1. a. Calculate $\det\left(\begin{bmatrix} 1 & 0 & 1 & 2 & 0 \\ 1 & 0 & 4 & 3 & 7 \\ 4 & 5 & 3 & 2 & 1 \\ 1 & 0 & 1 & 4 & 8 \\ 1 & 0 & 1 & 2 & 4 \end{bmatrix}\right)$.

Let A be the 5×5 matrix in the expression above. Then, using the Laplace expansion along the second column of A, we find

$\det(A) = 5(-1)^{2+3}\det\left(\begin{bmatrix} 1 & 1 & 2 & 0 \\ 1 & 4 & 3 & 7 \\ 1 & 1 & 4 & 8 \\ 1 & 1 & 2 & 4 \end{bmatrix}\right)$. Next, we subtract the first row of the matrix from all succeeding rows to obtain

$\det(A) = 5(-1)^{2+3}\det\left(\begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 3 & 1 & 7 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 4 \end{bmatrix}\right) = 5(-1)^{2+3}(1)(3)(2)(4) = -120.$ In the next to last step, we used the fact that the determinant of an upper (or lower) triangular matrix is the product of its diagonal entries.

b. Find the volume of the parallelepiped in $\mathbf{R}^3$, three of whose concurrent edges are described by the vectors $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, and $\begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix}$.

The volume is given by $\left|\det\begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & 0 \\ 1 & 2 & 2 \end{bmatrix}\right| = \left|\det\begin{bmatrix} 1 & 5 \\ 1 & 2 \end{bmatrix}\right| = |-3| = 3.$

c. What happens to the volume of the figure in $\mathbf{R}^3$ by subjecting it to the linear transformation $\vec{x} \mapsto \begin{bmatrix} 2 & 7 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 5 \end{bmatrix}\vec{x}$?

The volume of the figure is multiplied by the magnitude of the determinant of the matrix. In this case, the figure's volume is multiplied by $2 \cdot 3 \cdot 5 = 30$.

2. a. Find the determinant of the $n\times n$ matrix $\begin{bmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 2 & 2 & 3 & 4 & \cdots & n \\ 3 & 3 & 3 & 4 & \cdots & n \\ 4 & 4 & 4 & 4 & \cdots & n \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ n & n & n & n & \cdots & n \end{bmatrix}$.

Let A be the given matrix. Subtracting the first row A from each of its succeeding rows yields $\begin{bmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 2 & 1 & 0 & 0 & \cdots & 0 \\ 3 & 2 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ n-1 & n-2 & n-3 & n-4 & \cdots & 0 \end{bmatrix}$ whose determinant is the same as $\det(A)$. Now, using the Laplace expansion along the last column, we have $\det(A) = n(-1)^{n+1}\det\left(\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 2 & 1 & 0 & \cdots & 0 \\ 3 & 2 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n-1 & n-2 & n-3 & \cdots & 1 \end{bmatrix}\right)$. The last matrix is $(n–1)\times(n–1)$ and lower triangular. Its determinant is 1. So, $\det(A) = (-1)^{n+1}n$.

b. Suppose that $A = \begin{bmatrix} * & 100 & * & * \\ 100 & * & * & * \\ * & * & * & 100 \\ * & * & 100 & * \end{bmatrix}$ where the asterisks are positive integers less than 10 and they are not necessarily the same. Show that A is invertible by considering the size of the terms that sum to its determinant.

Each of the $4! = 24$ terms of $\det(A)$ is ± 1 times the product of 4 entries of the matrix, each from a different row and different column. The largest is $(100)^4 = 10^8$. All others have no more than two factors of 100 and two factors of 10.. Clearly, then, $\det(A) > 10^8 - (23)(10^2)(100^2) > 0$. So, A^{-1} exists because the determinant of A is not 0.

c. The cofactor matrix for the matrix A, i.e. the matrix consisting of the cofactors of A, is $C = \begin{bmatrix} -1 & -3 & 1 \\ -6 & 3 & -1 \\ 2 & -1 & -2 \end{bmatrix}$ and $\det(A) = -7$. What is A^{-1}?

Cramer's Formula tells us that $A^{-1} = \frac{1}{\det(A)} C^T = \frac{1}{7} \begin{bmatrix} 1 & 6 & -2 \\ 3 & -3 & 1 \\ -2 & 1 & 2 \end{bmatrix}$.

d. Assuming that the linear system $\begin{Bmatrix} ax + by = r \\ cx + dy = s \end{Bmatrix}$ has a unique solution, express the variable y as the ratio of two determinants in accordance with Cramer's Rule.

$$y = \frac{\det\begin{bmatrix} a & r \\ c & s \end{bmatrix}}{\det\begin{bmatrix} a & b \\ c & d \end{bmatrix}} = \frac{as - cr}{ad - bc}.$$

3. Given $A \in \mathbf{R}^{4\times 5}$, $A_{rref} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $\left(A^T\right)_{rref} = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$,

and $A[1\ 0\ 0\ 0\ 0]^T = [1\ 2\ 3\ 4]^T$. Determine each of the following.

[Notes: (1) The information provided here is insufficient to determine A. (2) Recall that elementary row operations may change the rows of a matrix but do not change their span.].

a. a basis for $\mathrm{Nul}(A)$.

From the Solution Algorithm applied to A_{rref}, we have the following basis for $\mathrm{Nul}(A)$: $\left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \right)$.

b. a basis for $\mathrm{Col}(A)$.

According to Note (2), the row vectors of A^T, which are the column vectors of A, have the same span as the row vectors of $\left(A^T\right)_{rref}$. But, the nonzero rows of $\left(A^T\right)_{rref}$ are linearly independent by observation and construction. So, they comprise a basis for the row vectors of A^T. Therefore, the transposes of the nonzero rows of $\left(A^T\right)_{rref}$ provide a basis for Col(A)

which is $\left(\begin{bmatrix}1\\0\\0\\2\end{bmatrix},\begin{bmatrix}0\\1\\0\\4\end{bmatrix},\begin{bmatrix}0\\0\\1\\-5\end{bmatrix}\right)$.

c. the solution set for $A\vec{x}=[1\ \ 2\ \ 3\ \ 4]^T$.

This is the null space of A shifted by the particular solution provided above, that is $[1\ \ 0\ \ 0\ \ 0\ \ 0]^T+\text{Nul}(A)$.

d. the solution set for $A\vec{x}=[1\ \ 0\ \ 0\ \ 1]^T$.

Since $[1\ \ 0\ \ 0\ \ 1]^T$ is clearly not in the span of our basis for Col(A), this set is empty.

4. Suppose that $A\in\mathbf{R}^{4\times 2}$ and $P=\frac{1}{3}\begin{bmatrix}2&1&0&-1\\1&1&-1&0\\0&-1&2&-1\\-1&0&-1&1\end{bmatrix}$ is the matrix for projection onto Col(A).

a. How are Col(A) and Col(P) related?

They are the same since the range of a projection is the set onto which it projects.

b. Determine Col(A).

Since the first two columns of P comprise a linearly independent pair and A has only two columns, rank(A) = 2. A basis for Col(A) is $\left(\begin{bmatrix}2\\1\\0\\-1\end{bmatrix},\begin{bmatrix}1\\1\\-1\\0\end{bmatrix}\right)$.

c. Determine Nul(A).

dim(Col(A)) = 2 and the Rank-Nullity Theorem tells us rank(A) = dim(Col(A)) + dim(Nul(A)). So, dim(Nul(A)) = 0 and Nul(A) = $\{\vec{0}\}$.

d. How many solutions do each of $A\vec{x}=\vec{b}$ and $A\vec{x}=\vec{c}$ have? Explain.

$\vec{b}$ does not belong to Col(A), so $A\vec{x}=\vec{b}$ has no solutions. $\vec{c}$ belongs to Col(A) and Nul(A) is trivial, so $A\vec{x}=\vec{c}$ has exactly one solution.

5. $L=(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ is a list of three vectors in $\mathbf{R}^{2012}$. Show that if L is linearly independent, then $L'=(\vec{v}_1+\vec{v}_2, \vec{v}_2+\vec{v}_3, \vec{v}_3+\vec{v}_1)$ is also linearly independent.

If L is linearly independent and $a(\vec{v}_1+\vec{v}_2)+b(\vec{v}_2+\vec{v}_3)+c(\vec{v}_3+\vec{v}_1)=\vec{0}$ for scalars a, b, and c, then $(a+c)\vec{v}_1+(a+b)\vec{v}_2+(b+c)\vec{v}_3=\vec{0}$. But, by the linear independence of L, this means that $a+c=a+b=b+c=0$ and this implies that $a = b = c = 0$ or that L' is linearly independent.

6. S is the subset of all vectors $\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$ in $\mathbf{R}^4$ for which $w+3y+z=0$ and $x+y+2z=0$. Show that S is a subspace of $\mathbf{R}^4$ and find a basis for S and dim(S).

The vectors $\vec{v}$ in S satisfy the matrix equation $\begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}\vec{v}=\vec{0}$. Since the above matrix is fully row-reduced, application of the Solution Algorithm yields $\vec{v}=\alpha\begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \end{bmatrix}+\beta\begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}=\alpha\,\vec{a}+\beta\,\vec{b}$ for any α and β in $\mathbf{R}$. Therefore, the solution set is the span of the two linearly independent vectors $\vec{a}$ and $\vec{b}$. So, S is the span of $\vec{a}$ and $\vec{b}$ and, since the span of any nonempty set of vectors is always a subspace, S is a subspace of $\mathbf{R}^4$, $(\vec{a}, \vec{b})$ is a basis for S, and dim(S) = 2.

7. $\mathbf{R}^{2\times 2}$ is the vector space consisting of all 2×2 matrices together with the usual rules for addition of matrices and multiplication of matrices by scalars. We know that dim($\mathbf{R}^{2\times 2}$) = 4. The trace of a matrix A, written tr(A) is the sum of its diagonal entries. Let V be the subset of $\mathbf{R}^{2\times 2}$ consisting of

all the matrices A in $\mathbf{R}^{2\times 2}$ such that $\mathrm{tr}(A) = 0$. Show that V is a subspace by showing it is the span of a finite list of linearly independent vectors in $\mathbf{R}^{2\times 2}$ and find $\dim(V)$.

$$V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a,b,c,d \in \mathbf{R} \text{ and } a+d=0 \right\} = \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \mid a,b,c \in \mathbf{R} \right\}$$

$$= \left\{ a\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \mid a,b,c \in \mathbf{R} \right\} = \mathrm{span}(L) \text{ where}$$

$L = \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right)$ is clearly nonempty, linearly independent and spanning in V, and so is a basis for V. $\dim(V) = 2$.

8. M is the plane in $\mathbf{R}^3$ described by the equation $x + 2y + 3z = 0$. T is the linear transformation on $\mathbf{R}^3$ that reverses all vectors normal to M and doubles all vectors in M. Our objective is to find A, the (standard) matrix for T so that $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbf{R}^3$

a. Find a basis $\mathcal{B} = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$ for $\mathbf{R}^3$ consisting of $\vec{v}_1$, a vector normal to M, plus two vectors $\vec{v}_2$ and $\vec{v}_3$, belonging to M.

A vector normal to M is given by the coefficients in the above equation, i.e. $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Vectors orthogonal to $\vec{v}_1$ lie in M and so they must satisfy the matrix equation $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \vec{v} = \vec{0}$. Since the matrix $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ is fully reduced, solutions to the matrix equation are obtained immediately from the Solution Algorithm. $\vec{v} = \alpha\vec{v}_1 + \beta\vec{v}_2 = \alpha\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + \beta\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$ for any α, β in $\mathbf{R}$.

b. What is the relationship between $\vec{x}$ and $[\vec{x}]_{\mathcal{B}}$ for any $\vec{x} \in \mathbf{R}^3$?

The coordinate transformation matrix is $S = [\vec{v}_1 \mid \vec{v}_2 \mid \vec{v}_3] = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ 3 & 0 & -1 \end{bmatrix}$ and so, $\vec{x} = S\,[\vec{x}]_{\mathcal{B}}$.

c. What is the matrix B that represents T relative to the basis $\mathcal{B}$, i.e., what is B so that $\left[T(\vec{x})\right]_{\mathcal{B}} = B[\vec{x}]_{\mathcal{B}}$ for all $\vec{x} \in \mathbf{R}^3$?

Since $T(\vec{v}_1) = -\vec{v}_1$, $T(\vec{v}_2) = 2\vec{v}_1$, $T(\vec{v}_3) = 2\vec{v}_3$, $B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

d. How is A determined from B? It is <u>not</u> necessary to calculate A.

$A = S\,B\,S^{-1}$. $\left(\text{Calculation shows that } A = \frac{1}{14}\begin{bmatrix} 25 & -6 & -9 \\ -6 & 16 & -18 \\ -9 & -18 & 1 \end{bmatrix} \right)$.

9. Find the basis $\mathfrak{B} = (\vec{v}_1, \vec{v}_2)$ for $\mathbf{R}^2$ if $\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathfrak{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 1 \end{bmatrix}_{\mathfrak{B}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Let $S = P_{\mathfrak{B}} = \underset{\mathcal{E} \leftarrow \mathfrak{B}}{P} = [\vec{v}_1 \,|\, \vec{v}_2]$ be the coordinate transformation matrix that maps from $\mathfrak{B}$-coordinates to standard coordinates. So, $\forall\ \vec{x} \in \mathbf{R}^2$ $\vec{x} = S\left[\vec{x}\right]_{\mathfrak{B}}$. According to the data given, $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = S\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 1 \end{bmatrix} = S\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

These two equations may be combined into one matrix equation

$\begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} = S\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$. So, $S = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}\frac{1}{1}\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} -7 & 5 \\ -1 & 1 \end{bmatrix}$

and $(\vec{v}_1, \vec{v}_2) = \left(-\begin{bmatrix} 7 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \end{bmatrix} \right)$.

10. T/F.

a. If $A, B \in \mathbf{R}^{n\times n}$, $\det(AB) = \det(A)\det(B)$.

True.

b. If $A, B \in \mathbf{R}^{n\times n}$, $\det(A+B) = \det(A) + \det(B)$.

False.

c. If $\vec{r}, \vec{s}, \vec{t} \in \mathbf{R}^3$, then $\vec{s} \mapsto \det[\vec{r} \,|\, \vec{s} \,|\, \vec{t}\,]$ is a linear function.

True.

d. Every nontrivial subspace of $\mathbf{R}^5$ contains infinitely many vectors.

True.

e. There are infinitely many different bases for $\mathbf{R}^5$.

True.

f. If, in $\mathbf{R}^n$, $(\vec{v}_1, \ldots, \vec{v}_p)$ is spanning and $(\vec{w}_1, \ldots, \vec{w}_q)$ is linearly independent in $\mathbf{R}^n$, then $p \le n \le q$.

False.

g. If $(\vec{v}_1, \ldots, \vec{v}_p)$ spans $\mathbf{R}^n$ and $a_1\vec{v}_1 + \ldots + a_p\vec{v}_p = b_1\vec{v}_1 + \ldots + b_p\vec{v}_p$, then $a_k = b_k$ for $k = 1, 2, \ldots, p$.

False.

h. If $(\vec{w}_1, \ldots, \vec{w}_q)$ is linearly independent in $\mathbf{R}^n$ and $\vec{x} \in \mathbf{R}^n$, then there are unique scalars $c_1, \ldots, c_q$ so that $\vec{x} = c_1\vec{w}_1 + \ldots + c_q\vec{w}_q$.

False.