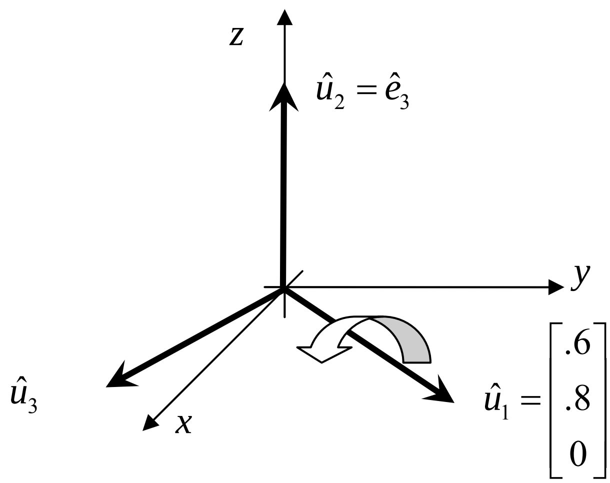
This test is due in class on Monday, 3 December 2007. You may receive no assistance ***whatsoever*** *from anyone not on your team. You may use any text, a calculator, Maple, Mathematica or any materials on the web and the notes of any member of your team. Explain and show every significant step.*

1. The object of this exercise is to find the matrix A that represents the linear transformation T that is a rotation by the angle $\pi/2$ about the line through the origin parallel to the unit vector $\begin{bmatrix} .6 \\ .8 \\ 0 \end{bmatrix}$. The rotation is counter-clockwise as viewed by an observer facing the origin and displaced from it by this vector. See the diagram below.



a. Choose the orthonormal basis $\mathcal{B} = (\hat{u}_1, \hat{u}_2, \hat{u}_3)$ shown above and evaluate $T(\hat{u}_1), T(\hat{u}_2)$ and $T(\hat{u}_3)$ in terms of $\hat{u}_1, \hat{u}_2$ and $\hat{u}_3$.

$\mathcal{B} = (\hat{u}_1, \hat{u}_2, \hat{u}_3) = \left(\begin{bmatrix} .6 \\ .8 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} .8 \\ -.6 \\ 0 \end{bmatrix} \right)$. By inspection , we see that

$T(\hat{u}_1) = \hat{u}_1,\ T(\hat{u}_2) = \hat{u}_3,\ T(\hat{u}_3) = -\hat{u}_2$.

b. Determine $[T(\hat{u}_1)]_{\mathcal{B}}, [T(\hat{u}_2)]_{\mathcal{B}}$ and $[T(\hat{u}_3)]_{\mathcal{B}}$ and, thereby, find the matrix B that represents T with respect to the $\mathcal{B}$-coordinates.

$[T(\hat{u}_1)]_{\mathcal{B}} = [\hat{u}_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $[T(\hat{u}_2)]_{\mathcal{B}} = [\hat{u}_3]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $[T(\hat{u}_3)]_{\mathcal{B}} = [-\hat{u}_2]_{\mathcal{B}} = -\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Therefore, $B = \left[[T(\hat{u}_1)]_{\mathscr{B}} \,|\, [T(\hat{u}_2)]_{\mathscr{B}} \,|\, [T(\hat{u}_3)]_{\mathscr{B}} \right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$.

c. Determine A.

To change back to standard coordinates, we need the coordinate transformation matrix $S = [\hat{u}_1 \,|\, \hat{u}_2 \,|\, \hat{u}_3] = \begin{bmatrix} .6 & 0 & .8 \\ .8 & 0 & -.6 \\ 0 & 1 & 0 \end{bmatrix}$ and its inverse. Since S is orthogonal, its inverse is $S^{-1} = S^T$. So, $A = S\,B\,S^T = \begin{bmatrix} .36 & .48 & .8 \\ .48 & .64 & -.6 \\ -.8 & .6 & 0 \end{bmatrix}$.

d. The following results are easily verified by direct computation. Explain why they are true.

i. $A^T A = I$

The result is expected since A must be an orthogonal matrix.

ii. $\det(A) = 1$

This result follows from the fact that A is a rotation, a proper orthogonal transformation that preserves parity or the handedness of coordinates.

iii. $A\hat{u}_1 = \hat{u}_1$

This expresses the fact that $\hat{u}_1$ is parallel to the rotation axis and so is left unchanged by the rotation.

iv. $\vec{v} \cdot A\vec{v} = 0$ for any $\vec{v} \in \operatorname{span}(\hat{u}_2, \hat{u}_3)$

Vectors in the plane of $\hat{u}_2$ and $\hat{u}_3$ are rotated by $\pi/2$ and so they are orthogonal to their images under this rotation.

v. $A^4 = I$

A represents a rotation about the axis parallel to the vector $\hat{u}_1$ by the angle $\pi/2$. So, the repetition of this linear transformation with itself four times corresponds to rotation by the angle $4(\pi/2) = 2\pi$ about the same axis. This is the identity transformation.

2. Let $A = \begin{bmatrix} 1 & 2 & 7 \\ 1 & 1 & 5 \\ 2 & 0 & 6 \\ 0 & 1 & 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 9 \\ 9 \\ 9 \\ 9 \end{bmatrix}$.

a. Show that $A\vec{x} = \vec{b}$ has no solution.

$$\text{rref}\left[A|\vec{b}\right]=\text{rref}\left[\begin{array}{ccc|c}1&2&7&9\\1&1&5&9\\2&0&6&9\\0&1&2&9\end{array}\right]=\left[\begin{array}{ccc|c}1&0&3&0\\0&1&2&0\\0&0&0&1\\0&0&0&0\end{array}\right].$$ Since the last column is a pivot column, the matrix equation has no solution. This is equivalent to the assertion $\vec{b}\notin \text{im}(A)$.

b. Let $\mathbf{S}$ be the set of all vectors $\vec{c}$ in $\mathbf{R}^4$ so that $A\vec{x}=\vec{c}$ has solutions. Describe $\mathbf{S}$ as explicitly and completely as possible.

$\mathbf{S}$ = im(A). From part a, we see that the first and second columns of A are pivot columns. That is, the first two column vectors of A comprise a basis for $\mathbf{S}$ and $\mathbf{S}=\text{span}\left(\begin{bmatrix}1\\1\\2\\0\end{bmatrix},\begin{bmatrix}2\\1\\0\\1\end{bmatrix}\right)$.

c. Let $\vec{c}_0$ be the vector in $\mathbf{S}$ that is closest to $\vec{b}$. Find $\vec{c}_0$.

$\vec{c}_0$ is the projection of $\vec{b}$ onto $\mathbf{S}$. The matrix representing projection onto $\mathbf{S}$ is given by $P=B\left(B^TB\right)^{-1}B^T$ where B is the 4×2 matrix whose columns are the basis vectors of $\mathbf{S}$. Of course, we cannot use A in this formula since its columns are linearly dependent. So,

$$P=\begin{bmatrix}1&2\\1&1\\2&0\\0&1\end{bmatrix}\left(\begin{bmatrix}1&1&2&0\\2&1&0&1\end{bmatrix}\begin{bmatrix}1&2\\1&1\\2&0\\0&1\end{bmatrix}\right)^{-1}\begin{bmatrix}1&1&2&0\\2&1&0&1\end{bmatrix}=\tfrac{1}{9}\begin{bmatrix}6&3&0&3\\3&2&2&1\\0&2&8&-2\\3&1&-2&2\end{bmatrix}.$$

Therefore, $c_0=P\vec{b}=\begin{bmatrix}12\\8\\8\\4\end{bmatrix}$.

d. Determine the set of all solutions to $A\vec{x}=\vec{c}_0$.

$$\text{rref}\left[A|\vec{c}_0\right]=\text{rref}\left[\begin{array}{ccc|c}1&2&7&12\\1&1&5&8\\2&0&6&8\\0&1&2&4\end{array}\right]=\left[\begin{array}{ccc|c}1&0&3&4\\0&1&2&4\\0&0&0&0\\0&0&0&0\end{array}\right].$$ So, according to the

Solution Algorithm, $\vec{x}=\begin{bmatrix}4\\4\\0\end{bmatrix}+t\begin{bmatrix}3\\2\\-1\end{bmatrix}=\begin{bmatrix}4+3t\\4+2t\\-t\end{bmatrix}$ for any $t\in\mathbf{R}$

e. Find the smallest solution to $A\vec{x}=\vec{c}_0$.

Let $f(t)=\|\vec{x}\|^2=(4+3t)^2+(4+2t)^2+(-t)^2$. f will be at a minimum when $0=f'(t)=6(4+3t)+4(4+2t)+2t$. This yields $t=-\frac{10}{7}$ and $\vec{x}=\frac{2}{7}\begin{bmatrix}-1\\4\\5\end{bmatrix}$.

3. Let $Q=\frac{1}{7}\begin{bmatrix}6&2&3\\2&3&-6\\-3&6&2\end{bmatrix}$.

a. Verify that Q is an orthogonal matrix with a positive determinant. Therefore, Q represents a rotation in $\mathbf{R}^3$.

$$Q^TQ=\tfrac{1}{7}\begin{bmatrix}6&2&-3\\2&3&6\\3&-6&2\end{bmatrix}\tfrac{1}{7}\begin{bmatrix}6&2&3\\2&3&-6\\-3&6&2\end{bmatrix}=\tfrac{1}{49}\begin{bmatrix}49&0&0\\0&49&0\\0&0&49\end{bmatrix}=I$$

$$\det(Q)=\tfrac{1}{7^3}\det\begin{bmatrix}6&2&3\\2&3&-6\\-3&6&2\end{bmatrix}=\tfrac{1}{343}\big(6(6+36)-2(4-18)+3(12+9)\big)=1$$

Since rotations are completely characterized by a rotation axis and a rotation angle, we seek to determine these next.

b. Show that 1 is an eigenvalue of Q and find the corresponding eigenspace, $E_1(A)$. What is the geometrical significance of $E_1(A)$?

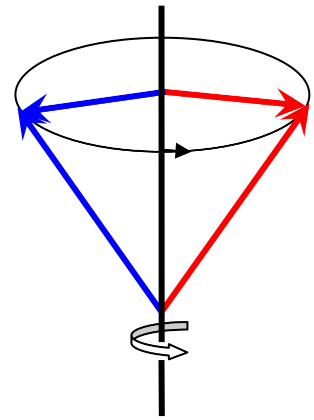
$$\det(Q-1I)=\det\left(\tfrac{1}{7}\begin{bmatrix}6&2&3\\2&3&-6\\-3&6&2\end{bmatrix}-\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right)=\det\tfrac{1}{7}\begin{bmatrix}-1&2&3\\2&-4&-6\\-3&6&-5\end{bmatrix}=0$$

This shows that 1 is an eigenvalue of Q. The corresponding eigenspace is

$$E_1(A)=\ker\tfrac{1}{7}\begin{bmatrix}-1&2&3\\2&-4&-6\\-3&6&-5\end{bmatrix}=\ker\begin{bmatrix}1&-2&0\\0&0&1\\0&0&0\end{bmatrix}=\operatorname{span}\begin{bmatrix}2\\1\\0\end{bmatrix}$$ which is the subspace of all eigenvectors of eigenvalue 1. These are the vectors parallel to the rotation axis.

c. Explain why the rotation angle is $\arccos\left((\vec{v}\cdot Q\vec{v})/\|\vec{v}\|^2\right)$ for any nonzero vector $\vec{v}\in\left(E_1(A)\right)^{\perp}$ but this formula does NOT give the rotation angle for any other vector. Compute the rotation angle.

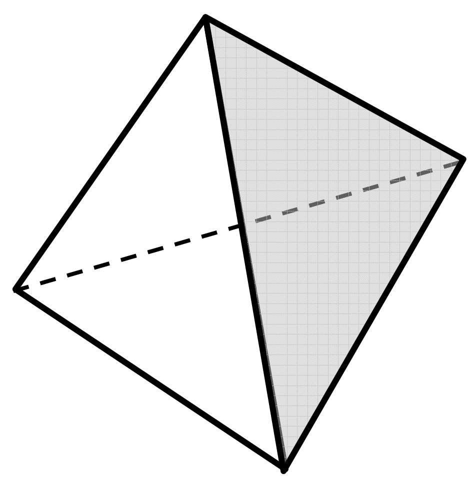
The angle between a vector and its image under a rotation is the rotation angle only if the vector is orthogonal to the rotation axis. The closer the vector is to the rotation axis, the smaller is the angle between it and its rotated image. Of course, for a vector parallel to the rotation axis, the vector and its rotated image are the same and the angle between them is 0. The diagram at the right illustrates this idea. In it, blue vectors are rotated around the black axis resulting in the red images.



If we choose, say, $\vec{v}=\begin{bmatrix}0 & 0 & 1\end{bmatrix}^T\in\left(E_1(A)\right)^{\perp}$, a vector orthogonal to the rotation axis, we find $\mathrm{arc\,cos}\left((\vec{v}\cdot Q\vec{v})/\|\vec{v}\|^2\right)=\mathrm{ar\,arccos}\left(\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\cdot\tfrac{1}{7}\begin{bmatrix}3\\-6\\2\end{bmatrix}\right)\Big/1^2\right)$

$=\arccos(2/7)\approx 1.28104.$

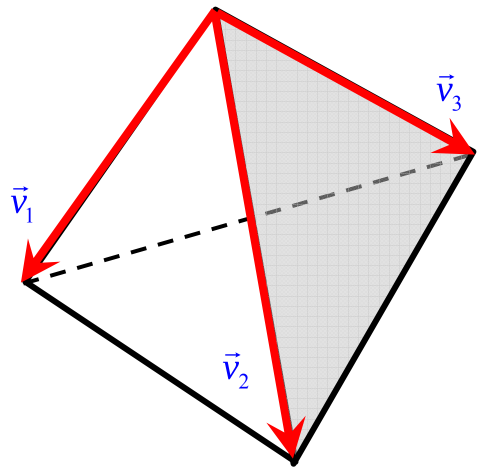
4. An n-sided pyramid is a polyhedron formed by connecting an n-sided closed plane polygon and a point, called the apex, not in the plane of the polygon by n-triangular faces ($n>2$). In other words, it is a conic solid with a polygonal base. The simplest is a triangular pyramid, also called a tetrahedron, shown at the right. It is defined by any four non-coplanar points. It is well known that the volume of a pyramid is one third of the product of its base area and its height.



a. Choose an apex and denote the three edges originating from it by $\vec{v}_1, \vec{v}_2,$ and $\vec{v}_3$. Express the volume of the tetrahedron in terms of a determinant involving these vectors. Explain why your volume formula is unaffected by any permutation of the vectors $\vec{v}_1, \vec{v}_2,$ and $\vec{v}_3$.

The volume V of this tetrahedron is $V=\tfrac{1}{6}\left|\det[\vec{v}_1\,|\,\vec{v}_2\,|\,\vec{v}_3]\right|$. A factor of one third enters the expression because of the pyramid volume formula and another factor of one half is introduced because the base is a triangle whose area is half that of the parallelogram with the same concurrent edges. Permuting the columns of the matrix above may change the sign of the determinant V is one sixth of the absolute value of the determinant.

b. Prove that the volume of this tetrahedron is the same whichever vertex is chosen as its apex. It is necessary to do this for only one other vertex.



Instead of choosing the "top" vertex as our apex, we choose the left vertex. The vectors emanating from this vertex are $-\vec{v}_1, \vec{v}_2 - v_1$, and $\vec{v}_3 - \vec{v}_1$. So, the volume formula is now $\frac{1}{6}\left|\det[-\vec{v}_1 \,|\, \vec{v}_2 - \vec{v}_1 \,|\, \vec{v}_3 - \vec{v}_1]\right|$. Changing the sign of the first column of the matrix does not change the value of this expression. Too, the determinant is unchanged by adding the new first column to the second and third columns. After making these manipulations, we arrive at the same expression as in V, above.

5. The three vectors $\begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$ comprise the concurrent edges of a 3-parallepiped in $\mathbf{R}^4$. Compute its 3-volume.

Since the three vectors are linearly independent, the 4×3 matrix A whose column vectors are the three vectors given above, the 3-volume is given by

$$\sqrt{\det(A^T A)} = \sqrt{\det\left(\begin{bmatrix} -1 & 0 & 1 & 1 \\ 1 & 1 & -1 & 0 \\ 1 & 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}\right)} = \sqrt{\det\begin{bmatrix} 3 & -2 & -2 \\ -2 & 3 & 2 \\ -2 & 2 & 3 \end{bmatrix}}$$

$$= \sqrt{7}.$$

6. $A = \frac{1}{3}\begin{bmatrix} -2 & 2 & 4 \\ 2 & -5 & 2 \\ 4 & 2 & -2 \end{bmatrix}$ is a real symmetric matrix. Therefore, A has 3 real eigenvalues, counting multiplicity and there is a rotation matrix Q such that $Q^T AQ$ is a diagonal matrix.

a. Find all eigenvalues of A.

The characteristic equation for A is

$$0 = \det(A - \lambda I) = \tfrac{1}{3}\det\begin{bmatrix} -2-3\lambda & 2 & 4 \\ 2 & -5-3\lambda & 2 \\ 4 & 2 & -2-3\lambda \end{bmatrix}$$

$= -(\lambda^3 + 3\lambda^2 - 4) = -(\lambda - 1)(\lambda + 2)^2$. Therefore, $\text{spec}(A) = (-2, -2, 1)$.

b. Find all eigenspaces of A.

$$E_{-2}(A) = \ker(A + 2I) = \ker\tfrac{1}{3}\begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix} = \ker\begin{bmatrix} 1 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ -1 \\ 0 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}\right) = \text{span}(\vec{v}_1, \vec{v}_2).$$

$$E_1(A) = \ker(A - I) = \ker\tfrac{1}{3}\begin{bmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{bmatrix} = \ker\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} = \text{span}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

$= \text{span}(\vec{v}_3)$.

The eigenvalue -2 has algebraic and geometric multiplicity 2 while the eigenvalue 1 has algebraic and geometric multiplicity 1. Therefore, A is diagonalizable. A has an eigenbasis for $\mathbf{R}^3$; in fact, since A is real symmetric, it is diagonalizable by a rotation.

c. Find the rotation matrix Q described above.

The two eigenspaces, $E_{-2}(A)$ and $E_1(A)$ are, as expected, orthogonal. However, the vectors $\vec{v}_1$ and $\vec{v}_2$ are not orthogonal. Application of the Gram-Schmidt process will give us an orthonormal eigenbasis for $\mathbf{R}^3$. We let $\hat{u}_1 = \vec{v}_1 / \|\vec{v}_1\| = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. So, $\vec{v}_2^{\perp} = \vec{v}_2 - (\vec{v}_2 \cdot \hat{u}_1)\hat{u}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -2 \\ \frac{1}{2} \end{bmatrix}$

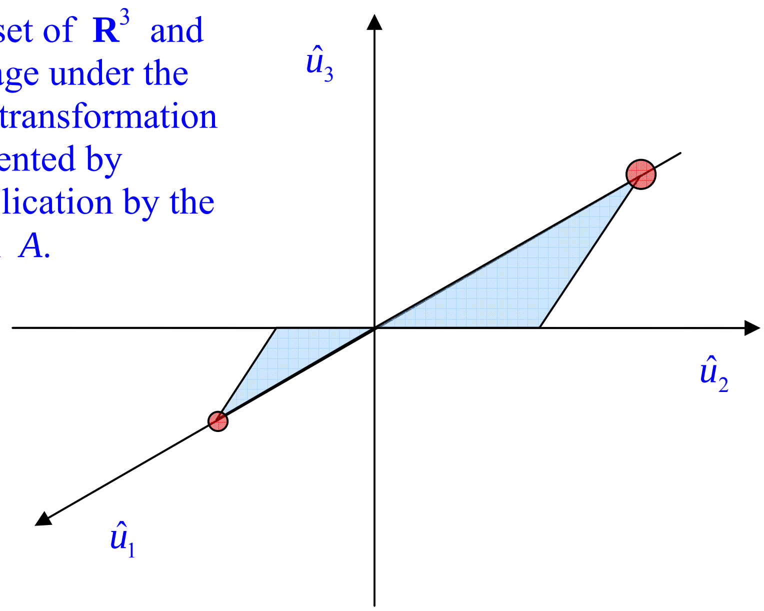
and $\hat{u}_2 = \vec{v}_2^{\perp} / \|\vec{v}_2^{\perp}\| = \frac{1}{\sqrt{18}}\begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix}$. Finally, $\hat{u}_3 = \vec{v}_3 / \|\vec{v}_3\| = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$. So, $(\hat{u}_1, \hat{u}_2, \hat{u}_3)$ is an orthonormal eigenbasis of $\mathbf{R}^3$ for the matrix A and we let

$Q = \left[-\hat{u}_1 \,|\, \hat{u}_2 \,|\, \hat{u}_3\right] = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{2}{3} \\ 0 & \frac{-4}{3\sqrt{2}} & \frac{1}{3} \\ \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{2}{3} \end{bmatrix}$. We chose $-\hat{u}_1$ instead of $\hat{u}_1$ in the first column of Q so that $\det(Q) = +1$. Q represents a rotation since it is a proper orthogonal matrix. Since Q is orthogonal, $Q^T = Q^{-1}$.

d. Provide a verbal description of the linear transformation represented by A. Choose the basis provided by the column vectors of Q for this purpose.

The matrix A represents a reflection across the line L through the origin parallel to $\hat{u}_3$ together with a doubling of distance from L. It is also equivalent to a rotation by π together with a stretch by a factor of 2 away from L. Points along L are left unaffected by this transformation. See figure below.

A subset of $\mathbf{R}^3$ and its image under the linear transformation represented by multiplication by the matrix A.



7. For each of the following, either provide a specific example of two real square matrices of the same size satisfying the properties described or prove that no such example exists.

a. A and B are invertible but their product AB is not invertible.

This is impossible. If A and B are invertible, $\det(A) \neq 0$ and $\det(B) \neq 0$. Therefore, $\det(AB) = \det(A)\det(B) \neq 0$ and so, $AB \neq 0$ and AB must also be invertible.

b. A and B are symmetric but their product AB is not symmetric.

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then, $AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Clearly, A and B are symmetric but their product is not.

c. A and B are diagonalizable but their product AB is not diagonalizable.

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & \frac{1}{2} \end{bmatrix}$. Then, $AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. A is not only diagonalizable, it is diagonal. B is upper triangular and $\operatorname{spec}(B) = (\frac{1}{2}, 1)$. So, having two distinct eigenvalues, B is diagonalizable. However, AB is a standard shear. It is not diagonalizable. 1 is the only eigenvalue of this matrix and its algebraic multiplicity is 2. On the other hand the eigenspace of this shear is $\operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$ whose dimension, the geometric multiplicity of the shear, is 1. AB is not similar to any diagonal matrix.