1. Provide complete and accurate definitions for each of the underlined and italicized terms by completing the sentences below.

a. The *span* of the list $(\vec{v}_1, \ldots, \vec{v}_p)$ of vectors in $\mathbf{R}^n$ is the set of all linear combinations of the vectors in the list; i.e., in set notation it is $\{\alpha_1\vec{v}_1 + \ldots + \alpha_p\vec{v}_p \mid \alpha_1, \ldots, \alpha_p \in \mathbf{R}\}$.

b. The list $(\vec{v}_1, \ldots, \vec{v}_p)$ of vectors in $\mathbf{R}^n$ is *linearly independent* if and only if none of its vectors is a linear combination of the others or only trivial linear combination of the vectors in the list is the zero vector in $\mathbf{R}^n$.

c. A nonempty subset $\mathbf{S}$ of $\mathbf{R}^n$ is a *subspace* of $\mathbf{R}^n$ if and only if $\mathbf{S}$ is closed under vector addition and closed under multiplication by scalars or $\mathbf{S}$ is closed under linear combinations.

d. A *basis* for a subspace $\mathbf{S}$ of $\mathbf{R}^n$ is a list of vectors in $\mathbf{S}$ that is linearly independent and spans $\mathbf{S}$.

e. The *kernel*, $\ker(A)$, of an $m\times n$ matrix A is the subset of all vectors $\vec{x}$ in $\mathbf{R}^n$ so that $A\vec{x} = \vec{0}$; i.e., in set notation it is $\{\vec{x} \in \mathbf{R}^n \mid A\vec{x} = \vec{0}\}$.

f. The *image*, $\operatorname{im}(A)$, of an $m\times n$ matrix A is the subset vectors $A\vec{x}$ in $\mathbf{R}^m$ for all $\vec{x}$ in $\mathbf{R}^n$; i.e., in set notation it is $\{A\vec{x} \mid \vec{x} \in \mathbf{R}^n\}$.

2. Consider the matrix equation $A\vec{x} = \vec{b}$ where A is a given $m\times n$ matrix and $\vec{b}$ is a given vector in $\mathbf{R}^m$. What does

a. $\operatorname{im}(A)$ tell us about the existence of solutions to $A\vec{x} = \vec{b}$?

Since the image of A contains all the values of $A\vec{x}$, the equation has no solutions unless $\vec{b}$ belongs to $\operatorname{im}(A)$. If $\vec{b}$ does not belong to $\operatorname{im}(A)$, there can be no $\vec{x}$ so that $A\vec{x} = \vec{b}$; in this case, the system is inconsistent.

b. $\ker(A)$ tell us about the uniqueness of solutions to $A\vec{x} = \vec{b}$?

Solutions to the equation are unique if and only if $\ker(A) = \{\vec{0}\}$. For, if $\vec{x}$ is a solution to the equation, $\vec{x} + \vec{k}$ is another solution for every nonzero $\vec{k} \in \ker(A)$ since $\vec{x} + \vec{k} \neq \vec{x}$ and $A(\vec{x} + \vec{k}) = A\vec{x} + A\vec{k} = \vec{b} + \vec{0} = \vec{b}$.

3. Given $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 1 & 3 & 2 \\ 1 & 0 & 1 & 2 \\ 3 & 1 & 4 & 5 \end{bmatrix}$ and $\operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, find bases for $\ker(A)$ and for $\operatorname{im}(A)$.

Examination of $\mathrm{rref}(A)$ reveals that the first and second column vectors of A comprise a linearly independent pair and the third and fourth columns are dependent on the first two. The same is true of the corresponding column vectors of A, so a basis for $\mathrm{im}(A)$ is $\left(\begin{bmatrix}1\\2\\1\\3\end{bmatrix},\begin{bmatrix}2\\1\\0\\1\end{bmatrix}\right)$. Using the Solution algorithm, we find that a basis for $\ker(A)$ is $\left(\begin{bmatrix}1\\1\\-1\\0\end{bmatrix},\begin{bmatrix}2\\-1\\0\\-1\end{bmatrix}\right)$. This is a basis for the set of vectors orthogonal to the row vectors of A or of $\mathrm{rref}(A)$. It is a basis for the subspace of vectors in $\mathbf{R}^4$ that satisfy the equation $A\vec{x}=\vec{0}$.

4. $\vec{v}_1=\begin{bmatrix}1\\2\\1\\0\end{bmatrix}$, $\vec{v}_2=\begin{bmatrix}1\\0\\-1\\2\end{bmatrix}$, and $\vec{v}_3=\begin{bmatrix}4\\5\\4\\3\end{bmatrix}$. Notice: $\vec{v}_1$ and $\vec{v}_2$ are orthogonal.

a. Determine the 4×4 matrices P_1 and P_2 that project onto the one dimensional subspaces $\mathbf{S}_1 = \mathrm{span}(\vec{v}_1)$ and $\mathbf{S}_2 = \mathrm{span}(\vec{v}_2)$, respectively.

Unit vectors parallel to $\vec{v}_1$ and $\vec{v}_2$ are $\hat{u}_1=\vec{v}_1/\sqrt{6}$ and $\hat{u}_2=\vec{v}_2/\sqrt{6}$. So,

$$P_1=\hat{u}_1\,\hat{u}_1^{\,T}=\tfrac{1}{6}\begin{bmatrix}1&2&1&0\\2&4&2&0\\1&2&1&0\\0&0&0&0\end{bmatrix} \text{ and } P_2=\hat{u}_2\,\hat{u}_2^{\,T}=\tfrac{1}{6}\begin{bmatrix}1&0&-1&2\\0&0&0&0\\-1&0&1&-2\\2&0&-2&4\end{bmatrix}.$$

b. Determine the 4×4 matrix, call it $P_{1,2}$, that projects onto the two dimensional subspace $\mathbf{S}_{1,2} = \mathrm{span}(\vec{v}_1,\vec{v}_2)$.

$$P_{1,2}=P_1+P_2=\tfrac{1}{6}\begin{bmatrix}2&2&0&2\\2&4&2&0\\0&2&2&-2\\2&0&-2&4\end{bmatrix}=\tfrac{1}{3}\begin{bmatrix}1&1&0&1\\1&2&1&0\\0&1&1&-1\\1&0&-1&2\end{bmatrix}.$$

c. Use $P_{1,2}$ to find the projection $\vec{v}_3^{\perp}$ of $\vec{v}_3$ <u>orthogonal</u> to $\mathbf{S}_{1,2}$.

$$\vec{v}_3^{\perp} = \vec{v}_3 - P_{1,2}\vec{v}_3 = \begin{bmatrix}4\\5\\4\\3\end{bmatrix} - \tfrac{1}{3}\begin{bmatrix}1&1&0&1\\1&2&1&0\\0&1&1&-1\\1&0&-1&2\end{bmatrix}\begin{bmatrix}4\\5\\4\\3\end{bmatrix} = \begin{bmatrix}4\\5\\4\\3\end{bmatrix} - \tfrac{1}{3}\begin{bmatrix}12\\18\\6\\6\end{bmatrix} = \begin{bmatrix}0\\-1\\2\\1\end{bmatrix}.$$

d. Determine an orthogonal basis for $\operatorname{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$.

Since $\vec{v}_3^{\perp}$ is orthogonal to $\vec{v}_1$ and $\vec{v}_2$ and is a linear combination of these vectors and $\vec{v}_3$, the desired basis is $(\vec{v}_1, \vec{v}_2, \vec{v}_3^{\perp})$.

5. Let A be the matrix that represents a stretch T by the factor 2 along the line through the origin parallel to $\begin{bmatrix}3\\4\end{bmatrix}$. Let $\mathfrak{B} = (\hat{u}_1, \hat{u}_2)$ where $\hat{u}_1 = \frac{1}{5}\begin{bmatrix}3\\4\end{bmatrix}$ and $\hat{u}_2 = \frac{1}{5}\begin{bmatrix}-4\\3\end{bmatrix}$. $\mathfrak{B}$ is an orthonormal basis for $\mathbf{R}^2$.

a. The relationship between any vector $\vec{x}$ in $\mathbf{R}^2$ and its $\mathfrak{B}$-coordinate vector $[\vec{x}]_{\mathfrak{B}}$ is determined by a coordinate change matrix S. What is S, what is that relationship, and what is S^{-1}?

$\vec{x} = S[\vec{x}]_{\mathfrak{B}}$. $S = \left[\hat{u}_1 \,|\, \hat{u}_2\right] = \frac{1}{5}\begin{bmatrix}3&-4\\4&3\end{bmatrix}$. $S^{-1} = S^T = \frac{1}{5}\begin{bmatrix}3&4\\-4&3\end{bmatrix}$. Note that S is an orthogonal matrix. In fact, it is a rotation.

b. What are $[\hat{u}_1]_{\mathfrak{B}}$ and $[\hat{u}_2]_{\mathfrak{B}}$?

$[\hat{u}_1]_{\mathfrak{B}} = \hat{e}_1 = \begin{bmatrix}1\\0\end{bmatrix}$ and $[\hat{u}_2]_{\mathfrak{B}} = \hat{e}_2 = \begin{bmatrix}0\\1\end{bmatrix}$. These equations are equivalent to the observations that $\vec{u}_1 = 1\vec{u}_1 + 0\vec{u}_2$ and $\vec{u}_2 = 0\vec{u}_1 + 1\vec{u}_2$.

c. Determine $T(\hat{u}_1), [T(\hat{u}_1)]_{\mathfrak{B}}, T(\hat{u}_2)$ and $[T(\hat{u}_2)]_{\mathfrak{B}}$.

We are given that T doubles vectors parallel to $\hat{u}_1$ and leaves unchanged those vectors that are orthogonal to $\hat{u}_1$. So,

$T(\hat{u}_1) = 2\hat{u}_1, \quad [T(\hat{u}_1)]_{\mathfrak{B}} = [2\hat{u}_1]_{\mathfrak{B}} = 2\hat{e}_1,$

$T(\hat{u}_2) = \hat{u}_2, \quad [T(\hat{u}_2)]_{\mathfrak{B}} = [\hat{u}_2]_{\mathfrak{B}} = \hat{e}_2.$

d. From part c, determine the matrix B that represents T relative to the basis $\mathfrak{B}$.

$$B = \left[[T(\hat{u}_1)]_{\mathfrak{B}} \,|\, [T(\hat{u}_2)]_{\mathfrak{B}}\right] = \begin{bmatrix}2&0\\0&1\end{bmatrix}.$$

e. What is the relationship between the matrices A and B?

$A\,S = S\,B$ or $A = S^{-1}B\,S$.

f. Calculate A.

$$A = \tfrac{1}{5}\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\tfrac{1}{5}\begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix} = \tfrac{1}{25}\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 6 & 8 \\ -4 & 3 \end{bmatrix} = \tfrac{1}{25}\begin{bmatrix} 34 & 12 \\ 12 & 41 \end{bmatrix}.$$

6. Choose either of the following problems.

a. $\mathcal{B} = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$ is a basis for a subspace $\mathbf{S}$ of $\mathbf{R}^4$. $\mathcal{C} = (\vec{w}_1, \vec{w}_2, \vec{w}_3)$ is a list of three vectors in $\mathbf{R}^4$. The 4×6 matrix $[\vec{w}_1 \mid \vec{w}_2 \mid \vec{w}_3 \mid \vec{v}_1 \mid \vec{v}_2 \mid \vec{v}_3]$ is fully row-reduced to obtain the matrix M. How can one determine, by inspection of M, whether $\mathcal{C}$ is also a basis for $\mathbf{S}$? Justify your assertions.

$\mathcal{C}$ is linearly independent if and only if the first three columns of M are pivot columns. $\mathcal{C}$ spans $\mathbf{S}$ if and only if every vector in $\mathbf{S}$, including those in $\mathcal{B}$, are linear combinations of those in $\mathcal{C}$ and this is true if and only if none of the last three columns of M are pivot columns. So, $\mathcal{C}$ is a basis for $\mathbf{S}$ if and only if M has the form $\begin{bmatrix} 1 & 0 & 0 & * & * & * \\ 0 & 1 & 0 & * & * & * \\ 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. The asterisks stand for any scalars (provided that the last three columns are linearly independent).

b. Prove that a linear system $A\vec{x} = \vec{b}$ cannot have exactly 2 different solutions.

If $\vec{v}$ and $\vec{w}$ are distinct solutions, $\vec{v} - \vec{w} \neq \vec{0}$ and $A(\vec{v} - \vec{w}) = A\vec{v} - A\vec{w}$ $= \vec{b} - \vec{b} = \vec{0}$. That is, the difference of two distinct solution vectors is a nonzero vector in $\ker(A)$. So, $\vec{v} + (\vec{v} - \vec{w}) = 2\vec{v} - \vec{w}$ is a third different solution. In fact, there are infinitely many different solutions. Others are $\vec{v} + \alpha(\vec{v} - \vec{w})$ where α is any scalar. So, as soon as we know there are two distinct solutions, we are assured that there are other infinitely many other distinct solutions.