SM261.1001 Special Problem 8 Solutions 20 April 2007

The objectives of this multipart exercise are to show that, in $\mathbf{R}^3$, the composite of two reflections is a rotation and to determine the axis and angle of this rotation. Recall that rotations and reflections are both orthogonal transformations and the determinant of a rotation is +1 while the determinant for a reflection is –1.

Let M be a plane through the origin in $\mathbf{R}^3$ and let $\hat{u}$ be a unit normal to M. Let P be the matrix for the projection onto $\hat{u}$. Let F be the matrix for reflection across M.

a. Explain why or demonstrate that $F = I - 2P$.

For any $\vec{x}$ in $\mathbf{R}^3$, $\vec{x} = P\vec{x} + (I-P)\vec{x}$ resolves $\vec{x}$ into its projections parallel to $\hat{u}$ and parallel to M. The reflection reverses the direction of the former and leaves the latter unchanged. Therefore, $F\vec{x} = FP\vec{x} + F(I-P)\vec{x} = -P\vec{x} + (I-P)\vec{x} = (I-2P)\vec{x}$. Since $\vec{x}$ is arbitrary, it follows that $F = I - 2P$.

b. Algebraically verify that $F = F^T$.

$F^T = (I - 2P)^T = I^T - 2P^T = I - 2P$ since I and P are both symmetric.

c. Algebraically verify that $F^2 = I$.

$F^2 = (I - 2P)^2 = I - 4P + 4P^2 = I - 4P + 4P = I$. Here, we used the identity $P^2 = P$ for the projection matrix P.

d. Algebraically verify that F is orthogonal.

From b and c above, we have $F^TF = FF = F^2 = I$. Hence F is orthogonal.

e. Now, let $\hat{u} = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}^T$. Show that

$$\det F = \det\left[\hat{e}_1 - 2u_1\hat{u} \,|\, \hat{e}_2 - 2u_2\hat{u} \,|\, \hat{e}_3 - 2u_3\hat{u}\right]$$

$$= \det\left[\hat{e}_1 | \hat{e}_2 | \hat{e}_3\right] - 2u_1\det\left[\hat{u} | \hat{e}_2 | \hat{e}_3\right] - 2u_2\det\left[\hat{e}_1 | \hat{u} | \hat{e}_3\right] - 2u_3\det\left[\hat{e}_1 | \hat{e}_2 | \hat{u}\right]$$

$$+4u_2u_3\det\left[\hat{e}_1 | \hat{u} | \hat{u}\right] + 4u_3u_1\det\left[\hat{u} | \hat{e}_2 | \hat{u}\right] + 4u_1u_2\det\left[\hat{u} | \hat{u} | \hat{e}_3\right]$$

$$-8u_1u_2u_2\det\left[\hat{u} | \hat{u} | \hat{u}\right] = -1.$$

The kth column of $F = I - 2P = I - 2\hat{u}\hat{u}^T$ is $F\hat{e}_k = (I - 2\hat{u}\hat{u}^T)\hat{e}_k = \hat{e}_k - 2\hat{u}\hat{u}^T\hat{e}_k = \hat{e}_k - 2\hat{u}(\hat{u}^T\hat{e}_k) = \hat{e}_k - 2u_k\hat{u}$. Therefore, $F = \left[\hat{e}_1 - 2u_1\hat{u} \,|\, \hat{e}_2 - 2u_2\hat{u} \,|\, \hat{e}_3 - 2u_3\hat{u}\right]$. Now, using the fact that determinants of matrices are linear in each of their columns, we obtain the second equality above. Now, $\det\left[\hat{e}_1 | \hat{u} | \hat{u}\right], \det\left[\hat{u} | \hat{e}_2 | \hat{u}\right], \det\left[\hat{u} | \hat{u} | \hat{e}_3\right], \det\left[\hat{u} | \hat{u} | \hat{u}\right]$ are all 0 because the columns of the matrices are evidently linearly dependent. This leaves $\det\left[\hat{e}_1 | \hat{e}_2 | \hat{e}_3\right] - 2u_1\det\left[\hat{u} | \hat{e}_2 | \hat{e}_3\right] - 2u_2\det\left[\hat{e}_1 | \hat{u} | \hat{e}_3\right] - 2u_3\det\left[\hat{e}_1 | \hat{e}_2 | \hat{u}\right]$ whose value is clearly $1 - 2u_1{}^2 - 2u_2{}^2 - 2u_3{}^2 = 1 - 2\|\hat{u}\|^2 = 1 - 2 = -1$.

Consider two different planes M_1 and M_2 through the origin with corresponding unit normals $\hat{u}_1$ and $\hat{u}_2$, projection matrices P_1 and P_2, and reflection matrices F_1 and F_2. Let the angle between the planes (or, equivalently, their normals) be ϕ. The matrix for the composite of the reflections is $F_2 F_1$.

f. Explain why $F_2 F_1$ is orthogonal, has determinant +1, and so is a rotation matrix.

We know that the product of orthogonal matrices is orthogonal and the determinant of a product of square matrices is the product of their determinants. So $\det(F_2 F_1) = \det(F_2)\det(F_1) = (-1)(-1) = +1$.

g. Algebraically verify that $\hat{u}_1 \times \hat{u}_2$ (or $\hat{u}_2 \times \hat{u}_1$) is parallel to the rotation axis for $F_2 F_1$. [Notice that this implies that $\hat{u}_1$ and $\hat{u}_2$ are each orthogonal to the rotation axis.]

$$F_2 F_1 \hat{u}_1 \times \hat{u}_2 = (I - 2\hat{u}_2 \hat{u}_2^T)(I - 2\hat{u}_1 \hat{u}_1^T)\hat{u}_1 \times \hat{u}_2$$
$$= (I - 2\hat{u}_2 \hat{u}_2^T)(\hat{u}_1 \times \hat{u}_2 - 2(\hat{u}_1 \cdot \hat{u}_1 \times \hat{u}_2)\hat{u}_1) = (I - 2\hat{u}_2 \hat{u}_2^T)(\hat{u}_1 \times \hat{u}_2)$$
$$= (I - 2\hat{u}_2 \hat{u}_2^T)(\hat{u}_1 \times \hat{u}_2) = \hat{u}_1 \times \hat{u}_2 - 2(\hat{u}_2 \cdot \hat{u}_1 \times \hat{u}_2)\hat{u}_2 = \hat{u}_1 \times \hat{u}_2.$$

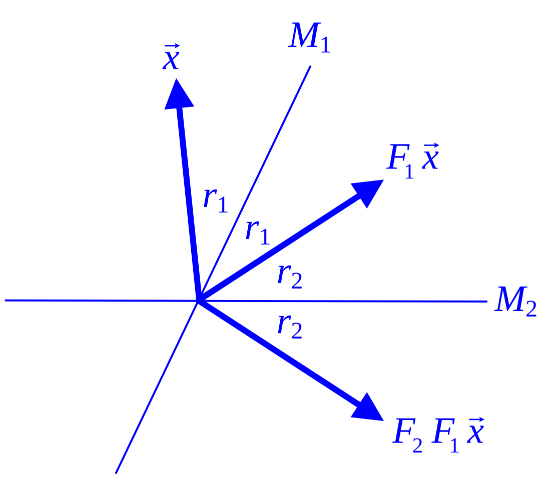
So, $\hat{u}_1 \times \hat{u}_2$ is left unchanged by this rotation; it must be parallel to the rotation axis.

h. Algebraically verify that the angle of rotation for $F_2 F_1$ is 2ϕ.

The angle of rotation, call it θ is the angle between any vector orthogonal to the rotation axis and its rotated image. Choose, $\hat{u}_1$ as a vector orthogonal to the rotation axis. Then, $\cos\theta = \hat{u}_1 \cdot F_2 F_1 \hat{u}_1$

$$= \hat{u}_1 \cdot (I - 2\hat{u}_2 \hat{u}_2^T)(I - 2\hat{u}_1 \hat{u}_1^T)\hat{u}_1 = \hat{u}_1 \cdot (I - 2\hat{u}_2 \hat{u}_2^T)(-\hat{u}_1) = -1 + 2\hat{u}_2 \cdot \hat{u}_1$$
$$= 2\cos\phi - 1 = \cos(2\phi) \Rightarrow \theta = 2\phi.$$

i. Draw a diagram to illustrate part h geometrically. The plane of this diagram should be that of $\hat{u}_1$ and $\hat{u}_2$ so that M_1 and M_2 are seen "edge-on".



Consider a vector $\vec{x}$ in the plane of $\hat{u}_1$ and $\hat{u}_2$. The angle between this vector and M_1 is denoted by r_1. The image of $\vec{x}$ after the first reflection is $F_1\vec{x}$. Denote the angle between $F_1\vec{x}$ and M_2 by r_2. After a second reflection across M_2, we arrive at $F_2 F_1 \vec{x}$. Then, $\phi = r_1 + r_2$ and

$$\theta = 2r_1 + 2r_2 = 2\phi.$$