Computational Control

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Convex Optimization

1. Quadratic Optimization Problem with Affine Constraint

Problem. Solve

$$\begin{aligned} \min_{x \in \mathbb{R}^2} \quad & \tfrac{1}{2} x^\top Q x + c^\top x \\ \text{s.t.} \quad & a^\top x = b, \end{aligned}$$

with

$$Q = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad c = \begin{bmatrix} -2 \\ -5 \end{bmatrix}, \quad a = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad b = 2.$$

Hint. The KKT conditions in this case are

$$\begin{bmatrix} Q & a \\ a^\top & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} -c \\ b \end{bmatrix}.$$

Since the constraint is an equality, λ is unrestricted in sign.

2. Discrete-time Lyapunov Equation

Problem. Solve the discrete-time Lyapunov equation

$$A^\top P A - P = -Q,$$

for

$$A = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}, \qquad Q = I.$$

Hint. The unknown is a symmetric matrix

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix},$$

and the condition $P \succeq 0$ for a 2×2 matrix means

$$p_{11} \geq 0, \quad p_{22} \geq 0, \quad p_{11} p_{22} - p_{12}^2 \geq 0.$$

3. Stability of LTI Systems

Problem. For each of the following matrices, determine whether the equilibrium at the origin of the system

$$x_{t+1} = A x_t$$

is *stable*, *marginally stable*, or *unstable*.

$$\begin{aligned} A_1 &= \begin{bmatrix} 0.5 & 0.2 \\ 0 & 0.4 \end{bmatrix}, & A_2 &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, & A_3 &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \\ A_4 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, & A_5 &= \begin{bmatrix} 0.7 & 0.6 \\ -0.2 & 0.7 \end{bmatrix}, & A_6 &= \begin{bmatrix} 1.2 & 0.5 \\ 0 & 1.1 \end{bmatrix}. \end{aligned}$$

4. Convexity of norms

Problem. For a norm $\|\cdot\|$ on $\mathbb{R}^n$ define its *dual norm* as $\|y\|_* = \sup_{\|x\| \leq 1} y^\top x$.

(a) Prove that the norm is convex.

(b) Prove that the dual norm $\|\cdot\|_*$ is convex.

5. Projections onto convex sets

Problem. Derive explicit *projections*

$$P_C(z) = \arg\min_{x \in C} \tfrac{1}{2}\|x - z\|_2^2$$

for the following convex sets.

(a) The ℓ_2 ball $C = \{x \mid \|x\|_2 \le r\}$.

(b) An affine subspace $C = \{x \mid Ax = b\}$ with A full row rank.

Hint. Consider each projection as a convex optimization problem and apply the KKT conditions.

6. Strong convexity, smoothness, and canonical inequalities

Problem. Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable.

(a) Show: f is μ-strongly convex ($\mu > 0$) if $x \mapsto f(x) - \frac{\mu}{2}\|x\|_2^2$ is convex.

(b) If f is L-smooth (i.e., ∇f is L-Lipschitz), prove the *descent lemma*:

$$f(y) \le f(x) + \langle \nabla f(x),\, y - x\rangle + \tfrac{L}{2}\|y - x\|_2^2.$$

