

Dr. J. Hermon

This assignment is due in Canvas at 23:59 on Friday, Dec 5th.

Late assignments are not accepted.

Submit a solution to 5 out of the 10 questions.

You can rely on Slutsky's Theorem in this assignment (see recommended problems).

1. Let $g : [0, \infty) \rightarrow [0, \infty)$ be such that $\lim_{x \rightarrow \infty} \frac{g(x)}{x} = \infty$. Let $(X_j : j \in J)$ be a collection of random variables satisfying that $\sup_{j \in J} \mathbb{E}[g(|X_j|)] < \infty$. Show that $(X_j : j \in J)$ is uniformly integrable.

Hint: $|x| \mathbb{1}_{|x| > M} \leq g(|x|) \delta_g(M)$ for all $M > 0$ for some appropriate $\delta_g : (0, \infty) \rightarrow (0, \infty)$.

2. For $n \in \mathbb{N}$ let σ_n be a random permutation of $[n] := \{1, 2, \dots, n\}$ picked uniformly at random (i.e., its distribution is the uniform distribution on the set of all $n!$ permutations of $[n]$). Let $X_n := |\{j \in [n] : \sigma_n(j) = j\}|$ be the number of fixed points of σ_n . Use the method of moments to show that $\sigma_n \Rightarrow \text{Pois}(1)$ (justify why it applies; you may look up online the MGF of the Poisson distribution).

You may rely without a proof on the fact that if $Y_n \sim \text{Bin}(n, \frac{1}{n})$ and $Y \sim \text{Pois}(1)$ then $\lim_{n \rightarrow \infty} \mathbb{E}[Y_n^k] = \mathbb{E}[Y^k]$ for all $k \in \mathbb{Z}_+$.

Hint: Use indicator decompositions of X_n and of Y_n . Expand X_n^k and of Y_n^k to a sum of monomials. Use the fact that $x^\ell = x$ for $x \in \{0, 1\}$ and $\ell \geq 1$.

3. (a) Suppose that the sequence of probability measures $(\mu_n)_{n=1}^\infty$ is tight. Show that their characteristic functions are uniformly equicontinuous, i.e., for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $|h| < \delta$ then for all n and all $t \in \mathbb{R}$

$$|\phi_n(t+h) - \phi_n(t)| < \varepsilon,$$

where ϕ_n is the characteristic function of μ_n .

Hint: You may rely on the inequality $|1 - e^{-iy}| \leq \min\{2, |y|\}$ for all $y \in \mathbb{R}$. Deduce from it that for all $x, h \in \mathbb{R}$, $|1 - e^{-ihx}| \leq |h|^{1/2} + 2\mathbb{1}_{|x| > |h|^{-1/2}}$.

- (b) Suppose that $\mu_n \Rightarrow \mu$. Denote the characteristic function of μ by ϕ and of μ_n by ϕ_n . Use part (a) to conclude that $\phi_n(t) \rightarrow \phi(t)$ uniformly on compact sets, i.e., for all $M > 0$ and $\varepsilon > 0$ there exists $N = N(\varepsilon, M)$ such that for all $n \geq N$ we have that $|\phi_n(t) - \phi(t)| < \varepsilon$ for all $t \in [-M, M]$.
- (c) Give an example to show that the convergence in (b) need not be uniform on the entire real line.

4. **Parts (i)–(v) of this question are unmarked** (you can skip them and rely on them):

Let $\lambda > 0$. Consider for all n a sequence $(p_{n,m} : 1 \leq m \leq N_n)$ of real numbers in $[0, 1]$ satisfying that (1) $\sum_{m=1}^{N_n} p_{n,m} = \lambda$ and (2) $\lim_{n \rightarrow \infty} p_*(n) = 0$, where $p_*(n) := \max_{1 \leq m \leq N_n} p_{n,m}$.

(i) Show that $\lim_{n \rightarrow \infty} \sum_{m=1}^{N_n} p_{n,m}^2 \leq \lim_{n \rightarrow \infty} p_*(n) \sum_{m=1}^{N_n} p_{n,m} = 0$.

(ii) Suppose that $(q_{n,m} : 1 \leq m \leq N_n)$ satisfy that for some $C > 0$ we have that $|q_{n,m} - p_{n,m}| \leq Cp_{n,m}^2$ for all $n \geq 1$ and all $m \in [N_n] := \{1, 2, \dots, N_n\}$. Show that $\lim_{n \rightarrow \infty} \sum_{m=1}^{N_n} q_{n,m} = \lambda$ and that $\lim_{n \rightarrow \infty} q_*(n) = 0$, where $q_*(n) := \max_{1 \leq m \leq N_n} q_{n,m}$.

(iii) Let $q_{n,m}$ be the solution to $\mathbb{P}[\text{Pois}(q_{n,m}) = 1] = q_{n,m} e^{-q_{n,m}} = p_{n,m}$.¹ Show that for some $C > 0$ we have for all $n \geq 1$ and all $m \in [N_n]$ that $|q_{n,m} - p_{n,m}| \leq Cp_{n,m}^2$ and $\mathbb{P}[\text{Pois}(q_{n,m}) \geq 2] = 1 - e^{-q_{n,m}} - q_{n,m} e^{-q_{n,m}} \leq Cp_{n,m}^2$.

(iv) Suppose that $\lambda_n > 0$ and $\lim_{n \rightarrow \infty} \lambda_n = \lambda$. Show that $\text{Pois}(\lambda_n) \Rightarrow \text{Pois}(\lambda)$.

¹Since $p_*(n) \rightarrow 0$, w.l.o.g. we may assume that $p_*(n) \leq e^{-1}$ for all n , and this ensures that such a solution $q_{n,m}$ exists and that it lies in $(0, 1]$.

(v) Let $q_{n,m}$ be as in (iii). For any fixed $n \geq 1$, let $(Y_{n,m} : m \in [N_n])$ be independent Poisson random variables, where $Y_{n,m} \sim \text{Pois}(q_{n,m})$ for all $n \geq 1$ and all $m \in [N_n]$. Show that $Y(n) := \sum_{m=1}^{N_n} Y_{n,m} \Rightarrow \text{Pois}(\lambda)$.

(vi) Show that

$$\lim_{n \rightarrow \infty} \mathbb{P} \left[\sum_{m=1}^{N_n} Y_{n,m} \mathbb{1}_{Y_{n,m} \geq 2} \neq 0 \right] = \lim_{n \rightarrow \infty} \mathbb{P}[Y_{n,m} \geq 2 \text{ for some } m \in [N_n]] = 0.$$

Deduce that $Z(n) := -\sum_{m=1}^{N_n} Y_{n,m} \mathbb{1}_{Y_{n,m} > 1} = -\sum_{m=1}^{N_n} Y_{n,m} \mathbb{1}_{Y_{n,m} \geq 2} \rightarrow 0$ in distribution, as $n \rightarrow \infty$. Use this and part (vi) to deduce that

$$Y(n) + Z(n) = \sum_{m=1}^{N_n} Y_{n,m} \mathbb{1}_{Y_{n,m} \leq 1} = \sum_{m=1}^{N_n} \mathbb{1}_{Y_{n,m}=1} \Rightarrow \text{Pois}(\lambda).$$

(vii) For all (fixed) $n \geq 1$ let $(X_{n,m} : 1 \leq m \leq N_n)$ be independent Bernoulli random variables with $\mathbb{P}[X_{n,m} = 1] = p_{n,m} = 1 - \mathbb{P}[X_{n,m} = 0]$ for all n and all $m \in [N_n] := \{1, 2, \dots, N_n\}$. Deduce from part (vi) that $X(n) := \sum_{m=1}^{N_n} X_{n,m} \rightarrow \text{Pois}(\lambda)$ in distribution as $n \rightarrow \infty$.

5. The distribution of a random variable X is called *infinitely divisible* if, for all $n \in \mathbb{N}$, there exists a sequence of independent and identically distributed random variables $Y_1^{(n)}, Y_2^{(n)}, \dots, Y_n^{(n)}$ such that X and $Y_1^{(n)} + Y_2^{(n)} + \dots + Y_n^{(n)}$ have the same distribution.²

Prove that the characteristic function ϕ of an infinitely divisible distribution is nonzero for all real t , i.e., that $\phi(t) \neq 0$ for all $t \in \mathbb{R}$.

Hint: Let ϕ_n be the characteristic function of $Y_i^{(n)}$. Let $\psi_n(t) = |\phi_n(t)|^2$ and $\psi(t) = |\phi(t)|^2$.

Step 1: Explain why these are also characteristic functions.

Step 2: Express $\psi_n(t)$ in terms of $\psi(t)$ and n and deduce from this expression that $\lim_{n \rightarrow \infty} \psi_n(t) = g(t)$ for all $t \in \mathbb{R}$ for some $g : \mathbb{R} \rightarrow \{0, 1\}$, where $g(t)$ has an expression involving $\psi(t)$ for all $t \in \mathbb{R}$.

Step 3: Use that expression to show that g is continuous at 0.

Step 4: Apply a corollary of the continuity theorem from lecture to deduce a useful fact about g . For intuition sake, observe that it is natural to expect that $Y_1^{(n)} \rightarrow 0$ in distribution as $n \rightarrow \infty$.

6. Let X be a random variable.

(i) Show that $\mathbb{E}[e^{sX} + e^{-sX}] = 2 \sum_{k=0}^{\infty} \frac{s^{2k} \mathbb{E}[X^{2k}]}{(2k)!}$ for all $s \in [0, \infty)$. In particular, $\mathbb{E}[e^{sX} + e^{-sX}] < \infty$ if and only if $\sum_{k=0}^{\infty} \frac{s^{2k} \mathbb{E}[X^{2k}]}{(2k)!} < \infty$.

(ii) Recall that by Stirling's approximation $\lim_{k \rightarrow \infty} \frac{\sqrt{2\pi k} (k/e)^k}{k!} = 1$. Use part (i) and Stirling's approximation to show that there exists $s > 0$ such that $M_X(t) < \infty$ for all $t \in [-s, s]$ if and only if $\sup_{k \in \mathbb{N}} \frac{(\mathbb{E}[X^{2k}])^{\frac{1}{2k}}}{2k} < \infty$, where $\mathbb{N} := \{1, 2, 3, \dots\}$.

7. Let $d \geq 2$. For notational ease, you may assume $d = 2$. The moment generating function $M_X : \mathbb{R}^d \rightarrow (0, \infty]$ of a random d -tuple $X = (X(1), X(2), \dots, X(d))$ is defined to be $M_X(t) = \mathbb{E}[e^{t \cdot X}]$, where $t = (t(1), t(2), \dots, t(d))$ and $t \cdot X = \sum_{j=1}^d t(j)X(j)$ is the standard inner product of t and X . Recall that $X = (X(1), X(2), \dots, X(d)) \stackrel{d}{=} Y = (Y(1), Y(2), \dots, Y(d))$ if and only if $t \cdot X \stackrel{d}{=} t \cdot Y$ for all $t \in \mathbb{R}^d$ and that $X_n = (X_n(1), X_n(2), \dots, X_n(d)) \Rightarrow X$ if and only if $t \cdot X_n \Rightarrow t \cdot X$ for all $t \in \mathbb{R}^d$.

Fix some norm $\|\cdot\|$ on \mathbb{R}^d . Throughout all parts of the question suppose that:

(*) There exists $\delta > 0$ such that $M_X(t) < \infty$ for all $t \in \mathbb{R}^d$ such that $\|t\| \leq \delta$.

You are encouraged to think about how condition (*) and the statement of part (c) below simplify in the special case that $X(1), X(2), \dots, X(d)$ are independent.

²Examples of infinitely divisible distributions include: normal, Cauchy, Gamma, Poisson, geometric.

(a) Suppose that there exists $\delta' > 0$ such that $M_X(t) = M_Z(t)$ for all $t \in \mathbb{R}^d$ such that $\|t\| \leq \delta'$. Show that $X \stackrel{d}{=} Z$. **Hint:** There is an easy and short solution.

(b) **Part (b) is unmarked – Do not submit a solution to it. You may rely on it below.** Show that $\left| \mathbb{E} \left[\prod_{j=1}^d X(j)^{k(j)} \right] \right| < \infty$ for all $k(1), \dots, k(d) \in \mathbb{Z}_+ := \{0, 1, 2, \dots\}$.

Hint: Use (*) and Part (i) of Question 6 to show that $X(j) \in L^p$ for all $p \in \mathbb{N}$, for all $j \in [d]$.

Then use Holder's inequality to get that $\left| \mathbb{E} \left[\prod_{j=1}^d X(j)^{k(j)} \right] \right|^d \leq \prod_{j=1}^d \mathbb{E} [|X(j)|^{k(j)d}]$.

(c) Suppose that for all $k(1), \dots, k(d) \in \mathbb{Z}_+$, $\mathbb{E} \left[\prod_{j=1}^d X_n(j)^{k(j)} \right]$ is finite for all n and that $\lim_{n \rightarrow \infty} \mathbb{E} \left[\prod_{j=1}^d X_n(j)^{k(j)} \right] = \mathbb{E} \left[\prod_{j=1}^d X(j)^{k(j)} \right]$. Show that³ $X_n \Rightarrow X$.

Hint: Apply the method of moments to $t \cdot X$ for any $t \in \mathbb{R}^d$.

8. This problem concerns the method of Monte Carlo integration, which is a method for the approximate evaluation of an integral $I = \int_0^1 f(x) dx$.

Let U_1, \dots, U_N be i.i.d. uniform random variables on the interval $(0, 1)$, and let

$$I_N = \frac{1}{N} [f(U_1) + \dots + f(U_N)].$$

Suppose that $\int_0^1 f(x)^2 dx < \infty$, and let $\sigma^2 = \text{Var} f(U_1) = \int_0^1 f(x)^2 dx - I^2$. Apply the central limit theorem to show that I_N converges to I as $N \rightarrow \infty$, in the sense that for all $x \in \mathbb{R}$

$$\lim_{N \rightarrow \infty} \mathbb{P} \left(|I_N - I| \leq \frac{\sigma x}{\sqrt{N}} \right) = \mathbb{P}(|Z| \leq x),$$

where Z is a standard normal random variable.

9. Let X_1, X_2, \dots be i.i.d. random variables with c.d.f. F . The *empirical c.d.f.* $\hat{F}_n : \mathbb{R} \rightarrow [0, 1]$ is defined by $\hat{F}_n(x) = \frac{1}{n} \sum_{k=1}^n \mathbb{1}_{\{X_k \leq x\}}$. Thus, for each $x \in \mathbb{R}$ and $n \geq 1$, $\hat{F}_n(x)$ is a random variable.

(a) For each $x \in \mathbb{R}$, prove that $\hat{F}_n(x) \rightarrow F(x)$ a.s. as $n \rightarrow \infty$.

(b) If x satisfies $0 < F(x) < 1$, prove that

$$\frac{\sqrt{n}(\hat{F}_n(x) - F(x))}{\sqrt{F(x)[1 - F(x)]}} \Rightarrow N(0, 1).$$

10. Let X_1, X_2, \dots be i.i.d. random variables with known mean m and unknown variance σ^2 , both finite and with $\sigma^2 \neq 0$. Let $W_n := \sum_{k=1}^n (X_k - m)^2$ and $U_n := \begin{cases} \sqrt{W_n} & \text{if } W_n > 0 \\ 1 & \text{if } W_n = 0 \end{cases}$. Prove that

$$\frac{S_n - nm}{U_n} \Rightarrow N(0, 1).$$

Recommended problems. The following problems from Rosenthal are recommended but are not to be handed in:

11.5.2, 11.5.4, 11.5.6, 11.5.12

9.5.2, 11.4.2, 11.5.18

For solutions to even-numbered problems see: <http://www.probability.ca/jeff/grprobbook.html>.

Additional recommended problems (**do not hand in**):

³In particular, this shows that if (*) holds and $\mathbb{E} \left[\prod_{j=1}^d Y(j)^{k(j)} \right] = \mathbb{E} \left[\prod_{j=1}^d X(j)^{k(j)} \right]$ for all $k(1), \dots, k(d) \in \mathbb{Z}_+$, then $X \stackrel{d}{=} Y = (Y(1), \dots, Y(d))$.

1. Prove Slutsky's Theorem: For every $n \in \mathbb{N}$ let X_n and Y_n be two random variables defined on the same probability space. Let X be a random variable on some probability space. Let $c \in \mathbb{R}$ be a constant. Suppose that $X_n \Rightarrow X$ and that $Y_n \Rightarrow c$. Then

(a) $X_n + Y_n \Rightarrow X + c$. (Consequently if $X_n \Rightarrow X$ and $Y_n := Z_n - X_n \Rightarrow 0$ then $Z_n = X_n + Y_n \Rightarrow X$.)

(b) $X_n Y_n \Rightarrow cX$.

(c) Suppose $c \neq 0$ and let z be an arbitrary non-zero real number. Then $X_n/Z_n \Rightarrow X/c$, where

$$Z_n := \begin{cases} Y_n & \text{if } Y_n \neq 0 \\ z & \text{if } Y_n = 0 \end{cases}.$$

2. Show that there exists $s > 0$ such that $M_X(t) < \infty$ for all $t \in [-s, s]$ if and only if there exist $C, c > 0$ such that $\mathbb{P}[|X| > r] \leq Ce^{-cr}$ for all $r \geq 0$.⁴

3. A collection of probability measures $(\mu_j : j \in J)$ on $(\mathbb{R}, \mathcal{B})$ is called uniformly integrable if for all $\varepsilon > 0$ there exists $M > 0$ such that $\int |x| \mathbb{1}_{|x| > M} d\mu_j(x) < \varepsilon$ for all $j \in J$. Show that $(\mu_j : j \in J)$ is uniformly integrable if and only if (i) $\sup_{j \in J} \int |x| d\mu_j(x) < \infty$ and (ii) $(\nu_j : j \in J)$ is tight, where for $j \in J$, ν_j is the probability measure on $(\mathbb{R}, \mathcal{B})$ satisfying that for every bounded Borel function $f : \mathbb{R} \rightarrow \mathbb{R}$

$$\int f(x) d\nu_j(x) = \frac{\int (1 + |x|) f(x) d\mu_j(x)}{\int (1 + |x|) d\mu_j(x)}.$$

Quote of the week: *I know of scarcely anything so apt to impress the imagination as the wonderful form of cosmic order expressed by the "Law of Frequency of Error." The law would have been personified by the Greeks and deified, if they had known of it. It reigns with serenity and in complete self-effacement, amidst the wildest confusion. The huger the mob, and the greater the apparent anarchy, the more perfect is its sway. It is the supreme law of Unreason. Whenever a large sample of chaotic elements are taken in hand and marshalled in the order of their magnitude, an unsuspected and most beautiful form of regularity proves to have been latent all along.*

Francis Galton⁵ describing the Central Limit Theorem [(1889) Natural Inheritance]

⁴If this conditions holds for all sufficiently large r for some $C, c > 0$ then it also holds for all $r \geq 0$ for some other $C, c > 0$. Thus it suffices to confirm it for sufficiently large r .

⁵Galton's reputation is tarnished by his work on eugenics. https://en.wikipedia.org/wiki/Francis_Galton