Cryptography and Architectures for Computer Security - Cheat Sheet

Information Theory

$Pr(C = c) = \sum_{k:c \in \{\mathbb{E}_k(m), \forall m \in \mathcal{M}\}} Pr(K = k)Pr(P = \mathbb{D}_k(c))$
Perfect secrecy $Pr(P = m|C = c) = Pr(P = m) \implies |\mathcal{K}| = |\mathcal{C}| = |\mathcal{P}|$
Entropy $H(X) = -\sum_{i=1}^{n} p_i \log_2 p_i \quad (p_i \log_2 p_i = 0 \text{ for } p_i = 0)$
$H(X, Y) = -\sum_{i=1}^{n} \sum_{j=1}^{m} Pr(X = x_i, Y = y_j) \log_2 Pr(X = x_i, Y = y_j)$
$H(X|Y = y) = -\sum_{i=1}^{n} \sum_{j=1}^{m} Pr(X = x_i|Y = y) \log_2 Pr(X = x_i|Y = y)$
$H(X|Y) = -\sum_{i=1}^{n} \sum_{j=1}^{m} Pr(Y = y_j)Pr(X = x_i, Y = y_j) \log_2 Pr(X = x_i|Y = y_j)$
$H(X) + H(Y) \geqslant H(X, Y)$; $H(X, Y) = H(Y) + H(X|Y)$; $H(X|Y) \leqslant H(X)$
Key equivocation $H(K|C) = H(P) + H(K) - H(C)$
Language redundancy $R_L = 1 - \frac{H_L}{\log_2 |\mathcal{M}|}$
Spurious keys $\bar{s_n} \geqslant \frac{|\mathcal{K}|}{|\mathcal{M}|^{nR_L}} - 1$
Unicity distance $n_0 \approx \frac{\log_2 |\mathcal{K}|}{R_L \log_2 |\mathcal{M}|}$

Symmetric Ciphers

Modes of Operation

ECB $c_i = \mathbb{E}_k(m_i)$
CBC $c_0 = IV$, $c_i = \mathbb{E}_k(m_i \oplus c_{i-1})$
CFB/OFB $ISR_0 = IV$, $OSR_i = \mathbb{E}_k(ISR_{i-1})$, $c_i = m_i \oplus$ j-th leftmost bits of OSR_i
CTR $ctr_i = IV + i$, $t_i = \mathbb{E}_k(ctr_i)$, $c_i = t_i \oplus m_i$

Cryptanalysis

Pile-up lemma $Pr(Z_1 \oplus \cdots \oplus Z_n = 0) = \frac{1}{2} + 2^{n-1} \prod_{i=1}^{n} \varepsilon_i$

Hash Functions

First preimage $Pr(m_i|d = h(m_i)) \approx \frac{q}{|D|}$
Second preimage $Pr(h(m_i) = h(m)) = \frac{q-1}{|D|}$
Collision $Pr(\text{no collisions}) = e^{-\frac{q(q-1)}{2|D|}} \implies q \leqslant 1.774\sqrt{|D|}$

Algebraic Structures

$\varphi(n) = |\{x \in \mathbb{N} : 1 \leqslant x \leqslant n - 1, gcd(n, x) = 1\}|$
$\varphi(nm) = \varphi(n)\varphi(m)$ for $gcd(n, m) = 1$, $\varphi(p^k) = p^k - p^{k-1}$
$\forall x \in \mathbb{Z}_n^* \quad x^{\varphi(n)} \equiv_n 1$, $x^{-1} \equiv_n x^{\varphi(n)-1}$
CRT $X = (\sum M_i M_i^{'} x_i) \mod N$, $M_i = \frac{N}{n_i}$, $M_i^{'} = M_i^{-1} \mod n_i$
$\sum_{d:d|n} N_d(p)d = p^n = \deg(x^{p^n} - x)$

$M_d(p) = \frac{\varphi(p^n-1)}{d}$
Irreducibility $gcd(f(x), x^{p^h}-1)$, $h \leqslant \lfloor \frac{\deg(f(x))}{2} \rfloor$

Elliptic Curves

For $\mathbb{K} = \mathbb{F}_p, p \geq 3$: $\Delta = 4a^3 + 27b^2 \neq 0$
$\mathbb{E}(\mathbb{F}_p) : y^2 = x^3 + ax + b$ for $a, b \in \mathbb{F}_p, p \geq 3$
$x_3 = (\frac{y_1-y_2}{x_1-x_2})^2 - x_1 - x_2$, $y_3 = (\frac{y_1-y_2}{x_1-x_2})(x_1 - x_3) - y_1$
$x_4 = (\frac{3x_1^2+a}{2y_1})^2 - 2x_1$, $y_4 = -y_1 + (\frac{3x_1^2+a}{2y_1})(x_1 - x_4)$
$-P_1 = (x_{1,-y_1})$
$\mathbb{E}(\mathbb{F}_{2^m}) : y^2 + xy = x^3 + ax^2 + b$ for $a, b \in \mathbb{F}_{2^m}, \Delta = b \neq 0$
$x_3 = (\frac{y_1+y_2}{x_1+x_2})^2 + (\frac{y_1+y_2}{x_1+x_2}) + x_1 + x_2 + a$, $y_3 = (\frac{y_1+y_2}{x_1+x_2})(x_1 + x_3) + x_3 + y_1$
$x_4 = x_1^2 + \frac{b}{x_1^2}$, $y_4 = x_1^2 + (x_1 + \frac{y_1}{x_1})x_3 + x_3$
$-P_1 = (x_1, x_1 + y_1)$

Public Key Cryptosystems

RSA

Keys $k_{pub} = \langle n, e \rangle$, $k_{priv} = \langle p, q, \varphi(n), d \rangle$
$n = p \cdot q$, $gcd(e, \varphi(n)) = 1$, $d = e^{-1} \mod \varphi(n)$
$c = m^{e \mod \varphi(n)} \mod n$, $m = c^{d \mod \varphi(n)} \mod n$
CRT $m_p \equiv_p c^{d \mod p-1}$, $m_q \equiv_q c^{d \mod q-1}$, $m \equiv_n m_p q(q^{-1} \mod p) + m_q p(p^{-1} \mod p)$

ElGamal

Keys $k_{pub} = \langle n, g, g^s \rangle$, $k_{priv} = \langle s \rangle$
$ctx = \langle \gamma, \delta \rangle = \langle g^l, ptx(g^s)^l \rangle$, $ptx = \gamma^{n-s}\delta$
$sign = \langle ptx, \langle \gamma, \delta \rangle \rangle = \langle ptx, g^l, l^{-1}(h(m) - s \cdot h(\gamma)) \mod n \rangle$

DSS-DSA

Keys $k_{pub} = \langle p, q, g, g^s \rangle$, $k_{priv} = (s)$; $q|(p-1), q = |\langle g \rangle|$

Montgomery Multiplication

Number Theoretical Cryptanalysis

Primality test

Fermat n is composite $\implies a^{n-1} \not\equiv_n 1$ with probability $> \frac{1}{2}$
Miller-Rabin $n - 1 = d2^r : a^d \not\equiv_n \pm 1$ and $a^{d2^r} \not\equiv -1 \implies n$ is composite $(r \leqslant s - 1)$

Factoring

Fermat $x = \lceil \sqrt{n} \rceil, y = x^2 - n$, until y is a perfect square $y = y + 2x + 1, x = x + 1$, then the factors are $x \pm \sqrt{y}$
Pollard's $\boldsymbol{\rho}$ pick a, b at random (e.g $x_0 = 2, x_i = x_{i-1}^2 + 1 \mod n)$), if $gcd(a - b, n) \neq 1$ the result is a factor
Pollard's p-1 p B-power-smooth, $a = 2^{B!}$, so $p = gcd(a - 1, n)$

DLog

Polig-Hellman for each prime factor $\eta = g^{\frac{n}{p}}$, $\gamma_i = \gamma_{i-1} g^{l_{i-1} p^{i-1}}$,$\delta_i = (\beta \gamma_i^{-1})^{\frac{n}{p^{i+1}}}$, $l_i = \log_\eta \delta_i$

Misc

Convert to power of 2 $x = 2^n \Leftrightarrow n = \frac{\ln(x)}{\ln(2)}$