Maths 721 Notes

2020

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1 Lecture 1

In the first half of this course we will cover three main topics:

- representations;
- modules;
- characters.

We will further see that representations and modules are essentially the same, and that modules and characters are essentially the same; and hence all three are essentially the same.

From now on G is a group.

1.1 Representations

Definition 1.1. A **representation** of a group G over a field F is a group homomorphism from G to $\mathrm{GL}(n, F)$, where n is the **degree** of the representation.

Explicitly, a representation is a function $\rho : G \to \mathrm{GL}(n, F)$ such that for all $g, h \in G$;

(i) $(gh)\rho = (g\rho)(h\rho)$,

(ii) $1_G\rho = I_n$,

(iii) $g^{-1}\rho = (g\rho)^{-1}$.

Note the use of the (incredibly shit) postfix function notation.

Example 1.2. Take D_4, the Dihedral group of order 8. It has the following group presentations

$$\begin{aligned} D_4 &= \langle a, b \mid a^4 = 1, b^2 = 1, a^b = a^{-1} \rangle \\ &\cong \langle (1\ 2\ 3\ 4), (1\ 4)(2\ 3) \rangle, \end{aligned}$$

where $a^b = bab^{-1}$ is conjugation of a by b. By defining the matrix subgroup

$$H = \left\langle A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\rangle$$

and defining $\rho : D_4 \to \mathrm{GL}(n, F)$ where $F = \mathbb{R}, \mathbb{C}$, by $a \mapsto A$ and $b \mapsto B$, and $a^i b^j \mapsto A^i B^j$ for $0 \le i \le 3$, and $0 \le j \le 1$. Hence we have ρ is a representation of D_4 over F. ◀

Example 1.3. Take $\mathbb{Q}_8$ the Quaternion group of order 8, which has the following group presentations

$$\begin{aligned} \mathbb{Q}_8 &= \langle a, b \mid a^4 = 1, a^2 = b^2, a^b = a^{-1} \rangle \\ &\cong \langle \bar{a} = (1\ 6\ 2\ 5)(3\ 8\ 4\ 7), \bar{b} = (1\ 4\ 2\ 3)(5\ 7\ 6\ 8) \rangle \end{aligned}$$

Define

$$H = \left\langle A = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\rangle \subset \mathrm{GL}(2, \mathbb{C}).$$

Then $\rho : \mathbb{Q}_8 \to \mathrm{GL}(2, \mathbb{C})$ defined by $a^k b^\ell \mapsto A^k B^\ell$ is a group representation of $\mathbb{Q}_8$ over $\mathbb{C}$ of degree 2. ◀

Definition 1.4. Let G be a group and define

$$\begin{aligned} \rho : G &\to \mathrm{GL}(n, F) \\ g\rho &= I_n \end{aligned}$$

for all $g \in G$. Then ρ is a representation, called the **trivial representation** of arbitrary of degree.

It follows from the trivial representation that any group G has a representation of an arbitrary degree.

Let $\rho : G \to \mathrm{GL}(n, F)$ be a group homomorphism, and take $T \in \mathrm{GL}(n, F)$. Then

$$(T^{-1}AT)(T^{-1}BT) = T^{-1}(AB)T.$$

Thus, given ρ define σ such that

$$g\sigma = T^{-1}(g\rho)T$$

for all $g \in G$. As for all $g, h \in G$, one has

$$\begin{aligned} (gh)\sigma &= T^{-1}\big((gh)\rho\big)T \\ &= T^{-1}(g\rho)(h\rho)T \\ &= T^{-1}(g\rho)TT^{-1}(h\rho)T \\ &= (g\sigma)(h\sigma), \end{aligned}$$

and so σ is a group homomorphism; and hence a representation.

Definition 1.5. Define

$$\rho : G \to \mathrm{GL}(m, F), \qquad \sigma : G \to \mathrm{GL}(n, F)$$

to both be representation of G over F. We say that ρ **is equivalent to** σ if $n = m$ and there exists $T \in \mathrm{GL}(n, F)$ such that $g\sigma = T^{-1}(g\rho)T$.

Proposition 1.6. *Equivalence of representations is an equivalence relation.*

Proof. Reflexivity is clear by taking $T = I_n$. For symmetry, take T to be its inverse. For transitivity, if

$$g\sigma = T^{-1}(g\rho)T, \qquad g\rho = S^{-1}(g\eta)S,$$

then

$$g\sigma = (ST)^{-1}(g\eta)(ST).$$

□

Definition 1.7. Define the **kernel** of the representation $\rho : G \to \mathrm{GL}(n, F)$ as $\ker \rho = \{g \in G \mid g\rho = I_n\}$.

Proposition 1.8. *The kernel of a representation of G is a normal subgroup of G; i.e.* $\ker \rho \lhd G$.

Proof. Suppose $g \in \ker \rho$ and $h \in G$ is arbitrary. Then

$$(hgh^{-1})\rho = (h\rho)(g\rho)(h^{-1}\rho) = (h\rho)I_n(h\rho)^{-1} = (h\rho)(h\rho)^{-1} = I_n,$$

and so $hgh^{-1} \in \ker \rho$. As $\ker \rho$ is closed under conjugation, it is a normal subgroup of G. □

Definition 1.9. We say ρ is a **faithful** representation of G if $\ker \rho = \{1_G\}$.

Example 1.10. For the trivial representation $\rho : G \to \mathrm{GL}(n, F)$ with $g \mapsto I_n$ for all $g \in G$, we have $\ker \rho = G$. Hence the representation is not faithful. ◀

Lemma 1.11. *Suppose G is a finite group, and ρ is a representation of G over F. Then ρ is faithful if, and only if,* $\operatorname{im} \rho \cong G$.

Proof. Immediate from the first isomorphism theorem. □

1.2 FG-Modules

Suppose G is a group, and $F = \mathbb{R}, \mathbb{C}$. Given $\rho : G \to \mathrm{GL}(n, F)$, with $V = F^n$, let $v = (\lambda_1, \ldots, \lambda_n) \in V$ for $\lambda_i \in F$ be a row vector. Moreover, note that $g\rho$ is an $n \times n$ matrix for all $g \in G$. Thus, we have $v \cdot (g\rho) \in V$, and satisfies the following properties:

(i) $v \cdot \big((gh)\rho\big) = v \cdot (g\rho)(h\rho)$;

(ii) $v \cdot (1_G\rho) = v$;

(iii) $(\lambda v) \cdot (g\rho) = \lambda\big(v \cdot (g\rho)\big)$;

(iv) $(u + v) \cdot (g\rho) = u \cdot (g\rho) + v \cdot (g\rho)$.

We often will omitted the $\cdot$ in the operation, and write $v(a\rho)$ for $v \cdot (a\rho)$.

Example 1.12. Recall D_4 and its given presentation from a previous example. We have

$$a\rho = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad b\rho = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

If $v = (\lambda_1, \lambda_2)$, then we have

$$v(a\rho) = (-\lambda_2, \lambda_1), \qquad v(b\rho) = (\lambda_1, -\lambda_2).$$

◀

Definition 1.13. Let V a vector space over the field $F = \mathbb{R}, \mathbb{C}$, and let G be a group. We say V is a FG**-module** if a multiplication $v \cdot g$ for $v \in V$, and $g \in G$ is defined such that:

(i) $v \cdot g \in V$;

(ii) $v \cdot (gh) = (v \cdot g) \cdot h$;

(iii) $v \cdot 1_G = v$;

(iv) $(\lambda v) \cdot g = \lambda(v \cdot g)$;

(v) $(u + v) \cdot g = u \cdot g + v \cdot g$.

This generalises the previous discussion from a matrix group to an arbitrary group.

Note that properties (i), (iv), and (v) imply that the map $v \mapsto v \cdot g$ is an endomorphism of V (a linear map from V to itself).

Definition 1.14. Suppose V is an FG-module and B is a basis for V. For $g \in G$, let $[g]_B$ denote the matrix of the endomorphism $v \mapsto v \cdot g$ of V relative to the basis B.

2 Lecture 2

Theorem 2.1. *Let $\rho : G \to GL(n, F)$ be a representation of G over F.*

(I) *If $V = F^n$ is an FG module and G acts on V by $v \cdot g = v(g\rho)$ there exists a basis B of V such that $g\rho = [g]_B$.*

(II) *The map $g \mapsto [g]_B$ is a representation for G over F.*

Proof. Choose the standard basis $B = [e_1, \ldots, e_n]$.

Since V is an FG-module we have $v(gh) = (vg)h$ for all $g, h \in G$ and $v \in V$. Thus $[gh]_B = [g]_B[h]_B$ so the map is a homomorphism. We now check that $[g]_B$ is invertable for all $g \in G$. We know $v \cdot 1_G = (vg)g^{-1}$ so $I_n = [g]_B[g^{-1}]_B$ and thus $[g]_B$ has an inverse. □

Example 2.2. Recall the representation of $G = D_4$ from a previous example. Define an FG-module $V = F^2$ with the action defined by taking vg to $v(g\rho)$.

$$\begin{aligned} v_1 &= (1, 0), \quad v_1 a = v_2, \quad v_1 b = v_1, \\ v_2 &= (0, 1), \quad v_1 a = -v_1, \quad v_1 b = -v_2. \end{aligned}$$

In this basis we recover our representation

$$a \mapsto [a]_B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad b \mapsto [b]_B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

◀

We now provide an equivalent basis-dependent definition for an FG-module.

Lemma 2.3. *Let V a vector space over the field $F = \mathbb{R}, \mathbb{C}$ with basis $B = [v_1, \ldots, v_n]$, and let G be a group. If a multiplication $v \cdot g$ for $v \in B$, and $g \in G$ is defined such that:*

(i) $v \cdot g \in V$;

(ii) $v \cdot (gh) = (v \cdot g) \cdot h$;

(iii) $v \cdot 1_G = v$;

(iv) $(\sum_{i=1}^{n} \lambda_i v_i) \cdot g = \sum_{i=1}^{n} \lambda_i (v_i \cdot g)$ *for all* $\lambda_i \in F$;

then V is an FG-module.

Definition 2.4. The trivial module of a group over F is a one dimensional vector space V over F such that $vg = v$ for all $v \in V$ and $g \in G$.

Definition 2.5. An FG-module is faithful if 1_G is the only $g \in G$ such that $vg = v$ for all $v \in V$.

Theorem 2.6. *Let V be an FG-module with basis B and ρ a representation of group G over F defined by taking $g \mapsto [g]_B$.*

(i) If B' is another basis of V then the map $g \mapsto [g]_{B'}$ is a representation of G equivalent to ρ.

(ii) If representation σ is equivalent to ρ then there exists basis B'' such that $\sigma(g) = [g]_{B''}$ for all $g \in G$.

Proof. Taking T to be the change of basis matrices, the two representations are equivalent. □

Example 2.7. Let $G = C_3 = \langle a \mid a^3 = 1 \rangle$ and representation $\rho : G \to GL(n, F)$ defined by

$$a \mapsto \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}.$$

We attempt to construct an FG-module with group action described by ρ. Take $V = F^2$ with basis $B = [v_1, v_2]$. Define the action of G on V by

$$v_1 a = v_2, \quad v_2 a = -v_1 - v_2.$$

Let us now choose alternate basis $B' = [u_1 = v_1, u_2 = v_1 + v_2]$. The action of G on this basis is described by

$$u_1 a = -u_1 + u_2, \quad u_2 a = -u_1.$$

This gives us a representation

$$a \mapsto [a]_{B'} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}.$$

To verify this construction we write our change of basis matrix as

$$T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

and verify that

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}.$$

◀

Definition 2.8. The permutation module of a group $G \leq S_n$ is an n-dimensional vector space V with basis $B = [v_1, \ldots, v_n]$ and action by G defined by

$$v_i g = v_{ig}$$

for all $g \in G$ where ig is the image of i under $g \in S_n$.

It follows from Caley's theorem that every group has a faithful FG-module.

Example 2.9. Take $G = S_4$ and pick $g = (1\ 2)$ and $h = (1\ 2\ 3\ 4)$. We have representations

$$[g]_B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad [h]_B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

◀

2.1 Module Reducibility

Definition 2.10. Let V be an FG-module. We call W a submodule of V if W is a vector subspace of V and W is closed under the action of G. We then write $W < V$.

Example 2.11. Let $G = C_3 = \langle (1\ 2\ 3) \rangle$ and V the permutation module of G with basis $B = [v_1, v_2, v_3]$. The subspace $W = \langle v_1 + v_2 + v_3 \rangle$ is a submodule but the subspace $U = \langle v_1 + v_2 \rangle$ is not.

For example, consider the action of $g = (1\ 2\ 3)$ on $v_1 + v_2 \in U$.

$$(v_1 + v_2)g = v_{1g} + v_{2g} = v_2 + v_3 \notin U$$

whereas G acts on W trivially. ◀

3 Lecture 3

3.1 Module and Representation Reducibility

For any module, it is clear that we have two trivial submodules: $0 < v$ and $V < V$. Where $0 = \{0\} \subset V$.

Definition 3.1. Let V be an FG-module. We say that V is irreducible if the only submodules of V are V and 0. Otherwise V is reducible

In 2.11 we showed that the permutation module of C_3 is reducible.

Definition 3.2. Let $\rho : G \to GL(n, F)$ be a representation. We say that ρ is irreducible if the corresponding FG-module (as constructed in 2.1) is irreducible. Otherwise ρ is reducible.

If an FG-module, V is reducible, that is, $0 < W < V$, $0 \neq W \neq V$. Let B_W be a basis for W. If we extend B_W to B a basis of V, then we get the following representation of G:

$$g \mapsto [g]_B = \begin{bmatrix} X_g & 0 \\ Y_g & Z_g \end{bmatrix} \tag{3.1}$$

Where the matrices X_g, Y_g and Z_g are some block matrices and 0 is a block of zeros and X_g has the dimensions $m \times m$ and, in this case, $\dim(W) = m$.

Proposition 3.3. *A representation $\rho : G \to GL(n, F)$ is reducible if and only if with respect to some basis, B, of F^n, $[g]_B$ has the form 3.1 for some $0 < m < \dim(V)$ for all $g \in G$. Then the maps $g \mapsto X_g$ and $g \mapsto Z_g$ are both representations of G.*

Proof. Suppose we have a presentation, $\rho : G \to GL(n, F)$ and a basis B of $V = F^n$ such that $[g]_B$ has the form 3.1 for every $g \in G$. Then consider the subspace $0 \subset W \subset V$ spanned by the first m elements of B. It is clear that $v[g]_B \in W$ for all $v \in W$. Therefore the module induced by ρ is reducible, so ρ is reducible. Now, if we have a reducible representation, then the argument above this proposition shows that with respect to any basis extending B_W, the matrices $[g]_B$ have the required form.
Now, using elementary block matrix multiplication, we get the following for $g, h \in G$:

$$\rho(g)\rho(h) = [g]_B[h]_B = \begin{bmatrix} X_g X_h & 0 \\ Y_g X_h + Z_g Y_h & Z_g Z_h \end{bmatrix} = [gh]_B = \rho(gh)$$

Therefore $X_{gh} = X_g X_h$ and $Z_{gh} = Z_g Z_h$, so the maps $g \mapsto X_g$ and $g \mapsto Z_g$ are both representations of G. $\square$

Problem 1. Prove that the example representation of D_8 of degree 2 over $\mathbb{R}$ or $\mathbb{C}$ is irreducible.

3.2 Group Algebras

Recall that an algebra over a field F is a vector space over F equipped with a bilinear product $A \times A \to A$ that is compatible with scalar multiplication.

Definition 3.4. The group algebra over a finite group G over a field F is an algebra[1] of dimension $n = |G|$ over $F = \mathbb{R}$ or $\mathbb{C}$ called FG, with basis $B = G = \{g_1, \ldots g_n\}$.

[1] See Lemma3.6

Where the algebra structure is given by the following for two arbitrary elements of FG, $u = \sum_{g \in G} \lambda_g g$, $v = \sum_{g \in G} \mu_g$, $\lambda_g, \mu_g \in F$ and $\nu \in F$:

1. $u + v = \sum_{i=1}^{n} (\lambda_i + \mu_i) g_i$

2. $\nu \cdot u = \sum_{i=1}^{n} (\nu \lambda_i) g_i$

3. $u \cdot v = \sum_{(h,g) \in G \times G} \lambda_g \mu_h (gh)$

This is clearly a vector space.

Example 3.5. Consider $G = C_3 = \{e, a, a^2\} = \langle a | a^3 = e \rangle$ and $F = \mathbb{R}$ or $\mathbb{C}$. Then if we let $u = e - a + 2a^2$, $v = \frac{1}{2}e + 5a$, then:

$$u + v = \frac{3}{2}e + 4a + 2a^2, \quad \frac{1}{3}u = \frac{1}{3}e - \frac{1}{3}a + \frac{2}{3}a^2, \quad uv = \frac{21}{2}e + \frac{9}{2}a - 4a^2$$

◀

Lemma 3.6. *Given a group algebra FG, $r, s, t \in FG$, $\lambda \in F$:*

1. $rs \in FG$

2. $(rs)t = r(st)$

3. $1_G r = r 1_G = r$

4. $(\lambda r)s = \lambda(rs)$

5. $(r + s)t = rt + st$

6. $r(s + t) = rs + rt$

7. $r0 = 0r = 0$

That is, FG is an associative algebra with unit

Proof. 1,3 and 7 are clear from the definition of FG, 4,5 and 6 follow from the distributive and associative laws of F and 2 follows from associativity in G. □

3.3 The Regular FG-module, FG

Problem 2. $V = FG$ is an FG-module with the group action defined by $v \cdot g = vg$ for $v \in FG$, $g \in G \subset FG$.

Definition 3.7. For a finite group G and $F = \mathbb{R}$ or $\mathbb{C}$, the regular FG-module is FG. The associated module, $g \mapsto [g]_B$ is called the regular representation.

Lemma 3.8. *FG is a faithful module for G over F*

Proof. If $vg = v$ for all $v \in FG$, then specifically, $hg = h$ for all $h \in G$, so $g = 1_G$. □

Example 3.9. For $C = C_3$, over the basis $B = G$, we get:

$$[e]_B = I_3, \quad [a]_B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad [a^2]_B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

◀

Now, if we have an FG-module, V, then FG acts on V in the following way:

$$v \cdot r = v \cdot \left(\sum_{g \in G} \mu_g g \right) = \sum_{g \in G} \mu_g (v \cdot g)$$

Lemma 3.10. *For $u, v \in V$, $\lambda \in F$, $r, s \in FG$:*

1. $vr \in FG$

2. $(vr)s = v(rs)$

3. $v1 = v$

4. $(\lambda v)r = \lambda(vr) = v(\lambda r)$

5. $(r + s)v = rv + sv$

6. $r(u + v) = ru + rv$

7. $r0 = v0 = 0$

Proof. 1,3 and the first part of 7 follow from V being an FG-module, the second equality of 7 follows from scalar multiplication by 0 in V. The following calculation:

$$\begin{aligned}
(\lambda v)r &= \sum_{g\in G} \mu_g((\lambda v)g) \\
&= \sum_{g\in G} \mu_g(\lambda(vg)) \\
&= \sum_{g\in G} (\lambda\mu_g)(vg) = v(\lambda r) \\
&= \lambda \sum_{g\in G} \mu_g(vg) \\
&= \lambda(vg)
\end{aligned}$$

proves 4. 6 follows from the linearity of the action of G on V. 5 follows from distributivity of scalar multiplication in V. Finally, to prove 2:

$$\begin{aligned}
v(rs) &= \sum_{(g,h)\in G\times G} (\mu_g \lambda_h (v(gh))) \\
&= \sum_{h\in G} \lambda_h \sum_{g\in G} \mu_g (gv)h \\
&= \sum_{h\in G} \lambda_h \left(\sum_{g\in G} \mu_g(gv) \right) h \\
&= \sum_{h\in G} \lambda_h (vr)h = (vr)s
\end{aligned}$$

□

4 Lecture 4

4.1 Homomorphisms

Definition 4.1. Let V and W be FG-modules. A *homomorphism* of FG-modules is a map $\sigma : V \to W$ which is a linear transformation and also satisfies $(vg)\sigma = (v\sigma)g$ for all $g \in G, v \in V$. The *kernel* and *image* are defined in the obvious way

Equivalently, it is a homomorphism of modules over the ring FG. Indeed:

Problem 3. Suppose $r \in FG$ is an element of the group algebra. Prove that $(vr)\sigma = (v\sigma)r$.

Lemma 4.2. *Let $\sigma : V \to W$ be a homomorphism of FG-algebras. Then the kernel and image of σ are submodules*

Proof. This is a matter of simple checking, which will be left to the reader. □

Example 4.3. Take $\sigma : V \to V$ to be $v \mapsto \lambda v$ for some $\lambda \in F^*$. Then $\ker \sigma = 0, \operatorname{im} \sigma = V$. ◀

Example 4.4. Let $G = S_n$ and $V = \langle v_1, ..., v_n \rangle$ be the permutation module for G over F, and let $W = \langle w \rangle$ be the trivial module. Now define $\sigma : V \to W$ by

$$\sum \lambda_i v_i \mapsto \sum \lambda_i w$$

Then $\ker \sigma = \{\sum \lambda_i v_i \mid \sum \lambda_i = 0\}$ and $\operatorname{im} \sigma = W$. ◀

Definition 4.5. A homomorphism of FG-modules is an isomorphism if it is bijective

Remark 1. In class we originally said "if the homomorphism has trivial kernel". However, this is definitely not correct because inclusions are always homomorphisms, but obviously not isomorphisms. ⧫

Lemma 4.6. *The inverse of an isomorphism is an isomorphism*

Proof. Once again, this is just an exercise in checking. The details will be left for the reader. □

Some rather obvious invariants of FG-modules (under isomorphism) are dimension and irreducibility.

Lemma 4.7. *V and W are isomorphic if and only if there exists bases $\mathcal{B}_1$ of V and $\mathcal{B}_2$ of W such that*

$$[g]_{\mathcal{B}_1} = [g]_{\mathcal{B}_2}$$

for all g.

Proof. Suppose firstly that V and W are isomorphic, and let $\sigma : V \to W$ be one such isomorphism. Let $\mathcal{B}_1 = \{v_1, ..., v_n\}$ be a basis for V. In particular, it is linearly independent, and it is easy to see that $\mathcal{B}_2 = \{v_1\sigma, ..., v_n\sigma\}$ is also linearly independent. Since V and W are isomorphic, they have the same dimension, and thus $\mathcal{B}_2$ is a basis for W. Since $(vg)\sigma = (v\sigma)g$ for all g and v, the action of g on the basis vectors of both bases are the same, and thus we conclude $[g]_{\mathcal{B}_1} = [g]_{\mathcal{B}_2}$.

Conversely, suppose that the latter hypothesis is satisfied. Let $\{v_1, ..., v_n\}$ be a basis for V and $\{w_1, ..., w_n\}$ be a basis for W. We define a bijective linear map $\sigma : V \to W$ such that $v_i\sigma = w_i$ for each i. Now observe that for each i, we have $v_i g = \lambda_1 v_1 + ... + \lambda_n v_n$ and $w_i g = \lambda_1 w_1 + ... + \lambda_n w_n$, where $(\lambda_1, ..., \lambda_n)$ is the i-th row of $[g]$. This means that

$$(v_i g)\sigma = (\lambda_1 v_1 + ... + \lambda_n v_n)\sigma = \lambda_1 v_1\sigma + ... + \lambda_n v_n\sigma = \lambda_1 w_1 + ... + \lambda_n w_n = w_i g = (v_i\sigma)g$$

and thus σ is a homomorphism of FG-modules. Since it is bijective, it is an isomorphism. □

Theorem 4.8. *Let V be an FG-module with basis $\mathcal{B}_1$ and W an FG-module with basis $\mathcal{B}_2$. Then $W \cong V$ if and only if $g \mapsto [g]_{\mathcal{B}_1}$ and $g \mapsto [g]_{\mathcal{B}_2}$ are equivalent.*

Proof. This follows from the previous Lemma and the fact that two matrices are conjugate (A and B are conjugate if $A = P^{-1}BP$ for some P) if and only if the linear transformations they define differ by a change of basis (that is they define the same transformation but with respect to different bases) $\square$

Example 4.9. Let $G = C_3 = \{e, a, a^2\}$. Let V be the regular representation, that is the natural representation induced by the module $FG = \langle e, a, a^2 \rangle$. Write $B := \{e, a, a^2\}$ as a basis for FG. Then

$$[a]_B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Now let W be the permutation module where $a = (1, 2, 3)$ and C_3 is considered a subgroup of S_3. Write $B' = \{v_1, v_2, v_3\}$ for the basis of W. Then

$$[a]_{B'} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Note that these two modules are isomorphic. ◀

Example 4.10. Let $G = D_4 = \langle a, b \mid a^4 = b^2 = 1, a^b = a^{-1} \rangle$. Now we can act on either F^4 or F^8. On F^4, we have the representation described in Example 1.2. On W, we have the regular representation. Clearly are not isomorphic. ◀

4.2 Sums

We now consider how modules behave with respect to direct sums. Let V be an FG-module and suppose $V = U \oplus W$, where U and W are submodules. Let $\mathcal{B}_1 = \{u_1, ..., u_n\}$ be a basis for U and $\mathcal{B}_2 = \{w_1, ..., w_m\}$ one for W, so that $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_\in$ is a basis for V. Then

$$[g]_{\mathcal{B}} = \begin{pmatrix} (g)_{\mathcal{B}_1} & 0 \\ 0 & [g]_{\mathcal{B}_2} \end{pmatrix}$$

(I have no idea why, but the moment I try use square brackets around that g it fucks up - Oliver).

Lemma 4.11. *Let V be an FG-module such that we have the decomposition*

$$V = \bigoplus_{i=1}^{n} U_i$$

Define the projection map $\pi_i : u_1 + u_2 + ... + u_n \mapsto u_i$. Then

(i) *π_i is a homomorphism*

(ii) $\pi_i \circ \pi_i = \pi_i$

Proof. Trivial □

Lemma 4.12. *Suppose we have a finite decomposition*

$$V = \sum U_i$$

where the U_i are irreducible. Then V is the direct sum of the subset of the U_i.

Proof. This follows from the fact that the intersection of two distinct irreducible modules is trivial (again, simple checking). □

We will now present an important result

Theorem 4.13 (Maschke's Theorem)**.** *Let G be a finite group, F a field of characteristic 0, V an FG-module and U a submodule. Then there exists some W such that $V = U \oplus W$.*

Proof. We first choose some W_1 such that $V = U \oplus W_1$ as vector spaces. Note that each $v \in V$ can be uniquely decomposed as $v = u + w$, where $u \in U, w \in W_1$. Now define the canonical projection $\sigma : V \to U$ where $v \mapsto u$. Clearly $\ker \sigma = W_1$ and $\operatorname{im} \sigma = U$. However, we note that σ is NOT necessarily a homomorphism of FG-modules. We modify it as follows: Define $\varphi : V \to V$ by

$$v \mapsto \frac{1}{|G|} \sum_{g \in G} v g \sigma g^{-1}$$

We claim that φ IS a homomorphism. Indee, suppose $x \in G, v \in V$. Then

$$\begin{aligned}
(xv)\varphi &= \frac{1}{|G|} \sum_{g \in G} (vx) g \sigma g^{-1} \\
&= \frac{1}{|G|} \sum_{h \in G} v h \sigma h^{-1} x \\
&= \left(\frac{1}{|G|} \sum_{h \in G} v h \sigma h^{-1} \right) x = (v\varphi)x
\end{aligned}$$

where the equality

$$\frac{1}{|G|} \sum_{g \in G} (vx) g \sigma g^{-1} = \frac{1}{|G|} \sum_{h \in G} v h \sigma h^{-1} x$$

follows from the change of variables $h = xg$. Clearly φ maps into U, and we now check it is a projection. Indeed, supposing $u \in U$ we have

$$\begin{aligned}
(u)\varphi &= \frac{1}{|G|} \sum_{g \in G} ug\sigma g^{-1} \\
&= \frac{1}{|G|} \sum_{g \in G} u\sigma gg^{-1} \\
&= \frac{1}{|G|} \sum_{g \in G} u \\
&= u
\end{aligned}$$

as desired.

Now clearly have $U = \operatorname{im} \varphi$ and define $W := \ker \varphi$. Then for each $v \in V$, write $u := v\varphi \in U$ and $w := v - u \in W$ so that $v = u + w$. It only remains to check that this is unique. To see this, suppose $u' + w' = v = u + w$. Then

$$u' = \varphi(u') = \varphi(v) = \varphi(u) = u$$

which implies the result. $\square$