Math 418/544 Assignment 5 October 10, 2025

Dr. J. Hermon

This assignment is due in Canvas at 11:59 p.m. on Friday, October 17.
Late assignments are not accepted.

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1. Let $X_1, X_2, \ldots$ be independent $N(0,1)$ (standard normal) random variables. Prove that

$$P\Big(\limsup_{n\to\infty} \frac{|X_n|}{\sqrt{\log n}} = \sqrt{2}\Big) = 1.$$

(You may use the fact that the cumulative distribution function Φ of the standard normal obeys $1-\Phi(x) \sim \frac{1}{x}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ as $x \to \infty$.)

Solution.

We observe from the definition of $\limsup$, that for any ω, we have that $\limsup \frac{|X_n(\omega)|}{\sqrt{\log n}} = \sqrt{2}$ if for any $\varepsilon > 0$

$$\omega \in \{\frac{|X_n|}{\sqrt{\log n}} > \sqrt{2} - \varepsilon \text{ i.o.}\} \setminus \{\frac{|X_n|}{\sqrt{\log n}} > \sqrt{2} + \varepsilon \text{ i.o.}\}.$$

Namely,

$$\{\limsup \frac{|X_n|}{\sqrt{\log n}} = \sqrt{2}\} = \bigcap_{m=1}^{\infty} \{\frac{|X_n|}{\sqrt{\log n}} > \sqrt{2} - \frac{1}{m} \text{ i.o.}\} \setminus \{\frac{|X_n|}{\sqrt{\log n}} > \sqrt{2} + \frac{1}{m} \text{ i.o.}\}.$$

For $c > 0$, and large n so that $c\sqrt{\log n}$ is large enough to use the given approximation for $1-\Phi(x)$, we have

$$\begin{aligned} P\left(\frac{|X_n|}{\sqrt{\log n}} > c\right) = P\left(|X_n| > c\sqrt{\log n}\right) &= 2(1 - \Phi\left(c\sqrt{\log n}\right)) \\ &= \frac{2}{c\sqrt{\log n}\sqrt{2\pi}} \exp\left(-c^2 \log n/2\right) \\ &= \frac{2}{c\sqrt{\log n}\sqrt{2\pi}} \left(\frac{1}{n}\right)^{c^2/2}. \end{aligned}$$

By Rudin chapter 3 theorems 3.28 and 3.29, we have that $\sum_{n=1}^{\infty} P\left(\frac{|X_n|}{\sqrt{\log n}} > c\right)$ diverges when $c \leq \sqrt{2}$ and converges when $c > \sqrt{2}$.

Now, we apply the Borel Cantelli lemma. Then,

$$\begin{aligned} c > \sqrt{2} &\Rightarrow \sum_{n=1}^{\infty} P\left(\frac{|X_n|}{\sqrt{\log n}} > c\right) < \infty \Rightarrow P\left(\frac{|X_n|}{\sqrt{\log n}} > c \text{ i.o.}\right) = 0 \\ c \leq \sqrt{2} &\Rightarrow \sum_{n=1}^{\infty} P\left(\frac{|X_n|}{\sqrt{\log n}} > c\right) = \infty \Rightarrow P\left(\frac{|X_n|}{\sqrt{\log n}} > c \text{ i.o.}\right) = 1. \end{aligned}$$

By continuity from below, this implies

$$P\left(\bigcap_{m=1}^{\infty} \frac{|X_n|}{\sqrt{\log n}} > \sqrt{2} - \frac{1}{m} \text{ i.o.}\right) = 1.$$

Thus,

$$P(\limsup \frac{|X_n|}{\sqrt{\log n}} = \sqrt{2}) \geq P\left(\bigcup_{m=1}^{\infty} \frac{|X_n|}{\sqrt{\log n}} > \sqrt{2} - \frac{1}{m} \text{ i.o.}\right) - P\left(\bigcup_{m=1}^{\infty} \frac{|X_n|}{\sqrt{\log n}} > \sqrt{2} + \frac{1}{m} \text{ i.o.}\right) = 1 - 0.$$

2. This problem shows that the conclusion of the second Borel–Cantelli Lemma holds under the assumption that the events are pairwise independent only. It is due to Erdős and Rényi, 1959. Suppose that the events $A_1, A_2, \dots$ are pairwise independent, i.e., that $P(A_i \cap A_j) = P(A_i)P(A_j)$. Let $S_n = \sum_{j=1}^n \mathbb{1}_{A_j}$ and $S_\infty = \sum_{j=1}^\infty \mathbb{1}_{A_j}$.
   (a) Prove that $\operatorname{Cov}(\mathbb{1}_{A_i}, \mathbb{1}_{A_j}) = P(A_i \cap A_j) - P(A_i)P(A_j)$.
   (b) Prove that
   $$\operatorname{Var}(S_n) = \sum_{j=1}^n (P(A_j) - P(A_j)^2) \leq \sum_{j=1}^n P(A_j).$$
   (c) Prove (Chebyshev) that
   $$P\big(S_n \leq \frac{1}{2}\sum_{j=1}^n P(A_j)\big) \leq \frac{4}{\sum_{j=1}^n P(A_j)}.$$
   (d) Prove that
   $$P\big(S_\infty \leq \frac{1}{2}\sum_{j=1}^n P(A_j)\big) \leq \frac{4}{\sum_{j=1}^n P(A_j)}.$$
   (e) Prove that if $\sum_{j=1}^\infty P(A_j) = \infty$ then $P(S_\infty < \infty) = 0$ and hence $P(A_n \text{ i.o.}) = 1$.

   Solution.

   (a) We have $E(\mathbb{1}_{A_i}) = P(A_i)$. By linearity of expectation:
   $$\begin{aligned}\operatorname{Cov}(\mathbb{1}_{A_i}, \mathbb{1}_{A_j}) &= E((\mathbb{1}_{A_i} - P(A_i))(\mathbb{1}_{A_j} - P(A_j))) \\ &= E(\mathbb{1}_{A_i}\mathbb{1}_{A_j}) - P(A_i)P(A_j).\end{aligned}$$
   But then, $\mathbb{1}_{A_i}\mathbb{1}_{A_j} = \mathbb{1}_{A_i \cap A_j}$. So, $\operatorname{Cov}(\mathbb{1}_{A_i}, \mathbb{1}_{A_j}) = P(A_i \cap A_j) - P(A_i)P(A_j)$.

   (b) We have:
   $$\begin{aligned}\operatorname{Var}(S_n) = \sum_{j=1}^n \operatorname{Var}(\mathbb{1}_{A_j}) + 2\sum_{j<i} \operatorname{Cov}(\mathbb{1}_{A_i}, \mathbb{1}_{A_j}) &= \sum_{j=1}^n E(\mathbb{1}_{A_j}^2) - E(\mathbb{1}_{A_j})^2 \\ &= \sum_{j=1}^n E(\mathbb{1}_{A_j}) - E(\mathbb{1}_{A_j})^2 = \sum_{j=1}^n P(A_j) - P(A_j)^2.\end{aligned}$$

   (c)
   $$\begin{aligned}P\left(S_n \leq \frac{1}{2}\sum_{j=1}^n P(A_j)\right) = P\left(\frac{1}{2}\sum_{j=1}^n P(A_j) \leq \sum_{j=1}^n P(A_j) - S_n\right) &= P\left(\frac{1}{2}\sum_{j=1}^n P(A_j) \leq E(S_n) - S_n\right) \\ &\leq P\left(\frac{1}{2}\sum_{j=1}^n P(A_j) \leq |S_n - E(S_n)|\right).\end{aligned}$$

Finally, we apply Chebyshev:

$$P\left(\frac{1}{2}\sum_{j=1}^{n} P(A_j) \le |S_n - E(S_n)|\right) \le \frac{4}{\left(\sum_{j=1}^{n} P(A_j)\right)^2}\mathrm{Var}(S_n) = \frac{4}{\sum_{j=1}^{n} P(A_j)}.$$

(d) We have that $S_\infty = S_n + \sum_{j=n+1}^{\infty} \mathbb{1}_{A_j}$, since $\mathbb{1}_{A_j} \ge 0$. Therefore, $S_n \le S_\infty$. However, this implies that for any $c \in \mathbb{R}$, $P(S_\infty \le c) \le P(S_n \le c)$, since if $\omega \in \{S_\infty \le c\}$ in other words $S_\infty(\omega) \le c$, then $S_n(\omega) \le S_\infty(\omega) \le c$ so that $\omega \in \{S_n \le c\}$. Therefore,

$$P\left(S_\infty \le \frac{1}{2}\sum_{j=1}^{n} P(A_j)\right) \le P\left(S_n \le \frac{1}{2}\sum_{j=1}^{n} P(A_j)\right) \le \frac{4}{\sum_{j=1}^{n} P(A_j)}.$$

(e) We show that $P(S_\infty \le N) = 0$ for all $N \in \mathbb{N}$. Since $\sum_{j=1}^{\infty} P(A_j) = \infty$, there exists some n such that $\frac{1}{2}\sum_{j=1}^{n} P(A_j) \ge N$. Likewise, for any $\varepsilon > 0$, there exists some n' such that $4/\sum_{j=1}^{n'} P(A_j) < \varepsilon$. Let $n'' = \max\{n, n'\}$. But then,

$$P(S_\infty < N) \le P(S_\infty \le \frac{1}{2}\sum_{j=1}^{n''} P(A_j)) \le \frac{4}{\sum_{j=1}^{n''} P(A_j)} < \varepsilon.$$

Since ε was arbitrary, $P(S_\infty < N) = 0$. By continuity from below, we have $P(S_\infty < \infty) = 0$.

3. Let $X \ge 0$ be a nonnegative random variable with $EX^2 < \infty$.

(a) Prove that

$$P(X > 0) \ge \frac{(EX)^2}{EX^2}.$$

Hint: Consider the random variable $X\mathbb{1}_{X>0}$.

(b) Prove that, for $\theta \in [0, 1]$,

$$P(X > \theta EX) \ge (1-\theta)^2 \frac{(EX)^2}{EX^2}.$$

Solution.

(a) We consider the hint. In particular, $X\mathbb{1}_{X>0} = X$. Applying the Cauchy Schwartz inequality,

$$E(X) = E(X\mathbb{1}_{X>0}) \le \sqrt{E(X^2)E(\mathbb{1}_{X>0}^2)} = \sqrt{E(X^2)P(\mathbb{1}_{X>0})}.$$

In conclusion, $P(X > 0) \ge E(X)^2/E(X^2)$.

(b) We apply a very similar application of the C.S. inequality. In particular, note that $E[X - E[\theta X]] \le E(X - E[\theta X])^+ = EX\mathbb{1}_{X \ge \theta X}$. Then:

$$(1-\theta)E[X] = E[X - E[\theta X]] \le EX\mathbb{1}_{X \ge \theta E[X]} \le \sqrt{EX^2 P(X \ge \theta E[X])}.$$

Rearranging, we get: $P(X \ge \theta E[X]) \ge (1-\theta)^2 E[X]^2/EX^2$.

4. Let $X_1, X_2, \ldots$ be i.i.d. with $EX_i^+ = \infty$ and $EX_i^- < \infty$. Let $S_n = X_1 + \cdots + X_n$. Prove that $n^{-1}S_n \to \infty$ a.s.
Hint: Apply SLLN to $X_{i,N} = \min\{X_i, N\}$.

Solution.

First, we see that $EX_{i,N}^- = EX_i^- < \infty$, and $EX_{i,N}^+ \leq E(N) \leq N$. Therefore, $EX_{i,N} < \infty$. Since $X_{i,N}$ are i.i.d., we can apply the SLLN to $X_{i,N}$. Let $\mu_N = EX_{i,N}$. Now, we have that $X_{i,N} \to X_i$ pointwise: indeed, for any $\omega \in \Omega$, we have that $X_i(\omega) < M$ for some $M \in \mathbb{N}$, and then $X_{i,N}(\omega) = X_i(\omega)$ for all $N > M$. By the Monotone Convergence Theorem, we have that: $EX_{i,N}^+ \to EX_i^+$ as $N \to \infty$. In particular, $\mu_N = EX_{i,N}^+ - EX_{i,N}^- \to \infty$ as $N \to \infty$.

Now, pointwise we have $X_i \geq \min\{X_i, N\}$. Therefore, at each point,

$$S_n = \sum_{j=1}^n X_j \geq \sum_{j=1}^n X_{j,N}.$$

In particular, $P(\lim_{n\to\infty} n^{-1}(\sum_{j=1}^n X_{j,N}) = \mu_N) = 1$ implies that $P(\lim_{n\to\infty} n^{-1} S_n \geq \mu_N) = 1$. But then, by continuity from above, since these sets are decreasing, and since $\mu_N \to \infty$ as $N \to \infty$, we have $P(\lim_{n\to\infty} n^{-1} S_n = \infty)$ or $n^{-1} S_n \to \infty$ a.s.

5. Let X and Y be random variables with finite mean. Suppose that for any $A \in \mathcal{F}$ we have $E[X\mathbb{1}_A] \leq E[Y\mathbb{1}_A]$. Prove that $X \leq Y$ a.s.

   Solution.

   Suppose for the sake of contradiction that $P(X - Y > 0) > 0$. Then, there exists some n such that $P\left(X - Y > \frac{1}{n}\right) > 0$, since otherwise we would have $P(X - Y > 0) = 0$ by continuity of probabilities. Then, there exists $\varepsilon > 0$ such that $P\left(X - Y > \frac{1}{n}\right) \geq \varepsilon$. Let $A = \{X - Y > \frac{1}{n}\}$. But then

   $$E[X\mathbb{1}_A] - E[Y\mathbb{1}_A] = E[X\mathbb{1}_A - Y\mathbb{1}_A] \geq \frac{1}{n} E[\mathbb{1}_A] = \frac{1}{n} P(A) \geq \frac{\varepsilon}{n} > 0.$$

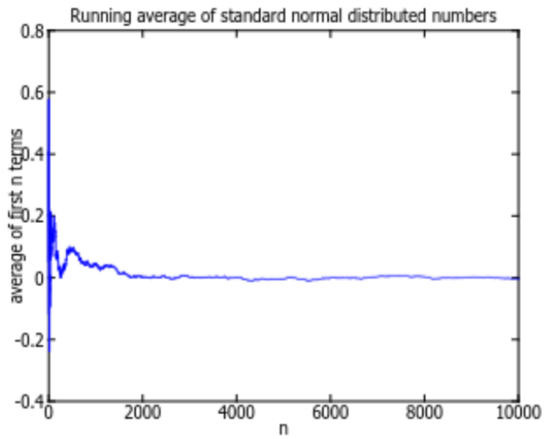
   where in the first step we used the fact that $X - Y > \frac{1}{n}$ and the order preserving property of expectation. However, this contradicts that $E[X\mathbb{1}_A] \leq E[Y\mathbb{1}_A]$, since this i the case for any $A \in \mathcal{F}$. Therefore, it must be the case that $P(X - Y > 0) = 0$, in other words, $P(X \leq Y)$ a.s.

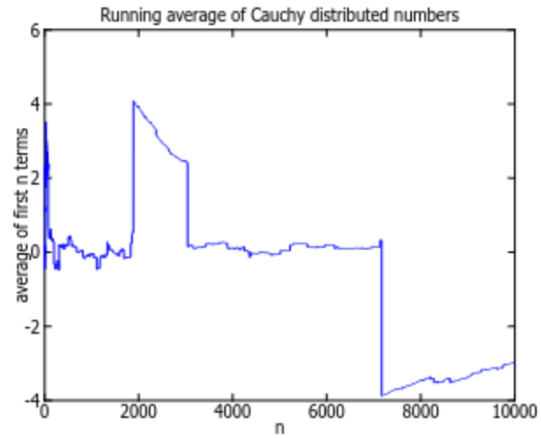
Recommended problems. The following problems from Rosenthal are recommended but are not to be handed in:

4.5.4, 4.5.7, 6.3.2, 6.3.4, 6.3.6., 5.5.6, 5.5.9.
For solutions to even-numbered problems see: `http://www.probability.ca/jeff/grprobbook.html`.

This problem (do not hand in) gives a probabilistic proof of Weierstrass's Theorem, due to Bernstein (1912): Let $f : [0,1] \to \mathbb{R}$ be a continuous function. For $x \in [0,1]$, let S_n have a $\mathrm{Bin}(n,x)$ distribution and set $X_n = n^{-1} S_n$. Prove that $q_n(x) = Ef(X_n)$ is a polynomial in x and that q_n converges uniformly to f on $[0,1]$.
Hint: Write $f(x) - q_n(x)$ as a sum over $m \in \{0, \dots, n\}$, divide the sum over m into $|\frac{m}{n} - x| \leq \delta$ and $|\frac{m}{n} - x| > \delta$, and use the Chebyshev inequality to bound the latter part of the sum.

Two simulations. (There is nothing for you to produce concerning this.) The standard normal with density $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ has mean zero. The Cauchy distribution with density $f(x) = \frac{1}{\pi} \frac{1}{1+x^2}$ has undefined mean (this distribution arose in Assignment 4 #1).





Quote of the week: *It is the mark of a truly intelligent person to be moved by statistics.*

George Bernard Shaw