1. Determine all the ways in which 40 bills may be chosen from among bills with denominations of \$1, \$5 and \$10 so their total value is \$120.

Let the number of singles, fives and tens be, respectively x_1, x_2, x_3. Then, our system has the form $A\vec{x}=\vec{b}$ where $A=\begin{bmatrix}1&1&1\\1&5&10\end{bmatrix}$ and $\vec{b}=\begin{bmatrix}40\\120\end{bmatrix}$. Row-reduction yields the following $\left[A\,|\,\vec{b}\right]=\left[\begin{array}{ccc|c}1&1&1&40\\1&5&10&120\end{array}\right]\leftrightarrow\left[\begin{array}{ccc|c}1&0&-\frac{5}{4}&20\\0&1&\frac{9}{4}&20\end{array}\right]$. So, $\vec{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}20\\20\\0\end{bmatrix}+s\begin{bmatrix}-\frac{5}{4}\\\frac{9}{4}\\-1\end{bmatrix}$ where s is any real. However, the numbers of each bill must be nonnegative integers. So, we set $s=-4n$ where n is a nonnegative integer and obtain $\vec{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}20+5n\\20-9n\\4n\end{bmatrix}$. The values allowed for n are 0, 1, and 2. Therefore, $\vec{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}20\\20\\0\end{bmatrix},\begin{bmatrix}25\\11\\4\end{bmatrix},\begin{bmatrix}30\\2\\8\end{bmatrix}$.

2. Consider a 3×4 matrix A. Determine A_{rref} with as much specificity as possible if

a. no pair of the column vectors of A is collinear but all are coplanar.

$A_{rref}=\begin{bmatrix}1&0&a&c\\0&1&b&d\\0&0&0&0\end{bmatrix}$ where a, b, c, d are nonzero reals and $\det\begin{bmatrix}a&c\\b&d\end{bmatrix}\neq 0$. This last condition assures that the last two column vectors are not collinear.

b. no triplet of the column vectors of A is coplanar.

$A_{rref}=\begin{bmatrix}1&0&0&a\\0&1&0&b\\0&0&1&c\end{bmatrix}$ where $abc\neq 0$. This last condition assures that the last column vector is not coplanar with any pair of the first three column vectors.

3. Solve $A\vec{x}=\begin{bmatrix}1\\1\\7\\0\end{bmatrix}$ completely if $A\begin{bmatrix}1\\2\\3\\4\end{bmatrix}=A\begin{bmatrix}3\\2\\1\\1\end{bmatrix}=A\begin{bmatrix}4\\1\\1\\2\end{bmatrix}=\begin{bmatrix}1\\1\\7\\0\end{bmatrix}$ and rank(A) = 2.

Subtracting the first from the second and the second from the third equations yields

$A\begin{bmatrix}2\\0\\-2\\-3\end{bmatrix} = A\begin{bmatrix}1\\-1\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$ and so, since rank(A) = 2, nullity(A) = 2 and we have a basis for ker(A). The complete solution set for the given equation is the sum of any particular solution plus any vector from the kernel, i.e. any vector in $\begin{bmatrix}1\\2\\3\\4\end{bmatrix} + \text{span}\left(\begin{bmatrix}2\\0\\-2\\-3\end{bmatrix}, \begin{bmatrix}1\\-1\\0\\1\end{bmatrix}\right)$.

4. a. How is im(A) related to the existence of solutions to the linear system $A\vec{x} = \vec{b}$?

There exists a solution to the system if and only if $\vec{b} \in \text{im}(A)$.

b. How is ker(A) related to the uniqueness of solutions to the system $A\vec{x} = \vec{b}$?

If a solution to the system exists, it is unique if and only if $\ker(A) = \{\vec{0}\}$.

5. a. What effect do elementary row operations have on linear relationships among the column vectors of a matrix?

There is no effect.

b. What effect do elementary row operations have on the span of the row vectors of a matrix?

There is no effect.

If $A \in \mathbf{R}^{6\times 6}$, $A_{rref} = \begin{bmatrix}1&0&1&0&1&-2\\0&1&0&0&-1&1\\0&0&0&1&1&1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix}$ and $A^T{}_{rref} = \begin{bmatrix}1&0&0&1&1&1\\0&1&0&-1&-4&3\\0&0&1&0&1&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix}$,

determine a basis for each of the following subspaces of $\mathbf{R}^6$.

c. im(A).

A basis for im(A) is a list of the (transposes) of the nonzero row vectors of $A^T{}_{rref}$, namely $([1\ 0\ 1\ 1\ 1\ 1]^T, [0\ 1\ 0\ -1\ -4\ 3]^T, [0\ 0\ 1\ 0\ 1\ 0]^T)$.

d. ker(A).

A basis for ker(A) is obtained from the nonpivot columns of A_{rref} through the Solution Algorithm, namely $([1\ 0\ -1\ 0\ 0\ 0]^T, [1\ -1\ 0\ -1\ 0\ 0]^T, [-2\ 1\ 0\ 1\ 0\ -1]^T)$.

6. a. Determine if $\begin{bmatrix}3\\4\\3\\5\end{bmatrix} \in \operatorname{span}\left(\begin{bmatrix}1\\1\\2\\1\end{bmatrix},\begin{bmatrix}1\\2\\-1\\3\end{bmatrix},\begin{bmatrix}1\\2\\3\\-1\end{bmatrix},\begin{bmatrix}4\\7\\3\\6\end{bmatrix}\right)$.

$\operatorname{rref}\left(\left[\begin{array}{cccc|c}1&1&1&4&3\\1&2&2&7&4\\2&-1&3&3&3\\1&3&-1&6&5\end{array}\right]\right)=\left[\begin{array}{cccc|c}1&0&0&1&2\\0&1&0&2&1\\0&0&1&1&0\\0&0&0&0&0\end{array}\right]$ reveals that the given vector is a linear combination of the first three in the list and so belongs to the span of the four.

b. Find a basis for and the dimension of the subspace of $\mathbf{R}^4$ consisting of all vectors of the form $\begin{bmatrix}2a-3b\\a+b\\-2a\\a+2b\end{bmatrix}$ where a and b are any reals.

$\left\{\begin{bmatrix}2a-3b\\a+b\\-2a\\a+2b\end{bmatrix} \mid a,b\in\mathbf{R}\right\}=\left\{a\begin{bmatrix}2\\1\\-2\\1\end{bmatrix}+b\begin{bmatrix}-3\\1\\0\\2\end{bmatrix} \mid a,b\in\mathbf{R}\right\}=\operatorname{span}\left(\begin{bmatrix}2\\1\\-2\\1\end{bmatrix},\begin{bmatrix}-3\\1\\0\\2\end{bmatrix}\right)$. Therefore, the two vectors inside the parens constitute a basis for this subspace and its dimension is 2.

7. Let $W=\operatorname{span}\left(\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}2\\1\\3\end{bmatrix}\right)$. Determine:

a. a basis for $W^{\perp}$.

$W^{\perp}=\ker\begin{bmatrix}1&1&1\\2&1&3\end{bmatrix}=\ker\begin{bmatrix}1&0&2\\0&1&-1\end{bmatrix}=\operatorname{span}\begin{bmatrix}2\\-1\\-1\end{bmatrix}=\operatorname{span}(\vec{v})$. $(\vec{v})$ is a basis for $W^{\perp}$.

b. an orthogonal basis for W.

Let $(\vec{w}_1,\vec{w}_2)=\left(\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}2\\1\\3\end{bmatrix}\right)$. Then, $\vec{w}_2^{\perp}=\vec{w}_2-\left(\vec{w}_2\cdot\frac{\vec{w}_1}{\|\vec{w}_1\|}\right)\frac{\vec{w}_1}{\|\vec{w}_1\|}=\begin{bmatrix}2\\1\\3\end{bmatrix}-\frac{1}{3}(6)\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}0\\-1\\1\end{bmatrix}$.

So, $(\vec{w}_1,\vec{w}_2^{\perp})$ is an orthogonal basis for W.

c. the matrix P for projection onto W.

Let $\hat{u} = \vec{v} / \|\vec{v}\| = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$. The matrix for projection onto $W^\perp$ is $P^\perp = \hat{u}\hat{u}^T$

$= \frac{1}{6} \begin{bmatrix} 4 & -2 & -2 \\ -2 & 1 & 1 \\ -2 & 1 & 1 \end{bmatrix}$. So, $P = I - P^\perp = \frac{1}{6} \begin{bmatrix} 2 & 2 & 2 \\ 2 & 5 & -1 \\ 2 & -1 & 5 \end{bmatrix}$.

8. Let $\vec{s} = [2\ 1\ 3]^T$ and $\vec{t} = [2\ 2\ 4]^T$ and let $\vec{x}$ be any vector in $\mathbf{R}^3$. Define the transformation $f(\vec{x}) = \det\left[\vec{s} \mid \vec{x} \mid \vec{t}\,\right]$.

a. Using this definition, explain why f is a *linear* transformation from $\mathbf{R}^3$ to $\mathbf{R}^1$.

The determinant of a matrix is linear in any column or row vector of the matrix.

b. Determine the standard matrix for f.

A is a 1×3 matrix whose kth column vector (in this case, the vector is a single number) is $f(\hat{e}_k)$, the image of the kth standard basis vector, under f. That is,

$$A = [f(\hat{e}_1) \mid f(\hat{e}_2) \mid f(\hat{e}_3)] = \left[\det\left[\vec{s} \mid \hat{e}_1 \mid \vec{t}\,\right] \mid \det\left[\vec{s} \mid \hat{e}_2 \mid \vec{t}\,\right] \mid \det\left[\vec{s} \mid \hat{e}_3 \mid \vec{t}\,\right]\right]$$

$$= \left[\det\begin{bmatrix} 2 & 1 & 2 \\ 1 & 0 & 2 \\ 3 & 0 & 4 \end{bmatrix} \Bigg| \det\begin{bmatrix} 2 & 0 & 2 \\ 1 & 1 & 2 \\ 3 & 0 & 4 \end{bmatrix} \Bigg| \det\begin{bmatrix} 2 & 0 & 2 \\ 1 & 0 & 2 \\ 3 & 1 & 4 \end{bmatrix}\right] = \begin{bmatrix} 2 & 2 & -2 \end{bmatrix}.$$

9. Suppose $A \in \mathbf{R}^{10\times 10}$ and the diagonal entries of A are odd positive integers and the off-diagonal entries are even positive integers. Explain why A is invertible.

The contribution to $\det(A)$ from the diagonal pattern is an odd integer. All other patterns contribute a positive or negative even integer value because the entries in such patterns always includes at least one even integer. Since the sum of an odd integer and any number of even integers cannot be zero, the determinant is not zero and the matrix is invertible.

10. $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ are vectors in $\mathbf{R}^3$. Compute $\left|\det[\vec{v}_1 \mid \vec{v}_2 \mid \vec{v}_3]\right|$, if $\|\vec{v}_1\| = 3$, $\|\vec{v}_2^{\,\perp}\| = 2$ and $\|\vec{v}_3^{\,\perp}\| = 5$; where $\vec{v}_2^{\,\perp}$ is the projection of $\vec{v}_2$ orthogonal to $\vec{v}_1$; and $\vec{v}_3^{\,\perp}$ is the projection of $\vec{v}_3$ orthogonal to both $\vec{v}_1$ and $\vec{v}_2$.

This absolute value of this determinant is the volume of the parallelepiped whose concurrent edges are given by the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$. This is the same as the volume of a rectangular box with side lengths are $\|\vec{v}_1\|, \|\vec{v}_2^{\,\perp}\|$, and $\|\vec{v}_3^{\,\perp}\|$. Its value is (3)(2)(5) = 30.

11. $T : \mathbf{R}^2 \to \mathbf{R}^2$ is a linear transformation that doubles all vectors in $\mathbf{R}^2$ parallel to the vector $[3\ 1]^T$ and reverses all vectors in $\mathbf{R}^2$ parallel to the vector $[1\ 2]^T$.

a. Determine a basis $\mathcal{B}$ for $\mathbf{R}^2$ relative to which the matrix for T is diagonal.

$\mathfrak{B} = \left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)$. $S = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$.

b. What is the $\mathfrak{B}$–matrix for T?

$B = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$.

c. What is the standard matrix for T?

$A = S\,B\,S^{-1} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \frac{1}{5} \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 13 & -9 \\ 6 & -8 \end{bmatrix}$.

d. What is $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$?

$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = \frac{1}{5} \begin{bmatrix} 13 & -9 \\ 6 & -8 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} .8 \\ -.4 \end{bmatrix}$.

12. Suppose that all the cadets at West Point have ice cream at evening meal every day. Washington Hall serves three flavors: chocolate, vanilla and strawberry. It is found that of cadets who choose chocolate one evening, 10% will choose vanilla and 10% will choose strawberry the next evening; of cadets who choose vanilla one evening, 20% will choose chocolate and 10% will choose strawberry the next evening; and of cadets who choose strawberry one evening, 30% will choose chocolate and 10% will choose vanilla the next evening. Let $x_1(k)$, $x_2(k)$ and $x_3(k)$ be the number of cadets who choose chocolate, vanilla and strawberry, respectively, for the kth evening. Here, $k = 0, 1, 2, ...$

a. Show that the circumstances described above may be summarized by a single vector equation of the form $\vec{x}(k+1) = A\vec{x}(k)$ where $\vec{x}(k) = [x_1(k)\;\; x_2(k)\;\; x_3(k)]^T$. Identify the matrix A.

$$\left\{ \begin{array}{l} x_1(k+1) = .8\,x_1(k) + .2\,x_2(k) + .3\,x_3(k) \\ x_2(k+1) = .1\,x_1(k) + .7\,x_2(k) + .1\,x_3(k) \\ x_3(k+1) = .1\,x_1(k) + .1\,x_2(k) + .6\,x_3(k) \end{array} \right\} \Rightarrow \quad \vec{x}(k+1) = A\vec{x}(k) \quad \text{where } A = \begin{bmatrix} .8 & .2 & .3 \\ .1 & .7 & .1 \\ .1 & .1 & .6 \end{bmatrix}$$

b. Use the vector equation found in part a to express $\vec{x}(k)$ in terms of A and $\vec{x}(0)$, the initial vector of cadet flavor choices for any k.

$\vec{x}(k) = A^k\,\vec{x}(0)$

c. Explain why 1 must be an eigenvalue of A as a consequence of the fact that the entries in each column of A must sum to 1. [Hint: Think A^T.]

If the entries in each column of A sum to 1, then $A^T \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ which shows that 1 is an eigenvalue of A^T. But this means that 1 is also an eigenvalue of A.

d. Determine spec(A).

The characteristic polynomial of A is $f_A(\lambda)=\lambda^3-2.1\lambda^2+1.4\lambda-.3$. Since 1 is a root, the characteristic polynomial is divisible by $\lambda-1$. So, $f_A(\lambda)=(\lambda-1)(\lambda^2-1.1\lambda+.3)=$ $f_A(\lambda)=(\lambda-1)(\lambda^2-1.1\lambda+.3)=(\lambda-1)(\lambda-.6)(\lambda-.5)$. Hence, spec($A$) = (1, .6, .5) = $(\lambda_1, \lambda_2, \lambda_3)$.

e. Find $\vec{x}(k)$ for any k, assuming $\vec{x}(0)=c_1\vec{v}_1+c_2\vec{v}_2+c_3\vec{v}_3$ for fixed constants c_1, c_2, and c_3 and fixed eigenvectors $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ corresponding to the distinct eigenvalues λ_1, λ_2 and $\lambda_3 \in \text{spec}(A)$.

From part b, we have $\vec{x}(k)=A^k\,\vec{x}(0)=c_1\vec{v}_1+c_2(.6)^k\vec{v}_2+(.5)^k c_3\vec{v}_3$.

f. Determine $E_1(A)$ and find the exact ratios of chocolate to vanilla to strawberry consumed each evening in Washington Hall as the number of days approach infinity.

$E_1(A)=\ker\begin{bmatrix}-.2 & .2 & .3\\ .1 & -.3 & .1\\ .1 & .1 & -.4\end{bmatrix}=\ker\begin{bmatrix}-2 & 2 & 3\\ 1 & -3 & 1\\ 0 & 0 & 0\end{bmatrix}=\ker\begin{bmatrix}1 & 0 & -\frac{11}{4}\\ 0 & 1 & -\frac{5}{4}\\ 0 & 0 & 0\end{bmatrix}=\text{span}\begin{bmatrix}11\\5\\4\end{bmatrix}$. So, the proportions of chocolate to vanilla to strawberry are 11:5:4 in the "long run." If the mess officer initially purchases offers 55% chocolate, 25% vanilla and 20% strawberry, those proportions will remain forever constant.

13. What is the distance between the subspace $\text{span}\left(\begin{bmatrix}1\\1\\-1\\1\end{bmatrix},\begin{bmatrix}1\\-1\\1\\1\end{bmatrix}\right)$ and $\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$?

Denote the vectors in the order of their appearance above by $\vec{v}_1, \vec{v}_2, \vec{v}_3$, respectively. Let P be the matrix for projection onto the subspace V spanned by $\vec{v}_1$ and $\vec{v}_2$. The vector in V closest to $\vec{v}_3$ is $P\vec{v}_3$ and the distance between them is $\|\vec{v}_3-P\vec{v}_3\|$. Since $\vec{v}_1$ and $\vec{v}_2$ are orthogonal, an orthonormal basis for V is

$(\hat{u}_1,\hat{u}_2)=\left(\frac{1}{2}\begin{bmatrix}1\\1\\-1\\1\end{bmatrix},\frac{1}{2}\begin{bmatrix}1\\-1\\1\\1\end{bmatrix}\right)$. So, $P\vec{v}_3=(\hat{u}_1\cdot\vec{v}_3)\hat{u}_1+(\hat{u}_2\cdot\vec{v}_3)\hat{u}_2=\frac{1}{2}\begin{bmatrix}1\\1\\-1\\1\end{bmatrix}+\frac{1}{2}\begin{bmatrix}1\\-1\\1\\1\end{bmatrix}=\begin{bmatrix}1\\0\\0\\1\end{bmatrix}$. So, $\|\vec{v}_3-P\vec{v}_3\|=\sqrt{2}$.

14. $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}\in\mathbf{R}^3$. Complete the table below. The entry in each cell is the scalar (or inner or dot) product of the vector to its left and the vector above it.

In view of the fact that $\vec{v}\cdot\vec{w}=\vec{w}\cdot\vec{v}$, the table must be symmetric. From this, we see that

$(\vec{a},\vec{b},\vec{d})$ is an orthonormal basis for $\mathbf{R}^3$. Therefore $\vec{c} = (\vec{c}\cdot\vec{a})\vec{a} + (\vec{c}\cdot\vec{b})\vec{b} + (\vec{c}\cdot\vec{d})\vec{d}$ and so, $\vec{c}\cdot\vec{c} = (\vec{c}\cdot\vec{a})^2 + (\vec{c}\cdot\vec{b})^2 + (\vec{c}\cdot\vec{d})^2 =$ $(3)^2$ + $(-2)^2$ + $(5)^2$ = 38. The calculated values are highlighted below.

	$\vec{a}$	$\vec{b}$	$\vec{c}$	$\vec{d}$
$\vec{a}$	1	0	**3**	**0**
$\vec{b}$	**0**	1	–2	**0**
$\vec{c}$	3	**–2**	**38**	5
$\vec{d}$	0	0	**5**	1

15. Let $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & a & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ where a is a real number. Determine:

a. the characteristic polynomial for A.

A is an upper triangular matrix and so, by inspection, $\operatorname{spec}(A) = (1,1,2,2)$. So, $f_A(\lambda) =$ $(\lambda-1)^2(\lambda-2)^2$.

b. $E_\lambda(A)$, the eigenspace of A for each λ in spec(A). Make clear the dependence on the value of a.

$$E_1(A) = \ker(A - 1I) = \ker\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & a & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \ker\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \left\{ \begin{array}{ll} \operatorname{span}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} & \text{if } a \neq 0 \\ \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right) & \text{if } a = 0 \end{array} \right\}$$

$$E_2(A)=\ker(A-2I)=\ker\begin{bmatrix}-1&1&0&0\\0&0&a&0\\0&0&-1&1\\0&0&0&0\end{bmatrix}=\ker\begin{bmatrix}1&-1&0&0\\0&0&a&0\\0&0&1&-1\\0&0&0&0\end{bmatrix}=\left\{\begin{array}{ll}\operatorname{span}\begin{bmatrix}1\\1\\0\\0\end{bmatrix} & \text{if } a\neq 0\\ \operatorname{span}\left(\begin{bmatrix}1\\1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\1\end{bmatrix}\right) & \text{if } a=0\end{array}\right\}$$

c. the values of a for which A diagonalizable and the diagonal matrices to which A is similar.

A is diagonalizable if and only if $a = 0$. In this case, A is similar to one of six diagonal matrices whose diagonal entries, in order, are (1, 1, 2, 2), (1, 2, 1, 2), (1, 2, 2, 1), (2, 1, 1, 2), (2, 1, 2, 1) and (2, 2, 1, 1).

16. Two variables, s and t, are related by an equation of the form $s=x_1t+x_2t^2$ for a pair of constants x_1 and x_2. Find the best (least-squares) choices of x_1 and x_2 consistent with the following data.

s	t
-1	1
0	1
1	2
2	2

Let $\vec{x}=[x_1\ \ x_2]^T$. Substitution into the form above yields the linear system $A\vec{x}=\vec{b}$ where $A=\begin{bmatrix}1&1\\1&1\\2&4\\2&4\end{bmatrix}$ and $\vec{b}=\begin{bmatrix}-1\\0\\1\\2\end{bmatrix}$. Clearly, $\vec{b}\notin\operatorname{im}(A)$. So, we seek a least-squares solution, i.e. a solution to the system $A^TA\vec{x}=A^T\vec{b}$. Since the columns of A are linearly independent, A^TA is invertible and the least-squares solution is $\vec{x}=(A^TA)^{-1}A^T\vec{b}=$

$$\vec{x}=(A^TA)^{-1}A^T\vec{b}=\begin{bmatrix}10&18\\18&34\end{bmatrix}^{-1}\begin{bmatrix}1&1&2&2\\1&1&4&4\end{bmatrix}\begin{bmatrix}-1\\0\\1\\2\end{bmatrix}=\tfrac{1}{16}\begin{bmatrix}34&-18\\-18&10\end{bmatrix}\begin{bmatrix}3\\5\end{bmatrix}=\tfrac{1}{16}\begin{bmatrix}12\\-4\end{bmatrix}=\tfrac{1}{4}\begin{bmatrix}3\\-1\end{bmatrix}.$$

The least-squares solution is, therefore, $s=x_1t+x_2t^2=\tfrac{1}{4}t(3-t)$.

17. For **10** of the assertions below, state whether **True** or **False**. No reason is required. A correct response will earn full credit; no response will earn no credit; and an incorrect response will earn *negative* credit.

T a. Any list of mutually orthogonal nonzero vectors in $\mathbf{R}^n$ is linearly independent.
Set the square of the magnitude of any LC of mutually orthogonal vectors to zero to show the LC must be trivial.

F b. If $\vec{u}$, $\vec{v}$ and $\vec{w} \in \mathbf{R}^5$, and $\vec{u}$ is a linear combination of $\vec{v}$ and $\vec{w}$, then $\vec{w}$ is a linear combination of $\vec{u}$ and $\vec{v}$.
Suppose the first and second vectors are collinear and the third is not.

T c. There is a real 2×2 matrix A other than I so that $A^{13} = I$.
Consider the matrix for rotation by $2\pi/13$.

F d. There is an invertible 10×10 matrix whose entries include 92 ones.
At least two rows of such a matrix will be identical consisting of all ones.

T e. All rotations through a fixed angle θ around axes through the origin in $\mathbf{R}^3$ are similar.
One of these rotations may be changed to the other by the coordinate change that rotates one axis to the other.

T f. The magnitudes of the entries of an orthogonal matrix never exceed 1.
Each column is a unit vector whose entries have a magnitude at most 1.

T g. Every invertible matrix is the product of an orthogonal matrix and an upper triangular matrix.
This is a consequence of the QR Theorem (or the Gram-Schmidt orthogonalization).

T h. For any $m\times n$ matrix A, $\operatorname{im}(A^T) = \left(\ker(A)\right)^{\perp}$.

This is an identity equivalent to $(\operatorname{im}(A))^{\perp} = \ker(A^T)$

T i. If A is an invertible square matrix, $\ker(A) = \ker(A^{-1})$.
For an invertible matrix, $\ker(A) = \ker(A^{-1})$ is trivial.

F j. There are $n\times n$ matrices A and B such that $AB = 0$ but $BA \neq 0$.
$AB = 0$ implies that $\det(AB) = \det(A)\det(B)$ which implies that $\det(A)$ or $\det(B) = 0$.

T k. If each of the column vectors of a square matrix A is a unit vector, $|\det(A)| \leq 1$.
The determinant of a matrix has a magnitude that is at most the product of the magnitudes of its columns.

T l. If A is an invertible $n\times n$ matrix, at least one of its submatrices, obtained by deleting one row and one column of A, is also invertible.
This is an immediate consequence of the Laplace expansion of a determinant.

F m. If $\vec{v}$ and $\vec{w}$ are eigenvectors of a square matrix A, then $\vec{v}+\vec{w}$ is also an eigenvector of A.
If the vectors correspond to different eigenvalues this will be false.

F n. If A is an invertible 3×3 matrix and B is a 3×4 matrix, then $\ker(AB) = \{\vec{0}\}$.
B has a nontrivial kernel and $\ker(AB)$ is larger.

T o. Suppose that T is a linear transformation on $\mathbf{R}^n$ and A is its standard matrix. Then, all n^2 entries of A are completely determined by the vectors $\left(T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_n)\right)$ where $\left(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\right)$ is any basis for $\mathbf{R}^n$.
The standard matrix for a LT is related by similarity to the matrix for the same LT in a different basis.

T p. If $A \in \mathbf{R}^{m\times n}$ and $B \in \mathbf{R}^{n\times p}$, then $\operatorname{im}(AB)$ is a subspace of $\operatorname{im}(A)$.
Every vector in $\operatorname{im}(AB)$ belongs to $\operatorname{im}(A)$.