Special Problem C Solutions

When a three-dimensional object is displayed graphically in the plane (on paper, a blackboard, a movie screen, or a computer monitor), points in $\mathbf{R}^3$ are mapped to points in $\mathbf{R}^2$. Here is a matrix for a linear transformation that does this.

$$A = \begin{bmatrix} -\frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix}.$$

We would like to determine the image of the unit cube under the linear transformation $\vec{x} \mapsto A\,\vec{x}$. The unit cube is $K = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mid 0 \le x_1, x_2, x_3 \le 1 \right\}$.

To obtain the image of K, it suffices to determine the images of its 8 vertices.

a. Explain why the image of an edge of K is the line segment in $\mathbf{R}^2$ that connects the images of the corresponding vertices of K in $\mathbf{R}^3$.

Let $T : \mathbf{R}^3 \to \mathbf{R}^2$ be the linear transformation defined by $T(\vec{x}) = A\,\vec{x}$ for any $\vec{x} \in \mathbf{R}^3$. The line segment between two vertices described by the vectors $\vec{a}$ and $\vec{b}$, is $\left\{ \vec{a} + t(\vec{b} - \vec{a}) \mid 0 \le t \le 1 \right\}$, as we saw in an earlier problem. The images of the points in this set is the set $\left\{ T\left(\vec{a} + s(\vec{b} - \vec{a}) \right) \mid 0 \le s \le 1 \right\}$. From the linearity of T, this is the same as $\left\{ T(\vec{a}) + s\left(T(\vec{b}) - T(\vec{a}) \right) \mid 0 \le s \le 1 \right\}$. But, this is the line segment connecting $T(\vec{a})$ and $T(\vec{b})$. So, the image, under a linear transformation T of a line segment connecting two points in one Euclidean space is a line segment connecting the images of those points in the co-domain of T.

b. Compute $A\,\vec{v}_0, A\,\vec{v}_1, A\,\vec{v}_2, A\,\vec{v}_3, A\,\vec{v}_4, A\,\vec{v}_5, A\,\vec{v}_6,$ and $A\,\vec{v}_7$ where the vectors $\vec{v}_0, \vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5, \vec{v}_6,$ and $\vec{v}_7$ represent the 8 vertices of K and use the result to sketch the image of K under the transformation $\vec{x} \mapsto A\,\vec{x}$. Also include the images of the coordinate axes in $\mathbf{R}^3$.

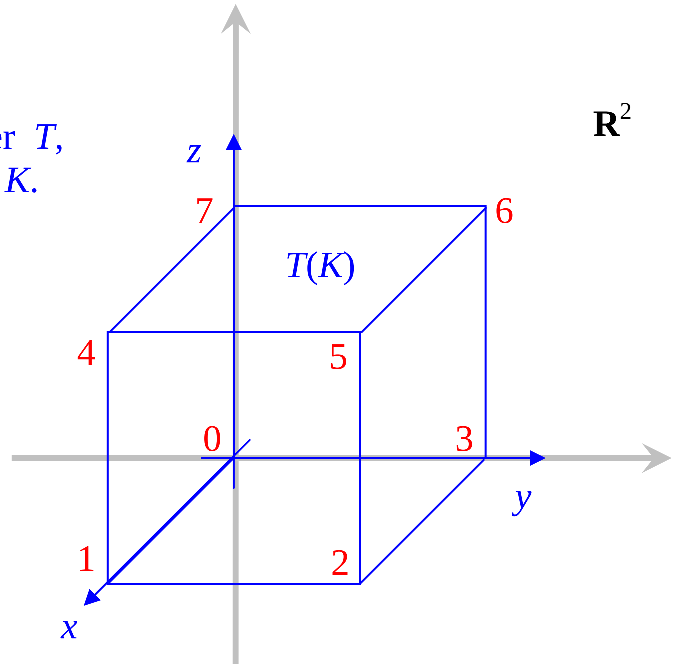
We compute the images of the vertices by premultiplying the matrix whose column vectors represent those 8 vertices by A.

$$A[\vec{v}_0 \mid \vec{v}_1 \mid \vec{v}_2 \mid \vec{v}_3 \mid \vec{v}_4 \mid \vec{v}_5 \mid \vec{v}_6 \mid \vec{v}_7] = \begin{bmatrix} -\frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\frac{1}{2} & \frac{1}{2} & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 1 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 1 & 1 & \frac{1}{2} & 1 \end{bmatrix}.$$ The column vectors of this last matrix are the images of the corresponding vertices of K in $\mathbf{R}^3$.

Special Problem C Solutions

The image, under T, of the unit cube K.



c. Find Nul(A), the null space of A. That is, determine all the vectors in $\mathbf{R}^3$ that are mapped to the zero vector of $\mathbf{R}^2$. What is the graphical significance of Nul(A) in this problem?

We determine Nul(A) by solving $A\vec{x} = \vec{0}$ or, equivalently, by finding A_{rref}.

We have $A = \begin{bmatrix} -\frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \end{bmatrix} = A_{rref}$ and so, using the Solution Algorithm, we deduce that $\vec{x} = c\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ for any real c. This describes a line ℓ in $\mathbf{R}^3$ through the origin and parallel to $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$. This is the line of view for the drawing above. The image of K may be obtained by projecting all of its points parallel to ℓ onto the plane in $\mathbf{R}^3$ that passes through the origin for which ℓ is its normal. All the points on ℓ are mapped onto the origin in $\mathbf{R}^3$ by T.