Math 418/544 — Assignment 2

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Instructions: This assignment is due in Canvas at 9:59 a.m. on Monday, September 22. Late assignments are not accepted.

1. **(Hat Problem)** Suppose n people remove their hats, mix them up, and then each chooses a hat uniformly at random. Find the probability p_n that nobody chooses their own hat, and show that $\lim_{n\to\infty} p_n = e^{-1}$.

   Hint: Let E_i denote the event that the i-th person selects their own hat, and apply the inclusion-exclusion principle given in Exercise 4.5.7.

   Solution: Let A_i denote the event that the i-th person selects their hat. Then, the probability that nobody chooses their own hat is given by $P\left(\bigcap_{i=1}^n A_i^c\right)$. First, we observe that $P(A_i) = \frac{1}{n}$, since the hats are chosen randomly (so that each sequence of hats have the same probability of ocurring... by symmetry, the number of sequences where person i has hat n is the same as the number of sequences where person i has hat j for any $1 \le j < n$). Likewise, symmetry bears witness to the fact that $P(A_{i_1} \cap \ldots \cap A_{i_k}) = \frac{(n-k)!}{n!}$. This is because, after the i_{k-1}th person has chosen his hat, there remain $n-k$ hats left to choose from, and the one remaining hat must be chosen to be the i_kth person's hat. After that, there are $(n-k)!$ possible sequences for the remaining hats, out of the $n!$ total possible sequences, giving us the probability $\frac{(n-k)!}{n!}$. Now, the inclusion-exclusion principle tells us that the probability that a single person gets the right hat, is then:

$$
\begin{aligned}
P\left(\bigcup_{i=1}^n A_i\right) &= \sum_{i=1}^n P(A_i) - \sum_{1\le i_1<i_2\le n} P(A_{i_1}\cap A_{i_2}) \\
&\quad + (-1)^{n-1} \sum_{1\le i_1<\ldots<i_{n-1}\le n} P(A_{i_1}\cap\ldots\cap A_{i_{n-1}}) + (-1)^n P\left(\bigcap_{i=1}^n A_i\right) \\
&= \sum_{i=1}^n \frac{1}{n} - \sum_{1\le i_1<i_2\le n} \frac{1}{n(n-1)} \\
&\quad + (-1)^{n-1} \sum_{1\le i_1<\ldots<i_{n-1}\le n} \frac{1}{n!} + (-1)^n \frac{1}{n!}.
\end{aligned}
$$

   First, we observe that the terms that are being summed are constant: independent of the indices of the sums. Next, we can see that the number of terms in each sum are given by $\binom{n}{k}$, where k is the number of indices in the sum. Indeed, if there are k indices, none of the indices can match (so their values are sampled from n without repetition), and since their ordering is fixed, we count each "choice" only once. This describes $\binom{n}{k}$ precisely.

Thus,

$$\begin{aligned} P\left(\bigcup_{i=1}^{n} A_i\right) &= \sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}\frac{(n-k)!}{n!} \\ &= \sum_{k=1}^{n}\frac{(-1)^{k+1}}{k!}. \end{aligned}$$

However, what we really need is $1 - P\left(\bigcup_{i=1}^{n} A_i\right) = P\left(\bigcap_{i=1}^{n} A_i^c\right)$. We compute:

$$p_n = 1 - P\left(\bigcup_{i=1}^{n} A_i\right) = 1 - \sum_{k=1}^{n}\frac{(-1)^{k+1}}{k!} = \sum_{k=0}^{n}\frac{(-1)^{k}}{k!}.$$

Then, $\lim_{n\to\infty} p_n = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(-1)^k}{k!} = e^{-1}$.

2. **(Borel σ-algebra)** The Borel σ-algebra $\mathcal{B}$ on $\mathbb{R}$ is by definition the σ-algebra generated by the open subsets of $\mathbb{R}$. Prove that the σ-algebra generated by the rational open intervals $\{(a,b) \subset \mathbb{R} \mid a, b \in \mathbb{Q}\}$ is the same as $\mathcal{B}$.

Solution: A basic result concerning sigma algebras is that $\sigma(A) = \sigma(B)$ if $A \subset \sigma(B)$ and $B \subset \sigma(A)$.

In this case, denote $A = \{(a,b) \subset \mathbb{R} | a, b \in \mathbb{Q}\}$, and B the set of open subsets of $\mathbb{R}$. Clearly, $A \subset B \subset \sigma(B)$. So, to show that $\sigma(A) = \sigma(B)$, we need only show that for any $B \subset \sigma(A)$.

To show this, let $X \in B$. Then, let

$$C = \bigcup_{\substack{Y \in \sigma(A) \\ Y \subset X}} Y.$$

This is a countable union, since the rational open intervals can be enumerated as $|\mathbb{Q} \times \mathbb{Q}|$ is countable. Since it is a countable union of rational open intervals, $C \in \sigma(A)$. By construction, $C \subset X$, since for any $x \in C$, $x \in Y$ for some $Y \in A$ such that $Y \subset X$. Now, suppose $x \in X$. Then, there exists some open ball $B(x,r) \subset X$ where $r > 0$. But then, by density of rationals there exists $q_1, q_2 \in \mathbb{Q}$ $x - r < q_1 < x < q_2 < x + r$, so that $(q_1, q_2) \subset B(x,r) \subset X$. Since (q_1, q_2) is a rational open interval, $(q_1, q_2) \in A$. Thus, (q_1, q_2) is in the union defining C, and so $(q_1, q_2) \subset C$. Therefore, $x \in C$, so $X \subset C$ and then $X = C$.

Since X was arbitrary, $B \subset \sigma(A)$. Therefore, $\sigma(B) = \sigma(A)$, in other words the sigma algebra generated by the set of rational open intervals is the Borel σ-algebra on $\mathbb{R}$.

3. **Rosenthal 2.7.5.** Suppose that $\Omega = \mathbb{N}$ is the set of positive integers, and $\mathcal{F}$ is the set of all subsets A such that either A or A^c is finite, and P is defined by $P(A) = 0$ if A is finite, and $P(A) = 1$ if A^c is finite.

   (a) Is $\mathcal{F}$ an algebra?

   (b) Is $\mathcal{F}$ a σ-algebra?

   (c) Is P finitely additive?

   (d) Is P countably additive on $\mathcal{F}$, meaning that if $A_1, A_2, \ldots \in \mathcal{F}$ are disjoint, and if it happens that $\bigcup_n A_n \in \mathcal{F}$, then $P(\bigcup_n A_n) = \sum_n P(A_n)$?

Solution:

(a) Yes. We check the requirements for $\mathcal{F}$ to be an algebra.

- $\emptyset, \Omega \in \mathcal{F}$. $\emptyset$ is finite, and $\Omega^c = \emptyset$. Therefore $\emptyset, \Omega \in \mathcal{F}$.
- Closed under complements: Suppose $A \in \mathcal{F}$. Then, one of A or A^c is finite, so one of A^c or $(A^c)^c = A$ is finite, so $A \in \mathcal{F}$.
- Closed under finite unions. Let $A_1, \ldots, A_n \in \mathcal{F}$, and denote $A = \bigcup_{i=1}^n A_i$. If all of $A_1, \ldots, A_n$ are finite, then $A = \bigcup_{i=1}^n A_i$ is finite, so A^c is infinite an d $A \in \mathcal{F}$. Now suppose any of A_i are infinite, and A_i^c is finite. Let $L = \max\{m \in A_i^c\}$. Then, for all $m > L$, $m \in A_i \subset A$. However, then $A^c \subset \{m \in \mathbb{N} | m \leq L\}$, which is finite. Therefore, $A \in \mathcal{F}$.

Therefore, $\mathcal{F}$ is an algebra.

(b) $\mathcal{F}$ is not a σ-algebra, because it is not countable additive. Consider the sets $\{2n\}$, where $n \in \mathbb{N}$. For each of these sets, $\mathbb{N} \setminus \{2n\}$ is infinite, so they are in $\mathcal{F}$. However, consider $A = \bigcup_{n=1}^{\infty} \{2n\} = \{2n | n \in \mathbb{N}\}$ (the even numbers). Then, $A^c = \{2n-1 | n \in \mathbb{N}\}$ (the odd numbers). However, both of these sets are infinite, so $A \notin \mathcal{F}$. Therefore, $\mathcal{F}$ is not closed under countable unions, so it is not a σ-algebra.

(c) Yes. Let $A_1, \ldots, A_n$ be disjoint. First, we claim that at most 1 of these sets can be infinite and in $\mathcal{F}$. Suppose not. Then, $A_i \cap A_j \neq \emptyset$, and both are infinite. However, both A_i^c and A_j^c are finite. Let $M_1 = \max A_i^c$, and $M_2 = \max A_j^c$. Then, this implies that $n > M_1 \Rightarrow n \in A_i$ and $n > M_2 \Rightarrow n \in A_j$, so that for instance $M_1 + M_2 + 1 \in A_i \cap A_j$, so that we arrive at a contradiction.

Now, if all $A_1, \ldots, A_n$ are finite, then their disjoint union is finite, and we have

$$P(\bigcup_{i=1}^n A_i) = 0 = \sum_{i=1}^n 0 = \sum_{i=1}^n A_i.$$

If one of $A_1, \ldots, A_n$ is infinite, then all the others are finite, and their union is infinite. Take i to be the index of the infinite set. Then, we have

$$P(\bigcup_{i=1}^n A_i) = 1 = \sum_{j \neq i} 0 + 1 = \sum_{j \neq i} P(A_j) + P(A_i) = \sum_{j=1}^n P(A_j).$$

(d) No, P is not countably additive on $\mathcal{F}$. We know that the sets $\{n\}$ for $n \in \mathcal{N}$ are disjoint, and each is finite and in $\mathcal{F}$. However,

$$P(\bigcup_{n=1}^{\infty}\{n\}) = P(\mathbb{N}) = 1 \neq 0 = \sum_{n=1}^{\infty} P(\{n\}) = \sum_{n=1}^{\infty} 0.$$

4. **Rosenthal 2.7.9.** Let $\mathcal{F}$ be a σ-algebra, and write $|\mathcal{F}|$ for the total number of subsets in $\mathcal{F}$. Prove that if $|\mathcal{F}| < \infty$ (i.e., if $\mathcal{F}$ consists of just a finite number of subsets), then $|\mathcal{F}| = 2^m$ for some $m \in \mathbb{N}$. [Hint: Consider those non-empty subsets in $\mathcal{F}$ which do not contain any other non-empty subset in $\mathcal{F}$. How can all subsets in $\mathcal{F}$ be "built up" from these particular subsets?]

Solution: We follow the hint. Let

$$S = \{A \in \mathcal{F} | \forall B \in \mathcal{F}, A \cap B \in \{A, \emptyset\}, A \neq \emptyset\}.$$

Since $\mathcal{F}$ is finite, we must have that S is finite as well. Now, consider $\mathcal{P}(S)$. Since it is a power set, we have that it has 2^n elements, where $n = |S|$.

We will show that the function $f : \mathcal{P}(S) \to \mathcal{F}$ defined by $f(X) = \bigcup_{A \in X} A$ is a bijection, which will show that $|\mathcal{F}|$.

- First, we note that this function is well defined: since, for any $X \in \mathcal{P}(S)$, the elements $A \in X$ are in $\mathcal{F}$, and $\mathcal{F}$ is closed under countable unions, it follows that $f(X) = \bigcup_{A \in X} A \in \mathcal{F}$.
- Now, to show that f is a bijection, we need only show that it is injective, since $\mathcal{F}$ and $\mathcal{P}(S)$ are finite sets. Let $X, Y \in \mathcal{P}(S)$. Suppose $f(X) = \bigcup_{A \in X} A = \bigcup_{B \in Y} B = f(Y)$. If $f(X) = \emptyset$, then $X = Y = \emptyset$, since no $A \in S$ is the empty set. Thus, suppose not. Then, there exists some $x \in \Omega$ such that $x \in f(X)$. Then, there exists some $A \in X$ such that $x \in A$, and $B \in Y$ such that $x \in B$, with $A, B \in S$. However, by the definition of S, this implies that $A = B$.

  Next, we show that

  $$f(X \setminus \{A\}) = \bigcup_{A' \in (X \setminus \{A\})} A' = \left(\bigcup_{A' \in X} A'\right) \setminus A.$$

  Suppose $x \in f(X \setminus \{A\})$. If $x \in A$, then we must have that $x \in A'$ for some $A' \in X \setminus \{A\}$. However, the definition of S gives us that $A' = A$ since the sets in S are pairwise disjoint, so we arrive at a contradiction. Therefore, $x \notin A$. Also, $x \in A'$ for some $A' \in X$. Therefore, $x \in \left(\bigcup_{A' \in X} A'\right) \setminus A$. Now, suppose $x \in \left(\bigcup_{A' \in X} A'\right) \setminus A$. Then, $x \in A'$ for some $A' \in X$, and $x \notin A$, so $A' \neq A$. Thus, $x \in f(X \setminus \{A\})$.

  Therefore, we can apply this result to see:

  $$f(X \setminus \{A\}) = (\cup_{A' \in X} A') \setminus A = (\cup_{B' \in Y} B') \setminus B = f(Y \setminus \{B\}).$$

But then, we can apply this procedure, to remove another set, A_1 from X, and get some $B_1 = A_1$ such that

$$f(X \setminus \{A, A_1\}) = (X \setminus A) \setminus A_1 = (X \setminus B) \setminus B_1 = f(Y \setminus \{B, B_1\}).$$

Since there are finitely many sets in X, we will eventually get that $f(X \setminus \{A_i\}_i) = f(\emptyset) = \emptyset = f(Y \setminus \{A_i\}_i)$. However, this implies that $Y \setminus \{A_i\}_i = \emptyset$. Thus, $Y \subset X = \{A_i\}_i$. However, WLOG, we can also show that $X \subset Y$. Therefore, $X = Y$ and f is injective. Thus, $\mathcal{F} = \mathcal{P}(S)$, and so has cardinality 2^m, as desired.

5. **Rosenthal 2.7.21.** Let λ be Lebesgue measure in dimension two, i.e. Lebesgue measure on $[0,1] \times [0,1]$. Let

$$A = \{(x, y) \in [0,1] \times [0,1] : \ y < x\}.$$

Prove that A is measurable with respect to λ, and compute $\lambda(A)$.

For the computation of $\lambda(A)$, you may find Proposition 3.3.1 to be useful.

Solution:

Consider the following sets:

$$A_n = \bigcup_{i=0}^{2^n - 1} (i2^{-n}, (i+1)2^{-n}] \times [0, i2^{-n}).$$

Intuitively, this consists of the Riemann rectangles below the graph of $y = x$, with rectangle width $\frac{1}{2^n}$. The idea here:

- each of these is a disjoint union of sets we know are Lebesgue measurable (rectangles of half-open intervals) whose measure we know
- $A_n \subset A_{n+1}$ so that we can compute $\lim\limits_{n\to\infty} \lambda(A_n) = \lambda\left(\bigcup_{n=1}^{\infty} A_n\right)$
- and last (but definitely not least) $A = \bigcup_{n=1}^{\infty} A_n$. Each of these claims we will check in turn.

Let us begin with the proofs of these statements.

- To see that A_n is a disjoint union, suppose WLOG that $i < j$. Then, for $(x_1, x_2) \in (i2^{-n}, (i+1)2^{-n}] \times [0, i2^{-n})$ and $(y_1, y_2) \in (j2^{-n}, (j+1)2^{-n}] \times [0, j2^{-n})$ we have $i2^{-n} < x_1 \leq (i+1)2^{-n} \leq j2^{-n} < y_1 \leq (j+1)2^{-n}$. In particular, $x_1 \neq y_1$, so we can be usre that $(x_1, x_2) \neq (y_1, y_2)$, and these two rectangles are disjoint. Since this holds for any pair i, j with $i \neq j$, all of the rectangles are pairwise disjoint.
- Next, we show that $A_n \subset A_{n+1}$. Let $x \in A_n$. There exists some $0 \leq i < 2^n$ such that $x \in (i2^{-n}, (i+1)2^{-n}] \times [0, i2^{-n})$. We compute that:

$$\begin{aligned}
&(i2^{-n}, (i+1)2^{-n}] \times [0, i2^{-n}) \\
&\subseteq (((2i)2^{-(n+1)}, (2i+1)2^{-(n+1)}] \times [0, (2i)2^{-(n+1)})) \cup \\
&\quad (((2i+1)2^{-(n+1)}, ((2i+1)+1)2^{-(n+1)}] \times [0, (2i+1)2^{-(n+1)})).
\end{aligned}$$

$0 \le i < 2^n$ implies $0 \le 2i < 2i+1 < 2^{n+1}$. Therefore, both of these rectangles in this last line are in A_{n+1}, so this set inclusion implies $x \in A_{n+1}$. Thus, we have that $A_n \subset A_{n+1}$.

- Next, we check that $A = \bigcup_{n=1}^{\infty} A_n$. Suppose $(x, y) \in [0,1] \times [0,1]$ and $y < x$. Then, there exists some n such that $y < x - 2^{-n}$. Since $(i2^{-n}, (i+1)2^{-n}]$, $0 \le i < 2^n$ is a partition of $(0,1]$, there exists some i such that $x \in (i2^{-n}, (i+1)2^{-n}]$. We cannot have $x = 0$, since then there is no way for $y < x$ since $y \ge 0$. But since $y < x - 2^{-n}$, the interval x sits inside implies $y < x - 2^{-n} < (i+1)2^{-n} - 2^{-n} = i2^{-n}$. Also, $y \ge 0$ since $y \in [0,1]$, so $y \in [0, i2^{-n})$. Therefore, $(x, y) \in (i2^{-n}, (i+1)2^{-n}] \times [0, i2^{-n}) \subset A_n$. Since this applies for all $(x, y) \in A$, we have $A \subset \bigcup_{n=1}^{\infty}$.

- Finally, we compute $\lambda(A)$. First, we compute:

$$\lambda(A_n) = \sum_{i=0}^{2^n-1} ((i+1)2^{-n} - i2^{-n})(i2^{-n}) = \sum_{i=0}^{2^n-1} i \cdot 2^{-n} \cdot 2^{-n} = (2^{-2n}) \cdot \frac{2^n(2^n-1)}{2}$$
$$= \frac{1}{2} - \frac{1}{2^{n+1}}.$$

We have:

$$\lambda(A) = \lambda\left(\bigcup_n A_n\right) = \lim_{n\to\infty} \lambda(A_n)$$
$$= \lim_{n\to\infty} \frac{1}{2} - \frac{1}{2^{n+1}} = \frac{1}{2}.$$

Therefore, A is lebesgue measurable, and has measure $\frac{1}{2}$.

Recommended (not to be handed in): 2.7.7, 2.7.14, 2.7.15, 2.7.19, 2.7.22. For solutions to even-numbered problems see:
`http://www.probability.ca/jeff/grprobbook.html`

"It is seen in this essay that the theory of probabilities is at bottom only common sense reduced to calculus; ..."
— Pierre Simon Laplace, *A Philosophical Essay on Probabilities*