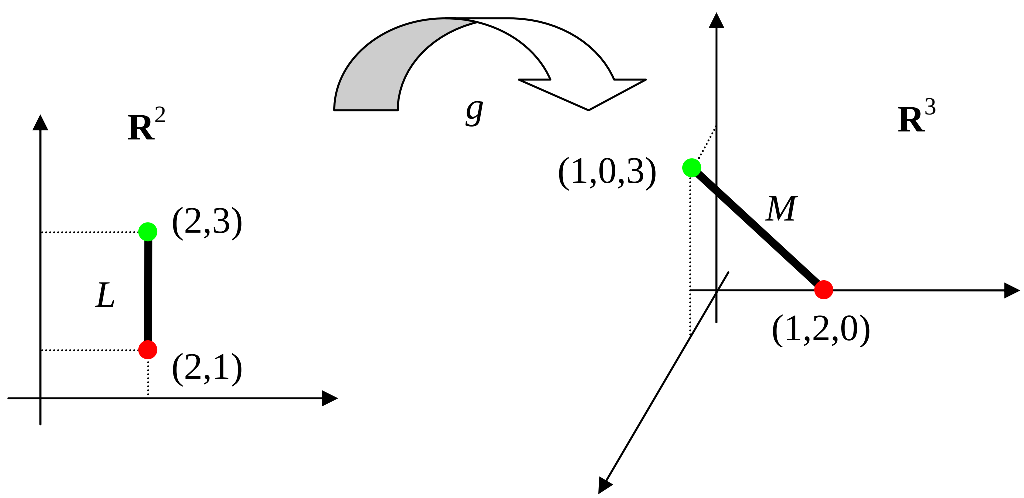
Practice Problems for Test 1

1. $\vec{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ are fixed vectors in $\mathbf{R}^n$ that are orthogonal. So, $\vec{a} \cdot \vec{b} = 0$. $f : \mathbf{R}^n \to \mathbf{R}^n$ is defined by $f(\vec{x}) = (\vec{b} \cdot \vec{x})\vec{a}$ for any $\vec{x}$ in $\mathbf{R}^n$.

   a. Demonstrate that f is linear.
   b. Show that $f(f(\vec{x})) = \vec{0}$ for any $\vec{x}$ in $\mathbf{R}^n$.
   c. Determine the matrix A for f.
   d. Part b implies that A has a certain algebraic property. What is it?
   e. Illustrate part d for $n = 3$, $\vec{a} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$.

2. The linear transformation $g : \mathbf{R}^2 \to \mathbf{R}^3$ maps the line segment L to the line segment M as shown in the diagram below.



Determine $g\left(\begin{bmatrix} r \\ s \end{bmatrix}\right)$ for any r and s in $\mathbf{R}$.

Solution to Practice Problems for Test 1

1. a. For any real r and s, and any vectors $\vec{x}$ and $\vec{y}$ in $\mathbf{R}^n$, we have

$$f(r\,\vec{x}+s\,\vec{y})=(\vec{b}\cdot(r\,\vec{x}+s\,\vec{y}))\vec{a}=r(\vec{b}\cdot\vec{x})\vec{a}+s(\vec{b}\cdot\vec{y})\vec{a}=r\,f(\vec{x})+s\,f(\vec{y}).$$

Here we used that fact that the scalar (dot) product is distributive over sums and multiples. This result shows that f is linear.

b. By direct substitution we find:

$$f(f(\vec{x}))=f((\vec{b}\cdot\vec{x})\vec{a})=(\vec{b}\cdot((\vec{b}\cdot\vec{x})\vec{a}))\vec{a}=((\vec{b}\cdot\vec{x})(\vec{b}\cdot\vec{a}))\vec{a}=0\,\vec{a}=\vec{0}.$$

To paraphrase this result, we have shown that the image under f of a linear combination of any two vectors is the corresponding linear combination of the images under f of the two vectors.

c. The kth column of A is the image of the kth standard basis vector in $\mathbf{R}^n$; i.e. $f(\hat{e}_k)=(\vec{b}\cdot\hat{e}_k)\vec{a}=(b_k)\vec{a}=\begin{bmatrix} a_1b_k \\ \vdots \\ a_nb_k \end{bmatrix}$. So, $A=\begin{bmatrix} a_1b_1 & \cdots & a_1b_n \\ \vdots & \ddots & \vdots \\ a_nb_1 & \cdots & a_nb_n \end{bmatrix}$.

d. $f(f(\vec{x}))=A(A\vec{x})=A^2\,\vec{x}$. So, the square of A must be the zero matrix, i.e. $A^2=A\,A=0_n$.

e. $A=\begin{bmatrix} 2 & -2 & 1 \\ 4 & -2 & 2 \\ 4 & -4 & 2 \end{bmatrix} \Rightarrow A\,A=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Note that this is rather different from the situation in $\mathbf{R}$ where, if the product of two scalars is 0, then at least one of the scalars must b zero.

2. Since g is linear, there is a 3×2 matrix B so that $g(\vec{x})=B\,\vec{x}$. From the diagram, we have $g\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)=B\begin{bmatrix} 2 \\ 1 \end{bmatrix}=\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $g\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right)=B\begin{bmatrix} 2 \\ 3 \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ or

$$B\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}=\begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 3 \end{bmatrix} \Rightarrow B=\begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}^{-1}=\begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 3 \end{bmatrix}\frac{1}{4}\begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}=\frac{1}{4}\begin{bmatrix} 2 & 0 \\ 6 & -4 \\ -3 & 6 \end{bmatrix}$$

Therefore, $g\left(\begin{bmatrix} r \\ s \end{bmatrix}\right)=B\begin{bmatrix} r \\ s \end{bmatrix}=\frac{1}{4}\begin{bmatrix} 2 & 0 \\ 6 & -4 \\ -3 & 6 \end{bmatrix}\begin{bmatrix} r \\ s \end{bmatrix}=\begin{bmatrix} \frac{1}{2}r \\ \frac{3}{2}r-s \\ -\frac{3}{4}r+\frac{3}{2}s \end{bmatrix}$.