Math 544 Assignment 1 September 8, 2025

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This assignment is due in Canvas at 23:59 a.m. on Friday, September 15.
Late assignments are not accepted.

1. In this problem, explain how you are counting—do not just write down an answer without explanation.

   (a) Compute the probability that a poker hand[1] contains:

   i. one pair ($aabcd$ with a, b, c, d distinct face values; answer: 0.4226)

   ii. two pairs ($aabbc$ with a, b, c distinct face values; answer: 0.04754).

   (b) Poker dice[2] is played by simultaneously rolling 5 dice. Compute the probabilities of the following outcomes:

   i. one pair ($aabcd$ with a, b, c, d distinct numbers; answer: 0.4630)

   ii. two pairs ($aabbc$ with a, b, c distinct numbers; answer: 0.2315).

Solution

(a) In both cases, we count the number of decks satisfying the required property, and then divide by the total number of decks to get the total probability.

i. First, we determine the number of ways we can count the values. We have 13 choices for the value of a, and $\binom{12}{3}$ choices for the values for b, c, d. Next, we count the number of suits for the different decks. We have $\binom{4}{2}$ choices for the suits of a, and 4 choices for each of b, c, d. Finally, we have $\binom{52}{5}$ decks in total to draw from. Thus, we get:

$$P(\text{One pair}) = \frac{13 \cdot \binom{12}{3} \cdot \binom{4}{2} \cdot 4^3}{\binom{52}{5}} = \frac{1,098,240}{2,598,960} = 0.4226.$$

ii. Similarly, we count values and then suits. For values, we have 13 choices for c, and then $\binom{12}{2}$ choices for the values of the pairs a, a and b, b. Then, we have 4 choices for the suit of c and $\binom{4}{2}^2$ choices for the suits of a, a and b, b. In total, this then gives us

$$P(\text{Two pairs}) = \frac{13 \cdot \binom{12}{2} \cdot 4 \cdot \binom{4}{2}^2}{\binom{52}{5}} = \frac{123,552}{2,598,960} = 0.04754.$$

(b) In this case, we don't have to count suits (since there are only values). For simplicity I will keep track of the orderings of the dice. There are then 6^5 different dice roll sequences.

i. We have $6 \cdot 5 \cdot 4 \cdot 3$ choices for the numbers, and $\binom{5}{2}$ choices for choosing which indices of the dice in the sequence are doubled. We compute:

$$P(\text{One pair}) = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot \binom{5}{2}}{6^5} = \frac{3600}{7776} = 0.4630.$$

[1] A poker hand consists of five cards drawn from a deck of 52 cards. The cards have 13 distinct values $2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A$, each in four suits called Hearts, Diamonds, Clubs, Spades.

[2] Each die is a cube with six sides labelled 1,2,3,4,5,6.

ii. We have $6 \cdot 5 \cdot 4$ choices for the values a, b, c. Next, choose which two dice show a ($\binom{5}{2}$ ways), and then which two of the remaining dice show b ($\binom{3}{2}$ ways). Since swapping a and b gives the same outcome, we divide by 2. Finally, the last die is c. Thus, we get:

$$P(\text{Two pairs}) = \frac{6 \cdot 5 \cdot 4 \cdot \frac{1}{2}\binom{5}{2}\binom{3}{2}}{6^5} = \frac{1800}{7776} = 0.2315.$$

2. Let $S = \{1, 2, \ldots, n\}$ and suppose that a pair of subsets (A, B) of S is chosen uniformly at random from all possible pairs of subsets. More precisely, the probability of choosing any specific pair is 2^{-2n}. Show that
$$P(A \subset B) = \left(\frac{3}{4}\right)^n.$$

Solution

We consider each element $x \in S$ independently. For a uniformly random pair (A, B), the membership of x in A and B is independent and each of the four outcomes

$$x \in A \cap B, \quad x \notin A \cup B, \quad x \in A \setminus B, \quad x \in B \setminus A$$

occurs with probability $1/4$.

The event $A \subset B$ fails iff $x \in A \setminus B$. Thus, for a fixed x,

$$P\big(x \in A \Rightarrow x \in B\big) = 1 - P(x \in A \setminus B) = 1 - \tfrac{1}{4} = \tfrac{3}{4}.$$

Since the choices for different x are independent, we obtain

$$P(A \subset B) = \Pi_{x \in S} P(x \in A \Rightarrow x \in B) = \left(\tfrac{3}{4}\right)^n.$$

3. Let Ω be a nonempty set and suppose that $\mathcal{F}$ is a collection of subsets of Ω such that $\Omega \in \mathcal{F} \subset 2^{\Omega}$.

   (a) Prove that $\mathcal{F}$ is an algebra if $A, B \in \mathcal{F}$ implies that $A \setminus B = A \cap B^c \in \mathcal{F}$.
   (For an algebra the condition of closure under countable unions in the definition of a σ-algebra is replaced by closure under finite unions.)

   (b) Suppose that $\mathcal{F}$ is closed under complements and finite *disjoint* unions. Show that $\mathcal{F}$ need not be an algebra.

Solution

(a) We check the requirements that $\mathcal{F}$ is an algebra.

- $\Omega \in \mathcal{F}$, by assumption.
- $\emptyset \in \mathcal{F}$ since $\emptyset = \Omega \setminus \Omega \in \mathcal{F}$.
- Closed under complements: for $A \in \mathcal{F}$, $A^c = \Omega \setminus A \in \mathcal{F}$.
- Closed under finite intersections: for $A, B \in \mathcal{F}$,

  $$A \cap B \;=\; A \setminus B^c \in \mathcal{F},$$

  since $B^c \in \mathcal{F}$ and $\mathcal{F}$ is closed under set difference. By induction, $\bigcap_{i=1}^n A_i \in \mathcal{F}$ for any finite family.
- Closed under finite unions: by De Morgan,

  $$A \cup B \;=\; (A^c \cap B^c)^c \in \mathcal{F}.$$

  Hence $\bigcup_{i=1}^n A_i \in \mathcal{F}$ for any finite family.

Therefore, $\mathcal{F}$ is an algebra.

(b) Consider the set $\Omega = \{1,2,3,4\}$, and the family $\mathcal{F} = \{\emptyset, \{1,2\}, \{3,4\}, \{1,4\}, \{2,3\}, \{1,2,3,4\}\}$. Aside from pairs containing the empty set, the only other disjoint unions are $\{1,2\} \dot\cup \{3,4\} = \Omega$ and $\{1,4\} \dot\cup \{2,3\} = \Omega$. Also, $\mathcal{F}$ is closed under complements: Indeed, we have, for non-empty sets, the following complementary pairs: $\{1,2\} = \{3,4\}^c$, and $\{1,4\} = \{2,3\}^c$. Therefore, $\mathcal{F}$ satisfies the requirements of this definition. However, $\mathcal{F}$ is not closed under finite intersections, so it is not an algebra, since $\{1,2\} \cap \{2,3\} = \{2\} \notin \mathcal{F}$.

(c) Consider $\Omega = \{1,2,3,4\}$ and

$$\mathcal{F} = \{\emptyset,\ \{1,2\},\ \{3,4\},\ \{1,4\},\ \{2,3\},\ \Omega\}.$$

Then $\mathcal{F}$ is closed under complements (e.g., $\{1,2\}^c = \{3,4\}$ and $\{1,4\}^c = \{2,3\}$). It is also closed under finite disjoint unions: aside from unions with $\emptyset$, the only nontrivial disjoint pairs are $\{1,2\}\dot\cup\{3,4\} = \Omega$ and $\{1,4\}\dot\cup\{2,3\} = \Omega$, both in $\mathcal{F}$.

However, $\mathcal{F}$ is not an algebra because it is not closed under finite intersections or finite unions. We can see that:

$$\{1,2\} \cap \{2,3\} = \{2\} \notin \mathcal{F} \quad \text{and} \quad \{1,2\} \cup \{2,3\} = \{1,2,3\} \notin \mathcal{F}.$$

Thus, closure under complements and finite disjoint unions does not imply that $\mathcal{F}$ is an algebra.

4. (a) Given an arbitrary nonempty collection of sets $\{E_\alpha : \alpha \in A\}$, prove that there is a smallest σ-algebra that contains every E_α. This σ-algebra is called the σ-algebra *generated* by $\{E_\alpha : \alpha \in A\}$.

   (b) Suppose that we have σ-algebras $\mathcal{F}_1$ and $\mathcal{F}_2$. Show (by counterexample) that the union $\mathcal{F}_1 \cup \mathcal{F}_2$ need not be a σ-algebra.

Solution

(a) Let $\Omega = \bigcup_{\alpha \in A} E_\alpha$ and set $S = \{E_\alpha : \alpha \in A\}$. There exists at least one σ-algebra on Ω containing S, namely 2^Ω.

Define

$$\mathcal{F} = \bigcap_{\substack{\mathcal{X} \text{ a } \sigma\text{-algebra on } \Omega \\ S \subset \mathcal{X}}} \mathcal{X}.$$

We check that $\mathcal{F}$ is a σ-algebra: clearly $\Omega, \emptyset \in \mathcal{F}$, since every σ-algebra on Ω must contain $\emptyset$ and Ω. If $A \in \mathcal{F}$, then $A \in \mathcal{X}$ for every such $\mathcal{X}$, hence $A^c \in \mathcal{X}$ for each $\mathcal{X}$, so $A^c \in \mathcal{F}$. If $A_1, A_2, \ldots \in \mathcal{F}$, then $A_i \in \mathcal{X}$ for every $\mathcal{X}$, and since each $\mathcal{X}$ is a σ-algebra, $\bigcup_{i=1}^\infty A_i \in \mathcal{X}$ for all $\mathcal{X}$; hence $\bigcup_{i=1}^\infty A_i \in \mathcal{F}$. Thus $\mathcal{F}$ is a σ-algebra containing S. Finally, if $\mathcal{G}$ is any σ-algebra on Ω with $S \subset \mathcal{G}$, then $\mathcal{G}$ appears in the intersection above, so $\mathcal{F} \subset \mathcal{G}$. Therefore $\mathcal{F}$ is the smallest σ-algebra containing $\{E_\alpha : \alpha \in A\}$ (the σ-algebra generated by this collection).

(b) If the whole space isn't fixed, we could take $\mathcal{F}_1 = \{\emptyset, \{1\}\}$ and $\mathcal{F}_2 = \{\emptyset, \{2\}\}$; then $\mathcal{F}_1 \cup \mathcal{F}_2 = \{\emptyset, \{1\}, \{2\}\}$ is not a σ-algebra (e.g., it is not closed under unions, and doesn't contain the whole space).

If we insist both σ-algebras are on the same sample space, take $\Omega = \{1,2,3\}$ and

$$\mathcal{F}_1 = \{\emptyset, \{1\}, \{2,3\}, \Omega\}, \qquad \mathcal{F}_2 = \{\emptyset, \{1,2\}, \{3\}, \Omega\}.$$

Both $\mathcal{F}_1$ and $\mathcal{F}_2$ are σ-algebras on Ω, but

$$\mathcal{F}_1 \cup \mathcal{F}_2 = \{\emptyset, \{1\}, \{2,3\}, \{1,2\}, \{3\}, \Omega\}$$

is not a σ-algebra: for example, $\{1,2\} \cap \{2,3\} = \{2\} \notin \mathcal{F}_1 \cup \mathcal{F}_2$. Hence the union of two σ-algebras need not be a σ-algebra.

5. Let $\mathcal{F}$ be the σ-algebra generated by an arbitrary nonempty collection of sets $\{E_\alpha : \alpha \in A\}$. Prove that for each $E \in \mathcal{F}$, there exists a countable subcollection $\{E_{\alpha_j} : j = 1, 2, \ldots\}$ (depending on E) such that E already belongs to the σ-algebra generated by this subcollection.
Hint: Consider the class of all sets with the asserted property and show that it is a σ-algebra containing each E_α.

Solution

We follow the hint. Let $\mathcal{C}$ be the collection of all sets E such that there exists a countable subcollection $\{E_{\alpha_j} : j = 1, 2, \ldots\}$ with E belonging to the σ-algebra generated by this subcollection.

Clearly, every E_α itself belongs to $\mathcal{C}$, since E_α is in the σ-algebra generated by $\{E_\alpha\}$.

Now we check that $\mathcal{C}$ is a σ-algebra:

- $\Omega, \emptyset \in \mathcal{C}$ provided that $\{E_\alpha : \alpha \in A\}$ is non-empty, since $\emptyset, \Omega$ are present in every σ-algebra including those generated by countably many sets.
- Closed under complements: if $E \in \mathcal{C}$, then there is a countable subcollection B with $E \in \mathcal{F}_B$, where $\mathcal{F}_B$ is the σ-algebra generated by B. Since $\mathcal{F}_B$ is a σ-algebra, $E^c \in \mathcal{F}_B$, hence $E^c \in \mathcal{C}$.
- Closed under countable unions: if $E_1, E_2, \ldots \in \mathcal{C}$, then for each i there is a countable B_i with $E_i \in \mathcal{F}_{B_i}$. The union $B = \bigcup_i B_i$ is countable, and $\mathcal{F}_B$ is a σ-algebra containing each E_i, hence also $\bigcup_i E_i$. Thus $\bigcup_i E_i \in \mathcal{C}$.

Therefore, $\mathcal{C}$ is a σ-algebra containing all E_α. By definition, the σ-algebra generated by $\{E_\alpha : \alpha \in A\}$ is the smallest such, so

$$\mathcal{F} \subset \mathcal{C}.$$

Thus, for each $E \in \mathcal{F}$ there exists a countable subcollection $\{E_{\alpha_j}\}$ such that E lies in the σ-algebra generated by that subcollection.

Recommended problems. Each assignment will include additional recommended problems, which are not to be handed in for marking. The following problems from Rosenthal are recommended but are not to be handed in:

2.7.2, 2.7.4, 2.7.6.
For solutions to even-numbered problems see: `http://www.probability.ca/jeff/grprobbook.html`.

Quote of the week: *The student of arithmetic who has mastered the first four rules of his art, and successfully striven with money sums and fractions, finds himself confronted by an unbroken expanse of questions known as problems. These are short stories of adventure and industry with the end omitted, and though betraying a strong family resemblance, are not without a certain element of romance.*

Stephen Leacock