Special Problem E Solution.

Let X and Y be the 3×3 matrices for rotation by $\pi/2$ about the x- and y-axes, respectively, in $\mathbf{R}^3$. Angles are regarded as positive if they are counterclockwise when viewed by an observer looking toward the origin from the positive half of the axis.

a. Compute X and Y and show that X and Y they do not commute. See if you can generalize this result to make a statement about pilots doing aerial maneuvers that involve pitch, roll and yaw.

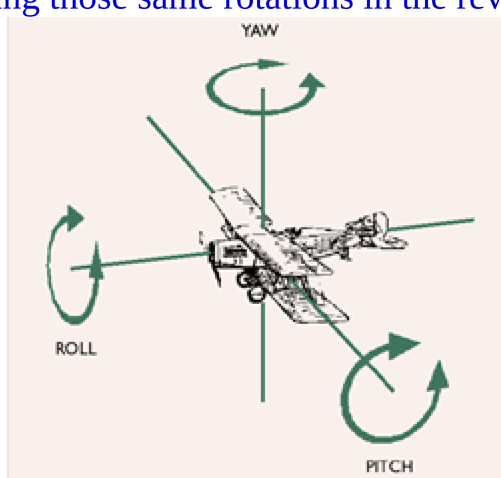
$X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$ and $Y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$. Each matrix was computed column by column by determining the images of the standard basis vectors under the rotations described.

$$X\,Y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \neq$$

$$Y\,X = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$$

So, the two matrices do not commute.

A plane has three principal axes of rotation. There is the axis parallel to its wings, the axis perpendicular to its fuselage, and the axis normal to both of those. Rotation about the first is called pitch; rotation about the second is called roll; and rotation about the third is called yaw. The same terminology is also applied to ships. Performing a roll and then a yaw, for example produces a result that is quite distinct from performing those same rotations in the reverse order.



The composite of two rotations in $\mathbf{R}^3$ is a rotation in $\mathbf{R}^3$. This may or may not be intuitive to you but it is true and we will prove this, later. Every rotation in $\mathbf{R}^3$ is characterized by its rotation axis and its rotation angle.

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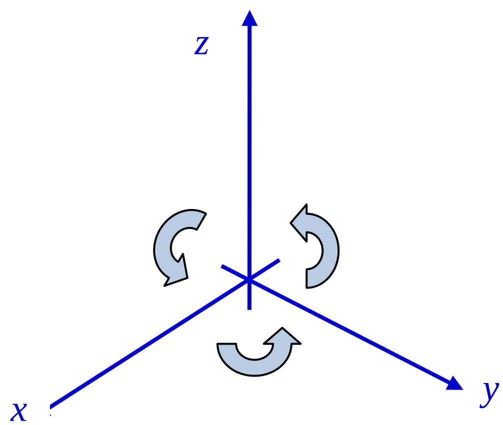
b. What are the rotation axis $\vec{v}$ and rotation angle θ for the rotation whose matrix is the product $X\,Y$? Pay attention to the order of multiplication here.

As we found in part a, $X\ Y = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. By examining the columns of this matrix, we deduce that this matrix represents the transformation $T : \mathbf{R}^3 \to \mathbf{R}^3$ such that $T(\vec{x}) = A\,\vec{x}$. T sends the *x*-axis onto the *y*-axis, the *y*-axis onto the *z*-axis, and the *z*-axis onto the *x*-axis. It is a rotation by $2\pi/3$ about the line through the origin parallel to the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.



The view toward the origin anti-parallel to the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \vec{u}$

c. If the rotation in part b is applied to an arbitrary vector $\vec{x}$ in $\mathbf{R}^3$, will the angle between $\vec{x}$ and its rotated image be θ? Discuss.

Let angle between $\vec{x}$ and its rotated image, $T(\vec{x})$, be α. α depends on the angle φ that $\vec{x}$ makes with the axis of rotation $\vec{u}$. For small φ, α will be small. It is only when $\varphi = \pi/2$ that $\alpha = \theta$.

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When $\vec{x}$ is rotated about $\vec{u}$ through the angle θ, its image is $T(\vec{x})$. The angle between $\vec{x}$ and $T(\vec{x})$ is α. Unless $\varphi = \pi / 2$, this is not θ; it is α.