1. For each of the following, indicate whether the statement is True or False by circling the T or F preceding the statement.

a. If the kernel of a matrix consists of the zero vector only, the matrix is invertible.

False. If the matrix is not square, it cannot be invertible. That the kernel is trivial shows that only the zero vector is orthogonal to all the row vectors of the matrix or, equivalently, that the list of column vectors is linearly independent.

b. The determinant of a diagonalizable square matrix is the product of its eigenvalues.

True. A diagonalizable matrix is similar to a diagonal matrix whose diagonal entries are the eigenvalues of the diagonalizable matrix.

c. An $n \times n$ matrix with n real eigenvalues is always diagonalizable.

False. If an eigenvalue is repeated and has geometric multiplicity less than its algebraic multiplicity, the matrix will not be diagonalizable.

d. If the 3×3 matrix A satisfies the equation $3A^2 - A = 2I$, A is invertible.

True. The equation may be rewritten as $A\frac{1}{2}(3A - I) = I$. Hence, $\frac{1}{2}(3A - I)$ is the inverse of A.

e. If the list $(\vec{u}, \vec{v}, \vec{w})$ of vectors in $\mathbf{R}^3$ is linearly dependent, then $\vec{u}$ must be a linear combination of $\vec{v}$ and $\vec{w}$.

False. For a counter-example, consider $\vec{u} = \hat{e}_1,\ \vec{v} = \hat{e}_2,\ \vec{w} = 2\hat{e}_2$

f. Linear combinations of eigenvectors of a square matrix are also eigenvectors of that matrix.

False. Linear combinations of eigenvectors corresponding to distinct eigenvalues are not, in general, eigenvectors.

2. Suppose that $\vec{r}, \vec{s}, \vec{t}$, and $\vec{u}$ are vectors in $\mathbf{R}^3$. Solve the matrix equation $\left[\vec{r} \mid \vec{s} \mid \vec{t}\right]\vec{x} = \vec{u}$ for $\vec{x}$ if $\det\left[\vec{r} \mid \vec{s} \mid \vec{t}\right] = 4$, $\det\left[\vec{s} \mid \vec{t} \mid \vec{u}\right] = 3$, $\det\left[\vec{t} \mid \vec{u} \mid \vec{r}\right] = 2$, and $\det\left[\vec{u} \mid \vec{r} \mid \vec{s}\right] = 1$.

Let $A = \left[\vec{r} \mid \vec{s} \mid \vec{t}\right]$ and let $\vec{b} = \vec{u}$. Then, the solution to $A\vec{x} = \vec{b}$ is given by Cramer's Rule, namely $x_k = \det(A_k)/\det(A)$ where A_k is the matrix obtained from A by replacing its kth column with $\vec{b}$. Employing the fact that the determinant of a matrix changes sign when two columns are interchanged, we have

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{\det\left[\vec{u} \mid \vec{s} \mid \vec{t}\right]}{\det\left[\vec{r} \mid \vec{s} \mid \vec{t}\right]} = \frac{-\det\left[\vec{s} \mid \vec{u} \mid \vec{t}\right]}{\det\left[\vec{r} \mid \vec{s} \mid \vec{t}\right]} = \frac{\det\left[\vec{s} \mid \vec{t} \mid \vec{u}\right]}{\det\left[\vec{r} \mid \vec{s} \mid \vec{t}\right]} = \frac{3}{4}$$

$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{\det\left[\vec{r} \mid \vec{u} \mid \vec{t}\right]}{\det\left[\vec{r} \mid \vec{s} \mid \vec{t}\right]} = \frac{-\det\left[\vec{t} \mid \vec{u} \mid \vec{r}\right]}{\det\left[\vec{r} \mid \vec{s} \mid \vec{t}\right]} = \frac{-2}{4} = -\frac{1}{2}$$

$$x_3 = \frac{\det(A_3)}{\det(A)} = \frac{\det\left[\vec{r} \mid \vec{s} \mid \vec{u}\right]}{\det\left[\vec{r} \mid \vec{s} \mid \vec{t}\right]} = \frac{-\det\left[\vec{u} \mid \vec{s} \mid \vec{r}\right]}{\det\left[\vec{r} \mid \vec{s} \mid \vec{t}\right]} = \frac{\det\left[\vec{u} \mid \vec{r} \mid \vec{s}\right]}{\det\left[\vec{r} \mid \vec{s} \mid \vec{t}\right]} = \frac{1}{4}$$

3. Suppose that $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ and $f : \mathbf{R}^3 \to \mathbf{R}^3$ is the linear transformation that doubles all vectors parallel to $\vec{v}$ and triples all vectors orthogonal to $\vec{v}$. Find the matrix for *f*.

Let P be the matrix for projection onto the 1-dimensional subspace spanned by $\vec{v}$ and let $\vec{x}$ be any vector in $\mathbf{R}^3$. Then, $\vec{x} = P\vec{x} + (I - P)\vec{x}$ resolves $\vec{x}$ into its vector components parallel and orthogonal to $\vec{v}$. So, $A\vec{x} = AP\vec{x} + A(I - P)\vec{x}$ $= 2P\vec{x} + 3(I - P)\vec{x} = (3I - P)\vec{x}$. Since $\vec{x}$ is arbitrary, $A = 3I - P$. Now, we can calculate the matrix for A. A unit vector parallel to $\vec{v}$ is $\hat{u} = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$. So, P = $\hat{u}\hat{u}^T$ and $A = 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 26 & -2 & -2 \\ -2 & 23 & -4 \\ -4 & -4 & 23 \end{bmatrix}$.

4. Consider the matrix $A = \frac{1}{7}\begin{bmatrix} 2 & 3 & 6 \\ 6 & 2 & -3 \\ -3 & 6 & -2 \end{bmatrix}$.

a. Verify that A is the matrix for a rotation.

Straightforward calculation shows that $\det(A)$ = +1 and A^TA = I. So, A is a proper orthogonal matrix and is, therefore a rotation.

b. Find the axis and the angle of this rotation.

The axis is characterized by the vectors left unchanged by the rotation. These are the vectors $\vec{x}$ such that $A\vec{x} = \vec{x}$ or $(A - 1I)\vec{x} = \vec{0}$, i.e. $\vec{x}$ belongs to

$$E_1(A) = \ker(A - I) = \ker\frac{1}{7}\begin{bmatrix} 2-7 & 3 & 6 \\ 6 & 2-7 & -3 \\ -3 & 6 & -2-7 \end{bmatrix} = \ker\begin{bmatrix} -5 & 3 & 6 \\ 6 & -5 & -3 \\ -3 & 6 & -9 \end{bmatrix}$$

$$= \ker\begin{bmatrix} 1 & -2 & 3 \\ 0 & 7 & -21 \\ 0 & 0 & 0 \end{bmatrix} = \ker\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} = \ker\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} = \operatorname{span}\begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$$

Now, a vector orthogonal to the rotation axis is, by inspection, $\vec{y} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

Therefore, the angle of rotation θ is given by $\cos\theta = (A\vec{y} \cdot \vec{y}) / \|y\|^2 =$

$$\left(\frac{1}{7} \begin{bmatrix} -1 \\ 4 \\ -9 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right) \frac{1}{2} = -\frac{5}{15} = -\frac{1}{3}, \text{ i.e. } \theta = \arccos(-\tfrac{1}{3}).$$

5. a. Let $\vec{v} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$. Find the matrix for the linear transformation $f : \mathbf{R}^3 \to \mathbf{R}^3$ defined by $f(\vec{w}) = \vec{v} \times \vec{w}$ for any $\vec{w}$ in $\mathbf{R}^3$.

Let A be the 3×3 matrix for f. The kth column vector of A, let's call it $\vec{a}_k$, is the image of the kth standard basis vector under f. So, we find

$$\left.\begin{aligned} \vec{a}_1 &= f(\hat{e}_1) = \vec{v} \times \hat{e}_1 = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix} \\ \vec{a}_2 &= f(\hat{e}_2) = \vec{v} \times \hat{e}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \\ \vec{a}_3 &= f(\hat{e}_3) = \vec{v} \times \hat{e}_3 = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} \end{aligned}\right\} \Rightarrow A = [\vec{a}_1 \mid \vec{a}_2 \mid \vec{a}_3] = \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & -1 \\ -3 & 1 & 0 \end{bmatrix}$$

b. If $\vec{v}$ is any nonzero vector in $\mathbf{R}^n$ and the transformation $g : \mathbf{R}^n \to \mathbf{R}^n$ is defined by $g(\vec{w}) = (\vec{v} \cdot \vec{w})\vec{w}$ for any $\vec{w}$ in $\mathbf{R}^n$, show that g is <u>non</u>linear.

Choose any vector $\vec{w}$ that is not orthogonal to $\vec{v}$. Then, g is not linear since $g(2\vec{w}) = (\vec{v} \cdot 2\vec{w})2\vec{w} = 4g(\vec{w}) \neq 2g(\vec{w})$.

6. a. Demonstrate that the subset of all vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbf{R}^2$ for which $xy \geq 0$ is <u>not</u> a subspace of $\mathbf{R}^2$.

$\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\-1\end{bmatrix}$ belong to the subset but $\begin{bmatrix}1\\0\end{bmatrix}+\begin{bmatrix}0\\-1\end{bmatrix}=\begin{bmatrix}1\\-1\end{bmatrix}$ does not. So, this subset is not closed under vector addition and is therefore not a subspace.

b. Prove that $\ker(T)$ is a subspace of the domain of T for any linear transformation $T:\mathbf{R}^n \to \mathbf{R}^m$.

If $\vec{x}$ and $\vec{y}$ belong to $\ker(T)$, $T(\vec{x})=T(\vec{y})=\vec{0}$ and, by the linearity of T, $T(\alpha\vec{x}+\beta\vec{y})=\alpha T(\vec{x})+\beta T(\vec{y})=\alpha\vec{0}+\beta\vec{0}=\vec{0}$ for any scalars α and β. So, linear combinations of vectors in $\ker(T)$ also belong to $\ker(T)$. Clearly, $\ker(T)$ is not empty because it contains $\vec{0}$. So, $\ker(T)$ is a subspace of $\mathbf{R}^n$.

7. Data is collected to determine the two parameters α and β in the relationship $s = \alpha + \beta t$. The following (t, s) data pairs are found: (1, 1), (2, 1), (2, 2). Find values of α and β that fit the data best in a least squares sense.

If the data truly fit a straight line, we would have $A\vec{x}=\vec{b}$ where

$A=\begin{bmatrix}1&1\\1&2\\1&2\end{bmatrix}$, $\vec{x}=\begin{bmatrix}\alpha\\\beta\end{bmatrix}$, $\vec{b}=\begin{bmatrix}1\\1\\2\end{bmatrix}$. However, it is evident that $b\notin \operatorname{im}(A)$ and that the columns of A are linearly independent. Therefore, the best (least squares) fit will be found by seeking a solution, instead, to $A^TA\vec{x}=A^T\vec{b}$. We find the solution to this equation to be $\vec{x}=(A^TA)^{-1}A^T\vec{b}=\left(\begin{bmatrix}1&1&1\\1&2&2\end{bmatrix}\begin{bmatrix}1&1\\1&2\\1&2\end{bmatrix}\right)^{-1}\begin{bmatrix}1&1&1\\1&2&2\end{bmatrix}\begin{bmatrix}1\\1\\2\end{bmatrix}=$

$\begin{bmatrix}3&5\\5&9\end{bmatrix}^{-1}\begin{bmatrix}1&1&1\\1&2&2\end{bmatrix}\begin{bmatrix}1\\1\\2\end{bmatrix}=\frac{1}{27-25}\begin{bmatrix}9&-5\\-5&3\end{bmatrix}\begin{bmatrix}4\\7\end{bmatrix}=\frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}$. So, the line we seek has the equation $s=\frac{1}{2}+\frac{1}{2}t$.

8. Let $(\vec{v}_1,\vec{v}_2)=\left(\begin{bmatrix}1\\1\\1\\1\end{bmatrix},\begin{bmatrix}5\\1\\5\\1\end{bmatrix}\right)$ and define the subspace $V=\operatorname{span}(\vec{v}_1,\vec{v}_2)$ of $\mathbf{R}^4$.

a. Determine the matrix P for the projection onto V.

Let $A=[\vec{v}_1\,|\,\vec{v}_2]$. Then, the columns of A are linealy independent and AA^T is an invertible matrix and the projection desired is given by

$$P=A(A^TA)^{-1}A^T=\begin{bmatrix}1&5\\1&1\\1&5\\1&1\end{bmatrix}\left(\begin{bmatrix}1&1&1&1\\5&1&5&1\end{bmatrix}\begin{bmatrix}1&5\\1&1\\1&5\\1&1\end{bmatrix}\right)^{-1}\begin{bmatrix}1&1&1&1\\5&1&5&1\end{bmatrix}$$

$$=\begin{bmatrix}1&5\\1&1\\1&5\\1&1\end{bmatrix}\begin{bmatrix}4&12\\12&52\end{bmatrix}^{-1}\begin{bmatrix}1&1&1&1\\5&1&5&1\end{bmatrix}=\frac{1}{4}\begin{bmatrix}1&5\\1&1\\1&5\\1&1\end{bmatrix}\begin{bmatrix}1&3\\3&13\end{bmatrix}^{-1}\begin{bmatrix}1&1&1&1\\5&1&5&1\end{bmatrix}$$

$$=\frac{1}{4}\begin{bmatrix}1&5\\1&1\\1&5\\1&1\end{bmatrix}\frac{1}{4}\begin{bmatrix}13&-3\\-3&1\end{bmatrix}\begin{bmatrix}1&1&1&1\\5&1&5&1\end{bmatrix}=\frac{1}{16}\begin{bmatrix}1&5\\1&1\\1&5\\1&1\end{bmatrix}\begin{bmatrix}-2&10&-2&10\\2&-2&2&-2\end{bmatrix}$$

$$=\frac{1}{16}\begin{bmatrix}8&0&8&0\\0&8&0&8\\8&0&8&0\\0&8&0&8\end{bmatrix}=\frac{1}{2}\begin{bmatrix}1&0&1&0\\0&1&0&1\\1&0&1&0\\0&1&0&1\end{bmatrix}.$$

b. If $\vec{w}=\begin{bmatrix}1\\0\\2\\3\end{bmatrix}$, for which $\vec{v}$ in V is $\|\vec{w}-\vec{v}\|$ least?

The vector in V that is closest to $\vec{w}$ is $P\vec{w}=\frac{3}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$.

9. If $A=\begin{bmatrix}3&2&1&2&2\\1&0&1&1&2\\2&2&0&1&0\\-1&2&-1&0&-2\end{bmatrix}$ then $A_{rref}=\begin{bmatrix}1&0&1&1&2\\0&1&-1&-\frac{1}{2}&-2\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$.

a. Determine a basis for both $\operatorname{im}(A)$ and $\ker(A)$.

Since the first and second columns are the only pivot columns, a basis for $\operatorname{im}(A)$ is $\left(\begin{bmatrix}3\\1\\2\\-1\end{bmatrix},\begin{bmatrix}2\\0\\2\\2\end{bmatrix}\right)$. The Solution Algorithm yields a basis for $\ker(A)$; it is $\left(\begin{bmatrix}1\\-1\\-1\\0\\0\end{bmatrix},\begin{bmatrix}1\\-\frac{1}{2}\\0\\-1\\0\end{bmatrix},\begin{bmatrix}2\\-2\\0\\0\\-1\end{bmatrix}\right)$.

b. For which $\vec{b}$ does the equation $A\vec{x}=\vec{b}$ have solutions?

The equations has solutions only for $\vec{b}\in\operatorname{im}(A)$.

c. Notice that $A\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}=\begin{bmatrix}10\\5\\5\\-2\end{bmatrix}$. Find all solutions to $A\vec{x}=\begin{bmatrix}10\\5\\5\\-2\end{bmatrix}$.

One solution is given. All the others differ from it by vectors in the kernel.

Therefore, $\vec{x}=\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}+\alpha\begin{bmatrix}1\\-1\\-1\\0\\0\end{bmatrix}+\beta\begin{bmatrix}1\\-\frac{1}{2}\\0\\-1\\0\end{bmatrix}+\gamma\begin{bmatrix}2\\-2\\0\\0\\-1\end{bmatrix}$ where α, β, γ are arbitrary reals.

10. Describe a step-by-step process and a criterion to unambiguously determine:

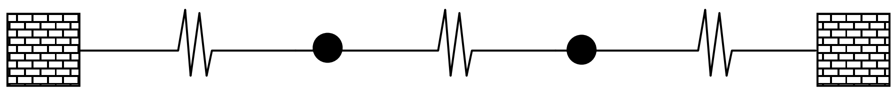
a. if a given vector $\vec{w}$ in $\mathbf{R}^n$ belongs to the span of a given list $(\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_p)$ of vectors in $\mathbf{R}^n$.

Let $A=\left[\vec{v}_1\,|\,\vec{v}_2\,|\ldots|\,\vec{v}_p\,|\,\vec{w}\right]$. Row-reduce A to A_{rref}. Then, $\vec{w}\in\operatorname{span}(\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_p)$ if and only if the last column of A is not a pivot column.

b. if a given list $(\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_p)$ of vectors in $\mathbf{R}^n$ is linearly independent.

Let $A = \left[\vec{v}_1 \,|\, \vec{v}_2 \,|\, \ldots \,|\, \vec{v}_p\right]$. Row-reduce A to obtain A_{rref}. Then, $(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p)$ is linearly independent if and only if every column of A is a pivot column.

11. Below is a diagram of a mechanical oscillator consisting of two particles and three springs. The masses of the two particles are each 1.0 kg. The springs are ideal and obey Hooke's Law. The middle spring constant is 1.5 kg/sec^2 and the two outside spring constants are each 1.0 kg/sec^2. Assume the oscillations occur on a frictionless table along the line between the particles and the fixed points.



$x_1(t)$ and $x_2(t)$ are the displacements of the left and right particles, respectively, relative to a configuration in which all three springs are relaxed. Newton's Second Law gives us a pair of homogeneous, second order, linear, ordinary differential equations. Using $\vec{x} = \left[x_1 \quad x_2\right]^T$, these equations may be written as one matrix differential equation $\vec{x}\,'' + A\vec{x} = \vec{0}$ where $A = \frac{1}{2}\begin{bmatrix} 5 & -3 \\ -3 & 5 \end{bmatrix}$. Use eigenvalue-eigenvector methods to find the general solution for $\vec{x}(t)$. The solution involves 4 arbitrary constants. Show how the constants are determined in terms of the initial positions and velocities of the particles.

First, we find the eigens of the matrix A. The eigenvalues are obtained by solving the eigenvalue equation. $0 = \det(A - \lambda I) = \det A = \det\left(\frac{1}{2}\begin{bmatrix} 5-2\lambda & -3 \\ -3 & 5-2\lambda \end{bmatrix}\right)$ $= \frac{1}{4}\left[(5-2\lambda)^2 - (-3)^2\right] = \frac{1}{4}\left[4\lambda^2 - 20\lambda + 25 - 9\right] = \lambda^2 - 5\lambda + 4 = (\lambda - 1)(\lambda - 4)$. So, spec($A$) = (1, 4). The eigenspaces are found next. $E_1(A) = \ker(A - 1I) =$ $\ker\left(\frac{1}{2}\begin{bmatrix} 5-2 & -3 \\ -3 & 5-2 \end{bmatrix}\right) = \ker\left(\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\right) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$. $E_4(A) = \ker(A - 4I) =$

$\ker\left(\frac{1}{2}\begin{bmatrix} 5-8 & -3 \\ -3 & 5-8 \end{bmatrix}\right) = \ker\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right) = \operatorname{span}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$. Since A is real and symmetric, it has two real eigenvalues and an orthogonal eigenbasis $\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$.

The scalar ODE $x'' + \omega^2 x = 0$ has the general solution

$x(t) = a\cos(\omega t) + b\sin(\omega t)$. Consequently, the general solution to the matrix ODE above is

$$\vec{x}(t) = \left(a_1\cos(t) + b_1\sin(t)\right)\begin{bmatrix}1\\1\end{bmatrix} + \left(a_4\cos(2t) + b_4\sin(t)\right)\begin{bmatrix}1\\-1\end{bmatrix}$$

The four constants a1, b1, a4, and b4 are determined by the initial conditions and the orthogonality of the eigenvectors. We have $\vec{x}(0) = a_1\begin{bmatrix}1\\1\end{bmatrix} + a_4\begin{bmatrix}1\\-1\end{bmatrix}$ and $\vec{x}\,'(0) = b_1\begin{bmatrix}1\\1\end{bmatrix} + 2b_4\begin{bmatrix}1\\-1\end{bmatrix}$. Therefore, $a_1 = \frac{1}{2}(x_1(0) + x_2(0))$, $b_1 = \frac{1}{2}(x_1(0) - x_2(0))$ and $a_4 = \frac{1}{2}(x'_1(0) + x'_2(0))$, $b_1 = \frac{1}{4}(x_1(0) - x_2(0))$.

12. Let $A = \dfrac{1}{2}\begin{bmatrix}3 & 1 & -1\\2 & 2 & -2\\1 & -1 & 1\end{bmatrix}$. Then, $A\begin{bmatrix}1 & 1 & 0\\1 & 0 & 1\\0 & 1 & 1\end{bmatrix} = \begin{bmatrix}2 & 1 & 0\\2 & 0 & 0\\0 & 1 & 0\end{bmatrix}$.

a. Determine $\operatorname{spec}(A)$, the list of eigenvalues of A.

The second matrix equation above can be read column by column. We have, by inspection, that $\operatorname{spec}(A) = (2, 1, 0)$.

b. Determine the eigenspaces $E_\lambda(A)$ for each of the eigenvalues λ.

Eigenspaces corresponding to this list of eigenvalues are also obtained by inspection. They are, respectively,

$$E_2(A) = \operatorname{span}\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right),\ E_1(A) = \operatorname{span}\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right),\ \text{and } E_0(A) = \operatorname{span}\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right).$$

c. Determine a matrix Q and its inverse Q^{-1} so that $Q^{-1}AQ$ is diagonal.

A diagonalizer for A is obtained by adjoining the three eigenvectors seen in the parentheses above as columns of Q. We have $Q = \begin{bmatrix}1 & 1 & 0\\1 & 0 & 1\\0 & 1 & 1\end{bmatrix}$. Its inverse is determined by row-reduction or a little trial and error. $Q^{-1} = \frac{1}{2}\begin{bmatrix}1 & 1 & -1\\1 & -1 & 1\\-1 & 1 & 1\end{bmatrix}$.

Therefore, $Q^{-1}AQ = D = \begin{bmatrix}2 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}$ and, correspondingly, $Q\,D\,Q^{-1} = A$.

d. Compute (all the entries of) A^n for any positive integer n.

From the last equation in c, we have $A^n = Q\,D^n\,Q^{-1} =$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2^n & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\tfrac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix} = \tfrac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2^n & 2^n & -2^n \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \tfrac{1}{2}\begin{bmatrix} 2^n+1 & 2^n-1 & -2^n+1 \\ 2^n & 2^n & -2^n \\ 1 & -1 & 1 \end{bmatrix}.$$