

Dr. J. Hermon

This assignment is due in Canvas at 23:59 p.m. on Monday, October 6.**Late assignments are not accepted.**

1. A searchlight is distance 1 from an infinitely long wall. Let Q denote the closest point on the wall and assume the searchlight scans along the wall so that at any given time, the angle Θ the beam of light makes is uniform on $(-\pi/2, \pi/2)$ (the point Q corresponds to angle 0). Let $X \in \mathbb{R}$ (positive or negative) be the position of the beam on the wall as measured from Q . Find the cumulative distribution function and density function for X . (*This is the Cauchy distribution; its expectation is undefined.*)

Solution

If the angle sampled from the uniform distribution $\omega \in (-\pi/2, \pi/2)$, then we have that the point on the wall is given by $z = \tan(\omega) \times (1\text{m})$. Thus, the random variable $Z(\omega) = \tan(\omega)$. To compute the CDF, we notice that the uniform distribution over $(-\pi/2, \pi/2)$ is $\frac{1}{\pi}\lambda$: the Lebesgue measure normalized on $(-\pi/2, \pi/2)$. We then compute:

$$\begin{aligned} F_Z(z) &= P(Z(\omega) \leq z) = P(\omega \leq \tan^{-1}(z)) \\ &= \frac{1}{\pi} \lambda((-\pi/2, \tan^{-1}(z))) \\ &= \frac{1}{\pi} \tan^{-1}(z) + \frac{1}{2}. \end{aligned}$$

The probability density function, by FTC, is given by

$$f_Z(z) = F'_Z(z) = \frac{1}{\pi} \frac{1}{x^2 + 1}.$$

2. Suppose that X, Y are independent standard normal random variables.¹ By differentiating the cumulative distribution function for the random variable $Z = X/Y$ to obtain the density of Z , show that Z has a standard Cauchy distribution.

Solution.

We follow the hint in the footnote. Now, $F_Z(z) = P(\frac{X}{Y} \leq z)$ is described by $x, y \in \mathbb{R}^2$, where $X(\omega) = x$ and $Y(\omega) = y$ such that $\frac{x}{y} \leq z$. This set, is then described by

$$B = (\{X \leq zY\} \cap \{Y > 0\}) \cup (\{X \geq zY\} \cap \{Y < 0\}).$$

We integrate over this region, with the pdf being $f_X(x) \cdot f_Y(y)$. When we differentiate, we use the Leibnitz integral rule:

$$\begin{aligned} F_Z(z) &= \int_{-\infty}^0 \int_{yz}^{\infty} f_X(x) f_Y(y) dx dy + \int_0^{\infty} \int_{-\infty}^{yz} f_X(x) f_Y(y) dx dy \\ f_Z(z) &= F'_Z(z) = \int_{-\infty}^0 -y f_X(yz) f_Y(y) dy + \int_0^{\infty} f_X(yz) f_Y(y) dy \\ &= 0. \end{aligned}$$

¹The independence has the consequence that the joint density g of X, Y is the product of the densities. In other words, for a Borel subset B of \mathbb{R}^2 ,

$$P((X, Y) \in B) = \int_B g(x, y) dx dy$$

with $g(x, y) = f_X(x) f_Y(y)$. We have not yet discussed integration over arbitrary Borel sets (this requires Lebesgue integration which we will discuss soon) but the sets B encountered in this assignment are nice enough that Riemann integration does the job.

Now, we substitute the pdf's for the normal distributions and compute each of these integrals:

$$\begin{aligned}\frac{1}{2\pi} \int_{-\infty}^0 (-y) \exp(-\frac{1}{2}(z^2 + 1)y^2) dy &= \frac{1}{2\pi} \int_{\infty}^0 (-1/2) \exp(-\frac{1}{2}(z^2 + 1)u) du \\ &= \frac{1}{2\pi} (-\frac{1}{2}) (\frac{-2}{z^2 + 1}) \left[\exp(-\frac{1}{2}(z^2 + 1)u) \right]_{\infty}^0 \\ &= \frac{1}{2\pi(z^2 + 1)}.\end{aligned}$$

$$\begin{aligned}\frac{1}{2\pi} \int_0^{\infty} y \exp(-\frac{1}{2}(z^2 + 1)y^2) dy &= \frac{1}{2\pi} \int_0^{\infty} (1/2) \exp(-\frac{1}{2}(z^2 + 1)u) du \\ &= \frac{1}{2\pi} (\frac{1}{2}) (\frac{-2}{z^2 + 1}) \left[\exp(-\frac{1}{2}(z^2 + 1)u) \right]_0^{\infty} \\ &= \frac{1}{2\pi(z^2 + 1)} (0 - 1) = -\frac{1}{2\pi(z^2 + 1)}.\end{aligned}$$

Thus,

$$f_Z(z) = \frac{1}{2\pi(z^2 + 1)} + \frac{1}{2\pi(z^2 + 1)} = \frac{1}{\pi(z^2 + 1)}.$$

3. Suppose that X, Y are independent $\text{Exp}(\lambda)$ random variables. By differentiating the cumulative distribution function for the random variable $Z = X - Y$, find the density function of Z (it is a Laplace distribution).

Similar to last problem, we fix z and determine the probability $P(Z \leq z)$. We consider two cases for z , $z \geq 0$ and $z < 0$. First, consider $z > 0$. In particular, this occurs when $X \leq Y + z$. Also, we recall that $X \geq 0$ and $Y \geq 0$ (since they are independent exponential random variables). We integrate over this region with the pdf being the joint pdf of X and Y .

$$\begin{aligned}F_Z(z) &= \int_0^{\infty} \int_0^{y+z} f_X(x) f_Y(y) dx dy \\ f_Z(z) &= F'_Z(z) = \int_0^{\infty} f_X(y+z) f_Y(y) dy.\end{aligned}$$

If $z \leq 0$, then instead, we can use the equivalent inequality $X + (-z) \leq Y$, where $-z > 0$. Then,

$$\begin{aligned}F_Z(z) &= \int_0^{\infty} \int_0^{x-z} f_X(x) f_Y(y) dy dx \\ f_Z(z) &= F'_Z(z) = \int_0^{\infty} f_X(x) f_Y(x-z) dx.\end{aligned}$$

We compute each of these integrals. For $z > 0$:

$$\begin{aligned}\int_0^{\infty} f_X(y+z) f_Y(y) dy &= \int_0^{\infty} \lambda^2 \exp(-\lambda(y+z)) \exp(-\lambda y) dy \\ &= \lambda^2 \exp(-\lambda z) \int_0^{\infty} \exp(-2\lambda y) dy \\ &= \lambda^2 \exp(-\lambda z) \frac{1}{2\lambda} = \frac{\lambda}{2} \exp(-\lambda z).\end{aligned}$$

Likewise, for $z < 0$, we compute a very similar integral to be $\frac{\lambda}{2} \exp(\lambda z)$. Putting these together, we get $f_Z(z) = F'_Z(z) = \frac{\lambda}{2} \exp(-\lambda|z|)$, as desired.

4. Suppose that X is a random variable whose cumulative distribution function F is continuous on \mathbb{R} . Let $Y = F(X)$. Prove that Y has a uniform distribution, i.e., that $F_Y(x) = x$ for $x \in (0, 1)$.

5. Let X_0, X_1, \dots be i.i.d. continuous random variables with cumulative distribution function

$$F(x) = \int_{-\infty}^x f(t) dt,$$

for some probability density function $f : \mathbb{R} \rightarrow [0, \infty)$. Let

$$N = \min\{n : X_n > X_0\}.$$

- (a) Show that the cumulative distribution function F_N of N is given by

$$F_N(n) = 1 - \frac{1}{n+1}$$

for $n \in \mathbb{N}$. (Of course $F_N(x) = 0$ for $x < 1$ and $F_N(x) = F_N(n)$ for $x \in (n, n+1)$.)

- (b) Show² that the cumulative distribution function of X_N is

$$F + (1 - F) \log(1 - F).$$

(Recall that $\log(1 - t) = -\sum_{n=1}^{\infty} \frac{t^n}{n}$ for $|t| < 1$; if $F = 1$ we interpret $(1 - F) \log(1 - F)$ as zero.)

Recommended problems (Do not hand in).

1. Suppose that X has a continuous density function f , that $P(\alpha \leq X \leq \beta) = 1$, and that g is a strictly increasing differentiable function on (α, β) . Show that the density of $g(X)$ is $f(g^{-1}(x))/g'(g^{-1}(x))$ if $x \in (g(\alpha), g(\beta))$ and otherwise is zero.
2. Use the result of (1) to determine the density of $aX + b$ for constants $a > 0$ and $b \in \mathbb{R}$.
3. Suppose X has a standard normal distribution. Use the result of (1) to compute the density of $\exp(X)$ (this is the lognormal distribution).
4. Suppose that X has continuous density function f . Find the density of X^2 . Let Z have a standard normal distribution, and find the density of Z^2 (this is a special case of the χ^2 -distribution).

Quote of the week: Probability theory is measure theory with a soul. — Mark Kac

²One way, not the only way, is to employ a higher-dimensional version of the previous footnote.