Computational Control

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System identification, data-driven predictive control, and behavioral system theory

1. Controllability of a Parameterized Single-Input System

Consider a single-input, discrete-time, LTI system described by the equation

$$x_{t+1} = \begin{bmatrix} 0 & 0 & 0 \\ a_1 & 0 & 0 \\ 0 & a_2 & 0 \end{bmatrix} x_t + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u_t.$$

Determine the controllability of the system as a function of the parameters $a_1, a_2 \in \mathbb{R}$.

Hint. Compute $\mathcal{C} = [B \; AB \; A^2B]$ and check its rank for different values of a_1, a_2.

2. Observability of a Parameterized Single-Output System

Consider a single-output, discrete-time, LTI system described by the equations

$$x_{t+1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_1 & a_2 & a_3 \end{bmatrix} x_t, \qquad y_t = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x_t.$$

Determine the observability of the system as a function of the parameters $a_1, a_2, a_3 \in \mathbb{R}$.

Hint. Compute $\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}$ and check its rank for different values of $a_1, a_2, a_3 \in \mathbb{R}$.

3. Markov Parameters and Coordinate Transformations

Consider a discrete-time LTI system described by the equations

$$x_{t+1} = Ax_t + Bu_t, \qquad y_t = Cx_t,$$

with $x_t \in \mathbb{R}^n$, $u_t \in \mathbb{R}^m$, and $y_t \in \mathbb{R}^p$. Consider the coordinate transformation

$$\xi_t = T^{-1} x_t,$$

where $T \in \mathbb{R}^{n \times n}$ is an invertible matrix, and define a new system described by the equations

$$\xi_{t+1} = T^{-1}AT\,\xi_t + T^{-1}B\,u_t, \qquad y_t = CT\,\xi_t.$$

Show that both systems have the same *Markov parameters* and hence the same input–output behavior.

Hint. Compare $\tilde{G}_k = \tilde{C}\tilde{A}^{k-1}\tilde{B}$ with $G_k = CA^{k-1}B$ using $\tilde{A} = T^{-1}AT$, $\tilde{B} = T^{-1}B$, and $\tilde{C} = CT$.

4. Realization of a Finite Impulse Response System

Consider a discrete-time LTI system described by the input–output relation

$$y_t = c_1 u_{t-1} + c_2 u_{t-2} + c_3 u_{t-3} + c_4 u_{t-4}.$$

Construct a state-space realization

$$x_{t+1} = Ax_t + Bu_t, \qquad y_t = Cx_t,$$

and determine conditions under which the realization is minimal.

Hint. Select a state vector encoding past input values and derive A, B, C accordingly.

5. Rank of Hankel Matrices from Exponential Sequences

Consider the sequences

$$v_t = a^t, \qquad t = 0, 1, 2, \ldots, 6, \quad a \in [0, 1],$$

and

$$w_t = \begin{bmatrix} \left(\frac{1}{2}\right)^t \\ \left(\frac{1}{3}\right)^t \end{bmatrix}, \qquad t = 0, 1, 2, 3, 4.$$

Determine the rank of the Hankel matrices $H_4(\mathbf{v})$ and $H_2(\mathbf{w})$.

Hint. How does the number of distinct geometric sequences in the signal affect the rank of its Hankel matrix?

6. Realization from Matrix-Valued Markov Parameters

Consider a discrete-time, single-input two-output (MISO) LTI system whose Markov parameters are

$$G_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad G_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad G_3 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad G_4 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Determine a minimal realization (A, B, C) satisfying $G_k = CA^{k-1}B$ for $k = 1, \ldots, 4$, and verify controllability and observability of your realization.

Hint. Use the shift pattern in G_k to infer the dimension of A and identify B as the input direction generating the first column of the sequence.

7. Derivation of the Finite-Horizon Input–Output Relation

Consider a discrete-time, LTI system described by the equations

$$x_{t+1} = Ax_t + Bu_t, \qquad y_t = Cx_t + Du_t.$$

By induction, show that, for any time horizon length $L \in \mathbb{N}$, any input–output trajectory over L steps satisfies

$$\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{L-1} \end{bmatrix} = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{L-1} \end{bmatrix}}_{\mathcal{O}_L} x_0 + \underbrace{\begin{bmatrix} D & 0 & 0 & \cdots & 0 \\ CB & D & 0 & \cdots & 0 \\ CAB & CB & D & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ CA^{L-2}B & CA^{L-3}B & \cdots & CB & D \end{bmatrix}}_{\mathcal{T}_L} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{L-1} \end{bmatrix}.$$

8. Basis of the Trajectory Subspace

Consider a single-input, discrete-time, LTI system described by the equation

$$x_{t+1} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} x_t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_t, \qquad y_t = \begin{bmatrix} 1 & 0 \end{bmatrix} x_t,$$

where $a_0, a_1 \in \mathbb{R}$.

Construct a basis for the subspace of all admissible input–output trajectories of length 4, i.e., the set of vectors

$$\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} \in \mathbb{R}^8 \quad \text{such that} \quad \exists\, x_0 \text{ satisfying the system equations.}$$

Hint. Identify a minimal set of independent columns of

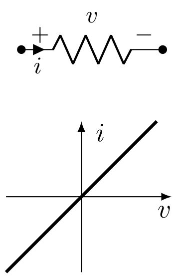
$$M_4 = \begin{bmatrix} I_4 & 0 \\ \mathcal{T}_4 & \mathcal{O}_4 \end{bmatrix}$$

whose span describes all admissible trajectories.

9.Behavioral Derivation of Physical Systems

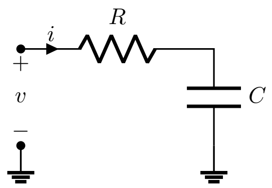
For each of the systems illustrated below, derive its *behavior*—that is, the set of all admissible signal pairs consistent with its governing physical laws. Define the time axis $\mathbb{T}$, the signal space $\mathbb{W}$, and the behavior as a subset of signal trajectories, e.g. $\mathcal{B} \subseteq (v, i)^{\mathbb{T}}$ or $\mathcal{B} \subseteq (x, \theta, u)^{\mathbb{T}}$.

(a) Ideal resistor



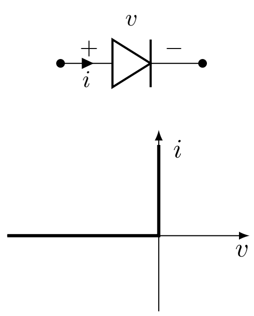
$$i = \frac{1}{R} v$$

(c) RC circuit



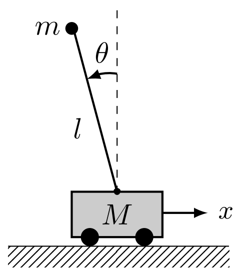
$$C\dot{v} = i - \frac{v}{R}$$

(b) Ideal diode



$$i \geq 0, \quad v \leq 0, \quad iv = 0$$

(d) Inverted pendulum on a cart



$$(M + m)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2 \sin\theta = u,$$

$$l\ddot{\theta} + g\sin\theta = -\ddot{x}\cos\theta.$$

For each system, classify the behavior as

$$\boxed{\text{linear / nonlinear} \quad \text{and} \quad \text{affine? cone? convex?}}$$

