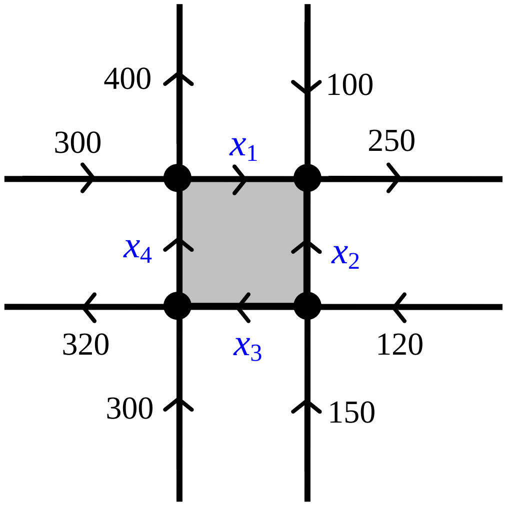
1. During a one hour period, the number of cars traversing certain one-way streets in a section of a city were carefully counted and noted on the diagram at the left. The total number of cars entering any of the four intersections matched the number leaving that intersection. The numbers of cars traversing the four streets on the boundary of the central square were not counted.



What were the numbers of cars traversing the four streets on the edges of the central square?

Let the number of cars that traversed the north edge of the square be x_1; let the number that traversed the east edge be x_2; let the number that traversed the south edge be x_3; and let the number that traversed the west edge be x_4. Using the principle of "conservation of cars" (number entering is number leaving or, to put it another way, the net number entering is zero), we have:

at the northwest intersection: $-x_1 + x_4 + 300 - 400 = 0$.

at the northeast intersection: $x_1 + x_2 + 100 - 250 = 0$.

at the southeast intersection: $-x_2 - x_3 + 120 + 150 = 0$.

at the southwest intersection: $x_3 - x_4 + 300 - 320 = 0$.

These equations are equivalent to the matrix equation $A\vec{x} = \vec{b}$ where

$$A = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} 100 \\ 150 \\ -270 \\ 20 \end{bmatrix}.$$ By Gauss elimination,

we find that $$[A\,|\,\vec{b}\,]_{rref} = \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -100 \\ 0 & 1 & 0 & 1 & 250 \\ 0 & 0 & 1 & -1 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \vec{x} = \begin{bmatrix} -100 \\ 250 \\ 20 \\ 0 \end{bmatrix} + k \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}.$$

Since each component of $\vec{x}$ must be a nonnegative integer (non-integer or negative numbers make no sense), the only allowed values for k are 100, 101, 102, …, and 250.

2. a. If the *ij*th entry of the $m\times n$ matrix A is a_{ij} and the *ij*th entry of the $n\times p$ matrix B is b_{ij}, express the rsth entry of the product matrix AB in terms of the entries of A and B. How many rows and columns are there in AB?

AB is an $m\times p$ matrix whose rsth entry is $a_{r1}b_{1s}+a_{r2}b_{2s}+\ldots+a_{rp}b_{ps}$.

b. Suppose the row vectors of the matrix A are $\vec{r}_1,\ldots,\vec{r}_m$ and the column vectors of A are $\vec{c}_1,\ldots,\vec{c}_n$ so that $A=\begin{bmatrix}\vec{r}_1\\ \hline \vdots \\ \hline \vec{r}_m\end{bmatrix}=[\vec{c}_1\mid\ldots\mid\vec{c}_n]$. Provide an expression for $A\vec{x}$ in terms of the row vectors of A and another expression for $A\vec{x}$ in terms of the column vectors of A. Here, $\vec{x}=\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}$.

$$A\vec{x}=\begin{bmatrix}\vec{r}_1\cdot\vec{x}\\ \hline \vdots \\ \hline \vec{r}_m\cdot\vec{x}\end{bmatrix}=x_1\vec{c}_1+\ldots+x_n\vec{c}_n.$$

c. Suppose that s is any positive integer, C is any $m\times n$ matrix, and $\vec{b}_1,\ldots,\vec{b}_s$ are any vectors in $\mathbf{R}^n$. Is the following formula always true?

$$C\left[\vec{b}_1\mid\cdots\mid\vec{b}_s\right]=\left[C\vec{b}_1\mid\cdots\mid C\vec{b}_s\right]$$

The formula is always true.

3. Suppose that $\vec{v}_1,\vec{v}_2,$ and $\vec{v}_3$ are three nonzero vectors in $\mathbf{R}^2$. Let M = $[\vec{v}_1\mid\vec{v}_2\mid\vec{v}_3]_{rref}$, the row-reduced echelon form of the matrix whose column vectors are $\vec{v}_1,\vec{v}_2,$ and $\vec{v}_3$.

a. Explain why no column vector of M is $\begin{bmatrix}0\\0\end{bmatrix}$.

By hypothesis, no column of $[\vec{v}_1\mid\vec{v}_2\mid\vec{v}_3]$ is $\begin{bmatrix}0\\0\end{bmatrix}$. Clearly, neither swapping two rows nor multiplying any row of $[\vec{v}_1\mid\vec{v}_2\mid\vec{v}_3]$ by a nonzero constant will result in a column of all zeros. But, adding any multiple of one row to another cannot result in zeros for both entries of any column. So, elementary row operations can produce a column of zeros if and only if the original matrix had a corresponding column consisting entirely of zeros.

b. Explain how M determines $\vec{v}_3$ as a linear combination of $\vec{v}_1$ and $\vec{v}_2$.

$\vec{v}_3$ is a linear combination of $\vec{v}_1$ and $\vec{v}_2$ if and only if there are real numbers x_1 and x_2 such that $\vec{v}_3 = x_1\vec{v}_1 + x_2\vec{v}_2$ and this is true if and only if the matrix equation $[\vec{v}_1 \,|\, \vec{v}_2]\,\vec{x} = \vec{v}_3$ has a solution $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. The solutions, if any, are determined by finding $M = [\vec{v}_1 \,|\, \vec{v}_2 \,|\, \vec{v}_3]_{rref}$.

c. Assume that $\vec{v}_1$ and $\vec{v}_2$ are non-collinear. What are the possible forms of M and what does each imply about $\vec{v}_3$ as a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

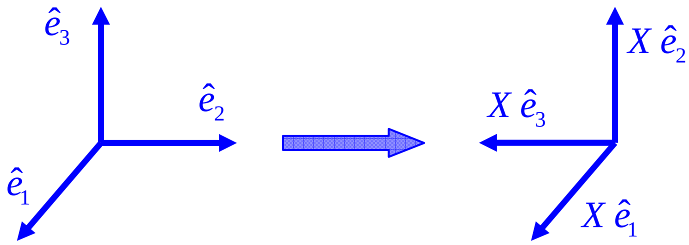
$M = \begin{bmatrix} 1 & 0 & r \\ 0 & 1 & s \end{bmatrix}$ where r and s are arbitrary reals $\Rightarrow$ $\vec{x} = \begin{bmatrix} r \\ s \end{bmatrix}$ and $\vec{v}_3 = r\vec{v}_1 + s\vec{v}_2$ is a unique linear combination of $\vec{v}_1$ and $\vec{v}_2$.

d. Now, assume that $\vec{v}_1$ and $\vec{v}_2$ are collinear. What are the possible forms of M and what does each imply about $\vec{v}_3$ as a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

One form is $M = \begin{bmatrix} 1 & r & s \\ 0 & 0 & 0 \end{bmatrix}$ where r and s are nonzero reals and $\vec{x} = \begin{bmatrix} s \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} r \\ -1 \end{bmatrix}$ $\Rightarrow$ $\vec{v}_3 = (s + \alpha r)\vec{v}_1 - \alpha\vec{v}_2$ for any α in $\mathbf{R}$. In this case, there are infinitely many linear combinations of $\vec{v}_1$ and $\vec{v}_2$ that sum to $\vec{v}_3$. This is the situation that obtains when all three vectors are collinear. The only other form is $M = \begin{bmatrix} 1 & r & 0 \\ 0 & 0 & 1 \end{bmatrix}$ with r a nonzero real $\Rightarrow$ there are no linear combinations of $\vec{v}_1$ and $\vec{v}_2$ that sum to $\vec{v}_3$. This situation obtains when $\vec{v}_3$ is not collinear with $\vec{v}_1$ and $\vec{v}_2$.

4. a. Determine, with the aid of diagrams, the 3×3 matrices that represent:
[The columns of the matrices are the results of applying the linear transformations to $\hat{e}_1$, $\hat{e}_2$ and $\hat{e}_3$]

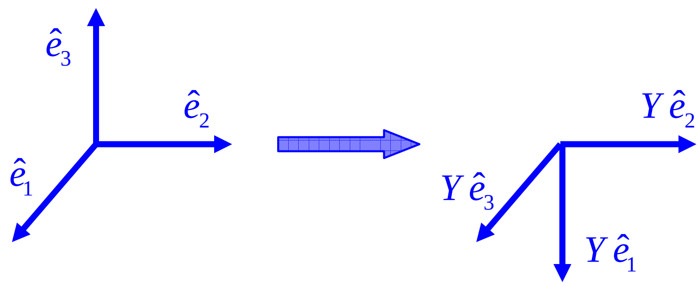
(i) rotation about the x-axis by $\frac{\pi}{2}$ [ccw when looking toward the origin from the point (1,0,0)].
Denote the transformation matrix by X.



So, $X=\begin{bmatrix}1&0&0\\0&0&-1\\0&1&0\end{bmatrix}$.

(ii)rotation about the *y*-axis by $\frac{\pi}{2}$ [ccw when looking toward the origin from the point (0,1,0)].

Denote the transformation matrix by Y.



So, $Y=\begin{bmatrix}0&0&1\\0&1&0\\-1&0&0\end{bmatrix}$.

b. Determine the matrix for the composite transformation that is the transformation in part a(ii) followed by the transformation in part a(i).

$$X\,Y=\begin{bmatrix}1&0&0\\0&0&-1\\0&1&0\end{bmatrix}\begin{bmatrix}0&0&1\\0&1&0\\-1&0&0\end{bmatrix}=\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$$

c. Describe, in geometrical terms, the composite linear transformation in part b by examining its matrix.

The column vectors of the matrix $X\,Y$ for this composite are of unit length and are mutually perpendicular. The transformation sends the *x*-axis to the *y*-axis, the *y*-axis to the *z*-axis and the *z*-axis to the *x*-axis. This is a rotation by $\frac{2\pi}{3}$ about the line through the origin parallel to $\begin{bmatrix}1\\1\\1\end{bmatrix}$.

5. Let $A=\frac{1}{3}\begin{bmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{bmatrix}$, $\vec{v}_1=\begin{bmatrix}1\\1\\1\end{bmatrix}$, $\vec{v}_2=\begin{bmatrix}-2\\1\\1\end{bmatrix}$, $\vec{v}_3=\begin{bmatrix}1\\-2\\1\end{bmatrix}$. Define $f:\mathbf{R}^3\to\mathbf{R}^3$ by $f(\vec{x})=A\vec{x}$ for any $\vec{x}\in\mathbf{R}^3$.

a. Show that $\vec{v}_1$ is perpendicular to both $\vec{v}_2$ and $\vec{v}_3$.

$\vec{v}_1\cdot\vec{v}_2=-2+1+1=0$ and $\vec{v}_1\cdot\vec{v}_3=1-2+1=0$.

b. What is $f(\alpha_1 \vec{v}_1)$ for any α_1 in $\mathbf{R}$?

By linearity, $f(\alpha_1 \vec{v}_1) = A\alpha_1 \vec{v}_1 = \alpha_1 A\vec{v}_1 = \vec{0}$. So, f "annihilates" all vectors parallel to $\vec{v}_1$.

c. What is $f(\alpha_2 \vec{v}_2 + \alpha_3 \vec{v}_3)$ for any α_2, α_3 in $\mathbf{R}$?

Since direct calculation shows that $A\vec{v}_2 = \vec{v}_2$ and $A\vec{v}_3 = \vec{v}_3$, we have, by linearity, $f(\alpha_2 \vec{v}_2 + \alpha_3 \vec{v}_3) = \alpha_2 A\vec{v}_2 + \alpha_3 A\vec{v}_3 = \alpha_2 \vec{v}_2 + \alpha_3 \vec{v}_3$. So, f leaves all the vectors that are linear combinations of $\vec{v}_2$ and $\vec{v}_3$ unchanged. This is the plane through the origin that is parallel to $\vec{v}_2$ and $\vec{v}_3$.

d. Describe f in geometrical terms.

f is projection onto the plane through the origin parallel to $\vec{v}_2$ and $\vec{v}_3$.

6. Calculate the 2×2 matrix that represents the linear transformation on $\mathbf{R}^2$ that doubles all vectors parallel to $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and halves all vectors perpendicular to $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$. Also, find the image of $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ under this transformation.

The unit vector parallel to $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is $\hat{u} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and the matrix for projection onto $\hat{u}$ is $P = \hat{u}\,\hat{u}^T = \frac{1}{5}\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$. Let A be the matrix for this linear transformation. For any vector $\vec{x} \in \mathbf{R}^2$, $\vec{x} = P\vec{x} + (I-P)\vec{x}$. So, by linearity, $A\vec{x} = AP\vec{x} + A(I-P)\vec{x} = 2P\vec{x} + \frac{1}{2}(I-P)\vec{x}$. Therefore,

$$A = 2P + \tfrac{1}{2}(I-P) = \tfrac{2}{5}\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} + \tfrac{1}{10}\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0.8 & -0.6 \\ -0.6 & 1.7 \end{bmatrix}.$$

Under this transformation, the image of $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is $A\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0.8 & -0.6 \\ -0.6 & 1.7 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -0.2 \\ 3.9 \end{bmatrix}$.

Here is an alternate approach to the same problem. By hypothesis, we are given that $A\begin{bmatrix} 1 \\ -2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$. A vector perpendicular to $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. We also know that $A\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$. Putting these two vector equations together as a single matrix equation, we have the following.

$A\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -4 & \frac{1}{2} \end{bmatrix}$. To solve for A, we multiply (on the right) both sides of the last equation by the inverse of $\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$. We obtain

$$A = \begin{bmatrix} 2 & 1 \\ -4 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}^{-1} = \tfrac{1}{2}\begin{bmatrix} 4 & 2 \\ -8 & 1 \end{bmatrix}\tfrac{1}{5}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \tfrac{1}{10}\begin{bmatrix} 8 & -6 \\ -6 & 17 \end{bmatrix} = \begin{bmatrix} 0.8 & -0.6 \\ -0.6 & 1.7 \end{bmatrix},$$

as before.

7. Solve for the matrix X if $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}^{-1} X \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

$$X = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}^{-1} = \left(\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\right)\tfrac{1}{3}\begin{bmatrix} 1 & 0 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 7 \\ 0 & 1 \end{bmatrix}.$$