1. Complete each of the following definitions.

a. The span, $\text{span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p)$, of the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ in $\mathbf{R}^n$ is
the set of all linear combinations of the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ or, in set notation, $\{c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_p\vec{v}_p \mid c_1, c_2, \ldots, c_p \in \mathbf{R}\}$.

b. The list $(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p)$ of vectors in $\mathbf{R}^n$ is linearly independent iff
only the trivial linear combination of these vectors sums to the zero vector, i.e. $c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_p\vec{v}_p = \vec{0}$ implies $c_1 = c_2 = \ldots = c_p = 0$.

c. A subset V of $\mathbf{R}^n$ is a subspace iff
the subset is nonempty and is closed under vector addition and multiplication by scalars.

d. A list $(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p)$ of vectors in a subspace V is a basis for V iff
the list in linearly independent and spans V.

e. A function $f : \mathbf{R}^n \to \mathbf{R}^m$ is linear iff
for every r, $s \in \mathbf{R}$ and every $\vec{x}, \vec{y} \in \mathbf{R}^n$, $f(r\,\vec{x} + s\,\vec{y}) = r\,f(\vec{x}) + s\,f(\vec{y})$.

2. a. State a theorem that provides the connection between a linear function $f : \mathbf{R}^n \to \mathbf{R}^m$ and its associated $m \times n$ matrix A.

$f : \mathbf{R}^n \to \mathbf{R}^m$ is linear iff there is an $m \times n$ matrix A such that $f(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbf{R}^n$. The kth column of A is $f(\hat{e}_k)$, the image of the kth standard basis vector.

b. State a theorem that provides the connection between the dimensions of the kernel and the image of a linear function $f : \mathbf{R}^n \to \mathbf{R}^m$.

This is the Rank-Nullity Theorem: $\dim(\text{im}(f)) + \dim(\ker(f)) = n$.

c. List 6 distinct properties of an $n \times n$ matrix A that are equivalent to the assertion that A is invertible (or nonsingular). For this purpose, two properties that differ only by the replacement of "row" for "column" are not regarded as distinct.

There exists an $n \times n$ matrix B so that $A\,B = I_n$ (or $B\,A = I_n$). The linear system $A\vec{x} = \vec{b}$ has a unique solution for all $\vec{b} \in \mathbf{R}^n$. $A_{rref} = I_n$. ($A_{cref} = I$.) The column (row) vectors of A are linearly independent in $\mathbf{R}^n$. The column (row) vectors of A span $\mathbf{R}^n$. The column (row) vectors (rows) of A comprise a basis for $\mathbf{R}^n$. $\det(A) \neq 0$. $\text{im}(A) = \mathbf{R}^n$. $\ker(A) = \{\vec{0}\}$. 0 is not an eigenvalue of A. $\text{Rank}(A) = n$. $\text{Nullity}(A) = 0$.

3. Consider the system $A\vec{x}=\vec{b}$ where $A=\begin{bmatrix}1 & 2 & 3\\ 1 & k & 4\\ 1 & 2 & k+2\end{bmatrix}$ and $\vec{b}=\begin{bmatrix}4\\6\\6\end{bmatrix}$.

$$[A|\vec{b}]=\left[\begin{array}{ccc|c}1 & 2 & 3 & 4\\ 1 & k & 4 & 6\\ 1 & 2 & k+2 & 6\end{array}\right]\leftrightarrow\left[\begin{array}{ccc|c}1 & 2 & 3 & 4\\ 0 & k-2 & 1 & 2\\ 0 & 0 & k-1 & 2\end{array}\right]$$

For which values of k does this system have

a. no solutions?

There are no solutions if $k = 1$ since, in this case, the last row is inconsistent, implying that $0 = .$

b. exactly one solution?

There is exactly one solution if $k \neq 1$ and $k \neq 2$. In this case, the first three columns are pivot columns.

c. exactly two solutions?

There are no values of k for which the system has exactly two solutions.

d. infinitely many solutions?

There are infinitely many solutions if $k = 2$. In this case, the last two rows are identical and the first and third columns are pivot columns.

4. Suppose that A is a 4×4 matrix whose rank is 2 and

$A\begin{bmatrix}1\\1\\1\\1\end{bmatrix}=A\begin{bmatrix}2\\1\\3\\4\end{bmatrix}=A\begin{bmatrix}3\\3\\4\\2\end{bmatrix}=\begin{bmatrix}1\\0\\1\\0\end{bmatrix}$. Completely solve the equation $A\vec{x}=\begin{bmatrix}2\\0\\2\\0\end{bmatrix}$.

Since three different vectors have the same image under multiplication by A, we can obtain vectors in $\ker(A)$ by taking differences of the given equations. We have $A\begin{bmatrix}1\\0\\2\\3\end{bmatrix}=A\begin{bmatrix}1\\2\\1\\-2\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$. So, $\left(\begin{bmatrix}1\\0\\2\\3\end{bmatrix},\begin{bmatrix}1\\2\\1\\-2\end{bmatrix}\right)$ is a pair of vectors in $\ker(A)$. This pair is clearly linearly independent. Moreover, since $\operatorname{rank}(A) = 2$, the Rank-Nullity Theorem tells us that $\operatorname{nullity}(A) =$

$4 - 2 = 2 = \dim(\ker(A))$ and so, this pair is a basis for $\ker(A)$. By inspection, one solution to $A\vec{x} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}$ is $\vec{x} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}$. Any other solution differs from this by a vector in the kernel. Consequently, the complete solution is $\vec{x} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} + r\begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix} + s\begin{bmatrix} 1 \\ 2 \\ 1 \\ -2 \end{bmatrix}$ where r and s are arbitrary reals.

5. Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \\ 0 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$.

a. Find the 3-volume of the 3-parallelopiped in $\mathbf{R}^4$ whose concurrent edges are described by $\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3$.

The volume of the 3-parallelopiped, ω, is given by $\sqrt{\det\left(A^T A\right)}$ where $A = \left[\vec{v}_1 \,|\, \vec{v}_2 \,|\, \vec{v}_3\right]$. Alternatively, we can compute the product of the lengths of the edges of a rectangular 3-parallelopiped of the same volume. Since the second edge vector above is orthogonal to the first, $\vec{v}_2^{\,\perp} = \vec{v}_2$, we need to apply the Gram-Schmidt process to find $\vec{v}_3^{\,\perp}$, the part of $\vec{v}_3$ orthogonal to $\vec{v}_1$ and $\vec{v}_2$. $\vec{v}_3^{\,\perp} = \vec{v}_3 - \left(\dfrac{\vec{v}_3 \cdot \vec{v}_1}{\|\vec{v}_1\|^2}\right)\vec{v}_1 - \left(\dfrac{\vec{v}_3 \cdot \vec{v}_2}{\|\vec{v}_2\|^2}\right)\vec{v}_2 = \vec{v}_3 - \tfrac{3}{9}\vec{v}_1 - \tfrac{3}{9}\vec{v}_2 = \tfrac{1}{3}\begin{bmatrix} 0 \\ 1 \\ 2 \\ -2 \end{bmatrix}$. So,

$\omega = \|\vec{v}_1\| \cdot \|\vec{v}_2^{\,\perp}\| \cdot \|\vec{v}_3^{\,\perp}\| = (3)(3)(1) = 9.$

b. Find the 4×4 matrix that represents projection onto the subspace of $\mathbf{R}^4$ spanned by $\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3$.

The matrix for projection onto the subspace V is $P = A(A^T A)^{-1} A^T$. We could calculate P from this formula. Alternatively, we can construct an orthonormal basis B for this subspace. We have $\hat{u}_1 = \vec{v}_1 / \|\vec{v}_1\| = \tfrac{1}{3}\vec{v}_1$,

$\hat{u}_2 = \vec{v}_2 / \|\vec{v}_2\| = \frac{1}{3}\vec{v}_2$, and $\hat{u}_3 = \vec{v}_3^{\perp} / \|\vec{v}_3^{\perp}\| = \vec{v}_3^{\perp}$. We have

$$B = (\hat{u}_1, \hat{u}_2, \hat{u}_3) = \left(\tfrac{1}{3}\begin{bmatrix} 1 \\ 0 \\ 2 \\ 2 \end{bmatrix}, \tfrac{1}{3}\begin{bmatrix} 2 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \tfrac{1}{3}\begin{bmatrix} 0 \\ 1 \\ 2 \\ -2 \end{bmatrix} \right) \text{ and so,}$$

$$P = \hat{u}_1\,\hat{u}_1^T + \hat{u}_2\,\hat{u}_2^T + \hat{u}_3\,\hat{u}_3^T = \tfrac{1}{9}\begin{bmatrix} 5 & 4 & 0 & 2 \\ 4 & 5 & 0 & -2 \\ 0 & 0 & 9 & 0 \\ 2 & -2 & 0 & 8 \end{bmatrix}.$$

6. S is the plane in $\mathbf{R}^3$ whose equation is $x + 2y + 3z = 0$.

a. Determine the matrix A for the linear transformation on $\mathbf{R}^3$ that reverses vectors normal to S and doubles vectors parallel to S.

A unit normal to S is $\hat{u} = \frac{1}{\sqrt{14}}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. The matrix for projection onto the one dimensional subspace $S^{\perp}$ of normal vectors is $P^{\perp} = \hat{u}\,\hat{u}^T = \frac{1}{14}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$.

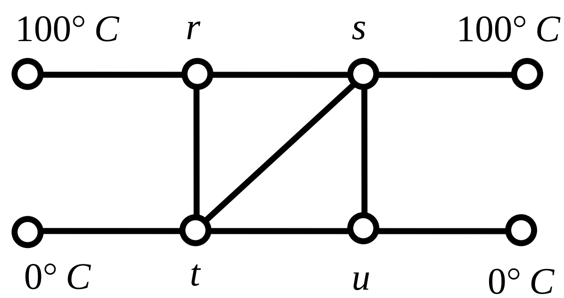
The matrix for projection onto S is $P = I - P^{\perp} = \frac{1}{14}\begin{bmatrix} 13 & -2 & -3 \\ -2 & 10 & -6 \\ -3 & -6 & 5 \end{bmatrix}$

The desired matrix is, therefore, $A = (-1)P^{\perp} + (2)P = \frac{1}{14}\begin{bmatrix} 25 & -6 & -9 \\ -6 & 16 & -18 \\ -9 & -18 & 1 \end{bmatrix}$.

b. Find the distance from the point $(1, 1, 1)$ to S.

The vector in S that is closest to $\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is its projection onto S, i.e. $P\,\vec{w}$. The distance between the two is $\|\vec{w} - P\,\vec{w}\| = \|P^{\perp}\,\vec{w}\| = \frac{6}{\sqrt{14}} \approx 1.60357$.

7. A grid with eight nodes is shown at the right. The four external nodes are maintained at fixed temperatures. The temperatures r, s, t and u at the four internal nodes are to be determined from the fact that the internal nodes are at thermal equilibrium and so their temperatures are the averages of the temperatures at the adjacent nodes. Two nodes are regarded as adjacent if they are connected by a line segment in the diagram. Formulate the problem as a linear system and solve for the temperatures at the internal nodes.

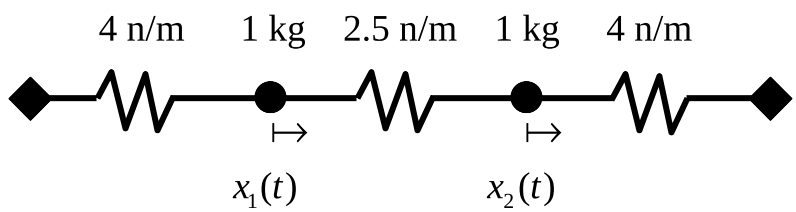


The equilibrium equations for the four internal nodes are as follows.

$$\left.\begin{aligned} r &= \tfrac{1}{3}(100+s+t) \\ s &= \tfrac{1}{4}(100+r+t+u) \\ t &= \tfrac{1}{4}(0+r+s+u) \\ u &= \tfrac{1}{3}(0+s+t) \end{aligned}\right\}\left\{\begin{aligned} &3r-s-t=100 \\ &-r+4s-t-u=100 \\ &-r-s+4t-u=0 \\ &-s-t+3u=0 \end{aligned}\right.$$

Let $\vec{x}=\begin{bmatrix} r \\ s \\ t \\ u \end{bmatrix}$, $A=\begin{bmatrix} 3 & -1 & -1 & 0 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ 0 & -1 & -1 & 3 \end{bmatrix}$, and $\vec{b}=\begin{bmatrix} 100 \\ 100 \\ 0 \\ 0 \end{bmatrix}$. Then, the scalar equations above are summarized by the single vector equation $A\vec{x}=\vec{b}$. We find, by row-reduction of $[A\,|\,\vec{b}]$ that $\vec{x}=\begin{bmatrix} r \\ s \\ t \\ u \end{bmatrix}=\begin{bmatrix} \frac{200}{3} \\ 60 \\ 40 \\ \frac{100}{3} \end{bmatrix}$.

8. Two point objects and three ideal springs are connected to one another and to two fixed external points along a straight line as diagrammed below. The displacements of the objects from their initial positions along this line are denoted by $x_1(t)$ and $x_2(t)$. Initially, each spring is relaxed and $x_1(0) = x_2(0) = 0$; the right object is at rest, $x_2'(0) = 0$; and the left object is struck so that $x_1'(0) = 1$. All units are MKS. Assume no other forces act on this system. Determine the position of each object for all time $t > 0$.



Applying Newton's Second Law of Motion to each object, we find that

$$\left.\begin{array}{l} x_1'' = -4x_1 - 2.5(x_1 - x_2) \\ x_2'' = +2.5(x_1 - x_2) - 4x_2 \end{array}\right\} \left\{\begin{array}{l} x_1'' + 6.5x_1 - 2.5x_2 = 0 \\ x_1'' - 2.5x_1 + 6.5x_2 = 0 \end{array}\right.$$

Now, let $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ and $A = \begin{bmatrix} \frac{13}{2} & -\frac{5}{2} \\ -\frac{5}{2} & \frac{13}{2} \end{bmatrix}$. Then, the coupled scalar ODEs above are abbreviated as the single vector ODE $\vec{x}'' + A\vec{x} = \vec{0}$.

The eigenvalues of A are obtained from the eigenvalue equation

$0 = \det(A - \lambda I) = (\frac{13}{2} - \lambda)^2 - (-\frac{5}{2})^2 = \lambda^2 - 13\lambda + 36 = (\lambda - 4)(\lambda - 9)$.

Hence, spec(A) = (4, 9). The eigenspaces are obtained from the eigenvector equations. $E_4(A) = \ker(A - 4I) = \ker\begin{bmatrix} \frac{5}{2} & -\frac{5}{2} \\ -\frac{5}{2} & \frac{5}{2} \end{bmatrix} = \operatorname{span}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$E_9(A) = \ker(A - 9I) = \ker\begin{bmatrix} -\frac{5}{2} & -\frac{5}{2} \\ -\frac{5}{2} & -\frac{5}{2} \end{bmatrix} = \operatorname{span}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Consequently,

$$\vec{x}(t) = (r_1\cos(2t) + s_1\sin(2t))\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (r_2\cos(3t) + s_2\sin(3t))\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Taking the initial conditions into account, we have the following.

$$\vec{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = (r_1)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (r_2)\begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow r_1 = r_2 = 0.$$

$$\vec{x}'(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = (2s_1)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (3s_2)\begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow s_1 = \tfrac{1}{4},\ s_2 = \tfrac{1}{6}.$$

Finally, $\vec{x}(t) = \frac{1}{4}\sin(2t)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{6}\sin(3t)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

9. Suppose that $A\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$ and $A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

From the above, spec(A) = (2, 3), $E_2(A) = \text{span}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $E_3(A) = \text{span}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Therefore, with $S = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$ $\left(S^{-1} = \tfrac{1}{5}\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \right)$ and $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, we have $A = S\,D\,S^{-1}$.

a. Let p be any real polynomial. Calculate the 2×2 matrix $p(A)$, expressing each of its four entries in terms of values of p.

$$p(A) = S\,p(D)\,S^{-1} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} p(2) & 0 \\ 0 & p(3) \end{bmatrix}\tfrac{1}{5}\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} = \tfrac{1}{5}\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2p(2) & -p(2) \\ -p(3) & 3p(3) \end{bmatrix}$$

$$= \tfrac{1}{5}\begin{bmatrix} 6p(2) - p(3) & -3p(2) + 3p(3) \\ 2p(2) - 2p(3) & -p(2) + 6p(3) \end{bmatrix}.$$

b. Determine the 2×2 matrix $\exp(A) = e^A$ by calculating each of its four entries.

$$\exp(A) = S\exp(D)S^{-1} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} e^2 & 0 \\ 0 & e^3 \end{bmatrix}\tfrac{1}{5}\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} = \tfrac{1}{5}\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2e^2 & -e^2 \\ -e^3 & 3e^3 \end{bmatrix}$$

$$= \tfrac{1}{5}\begin{bmatrix} 6e^2 - e^3 & -3e^2 + 3e^3 \\ 2e^2 - 2e^3 & -e^2 + 6e^3 \end{bmatrix} = \tfrac{1}{5}e^2\begin{bmatrix} 6 - e & -3 + 3e \\ 2 - 2e & -1 + 6e \end{bmatrix}.$$

10. Determine an equation for the straight line that fits the following data best in the least squares sense: (0, 1), (1, 0), (2, 1), and (3, 2).

Let the equation for the line desired be $s = \alpha + \beta t$ and let $\vec{x} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$. If a straight line fit the data, we would have a unique solution to the system

$A\vec{x} = \vec{b}$ where $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$. However, $\vec{b} \notin \text{im}(A)$ and so there is no conventional solution. We seek, instead, the Least Squares Solution by solving $A^T A\vec{x} = A^T\vec{b}$. The solution is $\vec{x} = \left(A^T A\right)^{-1} A^T\vec{b}$ =

$$\left(\begin{bmatrix}1&1&1&1\\0&1&2&3\end{bmatrix}\begin{bmatrix}1&0\\1&1\\1&2\\1&3\end{bmatrix}\right)^{-1}\begin{bmatrix}1&1&1&1\\0&1&2&3\end{bmatrix}\begin{bmatrix}1\\0\\1\\2\end{bmatrix}=\begin{bmatrix}4&6\\6&14\end{bmatrix}^{-1}\begin{bmatrix}4\\8\end{bmatrix}=\tfrac{2}{5}\begin{bmatrix}1\\1\end{bmatrix}$$

The Least Squares line has the equation $s = .4 + .4\,t$

11. Given that $A=\begin{bmatrix}1&2&0&3&5\\1&2&1&7&11\\2&4&-1&2&4\\1&2&2&11&17\end{bmatrix}$ and $A_{rref}=\begin{bmatrix}1&2&0&3&5\\0&0&1&4&6\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$,

a. Determine a basis for and the dimension of $\ker(A)$.

By the Solution Algorithm and inspection of A_{rref}, a basis for $\ker(A)$ is

$$\left(\begin{bmatrix}2\\-1\\0\\0\\0\end{bmatrix},\begin{bmatrix}3\\0\\4\\-1\\0\end{bmatrix},\begin{bmatrix}5\\0\\6\\0\\-1\end{bmatrix}\right)$$ and so, $\dim(\ker(A)) = 3$.

b. Determine a basis for and the dimension of $\operatorname{im}(A)$.

Inspection of A_{rref} reveals that the first and third column vectors of A are linearly independent and maximal. A basis for $\operatorname{im}(A)$ is

$$\left(\begin{bmatrix}1\\1\\2\\1\end{bmatrix},\begin{bmatrix}0\\1\\-1\\2\end{bmatrix}\right).$$ Therefore, $\dim(\operatorname{im}(A)) = 2$.

12. a. Prove that any list $(\vec{v}_1, \ldots, \vec{v}_p)$ of mutually orthogonal nonzero vectors in $\mathbf{R}^n$ is linearly independent.

Suppose that $c_1\vec{v}_1 + \ldots + c_p\vec{v}_p = \vec{0}$ for scalars $c_1, \ldots, c_p$. Now, take the inner product of both sides of this equation with $\vec{v}_1$. All terms on the left side, except for the first, are zero since $\vec{v}_j \cdot \vec{v}_k = 0$ whenever $j \neq k$. We are left with $c_1\|\vec{v}_1\|^2 = 0$. But, since $\hat{v}_1 \neq \vec{0}$, we conclude that $c_1 = 0$. The previous step can be repeated, replacing $\vec{v}_1$ with $\vec{v}_2$ and concluding that

$c_2 = 0$, and so on, until we deduce that all the scalar coefficients must be 0. Therefore, $(\vec{v}_1, \ldots, \vec{v}_p)$ is linearly independent.

b. Let $S = (\vec{w}_1, \ldots, \vec{w}_p)$ be any list of vectors in $\mathbf{R}^n$. Prove that $S^\perp$, the set of all vectors in $\mathbf{R}^n$ that are orthogonal to every vector in S, is a subspace of $\mathbf{R}^n$.

$S^\perp$ is not empty since it contains the zero vector. Now, suppose that $\vec{u}$ and $\vec{v}$ belong to $S^\perp$ and r and s are any scalars. Consider the linear combination $r\vec{u} + s\vec{v}$. The inner product of this vector with say, $\vec{w}_j$, is $(r\vec{u} + s\vec{v}) \cdot \vec{w}_j = r\vec{u} \cdot \vec{w}_j + s\vec{v} \cdot \vec{w}_j = r0 + s0 = 0$. Therefore, $r\vec{u} + s\vec{v} \in S^\perp$ and so $S^\perp$ is closed under linear combinations (and hence under vector addition and scalar multiplication). Therefore, $S^\perp$ is a subspace.

13. Let V be the subspace of $\mathbf{R}^4$ consisting of all vectors $\vec{v} = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$ such that $w + x + y + z = 0$ and $2w + x + 3y + z = 0$.

a. Find a basis for V.

$\vec{v}_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$ and $\vec{v}_2 = \begin{bmatrix} 2 & 1 & 3 & 1 \end{bmatrix}^T$ are vectors normal to the two planes whose equations are given above. Let $A = [\vec{v}_1 \,|\, \vec{v}_2]$. The subspace V consists of all the vectors orthogonal to both $\vec{v}_1$ and $\vec{v}_2$ and, therefore, to all vectors in their span. So, $V = \left(\mathrm{im}(A)\right)^\perp = \ker(A^T) = \ker\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 3 & 1 \end{bmatrix} =$

$\ker\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix} = \mathrm{span}\left(\begin{bmatrix} 2 \\ -1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}\right)$. So, a basis for V is $\left(\begin{bmatrix} 2 \\ -1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}\right)$.

b. Find a basis for $V^\perp$, the subspace of $\mathbf{R}^4$ consisting of all the vectors orthogonal to all the vectors in V.

$V^\perp = \left(\ker(A^T)\right)^\perp = \mathrm{im}(A)$ and so $\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix}\right)$ is a basis for $V^\perp$.

14. Suppose that V is a 3-dimensional subspace of $\mathbf{R}^5$ and $V^\perp$ is its orthogonal complement. P is the 5×5 matrix for orthogonal projection onto V and $P^\perp$ is the 5×5 matrix for orthogonal projection onto $V^\perp$.

a. Describe $\operatorname{im}(P)$ and $\ker(P)$.

The subspace onto which a projection matrix projects is its image. Therefore, $\operatorname{im}(P) = V$. The vectors annihilated by any matrix P are those in its kernel and, for a projection matrix those are the vectors orthogonal to the subspace onto which it projections, so $\ker(P) = V^\perp$.

b. What is the characteristic polynomial for P?

We know that projection matrices have only 1 and 0 in their spectrum. Since $\dim(V) = 3$ and $\dim(V^\perp) = 2$, we conclude that the eigenvalue 1 has algebraic and geometric multiplicity 3 while the eigenvalue 0 has algebraic and geometric multiplicity 2. Therefore, $f_A(\lambda) = \lambda^2(\lambda - 1)^3$.

c. What is the relationship between P and $P^\perp$?

There is more than one important relationship. $P + P^\perp = I_5$. Also, $P\,P^\perp = P^\perp P = 0_5$. These relationships were implicit in parts a and b.