1. The fully row-reduced echelon form of the matrix **A** is

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

T is the linear transformation on $\mathbf{R}^4$ such that $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$. Determine each of the following or state that it cannot be determined from the information given.

a. The matrix **A**.
b. A basis for the range of T.
c. A basis for the kernel of T.
d. The dimension of the range of T.
e. The dimension of the kernel of T.
f. The set of all vectors that are orthogonal to every row vector of **A**.
g. All linear combinations of the row vectors of **A** with resultant zero.
h. All the linear combinations of the column vectors of **A** that are zero.
i. The determinant of **A**.
j. All $\mathbf{x}$ for which $T(\mathbf{x}) = (1,2,3,4,5)$ if $T(1,1,1,1,1) = (1,2,3,4,5)$.

2. A square matrix **Q** is said to be *proper orthogonal* if and only if $\mathbf{Q}^T\mathbf{Q} = \mathbf{Q}\mathbf{Q}^T = \mathbf{I}$ and $\det(\mathbf{Q}) = +1$. It can be shown that a linear transformation on $\mathbf{R}^n$ whose matrix is proper orthogonal preserves the magnitude of vectors, the angles between them, and their relative orientation. Such transformations are rotations.

a. Show that the rows (or the columns) of a proper orthogonal 3×3 matrix **Q** are vectors of unit magnitude that are mutually orthogonal.

b. Verify that $\mathbf{Q} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$ is the matrix for a rotation in $\mathbf{R}^3$.

c. Think about the effect of a rotation on vectors that are parallel and vectors that are perpendicular to the axis of rotation. Now determine the axis of rotation and the angle of rotation for the matrix in part b.

3. Provide 8 different statements that do not include "column" or "transpose" and are equivalent to

The $n\times n$ matrix **A** is invertible (non-singular).

4. Suppose that $\mathbf{u} = (u_1, u_2, u_3)$ is a unit vector in $\mathbf{R}^3$.

a. Show that the matrix $\mathbf{P_u} = \begin{bmatrix} u_1 u_1 & u_1 u_2 & u_1 u_3 \\ u_2 u_1 & u_2 u_2 & u_2 u_3 \\ u_3 u_1 & u_3 u_2 & u_3 u_3 \end{bmatrix}$ represents projection along $\mathbf{u}$ in the standard basis.

b. Explain why $\mathbf{I} - \mathbf{P_u}$ represents projection orthogonal to $\mathbf{u}$.

c. Verify or explain: $\mathbf{P_u P_u} = \mathbf{P_u}$ and $(\mathbf{I} - \mathbf{P_u})(\mathbf{I} - \mathbf{P_u}) = \mathbf{I} - \mathbf{P_u}$.

d. Let $\mathbf{M}$ be a plane in $\mathbf{R}^3$ through the origin with unit normal $\mathbf{u}$. Show that reflection across $\mathbf{M}$ is the linear transformation whose matrix is $2\mathbf{P_u} - \mathbf{I}$.

e. Determine the matrix that represents reflection across the plane whose equation is $2x + y - 2z = 0$ and use it to find the image of the point $(1,-3,0)$ when reflected across this plane.

5. Let $\mathbf{S}$ be a subspace of dimension m in $\mathbf{R}^n$ and let $\mathbf{T}$ be the set of all vectors in $\mathbf{R}^n$ that are orthogonal to all the vectors in $\mathbf{S}$.

a. Show that $\mathbf{T}$ is a subspace of $\mathbf{R}^n$.

b. Show that the zero vector is the only vector that is common to both $\mathbf{S}$ and $\mathbf{T}$, i.e. $\mathbf{S} \cap \mathbf{T} = \{\mathbf{0}\}$.

c. Use part b to show that if $\mathbf{a} \in \mathbf{S}$, $\mathbf{b} \in \mathbf{T}$, and $\mathbf{a} + \mathbf{b} = \mathbf{0}$, then $\mathbf{a} = \mathbf{b} = \mathbf{0}$.

It is possible to show that every vector $\mathbf{x} \in \mathbf{R}^n$ is the sum of two vectors $\mathbf{s}$ and $\mathbf{t}$ where $\mathbf{s} \in \mathbf{S}$ and $\mathbf{t} \in \mathbf{T}$.

d. Use this last assertion and part c to show that for any $\mathbf{x} \in \mathbf{R}^n$, there is exactly one $\mathbf{s} \in \mathbf{S}$ and exactly one $\mathbf{t} \in \mathbf{T}$ so that $\mathbf{x} = \mathbf{s} + \mathbf{t}$. [Hint: suppose $\mathbf{x} = \mathbf{s}' + \mathbf{t}'$, where $\mathbf{s}' \in \mathbf{S}$ and $\mathbf{t} \in \mathbf{T}$.]

6. Suppose that $\mathbf{A}$ is a square $n \times n$ matrix.

a. If λ is an eigenvalue of $\mathbf{A}$, λ satisfies the scalar equation $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$. Why?

b. $\mathbf{A}$ has, at most, n distinct eigenvalues. Why?

c. Even if all entries of $\mathbf{A}$ are real, some eigenvalues of $\mathbf{A}$ may not be real. Why?

7. **A** is a real symmetric 3×3 matrix. It is known that 1 is an eigenvalue of **A**, and that $\mathbf{A}\begin{bmatrix}1\\-2\\2\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$, and $\mathbf{A}\begin{bmatrix}2\\2\\1\end{bmatrix}=\begin{bmatrix}4\\4\\2\end{bmatrix}$.

a. Determine all eigenvalues of **A**.

b. Determine all eigenvectors of **A** corresponding to each of its eigenvalues.

c. Compute $\mathbf{A}\begin{bmatrix}1\\2\\3\end{bmatrix}$.

8. The eigenvalues of a 3×3 matrix **A** are -1, 1, and 2 and eigenvectors corresponding to these eigenvalues are, respectively, $\begin{bmatrix}1\\1\\0\end{bmatrix}$, $\begin{bmatrix}1\\0\\1\end{bmatrix}$, and $\begin{bmatrix}1\\1\\1\end{bmatrix}$. Determine (all the entries of) $\mathbf{A}^n$ for any positive integer n.

9. If the $n\times n$ matrices **S** and **T** have a common set of eigenvectors $\mathbf{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and **B** is also basis for $\mathbf{R}^n$, prove that **S** and **T** commute. Do this by showing that $\mathbf{STx} = \mathbf{TSx}$ for every $\mathbf{x}$ in $\mathbf{R}^n$.

10. Suppose that f is a linear function from $\mathbf{R}^3$ to $\mathbf{R}^3$ and **b** is a fixed point in $\mathbf{R}^3$. Consider the solution sets for the equation $f(\mathbf{x}) = \mathbf{b}$. These solution sets are quite special. Not every subset of $\mathbf{R}^3$ can be a solution set of an equation of this kind. Describe all possible solution sets and be as specific as this limited information permits.

11. a. Construct a 3×3 matrix **A** that has the following properties. Its eigenvalues are 0, 1, and 4 and the eigenvectors corresponding to these eigenvalues are, respectively, $(1, 1, 1)$, $(1, 0, -1)$, and $(1, -2, 1)$.

b. Determine an invertible 3×3 matrix **Q** such that $\mathbf{Q}^{-1}\mathbf{AQ} = \mathbf{D}$ is a diagonal matrix and display **D**.

c. Find all entries of the matrix $\mathbf{A}^n$ for any positive integer n.

12. Every parabola with vertical symmetry axis in the plane has an equation of the form

$$ax^2 + bx + cy + d = 0$$

for some real coefficients a, b, c and d. Assuming that (x_1, y_1), (x_2, y_2), and (x_3, y_3) are three distinct points on this parabola, one also obtains:

$$ax_1^2 + bx_1 + cy_1 + d = 0$$

$$ax_2^2 + bx_2 + cy_2 + d = 0$$

$$ax_3^2 + bx_3 + cy_3 + d = 0$$

Now, we have 4 equations that are linear in the coefficients a, b, c and d.

a. What one would have to conclude about coefficients in the parabola equation if the determinant of $\begin{bmatrix} x^2 & x & y & 1 \\ x_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3 & y_3 & 1 \end{bmatrix}$ were <u>not</u> zero.

b. Use the results above to obtain an equation for the parabola containing the points (1, 0), (2, 1), and (3, 4).

13. Suppose that **A** and **B** are row-equivalent matrices. For each of the following assertions, state whether the assertion is **<u>T</u>**rue or **<u>F</u>**alse.

a. The row space of **A** is the same as the row space of **B**.

b. The columns space of **A** is the same as the column space of **B**.

c. The dimensions of the row spaces of **A** and **B** are the same.

d. The dimensions of the column spaces of **A** and **B** are the same.

e. If a linear combination of the rows of **A** is zero, the same linear combination of the corresponding rows of **B** is also zero.

f. If a linear combination of the columns of **A** is zero, the same linear combination, i.e. having the same coefficients, of the corresponding columns of **B** is also zero.

14. Give specific examples of each of the following.

a. A vector space **V** of dimension 4 that is not a subspace of any $\mathbf{R}^n$.

b. A basis **B** for your vector space **V**.

c. A two dimensional subspace of your vector space **V**.

d. A linear function f from $\mathbf{R}^4$ to your vector space **V** with kernel equal to the span of (1, 1, 0, 0).

e. The matrix of your function f with respect to the standard basis in $\mathbf{R}^4$ and your basis **B**.