1. Completely solve the linear system $A\vec{x} = \vec{b}$ for $\vec{x}$ if

a. $$rref([A|\vec{b}]) = \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 4 & 0 \\ 0 & 1 & 3 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

There is no solution since the fourth row implies 0 = 1.

b. $$rref([A|\vec{b}]) = \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 3 & 6 \\ 0 & 0 & 1 & 0 & 4 & 7 \\ 0 & 0 & 0 & 1 & 5 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

According to the Solution Algorithm,

$$\vec{x} = \begin{bmatrix} 6 \\ 0 \\ 7 \\ 8 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 3 \\ 0 \\ 4 \\ 5 \\ -1 \end{bmatrix} \text{ for any } \alpha \text{ and } \beta \text{ in } \mathbf{R}.$$

2. Suppose that $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ are any three vectors in $\mathbf{R}^n$, where n is any positive integer greater than 2. How can one determine, unambiguously, whether or not $\vec{v}_3$ is a linear combination of $\vec{v}_1$ and $\vec{v}_2$? Show that this problem can be recast as a standard problem and describe how to find the answer.

$\vec{v}_3$ is a linear combination of $\vec{v}_1$ and $\vec{v}_2$ if and only if there are reals x_1 and x_2 so that $x_1\vec{v}_1 + x_2\vec{v}_2 = \vec{v}_3$. Our problem is equivalent to determining whether $A\vec{x} = \vec{b}$ has a solution if $A = [\vec{v}_1 | \vec{v}_2]$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and $\vec{b} = \vec{v}_3$.

The answer is affirmative, if and only if the third column of $rref([A|\vec{b}])$ has no leading one.

3. A linear system $A\vec{x}=\vec{b}$, must have exactly none, one, or infinitely many solutions. Why can't there be, say, exactly two solutions? The following shows why.

a. Suppose that $\vec{v}$ and $\vec{w}$ are two distinct solutions to $A\vec{x}=\vec{b}$. What does that mean?

It means that $\vec{v}\neq\vec{w}$, $A\vec{v}=\vec{b}$, and $A\vec{w}=\vec{b}$.

b. Show that $\vec{v}+\alpha(\vec{v}-\vec{w})$ is a solution to $A\vec{x}=\vec{b}$ for every choice of α in $\mathbf{R}$ and each different choice of α gives a different solution.

$A\left(\vec{v}+\alpha(\vec{v}-\vec{w})\right)=A\vec{v}+\alpha(A\vec{v}-A\vec{w})=\vec{b}+\alpha(\vec{b}-\vec{b})=\vec{b}$. This shows that $\vec{v}+\alpha(\vec{v}-\vec{w})$ is a solution to $A\vec{x}=\vec{b}$ for each α in $\mathbf{R}$. Since $\vec{v}-\vec{w}\neq\vec{0}$, each value of α yields a different solution. So, if there are two distinct solutions, there are infinitely more solutions. There cannot be only two solutions.

4. Provide the missing items in the following statement of the central theorem in Chapter 2 of the text.

If $f:\mathbf{R}^n\to\mathbf{R}^m$ is a linear transformation, then $f(\vec{x})=$ $A\vec{x}$ for any $\vec{x}$ in $\mathbf{R}^n$ where A is an $m\times n$ matrix whose kth column is $f(\hat{e}_k)$ and $\hat{e}_k$ is the kth standard basis vector in $\mathbf{R}^n$.

5. f is a linear transformation on $\mathbf{R}^2$ such that

$$f\left(\begin{bmatrix}1\\1\end{bmatrix}\right)=\begin{bmatrix}1\\3\end{bmatrix} \text{ and } f\left(\begin{bmatrix}1\\2\end{bmatrix}\right)=\begin{bmatrix}4\\5\end{bmatrix}.$$

Find the matrix A corresponding to f.

$A=[f(\hat{e}_1)\,|\,f(\hat{e}_2)]$. $f(\hat{e}_1)$ and $f(\hat{e}_2)$ are found from the above.

$$f\left(\begin{bmatrix}1\\1\end{bmatrix}\right)=f(\hat{e}_1)+f(\hat{e}_2)=\begin{bmatrix}1\\3\end{bmatrix} \text{ and } f\left(\begin{bmatrix}1\\2\end{bmatrix}\right)=f(\hat{e}_1)+2f(\hat{e}_2)=\begin{bmatrix}4\\5\end{bmatrix}.$$

Subtracting the first from the second yields $f(\hat{e}_2)=\begin{bmatrix}4\\5\end{bmatrix}-\begin{bmatrix}1\\3\end{bmatrix}=\begin{bmatrix}3\\2\end{bmatrix}$.

Substitution gives $f(\hat{e}_1)=\begin{bmatrix}1\\3\end{bmatrix}-f(\hat{e}_2)=\begin{bmatrix}1\\3\end{bmatrix}-\begin{bmatrix}3\\2\end{bmatrix}=\begin{bmatrix}-2\\1\end{bmatrix}$. So, $A=\begin{bmatrix}-2&3\\1&2\end{bmatrix}$.

6. a. Determine the matrix P that corresponds to projection onto the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in $\mathbf{R}^2$.

The unit vector parallel to the given vector is $\hat{u} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Therefore,
$P = \begin{bmatrix} u_1u_1 & u_1u_2 \\ u_2u_1 & u_2u_2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

b. Describe, in geometric terms, the linear transformation to which the matrix $I - P$ corresponds.

$I - P$ corresponds to projection onto vectors perpendicular to $\hat{u}$.

c. Determine the matrix A that corresponds to the linear transformation on $\mathbf{R}^2$ that doubles all vectors parallel to $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and reverses the direction of all vectors perpendicular to $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Let f be the transformation described. $f(\vec{x}) = A\vec{x}$ for any $\vec{x}$ in $\mathbf{R}^2$. We resolve $\vec{x}$ into its projections parallel and perpendicular to $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. That is, $\vec{x} = \left[P + (I-P)\right]\vec{x}$. Multiplying by A, we find $A\vec{x} = A\left[P + (I-P)\right]\vec{x} = A[P\vec{x}] + A[(I-P)\vec{x}] = 2P\vec{x} - (I-P)\vec{x} = (3P - I)\vec{x}$. Since $\vec{x}$ is arbitrary,
$A = 3P - I = \frac{3}{5}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -2 & 6 \\ 6 & 7 \end{bmatrix}$.

7. Compute A^{-1} if $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 5 \end{bmatrix}$.

$$\left[A \mid I\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 3 & 1 & 5 & 0 & 0 & 1 \end{array}\right] \leftrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 1 & 2 & -3 & 0 & 1 \end{array}\right]$$
$$\leftrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right] \leftrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & -1 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right] = [I \mid A^{-1}]$$

So, $A^{-1} = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 1 \end{bmatrix}$.

8. If f is a linear transformation on $\mathbf{R}^n$ and f is also one-to-one and onto, f is invertible and so f^{-1} exists. This means that the equation $\vec{b} = f(\vec{x})$ has a unique solution for every $\vec{b}$ in $\mathbf{R}^n$ and it is $\vec{x} = f^{-1}(\vec{b})$. Too, for any $\vec{x}$ in $\mathbf{R}^n$, $f^{-1}(f(\vec{x})) = f(f^{-1}(\vec{x})) = \vec{x}$. In class, we showed that, for any vectors $\vec{u}$ and $\vec{v}$ in $\mathbf{R}^n$, $f^{-1}(\vec{u}+\vec{v}) = f^{-1}(\vec{u}) + f^{-1}(\vec{v})$. Complete the proof of the linearity of f^{-1} by showing that, for any real number α and any vector $\vec{w}$ in $\mathbf{R}^n$ $f^{-1}(\alpha\vec{w}) = \alpha f^{-1}(\vec{w})$.

$\vec{w} = f(\vec{z})$ for some unique $\vec{z}$ in $\mathbf{R}^n$. Also, $\vec{z} = f^{-1}(\vec{w})$. So, by linearity of f, $f^{-1}(\alpha\vec{w}) = f^{-1}(\alpha f(\vec{z})) = f^{-1}(f(\alpha\vec{z})) = \alpha\vec{z} = \alpha f^{-1}(\vec{w})$.

9. Exactly one of the following matrices represents a rotation about a line through the origin in $\mathbf{R}^3$ and exactly one represents a projection along a line through the origin in $\mathbf{R}^3$. Identify the matrix for the rotation and the matrix for the projection and provide convincing arguments for your choices.

$$A = \frac{1}{6}\begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{bmatrix}, \quad B = \frac{1}{7}\begin{bmatrix} 2 & 3 & 6 \\ 3 & 6 & 2 \\ 6 & 2 & 3 \end{bmatrix},$$

$$C = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ -2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}, \quad D = \frac{1}{5}\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}.$$

Rotations preserve the lengths of vectors and the angles between them. The column vectors of a rotation matrix are the images of the standard basis vectors under that rotation. So, the column vectors of the matrix for a rotation must be of unit length and they must be orthogonal to one another. Only C has these properties.

Projection onto a line always yields a vector collinear with the line. The column vectors of a projection matrix are the images of the standard basis vectors under the projection. Since these must be collinear, the column vectors of a matrix representing projection onto a line must be multiples of each other. Only A has this property.