect to some basis,  $B$ , of  $F^n$ ,  $[g]_B$  has the form 3.1 for some  $0 < m < \dim(V)$  for all  $g \in G$ . Then the maps  $g \mapsto X_g$  and  $g \mapsto Z_g$  are both representations of  $G$ .

*Proof.* Suppose we have a presentation,  $\rho : G \rightarrow GL(n, F)$  and a basis  $B$  of  $V = F^n$  such that  $[g]_B$  has the form 3.1 for every  $g \in G$ . Then consider the subspace  $0 \subset W \subset V$  spanned by the first  $m$  elements of  $B$ . It is clear that  $v[g]_B \in W$  for all  $v \in W$ . Therefore the module induced by  $\rho$  is reducible, so  $\rho$  is reducible. Now, if we have a reducible representation, then the argument above this proposition shows that with respect to any basis extending  $B_W$ , the matrices  $[g]_B$  have the required form.

Now, using elementary block matrix multiplication, we get the following for  $g, h \in G$ :

$$\rho(g)\rho(h) = [g]_B[h]_B = \begin{bmatrix} X_g X_h & 0 \\ Y_g X_h + Z_g Y_h & Z_g Z_h \end{bmatrix} = [gh]_B = \rho(gh)$$

Therefore  $X_{gh} = X_g X_h$  and  $Z_{gh} = Z_g Z_h$ , so the maps  $g \mapsto X_g$  and  $g \mapsto Z_g$  are both representations of  $G$ .  $\square$

**Problem 1.** Prove that the example representation of  $D_8$  of degree 2 over  $\mathbb{R}$  or  $\mathbb{C}$  is irreducible.

## 3.2 Group Algebras

Recall that an algebra over a field  $F$  is a vector space over  $F$  equipped with a bilinear product  $A \times A \rightarrow A$  that is compatible with scalar multiplication.

**Definition 3.4.** The group algebra over a finite group  $G$  over a field  $F$  is an algebra<sup>1</sup> of dimension  $n = |G|$  over  $F = \mathbb{R}$  or  $\mathbb{C}$  called  $FG$ , with basis  $B = G = \{g_1, \dots, g_n\}$ .

<sup>1</sup>See Lemma 3.6



Where the algebra structure is given by the following for two arbitrary elements of  $FG$ ,  $u = \sum_{g \in G} \lambda_g g$ ,  $v = \sum_{g \in G} \mu_g g$ ,  $\lambda_g, \mu_g \in F$  and  $\nu \in F$ :

- (i)  $u + v = \sum_{i=1}^n (\lambda_i + \mu_i) g_i$
- (ii)  $\nu \cdot u = \sum_{i=1}^n (\nu \lambda_i) g_i$
- (iii)  $u \cdot v = \sum_{(h,g) \in G \times G} \lambda_g \mu_h (gh)$

This is clearly a vector space.

**Example 3.5.** Consider  $G = C_3 = \{e, a, a^2\} = \langle a | a^3 = e \rangle$  and  $F = \mathbb{R}$  or  $\mathbb{C}$ . Then if we let  $u = e - a + 2a^2$ ,  $v = \frac{1}{2}e + 5a$ , then:

$$u + v = \frac{3}{2}e + 4a + 2a^2, \quad \frac{1}{3}u = \frac{1}{3}e - \frac{1}{3}a + \frac{2}{3}a^2, \quad uv = \frac{21}{2}e + \frac{9}{2}a - 4a^2$$

◀

**Lemma 3.6.** Given a group algebra  $FG$ ,  $r, s, t \in FG$ ,  $\lambda \in F$ :

- (I)  $rs \in FG$
- (II)  $(rs)t = r(st)$
- (III)  $1_G r = r 1_G = r$
- (IV)  $(\lambda r)s = \lambda(rs)$
- (V)  $(r + s)t = rt + st$
- (VI)  $r(s + t) = rs + rt$
- (VII)  $r0 = 0r = 0$

That is,  $FG$  is an associative algebra with unit

*Proof.* 1,3 and 7 are clear from the definition of  $FG$ , 4,5 and 6 follow from the distributive and associative laws of  $F$  and 2 follows from associativity in  $G$ .  $\square$

### 3.3 The Regular $FG$ -module, $FG$

**Problem 2.**  $V = FG$  is an  $FG$ -module with the group action defined by  $v \cdot g = vg$  for  $v \in FG$ ,  $g \in G \subset FG$ .

**Definition 3.7.** For a finite group  $G$  and  $F = \mathbb{R}$  or  $\mathbb{C}$ , the regular  $FG$ -module is  $FG$ . The associated module,  $g \mapsto [g]_B$  is called the regular representation.

**Lemma 3.8.**  $FG$  is a faithful module for  $G$  over  $F$

*Proof.* If  $vg = v$  for all  $v \in FG$ , then specifically,  $hg = h$  for all  $h \in G$ , so  $g = 1_G$ .  $\square$

**Example 3.9.** For  $C = C_3$ , over the basis  $B = G$ , we get:

$$[e]_B = I_3, \quad [a]_B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad [a^2]_B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

◀

Now, if we have an  $FG$ -module,  $V$ , then  $FG$  acts on  $V$  in the following way:

$$v \cdot r = v \cdot \left( \sum_{g \in G} \mu_g g \right) = \sum_{g \in G} \mu_g (v \cdot g)$$

**Lemma 3.10.** For  $u, v \in V$ ,  $\lambda \in F$ ,  $r, s \in FG$ :

- (I)  $vr \in FG$
- (II)  $(vr)s = v(rs)$
- (III)  $v1 = v$
- (IV)  $(\lambda v)r = \lambda(vr) = v(\lambda r)$
- (V)  $v(r + s) = vr + vs$
- (VI)  $(u + v)r = ur + vr$
- (VII)  $r0 = v0 = 0$

*Proof.* I, III and the first part of VII follow from  $V$  being an  $FG$ -module, the second equality of VII follows from scalar multiplication by 0 in  $V$ . The following calculation:

$$\begin{aligned} (\lambda v)r &= \sum_{g \in G} \mu_g ((\lambda v)g) \\ &= \sum_{g \in G} \mu_g (\lambda(vg)) \\ &= \sum_{g \in G} (\lambda \mu_g)(vg) = v(\lambda r) \\ &= \lambda \sum_{g \in G} \mu_g (vg) \\ &= \lambda(vg) \end{aligned}$$

---

proves IV. VI follows from the linearity of the action of  $G$  on  $V$ . V follows from distributivity of scalar multiplication in  $V$ . Finally, to prove II:

$$\begin{aligned}
v(rs) &= \sum_{(g,h) \in G \times G} (\mu_g \lambda_h(v(gh))) \\
&= \sum_{h \in G} \lambda_h \sum_{g \in G} \mu_g(gv)h \\
&= \sum_{h \in G} \lambda_h \left( \sum_{g \in G} \mu_g(gv) \right) h \\
&= \sum_{h \in G} \lambda_h(vr)h = (vr)s
\end{aligned}$$

□

## 4 Lecture 4

### 4.1 Homomorphisms

**Definition 4.1.** Let  $V$  and  $W$  be  $FG$ -modules. A *homomorphism* of  $FG$ -modules is a map  $\sigma : V \rightarrow W$  which is a linear transformation and also satisfies  $(vg)\sigma = (v\sigma)g$  for all  $g \in G, v \in V$ . The *kernel* and *image* are defined in the obvious way

Equivalently, it is a homomorphism of modules over the ring  $FG$ . Indeed:

**Problem 3.** Suppose  $r \in FG$  is an element of the group algebra. Prove that  $(vr)\sigma = (v\sigma)r$ .

**Lemma 4.2.** Let  $\sigma : V \rightarrow W$  be a homomorphism of  $FG$ -algebras. Then the kernel and image of  $\sigma$  are submodules

*Proof.* This is a matter of simple checking, which will be left to the reader. □

**Example 4.3.** Take  $\sigma : V \rightarrow V$  to be  $v \mapsto \lambda v$  for some  $\lambda \in F^*$ . Then  $\ker \sigma = 0, \text{im } \sigma = V$ . ◀

**Example 4.4.** Let  $G = S_n$  and  $V = \langle v_1, \dots, v_n \rangle$  be the permutation module for  $G$  over  $F$ , and let  $W = \langle w \rangle$  be the trivial module. Now define  $\sigma : V \rightarrow W$  by

$$\sum \lambda_i v_i \mapsto \sum \lambda_i w$$

Then  $\ker \sigma = \{ \sum \lambda_i v_i \mid \sum \lambda_i = 0 \}$  and  $\text{im } \sigma = W$ . ◀

**Definition 4.5.** A homomorphism of  $FG$ -modules is an *isomorphism* if it is bijective

**Remark 1.** In class we originally said "if the homomorphism has trivial kernel". However, this is definitely not correct because inclusions are always homomorphisms, but obviously not isomorphisms.  $\blacklozenge$

**Lemma 4.6.** *The inverse of an isomorphism is an isomorphism*

*Proof.* Once again, this is just an exercise in checking. The details will be left for the reader.  $\square$

Some rather obvious invariants of  $FG$ -modules (under isomorphism) are dimension and irreducibility.

**Lemma 4.7.**  *$V$  and  $W$  are isomorphic if and only if there exists bases  $\mathcal{B}_1$  of  $V$  and  $\mathcal{B}_2$  of  $W$  such that*

$$[g]_{\mathcal{B}_1} = [g]_{\mathcal{B}_2}$$

*for all  $g$ .*

*Proof.* Suppose firstly that  $V$  and  $W$  are isomorphic, and let  $\sigma : V \rightarrow W$  be one such isomorphism. Let  $\mathcal{B}_1 = \{v_1, \dots, v_n\}$  be a basis for  $V$ . In particular, it is linearly independent, and it is easy to see that  $\mathcal{B}_2 = \{v_1\sigma, \dots, v_n\sigma\}$  is also linearly independent. Since  $V$  and  $W$  are isomorphic, they have the same dimension, and thus  $\mathcal{B}_2$  is a basis for  $W$ . Since  $(vg)\sigma = (v\sigma)g$  for all  $g$  and  $v$ , the action of  $g$  on the basis vectors of both bases are the same, and thus we conclude  $[g]_{\mathcal{B}_1} = [g]_{\mathcal{B}_2}$ .

Conversely, suppose that the latter hypothesis is satisfied. Let  $\{v_1, \dots, v_n\}$  be a basis for  $V$  and  $\{w_1, \dots, w_n\}$  be a basis for  $W$ . We define a bijective linear map  $\sigma : V \rightarrow W$  such that  $v_i\sigma = w_i$  for each  $i$ . Now observe that for each  $i$ , we have  $v_i g = \lambda_1 v_1 + \dots + \lambda_n v_n$  and  $w_i g = \lambda_1 w_1 + \dots + \lambda_n w_n$ , where  $(\lambda_1, \dots, \lambda_n)$  is the  $i$ -th row of  $[g]$ . This means that

$$(v_i g)\sigma = (\lambda_1 v_1 + \dots + \lambda_n v_n)\sigma = \lambda_1 v_1\sigma + \dots + \lambda_n v_n\sigma = \lambda_1 w_1 + \dots + \lambda_n w_n = w_i g = (v_i\sigma)g$$

and thus  $\sigma$  is a homomorphism of  $FG$ -modules. Since it is bijective, it is an isomorphism.  $\square$

**Theorem 4.8.** *Let  $V$  be an  $FG$ -module with basis  $\mathcal{B}_1$  and  $W$  an  $FG$ -module with basis  $\mathcal{B}_2$ . Then  $W \cong V$  if and only if  $g \mapsto [g]_{\mathcal{B}_1}$  and  $g \mapsto [g]_{\mathcal{B}_2}$  are equivalent.*

*Proof.* This follows from the previous Lemma and the fact that two matrices are conjugate ( $A$  and  $B$  are conjugate if  $A = P^{-1}BP$  for some  $P$ ) if and only if the linear transformations they define differ by a change of basis (that is they define the same transformation but with respect to different bases)  $\square$

**Example 4.9.** Let  $G = C_3 = \{e, a, a^2\}$ . Let  $V$  be the regular representation, that is the natural representation induced by the module  $FG = \langle e, a, a^2 \rangle$ . Write  $B := \{e, a, a^2\}$  as a basis for  $FG$ . Then

$$[a]_B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Now let  $W$  be the permutation module where  $a = (1, 2, 3)$  and  $C_3$  is considered a subgroup of  $S_3$ . Write  $B' = \{v_1, v_2, v_3\}$  for the basis of  $W$ . Then

$$[a]_{B'} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Note that these two modules are isomorphic. ◀

**Example 4.10.** Let  $G = D_4 = \langle a, b \mid a^4 = b^2 = 1, a^b = a^{-1} \rangle$ . Now we can act on either  $F^4$  or  $F^8$ . On  $F^4$ , we have the representation described in Example 1.2. On  $W$ , we have the regular representation. Clearly are not isomorphic. ◀

## 4.2 Sums

We now consider how modules behave with respect to direct sums. Let  $V$  be an  $FG$ -module and suppose  $V = U \oplus W$ , where  $U$  and  $W$  are submodules. Let  $\mathcal{B}_1 = \{u_1, \dots, u_n\}$  be a basis for  $U$  and  $\mathcal{B}_2 = \{w_1, \dots, w_m\}$  one for  $W$ , so that  $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$  is a basis for  $V$ . Then

$$[g]_{\mathcal{B}} = \begin{pmatrix} [g]_{\mathcal{B}_1} & 0 \\ 0 & [g]_{\mathcal{B}_2} \end{pmatrix}$$

**Lemma 4.11.** *Let  $V$  be an  $FG$ -module such that we have the decomposition*

$$V = \bigoplus_{i=1}^n U_i$$

*Define the projection map  $\pi_i : u_1 + u_2 + \dots + u_n \mapsto u_i$ . Then*

- (I)  $\pi_i$  is a homomorphism
- (II)  $\pi_i \circ \pi_i = \pi_i$

*Proof.* Trivial □

**Lemma 4.12.** *Suppose we have a finite decomposition*

$$V = \sum U_i$$

*where the  $U_i$  are irreducible. Then  $V$  is the direct sum of some subset of the  $U_i$ .*

*Proof.* This follows from the fact that the intersection of two distinct irreducible modules is trivial (again, simple checking).  $\square$

We will now present an important result

**Theorem 4.13** (Maschke's Theorem). *Let  $G$  be a finite group,  $F$  a field of characteristic 0,  $V$  an  $FG$ -module and  $U$  a submodule. Then there exists some  $W$  such that  $V = U \oplus W$ .*

*Proof.* We first choose some  $W_1$  such that  $V = U \oplus W_1$  as vector spaces. Note that each  $v \in V$  can be uniquely decomposed as  $v = u + w$ , where  $u \in U, w \in W_1$ . Now define the canonical projection  $\sigma : V \rightarrow U$  where  $v \mapsto u$ . Clearly  $\ker \sigma = W_1$  and  $\text{im } \sigma = U$ . However, we note that  $\sigma$  is NOT necessarily a homomorphism of  $FG$ -modules. We modify it as follows: Define  $\varphi : V \rightarrow V$  by

$$v \mapsto \frac{1}{|G|} \sum_{g \in G} v g \sigma g^{-1}$$

We claim that  $\varphi$  IS a homomorphism. Indeed, suppose  $x \in G, v \in V$ . Then

$$\begin{aligned} (xv)\varphi &= \frac{1}{|G|} \sum_{g \in G} (vx) g \sigma g^{-1} \\ &= \frac{1}{|G|} \sum_{h \in G} v h \sigma h^{-1} x \\ &= \left( \frac{1}{|G|} \sum_{h \in G} v h \sigma h^{-1} \right) x = (v\varphi)x \end{aligned}$$

where the equality

$$\frac{1}{|G|} \sum_{g \in G} (vx) g \sigma g^{-1} = \frac{1}{|G|} \sum_{h \in G} v h \sigma h^{-1} x$$

follows from the change of variables  $h = xg$ . Clearly  $\varphi$  maps into  $U$ , and we now check it is a projection. Indeed, supposing  $u \in U$  we have

$$\begin{aligned} (u)\varphi &= \frac{1}{|G|} \sum_{g \in G} u g \sigma g^{-1} \\ &= \frac{1}{|G|} \sum_{g \in G} u \sigma g g^{-1} \\ &= \frac{1}{|G|} \sum_{g \in G} u \\ &= u \end{aligned}$$

as desired.

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Now clearly  $U = \text{im } \varphi$  and we define  $W := \ker \varphi$ . Then for each  $v \in V$ , write  $u := v\varphi \in U$  and  $w := v - u \in W$  so that  $v = u + w$ . It only remains to check that this is unique. To see this, suppose  $u' + w' = v = u + w$ . Then

$$u' = \varphi(u') = \varphi(v) = \varphi(u) = u$$

which implies the result.  $\square$

## 5 Lecture 5

We begin with consequences of Maschke's theorem.

**Example 5.1.** Let  $G = S_3$ , and  $V = \langle v_1, v_2, v_3 \rangle$  is the permutation module. Let  $U$  be the submodule of  $V$  defined as  $U = \langle v_1 + v_2 + v_3 \rangle < V$ . Suppose  $W_0 = \langle v_1, v_2 \rangle$ , so that  $V = U \oplus W_0$  as subspaces. Define a projection  $\phi : V \rightarrow U$  by  $v_1 \mapsto 0$ ,  $v_2 \mapsto 0$ , and  $v_3 \mapsto v_1 + v_2 + v_3$ . Further, define  $\theta : V \rightarrow V$ , as in proof of Maschke's theorem, so that

$$v\theta = \frac{1}{|G|} \sum_{g \in G} vg\theta g^{-1} = \frac{1}{6} \sum_{g \in S_3} vg\theta g^{-1}.$$

Consider the action of  $\theta$  on the basis elements  $v_i$ , then a short computation shows that

$$v_i\theta = \frac{1}{3}(v_1 + v_2 + v_3), \quad i = 1, 2, 3.$$

Moreover, we have

$$\ker \theta = \{v \in V : v\theta = 0\} = \left\{ \sum_{i=1}^3 \lambda_i v_i : \sum_{i=1}^3 \lambda_i = 0 \right\}.$$

Hence  $V = U \oplus \ker \theta$ , is a direct summand of submodules. Moreover, if  $\mathcal{B} = [v_1 + v_2 + v_3, v_1, v_2]$  is a basis for  $V$ , and  $\mathcal{B}' = [v_1 + v_2 + v_3, v_1 - v_2, v_2 - v_3]$  is another, one has

$$[g]_{\mathcal{B}} = \begin{pmatrix} * & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix}, \quad [g]_{\mathcal{B}'} = \begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}.$$

◀

In fact, it follows from Maschke's theorem that if we choose a basis  $\mathcal{B}$  for  $V$  such that

$$[g]_{\mathcal{B}} = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix},$$

then there exists a basis  $\mathcal{B}'$  for  $V$  such that

$$[g]_{\mathcal{B}'} = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}.$$

**Definition 5.2.** Let  $V$  be a  $FG$ -module,  $V$  is said to be **completely reducible** if  $V = U_1 \oplus \cdots \oplus U_r$  with each  $U_i$  an irreducible  $FG$ -module.

**Theorem 5.3.** Suppose  $G$  is a finite group, and  $F = \mathbb{R}, \mathbb{C}$ . Then every  $FG$ -module is completely reducible

*Proof.* Induction using Maschke's theorem.  $\square$

**Lemma 5.4.** Suppose  $G$  is a finite group,  $F = \mathbb{R}, \mathbb{C}$ , and  $V$  a  $FG$ -module. If  $U$  is a  $FG$ -submodule, then there exists a surjective  $FG$ -homomorphism from  $V$  onto  $U$ .

*Proof.* By Maschke's theorem, there exists a complementary submodule  $W$  to  $U$  such that  $V = U \oplus W$ . Thus, defining  $\pi : V \rightarrow U$  by  $u + w \mapsto u$  gives the result.  $\square$

**Example 5.5.** Take

$$G = \left\langle \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} \mid n \in \mathbb{Z} \right\rangle$$

and  $V = \mathbb{C}^2$ . Then  $V$  is not completely reducible.  $\blacktriangleleft$

**Example 5.6.** Let  $G = C_p = \langle a \mid a^p = 1 \rangle$  where  $p$  is prime, and take the representation

$$a^j \mapsto \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix}, \quad 0 \leq j \leq p-1$$

over the finite field  $F = \mathbb{Z}_p$ . If  $V = \langle v_1, v_2 \rangle$  is the  $FG$ -module of the representation and  $U = \langle v_2 \rangle$ , then there does not exist a submodule  $W < V$  such that  $V = U \oplus W$ .  $\blacktriangleleft$

The previous two examples show that the assumptions of Maschke's theorem are required, and cannot be relaxed.

## 5.1 Schur's Lemma

**Theorem 5.7** (Schur's lemma). Suppose  $V$  and  $W$  are irreducible  $\mathbb{C}G$ -modules.

- (I) If  $\theta : V \rightarrow W$  is a  $\mathbb{C}G$ -homomorphism, then  $\theta$  is either a  $\mathbb{C}G$ -isomorphism or the zero homomorphism.
- (II) If  $\theta : V \rightarrow V$  is a  $\mathbb{C}G$ -isomorphism, then  $\theta$  is scalar multiple of the identity endomorphism of  $V$ .

*Proof.* (I): Suppose that  $v\theta \neq 0$  for some  $v \in V$ , then the image of  $\theta$  is not trivial,  $\text{im } \theta \neq \{0\}$ . However,  $\text{im } \theta$  is a submodule of  $W$ , and so the irreducibility of  $W$  forces  $\text{im } \theta = W$ . Likewise, as the kernel of  $\theta$  is a submodule of  $V$ , but not all of  $V$ , we have  $\ker \theta = \{0\}$  as  $V$  is irreducible. Therefore,  $\theta$  is a bijective  $\mathbb{C}G$ -homomorphism, and so it is a  $\mathbb{C}G$ -isomorphism.



(II): Suppose  $\theta$  is a  $\mathbb{C}G$ -isomorphism. Then as  $\mathbb{C}$  is algebraically closed,  $\theta$  has an eigenvalue  $\lambda_v \in \mathbb{C}$  with  $v\theta = \lambda_v v$  for some  $v \in V$ . Now, as  $\ker(\theta - \lambda_v 1_V) \neq \{0\}$  is a submodule of  $V$ ,  $V$  being irreducible implies that  $\ker(\theta - \lambda_v 1_V) = V$ . Therefore,  $w(\theta - \lambda_v 1_V) = 0$  for all  $w \in V$ , and so  $\theta = \lambda 1_V$  as required.  $\square$

Further, we actually have a converse to the second statement of Schur's lemma.

**Proposition 5.8.** *Let  $V$  be a nontrivial  $\mathbb{C}G$ -module, and suppose that every  $\mathbb{C}G$ -homomorphism from  $V$  to  $V$  is a scalar multiple of the identity endomorphism of  $V$ . Then  $V$  is irreducible.*

*Proof.* Suppose  $V$  is reducible. Then there exists a nontrivial submodule  $U$  of  $V$  such that, by Maschke's theorem,  $V = U \oplus W$  with  $W$  also a submodule of  $V$ . Defining  $\pi : V \rightarrow V$  by  $u + w \mapsto u$  gives a  $\mathbb{C}G$ -homomorphism that is not a multiple of the identity endomorphism. A contradiction.  $\square$

We now interpret Schur's lemma as representation statement.

**Lemma 5.9.** *Let  $\rho : G \rightarrow \mathrm{GL}(n, \mathbb{C})$  be a representation. Then  $\rho$  is irreducible if, and only if, every  $n \times n$  matrix  $A$  which satisfies  $(g\rho)A = A(g\rho)$  for all  $g \in G$ , has the form  $A = \lambda I_n$ .*

*Proof.* Result follows from Schur's lemma and its partial converse.  $\square$

**Example 5.10.** Suppose  $G = C_3 = \langle a \mid a^3 = 1 \rangle$ , and  $\rho : G \rightarrow \mathrm{GL}(2, \mathbb{C})$  is the representation defined by

$$a\rho = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

Let

$$A = a\rho = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

Then it is clear that  $A(g\rho) = (g\rho)A$  for all  $g \in G$ . As  $A$  is not a scalar multiple of the identity, the representation is reducible.  $\blacktriangleleft$

**Example 5.11.** Suppose  $G = D_5 = \langle a, b \mid a^5 = 1, b^2 = 1, a^b = a^{-1} \rangle$ . Set  $\omega = e^{2\pi i/5}$ . Let  $\rho : G \rightarrow \mathrm{GL}(2, \mathbb{C})$  be the representation defined by

$$a \mapsto \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

If

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

---

commutes with  $a\rho$  and  $b\rho$ ; one calculates that

$$A = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} = \alpha I_n,$$

and so the representation is irreducible. ◀

**Lemma 5.12.** *Let  $G$  be a finite abelian group. Then every irreducible  $\mathbb{C}G$ -module has dimension 1.*

*Proof.* Choose  $x \in G$ , then  $v(gx) = v(xg)$  for all  $g \in G$ , and so  $v \mapsto vx$  is a  $\mathbb{C}G$ -homomorphism. It is actually an isomorphism with inverse  $v \mapsto vx^{-1}$ . Hence by Schur this isomorphism is a scalar multiple of the identity, say  $\lambda_x 1_V$ . Thus  $vx = \lambda_x 1_V$  for all  $v \in V$  and the group action by  $G$  is just usual scalar multiplication. This means that every subspace is a submodule. However,  $V$  is irreducible and so has no non-trivial submodules; which forces  $\dim V = 1$ . ◻

## 6 Lecture 6

Continuing our discussion of the representations of abelian groups, we provide a stronger theorem in which we construct these 1-dimensional representations by mapping group elements to roots of unity. But first recall the Fundamental Theorem of Abelian Groups which states that any finite abelian group  $G$  is isomorphic to the direct sum of cyclic groups

$$G \simeq C_{n_1} \oplus \cdots \oplus C_{n_r}$$

where  $n_i \mid n_{i+1}$  for all  $1 \leq i \leq r-1$ . Note that if  $C_i = \langle g_i \rangle$  then we can write  $G = \langle g_1, \dots, g_r \rangle$  and  $g_i$  has order  $n_i$ .

We define a homomorphism  $\rho$  from  $G$  to  $\mathbb{C}$  by taking  $g_i \mapsto \lambda_i$  where  $\lambda_i$  is the  $n_i$ -th root of unity. This defines a representation and is specified by roots of unity  $\lambda_1, \dots, \lambda_r$ . Thus for the representation  $\rho$  defined by roots of unity  $\lambda_1, \dots, \lambda_r$  we write  $\rho = \rho_{\lambda_1, \dots, \lambda_r}$ .

**Theorem 6.1.** *Suppose  $G \simeq C_{n_1} \times \cdots \times C_{n_r}$  for cyclic groups  $C_{n_i}$  of order  $n_i$ . The representation  $\rho_{\lambda_1, \dots, \lambda_r}$  of  $G$  is irreducible of degree 1. There are  $|G|$  many such representations and every irreducible representation of  $G$  over  $\mathbb{C}$  is equivalent to one of these.*

**Example 6.2.** Take  $G = \langle a \mid a^n = 1 \rangle$  and  $\omega = e^{2\pi i/n}$ . The irreducible representations of  $G$  are  $\rho_{\omega^j}$  for  $0 \leq j \leq n-1$  defined by

$$a^k \rho_{\omega^j} = \omega^{jk}, \quad 0 \leq k \leq n-1.$$

◀

## 6.1 Application of Schur's to $\mathbb{C}G$

**Definition 6.3.** If  $G$  is a finite group then we define the center of the group algebra  $Z(\mathbb{C}G)$  by

$$Z(\mathbb{C}G) = \{z \in \mathbb{C}G \mid zr = rz \quad \forall r \in \mathbb{C}G\}.$$

We now state some simple properties of the center:

1.  $Z(\mathbb{C}G)$  is a subspace of  $\mathbb{C}G$ .
2. If  $G$  is abelian then  $Z(\mathbb{C}G) = \mathbb{C}Z(G)$ .
3. If  $H$  is a normal subgroup of  $G$  then  $\sum_{h \in H} h \in Z(\mathbb{C}G)$ .

We provide a simple proof for the last statement.

*Proof.* Take  $z = \sum_{h \in H} h$  and  $g \in G$ . We then have

$$z^g = \sum_{h \in H} h^g = z.$$

because  $H$  is normal and hence fixed by conjugation of elements in  $G$ . □

**Example 6.4.** Take  $G = S_3 = \langle a = (1\ 2\ 3), b = (1\ 2) \rangle$ . Then  $\sum_{g \in G} g \in Z(\mathbb{C}G)$ . ◀

**Lemma 6.5.** Let  $V$  be an irreducible  $\mathbb{C}G$ -module and let  $z \in Z(\mathbb{C}G)$ . There exists  $\lambda \in \mathbb{C}$  such that  $vz = \lambda v$  for all  $v \in V$ .

*Proof.* For all  $r \in \mathbb{C}G$  and  $v \in V$  we know  $vrz = v zr$ . Thus the mapping  $v \mapsto vz$  is a  $\mathbb{C}G$ -homomorphism from  $V$  to  $V$ . The result then follows by Schur. □

**Remark 2.** Note that  $\mathbb{C}G$  is a faithful module. ♦

**Lemma 6.6.** If there exists a faithful, irreducible  $\mathbb{C}G$ -module then  $Z(G)$  is cyclic.

*Proof.* Let  $V$  be an irreducible, faithful  $\mathbb{C}G$ -module and take  $z \in Z(G) \subset Z(\mathbb{C}G)$ . There exists  $\lambda_z \in \mathbb{C}$  such that  $vz = \lambda_z v$  for all  $v \in V$ . But since  $V$  is faithful the mapping  $z \mapsto \lambda_z$  is injective from  $Z(G)$  to  $\mathbb{C}^\times$  and so  $Z(G) \simeq \{\lambda_z \mid z \in Z(G)\}$  is a finite subgroup of  $\mathbb{C}^\times$ . All of which are cyclic. □

**Example 6.7.**  $\mathbb{Z}_4$  has a faithful, irreducible representation by taking 1 to the 4-th root of unity. However  $\mathbb{Z}_2 \times \mathbb{Z}_2$  has no faithful irreducible representation as its center is not cyclic. ◀

**Remark 3.** Note that the converse is not true in general. There exists groups with cyclic centers but no faithful, irreducible representations. See Frobenius Groups. ♦

**Lemma 6.8.** Suppose  $G$  finite and every irreducible  $\mathbb{C}G$ -module has dimension 1. Then  $G$  is abelian.

*Proof.* Since  $\mathbb{C}G$  is a  $\mathbb{C}G$ -module and it is completely reducible, we can decompose it as follows

$$\mathbb{C}G = \bigoplus_{i=1}^{|G|} V_i$$

where each  $V_i$  is one-dimensional. For any  $v_i \in V_i$  and  $x, y \in G$ , note that

$$v_i xy = \lambda_x v_i y = \lambda_y \lambda_x v_i = \lambda_x \lambda_y v_i = v_i yx$$

Since the  $v_i$  form a basis for  $\mathbb{C}G$  this means  $vxy = v yx$  for all  $v \in \mathbb{C}G$  and  $x, y \in G$ . Since  $\mathbb{C}G$  is faithful, the result follows.  $\square$

## 6.2 The Group Algebra and Irreducible Modules

**Lemma 6.9.** *Let  $V, W$  be  $\mathbb{C}G$ -modules and  $\theta : V \rightarrow W$  is a  $\mathbb{C}G$ -homomorphism. There exists a submodule  $U < V$  such that  $V = \ker \theta \oplus U$  and  $U \simeq \text{im } \theta$*

*Proof.* We apply Maschke's theorem to  $\ker \theta < V$  and obtain submodule  $U < V$  such that  $V = \ker \theta \oplus U$ . Now define the map  $\bar{\theta} : U \rightarrow \text{im } \theta$  by  $u \mapsto u\theta$  which is a homomorphism because it is the restriction of  $\theta$  to a subspace. The kernel of  $\bar{\theta}$  is trivial as  $\ker \bar{\theta} = \ker \theta \cap U = \{0\}$ . And  $\text{im } \theta = \text{im } \bar{\theta}$  because if  $w \in \text{im } \theta$  then  $w = v\theta$  for some  $v \in V$  and  $v = k + u$  for some  $u \in U$  and  $k \in \ker \theta$ . We then have

$$w = v\theta = (u + k)\theta = u\theta = u\bar{\theta} \in \text{im } \bar{\theta}.$$

Finally, by the first isomorphism theorem we obtain

$$U \simeq U / \ker \bar{\theta} \simeq \text{im } \bar{\theta} = \text{im } \theta.$$

$\square$

## 7 Lecture 7

### 7.1 The Group Algebra and Irreducible Modules: Part II

**Definition 7.1.** For  $V$ , a  $\mathbb{C}G$ -module,  $U$  an irreducible  $\mathbb{C}G$ -module,  $U$  is called a composition factor for  $V$  if  $V$  has a  $\mathbb{C}G$ -submodule isomorphic to  $U$ .

**Lemma 7.2.** *Let  $V$  be a  $\mathbb{C}G$ -module, suppose we have a finite decomposition into into a direct sum of irreducible  $\mathbb{C}G$ -modules:*

$$V = \bigoplus_{i=1}^n U_i$$

*Then, if  $0 < U < V$  is an irreducible,  $\mathbb{C}G$ -submodule of  $V$ , then  $U \simeq U_i$  for some  $1 \leq i \leq n$ .*

*Proof.* Consider the maps  $\pi_i : U \rightarrow U_i$ , since each  $\pi_i : V \rightarrow U_i$  is a  $\mathbb{C}G$ -homomorphism, and since  $U$  is non-trivial, it is clear that not all the  $\pi_i$  are the zero map, let  $\pi_j$  be such a projection, then by Schur,  $\pi_j : U \rightarrow U_j$  is a  $\mathbb{C}G$ -isomorphism.  $\square$

**Theorem 7.3.** *Let the following be a decomposition of the regular representation into irreducible  $\mathbb{C}G$ -modules:*

$$\mathbb{C}G = \bigoplus_{i=1}^n U_i$$

*Then every irreducible  $\mathbb{C}G$ -module is isomorphic to  $U_i$  for some  $1 \leq i \leq n$*

*Proof.* Let  $W$  be an irreducible  $\mathbb{C}G$ -module, let  $0 \neq w \in W$ , then  $wG = \{wr : r \in \mathbb{C}G\}$  is a  $\mathbb{C}G$ -submodule of  $W$ . However,  $W$  is irreducible, so  $W = wG$ . Define  $\theta : \mathbb{C}G \rightarrow W$  by  $r\theta = wr$  for  $r \in \mathbb{C}G$ , then  $\theta$  is a linear  $\mathbb{C}G$ -homomorphism by 3.10 and  $\text{im } \theta = W$ . So  $\mathbb{C}G = \ker \theta \oplus U$ , where  $U \simeq \text{im } \theta \simeq W$  by 6.9. Now  $W$  is irreducible, so  $U$  is irreducible, so by 7.2,  $U \simeq U_i \simeq W$  for some  $1 \leq i \leq n$ .  $\square$

Now for a finite group  $G$ , we can characterise all irreducible  $\mathbb{C}G$ -modules by decomposing  $\mathbb{C}G$ .

**Example 7.4.** Let  $G = C_3 = \langle a | a^3 = 1 \rangle$ , then let  $\omega = e^{\frac{2\pi i}{3}}$  be a primitive 3rd root of unity and let the following be elements of  $\mathbb{C}G$ :

$$v_1 = 1 + a + a^2, \quad v_2 = 1 + \omega^2 a + \omega a^2, \quad v_3 = 1 + \omega a + \omega^2 a^2$$

We have  $(v_1)a = v_1$ ,  $(v_2)a = \omega + a + \omega^2 a^2 = \omega v_2$  and  $(v_3)a = \omega^2 + a + \omega a = \omega^2 v_3$ . Therefore,  $V_i = \langle v_i \rangle < \mathbb{C}G$  for  $i = 1, 2, 3$  are all irreducible submodules of  $\mathbb{C}G$ .

Furthermore, it is clear that  $\{v_1, v_2, v_3\}$  forms a basis for  $\mathbb{C}G$ , hence  $\mathbb{C}G = V_1 \oplus V_2 \oplus V_3$ .  $\blacktriangleleft$

**Example 7.5.** Let  $G = D_6 = \langle a, b | a^3 = 1, a^b = a^{-1} \rangle$ .  $\blacktriangleleft$

## 7.2 The Vector Space of $\mathbb{C}G$ -homomorphisms

**Definition 7.6.** Let  $V, W$  be  $\mathbb{C}G$ -modules, then define  $H = \text{Hom}_{\mathbb{C}G}(V, W) = \{\theta : V \rightarrow W : \theta \text{ is a } \mathbb{C}G \text{ homomorphism}\}$ .

We can then define addition and scalar multiplication operations on this set for  $\theta, \phi \in H$ ,  $\lambda \in \mathbb{C}$  as follows:

$$v(\theta + \phi) := v\theta + v\phi, \quad v(\lambda\theta) = \lambda(v\theta)$$

**Lemma 7.7.** *The space  $H$  with the operations defined above is a vector space over  $\mathbb{C}$ .*

*Proof.* Clear from the definition, note that the zero homomorphism is:  $v0 = 0_W$ .  $\square$

**Lemma 7.8.** *For irreducible  $\mathbb{C}G$ -modules,  $V, W$ ,*

$$\dim(H) = \begin{cases} 1 & V \simeq W \\ 0 & V \not\simeq W \end{cases}$$

*Proof.* If  $V \not\simeq W$ , then it follows that  $\theta = 0$  for all  $\theta \in H$  by Schur.

Now, suppose  $V \simeq W$  and let  $\theta : V \rightarrow W$  be an isomorphism. Now, let  $\phi \in H$ , we have that  $\phi\theta^{-1} : V \rightarrow V$  is a homomorphism, so by Schur,  $\phi\theta^{-1} = \lambda \text{Id}_V$ , so for  $v \in V$ ,  $v\phi\theta^{-1} = \lambda v \implies v\phi\theta^{-1}\theta = v\phi = (\lambda v)\theta = \lambda(v\theta)$ , therefore  $\phi = \lambda\theta$ , so  $\dim H = 1$ .  $\square$

**Proposition 7.9.** *Given  $V, V_1, V_2, W, W_1, W_2$   $\mathbb{C}G$ -modules,*

1.  $\dim \text{Hom}_{\mathbb{C}G}(V, W_1 \oplus W_2) = \dim \text{Hom}_{\mathbb{C}G}(V, W_1) + \dim \text{Hom}_{\mathbb{C}G}(V, W_2)$
2.  $\dim \text{Hom}_{\mathbb{C}G}(V_1 \oplus V_2, W) = \dim \text{Hom}_{\mathbb{C}G}(V_1, W) + \dim \text{Hom}_{\mathbb{C}G}(V_2, W)$

*Proof.* I will prove the first statement and the conversion of the proof of the second statement is left as a simple exercise.

Firstly, we define the projection homomorphisms:  $\pi_i : W_1 \oplus W_2 \rightarrow W_i$  defined by  $(w_1 + w_2)\pi_i = w_i$ . Now, if  $\theta \in \text{Hom}_{\mathbb{C}G}(V, W_1 \oplus W_2)$ , we have that  $\theta\pi_i \in \text{Hom}_{\mathbb{C}G}(V, W_i)$ .

Define a linear map  $f : \text{Hom}_{\mathbb{C}G}(V, W_1 \oplus W_2) \rightarrow \text{Hom}_{\mathbb{C}G}(V, W_1) \oplus \text{Hom}_{\mathbb{C}G}(V, W_2)$  by  $\theta \mapsto \theta\pi_1 \oplus \theta\pi_2$  then it is clear that the following map is an inverse of  $f$ :

$$v(\theta_1 \oplus \theta_2)f^{-1} = v\theta_1 + v\theta_2$$

therefore we have an isomorphism of vector spaces,  $\text{Hom}_{\mathbb{C}G}(V, W_1 \oplus W_2) \simeq \text{Hom}_{\mathbb{C}G}(V, W_1) \oplus \text{Hom}_{\mathbb{C}G}(V, W_2)$ , giving us  $\dim \text{Hom}_{\mathbb{C}G}(V, W_1 \oplus W_2) = \dim \text{Hom}_{\mathbb{C}G}(V, W_1) + \dim \text{Hom}_{\mathbb{C}G}(V, W_2) = \dim \text{Hom}_{\mathbb{C}G}(V_1, W) + \dim \text{Hom}_{\mathbb{C}G}(V_2, W)$   $\square$

**Corollary 7.9.1.** *For  $V_1, \dots, V_n, W_1, \dots, W_m$ , if  $V = \bigotimes_{i=1}^n V_i$  and  $W = \bigotimes_{i=1}^m W_i$*

$$\dim H = \sum_{i=1}^n \sum_{j=1}^m \dim \text{Hom}_{\mathbb{C}G}(V_i, W_j)$$

**Corollary 7.9.2.** *Suppose*

$$V = \bigoplus_{j=1}^n U_j$$

*Then for any irreducible module  $W$*

$$\dim \text{Hom}(V, W) = \dim \text{Hom}(W, V) = |\{j \mid U_j \cong W\}|$$

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**Example 7.10.** Recall that

$$\mathbb{C}D_3 = \bigoplus_{i=1}^4 U_i$$

where  $U_3 \cong U_4$ . Then

$$\dim \text{Hom}(\mathbb{C}D_3, U_3) = 2$$

◀

## 8 Lecture 8

We continue the discussion last lecture about  $\text{Hom}(U, V)$ .

**Lemma 8.1.** *Let  $V, W$  be two  $\mathbb{C}G$ -modules such that  $\text{Hom}(V, W)$  is nonzero. Then  $V$  and  $W$  share a composition factor.*

*Proof.* Suppose we have a morphism  $\theta : V \rightarrow W$ . Then there exists some  $v \in V$  such that  $v\theta \neq 0$ . Now  $v$  is contained in some irreducible submodule, say  $V_0$ . Then  $V_0\theta \cong V_0$ .  $\square$

**Lemma 8.2.** *Let  $U$  be a  $\mathbb{C}G$ -module. Then*

$$\dim \text{Hom}(\mathbb{C}G, U) = \dim U$$

.

*Proof.* Fix a basis  $\{u_1, \dots, u_n\}$  for  $U$  and define  $\varphi_i : \mathbb{C}G \rightarrow U$  as  $r \mapsto u_i r$ . We claim the  $\varphi_i$  form a basis for  $\text{Hom}(\mathbb{C}G, U)$ . Indeed, let  $\varphi \in \text{Hom}(\mathbb{C}G, U)$  and suppose

$$(1)\varphi = \sum_{i=1}^n \lambda_i u_i$$

Then for all  $r \in \mathbb{C}G$  we have

$$(r)\varphi = (1)\varphi r = \left( \sum_{i=1}^n \lambda_i u_i \right) r = \sum_{i=1}^n \lambda_i u_i r = \sum_{i=1}^n \lambda_i (r)\varphi_i$$

where the first equality follows from Problem 3  $\square$

**Theorem 8.3.** *Suppose*

$$\mathbb{C}G = \bigoplus_{j=1}^n V_j$$

*and  $U$  is an irreducible module. Then the number of  $j$  such that  $V_j \cong U$  is exactly  $\dim U$*

*Proof.* Combine Lemma 8.2 and Corollary 7.9.2  $\square$

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**Example 8.4.** Recall that

$$\mathbb{C}D_3 = \bigoplus_{i=1}^4 U_i$$

where  $U_3 \cong U_4$  but  $U_1 \not\cong U_2$ . Then  $U_1$  and  $U_2$  occur once whereas  $U_3$  occurs twice, consistently with the theorem.  $\blacktriangleleft$

**Theorem 8.5.** *Let  $V_1, \dots, V_n$  denote a complete set of irreducible modules that are pairwise non-isomorphic. Then*

$$\sum_{i=1}^n (\dim V_i)^2 = |G|$$

*Proof.* Suppose

$$\mathbb{C}G = \bigoplus_{j=1}^N U_j$$

where for each  $V_i$  there are exactly  $\dim V_i$  of the  $U_j$  isomorphic to  $V_i$ . Thus we have

$$|G| = \dim \mathbb{C}G = \sum_{i=1}^N \sum_{j=1}^{\dim V_i} \dim U_j = \sum_{i=1}^N \sum_{j=1}^{\dim V_i} \dim V_i = \sum_{i=1}^n (\dim V_i)^2$$

□