Name: Solutions	

1. Let $A=\begin{bmatrix}1&2\\1&2\\2&1\end{bmatrix}$ and $\vec{b}=\begin{bmatrix}1\\2\\3\end{bmatrix}$.

a. Explain why $A\vec{x}=\vec{b}$ has no solutions.

This equation has no solutions unless $\vec{b}\in\operatorname{im}(A)$. But, it is evident that $\vec{b}\notin\operatorname{im}(A)$ since the first two components of any vector in $\operatorname{im}(A)$ must be equal.

b. What property of A guarantees that $A^T A\vec{x}=A^T\vec{b}$ has a unique solution?

The square matrix $A^T A$ is invertible if and only if $\ker(A)=\{\vec{0}\}$. This is true in the present case since the two column vectors of A comprise a linearly independent pair.

c. Solve the equation in part b.

$$\vec{x}=(A^TA)^{-1}A^T\vec{b}=\left(\begin{bmatrix}1&1&2\\2&2&1\end{bmatrix}\begin{bmatrix}1&2\\1&2\\2&1\end{bmatrix}\right)^{-1}\begin{bmatrix}1&1&2\\2&2&1\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}$$

$$=\begin{bmatrix}6&6\\6&9\end{bmatrix}^{-1}\begin{bmatrix}9\\9\end{bmatrix}=\tfrac{9}{54-36}\begin{bmatrix}9&-6\\-6&6\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\tfrac{1}{2}\begin{bmatrix}3\\0\end{bmatrix}=\begin{bmatrix}\frac{3}{2}\\0\end{bmatrix}.$$

d. If $\vec{x}$ is the vector found in part c, how is $A\vec{x}$ related to $\vec{b}$?

$A\vec{x}$ is the projection of $\vec{b}$ onto $\operatorname{im}(A)$. Equivalently, $A\vec{x}$ is the vector in $\operatorname{im}(A)$ that is "closest" to $\vec{b}$. That is, if $\vec{y}$ is in $\operatorname{im}(A)$, $\|\vec{y}-\vec{b}\|$ is smallest when $\vec{y}=A\vec{x}$.

2. $\mathcal{B}=(\vec{v}_1,\vec{v}_2,\vec{v}_3)=\left(\begin{bmatrix}1\\2\\2\\4\end{bmatrix},\begin{bmatrix}-4\\2\\-2\\1\end{bmatrix},\begin{bmatrix}4\\3\\10\\5\end{bmatrix}\right)$ is a basis for the subspace W of $\mathbf{R}^4$. Notice that $\vec{v}_1$ and $\vec{v}_2$ are orthogonal.

a. Find an orthonormal basis $\mathcal{B}'=(\hat{u}_1,\hat{u}_2,\hat{u}_3)$ for W.

Let $\hat{u}_1 = \dfrac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{5}\begin{bmatrix} 1 \\ 2 \\ 2 \\ 4 \end{bmatrix}$ and $\hat{u}_2 = \dfrac{\vec{v}_2}{\|\vec{v}_2\|} = \frac{1}{5}\begin{bmatrix} -4 \\ 2 \\ -2 \\ 1 \end{bmatrix}$. The third basis vector in $\mathcal{B}'$ is obtained, following the Gram-Schmidt process, by removing from $\vec{v}_3$ its projections onto $\hat{u}_1$ and $\hat{u}_2$ and normalizing the result. We have

$$\vec{v}_3^{\perp} = \vec{v}_3 - (\vec{v}_3 \cdot \hat{u}_1)\hat{u}_1 - (\vec{v}_3 \cdot \hat{u}_1)\hat{u}_2 = \begin{bmatrix} 4 \\ 3 \\ 10 \\ 5 \end{bmatrix} - \tfrac{1}{25}(50)\begin{bmatrix} 1 \\ 2 \\ 2 \\ 4 \end{bmatrix} - \tfrac{1}{25}(-25)\begin{bmatrix} -4 \\ 2 \\ -2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 3 \\ 10 \\ 5 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 2 \\ 2 \\ 4 \end{bmatrix} + \begin{bmatrix} -4 \\ 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 4 \\ -2 \end{bmatrix}.$$ So, $\hat{u}_3 = \dfrac{\vec{v}_3^{\perp}}{\|\vec{v}_3^{\perp}\|} = \frac{1}{5}\begin{bmatrix} -2 \\ 1 \\ 4 \\ -2 \end{bmatrix}$.

b. Find a_1, a_2, and a_3 if $\begin{bmatrix} 5 \\ 0 \\ 4 \\ 3 \end{bmatrix} = a_1\hat{u}_1 + a_2\hat{u}_2 + a_3\hat{u}_3$ belongs to W.

Denote the vector on the left by $\vec{w}$. Then, the "Fourier coefficients" here are simply $a_k = \vec{w} \cdot \hat{u}_k$ for $k = 1, 2, 3$. We find, straightforwardly, that $a_1 = 5$, $a_2 = -5$, $a_3 = 0$

c. Prove that $\|b_1\hat{u}_1 + b_2\hat{u}_2 + b_3\hat{u}_3\|^2 = b_1^2 + b_2^2 + b_3^2$.

$$\|b_1\hat{u}_1 + b_2\hat{u}_2 + b_3\hat{u}_3\|^2 = (b_1\hat{u}_1 + b_2\hat{u}_2 + b_3\hat{u}_3) \cdot (b_1\hat{u}_1 + b_2\hat{u}_2 + b_3\hat{u}_3)$$
$$= b_1^2(\hat{u}_1 \cdot \hat{u}_1) + b_2^2(\hat{u}_2 \cdot \hat{u}_2) + b_3^2(\hat{u}_3 \cdot \hat{u}_3)$$
$$+ 2b_1b_2(\hat{u}_1 \cdot \hat{u}_2) + 2b_1b_3(\hat{u}_1 \cdot \hat{u}_3) + 2b_2b_3(\hat{u}_2 \cdot \hat{u}_3)$$
$$= b_1^2(1) + b_2^2(1) + b_3^2(1) + 2b_1b_2(0) + 2b_1b_3(0) + 2b_2b_3(0)$$
$$= b_1^2 + b_2^2 + b_3^2.$$

3. $\mathscr{B} = (\vec{v}_1, \vec{v}_2, \vec{v}_3) = \left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \\ 5 \end{bmatrix} \right)$ is a basis for the subspace W of $\mathbf{R}^4$.

a. Calculate the 4×4 matrix P that represents projection onto W. Proceed as follows. First determine a basis for $W^\perp$, the subspace of all vectors orthogonal to all the vectors in W. Recall that $W^\perp$ can be regarded as the kernel of a certain matrix. Next, from your basis for $W^\perp$, obtain the 4×4 matrix $P^\perp$ that represents projection onto $W^\perp$. Finally, calculate P.

Let $A = [\vec{v}_1 \,|\, \vec{v}_2 \,|\, \vec{v}_3] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \\ 1 & 3 & 5 \end{bmatrix}$. Then, $W^\perp = (\mathrm{im}(A))^\perp = \ker(A^T)$. We find

$$A^T = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 3 \\ 1 & 2 & 2 & 5 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 4 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} \Rightarrow$$

$W^\perp = \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix} \right)$. Therefore $(\hat{u}) = \left(\frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix} \right)$ is a normalized basis for $W^\perp$. Hence, the matrix for projection onto $W^\perp$ is $P^\perp = \hat{u}\hat{u}^T =$

$$\frac{1}{6}\begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 4 & -2 \\ -1 & 0 & -2 & 1 \end{bmatrix} \text{ and we obtain } P = I - P^\perp = \frac{1}{6}\begin{bmatrix} 5 & 0 & -2 & 1 \\ 0 & 6 & 0 & 0 \\ -2 & 0 & 2 & 2 \\ 1 & 0 & 2 & 5 \end{bmatrix}.$$

b. Without carrying out the calculation, describe an alternate and more direct means to obtain P. [Although more direct, the calculation takes longer.]

Let $A = [\vec{v}_1 \,|\, \vec{v}_2 \,|\, \vec{v}_3]$. Since the column vectors of A are linearly independent, $P = A(A^T A)^{-1} A^T$.

4. a. Evaluate $\det\left(\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 5 & 0 \end{bmatrix}\right)$.

Successive Laplace expansions along any row or column yield

$$\det\left(\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 5 & 0 \end{bmatrix}\right) = 1\cdot\det\left(\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 5 & 0 \end{bmatrix}\right) = 1\cdot 2\cdot\det\left(\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 5 & 0 \end{bmatrix}\right)$$

$$= 1\cdot 2\cdot 3\cdot\det\left(\begin{bmatrix} 0 & 4 \\ 5 & 0 \end{bmatrix}\right) = -1\cdot 2\cdot 3\cdot 4\cdot 5 = -5! = -120.$$

b. Find the 3-volume of the parallelepiped in $\mathbf{R}^4$, three of whose concurrent edges are described by the vectors $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

Let these vectors be the corresponding column vectors of the matrix A. Then, the parallelepiped has 3-volume given by

$$\sqrt{\det(A^T A)} = \sqrt{\det\left(\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\right)} = \sqrt{\det\begin{bmatrix} 2 & 1 & 2 \\ 1 & 3 & 3 \\ 2 & 3 & 4 \end{bmatrix}}$$

$$= \sqrt{\det\begin{bmatrix} 2 & 1 & 2 \\ 1 & 3 & 3 \\ 1 & 0 & 1 \end{bmatrix}} = \sqrt{\det\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix}} = \sqrt{2}.$$