1. a. Solve $A\vec{x}=\vec{b}$ completely if $[A|\vec{b}]_{rref}=\left[\begin{array}{ccccc|c} 1 & 3 & 0 & 0 & 1 & 5 \\ 0 & 0 & 1 & 0 & -2 & 6 \\ 0 & 0 & 0 & 1 & 1 & -7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$.

From the Solution Algorithm, we have

$$\vec{x}=\begin{bmatrix} 5 \\ 0 \\ 6 \\ -7 \\ 0 \end{bmatrix}+r\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}+s\begin{bmatrix} -1 \\ 0 \\ 2 \\ -1 \\ 1 \end{bmatrix}$$ where $r$ and $s$ are arbitrary real numbers.

b. Display the row-reduced echelon forms of all 3×4 matrices whose rank is 3.

We list the 3×4 rref matrices in which 3 of the 4 columns are pivot columns.

$$\begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{bmatrix},\begin{bmatrix} 1 & 0 & c & 0 \\ 0 & 1 & d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\begin{bmatrix} 1 & e & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$ Here, $a$, $b$, $c$, $d$, and $e$ are arbitrary reals.

2. a. Show that the transformation $f:\mathbf{R}\to\mathbf{R}^2$ defined by $f(x)=\begin{bmatrix} x^2 \\ x+1 \end{bmatrix}$ for any real $x$ is **not** linear.

It suffices to show that $f$ does not preserve sums or multiples. We demonstrate both.

$$f(ax)=\begin{bmatrix} (ax)^2 \\ (ax)+1 \end{bmatrix}=\begin{bmatrix} a^2x^2 \\ ax+1 \end{bmatrix}$$
$$\neq af(x)=a\begin{bmatrix} x^2 \\ x+1 \end{bmatrix}=\begin{bmatrix} ax^2 \\ ax+a \end{bmatrix}$$
if $a\neq 1$.

$$f(x+y)=\begin{bmatrix} (x+y)^2 \\ (x+y)+1 \end{bmatrix}=\begin{bmatrix} x^2+2xy+y^2 \\ x+y+1 \end{bmatrix}$$
$$\neq f(x)+f(y)=\begin{bmatrix} x^2 \\ x+1 \end{bmatrix}+\begin{bmatrix} y^2 \\ y+1 \end{bmatrix}=\begin{bmatrix} x^2+y^2 \\ x+y+2 \end{bmatrix}$$

b. Determine the matrix for the linear transformation $g:\mathbf{R}^2\to\mathbf{R}^3$ such that $g\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right)=\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $g\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right)=\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.

Since $g$ is linear, $g(\vec{x})=A\vec{x}$ for some 3×2 matrix $A$ and every $\vec{x}$ in $\mathbf{R}^2$.
So, the two equations above may be rewritten as

$A\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $A\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ or, more simply, $A\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}$.

Solving for $A$, we find

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}\frac{1}{2\cdot 3-(-1)\cdot 1}\begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 4 & 1 \\ 3 & -1 \\ 5 & 3 \end{bmatrix}$$

3. a. Determine the inverse of $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$.

$$[A\,|\,I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ 3 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \leftrightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right]$$

$$\leftrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 3 & -1 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right] \leftrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 6 & -1 & -1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right]$$

$$= [A\,|\,I]_{rref} = [I\,|\,A^{-1}] \quad \Rightarrow \quad A^{-1} = \begin{bmatrix} 6 & -1 & -1 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

b. Since the column vectors of $A$ are non-coplanar, every vector in $\mathbf{R}^3$ is a unique linear combination of them.

i. Express the vector $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ as a linear combination of the column vectors of $A$ (call them $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$, respectively) with unknown coefficients $x_1$, $x_2$, and $x_3$.

$\vec{b} = x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3$

ii. Rewrite this last equation in terms of $A$ and the vector $\vec{x}$ whose components are the coefficients to be determined.

$A\vec{x} = \vec{b}$

iii. Solve this last equation for $\vec{x}$ using part a.

$$\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} 6 & -1 & -1 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ -2 \end{bmatrix}$$

4. Let $\vec{v} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and let $f$ be the linear transformation on $\mathbf{R}^3$ that reverses the direction of all vectors parallel to $\vec{v}$ and doubles all vectors orthogonal to $\vec{v}$.

a. Determine the matrix for *f*.

We will let $A$ be the matrix for *f*. A unit vector parallel to $\vec{v}$ is $\hat{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and so, the matrix for projection onto $\hat{u}$ is $P = \begin{bmatrix} u_1u_1 & u_1u_2 & u_1u_3 \\ u_2u_1 & u_2u_2 & u_2u_3 \\ u_3u_1 & u_3u_2 & u_3u_3 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{bmatrix}$. For any vector $\vec{x}$ in $\mathbf{R}^3$, we have the following identity which expresses the resolution of $\vec{x}$ into vector components parallel and orthogonal to $\hat{u}$: $\vec{x} = P\vec{x} + (I-P)\vec{x}$.

Applying the transformation to both sides of this identity and using the linearity of *f*, we obtain $f(\vec{x}) = A\vec{x} = f(P\vec{x}) + f((I-P)\vec{x})$. But the description of $f$ above leads us to conclude that $f(P\vec{x}) = -P\vec{x}$ and $f((I-P)\vec{x}) = 2(I-P)\vec{x}$. Therefore, $A\vec{x} = -P\vec{x} + 2(I-P)\vec{x} = (2I-3P)\vec{x}$. Since $\vec{x}$ is arbitrary, we have

$$A = 2I - 3P = 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{3}{6}\begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3 & 1 & -2 \\ 1 & 3 & 2 \\ -2 & 2 & 0 \end{bmatrix}.$$

b. What does $f$ do to (i.e. what is the image, under *f*, of) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$?

$$f\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = A\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3 & 1 & -2 \\ 1 & 3 & 2 \\ -2 & 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$