Math 418/544 Assignment 4 September 29, 2025

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This assignment is due in Canvas at 23:59 p.m. on Monday, October 6.
Late assignments are not accepted.

1. A searchlight is distance 1 from an infinitely long wall. Let Q denote the closest point on the wall and assume the searchlight scans along the wall so that at any given time, the angle Θ the beam of light makes is uniform on $(-\pi/2, \pi/2)$ (the point Q corresponds to angle 0). Let $X \in \mathbb{R}$ (positive or negative) be the position of the beam on the wall as measured from Q. Find the cumulative distribution function and density function for X. *(This is the Cauchy distribution; its expectation is undefined.)*

   Solution

   If the angle sampled from the uniform distribution $\omega \in (-\pi/2, \pi/2)$, then we have that the point on the wall is given by $z = \tan(\omega) \times (1\text{m})$. Thus, the random variable $X(\omega) = \tan(\omega)$. To compute the CDF, we notice that the uniform distribution over $(-\pi/2, \pi/2)$ is $\frac{1}{\pi}\lambda$: the lebesgue measure normalized on $(-\pi/2, \pi/2)$. We then compute:

$$\begin{aligned} F_X(z) = P(X(\omega) \leq z) = P(\omega \leq \tan^{-1}(z)) \\ &= \frac{1}{\pi}\lambda((-\pi/2, \tan^{-1}(z))) \\ &= \frac{1}{\pi}\tan^{-1}(z) + \frac{1}{2}. \end{aligned}$$

   The probability density function, by FTC, is given by

$$f_X(z) = F_X'(z) = \frac{1}{\pi}\frac{1}{x^2+1}.$$

2. Suppose that X, Y are independent standard normal random variables.[1] By differentiating the cumulative distribution function for the random variable $Z = X/Y$ to obtain the density of Z, show that Z has a standard Cauchy distribution.

   Solution.

   We follow the hint in the footnote. Now, $F_Z(z) = P(\frac{X}{Y} \leq z)$ is described by $x, y \in \mathbb{R}^2$, where $X(\omega) = x$ and $Y(\omega) = y$ such that $\frac{x}{y} \leq z$. This set, is then described by

$$B = (\{X \leq zY\} \cap \{Y > 0\}) \cup (\{X \geq zY\} \cap \{Y < 0\}).$$

   We integrate over this region, with the pdf being $f_X(x) \cdot f_Y(y)$. When we differentiate, we use the Leibnitz integral rule:

$$\begin{aligned} F_Z(z) &= \int_{-\infty}^{0}\int_{yz}^{\infty} f_X(x)f_Y(y)dxdy + \int_{0}^{\infty}\int_{-\infty}^{yz} f_X(x)f_Y(y)dxdy \\ f_Z(z) = F'(z) &= \int_{-\infty}^{0} -yf_X(yz)f_Y(y)dy + \int_{0}^{\infty} f_X(yz)f_Y(y)dxdy. \end{aligned}$$

[1] The independence has the consequence that the joint density g of X, Y is the product of the densities. In other words, for a Borel subset B of $\mathbb{R}^2$,

$$P((X,Y) \in B) = \int_B g(x,y)\,dxdy$$

with $g(x,y) = f_X(x)f_Y(y)$. We have not yet discussed integration over arbitrary Borel sets (this requires Lebesgue integration which we will discuss soon) but the sets B encountered in this assignment are nice enough that Riemann integration does the job.

Now, we substitue the pdf's for the normal distributions and compute each of these integrals:

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\infty}^{0}(-y)\exp(-\frac{1}{2}(z^2+1)y^2)dy &= \frac{1}{2\pi}\int_{\infty}^{0}(-1/2)\exp(-\frac{1}{2}(z^2+1)u)du \\
&= \frac{1}{2\pi}(-\frac{1}{2})(\frac{-2}{z^2+1})\left[\exp(-\frac{1}{2}(z^2+1))\right]_{\infty}^{0} \\
&= \frac{1}{2\pi(z^2+1)}.
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2\pi}\int_{0}^{\infty}y\exp(-\frac{1}{2}(z^2+1)y^2)dy &= \frac{1}{2\pi}\int_{0}^{\infty}(1/2)\exp(-\frac{1}{2}(z^2+1)u)du \\
&= \frac{1}{2\pi}(\frac{1}{2})(\frac{-2}{z^2+1})\left[\exp(-\frac{1}{2}(z^2+1))\right]_{0}^{\infty} \\
&= \frac{1}{2\pi(z^2+1)}(0-1) = \frac{1}{2\pi(z^2+1)}.
\end{aligned}$$

Thus,

$$f_Z(z) = \frac{1}{2\pi(z^2+1)} + \frac{1}{2\pi(z^2+1)} = \frac{1}{\pi(z^2+1)}.$$

3. Suppose that X, Y are independent $\text{Exp}(\lambda)$ random variables. By differentiating the cumulative distribution function for the random variable $Z = X - Y$, find the density function of Z (it is a Laplace distribution).

   Similar to last problem, we fix z and determine the probability $P(Z \leq z)$. We consider two cases for z, $z \geq 0$ and $z < 0$. First, consider $z > 0$. In particular, this occurs when $X \leq Y + z$. Also, we recall that $X \geq 0$ and $Y \geq 0$ (since they are independent exponential random variables). We integrate over this region with the pdf being the joint pdf of X and Y.

$$\begin{aligned}
F_Z(z) &= \int_0^\infty \int_0^{y+z} f_X(x)f_Y(y)dxdy \\
f_Z(z) = F_Z'(z) &= \int_0^\infty f_X(y+z)f_Y(y)dy.
\end{aligned}$$

   If $z \leq 0$, then instead, we can use the equivalent inequality $X + (-z) \leq Y$, where $-z > 0$. Then,

$$\begin{aligned}
F_Z(z) &= \int_0^\infty \int_0^{x-z} f_X(x)f_Y(y)dydx \\
f_Z(z) = F'(z) &= \int_0^\infty f_X(x)f_Y(x-z)dx.
\end{aligned}$$

   We compute each of these integrals. For $z > 0$:

$$\begin{aligned}
\int_0^\infty f_X(y+z)f_Y(y)dy &= \int_0^\infty \lambda^2\exp(-\lambda(y+z))\exp(-\lambda y)dy \\
&= \lambda^2\exp(-\lambda z)\int_0^\infty \exp(-2\lambda y)dy \\
&= \lambda^2\exp(-\lambda z)\frac{1}{2\lambda} = \frac{\lambda}{2}\exp(-\lambda z).
\end{aligned}$$

   Likewise, for $z < 0$, we compute a very similar integral to be $\frac{\lambda}{2}\exp(\lambda z)$. Putting these together, we get $f_Z(z) = F_Z'(z) = \frac{\lambda}{2}\exp(-\lambda|z|)$, as desired.

4. Suppose that X is a random variable whose cumulative distribution function F is continuous on $\mathbb{R}$. Let $Y = F(X)$. Prove that Y has a uniform distribution, i.e., that $F_Y(x) = x$ for $x \in (0, 1)$.

   Solution.

We compute $F_Y(x)$. Let $0 < x < 1$. By definition,

$$F_Y(x) = P(Y \leq x) = P(F(X) \leq x).$$

Now, consider set $\{\omega : F(X(\omega)) \leq x\}$. Since F is continuous, there exists some $y \in \mathbb{R}$ such that $F(y) = x$. Also, since F is nondecreasing, and $\lim\limits_{y\to\infty} F(y) = 1$, and $x < 1$ the set of y such that $F(y) = x$ is bounded above. Now, let $y^* = \sup\{y \in \mathbb{R} : F(y) = x\}$. By continuity of F, we have that $F(y^*) = x$. In particular, if $X(\omega) > y^*$, then $F(X(\omega)) > x$, and since F is nondecreasing we have that if $X(\omega) \leq y^*$, then $F(X(\omega)) \leq x$.

Therefore,

$$\{\omega : F(X(\omega)) \leq x\} = \{\omega : X(\omega) \leq y^*\}.$$

However,

$$P(X \leq y^*) = F(y^*) = x.$$

Therefore, $F_Y(x) = x$, so Y has a uniform distribution.

5. Let $X_0, X_1, \ldots$ be i.i.d. continuous random variables with cumulative distribution function

$$F(x) = \int_{-\infty}^{x} f(t)\, dt,$$

for some probability density function $f : \mathbb{R} \to [0, \infty)$. Let

$$N = \min\{n : X_n > X_0\}.$$

(a) Show that the cumulative distribution function F_N of N is given by

$$F_N(n) = 1 - \frac{1}{n+1}$$

for $n \in \mathbb{N}$. (Of course $F_N(x) = 0$ for $x < 1$ and $F_N(x) = F_N(n)$ for $x \in (n, n+1)$.)

(b) Show[2] that the cumulative distribution function of X_N is

$$F + (1 - F)\log(1 - F).$$

(Recall that $\log(1-t) = -\sum_{n=1}^{\infty} \frac{t^n}{n}$ for $|t| < 1$; if $F = 1$ we interpret $(1-F)\log(1-F)$ as zero.)

Solution.

(a) We have that $\min\{n : X_n > X_0\} = n$ when $X_i \leq X_0$ where $1 \leq i \leq n-1$. This can then be described by the integral:

$$P(N > n) = \int_{-\infty}^{\infty} f(x)F(x)^n dx,$$

where $f(x)$ is the pdf of X_0. We can immediately solve this with a substitution $u = F(x)$, since $F(-\infty) = 0, F(\infty) = 1$ and $\frac{du}{dx} = f(x)$ to get

$$P(N > n) = \int_0^1 u^n du = \frac{1}{n+1}.$$

Then, $F_N(n) = P(N \leq n) = 1 - \frac{1}{n+1}$.

(b) In order to compute this probability, we condition on the value of N. In particular, the conditional probability $P(X_N \leq x | N = n) = F^{n+1}$, since $X_0, \ldots, X_n \leq x$, and the ordering

[2]One way, not the only way, is to employ a higher-dimensional version of the previous footnote.

of these random variables sizes is already taken care of in $N = n$. Then, we compute:

$$\begin{aligned}
P(X_N \leq x) &= \sum_{n=1}^{\infty} P(X_N \leq x | N = n) P(N = n) \\
&= \sum_{n=1}^{\infty} F^{n+1} \left(\frac{1}{n} - \frac{1}{n+1} \right) \\
&= F \sum_{n=1}^{\infty} \frac{F^n}{n} + F - \sum_{n=1}^{\infty} \frac{F^n}{n} \\
&= F(-\log(1-F)) + F + \log(1-F) \\
&= F + (1-F)\log(1-F),
\end{aligned}$$

as desired.

Recommended problems (Do not hand in).

1. Suppose that X has a continuous density function f, that $P(\alpha \leq X \leq \beta) = 1$, and that g is a strictly increasing differentiable function on (α, β). Show that the density of $g(X)$ is $f(g^{-1}(x))/g'(g^{-1}(x))$ if $x \in (g(\alpha), g(\beta))$ and otherwise is zero.
2. Use the result of (1) to determine the density of $aX + b$ for constants $a > 0$ and $b \in \mathbb{R}$.
3. Suppose X has a standard normal distribution. Use the result of (1) to compute the density of $\exp(X)$ (this is the lognormal distribution).
4. Suppose that X has continuous density function f. Find the density of X^2. Let Z have a standard normal distribution, and find the density of Z^2 (this is a special case of the χ^2-distribution).

   Quote of the week: Probability theory is measure theory with a soul. — Mark Kac