Computational Control

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Final exam

1. Control Design Strategies

Problem. Throughout the course, we have encountered several tools for designing controllers and policies:

LQR, standard/economic/robust MPC, data-driven MPC, policy/value iteration, Monte Carlo RL. . . .

Each method is suitable for different situations. Below are four scenarios inspired by real engineering and societal control problems. For each scenario, select *three* control design methods from the list above, ranked in order of priority as

1 = preferred, 2 = acceptable, 3 = fallback

and briefly justify your choices in at most $2-3$ sentences.

- **Smart building energy management** The heating and cooling system of a modern smart building must be operated. The linearized thermal model is reliable. Strict comfort and actuator constraints must be enforced. Energy prices vary predictably throughout the day. Violating constraints is not acceptable.
- **Macroeconomic policy** The federal bank chooses one of a few interventions each quarter (e.g., interest-rate adjustments). The economy is subject to unpredictable fluctuations. Its evolution reflects both policy decisions and external shocks. Long historical time series are available. The goal is to minimize a long-run cost representing inflation, unemployment, and volatility.
- **Automated insulin delivery** An insulin pump must regulate a patient's glucose level under moderately nonlinear and disturbance-rich physiology. A reliable first-principles model is difficult to obtain, because each patient is different. The device collects abundant patient-specific input–output data. Insulin delivery is safety-critical and subject to hard actuation limits.
- **Satellite attitude regulation** A small satellite must maintain a precise orientation for imaging tasks. A highly accurate linearized model is available. The actuators operate far from their operational limits. The goal is to minimize a quadratic infinite-horizon performance index capturing pointing error and fuel usage.

2. Stability of Linear Time-Invariant Systems

Problem. Consider the matrices

$$A_1 = \begin{bmatrix} 0.8 & 0.3 \\ -0.4 & 0.8 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1.05 & 0.1 \\ 0 & 0.9 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$
$$A_4 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \qquad A_5 = \begin{bmatrix} 0.6 & 0 \\ 0.1 & 0.7 \end{bmatrix}, \qquad A_6 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

For each discrete-time, linear, time-invariant (LTI) system

$$x_{t+1} = A_i x_t, \qquad i = 1, \ldots, 6,$$

determine if the equilibrium at the origin is *asymptotically stable*, *stable*, or *unstable*. Justify your answers.

Hint. The eigenvalues of a matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2},$$

are the roots of the characteristic polynomial

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0.$$

3. Economic MPC

Given the scalar system

$$x_{t+1} = x_t + u_t,$$

with $x_t \in \mathbb{R}$ and $u_t \in \mathbb{R}$, consider a one-step horizon economic MPC problem formulated as

$$\begin{aligned} \min_{\substack{u_0 \in \mathbb{R} \\ x_1 \in \mathbb{R}}} \quad & -x_1 + 0.5u_0 \\ \text{s.t} \quad & \begin{cases} x_1 = x_0 + u_0 \\ u_0 \in [0, 2] \\ x_1 \leq 5 \end{cases} \end{aligned}$$

Find all initial states for which the MPC control law is well defined. For those values, determine the MPC control law $u_0^\star(\cdot)$ and identify the regions of initial states where different constraints become active.

Hint. One may solve the problem using the KKT conditions. However, since the cost is strictly decreasing in u_0, the optimum is attained at one of the endpoints of its feasible interval.

4. Controllability of a Parameterized Single-Input System

Consider a single-input, discrete-time, LTI system described by the equation

$$x_{t+1} = \begin{bmatrix} 0 & 0 & 0 \\ a_1 & 0 & 0 \\ 0 & a_2 & 0 \end{bmatrix} x_t + \begin{bmatrix} b_1 \\ 0 \\ 0 \end{bmatrix} u_t.$$

Determine the controllability of the system as a function of the parameters $a_1, a_2 \in \mathbb{R}$ and $b_1 \in \mathbb{R}$.

Hint. Compute $\mathcal{C} = [B \ AB \ A^2B]$ and check its rank for different values of a_1, a_2, and b_1.

5. Elevator or Stairs?

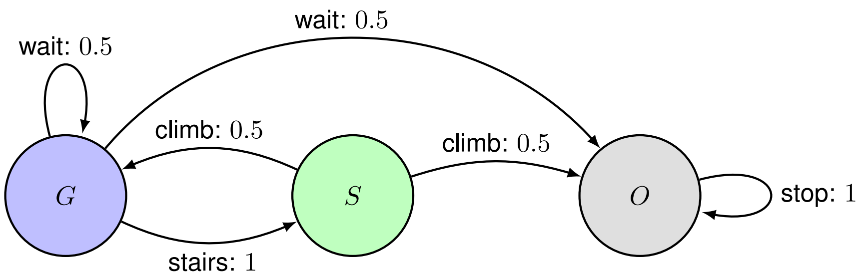
A commuter arrives at work and must decide how to reach the office: elevator or stairs? The decision-making process is modeled by a Markov decision process with state

$$x_t \in \{G, S, O\},$$

where G represents waiting for the elevator at the ground floor, S represents taking the stairs, and O represents having reached the office. At every time instant t, the available actions are

$$\begin{aligned} &\text{at } G: \quad u_t \in \{\text{wait, stairs}\}, \\ &\text{at } S: \quad u_t \in \{\text{climb}\}, \\ &\text{at } O: \quad u_t \in \{\text{stop}\}. \end{aligned}$$

The dynamics evolve according to the controlled Markov process shown in the diagram below.



The corresponding transition probabilities

$$P_{xx'}^u = \Pr(x_{t+1} = x' \mid x_t = x, \, u_t = u), \qquad x, x' \in \{G, S, O\},$$

are defined as follows:

- **At the ground floor** G the person can either wait for the elevator or take the stairs. However, waiting is a gamble: with probability $p = 0.5$ the elevator arrives and takes them to the office; otherwise they remain stuck at G. Choosing stairs commits them to the staircase. The transition probabilities are:

$$\begin{aligned} \text{wait}: &\quad P^{\text{wait}}_{G,G} = 0.5, \quad P^{\text{wait}}_{G,O} = 0.5, \\ \text{stairs}: &\quad P^{\text{stairs}}_{G,S} = 1. \end{aligned}$$

- **On the staircase** S the only action is $u_t =$ climb. However, the stairs are slippery: with probability $q = 0.5$ the person slips back to G; otherwise they reach the office O. The transition probabilities are:

$$P^{\text{climb}}_{S,G} = 0.5, \qquad P^{\text{climb}}_{S,O} = 0.5.$$

- **At the office** O the only available action is $u_t =$ stop, so

$$P^{\text{stop}}_{O,O} = 1.$$

The instantaneous costs are defined as

$$C^{\text{wait}}_G = 1, \qquad C^{\text{stairs}}_G = \gamma, \qquad C^{\text{climb}}_S = 1, \qquad C^{\text{stop}}_O = 0,$$

reflecting the inconvenience of waiting, the (discounted) time needed when taking the stairs, the effort of climbing, and the cost of being at the office, respectively.

Finally, consider the infinite-horizon control problem which seeks to minimize the cost

$$\mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t C^{u_t}_{x_t} \,\middle|\, x_0 = x\right],$$

with discount factor $\gamma \in (0, 1)$. Given a policy π, its associated value function is

$$V^\pi(x) = \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t C^{u_t}_{x_t} \,\middle|\, x_0 = x\right], \qquad x \in \{G, S, O\}, \quad u_t \sim \pi(x_t).$$

The action sets are finite, so we only consider *deterministic* stationary policies. The optimal value

$$V^* = \min_\pi V^\pi$$

satisfies the Bellman equation

$$V^*(x) = \min_u \left\{ C^u_x + \gamma \sum_{x'} P^u_{xx'} V^*(x') \right\}, \qquad x \in \{G, S, O\}.$$

Tasks.

(a) Write the Bellman equations for $V^*(G)$, $V^*(S)$, and $V^*(O)$.

(b) Find the value $V^\pi(G)$ of the stationary policies:
  - Always wait at G:

$$\pi(G) = \text{wait}, \qquad \pi(S) = \text{climb}, \qquad \pi(O) = \text{stop}.$$

  - Always take the stairs at G:

$$\pi(G) = \text{stairs}, \qquad \pi(S) = \text{climb}, \qquad \pi(O) = \text{stop}.$$

(c) Determine for which values of γ it is optimal to choose wait at state G, and for which values it is optimal to choose stairs.

