This morning, I asserted that a linear function $f:\mathbf{R}^n \to \mathbf{R}^m$ cannot be invertible unless $n = m$. Prove this.

Proceed as follows. First observe that f is linear means that there is an $m \times n$ [on the original, there was a typo with m and n transposed.] matrix A such that $f(\vec{x}) = A\vec{x}$ for every $\vec{x} \in \mathbf{R}^n$. Next, observe that f is invertible means that the equation $f(\vec{x}) = \vec{b}$ has a unique solution $\vec{x} \in \mathbf{R}^n$ for every $\vec{b} \in \mathbf{R}^m$. Now, using what you learned in Chapter 1, demonstrate the following.

(a) If $n > m$, then $f(\vec{x}) = \vec{b}$ cannot have a unique solution $\vec{x} \in \mathbf{R}^n$ for any $\vec{b} \in \mathbf{R}^m$.

In this case, the rank of A is, at most, m and so A must have at least one non-pivot column. Hence, either there are no solutions, if the last column of $[A|\vec{b}]_{rref}$ is a pivot column, or there are infinitely many solutions.

(b) If $n < m$, then there are some $\vec{b} \in \mathbf{R}^m$ for which $f(\vec{x}) = \vec{b}$ has no solution.

In this case, the rank of A is, at most n and so, A has at least one row of zeros. It is always* possible to choose $\vec{b}$ in $\mathbf{R}^m$ so that the last column of $[A|\vec{b}]_{rref}$ is a pivot column and $A\vec{x} = \vec{b}$ has no solutions.

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* This result has been alluded to before. Here is a proof. Suppose that $r_1, r_2, \ldots, r_p$ is a sequence of elementary row operations that changes A to A_{rref}. We'll write $r(B)$ to indicate that the elementary row operation r has been applied to the matrix B. So, $r_p\left(\ldots\left(r_2\left(r_1\left(A\right)\right)\right)\ldots\right) = A_{rref}$. Now, each elementary row operation has an inverse elementary row operation that "undoes" what the original did. We will denote the inverse of r by r^{-1}. If the first zero row of A_{rref} is the kth row, let $\vec{b} = r_1^{-1}\left(r_2^{-1}\left(\ldots r_p^{-1}\left(\hat{e}_k\right)\ldots\right)\right)$.

Applying the sequence of elementary row operations above to $[A|\vec{b}]$ yields

$$
\begin{aligned}
[A|b]_{rref} &= \left[r_p\left(\ldots\left(r_2\left(r_1\left(A\right)\right)\right)\ldots\right) | r_p\left(\ldots\left(r_2\left(r_1\left(\vec{b}\right)\right)\right)\ldots\right)\right] \\
&= \left[A_{rref} | r_p\left(\ldots\left(r_2\left(r_1\left(r_1^{-1}\left(r_2^{-1}\left(\ldots r_p^{-1}\left(\hat{e}_k\right)\ldots\right)\right)\right)\right)\right)\ldots\right)\right] \\
&= [A_{rref} | \hat{e}_k]
\end{aligned}
$$

But this means that the kth row contains a pivot in the last column.