Math 418/544 Assignment 6

Dr. J. Hermon

Due: Wednesday, October 29

Late assignments will not be accepted **Submit solutions for 5 out of the following 8 problems. The TA might only have time to grade 4 problems. Please indicate which 4 out of the 5 problems you submit you want graded (in case only 4 get graded).**

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I did problems 1-5. Please grade problems 1-4. Thank you!

1. For $p \geq 1$, let $\|X\|_p := (\mathbb{E}[|X|^p])^{1/p}$ and $\|X\|_\infty := \inf\{M : \mathbb{P}(|X| > M) = 0\}$.
   (a) Prove that $\|XY\|_1 \leq \|X\|_1 \|Y\|_\infty$.
   (b) Prove that $\|X\|_\infty = \lim_{p\to\infty} \|X\|_p$.

   Solution

   (a) Suppose that $\mathbb{P}(\|Y\|_\infty < |Y|) > 0$. If $\mathbb{P}\left(\|Y\|_\infty + \frac{1}{n} < |Y|\right) = 0$ for all $n \in \mathbb{N}$, then continuity of probabilities from below would give us $\mathbb{P}(\|Y\|_\infty < |Y|) = 0$, which is a contradiction. Therefore, we need to have $\mathbb{P}(\|Y\|_\infty < |Y|) = 0$. This implies that $\|Y\|_\infty \geq |Y|$ a.e., so that

   $$\|XY\|_1 \leq \mathbb{E}[|XY|] \leq \mathbb{E}[|X|\|Y\|_\infty] = \|Y\|_\infty \mathbb{E}[|X|] = \|Y\|_\infty \|X\|_1.$$

   (b) First, we show that $\lim_{p\to\infty} \|X\|_p \leq \|X\|_\infty$. Since $\|X\|_\infty \geq |X|$ a.e., it follows $(\|X\|_\infty)^p \geq |X|^p$ a.e. Therefore,

   $$\mathbb{E}[|X|^p] \leq \mathbb{E}[(\|X\|_\infty)^p] = (\|X\|_\infty)^p \Rightarrow \|X\|_p = \mathbb{E}[|X|^p]^{1/p} \leq \|X\|_\infty.$$

   Since p was arbitrary, we have that $\lim_{p\to\infty} \|X\|_p \leq \|X\|_\infty$.

   Now, we show that $\|X\|_\infty \leq \lim_{p\to\infty} \|X\|_p$. Let $\varepsilon > 0$. Then, there exists some M such that $M > \|X\|_\infty - \varepsilon$ and $\mathbb{P}(|X| > M) > \delta$ for some $\delta > 0$. Let $B = \{\omega : |X| > M\}$. Then,

   $$\|X\|_p = \mathbb{E}[|X|^p]^{1/p} \geq \mathbb{E}[|X|^p \mathbb{1}_{\omega\in B}]^{1/p} \geq \mathbb{E}[M^p \mathbb{1}_{\omega\in B}]^{1/p} \geq M\delta^{1/p} = M \exp\left(\frac{1}{p}\log\delta\right).$$

   Since the exponential function is continuous at 0, as $p \to \infty$, we know that $\exp\left(\frac{1}{p}\log\delta\right) \to 1$ and so $\lim_{p\to\infty} \|X\|_p \geq M = \|X\|_\infty - \varepsilon$. Since this applies for all $\varepsilon > 0$, we have that $\lim_{p\to\infty} \|X\|_p \geq \|X\|_\infty$.

   We conclude, that $\|X\|_\infty = \lim_{p\to\infty} \|X\|_p$.

2. Let $X_0 = (1, 0)$ and define $X_n \in \mathbb{R}^2$ inductively by declaring that X_{n+1} is chosen at random from the ball of radius $|X_n|$ centered at the origin; i.e. $X_{n+1}/|X_n|$ is uniformly distributed on the unit ball and independent of $X_1, \ldots, X_n$. Prove that $n^{-1} \log |X_n| \to c$ almost surely and compute c.

   Solution. We have that $X_{n+1}/|X_n|$ is uniformly distributed on the unit ball and independent of $X_1, \ldots, X_n$. Therefore, $X_1/|X_0|, \ldots, X_{n+1}/|X_n|$ are i.i.d. Since functions of independent random variables are independent, it follows that $\log(|X_1|/|X_0|), \ldots, \log(|X_{n+1}|/|X_n|)$ are i.i.d. Therefore, we can apply the strong law of large numbers to these variables. To do so, we must compute their mean. From the Calculus, in polar co-ordinates, we compute:

$$\begin{aligned}\frac{1}{\pi}\int_0^{2\pi}\int_0^1 \log(r) r dr &= \cancelto{0}{2\frac{1}{2}\log(r) r^2\Big|_0^1} - \int_0^1 r dr \\ &= -\frac{1}{2}.\end{aligned}$$

Both limits of $\log(r)r^2$ are zero: at $r=0$ beacuse of l'Hopital's rule, at $r=1$ since $\log(1)=0$.

Therefore, $\mathbb{E}[\log(|X_{n+1}/X_n|)] = -1/2$. Applying the S.L.L.N, we get $n^{-1}\sum_{k=1}^n \log(|X_n|/|X_{n-1}|) = \mathbb{E}[\log(|X_{n+1}|/|X_n|)] = -1/2$ a.s. However, we can see that this is a telescoping sum:

$$\begin{aligned}n^{-1}\sum_{k=1}^n \log(|X_n|/|X_{n-1}|) = n^{-1}\sum_{k=1}^n \log(|X_n|) - \log(|X_{n-1}|) &= \frac{1}{n}\left(\log(|X_n|) - \log(|X_0|)\right) \\ &= \frac{1}{n}\log(|X_n|) - \cancelto{0}{\frac{1}{n}\log(1)}.\end{aligned}$$

Therefore, $n^{-1}\log|X_n| \to -\frac{1}{2}$ a.s.

3. Each year you may invest in:
   - Bonds costing \$1 that are worth a at year's end.
   - Stocks worth a random amount $V \geq 0$.

   Investing a fixed proportion p in bonds yields $W_{n+1} = (ap + (1-p)V_n)W_n$. Suppose $V_1, V_2, \ldots$ are i.i.d. with $\mathbb{P}(V_n \in [\alpha, \beta]) = 1$ for some $0 < \alpha \leq \beta < \infty$.

   (a) Show there is a constant $c(p)$ such that $n^{-1}\log W_n \to c(p)$ a.s.

   (b) Show $c(p)$ is concave.

   (c) Examine $c'(0)$ and $c'(1)$ to determine conditions on V guaranteeing the optimal p lies in $(0,1)$.

   (d) Suppose $\mathbb{P}(V=1) = \mathbb{P}(V=4) = 1/2$. Find the optimal p as a function of a.

   Solution.

   (a) Let $c(p) = \mathbb{E}[\log(W_{n+1}/W_n)] = \mathbb{E}[\log(ap + (1-p)V)]$. Since $\{V_n\}_n$ are i.i.d. functions of V_n are also i.i.d. Therefore, we can apply the S.L.L.N., so that

   $$\lim_{n\to\infty} n^{-1}\sum_{k=1}^n \log(W_{k+1}/W_k) = \mathbb{E}[\log(ap + (1-p)V)] \text{ a.e..}$$

   Just like in question 2, this is a telescoping sum, so we get

   $$\begin{aligned}n^{-1}\sum_{k=1}^n \log(W_{k+1}/W_k) &= n^{-1}\log(W_{n+1}) - n^{-1}\log(W_0) \\ &= \frac{n^{-1}}{(n+1)^{-1}}(n+1)^{-1}\log(W_{n+1}) - n^{-1}\log(W_0) \\ &\to (n+1)^{-1}\log(W_{n+1}).\end{aligned}$$

   However, $\mathbb{E}[\log(W_0)]$ is a constant w.r.t. n, so the last term goes to zero as $n\to\infty$. Therefore,

   $$\lim_{n\to\infty} n^{-1}\log(W_n) = \mathbb{E}[\log(ap + (1-p)V)]\text{a.e.}$$

(b) Fix ω. Then, for $p(\lambda) = \lambda p_2 + (1-\lambda)p_1$, with $\lambda \in [0,1]$ and $p_1 < p_2$, we have

$$ap(\lambda) + (1-p(\lambda))V = \lambda(ap_2 + (1-p_2)V) + (1-\lambda)(ap_1 + (1-p_1)V).$$

Since log is concave, we have:

$$\log(\lambda(ap+(1-p)V)+(1-\lambda)(ap+(1-p)V)) \geq \lambda \log(ap_2+(1-p_2)V)+(1-\lambda)\log(ap_1+(1-p_1)V).$$

But, since ω was arbitrary, and $\mathbb{E}$ is monotone, we have:

$$\begin{aligned} c(p(\lambda)) = \mathbb{E}[\log(ap(\lambda) + (1-p(\lambda))V)] &\geq \lambda\mathbb{E}[\log(ap_2 + (1-p_2)V)] + (1-\lambda)\mathbb{E}[\log(ap_1 + (1-p_1)V)] \\ &= \lambda c(p_2) + (1-\lambda)c(p_1). \end{aligned}$$

Therefore, c is concave.

(c) Concavity tells us, that a sufficient condition for p lying in $(0,1)$, is that $c'(0) > 0 > c'(1)$. Elementary calculus of concave functions then tells us $c(p) \leq c(0)$ for all $p < 0$, and $c(p) \leq c(1)$ for all $p > 1$. Then, $c(p)$ will attain its maximum in $(0,1)$. We compute:

$$c'(p) = \mathbb{E}\left[\frac{a-V}{ap-(1-p)V}\right] \Rightarrow c'(0) = \mathbb{E}\left[\frac{a}{V} - 1\right], c'(1) = \mathbb{E}\left[1 - \frac{V}{a}\right].$$

So then, our requirements on V are that $\mathbb{E}[a/V] > 1$ and $\mathbb{E}[V/a] > 1$.

(d) We first compute $c(p)$. We have:

$$c(p) = \mathbb{E}[\log(ap + (1-p)V)] = \frac{1}{2}\log(ap + (1-p)) + \frac{1}{2}\log(ap + 4(1-p)).$$

We compute:

$$c'(p) = \frac{a-1}{2((a-1)p+1)} + \frac{a-4}{2((a-4)p+4)}.$$

Since concave, the max happens when $c'(p) = 0$. We compute:

$$\begin{aligned} (4-a)((a-1)p+1) &= (a-1)((a-4)p+4) \\ 4(1-a) + (4-a) &= 2(a-1)(a-4)p \\ p &= \frac{2}{4-a} + \frac{1}{2(1-a)}. \end{aligned}$$

Therefore, the optimum of $c(p)$ occurs at $p^* = \frac{2}{4-a} + \frac{1}{2(1-a)}$. We plug it into our expression for $c(p)$:

$$\begin{aligned} c(p^*) &= \frac{1}{2}\log\left(\frac{2a}{4-a} + \frac{a}{2(1-a)} + \left(1 - \frac{2}{4-a} - \frac{1}{2(1-a)}\right)\right) \\ &+ \frac{1}{2}\log\left(\frac{2a}{4-a} + \frac{a}{2(1-a)} + 4\left(1 - \frac{2}{4-a} - \frac{1}{2(1-a)}\right)\right) \\ &= \frac{1}{2}\log\left(\frac{9a^2}{4(a-1)(4-a)}\right). \end{aligned}$$

I used a symbolic calculator for the last part because I was too lazy to do the algebra. Next, we check the derivative at the edges to see when this optimimum holds. $c'(0) = \frac{5a-8}{8}$ which is greater than 0 when $a > 8/5$. When this doesn't hold, then the optimum is $p^* = 0$. $c'(1) = \frac{2a-5}{2a}$ which is smaller than 0 when $a < 5/2$. When this doesn't holld, the optimum is $p^* = 1$.

4. (a) Give an example of random sequences $\{Y_n\}$ and $\{N_n\}$ with $Y_n \to 0$ in probability and $N_n \to \infty$ a.s. but $Y_{N_n} \to 1$ a.s. *Hint:* Use the example from lecture $Y_n = \{x\} := x - \lfloor x \rfloor$ for $x \in [S_n, S_{n+1}]$, where $S_n = \sum_{k=0}^{n} 1/k$.

   (b) Suppose instead that $Y_n \to Y$ a.s. and $N_n \to \infty$ a.s. Prove that $Y_{N_n} \to Y$ a.s.

Solution.

(a) Define $\{Y_n\}_n$ as the sequence of intervals $\{[0,1], [0, \frac{1}{2}], [\frac{1}{2}, 1], [0, \frac{1}{3}], [\frac{1}{3}, \frac{2}{3}], [\frac{2}{3}, 1], [0, \frac{1}{4}] \ldots\}$.

Using a bit of calculus, we can actually compute what the n-th interval will be. There is one interval length 1, two length $\frac{1}{2}$, three length $\frac{1}{3}$, etc. Therefore, the first interval of length $\frac{1}{m}$ will have $\sum_{k=1}^{m-1} k = \frac{m(m-1)}{2}$ intervals before it. Conversely, if $n = \frac{m(m-1)}{2} + 1$, that is, the first interval of length $1/m$, then we solve for m in terms of n, by solving a quadratic, to get $m = \frac{\sqrt{8(n-1)+1}-1}{2}$. In particular, if we are at the j-th interval of width $\frac{1}{m}$, we proceed to compute $j = n - \left\lfloor \frac{\sqrt{8(n-1)+1}-1}{2} \right\rfloor$. Define $n(m,j) = \frac{m(m-1)}{2} + j$.

Now, we comute N_m. We define N_m pointwise, for each $\omega \in [0,1]$. Let $\omega \in [0,1]$. Since each group of intervals $\{[0, \frac{1}{m}], [\frac{1}{m}, \frac{2}{m}], \ldots, [\frac{m-1}{m}, 1]\}$ forms a partition of $[0,1]$, there exists one $j \in [0, m-1]$ such that $\omega \in [\frac{j}{m}, \frac{j+1}{m}]$. Call this $j(\omega)$. Let $N_m(\omega) = n(m, j(\omega))$. Clealry, this is increasing in m (since it is a quadratic with leading coefficient 1). Then, by construction, we have that for all $m \in \mathbb{N}$, $Y_{N_m(\omega)}(\omega) = 1$, for any $\omega \in [0,1]$. Thus, $Y_{N_n} \to 1$.

(b) Let $B = \{\omega | Y_n(\omega) \to Y\}$. Then, $P(B) = 1$. Let $C = \{\omega | N_n \to \infty\}$. The, $P(C) = 1$, so that $P(B \cap C) \geq P(B) + P(C) - P(B \cap C) = 1 + 1 - 1 = 1$. Let $\omega \in B \cap C$. Let $\varepsilon > 0$. Since $\omega \in B$, there exists some N_1 such that for all $n > N_1$, $|Y_n - Y| < \varepsilon$. But since $\omega \in C$, there exists some N_2 such that for all $n > N_2$, $N_n > N_1$. Therefore, for all $n > N_2$, we have that $|Y_{N_n(\omega)}(\omega) - Y| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we conclude that $Y_{N_n(\omega)}(\omega) \to Y(\omega)$. Since $\omega \in B \cap C$ was arbitrary, we conclude that on $B \cap C$, $Y_{N_n(\omega)}(\omega) \to Y(\omega)$. But since $P(B \cap C) = 1$, this means that $Y_{N_n} \to Y$ a.e.

5. (a) Show that $X_n \to X$ in probability if and only if every subsequence (X_{m_k}) has a further subsequence (X_{n_k}) with $X_{n_k} \to X$ a.s.

   (b) Suppose $X_n \to X$ in probability and $f : \mathbb{R} \to \mathbb{R}$ is continuous. Show that $f(X_n) \to f(X)$ in probability, and if f is bounded, $\mathbb{E}[f(X_n)] \to \mathbb{E}[f(X)]$.

Solution.

(a) $(\to)$ Suppose $X_n \to X$ in probability. Let X_{n_k} by any subsequence of X_n. In paticular, this implies that $X_{n_k} \to X$ in probability. Then, for all $\frac{1}{m}$, there exists some N_m such that for all $n_k > N_m$, $P(|X_{n_k} - X| > 1/m) < \frac{1}{2^m}$. Construct a new sequence, X_{n_m} as follows: $n_m < n_{m+1}$, and $n_m > N_m$. Let $A_m = \{\omega : |X_{n_m}(\omega) - X(\omega)| > \frac{1}{m}\}$. In particular, $P(\cup_{m=n} A_m) \leq \sum_{m=n}^{\infty} P(A_m) = \sum_{m=n}^{\infty} \frac{1}{2^m} = \frac{1}{2^{n-1}}$. Let $B = \cap_{n=1}^{\infty} \cup_{m=n} A_m$. By continuity of probabilities, we have that $P(B) = \lim\limits_{n \to \infty} P(\cup_{m=n} A_m) = \lim\limits_{n \to \infty} \frac{1}{2^{n-1}} = 0$. Thus, $P(B^c) = 1$. Let $\omega \in B^c$. Let $\varepsilon > 0$. Then, there exists some M such that for all $m > M$, $\varepsilon > \frac{1}{m}$. Since $\omega \in B^c$, this implies that $\omega \in \cup_{n=1}^{\infty} \cap_{m=n} A_m^c$. Thus, there exists some n such that $\omega \in A_m^c$ for all $m > n$. Then, by the definition of A_m, we have that $|X_{n_m}(\omega) - X(\omega)| < 1/m < \varepsilon$ for all $m > \max(n, M)$. But then, since $\varepsilon > 0$ was arbitrary, this means that $X_{n_m} \to X$ everywhere in B, which is a.e. Therefore, a subsequence of X_{n_k} exists that converges to X.

$(\leftarrow)$ Let $\{X_n\}$ be a seqence of random variables such that every subsequence $\{X_{n_k}\}$ of $\{X_n\}$ has a subsetqence that converges to X a.e. Suppose that X_n does not converge to X in probability. In particular, this implies that there exists some $\varepsilon > 0$ such that $\alpha = \liminf_{n \to \infty} P(|X_n - X| > \varepsilon) \neq \limsup_{n \to \infty} P(|X_n - X| > \varepsilon) = \beta$. From math 320, we have that there exists some subsequence $\{X_{n_k}\}$ such that $\lim_{k\to\infty} P(|X_{n_k} - X| > \varepsilon) = \alpha$, and another subsequence $\{X_{n_l}\}$ such that $\lim_{l\to\infty} P(|X_{n_l} - X| > \varepsilon) = \beta$. However, by the hypothesis, we have that $X_{n_l} \to X$ a.e., and $X_{n_k} \to X$ a.e. We proved in class (it is Theorem 5.2.3 in the text) that if a sequence converges a.e. then it converges in probability. Therefore, $\lim_{l\to\infty} P(|X_{n_l} - X| > \varepsilon) = 0 = \lim_{k\to\infty} P(|X_{n_k} - X| > \varepsilon) = 0$, which contradicts that $\alpha < \beta$. Therefore, the assumption that X_n does not converge to X in probability must be false, so $X_n \to X$ in probability.

(b) We seek to prove that $f(X_n) \to f(X)$ in probability. Fix $\varepsilon > 0$. Now, for each $n \in N$, define $A_n = \{x \in \mathbb{R} : f(N_{1/n}(x)) \subset N_\varepsilon(f(x))\}$ Since f is continuous everywhere, we have that $A_n \subset A_{n+1}$, and $\cup_{n=1}^{\infty} A_n = \mathbb{R}$. In particular, $\lim_{n\to\infty} P(X \in A_n) = 1$.

Let $\varepsilon^* > 0$. Then, there exists some n such that $P(X \in A_n) > 1 - \varepsilon^*/2$. Next, there is some $M \in \mathbb{N}$ such that for all $m > M$, $P(|X_m - X| < 1/n) > 1 - \varepsilon^*/2$. However, if $\omega \in \{X \in A_n\}$, and $\omega \in \{|X_n - X| < 1/n\}$, then by the definition of A_n, this implies $|f(X_n(\omega)) - f(X(\omega))| < \varepsilon$. In particular, this implies that for $m > M$,

$$\begin{aligned}
&P(|f(X_m) - f(X)| < \varepsilon) \\
&\geq P(\{X \in A_n\} \cap \{|X_m - X| > 1/n\}) \\
&= P(X \in A_n) + P(|X_m - X| > 1/n) - P(\{X \in A_n\} \cup \{|X_m - X| > 1/n\}) \\
&> 1 - \varepsilon^*/2 + 1 - \varepsilon^*/2 - 1 = 1 - \varepsilon^*.
\end{aligned}$$

In other words, there exists M such that for all $m > M$, $P(|f(X_n) - f(X)| \leq \varepsilon) > 1 - \varepsilon^*$. Since $\varepsilon^* > 0$ was arbitrary, this implies that $\lim_{n\to\infty} P(|f(X_n) - f(X)| > \varepsilon) = 0$. Since $\varepsilon > 0$ was arbitrary, this implies that $f(X_n) \to f(X)$ in probability.

Now, suppose f is bounded. In particular, there exists $M \in \mathbb{R}$ such that $|f| < M$. Let $\varepsilon > 0$. Then, since $f(X_n) \to f(X)$ in probability, $\lim_{n\to\infty} P(|f(X_n) - f(X)| < \varepsilon) = 0$ Let $B_n = \{|f(X_n) - f(X)| < \varepsilon\}$. Since f is bounded, if $\omega \in B_n^c$, then we have $|f(X_n(\omega)) - f(X)| \leq 2M$.

We can then see that:

$$\begin{aligned}
E[f(X_n)^+] &= E[f(X_n)^+ \mathbb{1}_{B_n^c}] + E[f(X_n)^+ \mathbb{1}_{B_n^c}] \\
&< 2MP(B_n^c) + E[(f(X)^+ + \varepsilon)\mathbb{1}_{B_n}] \\
&\leq 2MP(B_n^c) + E[f(X)^+] + \varepsilon.
\end{aligned}$$

Similarly, we can compute $E[f(X_n)^+] \geq E[f(X)^+] - \varepsilon - 2MP(B_n^c)$. But then, we can choose N such that for all $n > N$, $P(B_n^c) < \varepsilon^*$. So, if we take the limit, we have: $E[f(X)^+] - \varepsilon \leq \lim_{n\to\infty} E[f(X_n)^+] \leq E[f(X)^+] + \varepsilon$. However, since $\varepsilon > 0$ was arbitrary, we have equality. Doing the same for the negative component, and adding the results, we have $\lim_{n\to\infty} E[f(X_n)] = E[f(X)]$.

6. (**Renewal Theory**) Jobs arrive continuously at a busy server. The ith job's completion time is X_i, with X_i i.i.d., positive, and $\mathbb{E}[X_i] = \mu < \infty$. Let $T_n = X_1 + \cdots + X_n$ and $N_t = \sup\{n : T_n \leq t\}$. Prove that

$$\lim_{t\to\infty} \frac{N_t}{t} = \frac{1}{\mu} \quad \text{a.s.}$$

Hint: Use $T_{N_t} \leq t < T_{N_t+1}$ and recall the a.s. limit of T_n/n.

7. (**Hölder's Inequality**) For simplicity, you may take $k = 2$.

(a) Let $a_1, \dots, a_k > 0$ and $p_1, \dots, p_k \in (1, \infty)$ satisfy $\sum 1/p_j = 1$. Using convexity of $x \mapsto -\log x$, show

$$\prod_{j=1}^{k} a_j \leq \sum_{j=1}^{k} \frac{a_j^{p_j}}{p_j}.$$

(b) Deduce that for random variables $X_1, \dots, X_k$,

$$\Big\| \prod_{j=1}^{k} X_j \Big\|_1 \leq \prod_{j=1}^{k} \|X_j\|_{p_j}.$$

(c) Let $p, q \in (1, \infty)$ with $1/p + 1/q = 1$. Show

$$\|X\|_p = \max\{\mathbb{E}[XY] : \|Y\|_q \leq 1\}.$$

8. (**Completeness of L^2**) Show that if (X_n) is Cauchy in L^2, then $\exists X \in L^2$ with $\|X_n - X\|_2 \to 0$.
   (a) Show it suffices to find an a.s. and L^2 limit along a subsequence. Bt
   (b) Use the "2^{-k} subsequence trick" and Borel–Cantelli to find such a subsequence.
   (c) Verify the a.s. limit is also the L^2 limit and lies in L^2 (use Dominated Convergence).

Recommended Problems: Rosenthal 5.5.13, 6.3.2, 6.3.4, 7.4.4.
Solutions to even-numbered problems: `http://www.probability.ca/jeff/grprobbook.html`

More Recommended Problems:

A. Use Jensen's inequality to prove that $\|X\|_p \leq \|X\|_q$ for $1 \leq p \leq q \leq \infty$.

B. Let $X \sim \text{Poisson}(\lambda)$. Compute its moment generating function and use it to find $\mathbb{E}[X]$ and $\text{Var}(X)$.

When she was rid of the pretense of paper and pen, phrase-making and biography, she turned her attention in a more legitimate direction, though, strangely enough, she would rather have confessed her wildest dreams of hurricane and prairie than the fact that, upstairs, alone in her room, she rose early in the morning or sat up late at night to work at mathematics...

— **Virginia Woolf, *Night and Day***