Maths 721 Notes

2020

Contents

1 Lecture 1

In the first half of this course we will cover three main topics:

- representations;
- modules;
- characters.

We will further see that representations and modules are essentially the same, and that modules and characters are essentially the same; and hence all three are essentially the same.

From now on G is a group.

1.1 Representations

Definition 1.1. A **representation** of a group G over a field F is a group homomorphism from G to $\mathrm{GL}(n, F)$, where n is the **degree** of the representation.

Explicitly, a representation is a function $\rho : G \to \mathrm{GL}(n, F)$ such that for all $g, h \in G$;

(i) $(gh)\rho = (g\rho)(h\rho)$,

(ii) $1_G\rho = I_n$,

(iii) $g^{-1}\rho = (g\rho)^{-1}$.

Note the use of the (incredibly shit) postfix function notation.

Example 1.2. Take D_4, the Dihedral group of order 8. It has the following group presentations

$$\begin{aligned} D_4 &= \langle a, b \mid a^4 = 1, b^2 = 1, a^b = a^{-1} \rangle \\ &\cong \langle (1\ 2\ 3\ 4), (1\ 4)(2\ 3) \rangle, \end{aligned}$$

where $a^b = bab^{-1}$ is conjugation of a by b. By defining the matrix subgroup

$$H = \left\langle A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\rangle$$

and defining $\rho : D_4 \to \mathrm{GL}(n, F)$ where $F = \mathbb{R}, \mathbb{C}$, by $a \mapsto A$ and $b \mapsto B$, and $a^i b^j \mapsto A^i B^j$ for $0 \le i \le 3$, and $0 \le j \le 1$. Hence we have ρ is a representation of D_4 over F. ◀

Example 1.3. Take $\mathbb{Q}_8$ the Quaternion group of order 8, which has the following group presentations

$$\begin{aligned} \mathbb{Q}_8 &= \langle a, b \mid a^4 = 1, a^2 = b^2, a^b = a^{-1} \rangle \\ &\cong \langle \bar{a} = (1\ 6\ 2\ 5)(3\ 8\ 4\ 7), \bar{b} = (1\ 4\ 2\ 3)(5\ 7\ 6\ 8) \rangle \end{aligned}$$

Define
$$H = \left\langle A = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\rangle \subset \mathrm{GL}(2, \mathbb{C}).$$
Then $\rho : \mathbb{Q}_8 \to \mathrm{GL}(2, \mathbb{C})$ defined by $a^k b^\ell \mapsto A^k B^\ell$ is a group representation of $\mathbb{Q}_8$ over $\mathbb{C}$ of degree 2. ◀

Definition 1.4. Let G be a group and define
$$\begin{aligned} \rho : G &\to \mathrm{GL}(n, F) \\ g\rho &= I_n \end{aligned}$$
for all $g \in G$. Then ρ is a representation, called the **trivial representation** of arbitrary of degree.

It follows from the trivial representation that any group G has a representation of an arbitrary degree.

Let $\rho : G \to \mathrm{GL}(n, F)$ be a group homomorphism, and take $T \in \mathrm{GL}(n, F)$. Then
$$(T^{-1}AT)(T^{-1}BT) = T^{-1}(AB)T.$$
Thus, given ρ define σ such that
$$g\sigma = T^{-1}(g\rho)T$$
for all $g \in G$. As for all $g, h \in G$, one has
$$\begin{aligned} (gh)\sigma &= T^{-1}\big((gh)\rho\big)T \\ &= T^{-1}(g\rho)(h\rho)T \\ &= T^{-1}(g\rho)TT^{-1}(h\rho)T \\ &= (g\sigma)(h\sigma), \end{aligned}$$
and so σ is a group homomorphism; and hence a representation.

Definition 1.5. Define
$$\rho : G \to \mathrm{GL}(m, F), \qquad \sigma : G \to \mathrm{GL}(n, F)$$
to both be representation of G over F. We say that ρ **is equivalent to** σ if $n = m$ and there exists $T \in \mathrm{GL}(n, F)$ such that $g\sigma = T^{-1}(g\rho)T$.

Proposition 1.6. *Equivalence of representations is an equivalence relation.*

Proof. Reflexivity is clear by taking $T = I_n$. For symmetry, take T to be its inverse. For transitivity, if
$$g\sigma = T^{-1}(g\rho)T, \qquad g\rho = S^{-1}(g\eta)S,$$
then
$$g\sigma = (ST)^{-1}(g\eta)(ST).$$
□

Definition 1.7. Define the **kernel** of the representation $\rho : G \to \mathrm{GL}(n, F)$ as $\ker \rho = \{g \in G \mid g\rho = I_n\}$.

Proposition 1.8. *The kernel of a representation of G is a normal subgroup of G; i.e.* $\ker \rho \lhd G$.

Proof. Suppose $g \in \ker \rho$ and $h \in G$ is arbitrary. Then

$$(hgh^{-1})\rho = (h\rho)(g\rho)(h^{-1}\rho) = (h\rho)I_n(h\rho)^{-1} = (h\rho)(h\rho)^{-1} = I_n,$$

and so $hgh^{-1} \in \ker \rho$. As $\ker \rho$ is closed under conjugation, it is a normal subgroup of G. □

Definition 1.9. We say ρ is a **faithful** representation of G if $\ker \rho = \{1_G\}$.

Example 1.10. For the trivial representation $\rho : G \to \mathrm{GL}(n, F)$ with $g \mapsto I_n$ for all $g \in G$, we have $\ker \rho = G$. Hence the representation is not faithful. ◀

Lemma 1.11. *Suppose G is a finite group, and ρ is a representation of G over F. Then ρ is faithful if, and only if,* $\operatorname{im} \rho \cong G$.

Proof. Immediate from the first isomorphism theorem. □

1.2 FG-Modules

Suppose G is a group, and $F = \mathbb{R}, \mathbb{C}$. Given $\rho : G \to \mathrm{GL}(n, F)$, with $V = F^n$, let $v = (\lambda_1, \ldots, \lambda_n) \in V$ for $\lambda_i \in F$ be a row vector. Moreover, note that $g\rho$ is an $n \times n$ matrix for all $g \in G$. Thus, we have $v \cdot (g\rho) \in V$, and satisfies the following properties:

(i) $v \cdot \big((gh)\rho\big) = v \cdot (g\rho)(h\rho)$;

(ii) $v \cdot (1_G\rho) = v$;

(iii) $(\lambda v) \cdot (g\rho) = \lambda\big(v \cdot (g\rho)\big)$;

(iv) $(u + v) \cdot (g\rho) = u \cdot (g\rho) + v \cdot (g\rho)$.

We often will omitted the $\cdot$ in the operation, and write $v(a\rho)$ for $v \cdot (a\rho)$.

Example 1.12. Recall D_4 and its given presentation from a previous example. We have

$$a\rho = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad b\rho = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

If $v = (\lambda_1, \lambda_2)$, then we have

$$v(a\rho) = (-\lambda_2, \lambda_1), \qquad v(b\rho) = (\lambda_1, -\lambda_2).$$

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Definition 1.13. Let V a vector space over the field $F = \mathbb{R}, \mathbb{C}$, and let G be a group. We say V is a FG**-module** if a multiplication $v \cdot g$ for $v \in V$, and $g \in G$ is defined such that:

(i) $v \cdot g \in V$;

(ii) $v \cdot (gh) = (v \cdot g) \cdot h$;

(iii) $v \cdot 1_G = v$;

(iv) $(\lambda v) \cdot g = \lambda(v \cdot g)$;

(v) $(u + v) \cdot g = u \cdot g + v \cdot g$.

This generalises the previous discussion from a matrix group to an arbitrary group.

Note that properties (i), (iv), and (v) imply that the map $v \mapsto v \cdot g$ is an endomorphism of V (a linear map from V to itself).

Definition 1.14. Suppose V is an FG-module and B is a basis for V. For $g \in G$, let $[g]_B$ denote the matrix of the endomorphism $v \mapsto v \cdot g$ of V relative to the basis B.

2 Lecture 2

Theorem 2.1. *Let $\rho : G \to GL(n, F)$ be a representation of G over F.*

(I) *If $V = F^n$ is an FG module and G acts on V by $v \cdot g = v(g\rho)$ there exists a basis B of V such that $g\rho = [g]_B$.*

(II) *The map $g \mapsto [g]_B$ is a representation for G over F.*

Proof. Choose the standard basis $B = [e_1, \dots, e_n]$.

Since V is an FG-module we have $v(gh) = (vg)h$ for all $g, h \in G$ and $v \in V$. Thus $[gh]_B = [g]_B[h]_B$ so the map is a homomorphism. We now check that $[g]_B$ is invertable for all $g \in G$. We know $v \cdot 1_G = (vg)g^{-1}$ so $I_n = [g]_B[g^{-1}]_B$ and thus $[g]_B$ has an inverse. □

Example 2.2. Recall the representation of $G = D_4$ from a previous example. Define an FG-module $V = F^2$ with the action defined by taking vg to $v(g\rho)$.

$$\begin{aligned} v_1 &= (1, 0), \quad v_1 a = v_2, \quad v_1 b = v_1, \\ v_2 &= (0, 1), \quad v_1 a = -v_1, \quad v_1 b = -v_2. \end{aligned}$$

In this basis we recover our representation

$$a \mapsto [a]_B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad b \mapsto [b]_B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

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We now provide an equivalent basis-dependent definition for an FG-module.

Lemma 2.3. *Let V a vector space over the field $F = \mathbb{R}, \mathbb{C}$ with basis $B = [v_1, \ldots, v_n]$, and let G be a group. If a multiplication $v \cdot g$ for $v \in B$, and $g \in G$ is defined such that:*

(i) $v \cdot g \in V$;

(ii) $v \cdot (gh) = (v \cdot g) \cdot h$;

(iii) $v \cdot 1_G = v$;

(iv) $(\sum_{i=1}^n \lambda_i v_i) \cdot g = \sum_{i=1}^n \lambda_i (v_i \cdot g)$ *for all* $\lambda_i \in F$;

then V is an FG-module.

Definition 2.4. The trivial module of a group over F is a one dimensional vector space V over F such that $vg = v$ for all $v \in V$ and $g \in G$.

Definition 2.5. An FG-module is faithful if 1_G is the only $g \in G$ such that $vg = v$ for all $v \in V$.

Theorem 2.6. *Let V be an FG-module with basis B and ρ a representation of group G over F defined by taking $g \mapsto [g]_B$.*

(i) If B' is another basis of V then the map $g \mapsto [g]_{B'}$ is a representation of G equivalent to ρ.

(ii) If representation σ is equivalent to ρ then there exists basis B'' such that $\sigma(g) = [g]_{B''}$ for all $g \in G$.

Proof. Taking T to be the change of basis matrices, the two representations are equivalent. □

Example 2.7. Let $G = C_3 = \langle a \mid a^3 = 1 \rangle$ and representation $\rho : G \to GL(n, F)$ defined by

$$a \mapsto \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}.$$

We attempt to construct an FG-module with group action described by ρ. Take $V = F^2$ with basis $B = [v_1, v_2]$. Define the action of G on V by

$$v_1 a = v_2, \quad v_2 a = -v_1 - v_2.$$

Let us now choose alternate basis $B' = [u_1 = v_1, u_2 = v_1 + v_2]$. The action of G on this basis is described by

$$u_1 a = -u_1 + u_2, \quad u_2 a = -u_1.$$

This gives us a representation

$$a \mapsto [a]_{B'} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}.$$

To verify this construction we write our change of basis matrix as

$$T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

and verify that

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}.$$

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Definition 2.8. The permutation module of a group $G \leq S_n$ is an n-dimensional vector space V with basis $B = [v_1, \ldots, v_n]$ and action by G defined by

$$v_i g = v_{ig}$$

for all $g \in G$ where ig is the image of i under $g \in S_n$.

It follows from Caley's theorem that every group has a faithful FG-module.

Example 2.9. Take $G = S_4$ and pick $g = (1\ 2)$ and $h = (1\ 2\ 3\ 4)$. We have representations

$$[g]_B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad [h]_B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

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2.1 Module Reducibility

Definition 2.10. Let V be an FG-module. We call W a submodule of V if W is a vector subspace of V and W is closed under the action of G. We then write $W < V$.

Example 2.11. Let $G = C_3 = \langle (1\ 2\ 3) \rangle$ and V the presentation module of G with basis $B = [v_1, v_2, v_3]$. The subspace $W = \langle v_1 + v_2 + v_3 \rangle$ is a submodule but the subspace $U = \langle v_1 + v_2 \rangle$ is not.

For example, consider the action of $g = (1\ 2\ 3)$ on $v_1 + v_2 \in U$.

$$(v_1 + v_2)g = v_{1g} + v_{2g} = v_2 + v_3 \notin U$$

whereas G acts on W trivially. ◀