8 Linear Systems Theory

In this chapter we will review some properties of linear time-invariant systems. We consider their input/output relationship in the time domain, the impulse response, and the convolution theorem. We also review basic concepts of complex number theory; the Laplace transform, system's transfer function, and frequency domain analysis. More extended descriptions of this material can be found in many standard textbooks (Carlson, 1986; Poularikas and Seely, 1994; Oppenheim et al., 1996).

8.1 Linear Shift-Invariant Systems

In linear systems theory, a system is treated as a *black box* that does not reveal its internal states, and is characterized only by the relationship between its input and output (see Fig. 8.1). If a system has no internal stored energy, then its output response $y(t)$ is forced entirely by the input $x(t)$:

$$y(t) = F[x(t)] \tag{8.1.1}$$

where $F[\cdot]$ is the transfer function.

Linearity A system is linear if it obeys the two fundamental principles: **Homogeneity**, and **additivity**.

The principle of homogeneity states that output scales linearly with the input:

$$F[\alpha x(t)] = \alpha F[x(t)]. \tag{8.1.2}$$

Usually, linear systems theory is applied to time-varying signals. However the same methods can be applied to input and output signals that are distributed

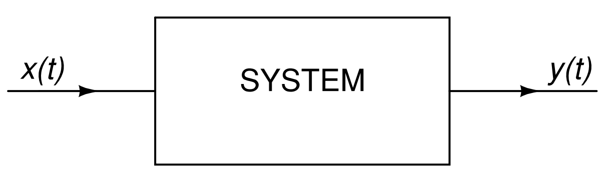


Figure 8.1
Typical black-box representation of a linear system. Its input is the signal $x(t)$ and its output is the signal $y(t)$.

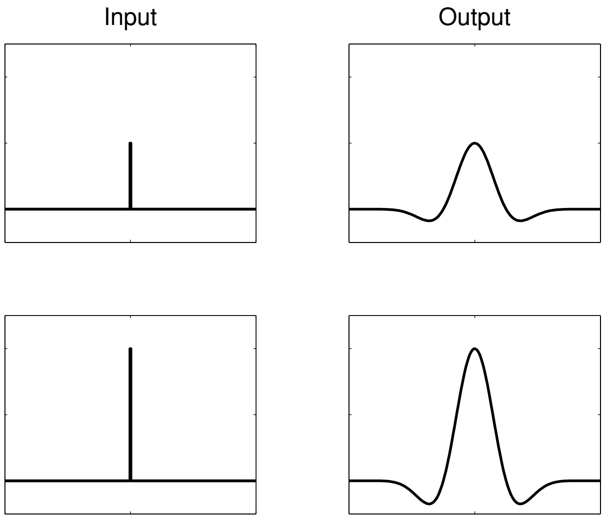


Figure 8.2
Graphical example of the homogeneity principle of a linear system. The signals in the left quadrants represent the system's input, and the signals in the right ones represent its output. An increase in the input signal causes a proportional increase in the output signal.

over *space* rather than *time*. For example, in Fig. 8.2, the input signal is a spatial unit impulse and the output is a spatial *Gabor* function (a Gaussian modulated by a cosine function)[1]

The principle of additivity states that if the input signal is composed of elementary signals, then the system's response is the composition of its responses to each of the elementary signals:

$$F[x_1(t)+x_2(t)+\cdots+x_n(t)] = F[x_1(t)]+F[x_2(t)]+\cdots+F[x_n(t)]. \quad (8.1.3)$$

Figure 8.3 shows a graphical example: If the response of the system to a spatial impulse is a Gabor function, and if the system's input signal is a linear combination of spatial unit impulses, then the system's response will be a linear combination of Gabor functions.

The principles of homogeneity and additivity taken together are commonly referred to as the *principle of superposition*, which state that a system is linear

1 Gabor functions are commonly used to model the (linear) response properties of a particular class of neurons in the visual cortex.

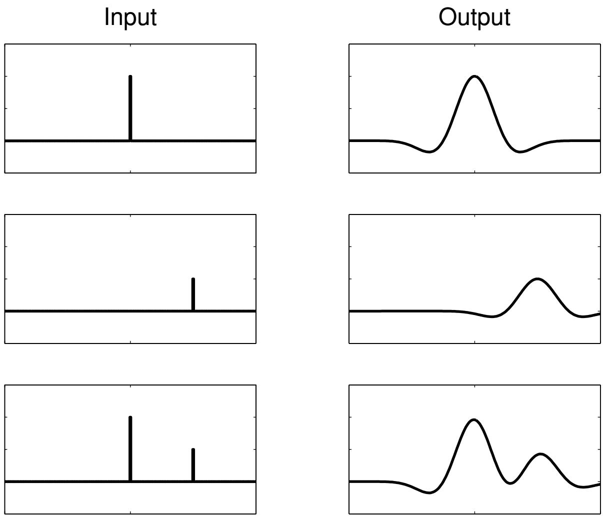


Figure 8.3
Graphical example of the additivity principle of a linear system. The signals in the left quadrants represent the system's input, and the signals in the right ones represent its output.

if

$$y(t) = \sum_k a_k F[x_k(t)] \tag{8.1.4}$$

for input $x(t) = \sum_k a_k x_k(t)$, and a_k constant for all k.

In other words, a system is linear if its response function F is a *linear operator*:

$$F\left[\sum_k a_k x_k(t)\right] = a_k \sum_k F[x_k(t)]. \tag{8.1.5}$$

Shift Invariance A system is said to be shift-invariant if its responses to identical stimuli shifted in time are also identical, except for the corresponding time shift (Fig. 8.4).

If a system is shift-invariant, then its response function is also shift-invariant: Given input signal $x(t)$, its time-shifted variant $x(t-\tau)$ will produce

$$F[x(t-\tau)] = y(t-\tau). \tag{8.1.6}$$

The system's output signal is unchanged, except for a time shift.

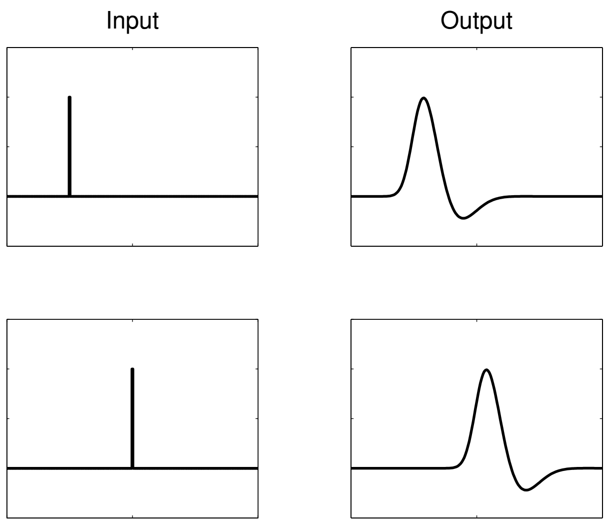


Figure 8.4
Graphical example of a time-invariant system's response.

Time invariance and linearity are two independent characteristics. *Not all linear systems are time-invariant and, similarly, not all time-invariant systems are linear.*

8.2 Convolution

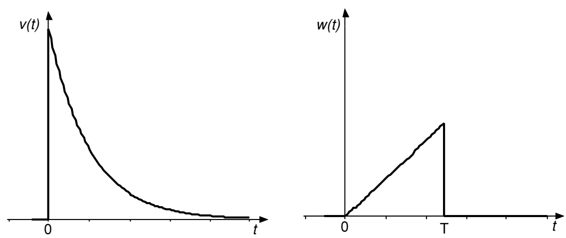
Convolution is an important mathematical operator used in linear systems analysis. The convolution of two time-varying signals, $v(t)$ and $w(t)$, is

$$v(t) * w(t) \equiv \int_{-\infty}^{+\infty} v(\lambda)w(t-\lambda)d\lambda \tag{8.2.1}$$

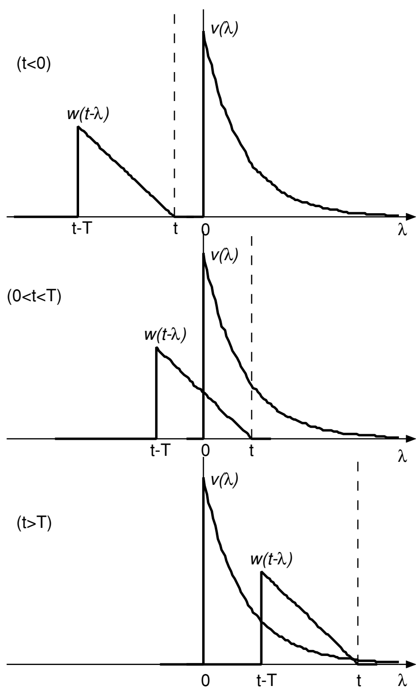
where λ is the integration variable, and t is the independent variable.

Figure 8.5 shows a graphical representation of the convolution process between signals $v(t)$ (Fig. 8.5(a)) and $w(t)$ (Fig. 8.5(b)) at three different time steps. The result of the convolution is shown in Fig. 8.6. Note how the overlap between the two curves is null for $t < 0$, increases for $0 < t < T$, peaks at $t = T$ and decreases again for $t > T$.

If the independent variable for both input signals is the same, then it can be omitted and we express the convolution between to the two signals $v(t)$ and



(a) (b)



(c)

Figure 8.5
Graphical representation of the convolution between $v(t)$ and $w(t)$ for three different values of t. Note that the integration variable λ in (c) is on the abscissae of the plots. Modified from Carlson, A. B. (1986).

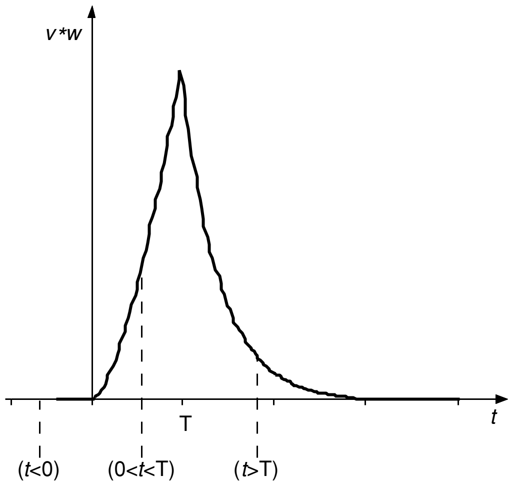


Figure 8.6
Result of the convolution between the two signals $v(t)$ and $w(t)$ of Fig. 8.5. The three dashed lines are at the three values of t used in Fig. 8.5.

$w(t)$ simply as $v * w$. The convolution operator is linear and has the following properties:

commutative:	$v * w$	$=$	$w * v$
associative:	$v * (w * z)$	$=$	$(v * w) * z$
distributive:	$v * (w + z)$	$=$	$(v * w) + (v * z)$

8.3 Impulses

The *unit impulse* or *Dirac delta function* $\delta(t)$ is not a function in the strict mathematical sense. It is defined by a set of assignment rules.

- If $v(t)$ is a continuous function at $t = 0$ then

$$\int_{t_1}^{t_2} v(t)\delta(t)dt = \begin{cases} v(0) & t_1 < 0 < t_2 \\ 0 & \text{otherwise.} \end{cases} \tag{8.3.1}$$

- If ϵ is an arbitrary small number

$$\int_{-\infty}^{+\infty} \delta(t)dt = \int_{-\epsilon}^{+\epsilon} \delta(t)dt = 1. \tag{8.3.2}$$

From these rules we can infer that $\delta(t)$ has unit area at $t = 0$ and that $\delta(t) = 0$, for all $t \neq 0$. We can also note that the Dirac delta function has no mathematical or physical meaning, unless it appears under the integral operator.

Impulse Integration Properties

When used in conjunction with the integral operator, the Dirac delta function has the following properties:

- Replication:

$$v(t) * \delta(t - \tau) = v(t - \tau) \tag{8.3.3}$$

- Sampling:

$$\int_{-\infty}^{+\infty} v(t)\delta(t - \tau)dt = v(\tau) \tag{8.3.4}$$

where $v(t)$ is a continuous time-varying signal.

Impulses in the Limit

There are many (proper mathematical) functions $\delta_{\epsilon}(t)$ that approach the Dirac delta function $\delta(t)$, in the limit:

$$\lim_{\epsilon \to 0} \delta_{\epsilon}(t) = \delta(t) \tag{8.3.5}$$

An example of such a function that is commonly used is

$$\delta_{\epsilon} = \frac{\sin(\frac{t}{\epsilon})}{t}. \tag{8.3.6}$$

Figure 8.7 shows how δ_{ϵ} of Eq. 8.3.6 approaches the Dirac delta function as ϵ decreases.

8.4 Impulse Response of a System

We can now use the notions of convolution and unit impulse to define the *impulse response* of a linear time-invariant system. If $y(t)$ is the system's response to its input $x(t)$ we can write

$$y(t) = F[x(t)]. \tag{8.4.1}$$

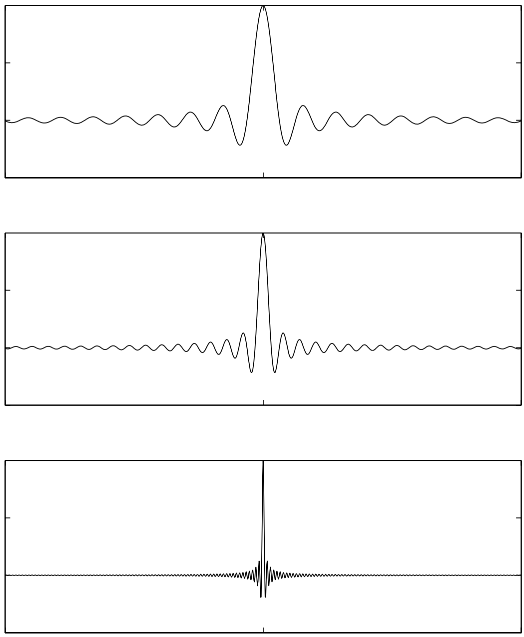


Figure 8.7
Plot of the function $\sin(t/\epsilon)/t$ for three decreasing values of ϵ.

If the input signal is the Dirac delta function ($x(t) = \delta(t)$), then the system's response to the unit impulse is defined as

$$h(t) \equiv F[\delta(t)]. \tag{8.4.2}$$

If $x(t)$ is continuous in time, the replication property of the unit impulse allows us to rewrite $x(t)$ as $x(t) * \delta(t)$. With this reformulation of the system's input signal, Eq. 8.4.1 becomes

$$y(t) = F\left[\int_{-\infty}^{+\infty} x(\lambda)\delta(t-\lambda)d\lambda\right]. \tag{8.4.3}$$

If the system is linear, Eq. 8.4.3 is equivalent to:

$$y(t) = \int_{-\infty}^{+\infty} x(\lambda)F\left[\delta(t-\lambda)\right]d\lambda \tag{8.4.4}$$

If we substitute Eq. 8.4.2 into Eq. 8.4.4, and if the system is time-invariant, then

$$y(t) = \int_{-\infty}^{+\infty} x(\lambda)h(t-\lambda)d\lambda = \int_{-\infty}^{+\infty} h(\lambda)x(t-\lambda)d\lambda. \tag{8.4.5}$$

This property of linear time-invariant systems is extremely powerful. It states that if a system's impulse response $h(t)$ is known, the response of the system to any arbitrary signal $x(t)$ can be computed simply by performing the convolution of its impulse response with the signal itself:

$$\boxed{y(t) = h(t) * x(t)}. \tag{8.4.6}$$

Step Response

We can define a system's *step response* in the same way we defined its impulse response. If the input signal $x(t)$ is the step function

$$u(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{8.4.7}$$

then we define its step response to be

$$g(t) \equiv F[u(t)]. \tag{8.4.8}$$

A first interesting property can be obtained by exploiting the system's impulse response (see Eq. 8.4.6):

$$g(t) = h(t) * u(t). \tag{8.4.9}$$

By applying the derivative operator to this equation, and noting that the derivative of the step function is the unit impulse, we obtain

$$\frac{d}{dt}g(t) = h(t) * \frac{d}{dt}u(t) = h(t) * \delta(t). \tag{8.4.10}$$

If we apply the unit impulse replication property (Eq. 8.3.3), then we obtain

$$h(t) = \frac{d}{dt}g(t). \tag{8.4.11}$$

Thus, a system's impulse response can be obtained by computing the derivative of its step response. This property is extremely useful in practical situations because unit impulses are impossible to generate with physical instruments but it is easy to generate waveforms that approximate ideal step functions. Consequently, a physical linear time-invariant system is characterized experimentally

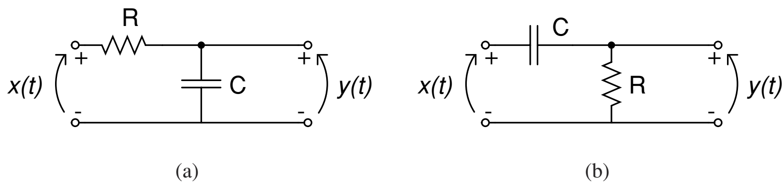


Figure 8.8
Resistor capacitor (RC) circuits. The signals $x(t)$ represent input voltages, and the signals $y(t)$ represent output voltages. (a) Integrator circuit; (b) Differentiator circuit.

by measuring its step response and then deriving its impulse response from Eq. 8.4.11.

8.5 Resistor-Capacitor Circuits

The resistor-capacitor (RC) circuits of Fig. 8.8 represent first order, linear, time-invariant systems. In both circuits, the input signal is $x(t)$ and the output signal is $y(t)$. The circuits of Fig. 8.8(a) and (b) are referred to as *RC integrator* and *RC differentiator* respectively. In this section we focus only on the properties of the RC integrator. The properties of the RC differentiator will be described in Chapter 9.

The integrator circuit of Fig. 8.8(a) is governed by the differential equation:

$$RC\frac{d}{dt}y(t) + y(t) = x(t). \qquad (8.5.1)$$

By solving Eq. 8.5.1 for a unit impulse input signal ($x(t) = \delta(t)$), we obtain the circuit's *impulse response*:

$$h(t) = \frac{1}{RC}\, e^{-t/RC} \cdot u(t) \qquad (8.5.2)$$

where u(t) is the step function. Similarly, solving Eq. 8.5.1 for a step input signal ($x(t) = u(t)$), we obtain the circuit's *step response*

$$g(t) = (1 - e^{-t/RC}) \cdot u(t). \qquad (8.5.3)$$

Figure 8.9 shows the impulse response and the step response. The value RC is defined as the system's *time-constant* and is often labeled τ. As pointed out in Section 8.4, the response of the circuit to an arbitrary input signal can be

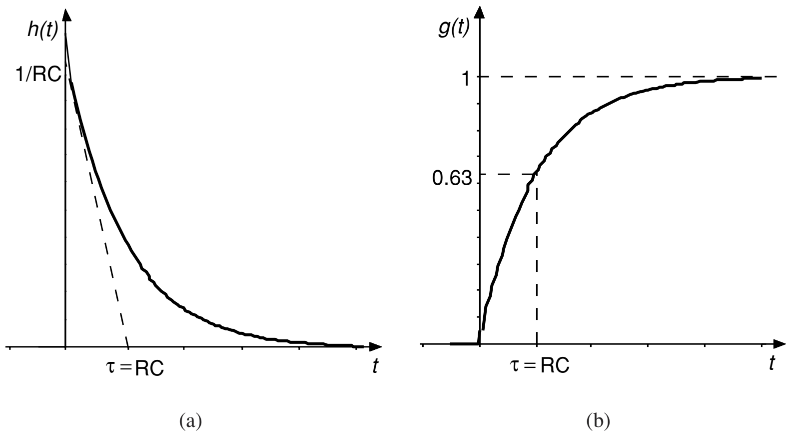


Figure 8.9
Impulse response (a) and step response (b) of an RC circuit.

obtained by the convolution between the input signal and the circuit's impulse response:

$$y(t) = x(t) * h(t) = \int_0^\infty \frac{1}{RC}\, e^{-\lambda/RC} \cdot x(t-\lambda)d\lambda. \tag{8.5.4}$$

8.6 Higher Order Equations

Time-domain analysis becomes increasingly difficult for higher order systems. Fortunately there is a *unified representation* in which any solution to a linear system can be expressed: **Exponentials with complex arguments.** All solutions to linear homogeneous (undriven) equations are of the form e^{st} where s is a *complex number* (see Fig. 8.10):

$$s = \sigma + j\omega = M\cos(\phi) + jM\sin(\phi) \tag{8.6.1}$$

where $j = \sqrt{-1}$, σ is the real part of the complex number, ω is the imaginary part, M represents its *magnitude*, and ϕ its *phase*. Magnitude and phase of a

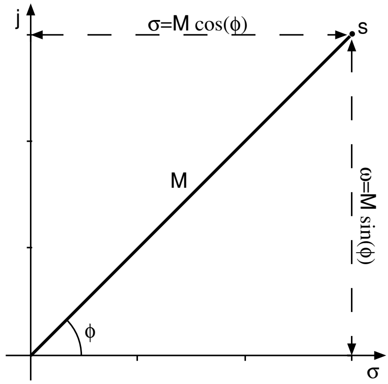


Figure 8.10
Complex number representation. The complex number s has magnitude M and phase ϕ. Its real part is σ and imaginary part is ω.

complex number obey the following relationships:

$$M = \sqrt{\sigma^2 + \omega^2} \tag{8.6.2}$$

$$\phi = \arctan\left(\frac{\omega}{\sigma}\right). \tag{8.6.3}$$

The magnitude of a complex number s is often denoted as $|s|$. Furthermore, applying the properties of complex exponentials, one can observe that

$$e^{j\phi} = \cos(\phi) + j\sin(\phi) \tag{8.6.4}$$

$$e^{-j\phi} = \cos(\phi) - j\sin(\phi). \tag{8.6.5}$$

It follows that s can be also written as

$$s = Me^{j\phi}. \tag{8.6.6}$$

These notations can be used to solve higher order differential equations. As an example, we consider the second order linear homogeneous equation

$$\frac{d^2}{dt^2}V + \alpha\frac{d}{dt}V + \beta V = 0. \tag{8.6.7}$$

Assume that e^{st} is an *eigenfunction*[2] and substitute for V:

$$s^2 e^{st} + \alpha s e^{st} + \beta e^{st} = 0. \tag{8.6.8}$$

Solving for s we obtain

$$s = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\beta}}{2}. \tag{8.6.9}$$

Consequently, if $\alpha^2 - 4\beta \geq 0$, s is real, otherwise s is a complex number. In practice, if Eq. 8.6.7 is a linear system, we could measure its response $V = e^{\,st}$ with a *real* instrument: but if e^{st} was a complex exponential, we would measure only its real component:

$$Re\{e^{st}\} = Re\left\{e^{(\sigma+j\omega)t}\right\} = e^{\sigma t} Re\{e^{j\omega t}\}. \tag{8.6.10}$$

So the measured response of the system would be

$$V_{meas} = e^{\sigma t} \cos \omega t. \tag{8.6.11}$$

Figure 8.11 shows the possible kinds of response of V_{meas} for different values of ω and σ. If $\sigma < 0$ all the solutions are stable and decay to zero with time. If $\sigma > 0$ all solutions are unstable and diverge with time. If $\sigma = 0$ the solutions are naturally stable (they neither decay, nor diverge). All physical *passive* linear systems will have stable solutions ($\sigma < 0$). The ω axis scales the oscillation frequency f of a solution ($\omega = 2\pi f$).

8.7 The Heaviside-Laplace Transform

By analyzing the example of the previous section (see Eq. 8.6.7) we can make the following observation: Any time we substitute the eigenfunction $e^{\,st}$ into a linear differential equation of order n, the following property obtains:

$$\frac{d^n}{dt^n} e^{st} = s^n e^{st}. \tag{8.7.1}$$

In other words:

We can consider s as an operator meaning *derivative* with respect to time. Similarly, we can view $\frac{1}{s}$ as the operator for *integration* with respect to time (Heaviside).

2 An eigenfunction is a nonzero solution of a second order linear homogenous differential equation

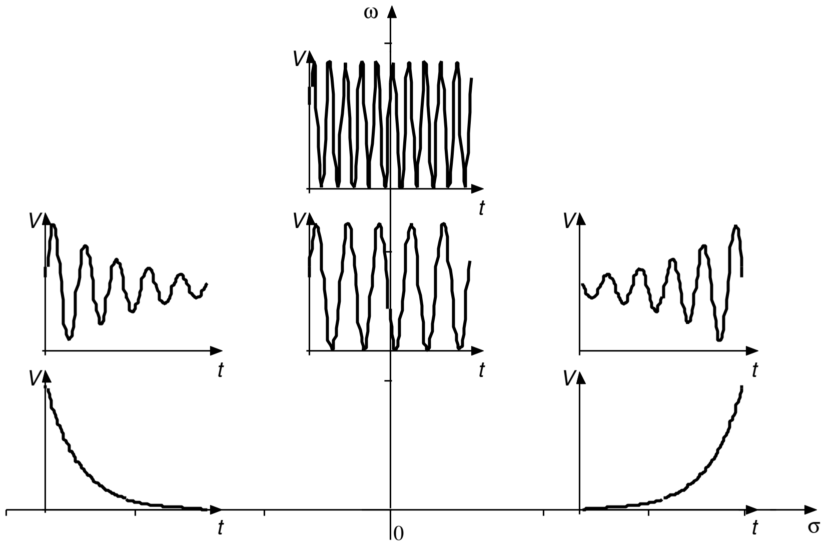


Figure 8.11
The possible kinds of *measured* responses for a first order linear system.

This observation was made by Heaviside, when trying to analyze analog circuits: but was also formalized by Laplace when he introduced the *Laplace Transform*. The Laplace transform is a useful operator that links functions that operate in the time domain with functions of complex variables:

$$\mathcal{L}[y(t)] = Y(s) \equiv \int_{-\infty}^{\infty} y(t)e^{-st}dt. \tag{8.7.2}$$

8.8 Linear System's Transfer Function

Now that we have introduced the concepts of convolution (Section 8.2), impulse response (Section 8.4), and the Laplace transform (Section 8.7), we can define a linear system's *transfer function*. It is a function defined in the complex domain:

$$H(s) \equiv \frac{Y(s)}{X(s)} \tag{8.8.1}$$

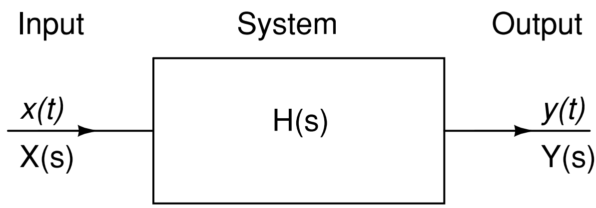


Figure 8.12
Typical representation of a linear system with input and output signals both in the time domain ($x(t), y(t)$) and in the Laplace domain ($X(s), Y(s)$).

where $Y(s)$ is the Laplace transform of the system's output $y(t)$ and $X(s)$ is the Laplace transform of the system's input $x(t)$ (see Fig. 8.12). Conversely, we can say that the output of any linear time-invariant system is determined by *multiplying* the system's transfer function with its input:

$$\boxed{Y(s) = H(s)X(s)}. \tag{8.8.2}$$

Transfer Function and Impulse Response

Consider the special case in which the system's input signal $x(t)$ is the unit impulse $x(t) = \delta(t)$. Its Laplace transform $X(s)$ is

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{-\infty}^{\infty} \delta(t)e^{-st}dt = 1. \tag{8.8.3}$$

In this case, following the definition of Eq. 8.8.1, the system's response in the complex plane is

$$Y(s) = H(s). \tag{8.8.4}$$

On the other hand, the system's response in the time domain is (by definition) its impulse response:

$$y(t) = h(t). \tag{8.8.5}$$

Because $Y(s)$ is the Laplace transform of $y(t)$, we can substitute Eq. 8.8.5 into Eq. 8.7.2:

$$Y(s) = \int_{-\infty}^{\infty} y(t)e^{-st}dt = \int_{-\infty}^{\infty} h(t)e^{-st}dt \tag{8.8.6}$$

and so

$$\boxed{H(s) = \int_{-\infty}^{\infty} h(\lambda)e^{-\lambda s}d\lambda = \mathcal{L}[h(t)]}. \tag{8.8.7}$$

The transfer function $H(s)$ is the Laplace transform of the impulse response $h(t)$.

Summary Given a linear time-invariant system with input $x(t)$, output $y(t)$, and impulse response $h(t)$:

$$\begin{aligned} y(t) =& x(t) * h(t) \\ Y(s) =& X(s)H(s) \end{aligned}$$

where

$$\begin{aligned} X(s) =& \mathcal{L}[x(t)] \\ Y(s) =& \mathcal{L}[y(t)] \\ H(s) =& \mathcal{L}[h(t)]. \end{aligned}$$

8.9 The Resistor-Capacitor Circuit (A Second Look)

Consider again the RC circuit of Fig. 8.8. As mentioned in Section 8.5, this circuit is governed by

$$\tau\frac{d}{dt}y(t) + y(t) = x(t) \tag{8.9.1}$$

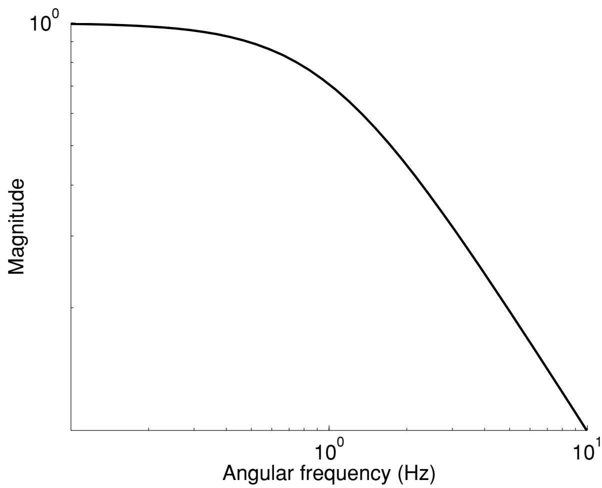
where $\tau = RC$.
In the complex domain, we have

$$Y(s)(\tau s + 1) = X(s). \tag{8.9.2}$$

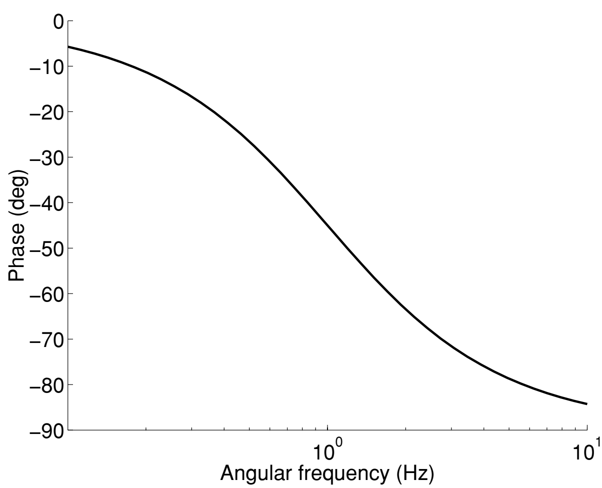
Therefore the circuit's transfer function is

$$H(s) = \frac{1}{1 + \tau s}. \tag{8.9.3}$$

Consider now how this circuit responds to sinusoidal signals of different *frequencies*. Sinusoids have a very special relationship to shift-invariant linear systems, such as the one we are analyzing. When a sinusoidal signal is applied



(a)



(b)

Figure 8.13
Bode plot of a first order linear system, such as the RC circuit of Fig. 8.8. (a) Magnitude, (b) Phase.

as input to a shift-invariant linear system, then its response will be another sinusoidal signal, with possibly a different amplitude and a different phase, but certainly with exactly the same frequency! That is, if the input is $x(t) = \sin(\omega t)$, the output will be $y(t) = A\sin(\omega t + \phi)$, where A and ϕ determine the scaling and shift.

When we analyze a system using sinusoidal signals of different frequencies, we are working in the frequency domain. In this domain $s = j\omega$ and the circuit's transfer function is

$$H(j\omega) = \frac{1}{1 + j\omega\tau}. \tag{8.9.4}$$

From this transfer function, we make two useful observations:

1. If the frequencies of the sinusoidal signals are small with respect to the circuit's time-constant ($\omega\tau \ll 1$), then the circuit's output will resemble its input ($Y(j\omega) \approx X(j\omega)$).
2. On the other hand, if the frequencies are large with respect to the circuit's time-constant ($\omega\tau \gg 1$), then

$$\frac{Y(j\omega)}{X(j\omega)} \approx \frac{1}{j\omega\tau}. \tag{8.9.5}$$

These observations are also reflected in the plots of the transfer function's magnitude and phase (Fig. 8.13). These plots are referred to as *Bode* plots and they are used to analyze the response of a dynamic system in terms of its transfer function. The magnitude of the transfer function is

$$|H(j\omega)| = \frac{1}{\sqrt{1 + (\omega\tau)^2}} \tag{8.9.6}$$

and its phase is

$$\phi = \arctan(-\omega\tau). \tag{8.9.7}$$

The frequency $\omega = \frac{1}{\tau}$ is defined as the *cutoff frequency* . In Fig. 8.13 it is set to one.

The RC circuit of Fig. 8.8 is a *low-pass filter*, because it allows sinusoidal signals with frequencies lower than the cutoff frequency to pass virtually unchanged. On the other hand, the frequency components of the input signals that are above the cutoff frequency are attenuated. The phase lag between the input and the output of the system increases with ω (see Fig. 8.13(b)) and saturates at $-90°$. Figure 8.14 shows experimental data measured from an RC

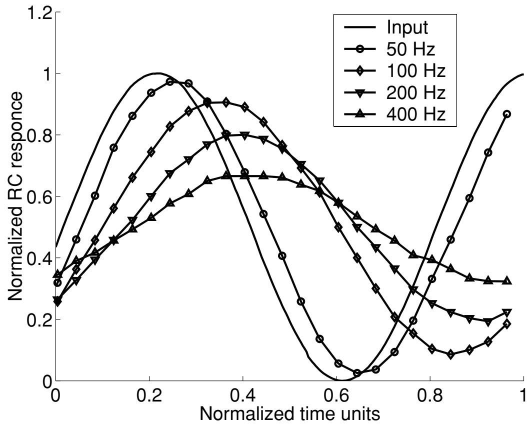


Figure 8.14
Response of an RC low-pass filter ($R = 10M\Omega$, $C = 1nF$) to input sinusoids of different frequencies. The input signals have been normalized to unity, and the outputs have been normalized with respect to the input. The time axis has also been normalized so that the responses to all the frequencies could be presented on the same graph.

lowpass filter with $R = 10M\Omega$ and $C = 1nF$. Sinuosoids of increasing frequency were applied to the circuit and the corresponding responses were measured. To show the effect of a range of input frequencies on the circuit's response, all the data are plotted on a normalized scale. The responses have been normalized with respect to the input and time has been normalized to unity. As expected, the output signal is attenuated as the input frequency increases; and the phase lag between the input and output signals increases with increasing frequency.

8.10 Low-Pass, High-Pass, and Band-Pass Filters

The RC circuit analyzed in the previous sections is the simplest example of a passive *filter*. Filters are typically used to alter the frequency spectrum of their input signals. Specifically, filters allow one or more frequency *bands* to pass unchanged (except for a multiplicative gain factor), whereas others are attenuated. Passive filters do not amplify the input signal, whereas active filters can also amplify the frequency components of the input signal. If a filter transmits low frequency components (from DC to a lower cutoff value ω_l), it is said to be a *low-pass* filter. If it transmits high frequency components (higher than a cutoff value ω_u), it is said to be a *high-pass* filter. Filters that transmit only frequency components between a lower cutoff and an upper cutoff are said to be *band-pass* filters.

The analysis of linear systems often assumes *ideal filters*. These filters have distortionless signal transmission over one or more frequency bands, and have zeros responses at all other frequencies. For example, the transfer function of an ideal *bandpass* filter is:

$$H(\omega) = \begin{cases} Ke^{-j\omega t_d} & \omega_l \leq |\omega| \leq \omega_u \\ 0 & \text{otherwise.} \end{cases} \tag{8.10.1}$$

Although ideal filters cannot be implemented in practice, their use in theoretical analysis simplifies the study of linear systems.