Special Problem 7 Solutions

7. It is possible that the image and kernel of a matrix can be the same. Of course, this means the matrix would have to be square.
Since the kernel is a subspace of the domain and the image is a subspace of the co-domain, if the two subspaces are the same, the domain and co-domain are the same. So, the hypothesis only makes sense if the matrix is square.

a. Using the Rank-Nullity Theorem, show that, for an $n \times n$ matrix A, $\ker(A) = \operatorname{im}(A)$ implies that n must be a positive even integer. [Consequently, there are no 3×3 matrices whose kernel and image is the same subspace.]
Let k be the dimension of $\ker(A) = \operatorname{im}(A)$. So, their dimensions, the nullity and rank of A are both k. This is a non-negative integer and the Rank-Nullity Theorem tells us that $n = k + k = 2k$. Therefore, n must be an even positive integer.

b. Show that if $\ker(A) = \operatorname{im}(A)$, A cannot be the zero matrix yet its square, A^2, is the zero matrix.
For any vector $\vec{x}$ in $\mathbf{R}^n$, $A\vec{x}$ is a vector in $\operatorname{im}(A)$. Since the image and kernel are the same for A, $A\vec{x}$ is also a vector in $\operatorname{im}(A)$. Therefore, $A\vec{x}$ belongs to $\ker(A)$ and so, $A(A\vec{x}) = A^2\vec{x} = \vec{0}$. Since $\vec{x}$ is arbitrary, it follows that A^2 is the zero matrix. Now, A cannot be the zero matrix because that would imply $\ker(A) = \mathbf{R}^n$ but $\operatorname{im}(A) = \{\vec{0}\}$.

c. Find a 2×2 matrix such that $\ker(A) = \operatorname{im}(A)$. [Here is another example in which matrices differ from ordinary real numbers. We know that the square of a real number is zero if and only if that number is zero.]
Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then, $A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+bc & (a+d)b \\ (a+d)c & bc+d^2 \end{bmatrix}$ and we have $a^2+bc=0$, $(a+d)b=0$, $(a+d)c=0$, $bc+d^2=0$ and so, A must be a matrix of the form $\begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$ where a and b are any real numbers but not both are 0. For example, if we set $a = b = 0$, we get $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$. It is easy to check that $\ker(A) = \operatorname{im}(A) = \operatorname{span}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$ and $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.