Math 418/544 Assignment 7 November 5, 2025

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This assignment is due in Canvas at 23:59 p.m. on Monday, November 17.
Late assignments are not accepted. This assignment is longer than other assignments and will have a double weight. If this is your lowest graded assignment it would still be omitted from your HW assignment average (despite having a double weight).

1. A collection of random variables $(X_n : n \in \mathcal{I})$ (defined on the same probability space) is called *uniformly integrable* if for all ε there exists some $M = M(\varepsilon)$ such that $\mathbb{E}[|X_n|\mathbb{1}_{|X_n|>M}] < \varepsilon$ for all $n \in \mathcal{I}$. Prove that $(X_n : n \in \mathcal{I})$ is uniformly integrable if and only if $\sup_{n\in\mathcal{I}} \mathbb{E}[|X_n|] < \infty$ and for any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that $E[|X_n|\mathbb{1}_A] < \varepsilon$ for $n \in \mathcal{I}$, for all $A \in \mathcal{F}$ with $\mathbb{P}(A) < \delta$.

2. Suppose that $X_1, X_2, \dots$ are random variables with $X_1 \geq X_2 \geq X_3 \geq \cdots \geq 0$, with $X_n(\omega) \to X(\omega)$ for all $\omega \in \Omega$, and with $\mathbb{E}X_1 < \infty$.
   (a) Prove that $\lim_{n\to\infty} \mathbb{E}X_n = \mathbb{E}X$.
   (b) Give an example to show that the conclusion in part (a) need not hold if $\mathbb{E}X_1 = \infty$.

3. Let X have a Bernoulli($\frac{1}{2}$) distribution and define the rate function $I(z) = \sup_{s\in\mathbb{R}}[sz - \log M_X(s)]$. Prove that
$$I(z) = \begin{cases} \log 2 + z\log z + (1-z)\log(1-z) & (z \in [0,1]) \\ \infty & \text{otherwise.} \end{cases}$$
   (For the case $z > \frac{1}{2}$ where we used this rate function in class for the coin flipping example, note that the supremum over $s \in \mathbb{R}$ turns out to be the same as the supremum over $s > 0$ as appears in Theorem 9.3.4.)

4. Use Theorem 9.3.4 to obtain a numerical upper bound for the probability that the proportion of Heads in n coin flips (fair coin) is 0.6 or higher, for the cases $n = 10, 100, 1000$.

5. (a) If $X_n \Rightarrow X$ and $Y_n \Rightarrow c$, where c is a constant (and for all n, X_n and Y_n are defined on the same probability space), then $X_n + Y_n \Rightarrow X + c$. (Consequently if $X_n \Rightarrow X$ and $Z_n - X_n \Rightarrow 0$ then $Z_n \Rightarrow X$.)
   (b) Prove or disprove that, more generally, if $X_n \Rightarrow X$ and $Y_n \Rightarrow Y$ then $X_n + Y_n \Rightarrow X + Y$.

6. Let $X, X_1, X_2, \dots$ be integer-valued discrete random variables. Show that $X_n \Rightarrow X$ if and only if $\lim_{n\to\infty} P(X_n = m) = P(X = m)$ for all integers m.

7. Let $\phi(t)$ be the characteristic function of the random variable X. Suppose that $|\phi(t_0)| = 1$ for some $t_0 \neq 0$. Prove there exist $a, b \in \mathbb{R}$ such that $P(X \in a + b\mathbb{Z}) = 1$.

8. Using characteristic functions,[1] prove the following:
   (a) Suppose X_i are independent with $N(0, \sigma_i^2)$ distributions. Let $S_n = X_1 + \cdots + X_n$. Then S_n has distribution $N(0, \sum_{i=1}^n \sigma_i^2)$. In particular, if Z_i has a standard normal $N(0,1)$ distribution then $\frac{1}{\sqrt{n}}(Z_1 + \cdots + Z_n)$ also has a standard normal distribution.
   (b) Suppose X_i are independent Cauchy random variables (density $f(x) = \frac{1}{\pi}\frac{1}{1+x^2}$, $x \in \mathbb{R}$). Let $S_n = X_1 + \cdots + X_n$. Then $\frac{1}{n}S_n$ has a Cauchy distribution.[2]

[1] You can look up the characteristic functions for the normal, Cauchy and exponential random variables, it is not necessary to perform the integrals yourself.
[2] Cf. the simulation depicted in Assignment 5.

(c) Recall that a Geometric(p) random variable has p.m.f. $g(k) = (1-p)^{k-1}p$ for $k \in \mathbb{N}$. Consider the following variation: X_n has p.m.f. $P(X_n = k/n) = (1 - \lambda/n)^{k-1}(\lambda/n)$ for $k \in \mathbb{N}$, with $\lambda > 0$. Apply the Continuity Theorem[3] to prove that X_n converges weakly to an Exp(λ) random variable.[4]

9. Let $Y_1, Y_2, \ldots$ be i.i.d. random variables (on the same probability space), each of which takes any value in $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ with equal probability $\frac{1}{10}$. Let

$$X_n = \sum_{i=1}^{n} \frac{Y_i}{10^i}.$$

Apply the Continuity Theorem to prove that X_n converges in distribution to a uniform random variable on $[0, 1]$.

Recommended problems. The following problems from Rosenthal are recommended but are not to be handed in:

9.5.4, 9.5.10, 9.5.15, 9.5.16, 10.3.1, 10.3.2, 10.3.3, 10.3.4, 10.3.6. 10.3.8, 10.3.9, 10.3.10.

For solutions to even-numbered problems see: `http://www.probability.ca/jeff/grprobbook.html`.

More recommended problems (not to be handed in):

A) Let X be a Poisson random variable with parameter $\lambda > 0$. Compute the moment generating function of X and differentiate it to compute the mean and variance of X.

B) Let $\alpha, \lambda > 0$. A random variable X has a Gamma(α, λ) distribution if its density function is given by

$$f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)}\lambda^\alpha x^{\alpha-1}e^{-\lambda x} & (x \geq 0) \\ 0 & (x < 0). \end{cases}$$

Here $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$. Recall that $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$. Note that an Exp(λ) random variable has the same distribution as a Gamma($1, \lambda$) random variable.

(a) Prove that the sum of n i.i.d. Exp(λ) random variables has a Gamma(n, λ) distribution in two ways: by induction on n using a calculation of the density function using the convolution formula and by comparison of moment generating functions or of characteristic functions.[5]

(b) Prove in two ways (as above) that the sum of independent Gamma(α, λ) and Gamma(β, λ) random variables is Gamma($\alpha + \beta, \lambda$).
A useful fact: $\int_0^x t^{r-1}(x-t)^{s-1}dt = x^{r+s-1}\frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}$ for $r, s, x > 0$.

Quote of the week: *"The other girls study mathematics," Ai-ming said, trying again.*

"That's what we need!" the vendor said, smacking her chopsticks against the metal pot. "Real numbers. Without real numbers, how can we fix our economy, make plans, understand what we need? Young lady, I don't mean to be rude but you should really think about studying mathematics, too."

"I will."

Madeleine Thien in *Do not say we have nothing*

Second quote of the week: *Mathematicians are like Frenchmen: whatever you say to them they translate into their own language and forthwith it is something entirely different.* J.W. Goethe

[3] We will not finish the proof of the Continuity Theorem in class until November 17, but its statement in Theorem 11.1.14 (appearing in the lecture notes) is easy to understand and apply.

[4] It is straightforward to prove this via the characterization of convergence in distribution in terms of convergence of the CDFs to the CDF of the limit. This is also the case for question 9.

[5] We have not proved it, but it is the case that if two random variables have the same moment generating function, which is defined on a neighbourhood of the origin, then the two random variables have the same distribution (see Theorem 11.4.3). A related statement that we will see sooner in class is Corollary 11.1.7.