Math 418/544 Assignment 8 November 25, 2024

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This assignment is due in Canvas at 23:59 on Friday, Dec 5th.
Late assignments are not accepted.
Submit a solution to 5 out of the 10 questions.

We solve problems 1, 2, 3, 8, 10

1. Let $g : [0,\infty) \to [0,\infty)$ be such that $\lim_{x\to\infty} \frac{g(x)}{x} = \infty$. Let $(X_j : j \in J)$ be a collection of random variables satisfying that $\sup_{j\in J} \mathbb{E}[g(|X_j|)] < \infty$. Show that $(X_j : j \in J)$ is uniformly integrable.

   Hint: $|x|\mathbb{1}_{|x|>M} \le g(|x|)\delta_g(M)$ for all $M > 0$ for some appropriate $\delta_g : (0,\infty) \to (0,\infty)$.

   Solution. Let $\varepsilon > 0$. Then, $C = \sup_{j\in J} \mathbb{E}[g(|X_j|)]$, so $C < \infty$. Let $L = C/\varepsilon$. Since $\lim_{x\to\infty} \frac{g(x)}{x} = \infty$, we have that there exists some $M \in \mathbb{R}$ such that for all $x > M$, $g(x)/x > L$. However, this implies

   $$|x|\mathbb{1}_{|x|>M} < \frac{1}{L} g(|x|)\mathbb{1}_{|x|>M} \le \frac{1}{L} g(|x|),$$

   where the last inequality follows because $g \ge 0$. Then, for all $j \in J$, we have

   $$\mathbb{E}[|X_j|\mathbb{1}_{|X_j|>M}] < \frac{1}{L}\mathbb{E}[g(|X_j|)] \le \frac{C}{L} < \varepsilon,$$

   so that $\sup_{j\in J} \mathbb{E}[|X_j|\mathbb{1}_{|X_j|>M}] < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, it follows that

   $$\lim_{M\to\infty} \sup_{j\in J} \mathbb{E}[|X_j|\mathbb{1}_{|x|>M}] < \varepsilon,$$

   so that $(X_j : j \in J)$ is uniformly integrable.

2. For $n \in \mathbb{N}$ let σ_n be a random permutation of $[n] := \{1, 2, \ldots, n\}$ picked uniformly at random (i.e., its distribution is the uniform distribution on the set of all $n!$ permutation of $[n]$). Let $X_n := |\{j \in [n] : \sigma_n(j) = j\}|$ be the number of fixed points of σ_n. Use the method of moments to show that $X_n \Rightarrow \mathrm{Pois}(1)$ (justify why it applies; you may look up online the MGF of the Poisson distribution).
   You may rely without a proof on the fact that if $Y_n \sim \mathrm{Bin}(n, \frac{1}{n})$ and $Y \sim \mathrm{Pois}(1)$ then $\lim_{n\to\infty} \mathbb{E}[Y_n^k] = \mathbb{E}[Y^k]$ for all $k \in \mathbb{Z}_+$.

   Hint: Use indicator decompositions of X_n and of Y_n. Expand X_n^k and of Y_n^k to a sum of monomials. Use the fact that $x^\ell = x$ for $x \in \{0,1\}$ and $\ell \ge 1$.

   Solution.

   We compute the moments of X_n and Y_n, and then show that they both converge to the same value. As we are given that the moments of Y_n converge to the moments of Y, it follows that the moments of X_n converge to the moments of Y.

   Consider the k-th moment. First, we take the hint. Notice that $X_n = \sum_{j=1}^n I_j$, where $I_j =$

$\mathbb{1}_{\{\sigma_n(j)=j\}}$. Then, for X_n, by linearity of expectation we have:

$$\begin{aligned}\mathbb{E}[X_n^k] &= \mathbb{E}\left[\left(\sum_{j=1}^{n} I_j\right)^k\right] \\ &= \sum_{\substack{(j_1,\ldots,j_k) \\ j_i\in\{1,\ldots,n\}}} \mathbb{E}\left[\prod_{i=1}^{k} I_{j_i}\right]\end{aligned}$$

Here, we have every combination of $j_1, \ldots, j_k$ such that $j_i \in \{1, \ldots, n\}$. Now, since $I_j \cdot I_j = I_j$, we have that for any sequence containing distince $j_1, \ldots, j_r$, we will then have

$$I_{j_1}^{n_1} \cdot \ldots \cdot I_{j_r}^{n_r} = I_{j_1} \cdot \ldots \cdot I_{j_r}, \sum_{i=1}^{r} n_i = k.$$

Also, we compute:

$$\mathbb{E}[I_{j_1} \cdot \ldots \cdot I_{j_r}] = \frac{(n-r)!}{n!},$$

since if we have r elements fixed, there are $(n-r)!$ permutations that permute the remaining elements. If we fix some distinct set $j_1, \ldots, j_r$, we see that there is some unique number $n(r, k)$, the number of terms which are a product of k of these I_j, if the product contains r distinct factors $I_{j_1}, \ldots, I_{j_r}$. Next, if there are n total terms in the sum X_n, then we can see that there will be $n(n-1) \cdot \ldots \cdot (n-r+1) = \frac{n!}{(n-r)!}$ different choices for r distinct factors I_j. Then, we group the terms in the sum (**??**) based on how many distinct

$$\begin{aligned}\sum_{\substack{(j_1,\ldots,j_k) \\ j_i\in\{1,\ldots,n\}}} \mathbb{E}\left[\prod_{i=1}^{k} I_{j_i}\right] &= \sum_{r=1}^{k} n(r,k) \sum_{j_1<\ldots<j_r} \mathbb{E}[I_{j_1} \cdot \ldots \cdot I_{j_r}] \\ &= \sum_{r=1}^{k} n(r,k) \sum_{j_1<\ldots<j_r} \frac{(n-r)!}{n!} \\ &= \sum_{r=1}^{k} n(r,k) \frac{n!}{(n-r)!} \frac{(n-r)!}{n!} = \sum_{r=1}^{k} n(r,k).\end{aligned}$$

Notice, that this is independent of n, so we have that $\lim_{n\to\infty} \mathbb{E}[X_n^k] = \sum_{r=1}^{k} n(r,k)$.

Now, we do a similar computation for the Binomial random variable $B(n, 1/n)$ random variable. We have that $B(n, 1/n) = \sum_{j=1}^{n} B_j$, where $B_j(1/n)$ is the bernoilli random variable with probability $1/n$ of success. Then, we have $B_j(1/n)_j^n = B_j(1/n)$, and so for $j_1, \ldots, j_r$ distinct, we have by independence of these bernoilli trials, that

$$\mathbb{E}[B_{j_1}(1/n)^{n_{j_1}} \cdot \ldots \cdot B_{j_r}(1/n)^{n_{j_r}}] = \mathbb{E}[B_{j_1}] \cdot \ldots \cdot \mathbb{E}[B_{j_r}(1/n)] = (1/n)^r.$$

Then, a similar computation as for the X_n moment calculation gives us

$$\mathbb{E}[Y_n^k] = \sum_{r=1}^{k} n(r,k) \sum_{j_1<\ldots<j_r} \left(\frac{1}{n}\right)^r = \sum_{r=1}^{k} n(r,k) \frac{n!}{(n-r)!} \frac{1}{n^r}.$$

But then as $n \to \infty$, we have that $\frac{n!}{(n-r)!} \to n^r$, so $E[Y_n^k] \to \sum_{r=1}^{k} n(r,k) = E[X_n^k]$.

Now, since the MGF of the poisson (1) distribution is $\exp(e^t - 1)$, we have that for any $s_0 > 0$, $\exp(e^s - 1)$ is finite for all $|s| < s_0$. Then, by theorem 11.4.3, the Poisson distribution is determined by its moments. Therefore, if a sequence of random variables is such that its moments converge to the moments of the Poisson distribution, then the random variables converge to the Poisson distribution in distribution. Since we are given that $\mathbb{E}[Y_n^k] \to \mathbb{E}[\text{Pois}(1)^k]$, and $\mathbb{E}[X_n^k] \to \lim_{n\to\infty} \mathbb{E}[Y_n^k]$, we have that $\mathbb{E}[X_n^k] \to \mathbb{E}[\text{Pois}(1)^k]$, so X_n converges to Pois(1) in distribution.

3. (a) Suppose that the sequence of probability measures $(\mu_n)_{n=1}^{\infty}$ is tight. Show that their characteristic functions are uniformly equicontinuous, i.e., for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $|h| < \delta$ then for all n and all $t \in \mathbb{R}$

$$|\phi_n(t+h) - \phi_n(t)| < \infty\,,$$

where ϕ_n is the characteristic function of μ_n.

Hint: You may rely on the inequality $|1 - e^{-iy}| \leq \min\{2, |y|\}$ for all $y \in \mathbb{R}$. Deduce from it that for all $x, h \in \mathbb{R}$, $|1 - e^{-ihx}| \leq |h|^{1/2} + 2\mathbb{1}_{|x|>|h|^{-1/2}}$.

(b) Suppose that $\mu_n \Rightarrow \mu$. Denote the characteristic function of μ by ϕ and of μ_n by ϕ_n. Use (a) to conclude that $\phi_n(t) \to \phi(t)$ uniformly on compact sets, i.e., for all $M > 0$ and $\varepsilon > 0$ there exists $N = N(\varepsilon, M)$ such that for all $n \geq N$ we have that $|\phi_n(t) - \phi(t)| < \varepsilon$ for all $t \in [-M, M]$.

(c) Give an example to show that the convergence in (b) need not be uniform on the entire real line.

Solution.

(a) We compute:

$$\begin{aligned}|\phi_n(t+h) - \phi_n(t)| &= \left|\int \exp(i(t+h)x) - \exp(itx)\mu_n(dx)\right| \\ &\leq \int |\exp(i(t+h)x)(1 - \exp(-ihx))|\,\mu_n(dx) = \int |1 - \exp(-ihx)|\,\mu_n(dx).\end{aligned}$$

Now, we are given that $|1 - e^{ihx}| \leq \min\{2, |hx|\}$. Then, consider that for $|x| < \frac{1}{|h|^{1/2}}$, we have that $|hx| < |h|^{1/2}$. But then, we have:

$$\min\{|hx|, 2\} \leq |h|^{1/2}\mathbb{1}_{|x|<\frac{1}{|h|^{1/2}}} + \mathbb{1}_{|x|\geq\frac{1}{|h|^{1/2}}} \leq |h|^{1/2} + \mathbb{1}_{|x|\geq\frac{1}{|h|^{1/2}}}.$$

We can then split this integral apart:

$$\int |1 - \exp(-ihx)|\mu_n(dx) \leq \int |h|^{1/2}\mu_n(dx) + \int \mathbb{1}_{|x|\geq\frac{1}{|h|^{1/2}}}\mu_n(dx).$$

Let $\varepsilon > 0$. Now, since $(\mu_n)_{n=1}^{\infty}$ is tight, we have that there exists some R such that $\int \mathbb{1}_{|x|\geq R}\mu_n(dx) < \varepsilon/2$. Then, let $\delta < \min\{(\varepsilon/2)^2, \frac{1}{R^2}\}$. For all $|h| < \delta$, we then have

$$\int |h|^{1/2}\mu_n(dx) + \int \mathbb{1}_{|x|\geq\frac{1}{|h|^{1/2}}}\mu_n(dx) < \varepsilon/2 + \int \mathbb{1}_{|x|\geq R}\mu_n(dx) \leq \varepsilon.$$

Therefore, the characteristic functions are uniformly euquicontinuous, since t, ε and we have produced δ as required.

(b) Let $\varepsilon > 0$. Since equicontinuous, there exists some δ such that for all $t \in [-M, M]$, and all n, $|\phi_n(t+h) - \phi_n(t)| < \varepsilon/3$ for all $|h| < \delta$.

Since $[-M, M]$ is compact, and $\phi(t)$ is continuous,we have that $\phi(t)$ is uniformly continuous on $[-M, M]$ (this is a basic result from Math 321). Therefore, there exists some δ' such that $|\phi(t+h) - \phi(t)| < \varepsilon/3$ for all $|h| < \delta$.

Now, since $\phi_n(t) \to \phi(t)$ pointwise, we have that each each t there exists some L such that for all $n > L$, $|\phi_n(t) - \phi(t)| < \varepsilon/3$. Call this number $L(t)$. Let $\delta'' = \min\{\delta, \delta'\}$. Then, consider the set of open sets $\{N(t, \delta'')\}, t \in [-M, M]$. This is clearly an open cover of $[-M, M]$, since it contains the balls centered at each $t \in [-M, M]$. Since $[-M, M]$ is compact, there is a finite open subcover $\{N(t_1, \delta''), \ldots, N(t_s, \delta'')\}$. Let $L = \max\{L(t_1), \ldots, L(t_s)\}$. In particular, for all $n > L$, $|\phi_n(t_i) - \phi(t_i)| < \varepsilon/3$ for any $1 \leq i \leq s$.

Putting it all together, let $t \in [-M, M]$. Let $n > L$. Then, there exists some $1 \leq i \leq s$ such that $t \in N(t_i, \delta'')$.

Since $|t_i - t| < \delta$, uniform equicontinuity of ϕ_n means that we have

$$|\phi_n(t_i) - \phi_n(t)| < \varepsilon/3.$$

Since $n > L$, we have

$$|\phi_n(t_i) - \phi(t_i)| < \varepsilon/3.$$

Finally, since $|t_i - t| < \delta'$, we have

$$|\phi(t) - \phi(t_i)| < \varepsilon/3.$$

By triangle inequality,

$$|\phi_n(t) - \phi(t)| = |\phi_n(t) - \phi_n(t_i) + \phi_n(t_i) - \phi(t_i) + \phi(t_i) - \phi(t)| \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

Therefore, $\phi_n \to \phi$ uniformly on compact sets.

(c) For an example where converge in (b) need not be uniform on the entire real line, we need to find a distributtion $\mu_n \Rightarrow \mu$ such that $(\mu_n)_{n=1}^{\infty}$ is tight, but $\phi_n \to \phi$ doesn't converge uniformly on the entire real line.

Let μ_n be the normal distribution, with mean 0 and standard deviation $\frac{1}{n}$. First, we show that these distributions are tight. For μ_1, we have that the CDF is given by $\Phi(x)$, since μ_1 is the standard normal distribution. Let $\varepsilon > 0$. Since $\lim\limits_{x \to -\infty} \Phi(x) = 0$, there must be some R such that $\Phi(-R) < \varepsilon/2$, and so $1 - \Phi(R) < \varepsilon/2$. But then, for all $n > 1$, the CDF of μ_n is given by $\Phi(x/(1/n)) = \Phi(nx)$. But then, we have that $\Phi(n(-R)) < \Phi(-R) < \varepsilon/2$, and similarly, $1 - \Phi(nR) < \Phi(R) < \varepsilon/2$.

Therefore, for all n, we have that there exists R such that $\int_{|x|>R} \mu_n(dx) < \varepsilon$, so (μ_n) is tight.

Next, we observe that the probability desnity function of μ_n is given by $f_n(x) = \frac{n}{\sqrt{2\pi}} \exp(-n^2x^2/2)$. We computed the characterisitc function of the normal distribution in class: $\phi_n(t) = \exp(-t^2/2n^2)$ We can see that, as $n \to \infty$, $\phi_n(t) = 1$. However, this clearly does not converge uniformly, since for any n, by taking $t \to \infty$, we can find some t' sufficiently large so that $|1 - \exp(-(t')^2/2n^2)| > \frac{1}{2}$. Therefore, $\phi_n(t) \to 1$, but not uniformly.

4. **Parts (i)–(v) of this question are unmarked** (you can skip them and rely on them): Let $\lambda > 0$. Consider for all n a sequence $(p_{n,m} : 1 \leq m \leq N_n)$ of real numbers in $[0, 1]$ satisfying that (1) $\sum_{m=1}^{N_n} p_{n,m} = \lambda$ and (2) $\lim\limits_{n\to\infty} p_*(n) = 0$, where $p_*(n) := \max_{1 \leq m \leq N_n} p_{n,m}$.

(i) Show that $\lim\limits_{n\to\infty} \sum_{m=1}^{N_n} p_{n,m}^2 \leq \lim\limits_{n\to\infty} p_*(n) \sum_{m=1}^{N_n} p_{n,m} = 0$.

(ii) Suppose that $(q_{n,m} : 1 \leq m \leq N_n)$ satisfy that for some $C > 0$ we have that $|q_{n,m} - p_{n,m}| \leq Cp_{n,m}^2$ for all $n \geq 1$ and all $m \in [N_n] := \{1, 2, \ldots, N_n\}$. Show that $\lim\limits_{n\to\infty} \sum_{m=1}^{N_n} q_{n,m} = \lambda$ and that $\lim\limits_{n\to\infty} q_*(n) = 0$, where $q_*(n) := \max_{1 \leq m \leq N_n} q_{n,m}$.

(iii) Let $q_{n,m}$ be the solution to $\mathbb{P}[\mathrm{Pois}(q_{n,m}) = 1] = q_{n,m}e^{-q_{n,m}} = p_{n,m}$.[1] Show that for some $C > 0$ we have for all $n \geq 1$ and all $m \in [N_n]$ that $|q_{n,m} - p_{n,m}| \leq Cp_{n,m}^2$ and $\mathbb{P}[\mathrm{Pois}(q_{n,m}) \geq 2] = 1 - e^{-q_{n,m}} - q_{n,m}e^{-q_{n,m}} \leq Cp_{n,m}^2$.

[1] Since $p_*(n) \to 0$, w.l.o.g. we may assume that $p_*(n) \leq e^{-1}$ for all n, and this ensures that such a solution $q_{n,m}$ exists and that it lies in $(0, 1]$.

(iv) Suppose that $\lambda_n > 0$ and $\lim_{n\to\infty} \lambda_n = \lambda$. Show that $\text{Pois}(\lambda_n) \Rightarrow \text{Pois}(\lambda)$.

(v) Let $q_{n,m}$ be as in (iii). For any fixed $n \geq 1$, let $(Y_{n,m} : m \in [N_n])$ be independent Poisson random variables, where $Y_{n,m} \sim \text{Pois}(q_{n,m})$ for all $n \geq 1$ and all $m \in [N_n]$. Show that $Y(n) := \sum_{m=1}^{N_n} Y_{n,m} \Rightarrow \text{Pois}(\lambda)$.

(vi) Show that

$$\lim_{n\to\infty} \mathbb{P}\left[\sum_{m=1}^{N_n} Y_{n,m}\mathbb{1}_{Y_{n,m}\geq 2} \neq 0\right] = \lim_{n\to\infty} \mathbb{P}[Y_{n,m} \geq 2 \text{ for some } m \in [N_n]] = 0\,.$$

Deduce that $Z(n) := -\sum_{m=1}^{N_n} Y_{n,m}\mathbb{1}_{Y_{n,m}>1} = -\sum_{m=1}^{N_n} Y_{n,m}\mathbb{1}_{Y_{n,m}\geq 2} \to 0$ in distribution, as $n \to \infty$. Use this and part (vi) to deduce that

$$Y(n) + Z(n) = \sum_{m=1}^{N_n} Y_{n,m}\mathbb{1}_{Y_{n,m}\leq 1} = \sum_{m=1}^{N_n} \mathbb{1}_{Y_{n,m}=1} \Rightarrow \text{Pois}(\lambda)\,.$$

(vii) For all (fixed) $n \geq 1$ let $(X_{n,m} : 1 \leq m \leq N_n)$ be independent Bernoulli random variables with $\mathbb{P}[X_{n,m} = 1] = p_{n,m} = 1 - \mathbb{P}[X_{n,m} = 0]$ for all n and all $m \in [N_n] := \{1, 2, \ldots, N_n\}$. Deduce from part (vi) that $X(n) := \sum_{m=1}^{N_n} X_{n,m} \to \text{Pois}(\lambda)$ in distribution as $n \to \infty$.

5. The distribution of a random variable X is called *infinitely divisible* if, for all $n \in \mathbb{N}$, there exists a sequence of independent and identically distributed random variables $Y_1^{(n)}, Y_2^{(n)}, \ldots, Y_n^{(n)}$ such that X and $Y_1^{(n)} + Y_2^{(n)} + \cdots + Y_n^{(n)}$ have the same distribution.[2]

   Prove that the characteristic function ϕ of an infinitely divisible distribution is nonzero for all real t, i.e., that $\phi(t) \neq 0$ for all $t \in \mathbb{R}$.
   Hint: Let ϕ_n be the characteristic function of $Y_i^{(n)}$. Let $\psi_n(t) = |\phi_n(t)|^2$ and $\psi(t) = |\phi(t)|^2$.
   Step 1: Explain why these are also characteristic functions.
   Step 2: Express $\psi_n(t)$ in terms of $\psi(t)$ and n and deduce from this expression that $\lim_{n\to\infty} \psi_n(t) = g(t)$ for all $t \in \mathbb{R}$ for some $g : \mathbb{R} \to \{0, 1\}$, where $g(t)$ has an expression involving $\psi(t)$ for all $t \in \mathbb{R}$.
   Step 3: Use that expression to show that g is continuous at 0.
   Step 4: Apply a corollary of the continuity theorem from lecture to deduce a useful fact about g.
   For intuition sake, observe that it is natural to expect that $Y_1^{(n)} \to 0$ in distribution as $n \to \infty$.

6. Let X be a random variable.

   (i) Show that $\mathbb{E}[e^{sX} + e^{-sX}] = 2\sum_{k=0}^{\infty} \frac{s^{2k}\mathbb{E}[X^{2k}]}{(2k)!}$ for all $s \in [0, \infty)$. In particular, $\mathbb{E}[e^{sX} + e^{-sX}] < \infty$ if and only if $\sum_{k=0}^{\infty} \frac{s^{2k}\mathbb{E}[X^{2k}]}{(2k)!} < \infty$.

   (ii) Recall that by Stirling's approximation $\lim_{k\to\infty} \frac{\sqrt{2\pi k}(k/e)^k}{k!} = 1$. Use part (i) and Stirling's approximation to show that there exists $s > 0$ such that $M_X(t) < \infty$ for all $t \in [-s, s]$ if and only if $\sup_{k\in\mathbb{N}} \frac{\left(\mathbb{E}[X^{2k}]\right)^{\frac{1}{2k}}}{2k} < \infty$, where $\mathbb{N} := \{1, 2, 3, \ldots\}$. (ii) Recall that by Stirling's approximation $\lim_{k\to\infty} \frac{\sqrt{2\pi k}(k/e)^k}{k!} = 1$. Use part (i) and Stirling's approximation to show that there exists $s > 0$ such that $M_X(t) < \infty$ for all $t \in [-s, s]$ if and only if $\sup_{k\in\mathbb{N}} \frac{\left(\mathbb{E}[X^{2k}]\right)^{\frac{1}{2k}}}{2k} < \infty$, where $\mathbb{N} := \{1, 2, 3, \ldots\}$.

7. Let $d \geq 2$. For notational ease, you may assume $d = 2$. The moment generating function $M_X : \mathbb{R}^d \to (0, \infty]$ of a random d-tuple $X = (X(1), X(2), \ldots, X(d))$ is defined to be $M_X(t) = \mathbb{E}\left[e^{t\cdot X}\right]$, where $t = (t(1), t(2), \ldots, t(d))$ and $t \cdot X = \sum_{j=1}^{d} t(j)X(j)$ is the standard inner product of t and X. Recall that $X = (X(1), X(2), \ldots, X(d)) \stackrel{d}{=} Y = (Y(1), Y(2), \ldots, Y(d))$ if and only if $t \cdot X \stackrel{d}{=} t \cdot Y$ for all $t \in \mathbb{R}^d$ and that $X_n = (X_n(1), X_n(2), \ldots, X_n(d)) \Rightarrow X$ if and only if $t \cdot X_n \Rightarrow t \cdot X$ for all $t \in \mathbb{R}^d$.

[2] Examples of infinitely divisible distributions include: normal, Cauchy, Gamma, Poisson, geometric.

Fix some norm $\|\cdot\|$ on $\mathbb{R}^d$. Throughout all parts of the question suppose that:
(*) There exists $\delta > 0$ such that $M_X(t) < \infty$ for all $t \in \mathbb{R}^d$ such that $\|t\| \le \delta$.

You are encouraged to think about how condition (*) and the statement of part (c) below simplify in the special case that $X(1), X(2), \ldots, X(d)$ are independent.

(a) Suppose that there exists $\delta' > 0$ such that $M_X(t) = M_Z(t)$ for all $t \in \mathbb{R}^d$ such that $\|t\| \le \delta'$. Show that $X \stackrel{d}{=} Z$. **Hint:** There is an easy and short solution.

(b) **Part (b) is unmarked – Do not submit a solution to it. You may rely on it below.** Show that $\left|\mathbb{E}\left[\prod_{j=1}^d X(j)^{k(j)}\right]\right| < \infty$ for all $k(1), \ldots, k(d) \in \mathbb{Z}_+ := \{0, 1, 2, \ldots\}$.
Hint: Use (*) and Part (i) of Question 6 to show that $X(j) \in L^p$ for all $p \in \mathbb{N}$, for all $j \in [d]$. Then use Holder's inequality to get that $\left|\mathbb{E}\left[\prod_{j=1}^d X(j)^{k(j)}\right]\right|^d \le \prod_{j=1}^d \mathbb{E}\left[|X(j)|^{k(j)d}\right]$.

(c) Suppose that for all $k(1), \ldots, k(d) \in \mathbb{Z}_+$, $\mathbb{E}\left[\prod_{j=1}^d X_n(j)^{k(j)}\right]$ is finite for all n and that $\lim_{n\to\infty} \mathbb{E}\left[\prod_{j=1}^d X_n(j)^{k(j)}\right] = \mathbb{E}\left[\prod_{j=1}^d X(j)^{k(j)}\right]$. Show that[3] $X_n \Rightarrow X$.

Hint: Apply the method of moments to $t \cdot X$ for any $t \in \mathbb{R}^d$.

8. This problem concerns the method of Monte Carlo integration, which is a method for the approximate evaluation of an integral $I = \int_0^1 f(x)\mathrm{d}x$.
Let $U_1, \ldots, U_N$ be i.i.d. uniform random variables on the interval $(0, 1)$, and let

$$I_N = \frac{1}{N}[f(U_1) + \cdots + f(U_N)].$$

Suppose that $\int_0^1 f(x)^2\mathrm{d}x < \infty$, and let $\sigma^2 = \mathrm{Var} f(U_1) = \int_0^1 f(x)^2\mathrm{d}x - I^2$. Apply the central limit theorem to show that I_N converges to I as $N \to \infty$, in the sense that for all $x \in \mathbb{R}$

$$\lim_{N\to\infty} \mathbb{P}\left(|I_N - I| \le \frac{\sigma x}{\sqrt{N}}\right) = \mathbb{P}(|Z| \le x),$$

where Z is a standard normal random variable.

Solution. Consider the random variable $S_N = f(U_1) + \ldots + f(U_N)$. Then, $m = f(U_i) = I$. By Corollary 11.2.3 (an extension of the CLT) in the text, we have

$$\lim_{n\to\infty} P\left(\frac{S_n - nm}{\sqrt{nv}} \le x\right) = \Phi(x)$$

if S_n is the sum of random variables X_n with mean m and variance v. In this case, we have $S_n = NI_N$, $m = I$, and $\sqrt{v} = \sigma$. We then substitute:

$$\lim_{n\to\infty} P\left(\frac{NI_N - NI}{\sqrt{Nv}} \le x\right) = \lim_{n\to\infty} P\left(I_N - I \le \frac{\sigma x}{\sqrt{N}}\right) = \Phi(x) = P(Z \le x).$$

Therefore, the CDFs are the same, and we get:

$$\lim_{n\to\infty} P\left(|I_N - I| \le \frac{\sigma x}{\sqrt{N}}\right) = P(|Z| \le x).$$

9. Let $X_1, X_2, \ldots$ be i.i.d. random variables with c.d.f. F. The *empirical c.d.f.* $\widehat{F}_n : \mathbb{R} \to [0, 1]$ is defined by $\widehat{F}_n(x) = \frac{1}{n}\sum_{k=1}^n \mathbb{1}_{\{X_k \le x\}}$. Thus, for each $x \in \mathbb{R}$ and $n \ge 1$, $\widehat{F}_n(x)$ is a random variable.

[3] In particular, this shows that if (*) holds and $\mathbb{E}\left[\prod_{j=1}^d Y(j)^{k(j)}\right] = \mathbb{E}\left[\prod_{j=1}^d X(j)^{k(j)}\right]$ for all $k(1), \ldots, k(d) \in \mathbb{Z}_+$, then $X \stackrel{d}{=} Y = (Y(1), \ldots, Y(d))$.

(a) For each $x \in \mathbb{R}$, prove that $\widehat{F}_n(x) \to F(x)$ a.s. as $n \to \infty$.

(b) If x satisfies $0 < F(x) < 1$, prove that

$$\frac{\sqrt{n}(\widehat{F}_n(x) - F(x))}{\sqrt{F(x)[1-F(x)]}} \Rightarrow N(0,1)\,.$$

10. Let $X_1, X_2, \ldots$ be i.i.d. random variables with known mean m and unknown variance σ^2, both finite and with $\sigma^2 \neq 0$. Let $W_n := \sum_{k=1}^n (X_k - m)^2$ and $U_n := \begin{cases} \sqrt{W_n} & \text{if } W_n > 0 \\ 1 & \text{if } W_n = 0 \end{cases}$. Prove that

$$\frac{S_n - nm}{U_n} \Rightarrow N(0,1)\,.$$

Solution.

By CLT, we have that

$$\frac{S_n - nm}{\sqrt{n}\sigma} \Rightarrow N(0,1).$$

Now, by the strong law of large numbers, which applies since we are dealing with i.i.d. R.V.s with $(X_k - m)^2$ having finite expectation, we have

$$\frac{W_n}{n} \to \sigma^2 \quad a.s.$$

Since $\sqrt{x}$ is a continuous function, we have that the limit

$$\sqrt{W_n/n} \to \sigma \quad a.s.$$

Now,

$$U_n/\sqrt{n} = \begin{cases} \sqrt{W_n}/\sqrt{n} & \text{if } W_n > 0 \\ 1/\sqrt{n} & \text{if } W_n = 0 \end{cases}.$$

Now, since $P(W_n = 0) = 0$ as $n \to \infty$, we have that $U_n/\sqrt{n} \to \sqrt{W_n}/\sqrt{n}$ a.s. Therefore, $U_n\sigma$.

Then, we apply Slutsky's theroem to $X_n = \frac{S_n - nm}{\sqrt{n}\sigma}$ and $Y_n = U_n/\sqrt{n}$. Since U_n is never zero, Y_n is never zero either. Then,

$$\frac{S_n - nm}{U_n} = \frac{S_n - nm}{\sqrt{n}\sigma U_n/\sqrt{n}} = \sigma N(0,1)/\sigma = N(0,1).$$

Recommended problems. The following problems from Rosenthal are recommended but are not to be handed in:

11.5.2, 11.5.4, 11.5.6, 11.5.12

9.5.2, 11.4.2, 11.5.18
For solutions to even-numbered problems see: `http://www.probability.ca/jeff/grprobbook.html`.

Additional recommended problems (**do not hand in**):

1. Prove Slutsky's Theorem: For every $n \in \mathbb{N}$ let X_n and Y_n be two random variables defined on the same probability space. Let X be a random variable on some probability space. Let $c \in \mathbb{R}$ be a constant. Suppose that $X_n \Rightarrow X$ and that $Y_n \Rightarrow c$. Then
(a) $X_n+Y_n \Rightarrow X+c$. (Consequently if $X_n \Rightarrow X$ and $Y_n := Z_n - X_n \Rightarrow 0$ then $Z_n = X_n+Y_n \Rightarrow X$.)
(b) $X_nY_n \Rightarrow cX$.
(c) Suppose $c \neq 0$ and let z be an arbitrary non-zero real number. Then $X_n/Z_n \Rightarrow X/c$, where $Z_n := \begin{cases} Y_n & \text{if } Y_n \neq 0 \\ z & \text{if } Y_n = 0 \end{cases}$.

2. Show that there exists $s > 0$ such that $M_X(t) < \infty$ for all $t \in [-s, s]$ if and only if there exist $C, c > 0$ such that $\mathbb{P}[|X| > r] \leq Ce^{-cr}$ for all $r \geq 0$.[4]

3. A collection of probability measures $(\mu_j : j \in J)$ on $(\mathbb{R}, \mathcal{B})$ is called uniformly integrable if for all $\varepsilon > 0$ there exists $M > 0$ such that $\int |x|\mathbb{1}_{|x|>M} \mathrm{d}\mu_j(x) < \varepsilon$ for all $j \in J$. Show that $(\mu_j : j \in J)$ is uniformly integrable if and only if (i) $\sup_{j \in J} \int |x| \mathrm{d}\mu_j(x) < \infty$ and (ii) $(\nu_j : j \in J)$ is tight, where for $j \in J$, ν_j is the probability measure on $(\mathbb{R}, \mathcal{B})$ satisfying that for every bounded Borel function $f : \mathbb{R} \to \mathbb{R}$
$$\int f(x)\mathrm{d}\nu_j(x) = \frac{\int (1+|x|)f(x)\mathrm{d}\mu_j(x)}{\int (1+|x|)\mathrm{d}\mu_j(x)} .$$

Quote of the week: *I know of scarcely anything so apt to impress the imagination as the wonderful form of cosmic order expressed by the "Law of Frequency of Error." The law would have been personified by the Greeks and deified, if they had known of it. It reigns with serenity and in complete self-effacement, amidst the wildest confusion. The huger the mob, and the greater the apparent anarchy, the more perfect is its sway. It is the supreme law of Unreason. Whenever a large sample of chaotic elements are taken in hand and marshalled in the order of their magnitude, an unsuspected and most beautiful form of regularity proves to have been latent all along.*

Francis Galton[5] describing the Central Limit Theorem [(1889) Natural Inheritance]

[4] If this conditions holds for all sufficiently large r for some $C, c > 0$ then it also holds for all $r \geq 0$ for some other $C, c > 0$. Thus it suffices to confirm it for sufficiently large r.
[5] Galton's reputation is tarnished by his work on eugenics. `https://en.wikipedia.org/wiki/Francis_Galton`