

Computational Control

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Optimal and predictive control

1. Finite-Horizon LQR

Consider the scalar system

$$x_{t+1} = x_t + 2u_t,$$

with $x_t \in \mathbb{R}$ and $u_t \in \mathbb{R}$, and the corresponding finite-horizon LQR problem with cost

$$\sum_{t=0}^1 x_t^2 + u_t^2.$$

and zero terminal cost.

Determine the sequence of solutions to the Riccati difference equation, together with the associated optimal feedback gains and inputs.

2. Infinite-Horizon LQR

Consider the scalar system

$$x_{t+1} = x_t + 2u_t,$$

with $x_t \in \mathbb{R}$ and $u_t \in \mathbb{R}$, and the corresponding infinite-horizon LQR problem with cost

$$\sum_{t=0}^{\infty} x_t^2 + u_t^2.$$

Determine the algebraic Riccati equation solution $P_{\infty} \succeq 0$ and the corresponding optimal feedback gain Γ_{∞} .

3. Riccati equations over finite and infinite time horizons

Consider the scalar system

$$x_{t+1} = x_t + u_t,$$

with $x_t \in \mathbb{R}$ and $u_t \in \mathbb{R}$, the corresponding finite-horizon LQR problem with cost

$$\sum_{t=0}^T x_t^2 + u_t^2$$

and zero terminal cost, and the corresponding infinite-horizon LQR problem with cost

$$\sum_{t=0}^{\infty} x_t^2 + u_t^2.$$

- Solve the difference Riccati equation equation over a time horizon of length $T = 3$ with zero terminal cost.
- Solve the algebraic Riccati equation (over an infinite time horizon).
- Compute the corresponding feedback gains Γ_0 (at $t = 0$) and Γ_{∞} . What is their absolute difference?

4. Explicit MPC

Given the scalar system

$$x_{t+1} = x_t + u_t,$$

with $x_t \in \mathbb{R}$ and $u_t \in \mathbb{R}$, consider a one-step horizon MPC problem formulated as

$$\begin{aligned} & \min_{\substack{u_0 \in \mathbb{R} \\ x_1 \in \mathbb{R}}} u_0^2 + x_1^2 \\ \text{s.t. } & \begin{cases} x_1 = x_0 + u_0 \\ u_0 \in [-1, 1] \end{cases} \end{aligned}$$

Determine the corresponding MPC control law $u_0^*(\cdot)$ by applying the Karush–Kuhn–Tucker (KKT) conditions to the optimization problem, and identify the regions of initial states where different constraints become active.

5. Economic MPC

Given the scalar system

$$x_{t+1} = x_t + u_t,$$

with $x_t \in \mathbb{R}$ and $u_t \in \mathbb{R}$, consider a one-step horizon economic MPC problem formulated as

$$\begin{aligned} & \min_{\substack{u_0 \in \mathbb{R} \\ x_1 \in \mathbb{R}}} -x_1 + 0.5u_0 \\ \text{s.t. } & \begin{cases} x_1 = x_0 + u_0 \\ u_0 \in [0, 2] \\ x_1 \leq 5 \end{cases} \end{aligned}$$

Find all initial states for which the MPC control law is well defined. For those values, determine the MPC control law $u_0^*(\cdot)$ and identify the regions of initial states where different constraints become active.

Hint. One may solve the problem using the KKT conditions. However, since the cost is strictly decreasing in u_0 , the optimum is attained at one of the endpoints of its feasible interval.

6. Robust MPC

Consider the scalar system

$$x_{t+1} = x_t + u_t + w_t,$$

with $x_t \in \mathbb{R}$, $u_t \in \mathbb{R}$, and additive disturbance $w_t \in [-\delta, \delta]$, with $\delta > 0$ known. Suppose we want to apply a finite-horizon MPC controller with horizon $N = 3$ with state constraint $x_t \geq 0$, ignoring disturbances in the prediction model (that is, using the nominal model), but we require the true disturbed trajectory to satisfy the state constraints

$$x_{t+k} \leq 5 \quad \text{for } k = 1, 2, 3,$$

for all disturbance sequences $w_t, w_{t+1}, w_{t+2} \in [-\delta, \delta]$.

What constraints should be imposed on the nominal predicted states $\hat{x}_{t+1}, \hat{x}_{t+2}, \hat{x}_{t+3}$ in the MPC optimization problem to ensure robust satisfaction of $x_{t+k} \leq 5$ for $k = 1, 2, 3$? Express the constraints in terms of δ , and provide a justification.

Hint. At each prediction step k , bound the worst-case accumulated disturbance. Use

$$x_{t+k} = \hat{x}_{t+k} + \underbrace{\sum_{j=0}^{k-1} w_{t+j}}_{\text{disturbance accumulation}}.$$

7. Infinite-Horizon LQR with Non-Detectable Unstable Mode

Consider the linear system

$$x_{t+1} = Ax_t + Bu_t, \quad y_t = Cx_t,$$

with $x_t \in \mathbb{R}^2$, $u_t \in \mathbb{R}$, $y_t \in \mathbb{R}$, and

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Consider the corresponding infinite-horizon LQR problem with cost

$$\sum_{t=0}^{\infty} u_t^2 + y_t^2.$$

- (i) Find the solution $P_{\infty} \succeq 0$ of algebraic Riccati equation and the corresponding optimal feedback gain Γ_{∞} .
- (ii) Assess if the optimal control u_t^* is stabilizing by inspecting the eigenvalues of $A + B\Gamma_{\infty}$.
- (iii) Determine for which initial states the minimal cost is equal to zero.

Hint. To solve algebraic Riccati equation, assume

$$P_{\infty} = \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix}.$$

and solve for the scalar $p \geq 0$. Note also that the pair (A, B) is reachable, but (A, C) is not observable. In particular, the mode corresponding to the eigenvalue $\lambda = 3$ is unobservable.

8. LQR Value Function as a Lyapunov Function (Bonus)

Consider the linear system

$$x_{t+1} = Ax_t + Bu,$$

with $x_t \in \mathbb{R}^n$ and $u_t \in \mathbb{R}^m$, and the corresponding infinite-horizon LQR problem with cost

$$\sum_{t=0}^{\infty} x_t^\top Q x_t + u_t^\top R u_t.$$

with $Q \succ 0$ and $R \succ 0$. Let $P \succ 0$ solve the algebraic Riccati equation and define the value function

$$V(x_t) = x_t^\top P x_t.$$

Show that $V(\cdot)$ is a Lyapunov function for the closed-loop system under optimal feedback $u_t = \Gamma x_t$.

Hint. Use Bellman's optimality:

$$V(x_t) = x_t^\top Q x_t + u_t^\top R u_t + V(x_{t+1}),$$

and rearrange to show $V(x_{t+1}) - V(x_t) < 0$ for all $x_t \neq 0$.