Special Problem H solution

Let M be the plane in $\mathbf{R}^3$ whose equation is $2x_1 - 6x_2 + 3x_3 = 0$. Consider the linear transformation $f : \mathbf{R}^3 \to \mathbf{R}^3$ that reverses the direction of all vectors normal to M and doubles all vectors that are parallel to M. Determine the standard matrix for f and calculate $f\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right)$ for any reals a, b, and c.

Let A be the standard matrix for f. Then, $f(\vec{x}) = A\vec{x}$ for any $\vec{x} \in \mathbf{R}^3$. We now proceed to obtain A. A unit vector normal to M is $\hat{u} = \frac{1}{7}\begin{bmatrix} 2 \\ -6 \\ 3 \end{bmatrix}$ and so, the matrix for projection along the direction normal to M is $P = \hat{u}\,\hat{u}^T = \frac{1}{49}\begin{bmatrix} 4 & -12 & 6 \\ -12 & 36 & -18 \\ 6 & -18 & 9 \end{bmatrix}$.

Therefore, the standard matrix for projection onto M is $I - P$. We find

$$A = -P + 2(I - P) = 2I - 3P = \frac{1}{49}\begin{bmatrix} 98 & 0 & 0 \\ 0 & 98 & 0 \\ 0 & 0 & 98 \end{bmatrix} - \frac{1}{49}\begin{bmatrix} 12 & -36 & 18 \\ -36 & 108 & -54 \\ 18 & -54 & 27 \end{bmatrix}$$

$= \frac{1}{49}\begin{bmatrix} 86 & 36 & -18 \\ 36 & -10 & 54 \\ -18 & 54 & 71 \end{bmatrix}$. So, $f\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = A\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{1}{49}\begin{bmatrix} 86a + 36b - 18c \\ 36a - 10b + 54c \\ -18a + 54b + 71c \end{bmatrix}$.