1. Recast each problem below as a problem in finding a solution $\vec{x}$ to the vector equation $A\vec{x}=\vec{b}$, where A is a matrix and $\vec{x}$ and $\vec{b}$ are vectors. For each, define identify A, $\vec{x}$ and $\vec{b}$. Do not solve.

a. Determine if $\begin{bmatrix}7\\5\\10\\6\end{bmatrix}$ is a linear combination of $\begin{bmatrix}3\\1\\2\\2\end{bmatrix}, \begin{bmatrix}1\\2\\3\\1\end{bmatrix}$, and $\begin{bmatrix}2\\3\\3\\1\end{bmatrix}$.

If we label the first vector as $\vec{w}$ and the triplet of vectors $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$, then this problem is equivalent to determining whether there are scalar coefficients x_1, x_2 and x_3 so that $\vec{w}=x_1\vec{v}_1+x_2\vec{v}_2+x_3\vec{v}_3=\left[\vec{v}_1\,|\,\vec{v}_2\,|\,\vec{v}_3\right]\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$.

Let $A=\left[\vec{v}_1\,|\,\vec{v}_2\,|\,\vec{v}_3\right]=\begin{bmatrix}3&1&2\\1&2&3\\2&3&3\\2&1&1\end{bmatrix}$, $\vec{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$, and $\vec{b}=\vec{w}=\begin{bmatrix}7\\5\\10\\6\end{bmatrix}$ and we must determine whether $A\vec{x}=\vec{b}$ has any solutions.

b. Find a vector in $\mathbf{R}^3$ whose scalar products with $\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}2\\3\\1\end{bmatrix}$, and $\begin{bmatrix}4\\1\\2\end{bmatrix}$ are, respectively, 3, 4, and 5.

Let $\vec{x}$ be the vector sought, let $\vec{b}$ be the 3-vector whose components are the specified scalar products, and let A be the matrix whose rows are the specified vectors. So, this problem is equivalent to seeking the solutions $\vec{x}$ to $A\vec{x}=\vec{b}$ where $A=\begin{bmatrix}1&1&2\\2&3&1\\4&1&2\end{bmatrix}$ and $\vec{b}=\begin{bmatrix}3\\4\\5\end{bmatrix}$.

c. $F=\frac{1}{3}\begin{bmatrix}2&1&-2\\1&2&2\\-2&2&-1\end{bmatrix}$ is the matrix for a reflection across a plane through the origin in $\mathbf{R}^3$. Find all vectors normal (perpendicular) to this plane by using the fact that the image of a normal vector $\vec{x}$ is $-\vec{x}$.

We wish to solve $F\,\vec{x}=-\vec{x}=-I\,\vec{x}$ for $\vec{x}$ or, equivalently, the equation

$F\vec{x}+I\vec{x}=(F+I)\vec{x}=\vec{0}$. So, we let $A=F+I=\frac{1}{3}\begin{bmatrix} 5 & 1 & -2 \\ 1 & 5 & 2 \\ -2 & 2 & 4 \end{bmatrix}$ and $\vec{b}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and solve $A\vec{x}=\vec{b}$.

2. Given $[A|\vec{b}]_{rref}$ in each case below, solve $A\vec{x}=\vec{b}$ completely.

a. $[A|\vec{b}]_{rref}=\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$.

We have a single (unique) solution $\vec{x}=\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

b. $[A|\vec{b}]_{rref}=\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$.

This system has no solution since the last column is a pivot column. Alternately, we note that the last row is equivalent to the inconsistent equation $0 = 1$.

c. $[A|\vec{b}]_{rref}=\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 3 & 6 \\ 0 & 0 & 1 & 0 & 4 & 7 \\ 0 & 0 & 0 & 1 & 5 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$.

$\vec{x}=\begin{bmatrix} 6 \\ 0 \\ 7 \\ 8 \\ 0 \end{bmatrix}+\alpha\begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}+\beta\begin{bmatrix} 3 \\ 0 \\ 4 \\ 5 \\ -1 \end{bmatrix}$, where α and β are any reals.

3. Consider the equation $A\vec{x}=\vec{b}$ where A is an $m\times n$ matrix, $\vec{x}$ is in $\mathbf{R}^n$ and $\vec{b}$ is in $\mathbf{R}^m$. Show that this equation cannot have a finite number of solutions greater than 1 by proceeding as follows.

a. First, define what is meant by: $\vec{w}$ is a solution of the equation above.

$A\vec{w}=\vec{b}$.

b. Suppose that $\vec{u}$ and $\vec{v}$ are distinct solutions to $A\vec{x}=\vec{b}$. Let $\vec{z}$ be the difference between $\vec{v}$ and $\vec{u}$. Note that $\vec{z}\neq\vec{0}$. What is $A(\alpha\vec{z})$ for any α in $\mathbf{R}$?

$A(\alpha\vec{z})=\alpha A\vec{z}=\alpha(A\vec{u}-A\vec{v})=\alpha(\vec{b}-\vec{b})=\alpha\vec{0}=\vec{0}.$

c. Now, show that $\{\vec{u}+\alpha\vec{z}\mid\alpha\in\mathbf{R}\}$ is a set with infinitely many (different) vectors each of which is a solution to $A\vec{x}=\vec{b}$.

Since $A\vec{u}=\vec{b}$ and $A(\alpha\vec{z})=\vec{0}$, we have $A(\vec{u}+\alpha\vec{z})=A\vec{u}+A(\alpha\vec{z})=\vec{b}+\vec{0}=\vec{b}$ for all real α. Since $\vec{z}\neq\vec{0}$, distinct α's give distinct solutions.

4. a. Suppose that $f:\mathbf{R}^2\to\mathbf{R}^2$ is linear, f triples vectors parallel to $\begin{bmatrix}1\\2\end{bmatrix}$ and doubles vectors parallel to $\begin{bmatrix}3\\2\end{bmatrix}$. Find $f\begin{bmatrix}x\\y\end{bmatrix}$ for any real x and y.

Let A be the 2×2 matrix corresponding to f. Then, $A\begin{bmatrix}1&3\\2&2\end{bmatrix}=\begin{bmatrix}3&6\\6&4\end{bmatrix}$.

So, $A=\begin{bmatrix}3&6\\6&4\end{bmatrix}\begin{bmatrix}1&3\\2&2\end{bmatrix}^{-1}=\begin{bmatrix}3&6\\6&4\end{bmatrix}\frac{1}{2-6}\begin{bmatrix}2&-3\\-2&1\end{bmatrix}=\frac{-1}{4}\begin{bmatrix}-6&-3\\4&-14\end{bmatrix}=\frac{1}{4}\begin{bmatrix}6&3\\-4&14\end{bmatrix}$.

Therefore, $f\begin{bmatrix}x\\y\end{bmatrix}=A\begin{bmatrix}x\\y\end{bmatrix}=\frac{1}{4}\begin{bmatrix}6&3\\-4&14\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}\frac{3}{2}x+\frac{3}{4}y\\-x+\frac{7}{2}y\end{bmatrix}$.

b. Solve for the matrix X if $\begin{bmatrix}1&1\\1&3\end{bmatrix}X^{-1}\begin{bmatrix}1&1\\1&3\end{bmatrix}^{-1}=\begin{bmatrix}5&2\\-3&2\end{bmatrix}^{-1}$.

Inverting both sides, we have $\begin{bmatrix}1&1\\1&3\end{bmatrix}X\begin{bmatrix}1&1\\1&3\end{bmatrix}^{-1}=\begin{bmatrix}5&2\\-3&2\end{bmatrix}$. Now, multiplying on the left by $\begin{bmatrix}1&1\\1&3\end{bmatrix}^{-1}$ and on the right by $\begin{bmatrix}1&1\\1&3\end{bmatrix}$, we get

$$X = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 5 & 2 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} = \tfrac{1}{2} \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 15 \\ -4 & -4 \end{bmatrix}.$$

5. For each assertion below, indicate whether the assertion is true (***T***) or false (***F***). It is not necessary to show any work. However, a correct response earns full credit, an incorrect response earns negative half credit, and no response earns no credit.

a. A linear system of 7 equations in 8 variables always has infinitely many solutions.

False.. It may have no solutions.

b. A 4×4 matrix of rank 4 is always invertible.

True. This assertion is equivalent to the assertion that the matrix is row-reducible to the identity.

c. Given any three nonzero vectors in $\mathbf{R}^2$, no two of which are collinear, any one of them is a always a linear combination of the other two.

True. This was established by a geometrical argument in class and in homework. Any pair of non-collinear vectors in $\mathbf{R}^2$ spans $\mathbf{R}^2$.

d. The function $f : \mathbf{R}^2 \to \mathbf{R}^2$ defined by $f\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ 1 \end{bmatrix}$ is linear.

False. Clearly, $f\left(\vec{0}\right) \neq \vec{0}$. It is also easy to show that, under f, the image of a sum is not the corresponding sum of the images and the image of a multiple is not that multiple of the image.

e. There is a 2×2 matrix A, different from the identity matrix, such that A^{2008} is the identity matrix.

True. There are many such matrices. For example, let $A = -I$ or let A be a reflection across any line in $\mathbf{R}^2$, or let A be the matrix for rotation by $\frac{2\pi}{2008}$, or let A be the matrix for rotation by $\frac{2\pi N}{2008}$ where N is any integer.

f. f and g are two linear transformations defined by $f(\vec{x}) = A\vec{x}$ and $g(\vec{x}) = B\vec{x}$ where A and B are 3×3 matrices. The composite function $f \circ g$ is also a linear transformation and its matrix is AB.

True. This is precisely the definition for the matrix of a composite.

g. No invertible 10×10 matrix can have more than 90 ones among its entries.

False. A 10×10 matrix with 92 or more ones would have 8 or fewer non-one entries. If the matrix had 8 or fewer non-one entries, these entries could be distributed over at most 8 rows. That would mean that, at least two rows would consist of all ones. Subtracting one of these rows of all ones from the other would result in a row of all zeros. Therefore, the rref of such a matrix would have a row of zeros. Such a matrix would not be row-equivalent to the identity and it would therefore not be invertible. Among the 10×10 matrices with exactly 91 ones are the following.

$$A=\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

It is not hard to see that $A_{rref} = I_{10}$ and so A is invertible. [Can you see how to show this? Hint: Begin by subtracting the second row from all the other rows.]