FALL 2011 FINAL EXAM FOR SM261
1330 DECEMBER 17, 2011

SHOW ALL WORK

PART 1. On this part (problems 1-5) you may **NOT** use a calculator. Show all your work.

1. Complete the following definitions.

a. A function T from $\mathbb{R}^m$ to $\mathbb{R}^n$ is called a linear transformation if...

b. A vector $\vec{b}$ in $\mathbb{R}^n$ is called a linear combination of the vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$ in $\mathbb{R}^n$ if...

c. A set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$ in a subspace V of $\mathbb{R}^n$ forms a basis for V if...

d. If V is a subspace of $\mathbb{R}^n$, then the dimension of V is...

2. Consider the linear system of equations

$$\begin{aligned} x + 2y - z &= 3 \\ 3x + y + 2z &= 14 \\ 2x - 6y + 8z &= 16 \end{aligned}$$

a. If we write the system in the matrix form $A\vec{v} = \vec{b}$, what are A, $\vec{v}$ and $\vec{b}$?

b. Find the reduced row echelon form of the augmented matrix $[A|\,\vec{b}\,]$ for this system. Show all steps.

c. Describe all solutions to the system.

d. Is the set of all solutions a subspace of $\mathbb{R}^3$? Explain.

3. Find the determinant of the matrix

$$\begin{bmatrix} 0 & 3 & 2 & 0 \\ 3 & 5 & 0 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 3 & 0 & 0 \end{bmatrix}.$$

4. Let $A = \left[\begin{smallmatrix} 2 & 1 \\ 4 & 3 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 3 & 2 \\ 1 & 2 \end{smallmatrix}\right]$. Find AB^{-1}.

5. Let S be the subspace of $\mathbb{R}^2$ defined by $S = \{\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \mathbb{R}^2 | y = 3x\}$.

a. Find 3 numbers b, c, d so that $\operatorname{im}\left[\begin{smallmatrix} 3 & b \\ c & d \end{smallmatrix}\right] = \mathbb{R}^2$.

b. Find 3 numbers b, c, d so that $\operatorname{im}\left[\begin{smallmatrix} 3 & b \\ c & d \end{smallmatrix}\right] = S$.

c. Find 3 numbers b, c, d so that $\ker\left[\begin{smallmatrix} 3 & b \\ c & d \end{smallmatrix}\right] = S$.

This is the end of Part 1. Turn in your answers and proceed to Part 2.

PART 2. You may use a calculator on this part. Show all your work.

6. Define the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^2$ by

$$T \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

Find the matrix of T with respect to the standard bases for $\mathbb{R}^4$ and $\mathbb{R}^2$.

7. Consider the following matrices:

a. $\left[\begin{smallmatrix} 5 & 0 \\ 0 & 5 \end{smallmatrix}\right]$ **b.** $\begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}$ **c.** $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$ **d.** $\begin{bmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix}$

Identify each of the 4 corresponding linear transformations as a rotation, a projection, a scaling or a reflection.

8. Let A be the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 3 & 6 & 9 & 6 & 3 \\ 1 & 2 & 4 & 1 & 2 \\ 2 & 4 & 9 & 1 & 2 \end{bmatrix}.$$

We have

$$\operatorname{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 5 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

a. Find a basis for $\ker(A)$. Find $\dim(\ker(A))$.

b. Find a basis for $\operatorname{im}(A)$. Find $\operatorname{rank}(A)$.

9. Let $V = \operatorname{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$, where $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 3 \\ 5 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 20 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, and $\vec{v}_4 = \begin{bmatrix} 0 \\ 2 \\ -1 \\ -2 \end{bmatrix}$.

a. Find a basis for V.

b. Is the vector $\begin{bmatrix} 1 \\ 6 \\ -1 \\ -3 \end{bmatrix}$ in V? Explain.

10. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation which satisfies $T\left[\begin{smallmatrix}2\\4\end{smallmatrix}\right] = \left[\begin{smallmatrix}2\\1\end{smallmatrix}\right]$ and $T\left[\begin{smallmatrix}1\\3\end{smallmatrix}\right] = \left[\begin{smallmatrix}1\\5\end{smallmatrix}\right]$. Find the matrix of T relative to the standard basis $\{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\}$.

11. Let V be the subspace of $\mathbb{R}^4$ with basis $\left\{ \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 2 \\ 3 \end{bmatrix} \right\}$.

a. Find an orthonormal basis for V.

b. Find the matrix of the orthogonal projection onto V.

12. Using least squares, fit a function $f(t) = a + bt$ to the data points $(0, 2)$, $(1, 2)$, $(1, 4)$.

13. Let P be the parallelepiped in $\mathbb{R}^3$ defined by the vectors

$$\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}.$$

a. Find the volume of P.

b. Let T be the linear tranformation whose matrix is $A = \left[\begin{smallmatrix} 3 & 4 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{smallmatrix}\right]$. Find the volume of $T(P)$.

14. True or False. If true, give a reason. If false, give a reason or a counterexample.

a. If S and T are linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$, then the set of all vectors $\vec{v}$ in $\mathbb{R}^n$ such that $S(\vec{v}) = T(\vec{v})$ is a subspace of $\mathbb{R}^n$.

b. If the determinant of a 5×5 matrix A is 4, then $1 \leq \text{rank}(A) \leq 4$.

c. The matrix of an orthogonal projection onto a subspace in $\mathbb{R}^n$ is a symmetric matrix.

d. If V is a 3-dimensional subspace of $\mathbb{R}^n$ and $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ is a basis of V, then $\{\vec{v_1}, \vec{v_1} + \vec{v_2}, \vec{v_1} + \vec{v_2} + \vec{v_3}\}$ is also a basis for V.

e. If two matrices have the same characteristic polynomial, then they are similar.

In Problems 15-17, do all calculations by hand and show all steps. You may check your work with a calculator.

15. Let A be the matrix

$$\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}.$$

a. Find the characteristic polynomial $f_A(\lambda)$ of A.

b. Find the eigenvalues of A.

c. For each eigenvalue λ, find a basis for the eigenspace E_λ.

d. Is A diagonalizable? If so, find an eigenbasis for A and a diagonal matrix D which is similar to A. If not, give your reasons.

16. The eigenvalues of

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 5 \\ 0 & 1 & -2 \end{bmatrix}$$

are $\lambda = -3, 3, 3$. Is A diagonalizable? If so, find an eigenbasis for A and a diagonal matrix D which is similar to A. If not, give your reasons.

17. The eigenvalues of

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

are $\lambda = 2, 3, 3$. Is A diagonalizable? If so, find an eigenbasis for A and a diagonal matrix D which is similar to A. If not, give your reasons.

END OF EXAM