SM261 FINAL EXAMINATION
14 DECEMBER 2006

PART ONE: CALCULATORS ARE NOT PERMITTED

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1. Let $A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 2 \end{bmatrix}$ and let $B = \begin{bmatrix} 1 & 2 \\ 1 & 4 \\ 1 & 7 \end{bmatrix}$.

a. Calculate AB.
b. Calculate $B^T A^T$.

2. Let $C = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 4 \\ 1 & 1 & 2 \end{bmatrix}$. Find C^{-1}.

3. Find all solutions to the following system of equations. Write your solutions in vector form.

$$\begin{aligned} x_1 + x_2 - x_3 - x_4 + x_5 &= 2 \\ 2x_1 + 2x_2 - x_3 - x_4 + x_5 &= -1 \\ 4x_1 + 4x_2 - 3x_3 - x_4 + 3x_5 &= 3 \end{aligned}$$

4. Identify the redundant vectors among the vectors in the list below.

$$\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 5 \\ 0 \end{bmatrix} \right).$$

5. Use row reduction techniques to find det(A) if $A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix}$.

6. Let T be the linear transformation determined by $T(\vec{e}_1) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $T(\vec{e}_2) = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$.

a. Find the matrix of T with respect to the standard basis $\{\vec{e}_1, \vec{e}_2\}$.

b. Find the matrix of T with respect to the basis $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

c. Is T an orthogonal linear transformation? Explain.

7. Let A be the matrix $\begin{bmatrix} 16 & 9 \\ -4 & 4 \end{bmatrix}$.

a. Find all of the eigenvalues of the matrix A.

b. For one of the eigenvalues of the matrix A compute the corresponding eigenspace.

8. Use Cramer's Rule to find the solutions to the system

$$\begin{aligned} 2x &+ y &= 4 \\ 3x &+ 10y &= 3. \end{aligned}$$

Show your work.

END OF PART ONE

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PART TWO: CALCULATORS ARE PERMITTED

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1. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 4 & 1 & 5 \\ 2 & 4 & 2 & 6 \\ 1 & 2 & 2 & 4 \end{bmatrix}$.

a. Find a basis for im(A).

b. Find a basis for ker(A).

2. Suppose $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ are non-zero vectors in R^3 that are orthogonal to each other, i.e. $0 = \vec{v}_1 \cdot \vec{v}_2 = \vec{v}_1 \cdot \vec{v}_3 = \vec{v}_2 \cdot \vec{v}_3$.

a. Explain why the three vectors are linearly independent.

b. Explain, using part a, why the three vectors form a basis for R^3.

3. I have *17* bills in my pocket (*1*'s, *5*'s, and *10*'s) whose total value is *$77*. How many of each type of bill do I have? (Use techniques from this course to find all solutions.)

4. Let A be a 10×10 invertible matrix. Explain your answers to the following.

a. What does it mean for A to be invertible?

b. What are the possible values of the rank of A?

c. What are the possible values of the nullity of A?

d. What are the possible values of det(A)?

e. Explain why for any 10×1 vector $\vec{b}$ the equation $A\vec{x} = \vec{b}$ is consistent, i.e. has a solution.

5. Suppose an $n \times n$ matrix A satisfies the matrix equation $A^2 + 2A = I$, where I is the $n \times n$ identity matrix. Show that A is invertible.

6. Suppose A is a 3×8 matrix.

a. What are the possible values of the rank of A?

b. What are the possible values of the nullity of A?

c. What are the possible values of the sum of the rank and nullity of A?

7. a. Given a subspace V of R^n, define $V^\perp$ and explain why it is a subspace (of R^n).

b. Let V be the subspace of R^3 with basis $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$. Find a basis for $V^\perp$.

8. Let V be the subspace of R^4 spanned by the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 2 \\ 2 \\ 0 \end{bmatrix}$.

a. Use the Gram-Schmidt method to find an orthonormal basis for V.

b. Find $proj_V \left(\begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right)$, the projection of the vector $\begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ onto V.

9. Suppose $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, and $\vec{v}_4$ are the *rows* of a 4×4 matrix A, i.e., $A = \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vec{v}_3 \\ \vec{v}_4 \end{bmatrix}$.

Suppose also that det(A) = 2. Find the determinants of the following matrices. Explain your answers.

a. $\begin{bmatrix} \vec{v}_3 \\ \vec{v}_2 \\ \vec{v}_1 \\ \vec{v}_4 \end{bmatrix}$ b. $\begin{bmatrix} \vec{v}_1 + 3\vec{v}_2 \\ \vec{v}_2 \\ 4\vec{v}_3 \\ \vec{v}_4 \end{bmatrix}$ c. $\begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vec{v}_2 \\ \vec{v}_4 \end{bmatrix}$

10. A matrix A has eigenvalues *2* and *3*.

a. Show that if $\vec{v}$ is an eigenvector of A then it is also an eigenvector of A^2. What are the eigenvalues of A^2?

b. Show that if $\vec{v}$ is an eigenvector of A then it is also an eigenvector of A^{-1}. What are the eigenvalues of A^{-1}?

11. Find the best (least squares) fit $y = c_0 + c_1 t$ to the data $(t, y) =$ $(1,-1)$, $(2,1)$, and $(3,4)$.

12. Let T be the linear transformation from R^2 to R^2 which is the projection onto the line $y = x$. Let A be the matrix of the linear transformation T.

a. Find A.

b. Find the eigenvalues and eigenvectors of the matrix A.

c. Use part b to find an invertible matrix S and a diagonal matrix D so that $S^{-1}AS = D$.

END OF PART TWO