FALL 2005 FINAL EXAM FOR SM261
1330 DECEMBER 16, 2005

SHOW ALL WORK

PART 1. On this part you may **NOT** use a calculator. Show all your work.

1. Complete the following definitions.

a. A function T from $\mathbb{R}^m$ to $\mathbb{R}^n$ is called a linear transformation if...
b. The kernel of a linear transformation T is...
c. A vector $\vec{b}$ in $\mathbb{R}^n$ is called a linear combination of the vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\}$ in $\mathbb{R}^n$ if...
d. A set of vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\}$ in a subspace V of $\mathbb{R}^n$ forms a basis for V if...
e. If V is a subspace of $\mathbb{R}^n$, then the dimension of V is...

2. Consider the linear system of equations

$$\begin{aligned} x - y + 2z &= 3 \\ 3x + 4y - z &= 2 \\ 2x + 12y - 10z &= -8 \end{aligned}$$

a. If we write the system in the matrix form $A\vec{v} = \vec{b}$, what are A, $\vec{v}$ and $\vec{b}$?
b. Find the reduced row echelon form of the augmented matrix $[A|\,\vec{b}\,]$ for this system. Show all steps.
c. Solve the system.

d. Is the solution set a subspace of $\mathbb{R}^3$? Explain. If so, what is its dimension?
e. Is the solution set of the corresponding homogenous system $Av = 0$ a subspace of $\mathbb{R}^3$? Explain. If so, what is its dimension?

3. Find the determinant of the matrix

$$\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 3 & 1 & 2 \\ 3 & 5 & 0 & 3 \end{bmatrix}.$$

4. Let $A = \left[\begin{smallmatrix} 4 & 2 \\ 3 & 2 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 2 & 1 \\ 3 & 3 \end{smallmatrix}\right]$.

a. Find AB^{-1}.
b. Does $\mathrm{im}(A)=\mathrm{im}(B)$? Explain.
c. Does $\ker(A)=\ker(B)$? Explain.

This is the end of Part 1. Turn in your answers and proceed to Part 2.

PART 2. You may use a calculator on this part. Show all your work.

5. Let $T_1 : \mathbb{R}^3 \to \mathbb{R}^3$ be the projection onto the plane $x + y + z = 0$. The matrix of T_1 is

$$\frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

Let $T_2 : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by $T_2\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x-y+2z \\ 2x+y-3z \end{bmatrix}$. Let S be the linear transformation given by $S(\vec{v}) = T_2(T_1(\vec{v}))$. Find the matrix of S.

6. Let $\vec{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be a vector in $\mathbb{R}^3$. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by $T(\vec{v}) = \vec{v} \times \vec{w}$ (cross product). Find the matrix of T.

7. Let A be the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 3 & 6 & 9 & 6 & 3 \\ 1 & 2 & 4 & 1 & 2 \\ 2 & 4 & 9 & 1 & 2 \end{bmatrix}.$$

We have

$$\operatorname{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 5 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

a. Find a basis for $\ker(A)$.
b. Find $\dim(\ker(A))$.
c. Find a basis for $\operatorname{im}(A)$.
d. Find $\operatorname{rank}(A)$.

8. Let $V = \operatorname{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$, where $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 3 \\ 5 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 20 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, and $\vec{v}_4 = \begin{bmatrix} 0 \\ 2 \\ -1 \\ -2 \end{bmatrix}$.

a. Find a basis for V.

b. Is the vector $\begin{bmatrix} 1 \\ 6 \\ -1 \\ -3 \end{bmatrix}$ in V? Explain.

9. Let V be the subspace of $\mathbb{R}^4$ which consists of all vectors $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ such that $x - y + 2z - 3w = 0$. Find a basis for V.

10. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x+2y \\ x-3y \end{bmatrix}$. Let $\mathcal{B}$ be the basis of $\mathbb{R}^2$ given by $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}$. Find the $\mathcal{B}$-matrix of T.

11. Let V be the subspace of $\mathbb{R}^4$ with basis $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$. Use the Gram-Schmidt procedure to find an orthonormal basis for V.

12. Find the matrix of the orthogonal projection onto the subspace V of $\mathbb{R}^3$ which is spanned by the vector $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$.

13. Find the least-squares solution of the system $A\vec{x} = \vec{b}$, where $A = \begin{bmatrix} 1 & 1 \\ 2 & 8 \\ 1 & 5 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$.

14. Consider the following matrices:

a. $\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$ **b.** $\begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}$ **c.** $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$ **d.** $\begin{bmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix}$

Identify each of the 4 corresponding linear transformations as a a rotation, a projection, a scaling or a reflection.

In Problems 15-17, do all calculations by hand and show all steps. You may check your work with a calculator.

15. Let A be the matrix

$$\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}.$$

a. Find the characteristic polynomial $f_A(\lambda)$ of A.

b. Find the eigenvalues of A. Find the algebraic multiplicity of each eigenvalue.

c. For each eigenvalue λ, find a basis for the eigenspace E_λ.

d. Is A diagonalizable? If so, find an eigenbasis for A and a diagonal matrix D which is similar to A. If not, give your reasons.

16. Repeat Problem 15 for

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 5 \\ 0 & 1 & -2 \end{bmatrix}.$$

17. Repeat Problem 15 for

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

18. Let P be the parallelepiped in $\mathbb{R}^3$ formed with the vectors

$$\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}.$$

a. Find the 3-volume of P.

b. Let T be the linear tranformation whose matrix is $A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 5 & 1 \\ 2 & 1 & 0 \end{bmatrix}$. Find the 3-volume of $T(P)$.

19. Use Cramer's Rule to solve the linear algebraic system below. No credit for other methods.

$$\begin{aligned} 2x - 3y + z &= 1 \\ x + y - z &= 0 \\ 3x + 2y - 2z &= 0. \end{aligned}$$

20. True or False. If true, give a reason. If false, give a reason or a counterexample.

a. If an invertible matrix A is diagonalizable, then A^{-1} is also diagonalizable.

b. If the determinant of a 5×5 matrix is 5, then the rank of the matrix is 5.

c. The matrix of an orthogonal projection onto a subspace in $\mathbb{R}^n$ is an orthogonal matrix.

d. If $\mathcal{B} = \{\vec{v_1}, \ldots, \vec{v_n}\}$ is a basis of $\mathbb{R}^n$ and V is a subspace of $\mathbb{R}^n$, then some subset of $\mathcal{B}$ is a basis of V.

e. If S and T are linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$, then the set of all vectors $\vec{v}$ in $\mathbb{R}^n$ such that $S(\vec{v}) = T(\vec{v})$ is a subspace of $\mathbb{R}^n$.

END OF TEST