SM261 FINAL EXAMINATION
14 DECEMBER 2006

PART ONE: CALCULATORS ARE NOT PERMITTED

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1. Let $A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 2 \end{bmatrix}$ and let $B = \begin{bmatrix} 1 & 2 \\ 1 & 4 \\ 1 & 7 \end{bmatrix}$.

   a. Calculate AB.

   $$AB = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 4 \\ 1 & 7 \end{bmatrix} = \begin{bmatrix} 6 & 28 \\ 4 & 22 \end{bmatrix}$$

   b. Calculate $B^T A^T$.

   $$B^T A^T = (AB)^T = \begin{bmatrix} 6 & 28 \\ 4 & 22 \end{bmatrix}^T = \begin{bmatrix} 6 & 4 \\ 28 & 22 \end{bmatrix}.$$

2. Let $C = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 4 \\ 1 & 1 & 2 \end{bmatrix}$. Find C^{-1}.

$$[C \mid I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 4 & 4 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \leftrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & -2 & -2 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \end{array}\right]$$

$$\leftrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 1 & 1 & 0 & -1 \end{array}\right] \leftrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & -\frac{1}{2} & 0 \end{array}\right]$$

$$\leftrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & -\frac{1}{2} & 0 \end{array}\right] \leftrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & \frac{1}{2} & 2 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & -1 \\ 0 & 0 & 1 & 1 & -\frac{1}{2} & 0 \end{array}\right]$$

$$= [C \mid I]_{rref} = [I \mid C^{-1}] \quad \Rightarrow \quad C^{-1} = \begin{bmatrix} -2 & \frac{1}{2} & 2 \\ 0 & \frac{1}{2} & -1 \\ 1 & -\frac{1}{2} & 0 \end{bmatrix}$$

3. Find all solutions to the following system of equations. Write your solutions in vector form.

$$\begin{array}{rcrcrcrcrcr} x_1 & + & x_2 & - & x_3 & - & x_4 & + & x_5 & = & 2 \\ 2x_1 & + & 2x_2 & - & x_3 & - & x_4 & + & x_5 & = & -1 \\ 4x_1 & + & 4x_2 & - & 3x_3 & - & x_4 & + & 3x_5 & = & 3 \end{array}$$

The system is equivalent to the single matrix equation $A\vec{x}=\vec{b}$ whose adjoined matrix is $\left[A|\vec{b}\right]=\left[\begin{array}{ccccc|c} 1 & 1 & -1 & -1 & 1 & 2 \\ 2 & 2 & -1 & -1 & 1 & -1 \\ 4 & 4 & -3 & -1 & 3 & 3 \end{array}\right]$. Row reduction yiels

$$\left[A|\vec{b}\right] \leftrightarrow \left[\begin{array}{ccccc|c} 1 & 1 & -1 & -1 & 1 & 2 \\ 0 & 0 & 1 & 1 & -1 & -5 \\ 0 & 0 & 1 & 3 & -1 & -5 \end{array}\right] \leftrightarrow \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & -3 \\ 0 & 0 & 1 & 1 & -1 & -5 \\ 0 & 0 & 0 & 2 & 0 & 0 \end{array}\right]$$

$$\leftrightarrow \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & -3 \\ 0 & 0 & 1 & 1 & -1 & -5 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right] \leftrightarrow \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & -1 & -5 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right] = \left[A|\vec{b}\right]_{rref}.$$

So, $\vec{x}=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}=\begin{bmatrix} -3 \\ 0 \\ -5 \\ 0 \\ 0 \end{bmatrix}+s\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}+t\begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ -1 \end{bmatrix}$ where s and t are arbitrary reals.

4. Identify the redundant vectors among the vectors in the list below.

$$\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 5 \\ 0 \end{bmatrix}\right).$$

The second and fifth vectors are redundant: the second is 2 times the first and the fifth is the sum of 3 times the first, 4 times the third, and 5 times the fourth.

5. Use row reduction techniques to find det(A) if $A=\begin{bmatrix}1&2&0&0\\3&2&1&0\\0&1&1&1\\0&0&2&1\end{bmatrix}$.

$$\det(A)=\begin{vmatrix}1&2&0&0\\3&2&1&0\\0&1&1&1\\0&0&2&1\end{vmatrix}=\begin{vmatrix}1&2&0&0\\0&-4&1&0\\0&1&1&1\\0&0&2&1\end{vmatrix}=\begin{vmatrix}1&2&0&0\\0&0&5&4\\0&1&1&1\\0&0&2&1\end{vmatrix}=\begin{vmatrix}1&2&0&0\\0&0&5&4\\0&1&1&1\\0&0&2&1\end{vmatrix}$$

$$=\begin{vmatrix}1&2&0&0\\0&1&1&1\\0&0&2&1\\0&0&5&4\end{vmatrix}=\begin{vmatrix}1&2&0&0\\0&1&0&1\\0&0&1&1\\0&0&1&4\end{vmatrix}=\begin{vmatrix}1&2&0&0\\0&1&0&1\\0&0&1&1\\0&0&0&3\end{vmatrix}=3.$$

6. Let T be the linear transformation determined by $T(\vec{e}_1)=\begin{bmatrix}3\\4\end{bmatrix}$ and $T(\vec{e}_2)=\begin{bmatrix}-4\\3\end{bmatrix}$.

a. Find the matrix of T with respect to the standard basis $\{\vec{e}_1,\vec{e}_2\}$.

$T(\vec{x})=A\vec{x}$ where $A=\left[T(\vec{e}_1)\mid T(\vec{e}_2)\right]=\begin{bmatrix}3&-4\\4&3\end{bmatrix}$.

b. Find the matrix of T with respect to the basis $\left\{\begin{bmatrix}2\\1\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix}\right\}$.

Define the change of coordinate matrix S by $S=\begin{bmatrix}2&1\\1&1\end{bmatrix}$. Then, B the matrix of T in the basis $\mathfrak{B}=\left(\begin{bmatrix}2\\1\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix}\right)$ is given by $[T(\vec{x})]_{\mathfrak{B}}=B[\vec{x}]_{\mathfrak{B}}$ and $B=S^{-1}AS=\begin{bmatrix}2&1\\1&1\end{bmatrix}^{-1}\begin{bmatrix}3&-4\\4&3\end{bmatrix}\begin{bmatrix}2&1\\1&1\end{bmatrix}=\begin{bmatrix}1&-1\\-1&2\end{bmatrix}\begin{bmatrix}2&-1\\11&7\end{bmatrix}=\begin{bmatrix}-9&-8\\20&15\end{bmatrix}$.

c. Is T an orthogonal linear transformation? Explain.

T is not an orthogonal transformation since the columns of A do not constitute an orthonormal basis for R^2. Equivalently, the images of the standard basis vectors are not vectors of unit length.

7. Let A be the matrix $\begin{bmatrix} 16 & 9 \\ -4 & 4 \end{bmatrix}$.

   a. Find all of the eigenvalues of the matrix A.

   $$0 = \det(A - \lambda I) = \det\begin{bmatrix} 16-\lambda & 9 \\ -4 & 4-\lambda \end{bmatrix} = (16-\lambda)(4-\lambda)+36$$

   $$= \lambda^2 - 20 + \lambda + 64 + 36 = \lambda^2 - 20 + \lambda + 100 = (\lambda - 10)^2 \Rightarrow$$

   $\text{spec}(A) = (10, 10)$. 10 is the only eigenvalue; it is repeated twice.

   b. For one of the eigenvalues of the matrix A compute the corresponding eigenspace.

   $$E_{10}(A) = \ker(A - 10I) = \ker\begin{bmatrix} 6 & 9 \\ -4 & -6 \end{bmatrix} = \ker\begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix} = \text{span}\begin{bmatrix} 3 \\ -2 \end{bmatrix}.$$

8. Use Cramer's Rule to find the solutions to the system

$$\begin{array}{rcrcr} 2x & + & y & = & 4 \\ 3x & + & 10y & = & 3 \end{array}$$

$$x = \frac{\det\begin{bmatrix} 4 & 1 \\ 3 & 10 \end{bmatrix}}{\det\begin{bmatrix} 2 & 1 \\ 3 & 10 \end{bmatrix}} = \frac{37}{17}, \qquad y = \frac{\det\begin{bmatrix} 2 & 4 \\ 3 & 3 \end{bmatrix}}{\det\begin{bmatrix} 2 & 1 \\ 3 & 10 \end{bmatrix}} = \frac{-6}{17}$$

END OF PART ONE

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PART TWO: CALCULATORS ARE PERMITTED

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1. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 4 & 1 & 5 \\ 2 & 4 & 2 & 6 \\ 1 & 2 & 2 & 4 \end{bmatrix}$.

For this problem, it is necessary to determine the rref of A.

$$A_{rref} = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

a. Find a basis for im(A).

The pivot columns of A are its first and third. So, a basis for im(A) is $\left(\begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix} \right)$.

b. Find a basis for ker(A).

A basis for the solution space of $A\vec{x} = \vec{0}$ is determined by the nonpivot columns of A_{rref}; we have the basis $\left(\begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right)$

2. Suppose $\vec{v}_1, \vec{v}_2,$ and $\vec{v}_3$ are non-zero vectors in R^3 that are orthogonal to each other, i.e. $0 = \vec{v}_1 \cdot \vec{v}_2 = \vec{v}_1 \cdot \vec{v}_3 = \vec{v}_2 \cdot \vec{v}_3$.

a. Explain why the three vectors are linearly independent.

$\vec{0} = a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 \quad \Rightarrow \quad \vec{v}_k \cdot \vec{0} = 0 = \vec{v}_k \cdot (a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3) = a_k(\vec{v}_k \cdot \vec{v}_k)$

$\Rightarrow \quad 0 = a_k \left\|\vec{v}_k\right\|^2 \quad \Rightarrow \quad a_k = 0$ for $k = 1, 2, 3$. Only the trivial linear combination of the three vectors sums to the zero vector.

b. Explain, using part a, why the three vectors form a basis for R^3.

Any three linearly independent vectors in R^3 comprise a basis for R^3.

3. I have *17* bills in my pocket (*1*'s, *5*'s, and *10*'s) whose total value is *$77*. How many of each type of bill do I have? (Use techniques from this course to find all solutions.)

Let x_1, x_2, and x_3 be the number of singles, fives, and tens, respectively. Then,
$$\begin{Bmatrix} x_1 + x_2 + x_3 = 17 \\ x_1 + 5x_2 + 10x_3 = 77 \end{Bmatrix}$$

This pair of scalar equations is equivalent to the single matrix equation

$A\vec{x}=\vec{b}$ where $A=\begin{bmatrix}1 & 1 & 1\\ 1 & 5 & 10\end{bmatrix}$, $\vec{x}=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$, and $\vec{b}=\begin{bmatrix}17\\ 77\end{bmatrix}$. Then,

$$[A\,|\,\vec{b}]=\left[\begin{array}{ccc|c}1 & 1 & 1 & 17\\ 1 & 5 & 10 & 77\end{array}\right]\leftrightarrow[A\,|\,\vec{b}]_{rref}=\left[\begin{array}{ccc|c}1 & 0 & -\frac{5}{4} & 2\\ 0 & 1 & \frac{9}{4} & 15\end{array}\right]\Rightarrow \vec{x}=\begin{bmatrix}2\\ 15\\ 0\end{bmatrix}+s\begin{bmatrix}\frac{5}{4}\\ -\frac{9}{4}\\ 1\end{bmatrix}$$

where s must be a real. However, to obtain nonnegative solutions, we must stipulate that $s = 4k$ where k is a nonnegative integer to obtain integer solutions. So,

$\vec{x}=\begin{bmatrix}2\\ 15\\ 0\end{bmatrix}+k\begin{bmatrix}5\\ -9\\ 4\end{bmatrix}$. Only $k = 0$ and 1 yield sensible solutions and they are

$\vec{x}=\begin{bmatrix}2\\ 15\\ 0\end{bmatrix}$ and $\vec{x}=\begin{bmatrix}7\\ 6\\ 4\end{bmatrix}$.

4. Let A be a 10×10 invertible matrix.
   a. What does it mean for A to be invertible?
      There is a 10×10 matrix B such that $AB = BA = I$.
   b. What are the possible values of the rank of A?
      The rank of A is the number of pivot columns of A. Since A is invertible, rref(A) = I. So, the rank of A must be 10.
   c. What are the possible values of the nullity of A?.
      For a 10×10 matrix A, rank(A) + nullity(A) = 10. So, nullity(A) = 0.
   d. What are the possible values of det(A)?
      det(A) is any real other than 0.
   e. Explain why for any 10×1 vector $\vec{b}$ the equation $A\vec{x}=\vec{b}$ is consistent, i.e. has a solution.
      Since A^{-1} exists, $A\vec{x}=\vec{b}$ has the unique solution $\vec{x}=A^{-1}\vec{b}$.

5. Suppose an $n\times n$ matrix A satisfies the matrix equation $A^2+2A=I$, where I is the $n\times n$ identity matrix. Show that A is invertible.

   The equation may be rewritten as $A(A+2I)=I$. This reveals that the matrix $A+2I$ is the inverse of A.

6. Suppose A is a 3×8 matrix.
   a. What are the possible values of the rank of A?
      rank(A) is the number of pivots of A and this is, at most, the smaller of the number of its rows and columns. So rank(A) = 0, 1, 2, 3.
   b. What are the possible values of the nullity of A?
      By the Rank-Nullity Theorem, rank(A) + nullity(A) = 8, so the possible values for nullity(A) are 5, 6, 7, or 8.
   c. What are the possible values of the sum of the rank and nullity of A?
      Referring back to part b, rank(A) + nullity(A) = 8.

7. a. Given a subspace V of R^n, define $V^\perp$ and explain why it is a subspace (of R^n).
      $V^\perp$ is the set of all vectors in R^n that are orthogonal to every vector in V. It is a subspace because every linear combination of two vectors orthogonal to every vector in V is also orthogonal to every vector in V.

   b. Let V be the subspace of R^3 with basis $\left\{\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$. Find a basis for $V^\perp$.

      If $A=\begin{bmatrix}1&0\\1&1\\0&1\end{bmatrix}$, $V^\perp=\ker(A^T)=\ker\begin{bmatrix}1&1&0\\0&1&1\end{bmatrix}=\ker\begin{bmatrix}1&0&-1\\0&1&1\end{bmatrix}$

$=\operatorname{span}\begin{bmatrix}1\\-1\\1\end{bmatrix}$. So, $\left(\begin{bmatrix}1\\-1\\1\end{bmatrix}\right)$ is a basis for $V^\perp$.

8. Let V be the subspace of R^4 spanned by the vectors $\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$ and $\begin{bmatrix}0\\2\\2\\0\end{bmatrix}$.

   a. Use the Gram-Schmidt method to find an orthonormal basis for V.

We let $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Then, $\hat{u}_1 = \vec{v}_1 / \|\vec{v}_1\| = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. If $\vec{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 0 \end{bmatrix}$, then

$\vec{v}_2^{\perp} = \vec{v}_2 - (\vec{v}_2 \cdot \hat{u}_1)\hat{u}_1 = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 0 \end{bmatrix} - (2)\frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$. So, $\hat{u}_2 = \vec{v}_2^{\perp} / \|\vec{v}_2^{\perp}\| = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$ and $(\hat{u}_1, \hat{u}_2)$ is the desired orthonormal basis for V.

b. Find $proj_V\left(\begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right)$, the projection of the vector $\begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ onto V.

Let $\vec{w} = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Then, $proj_V(\vec{w}) = (\vec{w} \cdot \hat{u}_1)\hat{u}_1 + (\vec{w} \cdot \hat{u}_2)\hat{u}_2$

$= (2)\frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + (-2)\frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix}$.

9. Suppose $\vec{v}_1, \vec{v}_2, \vec{v}_3,$ and $\vec{v}_4$ are the *rows* of a 4×4 matrix *A*, i.e., *A* = $\begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vec{v}_3 \\ \vec{v}_4 \end{bmatrix}$. Suppose also that det(*A*) = 2. Find the determinants of the following matrices. Explain your answers.

a. $M = \begin{bmatrix} \vec{v}_3 \\ \vec{v}_2 \\ \vec{v}_1 \\ \vec{v}_4 \end{bmatrix}$ b. $M = \begin{bmatrix} \vec{v}_1 + 3\vec{v}_2 \\ \vec{v}_2 \\ 4\vec{v}_3 \\ \vec{v}_4 \end{bmatrix}$ c. $M = \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vec{v}_2 \\ \vec{v}_4 \end{bmatrix}$

a det(M) = –2. Swapping two rows changes the sign of the determinant.

b. det(M) = 8. Adding any multiple of one row of a matrix to another row has no effect on the determinant. Multiplying a row of a matrix by a constant multiplies the determinant by that constant.

c. det(M) = 0. The second and third rows are identical and so this matrix is not invertible. Equivalently, subtracting the second from the third row results in a matrix whose third row is all zeros.

10. A matrix A has eigenvalues 2 and 3.

a. Show that if $\vec{v}$ is an eigenvector of A then it is also an eigenvector of A^2. What are the eigenvalues of A^2?

If $\vec{v}$ is an eigenvector of A with eigenvalue λ, then

$$A\vec{v} = \lambda\vec{v} \quad \Rightarrow \quad AA\vec{v} = A\lambda\vec{v} = \lambda A\vec{v} = \lambda\lambda\vec{v} = \lambda^2\vec{v} \quad \Rightarrow \quad A^2 v = \lambda^2\vec{v}$$

So, $\vec{v}$ is an eigenvector of A^2 with eigenvalue λ^2. The eigenvalues of A are therefore 4 and 9.

b. Show that if $\vec{v}$ is an eigenvector of A then it is also an eigenvector of A^{-1}. What are the eigenvalues of A^{-1}?

First notice that, since 0 is not an eigenvalue of A, A is invertible. If $\vec{v}$ is an eigenvector of A with eigenvalue λ, then

$$A\vec{v} = \lambda\vec{v} \quad \Rightarrow \quad A^{-1}A\vec{v} = \vec{v} = A^{-1}\lambda\vec{v} = \lambda A^{-1}\vec{v} \quad \Rightarrow \quad A^{-1}v = \lambda^{-1}\vec{v}.$$

So, $\vec{v}$ is an eigenvector of A^{-1} with eigenvalue λ^{-1}. The eigenvalues of A^{-1} are therefore 1/2 and 1/3.

11. Find the best (least squares) fit $y = c_0 + c_1 t$ to the data $(t, y) = (1,–1)$ (2,1), and (3,4).

If there were a line that fit the data exactly, it would satisfy the three scalar equations $\left\{ \begin{aligned} -1 &= c_0 + 1c_1 \\ 1 &= c_0 + 2c_1 \\ 4 &= c_0 + 3c_1 \end{aligned} \right\}$ or, equivalently, the single matrix equation

$A\vec{x}=\vec{b}$ where $A=\begin{bmatrix}1&1\\1&2\\1&3\end{bmatrix}$, $\vec{x}=\begin{bmatrix}c_0\\c_1\end{bmatrix}$, $\vec{b}=\begin{bmatrix}-1\\1\\4\end{bmatrix}$. However, this equation has no solution since $\vec{b}\notin \operatorname{im}(A)$. Instead, we solve $A^TA\vec{x}=A^T\vec{b}$. Since the columns of A are linearly independent, we have a unique solution to this normalized equation. It is

$$\vec{x}=\begin{bmatrix}c_0\\c_1\end{bmatrix}=(A^TA)^{-1}A^T\vec{b}=\left(\begin{bmatrix}1&1\\1&2\\1&3\end{bmatrix}^T\begin{bmatrix}1&1\\1&2\\1&3\end{bmatrix}\right)^{-1}\begin{bmatrix}1&1\\1&2\\1&3\end{bmatrix}^T\begin{bmatrix}-1\\1\\4\end{bmatrix}$$

$$=\left(\begin{bmatrix}1&1&1\\1&2&3\end{bmatrix}\begin{bmatrix}1&1\\1&2\\1&3\end{bmatrix}\right)^{-1}\begin{bmatrix}1&1&1\\1&2&3\end{bmatrix}\begin{bmatrix}-1\\1\\4\end{bmatrix}=\begin{bmatrix}3&6\\6&14\end{bmatrix}^{-1}\begin{bmatrix}4\\13\end{bmatrix}$$

$$=\tfrac{1}{6}\begin{bmatrix}14&-6\\-6&3\end{bmatrix}\begin{bmatrix}4\\13\end{bmatrix}=\begin{bmatrix}-22/6\\15/6\end{bmatrix}=\begin{bmatrix}-11/3\\3/2\end{bmatrix}\Rightarrow y=-\tfrac{11}{3}+\tfrac{3}{2}t.$$

12. Let T be the linear transformation from R^2 to R^2 which is the projection onto the line $y=x$. Let A be the matrix of the linear transformation T.

a. Find A.

A unit vector parallel to the line is $\hat{u}=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$ and so the matrix that represents T is $A=\hat{u}\,\hat{u}^T=\frac{1}{2}\begin{bmatrix}1&1\\1&1\end{bmatrix}$

b. Find the eigenvalues and eigenvectors of the matrix A.

The eigenvalues of A are determined by the characteristic equation

$$0=\det(A-\lambda I)=\det\begin{bmatrix}\frac{1}{2}-\lambda&\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}-\lambda\end{bmatrix}=(\tfrac{1}{2}-\lambda)^2-\tfrac{1}{4}=\lambda^2-\lambda+\tfrac{1}{4}-\tfrac{1}{4}$$

$$=\lambda^2-\lambda=\lambda(\lambda-1)\quad\Rightarrow\quad \operatorname{spec}(A)=(0,1).$$

The eigenspaces are determined as follows.

$$E_0(A) = \ker(A - 0I) = \ker\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \ker\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \operatorname{span}\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

$$E_1(A) = \ker(A - 1I) = \ker\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \ker\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = \operatorname{span}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The results here are intuitive because the projection leaves vectors parallel to the line unaffected while it annihilates vectors orthogonal to the line.

c. Use part b to find an invertible matrix S and a diagonal matrix D so that $S^{-1}AS = D.$

Choosing one nonzero vector from each eigenspace, e.g. $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, we set $S = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. Then, $D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, a diagonal matrix whose diagonal entries are the corresponding eigenvalues for the two basis vectors we chose above. Now, we have $S^{-1}AS = D$.

END OF PART TWO