# FALL 2012 FINAL EXAM FOR SM261
# 1330 DECEMBER 18, 2012

**SHOW ALL WORK**

**PART 1.** On this part (problems 1-5) you may **NOT** use a calculator.

**1.** Complete the following definitions.

**a.** If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ are vectors in $\mathbb{R}^n$, then $\text{span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m)$ is...

**b.** If $A$ is an $n \times n$ matrix, an eigenvector of $A$ is ...

**c.** If $A$ is an $n \times m$ matrix, then a least-squares solution of the linear system $A\vec{x} = \vec{b}$ is...

**2.** Define $T : \mathbb{R}^4 \to \mathbb{R}^3$ by $T\left[\begin{smallmatrix} x \\ y \\ z \\ w \end{smallmatrix}\right] = \left[\begin{smallmatrix} x-2y+z-3w \\ y+z+w \\ x-y+2z-2w \end{smallmatrix}\right]$.

**a.** Find a matrix $A$ so that $T(\vec{v}) = A\vec{v}$ for all vectors $\vec{v} \in \mathbb{R}^4$.

**b.** Find the reduced row-echelon form of the augmented matrix of the linear system $A\vec{v} = \left[\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}\right]$. Show all steps.

**c.** Describe all solutions to the linear system $A\vec{v} = \left[\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}\right]$.

**d.** Is every linear combination of solutions to the linear system $A\vec{v} = \left[\begin{smallmatrix}1\\2\\3\end{smallmatrix}\right]$ also a solution? Explain.

**3.** Find the determinant of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 2 & 1 & 0 \\ 0 & 4 & 3 & 0 \\ 3 & 0 & 0 & 4 \end{bmatrix}.$$

**4.** Let $V$ be the subspace of $\mathbb{R}^3$ defined by $V = \{\left[\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right] \in \mathbb{R}^3 | x + 2y + 3z = 0\}$.
Find a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ so that $\text{im}(T) = V$.

**5.** Let $T\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} x+4y \\ 3x+2y \end{smallmatrix}\right]$. Let $S = \{\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \mathbb{R}^2 | 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$.

Sketch $S$ and $T(S)$.

This is the end of Part 1. Turn in your answers and proceed to Part 2.

**PART 2.** You may use a calculator on this part. Show all your work.

**6.** Define the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ by $T(\vec{e_k}) = \left[\begin{smallmatrix} 1 \\ k \\ k^2 \end{smallmatrix}\right]$ for $k = 1, 2, 3, 4$. Find the matrix of $T$.

**7.** Let $T$ and $U$ be linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$ defined as follows. $T$ multiplies each vector by the scalar $\sqrt{2}$. $U$ is a counter-clockwise rotation by $45°$. Find the matrix of the linear transformation $(T \circ U)(\vec{v}) = T(U(\vec{v}))$.

**8.** Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which is reflection through the line $x + \sqrt{3}y = 0$. Find the matrix of $T$.

**9.** Let $A$ be the matrix

$$\begin{bmatrix} 1 & 2 & 2 & 3 & 3 \\ 3 & 6 & 6 & 9 & 9 \\ 1 & 4 & 6 & 1 & -1 \\ 1 & 1 & 2 & 2 & 3 \end{bmatrix}.$$

We have

$$\mathrm{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 3 & 5 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

**a.** Find a basis for $\ker(A)$. Find $\dim(\ker(A))$.

**b.** Find a basis for $\mathrm{im}(A)$. Find $\mathrm{rank}(A)$.

**10.** Let $V = \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$, where $\vec{v}_1 = \begin{bmatrix} 1 \\ 7 \\ 2 \\ 5 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 3 \\ -4 \\ 1 \\ 5 \end{bmatrix}$, and $\vec{v}_4 = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 4 \end{bmatrix}$.

**a.** Find a basis for $V$.

**b.** Is the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ in $V$? Explain.

**11.** Find a basis $\mathcal{B}$ of $\mathbb{R}^2$ such that the $\mathcal{B}-$matrix of $T(\vec{v}) = \begin{bmatrix} -5 & -9 \\ 4 & 7 \end{bmatrix} \vec{v}$ is $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

**12.** Let $V$ be the subspace of $\mathbb{R}^4$ with basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Use the Gram-Schmidt process to find an orthonormal basis for $V$.

**13.** Let $V$ be the subspace of $\mathbb{R}^3$ with orthonormal basis $\left\{ \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}, \begin{bmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix} \right\}$.

Find the matrix of the orthogonal projection onto $V$.

**14.** Using least squares, fit a function $f(t) = a + bt$ to the data points $(0, 1)$, $(1, 3)$, $(1, 5)$.

**15.** Let $P$ be the parallelepiped in $\mathbb{R}^3$ defined by the vectors

$$\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

**a.** Find the volume of $P$.

**b.** Let $T$ be the linear tranformation given by $T\left[\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right] = \left[\begin{smallmatrix} 2x-y \\ y+3z \\ x-y+z \end{smallmatrix}\right]$. Find the volume of $T(P)$.

**16.** True or False. If true, give a reason. If false, give a reason or a counterexample.

**a.** If $T : \mathbb{R}^7 \to \mathbb{R}^6$ is a linear transformation, then $\ker(T) \neq 0$.

**b.** If $\{\vec{v_1}, \vec{v_2}\}$ is a set of two linearly independent vectors in $\mathbb{R}^4$, then the set $\{c_1\vec{v_1} + c_2\vec{v_2}\}$ of all linear combinations forms a basis of a two dimensional subspace of $\mathbb{R}^4$.

**c.** The matrix of an orthogonal projection onto a subspace in $\mathbb{R}^n$ is an orthogonal matrix.

**d.** All diagonalizable matrices are invertible.

**e.** If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation with $T(\vec{v}) = \left[\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right]$ and $T(\vec{w}) = \left[\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}\right]$, then $T(\vec{v} + 2\vec{w}) = \left[\begin{smallmatrix} 4 \\ 2 \end{smallmatrix}\right]$.

**In Problems 17-18, do all calculations by hand and show all steps. You may check your work with a calculator.**

**17.** Let $A$ be the matrix

$$\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}.$$

**a.** Find the characteristic polynomial $f_A(\lambda)$ of $A$.

**b.** Find the eigenvalues of $A$.

**c.** For each eigenvalue $\lambda$, find a basis for the eigenspace $E_\lambda$.

**d.** Is $A$ diagonalizable? If so, find an eigenbasis for $A$ and a diagonal matrix $D$ which is similar to $A$. If not, give your reasons.

**18.** Let

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & -2 \\ 0 & 3 & -1 \end{bmatrix}.$$

$A$ and $B$ each have eigenvalues $\lambda = 1, 2, 2$. Determine which of these matrices is diagonalizable. For those which are, find an eigenbasis and a diagonal matrix $D$ which is similar. For those which are not, give your reasons.

**END OF EXAM**