1. Consider the linear system $A\vec{x} = \vec{b}$ where A is a 4×4 matrix, $\vec{b} \in \mathbf{R}^4$,

P is the 4×4 matrix for projection onto $\operatorname{im}(A)$, and $Q = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

is the matrix for projection onto $\ker(A)$.

a. Describe a simple test, involving only P and $\vec{b}$ to determine if the linear system above has any solutions for a given $\vec{b}$.

Solutions to the linear system exist if and only if $\vec{b} \in \operatorname{im}(A)$ if and only if $\vec{b} = P\vec{b}$.

b. Determine a basis for and the dimension of $\ker(A)$.

The column vectors of Q span the image of Q which is $\ker(A)$. So, a basis for $\ker(A)$ is, by inspection, $\left(\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right)$ and $\dim(\ker(A)) = 2$.

c. Find all solutions to the linear system above if one solution is $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$.

The solution vectors are $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ where α and $\beta \in \mathbf{R}$.

2. Let $S = \left(\begin{bmatrix} 1 \\ 1 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 4 \\ 3 \end{bmatrix} \right)$.

a. Find a basis for and the dimension of $S^{\perp}$.

We let A be the matrix whose column vectors are those in S. Then,

$$S^{\perp}=\left(\mathrm{im}(A)\right)^{\perp}=\ker(A^T)=\ker\begin{bmatrix}1&1&3&3\\1&2&5&6\\2&1&4&3\end{bmatrix}=\ker\begin{bmatrix}1&1&3&3\\0&1&2&3\\0&-1&-2&-3\end{bmatrix}$$

$$=\ker\begin{bmatrix}1&0&1&0\\0&1&2&3\\0&0&0&0\end{bmatrix}.\ \text{So, }\left(\vec{w}_1,\vec{w}_2\right)=\left(\begin{bmatrix}1\\2\\-1\\0\end{bmatrix},\begin{bmatrix}0\\3\\0\\-1\end{bmatrix}\right)\text{ is a basis for } S^{\perp}$$

according to the Solution Algorithm and $\dim(S^{\perp})=2$.

b. From a, find an orthonormal basis for $S^{\perp}$.

Let $\hat{u}_1=\vec{w}_1/\|\vec{w}_1\|=\frac{1}{\sqrt{6}}\begin{bmatrix}1\\2\\-1\\0\end{bmatrix}$. Then, $\vec{w}_2{}^{\perp}=\vec{w}_2-(\vec{w}_2\cdot\hat{u}_1)\hat{u}_1=\begin{bmatrix}0\\3\\0\\-1\end{bmatrix}-\begin{bmatrix}1\\2\\-1\\0\end{bmatrix}$

$=\begin{bmatrix}-1\\1\\1\\-1\end{bmatrix}$ and we let $\hat{u}_2=\vec{w}_2{}^{\perp}/\|\vec{w}_2{}^{\perp}\|=\frac{1}{2}\begin{bmatrix}-1\\1\\1\\-1\end{bmatrix}$. $\left(\hat{u}_1,\hat{u}_2\right)$ is the desired basis.

c. From b, find the 4×4 matrix P that represents projection onto $S^{\perp}$.

$P=\hat{u}_1\hat{u}_1{}^T+\hat{u}_2\hat{u}_2{}^T$ (alternately, $P=\left[\hat{u}_1\,|\,\hat{u}_2\right]\left[\hat{u}_1\,|\,\hat{u}_2\right]^T$) $=$

$$\frac{1}{6}\begin{bmatrix}1&2&-1&0\\2&4&-2&0\\-1&-2&1&0\\0&0&0&0\end{bmatrix}+\frac{1}{4}\begin{bmatrix}1&-1&-1&1\\-1&1&1&-1\\-1&1&1&-1\\1&-1&-1&1\end{bmatrix}=\frac{1}{12}\begin{bmatrix}5&1&-5&3\\1&11&-1&-3\\-5&-1&5&-3\\3&-3&-3&3\end{bmatrix}.$$

d. From c, find the vector in $S^{\perp}$ that is closest to $\hat{e}_1$.

The vector in $S^{\perp}$ closest to $\hat{e}_1$ is $\hat{e}_1$'s projection onto $S^{\perp}$, $P\hat{e}_1=\frac{1}{12}\begin{bmatrix}5\\1\\-5\\3\end{bmatrix}$.

3. $\mathscr{B}=\left(\hat{u}_1,\ \hat{u}_2,\ \hat{u}_3\right)=\left(\frac{1}{3}\begin{bmatrix}1\\2\\2\end{bmatrix},\ \frac{1}{3}\begin{bmatrix}2\\1\\-2\end{bmatrix},\ \frac{1}{3}\begin{bmatrix}-2\\2\\-1\end{bmatrix}\right)$ is an orthonormal basis for

$\mathbf{R}^3$. $T:\mathbf{R}^3 \to \mathbf{R}^3$ is linear and $T(\vec{u}_1)=\vec{u}_1$, $T(\vec{u}_2)=2\vec{u}_2$, $T(\vec{u}_3)=3\vec{u}_3$.

a. Find the matrix B for T in $\mathfrak{B}$-coordinates. In other words, find B so that $[T(\vec{x})]_{\mathfrak{B}} = B[\vec{x}]_{\mathfrak{B}}$ for any $\vec{x}\in\mathbf{R}^3$.

$$B=\left[[T(\hat{u}_1)]_{\mathfrak{B}}\,|\,[T(\hat{u}_2)]_{\mathfrak{B}}\,|\,[T(\hat{u}_3)]_{\mathfrak{B}}\right]=\left[[\hat{u}_1]_{\mathfrak{B}}\,|\,[2\hat{u}_2]_{\mathfrak{B}}\,|\,[3\hat{u}_3]_{\mathfrak{B}}\right]=\left[\hat{e}_1\,|\,2\hat{e}_2\,|\,3\hat{e}_3\right]$$
$$=\begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}.$$

b. Find the matrix A for T in standard coordinates. In other words, find A so that $T(\vec{x})=A\vec{x}$ for any $\vec{x}\in\mathbf{R}^3$.

The coordinate transformation matrix is $S=\left[\hat{u}_1\,|\,\hat{u}_2\,|\,\hat{u}_3\right]=\frac{1}{3}\begin{bmatrix}1&2&-2\\2&1&2\\2&-2&-1\end{bmatrix}$

and, since the vectors in $\mathfrak{B}$ are orthonormal, S is orthogonal. So, $S^{-1}=S^T$ and $A=SBS^{-1}=SBS^T=$

$$\frac{1}{3}\begin{bmatrix}1&2&-2\\2&1&2\\2&-2&-1\end{bmatrix}\begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}\frac{1}{3}\begin{bmatrix}1&2&2\\2&1&-2\\-2&2&-1\end{bmatrix}=\frac{1}{3}\begin{bmatrix}7&-2&0\\-2&-2&-2\\0&-2&5\end{bmatrix}.$$

4. Determine the straight line in $\mathbf{R}^2$ that fits the following data best in the least squares sense: (0, 1), (1, 0), (1, 3).

We seek the parameters r and s for the equation $y = r\,x + s$.

$\left\{\begin{matrix}1=r\,0+s\\0=r\,1+s\\3=r\,1+s\end{matrix}\right\}$ or $\vec{b}=A\vec{x}$ where $A=\begin{bmatrix}0&1\\1&1\\1&1\end{bmatrix}$, $\vec{x}=\begin{bmatrix}r\\s\end{bmatrix}$ and $\vec{b}=\begin{bmatrix}1\\0\\3\end{bmatrix}$. There is no solution to the equation $\vec{b}=A\vec{x}$ since $\vec{b}\notin \operatorname{im}(A)$; so we seek instead, to solve $A^T\vec{b}=A^TA\vec{x}$. We obtain $\vec{x}=\begin{bmatrix}r\\s\end{bmatrix}=(A^TA)^{-1}A^T\vec{b}=$

$$\left(\begin{bmatrix}0&1&1\\1&1&1\end{bmatrix}\begin{bmatrix}0&1\\1&1\\1&1\end{bmatrix}\right)^{-1}\begin{bmatrix}0&1&1\\1&1&1\end{bmatrix}\begin{bmatrix}1\\0\\3\end{bmatrix}=\begin{bmatrix}2&2\\2&3\end{bmatrix}^{-1}\begin{bmatrix}3\\4\end{bmatrix}=\frac{1}{2}\begin{bmatrix}3&-2\\-2&2\end{bmatrix}\begin{bmatrix}3\\4\end{bmatrix}=\begin{bmatrix}\frac{1}{2}\\1\end{bmatrix}.$$

So, the equation for the straight line that fits this data "best" is $y=\frac{1}{2}r+1$.

5. True (T) or False (F)? Circle one or none. [Negative half credit is earned for incorrect responses.]

a. ***False*** The intersection of the image and kernel of a 4×3 matrix is a subspace.
The kernel is a subspace of $\mathbf{R}^3$ and the image is a subspace of $\mathbf{R}^4$, so their intersection is empty.

b. ***False*** If the rank of a 5×4 matrix is 3, its nullity is 2.
Rank + nullity is the dimension of the domain which is 4.

c. ***False*** If $T:\mathbf{R}^3 \to \mathbf{R}^3$ is linear and $T(\hat{e}_1), T(\hat{e}_2), T(\hat{e}_3)$ each have unit length, then T is an orthogonal transformation.
Preserving the lengths of the basis vectors does not guarantee preservation of the lengths of <u>all</u> vectors. That is because the magnitude of a vector is not a linear property. For example, if $T(\hat{e}_1) = T(\hat{e}_2) = T(\hat{e}_3) = \hat{e}_1$, T is clearly not orthogonal since, for example, $\|T(\hat{e}_1 + \hat{e}_2)\| = \|T(\hat{e}_1) + T(\hat{e}_2)\| = \|2\hat{e}_1\| = 2$ but $\|\hat{e}_1 + \hat{e}_2\| = \sqrt{2}$.

d. ***False*** If A and B are both symmetric 3×3 matrices, then AB is also a symmetric 3×3 matrix.
$(AB)^T = B^T A^T = BA$ which differs from AB unless A and B commute.

e. ***True*** If the column vectors of a 4×3 matrix A are orthonormal, $A^T A$ is the 3×3 identity matrix.
In this case, $A = [\hat{u}_1 | \hat{u}_2 | \hat{u}_3]$ where $\hat{u}_j \cdot \hat{u}_k = \hat{u}_j^T \hat{u}_k = \delta_{jk}$. So,

$$A^T A = \begin{bmatrix} \hat{u}_1^T \\ \hline \hat{u}_2^T \\ \hline \hat{u}_3^T \end{bmatrix} [\hat{u}_1 | \hat{u}_2 | \hat{u}_3] = \begin{bmatrix} \hat{u}_1^T \hat{u}_1 & \hat{u}_1^T \hat{u}_2 & \hat{u}_1^T \hat{u}_3 \\ \hat{u}_2^T \hat{u}_1 & \hat{u}_2^T \hat{u}_2 & \hat{u}_2^T \hat{u}_3 \\ \hat{u}_3^T \hat{u}_1 & \hat{u}_3^T \hat{u}_2 & \hat{u}_3^T \hat{u}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$