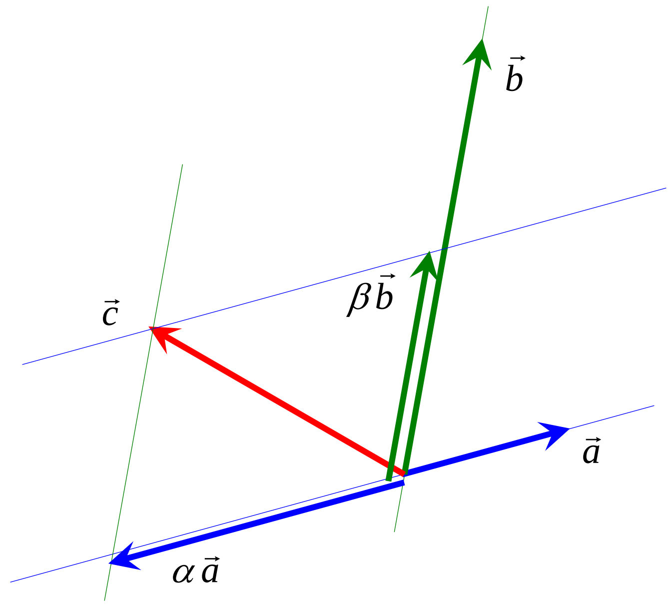
Special Problems 5 and 6 Solutions

5. Demonstrate, by using a careful geometrical construction with your parallel rulers, that the vector $\vec{c}$ is a linear combination of the two non-collinear vectors $\vec{a}$ and $\vec{b}$ in the diagram below. What can be said about the coefficients in this linear combination?



We drew blue and green lines parallel, respectively, to the vectors $\vec{a}$ and $\vec{b}$ and through the head of the vector $\vec{c}$. From these lines, we constructed the vectors αa and $\beta\vec{b}$. These vectors are collinear with $\vec{a}$ and $\vec{b}$ and, by the parallelogram rule for adding vectors, they clearly sum to $\vec{c}$. Moreover, by inspection, $\alpha < -1$ and $0 < \beta < 1$.

6. Given an $m \times n$ matrix A and a vector $\vec{b}$ in $\mathbf{R}^n$, we can solve the equation $A\vec{x} = \vec{b}$ for $\vec{x}$ by standard row-reduction. Several kinds of problems in linear algebra may be reformulated so that their solutions amount to solving an equation of this kind. Demonstrate this by showing how to reformulate the following problems. Discuss.

a. Given $s + 1$ vectors $\vec{w}, \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_s$ in $\mathbf{R}^n$, determine if $\vec{w} \in \mathrm{span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_s)$.

We seek scalars $x_1, x_2, \ldots, x_s$ so that $\vec{w} = x_1\vec{v}_1 + x_2\vec{v}_2 + \ldots + x_s\vec{v}_s$. Let A be the matrix whose columns are the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_s$. So $A = [\vec{v}_1 \mid \vec{v}_2 \mid \cdots \mid \vec{v}_s]$.

Now, let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_s \end{bmatrix}$ and let $\vec{b} = \vec{w}$. Then, the given problem is equivalent to solving $A\vec{x}=\vec{b}$.

b. Given m vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ in $\mathbf{R}^n$ where $n \geq m$, determine if the m vectors are linearly independent.

The list of m vectors is linearly independent if and only if there are scalars, $x_1, x_2, \ldots, x_m$, not all of them 0, so that $\vec{0}=x_1\vec{v}_1+x_2\vec{v}_2+\ldots+x_m\vec{v}_m$. As in part a, we choose A to be the matrix whose columns are the given vectors and we let $\vec{x}$ be the vector whose components are the scalars. Then, the list of vectors is linearly independent if and only if the equation $A\vec{x}=\vec{0}$ has a solution other than $\vec{x}=\vec{0}$. Notice, too, that the problem of finding whether a list of vectors is linearly independent or not is equivalent to determining whether the matrix whose columns is the given list has a nontrivial kernel.