1. Consider the linear system $\left\{\begin{array}{rcrcrcrcr} x & + & 2y & & & - & w & = & 0 \\ 2x & + & 6y & - & 3z & - & 3w & = & 3 \\ 3x & + & 10y & + & kz & - & 5w & = & 2 \end{array}\right\}$.

The system has the form $A\vec{x}=\vec{b}$. So, we apply row operations to $\left[A\,|\,\vec{b}\right]$

$$=\left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 0 \\ 2 & 6 & -3 & -3 & 3 \\ 3 & 10 & k & -5 & 2 \end{array}\right] \leftrightarrow \left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 0 \\ 0 & 2 & -3 & -1 & 3 \\ 0 & 4 & k & -2 & 2 \end{array}\right] \leftrightarrow \left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 0 \\ 0 & 2 & -3 & -1 & 3 \\ 0 & 0 & k+6 & 0 & -4 \end{array}\right].$$

For which real values of k does the system have

a. a unique solution?

There is never a unique solution since there is at least one nonpivot column of the coefficient matrix A for any value of k.

b. infinitely many solutions?

If $k \neq -6$, there are infinitely many solutions since the fourth column is a non-pivot column.

c. no solutions?

If $k = -6$, there are no solutions and the system is inconsistent since the last row in the last matrix above is equivalent to the illogical assertion that $0 = -4$.

2. Consider the matrix $A=\begin{bmatrix} 1 & 1 & -1 \\ 4 & 5 & 0 \\ 5 & 8 & 7 \\ -1 & -2 & -3 \end{bmatrix}$. Given that $\operatorname{rref}(A)=\begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$,

a. find a basis for $\operatorname{im}(A)$. What is the dimension of $\operatorname{im}(A)$?

By inspection of $\operatorname{rref}(A)$, we deduce that the first two columns of A comprise a linearly independent pair and the third column of A is a linear of the first two. So, a basis for $\operatorname{im}(A)$ is $\left(\begin{bmatrix} 1 \\ 4 \\ 5 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 8 \\ -2 \end{bmatrix}\right)$ and $\dim(\operatorname{im}(A)) = 2$.

b. find a basis for $\ker(A)$. What is the dimension of $\ker(A)$?

Applying the Solution Algorithm to $\operatorname{rref}(A)$, we find a basis for $\ker(A)$ to be $\left(\begin{bmatrix} -5 \\ 4 \\ -1 \end{bmatrix}\right)$ and it follows that $\dim(\ker(A)) = 1$.

3. State whether each of the following statements is true or false. Justify your answer.

a. If A is a 3×5 matrix, then there must exist at least two linearly independent vectors in ker(A).

True. rank(A) $\leq$ min(3,5) = 3 and, by the Rank-Nullity Theorem, dim(ker(A)) = nullity(A) $\geq 5 - 3 = 2$. So, ker(A) has at least two linearly independent vectors.

b. If B is a 4×3 matrix and the system $B\vec{x}=\vec{0}$ has a unique solution, then for every vector $\vec{b}$, the system $B\vec{x}=\vec{b}$ also has a unique solution.

False. $B\vec{x}=\vec{b}$ may have no solution at all if $\vec{b}\notin \mathrm{im}(B)$.

c. If C is a 4×3 matrix and, for some vector $\vec{c}$, the system $C\vec{x}=\vec{c}$ has a unique solution, then the system $C\vec{x}=\vec{0}$ also has a unique solution.

True. If $C\vec{x}=\vec{c}$ has a unique solution for some $\vec{c}$, we may conclude that $\vec{c}\in \mathrm{im}(C)$ and that $\ker(C)$ is trivial. Since $\vec{0}_4\in \mathrm{im}(C)$, $C\vec{x}=\vec{0}_4$ is consistent and since ker(C) is trivial, that solution is unique; it is $\vec{0}_3$.

4. Let $T:\mathbf{R}^2\to\mathbf{R}^3$ be a linear transformation such that

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}1\\2\\0\end{bmatrix} \text{ and } T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}1\\-1\\1\end{bmatrix}.$$

Consider also the linear transformation $S:\mathbf{R}^3\to\mathbf{R}^2$ given by

$$S\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)=\begin{bmatrix}2x-y-4z\\x-z\end{bmatrix}.$$

a. Find the matrix A of T and the matrix B of S.

The images, under T, of the standard basis vectors are the corresponding columns of A. Therefore, $A=\begin{bmatrix}1&1\\2&-1\\0&1\end{bmatrix}$. From the definition of S, $B=\begin{bmatrix}2&-1&-4\\1&0&-1\end{bmatrix}$.

b. Show that the map $Q:\mathbf{R}^2\to\mathbf{R}^2$ given by $Q(\vec{x})=S(T(\vec{x}))$ is a rotation in the plane and determine the angle of the rotation.

Let C be the matrix of Q which, by definition, is the composite of T followed by

S. Therefore, $C = BA = \begin{bmatrix} 2 & -1 & -4 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. This belongs to the family of rotation matrices $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Here, the rotation angle is $\theta = \pi/2$.

5. Consider the vectors $\vec{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ in $\mathbf{R}^2$.

a. Explain why the set $\mathfrak{B} = \{\vec{v}_1, \vec{v}_2\}$ is a basis for $\mathbf{R}^2$.

By inspection, $\vec{v}_1$ and $\vec{v}_2$ are a linearly independent pair and so comprise a basis.

b. Find the vector $\vec{y}$ in $\mathbf{R}^2$ whose coordinate vector with respect to $\mathfrak{B}$ is $[\vec{y}]_{\mathfrak{B}} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

Let $S = [\vec{v}_1 \,|\, \vec{v}_2] = \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}$. Then, $\vec{y} = S\,[\vec{y}]_{\mathfrak{B}} = \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$.

c. Let $\vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Find $[\vec{x}]_{\mathfrak{B}}$.

$$[\vec{x}]_{\mathfrak{B}} = S^{-1}\vec{x} = \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \tfrac{1}{-2} \begin{bmatrix} 0 & -1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

d. Suppose that $T : \mathbf{R}^2 \to \mathbf{R}^2$ is a linear transformation such that

$$T(\vec{v}_1) = \vec{v}_1 - 2\vec{v}_2 \quad \text{and} \quad T(\vec{v}_2) = \vec{v}_1.$$

Find:

i. The matrix B of T with respect to the basis $\mathfrak{B}$.

$$B = \left[[T(\vec{v}_1)]_{\mathfrak{B}} \,|\, [T(\vec{v}_2)]_{\mathfrak{B}} \right] = \left[[\vec{v}_1 - 2\vec{v}_2]_{\mathfrak{B}} \,|\, [\vec{v}_1]_{\mathfrak{B}} \right] = [\hat{e}_1 - 2\hat{e}_2 \,|\, \hat{e}_1] = \begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix}.$$

ii. The standard matrix A of T.

$$A = S\,B\,S^{-1} = \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix} \tfrac{1}{2} \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix}.$$

6. Let V be the subspace of $\mathbf{R}^4$ spanned by the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix}$.

a. Find an orthonormal basis for V.

Let $\vec{v}_1$ and $\vec{v}_2$ be the given vectors, respectively. Applying the Gram-Schmidt orthogonalization, we find the following. $\hat{u}_1 = \vec{v}_1 / \|\vec{v}_1\| = \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$. Then,

$$\vec{v}_2^{\perp} = \vec{v}_2 - (\vec{v}_2 \cdot \hat{u}_1)\hat{u}_1 = \begin{bmatrix}2\\0\\0\\2\end{bmatrix} - (2)\tfrac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}1\\-1\\-1\\1\end{bmatrix} \text{ and } \hat{u}_2 = \vec{v}_2^{\perp} / \|\vec{v}_2^{\perp}\| = \tfrac{1}{2}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}.$$

So, our orthonormal basis for V is $\mathfrak{B} = (\hat{u}_1, \hat{u}_2)$.

b. Find the matrix P of the orthogonal projection onto V.

$$P = \hat{u}_1\, \hat{u}_1^T + \hat{u}_2\, \hat{u}_2^T = \tfrac{1}{4}\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix} + \tfrac{1}{4}\begin{bmatrix}1&-1&-1&1\\-1&1&1&-1\\-1&1&1&-1\\1&-1&-1&1\end{bmatrix} = \tfrac{1}{2}\begin{bmatrix}1&0&0&1\\0&1&1&0\\0&1&1&0\\1&0&0&1\end{bmatrix}.$$

c. Find the orthogonal projection of the vector $\begin{bmatrix}0\\0\\0\\4\end{bmatrix}$ onto V.

The projection of $\begin{bmatrix}0\\0\\0\\4\end{bmatrix}$ onto V is $P\begin{bmatrix}0\\0\\0\\4\end{bmatrix} = \begin{bmatrix}2\\0\\0\\2\end{bmatrix}$.

d. Find the matrix of the orthogonal projection onto $V^{\perp}$.

The projection matrix is $P^{\perp} = I - P = \frac{1}{2}\begin{bmatrix}1&0&0&-1\\0&1&-1&0\\0&-1&1&0\\-1&0&0&1\end{bmatrix}$.

7. Let W be the subspace of $\mathbf{R}^3$ consisting of all vectors perpendicular to $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

a. Find a basis for W.

Let $A = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. Then, $\operatorname{im}(A)$ is the span of this vector and

$W = \left(\operatorname{im}(A)\right)^{\perp} = \ker(A^T) = \ker[1 \ \ -2 \ \ 1] = \operatorname{span}\left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right)$. In the last step, we used the Solution Algorithm. The desired basis is $\left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right)$.

b. Find a basis for $W^{\perp}$.

$W^{\perp} = \left(\operatorname{im}(A)\right)^{\perp\perp} = \operatorname{im}(A) = \operatorname{span}\left(\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right)$ and so, a basis for $W^{\perp}$ is $\left(\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right)$.

c. Suppose that $T : \mathbf{R}^3 \to \mathbf{R}^3$ is the reflection across W and let A be the matrix for T. Find all the eigenvalues and a basis for each eigenspace of A.

The eigenvalues of A are $+1$ and -1. W is left invariant by reflection across it. This 2-dimensional plane contains all the eigenvectors of A with eigenvalue 1. That is, $E_1(A) = W$. The normal line to W is $W^{\perp}$ and it is the eigenspace for the eigenvalue -1. Every vector in $W^{\perp}$ is reversed by the reflection. So, $E_{-1}(A) = W^{\perp}$.

d. Find an invertible matrix S and a diagonal matrix D such that $A = S\,D\,S^{-1}$.

A diagonalizer S for A is any 3×3 matrix whose columns consist of an eigenbasis.

So, we choose $S = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -2 \\ 0 & -1 & 1 \end{bmatrix}$ and then $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

e. Explain why A is invertible and compute A^{-1}.

Reflections are their own inverses since $A\,A = I$. Therefore, $A^{-1} = S\,D\,S^{-1}$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -2 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -2 \\ 0 & -1 & 1 \end{bmatrix}^{-1} = \tfrac{1}{3} \begin{bmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{bmatrix}.$$

8. Determine whether each of the following statements is true or false. Justify your answers.

a. The function $T : \mathbf{R}^2 \to \mathbf{R}^2$ defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ 1-x \end{bmatrix}$ is a linear transformation.

False. T is not linear, since T does not map the zero vector to the zero vector $T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. We could also verify that T does not preserve linear combinations.

b. If the non-zero vectors $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ satisfy the relation $\vec{v}_1 + 2\vec{v}_2 + 3\vec{v}_3 = 4\vec{v}_1 + 5\vec{v}_2 + 6\vec{v}_3$ then they must be linearly dependent.

True. Subtracting the left from the right side, we find $\vec{v}_1 + \vec{v}_2 + \vec{v}_3 = \vec{0}$. Hence, there is a nontrivial linear combination of these vectors that is the zero vector.

c. If A is a 4×4 matrix and $\det(A)$. = 4, then the rank of A must be 4.

True. Since the determinant is nonzero, the matrix is invertible and so its columns comprise a linearly independent quartet.

d. If A is a 2×2 matrix with eigenvalues 1 and 0, we must have $A^2 = A$.

True. Since the eigenvalues are distinct, the matrix is diagonalizable and so, A is similar to the diagonal matrix $D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. That is, there is an invertible matrix S so that $A = S^{-1} D S$. Consequently, $A^2 = S^{-1} D^2 S = S^{-1} D S = A$.

9. Suppose that $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ are the rows of a 3×3 matrix A; i.e. $A = \begin{bmatrix} \vec{v}_1 \\ \hline \vec{v}_2 \\ \hline \vec{v}_3 \end{bmatrix}$.

Suppose also that $\det(A) = 2$ and that B is a 3×3 matrix with $\det(B) = -3$. Compute the following, giving reasons for your answers.

a. $\det\left(\begin{bmatrix}\vec{v}_3\\ \hline \vec{v}_1\\ \hline \vec{v}_2\end{bmatrix}\right)=(-1)\det\left(\begin{bmatrix}\vec{v}_1\\ \hline \vec{v}_3\\ \hline \vec{v}_2\end{bmatrix}\right)=(-1)^2\det\left(\begin{bmatrix}\vec{v}_1\\ \hline \vec{v}_2\\ \hline \vec{v}_3\end{bmatrix}\right)=2.$ Each row swap introduces a factor of (-1).

b. $\det\left(\begin{bmatrix}\vec{v}_2-2\vec{v}_1\\ \hline \vec{v}_2\\ \hline \vec{v}_1-\vec{v}_3\end{bmatrix}\right)=\det\left(\begin{bmatrix}-2\vec{v}_1\\ \hline \vec{v}_2\\ \hline \vec{v}_1-\vec{v}_3\end{bmatrix}\right)=(-2)\det\left(\begin{bmatrix}\vec{v}_1\\ \hline \vec{v}_2\\ \hline \vec{v}_1-\vec{v}_3\end{bmatrix}\right)=(-2)\det\left(\begin{bmatrix}\vec{v}_1\\ \hline \vec{v}_2\\ \hline -\vec{v}_3\end{bmatrix}\right)$
$=(-1)(-2)\det\left(\begin{bmatrix}\vec{v}_1\\ \hline \vec{v}_2\\ \hline \vec{v}_3\end{bmatrix}\right)=4.$ Here, we used linearity of the determinant in each row.

c. $\det(3A)=3^3\det(A)=54$. Multiplying a 3×3 matrix by 3 multiplies each row by 3 and linearity of the determinant in each row yields the result.

d. $\det(B\,A\,B^T)=\det(B)\det(A)\det(B^T)=\det(B)\det(A)\det(B)=(-3)(2)(-3)=18.$

e. $\operatorname{rank}(A)=3$ since $\det(A)=4\neq0$, A is an invertible 3×3 matrix.

10. Find the equation of the straight line $y=a\,x+b$ that best fits (in the least squares sense) the points $(1,1)$, $(2,-1)$ and $(3,2)$.
The corresponding linear system

$A\,\vec{p}=\vec{c}$ where $A=\begin{bmatrix}1&1\\2&1\\3&1\end{bmatrix}$, $\vec{p}=\begin{bmatrix}a\\b\end{bmatrix}$ and $\vec{c}=\begin{bmatrix}1\\-1\\2\end{bmatrix}$ has no solution since $\vec{c}\notin\operatorname{im}(A)$. Instead, we seek the solution to $A^TA\,\vec{p}=A^T\vec{c}$. Since the columns of A are linearly independent, A^TA will be invertible and so the solution we seek is

$$\begin{bmatrix}a\\b\end{bmatrix}=(A^TA)^{-1}A^T\vec{c}=\left(\begin{bmatrix}1&2&3\\1&1&1\end{bmatrix}\begin{bmatrix}1&1\\2&1\\3&1\end{bmatrix}\right)^{-1}\begin{bmatrix}1&2&3\\1&1&1\end{bmatrix}\begin{bmatrix}1\\-1\\2\end{bmatrix}=\begin{bmatrix}14&6\\6&3\end{bmatrix}^{-1}\begin{bmatrix}5\\2\end{bmatrix}$$

$$=\tfrac{1}{6}\begin{bmatrix}3&-6\\-6&14\end{bmatrix}\begin{bmatrix}5\\2\end{bmatrix}=\tfrac{1}{6}\begin{bmatrix}3\\-2\end{bmatrix}=\begin{bmatrix}\frac12\\-\frac13\end{bmatrix}.$$ The best fit equation is $y=\tfrac12x-\tfrac13$.

11. Consider the system of equations $\left\{\begin{array}{rcrcl}a\,x&-&4\,y&=&1\\9\,x&+&a\,y&=&3\end{array}\right\}$ where a is a parameter.

a. Prove that, for each value of a, this system has a unique solution.

This system is of the form $A\vec{x}=\vec{b}$ where $A=[\vec{v}_1 \mid \vec{v}_2]=\begin{bmatrix} a & -4 \\ 9 & a \end{bmatrix}$ and $\vec{b}=\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

Since $\det(A) = a^2+36>0$, A is invertible and the system has a unique solution.

b. Use Cramer's Rule to solve the system for each value of a.

$$x=\frac{\det\left(\left[\vec{b} \mid \vec{v}_2\right]\right)}{\det\left(\left[\vec{v}_1 \mid \vec{v}_2\right]\right)}=\frac{\det\left(\begin{bmatrix} 1 & -4 \\ 3 & a \end{bmatrix}\right)}{a^2+36}=\frac{a+12}{a^2+36},$$

$$y=\frac{\det\left(\left[\vec{v}_1 \mid \vec{b}\right]\right)}{\det\left(\left[\vec{v}_1 \mid \vec{v}_2\right]\right)}=\frac{\det\left(\begin{bmatrix} a & 1 \\ 9 & 3 \end{bmatrix}\right)}{a^2+36}=\frac{3a-9}{a^2+36}.$$

12. Let $A=\begin{bmatrix} .50 & .25 \\ .50 & .75 \end{bmatrix}$.

a. Find an invertible matrix S and a diagonal matrix D such that $A=SDS^{-1}$.

The eigenvalues of A are found by solving

$0=\det(A-\lambda I)=(.5-\lambda)(.75-\lambda)-(.5)(.25)=\lambda^2-1.25\lambda+.375-.125=$

$=\lambda^2-1.25\lambda+.25=(\lambda-1)(\lambda-.25)$. So, $\operatorname{spec}(A)=(1,\tfrac{1}{4})$.

The eigenspaces of A are found next.

$$E_1(A)=\ker(A-1I)=\ker\left(\begin{bmatrix} -.5 & .25 \\ .5 & -.25 \end{bmatrix}\right)=\ker\left(\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}\right)=\operatorname{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right).$$

$$E_{\frac{1}{4}}(A)=\ker\left(A-\tfrac{1}{4}I\right)=\ker\left(\begin{bmatrix} .25 & .25 \\ .50 & .50 \end{bmatrix}\right)=\ker\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right)=\operatorname{span}\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right).$$

So, we choose $S=\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$ and $D=\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$. Note $S^{-1}=\frac{1}{3}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$.

b. Compute the matrix A^n for any positive integer n.

$$A^n=(SDS^{-1})^n=SD^nS^{-1}=\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{bmatrix}^n\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}^{-1}=$$

$$\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4^{-n} \end{bmatrix}\frac{1}{3}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}=\frac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2\cdot 4^{-n} & 4^{-n} \end{bmatrix}=\frac{1}{3}\begin{bmatrix} 1+2\cdot 4^{-n} & 1-4^{-n} \\ 2-4^{-n} & 2+4^{-n} \end{bmatrix}.$$

c. Compute $\lim_{n\to\infty} A^n$.

Since $4^{-n} \to 0$ as $n \to \infty$, $\lim_{n\to\infty} A^n = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$.

d. Find a matrix B such that $B^2 = A$.

Although we found a formula for A^n in part b above that we knew was true for positive integer values of n, it is easy to see that, because the eigenvalues of A are both nonnegative, it is also valid when $n = \frac{1}{2}$. So, $B = \sqrt{A} = \sqrt{S\,D\,S^{-1}} = S\,\sqrt{D}\,S^{-1}$

$$B = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\frac{1}{3}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & \frac{1}{2} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1+1 & 1-\frac{1}{2} \\ 2-1 & 2+\frac{1}{2} \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 2 & \frac{1}{2} \\ 1 & \frac{5}{2} \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 4 & 1 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{1}{6} \\ \frac{1}{3} & \frac{5}{6} \end{bmatrix}.$$ It is easy to check that $B^2 = A$.

In this calculation, we chose the positive square roots of the two eigenvalues of A to compute $\sqrt{D}$. We would have gotten different results had we chosen one or both of the negative square roots. So, in fact, there are three other distinct matrices whose squares are A.