name

Show all work (*except for True/False problem*).

Choose any **15** of the 16 problems.
Place an "X" next to the number
for the problem you choose not to do.

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1. Let A be a real $n \times n$ matrix. List up to 7 different properties of A that are equivalent to A being invertible (nonsingular). You may not use the words *column* or *transpose* or synonyms or symbols for them.

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2. Let A be any matrix and let A_{rref} be its row-reduced echelon form. For each of the following assertions, state whether ***True*** or ***False***. [Half credit penalty for an incorrect response.]

a. The row space of A and the row space of A_{rref} are the same.

b. The column space of A and the column space of A_{rref} are the same.

c. Linear relationships among the row vectors of A and linear relationships among the corresponding row vectors of A_{rref} are the same.

d. Linear relationships among the column vectors of A and linear relationships among the corresponding column vectors of A_{rref} are the same.

3. Suppose that $A = \left[\vec{v}_1 \mid \vec{v}_2 \mid \vec{v}_3 \mid \vec{v}_4 \right]$ and that its column vectors belong to $\mathbf{R}^3$. Determine A_{rref} with as much specificity as possible if

a. no three of the four column vectors of A are coplanar

b. no two of the four column vectors of A are collinear but all four are coplanar.

4. List all the ways 13 bills may be chosen from among the denominations \$1, \$2, \$5, and \$10 so that their total value is \$26.

5. For each of the following, provide a specific example of a real 3×3 matrix A or give a convincing but succinct argument why no such matrix exists.

a. A is invertible but A^2 is not invertible.

b. A has no zero entries and $A^3 = A$.

c. A is not the identity and $A^3 = I$.

d. $\mathrm{Col}(A) = \mathrm{Nul}(A)$.

6. a. Prove that the intersection of any two subspaces of any vector space is itself a subspace.

b. Give an example of two subspaces of $\mathbf{R}^2$ whose union is not a subspace and prove that the union is not a subspace.

7. $\vec{w}=\begin{bmatrix}3\\4\\3\\5\end{bmatrix}$ and $L=(\vec{v}_1,\vec{v}_2,\vec{v}_3,\vec{v}_4)=\left(\begin{bmatrix}1\\1\\2\\1\end{bmatrix},\begin{bmatrix}1\\2\\-1\\3\end{bmatrix},\begin{bmatrix}1\\2\\3\\-1\end{bmatrix},\begin{bmatrix}4\\7\\3\\6\end{bmatrix}\right)$.

Determine if L is a basis for $\mathbf{R}^4$ and determine if $\vec{w}\in\text{span}(L)$ by row-reducing a single matrix. Discuss.

8. Let $A = \begin{bmatrix} 2 & 6 & 1 \\ 3 & 7 & 2 \\ 2 & 6 & 1 \\ 4 & 8 & 3 \end{bmatrix}$.

a. Determine a basis for and the dimension of Col(A).

b. Determine a basis for and the dimension of Nul(A).

c. What does Col(A) tell us about solutions to the equation $A\vec{x} = \vec{b}$?

d. What does Nul(A) tell us about solutions to the equation $A\vec{x} = \vec{b}$?

9. Theory predicts that the electrical resistivity r of silver doped with trace amounts of silicon is given by $r = x_1 c + x_2 c^2$ where c is the concentration of silicon in silver and c is restricted to the interval [1, 2]. Find the best (least squares) choice of the coefficients x_1 and x_2 using results from an experiment that yielded the following (c, r)-pairs: (1, 1), (1, 2), (2, 2), and (2, 3).

10. The populations at time t of two competing insect species are $x_1(t)$ and $x_2(t)$. They satisfy $\dfrac{d\,x_1(t)}{d\,t} - x_1(t) + 2x_2(t) = 0$ and $\dfrac{d\,x_2(t)}{d\,t} + 2x_1(t) - x_2(t) = 0$.

a. Rewrite this pair of coupled, homogeneous, first-order ODEs as a single first order matrix-vector ODE.

b. Using eigenvector-eigenvalue methods, find the general solution to the above equations.

c. Compute $x_1(t)$ and $x_2(t)$ if $x_1(0) = 2$ and $x_2(0) = 3$

d. In this model, one of the species eventually becomes extinct. At what time does this occur for the initial conditions given above?

11. Let $\hat{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ be a fixed unit vector and let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be an arbitrary vector in $\mathbf{R}^3$ and define the function $f : \mathbf{R}^3 \to \mathbf{R}^3$ by $f(\vec{x}) = \hat{u} \times \vec{x}$. The symbol $\times$ denotes the usual cross or vector product.

a. Explain why f is linear.

b. Find the 3×3 matrix A so that $f(\vec{x}) = A\vec{x}$ for any $\vec{x} \in \mathbf{R}^3$.

c. Now determine the matrix B so that $B\vec{x} = \hat{u} \times (\hat{u} \times \vec{x})$ for any $\vec{x} \in \mathbf{R}^3$.

d. Show that the matrix B is simply related to the matrix for projection onto the one-dimensional subspace of $\mathbf{R}^3$ spanned by $\hat{u}$.

12. $\boldsymbol{W}$ is the plane through the origin parallel to both $\vec{a} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 4 \\ 4 \\ 3 \end{bmatrix}$.

Compute the matrix P for orthogonal projection onto $\boldsymbol{W}$ in <u>three</u> distinct ways.

a. by using an orthonormal basis for $\boldsymbol{W}$ to construct P.

b. by finding the orthogonal complement $\boldsymbol{W}^{\perp}$ of $\boldsymbol{W}$.

c. by using a single formula involving $\left[\vec{a} \,|\, \vec{b} \right]$.

13. Let $\vec{w}=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$ and $(\vec{v}_1,\vec{v}_2)=\left(\begin{bmatrix}1\\1\\-1\\1\end{bmatrix},\begin{bmatrix}1\\-1\\1\\1\end{bmatrix}\right)$.

a. Find the distance from the vector $\vec{w}$ to the subspace $\operatorname{span}(\vec{v}_1,\vec{v}_2)$ in $\mathbf{R}^4$.

b. Find the area of the parallelogram in $\mathbf{R}^4$ two of whose concurrent edges are described by the vectors $\vec{v}_1$ and $\vec{v}_2$.

14. $T:\mathbf{R}^2 \to \mathbf{R}^2$ is a linear transformation that

- doubles all vectors in $\mathbf{R}^2$ parallel to the line with equation $x_1 + 2x_2 = 0$ and
- triples all vectors in $\mathbf{R}^2$ parallel to the line with equation $2x_1 - 3x_2 = 0$.

a. Determine a basis $\mathcal{B}$ for $\mathbf{R}^2$ relative to which the matrix for T is diagonal.

b. What is the $\mathcal{B}$–matrix for T?

c. What is the standard matrix for T?

d. What is $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$?

15. Suppose that $\boldsymbol{M}$ is a plane through the origin in $\mathbf{R}^3$ and F is the standard matrix for reflection across $\boldsymbol{M}$.

a. Describe, with as much specificity as possible, the eigenvalues and eigenspaces of F. Determine the algebraic and geometric multiplicities for each eigenvalue and specify the characteristic polynomial for F.

b. Suppose that $\boldsymbol{N}$ is a plane through the origin in $\mathbf{R}^3$ that is different from $\boldsymbol{M}$ and G is the matrix for reflection across $\boldsymbol{N}$. Then, F and G are similar. In fact, explain why there is an orthogonal matrix Q such that $F = QGQ^{-1}$.

16. a. If A is a 3×3 invertible matrix, what geometrical information does the value of $|\det(A)|$ convey about the column vectors of A?

b. If A is a 3×3 invertible matrix, what geometrical information does the value of $|\det(A)|$ convey about images of regions under the linear transformation $\vec{x} \mapsto A\,\vec{x}$?

c. If $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} j \\ k \\ l \end{bmatrix}$ where the coefficient matrix $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ is invertible, express x_3 as the ratio of the determinants of two 3×3 matrices. It is not necessary to evaluate the determinants.