SM261.1001 Special Problems 2 Solutions 29 January 2008

1. Let R_θ be the 2×2 matrix that represents counter-clockwise rotation in the plane about the origin by the angle θ. Verify: $R_\alpha\ R_\beta = R_\beta\ R_\alpha = R_{\alpha+\beta}$.

$$R_\alpha\ R_\beta = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\left(\sin\alpha\cos\beta + \cos\alpha\sin\beta\right) \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & \cos\alpha\cos\beta - \sin\alpha\sin\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix} = R_{\alpha+\beta}$$

$$R_\beta\ R_\alpha = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

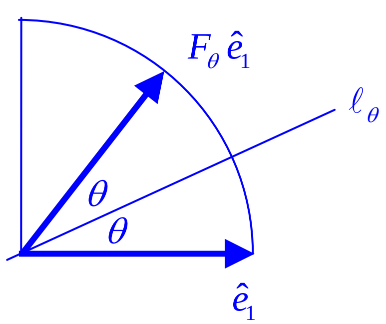
$$= \begin{bmatrix} \cos\beta\cos\alpha - \sin\beta\sin\alpha & -\left(\sin\beta\cos\alpha + \cos\beta\sin\alpha\right) \\ \sin\beta\cos\alpha + \cos\beta\sin\alpha & \cos\beta\cos\alpha - \sin\beta\sin\alpha \end{bmatrix}$$

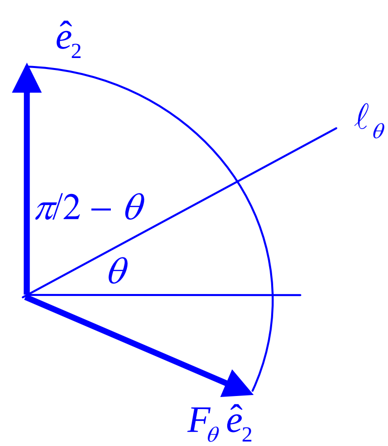
$$= \begin{bmatrix} \cos(\beta+\alpha) & -\sin(\beta+\alpha) \\ \sin(\beta+\alpha) & \cos(\beta+\alpha) \end{bmatrix} = R_{\beta+\alpha}$$

Of course, $R_{\alpha+\beta} = R_{\beta+\alpha}$. In each of the computations above, we employed the trigonometric formulae for the cosine or sine of a sum; this occurred after the third equality.

2. Let ℓ_θ be the line through the origin in $\mathbf{R}^2$ whose angle with the 1-axis (the x-axis, if you prefer) is θ.

a. Determine the 2×2 matrix F_θ that represents reflection across ℓ_θ. Proceed by finding (with the aid of diagrams and trigonometry) the images of the standard basis vectors under this reflection.





$$F_\theta\,\hat{e}_1=\begin{bmatrix}\cos(2\theta)\\ \sin(2\theta)\end{bmatrix}\qquad F_\theta\,\hat{e}_2=\begin{bmatrix}\cos\left(\theta-(\pi/2-\theta)\right)\\ -\sin\left(\theta-(\pi/2-\theta)\right)\end{bmatrix}=\begin{bmatrix}\sin(2\theta)\\ -\cos(2\theta)\end{bmatrix}$$

Consequently, $F_\theta=\begin{bmatrix}\cos(2\theta) & \sin(2\theta)\\ \sin(2\theta) & -\cos(2\theta)\end{bmatrix}$.

b. Compute and simplify the matrix A that represents the composite linear transformation that is reflection across ℓ_α followed by reflection across ℓ_β.

The matrix for the composite transformation is

$$F_\beta F_\alpha=\begin{bmatrix}\cos(2\beta) & \sin(2\beta)\\ \sin(2\beta) & -\cos(2\beta)\end{bmatrix}\begin{bmatrix}\cos(2\alpha) & \sin(2\alpha)\\ \sin(2\alpha) & -\cos(2\alpha)\end{bmatrix}$$

$$=\begin{bmatrix}\cos(2\beta)\cos(2\alpha)+\sin(2\beta)\sin(2\alpha) & \cos(2\beta)\sin(2\alpha)-\sin(2\beta)\cos(2\alpha)\\ \sin(2\beta)\cos(2\alpha)-\cos(2\beta)\sin(2\alpha) & \sin(2\beta)\sin(2\alpha)+\cos(2\beta)\cos(2\alpha)\end{bmatrix}$$

$$=\begin{bmatrix}\cos(2(\beta-\alpha)) & -\sin(2(\beta-\alpha))\\ \sin(2(\beta-\alpha)) & \cos(2(\beta-\alpha))\end{bmatrix}$$

We have used trigonometric identities for the sine and cosine of a sum (or difference) of angles after the third equality above.

c. Describe, as succinctly and completely as possible, the transformation represented by A.

The matrix in part b is $R_{2(\beta-\alpha)}$ which is rotation by twice the difference of the angle, $\beta-\alpha$. This is twice the angle measured ccw from the first to the second line of reflection. In other words, two successive reflections across lines through the origin is a rotation about the origin ! We might have expected this because of the following reasoning. Each reflection preserves the lengths of vectors and the angle between them. So, a composite of two reflections should do the same thing. On the other hand, a reflection reverses the relative orientation of one vector relative to another and two reflections will therefore preserve the orientation. So, the composite of two reflections will preserve lengths of vectors, angles between vectors, and the orientation of one vector relative to another. The only linear transformation that we have encountered that does this is a rotation. The calculation above confirms that the composite is, indeed, a rotation.