1. Let A be a real $n\times n$ matrix. List up to 7 different properties of A that are equivalent to A being invertible (nonsingular). You may not use the words <u>column</u> or <u>transpose</u> or synonyms or symbols for them.

(1) There is an $n\times n$ matrix B such that $AB = I$ (or $BA = I$).

(2) $\det(A) \neq 0$.

(3) The row vectors of A are linearly independent.

(4) The row vectors of A span $\mathbf{R}^3$ (or $\mathrm{Row}(A) = \mathbf{R}^3$).

(5) $A_{rref} = I$.

(6) $0 \notin \mathrm{spec}(A)$.

(7) $\mathrm{Nul}(A) = \{\vec{0}\}$.

(8) The equation $A\vec{x} = \vec{b}$ has a unique solution for every $\vec{b} \in \mathbf{R}^n$

2. Let A be any matrix and let A_{rref} be its row-reduced echelon form. For each of the following assertions, state whether ***True*** or ***False***. [Half credit penalty for an incorrect response.]

a. The row space of A and the row space of A_{rref} are the same.

True. The row vectors of A_{rref} are nontrivial linear combinations of the row vectors of A.

b. The column space of A and the column space of A_{rref} are the same.

False. If A consisted of a single column vector with two or more nonzero entries, A_{rref} would be $\hat{e}_1$.

c. Linear relationships among the row vectors of A and linear relationships among the corresponding row vectors of A_{rref} are the same.

False. If A consisted of two identical nonzero rows, the second row of A_{rref} would be a zero row.

d. Linear relationships among the column vectors of A and linear relationships among the corresponding column vectors of A_{rref} are the same.

True. Elementary row operations on a matrix do not affect linear relationships among the columns of a matrix.

3. Suppose that $A = [\vec{v}_1 \,|\, \vec{v}_2 \,|\, \vec{v}_3 \,|\, \vec{v}_4]$ and that its column vectors belong to $\mathbf{R}^3$. Determine A_{rref} with as much specificity as possible if

a. no three of the four column vectors of A are coplanar

$$A_{rref} = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{bmatrix}$$ where none of a, b, or c is 0 (i.e. $abc \neq 0$).

b. no two of the four column vectors of A are collinear but all four are coplanar.

$A_{rref} = \begin{bmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \\ 0 & 0 & 0 & 0 \end{bmatrix}$ where the third and fourth column vectors may not be proportional to each other nor proportional to the first or second column vectors (i.e. $abcd \neq 0$ and $ad - bc \neq 0$).

4. List all the ways 13 bills may be chosen from among the denominations \$1, \$2, \$5, and \$10 so that their total value is \$26.

Let x_1, x_2, x_3, x_4 be, respectively, the number of \$1, \$2, \$5, and \$10 bills and also let $\vec{x} = [x_1 \quad x_2 \quad x_3 \quad x_4]^T$, $\vec{b} = [13 \quad 26]^T$ and $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 5 & 10 \end{bmatrix}$. We wish to solve $A\vec{x} = \vec{b}$ subject to the constraint that each of the components of $\vec{x}$ be non-negative integers. So, we row-reduce: $\left[A\,|\,\vec{b}\right] = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 13 \\ 1 & 2 & 5 & 10 & 26 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 13 \\ 0 & 1 & 4 & 9 & 13 \end{array}\right] \sim$
$\left[\begin{array}{cccc|c} 1 & 0 & -3 & -8 & 0 \\ 0 & 1 & 4 & 9 & 13 \end{array}\right] = \left[A\,|\,\vec{b}\right]_{rref}$. According to the Solution Algorithm, we have

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 13 \\ 0 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -3 \\ 4 \\ -1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -8 \\ 9 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 13 \\ 0 \\ 0 \end{bmatrix} + j \begin{bmatrix} 3 \\ -4 \\ 1 \\ 0 \end{bmatrix} + k \begin{bmatrix} 8 \\ -9 \\ 0 \\ 1 \end{bmatrix}$$

where j and k must be non-negative integers. The values of j and k that yield allowable values for the components of $\vec{x}$ are the following: (j, k) = (0, 0), (0, 1), (1, 0), (1,1), (2,0), (3,0).

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 13 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ 4 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 11 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 5 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 9 \\ 1 \\ 3 \\ 0 \end{bmatrix}.$$

We have found 6 different combinations.

5. For each of the following, provide a specific example of a real 3×3 matrix A or give a convincing but succinct argument why no such matrix exists.

a. A is invertible but A^2 is not invertible.

There exists no such A. If A is invertible A^{-1} exists and $(A^{-1})^2$ is clearly the inverse of A^2 since $(A^2)(A^{-1})^2 = A\,(A\,A^{-1})\,A^{-1} = A\,(\,I\,)\,A^{-1} = A\,A^{-1} = I.$

b. A has no zero entries and $A^3 = A$.

Choose A to be any projection matrix with non-zero entries, e.g. the matrix for projection onto the vector $\begin{bmatrix}1\\1\\1\end{bmatrix}$. Then, $A = \frac{1}{3}\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$ and $A^3 = A(A^2) = AA = A$.

c. A is not the identity and $A^3 = I$.

Three consecutive rotations by $2\pi/3$ around the same axis is the identity. Let

$$A = \begin{bmatrix}\cos(\frac{2\pi}{3}) & -\sin(\frac{2\pi}{3}) & 0\\ \sin(\frac{2\pi}{3}) & \cos(\frac{2\pi}{3}) & 0\\ 0 & 0 & 1\end{bmatrix} = -\frac{1}{2}\begin{bmatrix}1 & \sqrt{3} & 0\\ -\sqrt{3} & 1 & 0\\ 0 & 0 & -2\end{bmatrix}.$$

d. Col(A) = Nul(A).

There is no such matrix. If there were, the Rank-Nullity Theorem would tell us that dim(Col(A)) + dim(Nul(A)) = 2 dim(Col(A)) = 2 dim(Nul(A)) = 3, implying that the dimensions of Col(A) and Nul(A) were 3/2, an impossibility.

6. a. Prove that the intersection of any two subspaces of any vector space is itself a subspace.

Let $\boldsymbol{U}$ and $\boldsymbol{V}$ be subspaces of a vector space $\boldsymbol{W}$. $\boldsymbol{U}\cap\boldsymbol{V}$ is a non-empty subset of $\boldsymbol{W}$ since $\boldsymbol{U}$ and $\boldsymbol{V}$ contain $\vec{0}$ and so their intersection contains $\vec{0}$. If $\vec{a},\vec{b}\in\boldsymbol{U}\cap\boldsymbol{V}$, then $\vec{a},\vec{b}\in\boldsymbol{U}$ and $\vec{a},\vec{b}\in\boldsymbol{V}$. Moreover, $\vec{a}+\vec{b}\in\boldsymbol{U}$ and $\vec{a}+\vec{b}\in\boldsymbol{V}$ because each of the subspaces is closed under vector addition. So, $\boldsymbol{U}\cap\boldsymbol{V}$ is also closed under vector addition. Additionally, for any scalar s, if $\vec{a}\in\boldsymbol{U}\cap\boldsymbol{V}$, $\vec{a}\in\boldsymbol{U}$ and $\vec{a}\in\boldsymbol{V}$. So, $s\vec{a}\in\boldsymbol{U}$ and $s\vec{a}\in\boldsymbol{V}$. Therefore, $\vec{a}\in\boldsymbol{U}\cap\boldsymbol{V}$, proving that $\boldsymbol{U}\cap\boldsymbol{V}$ is also closed under multiplication by scalars. $\boldsymbol{U}\cap\boldsymbol{V}$ is a subspace.

b. Give an example of two subspaces of $\mathbf{R}^2$ whose union is not a subspace and prove that the union is not a subspace.

Let $\boldsymbol{U}_1 = \text{span}(\hat{e}_1)$ and $\boldsymbol{U}_2 = \text{span}(\hat{e}_2)$. These are the coordinate axes in $\mathbf{R}^2$ and they are each one-dimensional subspaces of $\mathbf{R}^2$. Although $\hat{e}_1$ and $\hat{e}_2$ both belong to $\boldsymbol{U}_1\cup\boldsymbol{U}_2$, $\hat{e}_1+\hat{e}_2$ does not belong to $\boldsymbol{U}_1\cup\boldsymbol{U}_2$. So, $\boldsymbol{U}_1\cup\boldsymbol{U}_2$ is not a subspace of $\mathbf{R}^2$.

7. $\vec{w} = \begin{bmatrix}3\\4\\3\\5\end{bmatrix}$ and $L = (\vec{v}_1,\vec{v}_2,\vec{v}_3,\vec{v}_4) = \left(\begin{bmatrix}1\\1\\2\\1\end{bmatrix},\begin{bmatrix}1\\2\\-1\\3\end{bmatrix},\begin{bmatrix}1\\2\\3\\-1\end{bmatrix},\begin{bmatrix}4\\7\\3\\6\end{bmatrix}\right)$.

Determine if L is a basis for $\mathbf{R}^4$ and determine if $\vec{w}\in\text{span}(L)$ by row-reducing a <u>single</u> matrix. Discuss.

Let $A = [\vec{v}_1 \,|\, \vec{v}_2 \,|\, \vec{v}_3 \,|\, \vec{v}_4 \,|\, \vec{w}]$. Then, $A = \begin{bmatrix} 1 & 1 & 1 & 4 & 3 \\ 1 & 2 & 2 & 7 & 4 \\ 2 & -1 & 3 & 3 & 3 \\ 1 & 3 & -1 & 6 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = A_{rref}$.

So, we see that L is not linearly dependent since $\vec{v}_4 = \vec{v}_1 + 2\vec{v}_2 + \vec{v}_3$ and so L does not span $\mathbf{R}^4$. However, we also see that $\vec{w} = 2\vec{v}_1 + \vec{v}_2 \in \text{span}(L)$.

8. Let $A = \begin{bmatrix} 2 & 6 & 1 \\ 3 & 7 & 2 \\ 2 & 6 & 1 \\ 4 & 8 & 3 \end{bmatrix}$.

a. Determine a basis for and the dimension of $\text{Col}(A)$.

$A = \begin{bmatrix} 2 & 6 & 1 \\ 3 & 7 & 2 \\ 2 & 6 & 1 \\ 4 & 8 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{5}{4} \\ 0 & 1 & -\frac{1}{4} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = A_{rref}$. So, it is clear that the first two column vectors of A are linearly dependent while the third is in their span. Therefore, $\left(\begin{bmatrix} 2 \\ 3 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 6 \\ 7 \\ 6 \\ 8 \end{bmatrix} \right)$ is a basis for $\text{Col}(A)$ and $\dim(\text{Col}(A)) = 2$.

b. Determine a basis for and the dimension of $\text{Nul}(A)$.

Referring back to A_{rref} in part a and, using the Solution Algorithm, we deduce that $\left(\begin{bmatrix} 5 \\ -1 \\ -4 \\ 0 \end{bmatrix} \right)$ is a basis for $\text{Nul}(A)$ and $\dim(\text{Nul}(A)) = 1$.

c. What does $\text{Col}(A)$ tell us about solutions to the equation $A\vec{x} = \vec{b}$?

The equation has no solutions unless $\vec{b} \in \text{Col}(A)$.

d. What does $\text{Nul}(A)$ tell us about solutions to the equation $A\vec{x} = \vec{b}$?

If $\vec{b} \in \text{Col}(A)$, the equation has infinitely many solutions since $\text{Nul}(A)$ is nontrivial.

9. Theory predicts that the electrical resistivity r of silver doped with trace amounts of silicon is given by $r = x_1 c + x_2 c^2$ where c is the concentration of silicon in silver and c lies in the interval [1, 2]. Find the best (least squares) choice of the coefficients x_1 and x_2 using results from an experiment that yielded the following (c, r)-pairs: (1, 1), (1, 2), (2, 2), and (2, 3).

We are seeking a solution to $A\vec{x} = \vec{b}$ where $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 2 & 4 \\ 2 & 4 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 3 \end{bmatrix}$. But, there is none since $\vec{b} \notin \mathrm{Col}(A)$. So, instead, we solve $A^T A\vec{x} = A^T\vec{b}$. The solution is

$$\vec{x} = \left(A^T A\right)^{-1} A^T\vec{b} = \left(\begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 4 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 2 & 4 \\ 2 & 4 \end{bmatrix}\right)^{-1}\begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 4 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 10 & 18 \\ 18 & 34 \end{bmatrix}^{-1}\begin{bmatrix} 13 \\ 23 \end{bmatrix}$$

$= \frac{1}{8}\begin{bmatrix} 17 & -9 \\ -9 & 5 \end{bmatrix}\begin{bmatrix} 13 \\ 23 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 14 \\ -2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 7 \\ -1 \end{bmatrix}$. The Least Squares fit is $r = \frac{1}{4}(7c - c^2)$.

10. The populations at time t of two competing insect species are $x_1(t)$ and $x_2(t)$. They satisfy $\dfrac{d\,x_1(t)}{d\,t} - x_1(t) + 2x_2(t) = 0$ and $\dfrac{d\,x_2(t)}{d\,t} + 2x_1(t) - x_2(t) = 0$.

a. Rewrite this pair of coupled, homogeneous, first-order ODEs as a single first order matrix-vector ODE.

Set $A = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$. Then, our coupled system is $\vec{x}\,'(t) + A\,\vec{x}(t) = \vec{0}$.

b. Using eigenvector-eigenvalue methods, find the general solution to the above equations.

From the characteristic equation, $0 = \det(A - \lambda I)$, we find the eigenvalues. spec(A) = (–3, 1) and $E_1(A) = \mathrm{Nul}(A - I) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ and $E_{-3}(A) = \mathrm{Nul}(A + 3I) = \mathrm{span}\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$.

Since A is real symmetric, the eigenspaces are orthogonal. Since the scalar ODE $y'(t) + \lambda y(t) = 0$ has the general solution $y(t) = c\,e^{-\lambda t}$, $\vec{x}(t) = c_1 e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{3t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

c. Compute $x_1(t)$ and $x_2(t)$ if $x_1(0) = 2$ and $x_2(0) = 3$

$$\vec{x}(0)=\begin{bmatrix}2\\3\end{bmatrix}=c_1\begin{bmatrix}1\\1\end{bmatrix}+c_2\begin{bmatrix}-1\\1\end{bmatrix} \Rightarrow c_1=\tfrac{5}{2}, c_2=\tfrac{1}{2} \Rightarrow \begin{Bmatrix}x_1(t)=\frac{5}{2}e^{-t}-\frac{1}{2}e^{3t}\\ x_2(t)=\frac{5}{2}e^{-t}+\frac{1}{2}e^{3t}\end{Bmatrix}.$$

d. In this model, one of the species eventually becomes extinct. At what time does this occur for the initial conditions given above?

Species 1 reaches extinction when $0=x_1(t)=\frac{5}{2}e^{-t}-\frac{1}{2}e^{3t} \Rightarrow e^{4t}=5$ or $t=\frac{1}{4}\ln 5\cong .402$.

11. Let $\hat{u}=\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}$ be a fixed unit vector and let $\vec{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ be an arbitrary vector in $\mathbf{R}^3$ and define the function $f:\mathbf{R}^3\to\mathbf{R}^3$ by $f(\vec{x})=\hat{u}\times\vec{x}$. The symbol $\times$ denotes the usual cross or vector product.

a. Explain why f is linear.

For any $a,b\in\mathbf{R}$ and any $\vec{x},\vec{y}\in\mathbf{R}^3$, $f(a\vec{x}+b\vec{y})=\hat{u}\times(a\vec{x}+b\vec{y})=a\hat{u}\times\vec{x}+b\hat{u}\times\vec{y}=$ $a\,f(\vec{x})+b\,f(\vec{y})$.

\+ b. Find the 3×3 matrix A so that $f(\vec{x})=A\vec{x}$ for any $\vec{x}\in\mathbf{R}^3$.

$A=\left[f(\hat{e}_1)\,|\,f(\hat{e}_2)\,|\,f(\hat{e}_3)\right]$. That is, the column vectors of A are the images of the corresponding standard basis vectors of $\mathbf{R}^3$. $f(\hat{e}_1)=\hat{u}\times\hat{e}_1=(u_1\hat{e}_1+u_2\hat{e}_2+u_3\hat{e}_3)\times\hat{e}_1=$

$u_1\hat{e}_1\times\hat{e}_1+u_2\hat{e}_2\times\hat{e}_1+u_3\hat{e}_3\times\hat{e}_1=-u_2\hat{e}_3+u_3\hat{e}_2=\begin{bmatrix}0\\u_3\\-u_2\end{bmatrix}$. The second and third column vectors of A are computed similarly. We find $A=\begin{bmatrix}0&-u_3&u_2\\u_3&0&-u_1\\-u_2&u_1&0\end{bmatrix}$.

c. Now determine the matrix B so that $B\vec{x}=\hat{u}\times(\hat{u}\times\vec{x})$ for any $\vec{x}\in\mathbf{R}^3$.

B is the standard matrix that corresponds to the composite of f with itself, i.e. $f\circ f$.

$$B=A^2=\begin{bmatrix}0&-u_3&u_2\\u_3&0&-u_1\\-u_2&u_1&0\end{bmatrix}\begin{bmatrix}0&-u_3&u_2\\u_3&0&-u_1\\-u_2&u_1&0\end{bmatrix}=\begin{bmatrix}-(u_2^2+u_3^2)&u_1u_2&u_1u_3\\u_2u_1&-(u_3^2+u_1^2)&u_2u_3\\u_3u_1&u_3u_2&-(u_1^2+u_2^2)\end{bmatrix}.$$

d. Show that the matrix B is simply related to the matrix for projection onto the one-dimensional subspace of $\mathbf{R}^3$ spanned by $\hat{u}$.

Since $u_1^2+u_2^2+u_3^2=1$, $B=\begin{bmatrix}u_1^2-1&u_1u_2&u_1u_3\\u_2u_1&u_2^2-1&u_2u_3\\u_3u_1&u_3u_2&u_3^2-1\end{bmatrix}=P-I=-P^{\perp}$ where $P=\hat{u}\hat{u}^T$.

12. $\boldsymbol{W}$ is the plane through the origin parallel to both $\vec{a}=\begin{bmatrix}1\\2\\2\end{bmatrix}$ and $\vec{b}=\begin{bmatrix}4\\4\\3\end{bmatrix}$.

Compute the matrix P for orthogonal projection onto $\boldsymbol{W}$ in <u>three</u> distinct ways.

a. by using an orthonormal basis for $\boldsymbol{W}$ to construct P.

Let $\hat{u}_1=\dfrac{\vec{a}}{\|\vec{a}\|}=\frac{1}{3}\begin{bmatrix}1\\2\\2\end{bmatrix}$. Then, $\vec{b}^{\perp}=\vec{b}-(\vec{b}\cdot\hat{u}_1)\hat{u}_1=\begin{bmatrix}4\\4\\3\end{bmatrix}-\left(\begin{bmatrix}4\\4\\3\end{bmatrix}\cdot\frac{1}{3}\begin{bmatrix}1\\2\\2\end{bmatrix}\right)\frac{1}{3}\begin{bmatrix}1\\2\\2\end{bmatrix}=\begin{bmatrix}2\\0\\-1\end{bmatrix}$ and

$\hat{u}_2=\dfrac{\vec{b}^{\perp}}{\|\vec{b}^{\perp}\|}=\frac{1}{\sqrt{5}}\begin{bmatrix}2\\0\\-1\end{bmatrix}$. $(\hat{u}_1,\hat{u}_2)$ is an orthonormal basis for $\boldsymbol{W}$ and the matrix for projection onto $\boldsymbol{W}$ is

$$P=\hat{u}_1\hat{u}_1^{T}+\hat{u}_2\hat{u}_2^{T}=\frac{1}{9}\begin{bmatrix}1&2&2\\2&4&4\\2&4&4\end{bmatrix}+\frac{1}{5}\begin{bmatrix}4&0&-2\\0&0&0\\-2&0&1\end{bmatrix}=\frac{1}{45}\begin{bmatrix}41&10&-8\\10&20&20\\-8&20&29\end{bmatrix}.$$

b. by finding the orthogonal complement $\boldsymbol{W}^{\perp}$ of $\boldsymbol{W}$.

$$\boldsymbol{W}^{\perp}=\left(\operatorname{Col}\left[\vec{a}\,|\,\vec{b}\right]\right)^{\perp}=\operatorname{Nul}\left(\left[\vec{a}\,|\,\vec{b}\right]^{T}\right)=\operatorname{Nul}\begin{bmatrix}1&2&2\\4&4&3\end{bmatrix}=\operatorname{Nul}\begin{bmatrix}1&0&-\frac{1}{2}\\0&1&\frac{5}{4}\end{bmatrix}=\operatorname{span}\begin{bmatrix}2\\-5\\4\end{bmatrix}$$

is the 1-dimensional space orthogonal to $\boldsymbol{W}$. A unit vector that spans this space is

$\hat{u}_3=\frac{1}{\sqrt{45}}\begin{bmatrix}2\\-5\\4\end{bmatrix}$. So, $P=I-\hat{u}_3\hat{u}_3^{T}=I-\frac{1}{45}\begin{bmatrix}4&-10&8\\-10&25&-20\\8&-20&16\end{bmatrix}=\frac{1}{45}\begin{bmatrix}41&10&-8\\10&20&20\\-8&20&29\end{bmatrix}$.

c. by using a single formula involving the matrix $\left[\vec{a}\,|\,\vec{b}\right]$.

$$P=\left[\vec{a}\,|\,\vec{b}\right]\left(\left[\vec{a}\,|\,\vec{b}\right]^{T}\left[\vec{a}\,|\,\vec{b}\right]\right)^{-1}\left[\vec{a}\,|\,\vec{b}\right]^{T}=\begin{bmatrix}1&4\\2&4\\2&3\end{bmatrix}\left(\begin{bmatrix}1&2&2\\4&4&3\end{bmatrix}\begin{bmatrix}1&4\\2&4\\2&3\end{bmatrix}\right)^{-1}\begin{bmatrix}1&2&2\\4&4&3\end{bmatrix}$$

$$=\frac{1}{45}\begin{bmatrix}41&10&-8\\10&20&20\\-8&20&29\end{bmatrix}.$$

13. Let $\vec{w}=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$ and $(\vec{v}_1,\vec{v}_2)=\left(\begin{bmatrix}1\\1\\-1\\1\end{bmatrix},\begin{bmatrix}1\\-1\\1\\1\end{bmatrix}\right)$.

a. Find the distance from the vector $\vec{w}$ to the subspace $\operatorname{span}(\vec{v}_1,\vec{v}_2)$ in $\mathbf{R}^4$.

Since $\vec{v}_1$ and $\vec{v}_2$ are orthogonal, the sum of the projections onto each is the projection P onto their span. So, let $\hat{u}_1$ and $\hat{u}_2$ be the unit vectors corresponding to $\vec{v}_1$ and $\vec{v}_2$, respectively. Then, $P=\hat{u}_1\hat{u}_1^T+\hat{u}_2\hat{u}_2^T=\frac{1}{4}\begin{bmatrix}1&1&-1&1\\1&1&-1&1\\-1&-1&1&-1\\1&1&-1&1\end{bmatrix}+\frac{1}{4}\begin{bmatrix}1&-1&1&1\\-1&1&-1&-1\\1&-1&1&1\\1&-1&1&1\end{bmatrix}=$

$\frac{1}{2}\begin{bmatrix}1&0&0&1\\0&1&-1&0\\0&-1&1&0\\1&0&0&1\end{bmatrix}$ and so, $P^{\perp}=I-P=\frac{1}{2}\begin{bmatrix}1&0&0&-1\\0&1&1&0\\0&1&1&0\\-1&0&0&1\end{bmatrix}$. The distance from $\vec{w}$ to the subspace is $\|\vec{w}-P\vec{w}\|=\|(I-P)\vec{w}\|=\|P^{\perp}\vec{w}\|=\left\|\begin{bmatrix}0\\1\\1\\0\end{bmatrix}\right\|=\sqrt{2}$.

b. Find the area of the parallelogram in $\mathbf{R}^4$ two of whose concurrent edges are described by the vectors $\vec{v}_1$ and $\vec{v}_2$.

The parallelogram is, in fact, a square in $\mathbf{R}^4$ since the edges are orthogonal and so, the area is simply $\|\vec{v}_1\|\cdot\|\vec{v}_2\|=2\cdot 2=4$.

14. $T:\mathbf{R}^2\to\mathbf{R}^2$ is a linear transformation that

- doubles all vectors in $\mathbf{R}^2$ parallel to the line with equation $x_1+2x_2=0$ and
- triples all vectors in $\mathbf{R}^2$ parallel to the line with equation $2x_1-3x_2=0$.

a. Determine a basis $\mathcal{B}$ for $\mathbf{R}^2$ relative to which the matrix for T is diagonal.

Vectors parallel to the two lines are, respectively, $\vec{v}_1=\begin{bmatrix}2\\-1\end{bmatrix}$ and $\vec{v}_2=\begin{bmatrix}3\\2\end{bmatrix}$ and these are the basis vectors for $\mathbf{R}^2$ we choose. That is, $\mathcal{B}=(\vec{v}_1,\vec{v}_2)$.

b. What is the $\mathcal{B}$–matrix for T?

$B = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

c. What is the standard matrix for T?

The change of coordinate matrix is $S = [\vec{v}_1 \,|\, \vec{v}_2] = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$ and so, the standard matrix for T is $A = SBS^{-1} = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\frac{1}{7}\begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 4 & -6 \\ 3 & 6 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 17 & 6 \\ 2 & 18 \end{bmatrix}$.

d. What is $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$?

$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 17 & 6 \\ 2 & 18 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 23 \\ 20 \end{bmatrix}$.

15. Suppose that $\boldsymbol{M}$ is a plane through the origin in $\mathbf{R}^3$ and F is the standard matrix for reflection across $\boldsymbol{M}$.

a. Describe, with as much specificity as possible, the eigenvalues and eigenspaces of F. Determine the algebraic and geometric multiplicities for each eigenvalue and specify the characteristic polynomial for F.

Reflection across $\boldsymbol{M}$ will leave unaffected all vectors in $\boldsymbol{M}$ and it will reverse all vectors normal to $\boldsymbol{M}$. All other nonzero vectors leave their span when reflected. So, $\operatorname{spec}(F) = (+1, -1)$, $E_{+1}(F) = \boldsymbol{M}$, $E_{-1}(F) = \boldsymbol{M}^{\perp}$. So, the geometric multiplicities of $+1$ is 2 and the geometric multiplicity of -1 is 1. These are also the algebraic multiplicities of these eigenvalues. Consequently, the characteristic polynomial of F is $\chi_F(\lambda) = (\lambda - 1)^2(\lambda + 1) = \lambda^3 - \lambda^2 - \lambda + 1$.

b. Suppose that $\boldsymbol{N}$ is a plane through the origin in $\mathbf{R}^3$ that is different from $\boldsymbol{M}$ and G is the matrix for reflection across $\boldsymbol{N}$. Then, F and G are similar. In fact, explain why there is an orthogonal matrix Q such that $F = QGQ^{-1}$.

A rotation can bring $\boldsymbol{M}$ into coincidence with $\boldsymbol{N}$. This rotation also brings the normal to $\boldsymbol{M}$ into coincidence with the normal to $\boldsymbol{N}$. The axis of the rotation is the line of intersection of $\boldsymbol{M}$ and $\boldsymbol{N}$ and the angle for the rotation is that between $\boldsymbol{M}$ and $\boldsymbol{N}$. Consequently, the matrices F and G are similar. If Q is the matrix for this rotation, it is orthogonal and $F = QGQ^{-1}$

16. a. If A is a 3×3 invertible matrix, what geometrical information does the value of $|\det(A)|$ convey about the column vectors of A?

This is the volume of the parallelipiped whose concurrent edges are the column vectors of A.

b. If A is a 3×3 invertible matrix, what geometrical information does the value of $|\det(A)|$ convey about images of regions under the linear transformation $\vec{x} \mapsto A\vec{x}$?

This is the factor by which the volume of any region of $\mathbf{R}^3$ is multiplied when subject to this transformation.

c. If $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} j \\ k \\ l \end{bmatrix}$ where the coefficient matrix $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ is invertible, express x_3 as the ratio of the determinants of two 3×3 matrices. It is not necessary to evaluate the determinants.

According to Cramer's Rule, $x_3 = \dfrac{\det\begin{bmatrix} a & b & j \\ d & e & k \\ g & h & l \end{bmatrix}}{\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}}$.