1. Let $A = \frac{1}{3}\begin{bmatrix} -2 & 2 & 4 \\ 2 & -5 & 2 \\ 4 & 2 & -2 \end{bmatrix}$. By straightforward calculation one verifies that $A\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$, $A\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 0 \end{bmatrix}$, and $A\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$. Determine an orthogonal matrix Q and a diagonal matrix D so that $Q^TAQ = D$.

We are given that -2 and 1 are eigenvalues of A with eigenspaces

$$E_{-2}(A) = \operatorname{span}(\vec{v}_1, \vec{v}_2) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}\right) \text{ and } E_1(A) = \operatorname{span}(\vec{v}_3) = \operatorname{span}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}.$$

The eigenvalue -2 has algebraic and geometric multiplicity 2 and the eigenvalue 1 has algebraic and geometric multiplicity 1. A is diagonalizable as expected since it is real and symmetric. In fact, $(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ is a basis for $\mathbf{R}^3$ consisting of eigenvectors of A. So, A is similar to the diagonal matrix $D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. A diagonalizer for A is the matrix $[\vec{v}_1 \mid \vec{v}_2 \mid \vec{v}_3]$.

However, although $\vec{v}_3$ is orthogonal to both $\vec{v}_1$ and $\vec{v}_2$, $\vec{v}_1$ and $\vec{v}_2$ are not orthogonal and none of the three vectors is of unit magnitude. This is easily remedied by using Gram-Schmidt to find an orthonormal basis for $\operatorname{span}(\vec{v}_1, \vec{v}_2)$. Separately, we can normalize $\vec{v}_3$. We let

$\hat{u}_1 = \vec{v}_1 / \|\vec{v}_1\| = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. We find $\vec{v}_2^{\perp} = \vec{v}_2 - (\vec{v}_2 \cdot \hat{u}_1)\hat{u}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -2 \\ \frac{1}{2} \end{bmatrix}$

and $\hat{u}_2 = \vec{v}_2^{\perp} / \|\vec{v}_2^{\perp}\| = \frac{1}{\sqrt{18}}\begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix}$. Finally, $\hat{u}_3 = \vec{v}_3 / \|\vec{v}_3\| = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$. So, $(\hat{u}_1, \hat{u}_2, \hat{u}_3)$ is an orthonormal basis for $\mathbf{R}^3$ consisting of eigenvectors of A and

$$Q = [\hat{u}_1 \mid \hat{u}_2 \mid \hat{u}_3] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{2}{3} \\ 0 & \frac{-4}{3\sqrt{2}} & \frac{1}{3} \\ -\frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{2}{3} \end{bmatrix}$$ is the orthogonal diagonalizer for A.

2. Let $A = \begin{bmatrix} -9 & 20 \\ -6 & 13 \end{bmatrix}$. Express each entry of A^n as a function of the non-negative integer n.

First, we determine the eigenvalues and corresponding eigespaces of A.

$0 = \det(A - \lambda I) = (-9-\lambda)(13-\lambda) - (-6)(20) = \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3)$

So, spec(A) = (1, 3). $\mathrm{E}_1(A) = \ker(A - 1I) = \ker \begin{bmatrix} -10 & 20 \\ -6 & 12 \end{bmatrix} = \operatorname{span} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

$\mathrm{E}_3(A) = \ker(A - 3I) = \ker \begin{bmatrix} -12 & 20 \\ -6 & 9 \end{bmatrix} = \operatorname{span} \begin{bmatrix} 5 \\ 3 \end{bmatrix}$. A diagonalizer for A is $S = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ whose columns are the eigenbasis vectors found above. So, A is similar to $B = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ whose diagonal entries are the corresponding eigenvalues of A. We are now prepared to calculate A^n.

$$A^n = (S\,B\,S^{-1})^n = S\,B^n\,S^{-1} = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3^n \end{bmatrix} \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 - 5 \cdot 3^n & -10 + 10 \cdot 3^n \\ 3 - 3 \cdot 3^n & -5 + 6 \cdot 3^n \end{bmatrix}$$

3. An idealized ecosystem has two species: predator and prey. The prey species has an inexhaustible supply of food and, in the absence of the predator species, its population grows exponentially. The population of the predator species, on the other hand, decreases exponentially in the absence of the prey on which it depends for its sustenance. Denoting the population (in millions) on day t as $x_1(t)$ for the predator species and $x_2(t)$ for the prey species, the evolution of this ecosystem is described by

$$x_1(t+1) = .4\,x_1(t) + .3\,x_2(t)$$

$$x_2(t+1) = -.2\,x_1(t) + 1.1\,x_2(t)$$

a. Determine $x_1(t)$ and $x_2(t)$ for all positive integers t if $x_1(0) = 11$ and $x_2(0) = 7$.

Let $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$. The above equations abbreviate to $\vec{x}(t+1) = A\,\vec{x}(t)$ where $A = \begin{bmatrix} .4 & .3 \\ -.2 & 1.1 \end{bmatrix}$. The solution is $\vec{x}(t) = A^t\,\vec{x}(0)$. The eigens of A are now found. $0 = \det(A - \lambda I) = (.4-\lambda)(1.1-\lambda) + .06 = (\lambda - .5)(\lambda - 1)$ implies spec(A) = (.5, 1). So,

$$\mathrm{E}_{.5}(A) = \mathrm{Nul}(A - .5I) = \mathrm{Nul}\begin{bmatrix} -.1 & .3 \\ -.2 & .6 \end{bmatrix} = \mathrm{Nul}\begin{bmatrix} -1 & 3 \\ 0 & 0 \end{bmatrix} = \mathrm{span}\begin{bmatrix} 3 \\ 1 \end{bmatrix} \text{ and}$$

$$\mathrm{E}_{1}(A) = \mathrm{Nul}(A - 1I) = \mathrm{Nul}\begin{bmatrix} -.6 & .3 \\ -.2 & .1 \end{bmatrix} = \mathrm{Nul}\begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} = \mathrm{span}\begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

An eigenbasis for $\mathbf{R}^2$ is $\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$ and a diagonalizer for A is $S = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$ with inverse $S^{-1} = \frac{1}{5}\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$. Since $\vec{x}(0) = \begin{bmatrix} 11 \\ 7 \end{bmatrix} = 3\begin{bmatrix} 3 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ [the coefficients 3 and 2 are the components of $S^{-1}\vec{x}(0)$], we have

$$\vec{x}(t) = A^t\begin{bmatrix} 11 \\ 7 \end{bmatrix} = 3A^t\begin{bmatrix} 3 \\ 1 \end{bmatrix} + 2A^t\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 3\cdot(.5)^t\begin{bmatrix} 3 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

b. What are $\lim\limits_{t\to\infty} x_1(t)$ and $\lim\limits_{t\to\infty} x_2(t)$?

$$\begin{bmatrix} \lim\limits_{t\to\infty} x_1(t) \\ \lim\limits_{t\to\infty} x_2(t) \end{bmatrix} = \lim_{t\to\infty}\vec{x}(t) = \lim_{t\to\infty}\left(A^t\begin{bmatrix} 11 \\ 7 \end{bmatrix}\right) = \lim_{t\to\infty}\left(3\cdot 2^{-t}\begin{bmatrix} 3 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} \in E_1(A).$$

c. Describe all initial conditions for which the ratio of $x_1(t)$ to $x_2(t)$ remains constant in time.

Writing $\vec{x}(0) = c_1\begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, we have, $\vec{x}(t) = (.5)^t c_1\begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. So, $x_1(t)/x_2(t) = \left(3c_1(.5)^t + c_2\right)/\left(c_1(.5)^t + 2c_2\right)$. This ratio is constant in time, if and only if one of c_1 or c_2 is 0, i.e. if and only if $\vec{x}(0)$ is an eigenvector of A.

4. Let $A = \begin{bmatrix} -3 & 2 & 1 \\ -5 & 4 & 1 \\ -5 & 2 & 3 \end{bmatrix}$.

a. Calculate $A\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $A\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, and $A\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$.

$$A\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}=0\begin{bmatrix}1\\1\\1\end{bmatrix},\ A\begin{bmatrix}1\\2\\1\end{bmatrix}=\begin{bmatrix}2\\4\\2\end{bmatrix}=2\begin{bmatrix}1\\2\\1\end{bmatrix},\ \text{and } A\begin{bmatrix}1\\1\\3\end{bmatrix}=\begin{bmatrix}2\\2\\6\end{bmatrix}=2\begin{bmatrix}1\\1\\3\end{bmatrix}$$

b. From part a, determine the spectrum of A.

From the calculations, 0, 2, and 2 are eigenvalues of A and so spec(A) = (0, 2, 2).

c. From part a, determine an eigenbasis of A for $\mathbf{R}^3$.

The three vectors multiplied by A in part a are eigenvectors of A. Moreover, they are linearly independent and span $\mathbf{R}^3$. So,

$$(\vec{v}_1, \vec{v}_2, \vec{v}_3)=\left(\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}1\\2\\1\end{bmatrix},\begin{bmatrix}1\\1\\3\end{bmatrix}\right)$$ is an eigenbasis of A for $\mathbf{R}^3$.

d. Provide a nonsingular 3×3 matrix S such that $S^{-1}AS=D$ is a diagonal matrix.

$$S=[\vec{v}_1\,|\,\vec{v}_2\,|\,\vec{v}_3]=\begin{bmatrix}1&1&1\\1&2&1\\1&1&3\end{bmatrix}.$$

5. For each of the following statements, A, B, and C are $n\times n$ matrices where $n>1$. Indicate whether the statement is True (**T**) or False (**F**) and provide an explanation.

a. If 0 is an eigenvalue of A, then A is singular (not invertible).

True. 0 is an eigenvalue of A if and only if 0 = det(A – 0I) = det(A).

b. If A is similar to B and B is similar to C, then A is similar to C.

True. Similarity is a transitive relationship.

c. Similar matrices have the same eigenvectors.

False. $A=\begin{bmatrix}1&0\\0&2\end{bmatrix}$ is similar to $B=\begin{bmatrix}1&1\\0&2\end{bmatrix}$ are similar but

$$E_2(A)=\operatorname{span}\begin{bmatrix}0\\1\end{bmatrix}\ \text{and}\ E_2(B)=\operatorname{span}\begin{bmatrix}1\\1\end{bmatrix}.$$

d. If 1 is the only eigenvalue of A and its algebraic multiplicity is n, then A is the identity matrix.

False. Let $A=\begin{bmatrix}1&1\\0&1\end{bmatrix}\neq I$.

e. The characteristic polynomials of A and A^T are the same.

True. $\det(\lambda I - A) = \det\left((\lambda I - A)^T\right) = \det(\lambda I - A^T)$.

f. If $\vec{v}$ is an eigenvector of both A and B, then $\vec{v}$ is also an eigenvector of both $A + B$ and $A\,B$.

True. If $A\vec{v} = \alpha\vec{v}$ and $B\,\vec{v} = \beta\vec{v}$ then,

$$(A+B)\vec{v} = A\,\vec{v} + B\vec{v} = \alpha\,\vec{v} + \beta\,\vec{v} = (\alpha+\beta)\vec{v}$$

$$(AB)\vec{v} = A(B\vec{v}) = A(\beta\,\vec{v}) = \beta(A\vec{v}) = (\beta\alpha)\vec{v} = (\alpha\beta)\vec{v}$$

g. Every eigenvector of A belongs to $\text{Col}(A)$ or to $\text{Nul}(A)$.

True. If $\vec{v}$ is an eigenvector with nonzero eigenvalue λ, then $A(1/\lambda)\vec{v} = (\lambda/\lambda)\vec{v} = \vec{v}$ shows that $\vec{v}$ is in $\text{Col}(A)$. On the other hand, if $\vec{v}$ is an eigenvector with zero eigenvalue, $A\vec{v} = 0\vec{v} = \vec{0}$ shows that $\vec{v}$ belongs to $\text{Nul}(A)$.

h. If A and B are diagonalizable, then $A\,B$ is diagonalizable.

False. $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & \frac{1}{2} \end{bmatrix}$ are diagonalizable since they have a pair of distinct eigenvalues. In fact, A is diagonal; but, $AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is a non-diagonalizable shear.

6. Suppose that M is a plane through the origin in $\mathbf{R}^3$ and N is the subspace of $\mathbf{R}^3$ that is orthogonal to M, i.e. $M^\perp = N$. Now, let F be the standard matrix for reflection across M.

a. What are $\dim(M)$ and $\dim(N)$?

N is the subspace of $\mathbf{R}^3$ spanned by any nonzero vector normal (orthogonal) to M. Clearly, $\dim(M) = 2$ and $\dim(N) = 1$.

b. What are the eigenvalues of F and what are their algebraic and geometric multiplicities?

M and N are the eigenspaces of F corresponding to the eigenvalues 1 and -1, respectively. 1 has geometric multiplicity 2 and so it must have algebraic multiplicity 2. -1 has geometric and algebraic multiplicity 1

c. What is the characteristic polynomial $\chi_F(\lambda)$ of F?

Since we know the eigenvalues and their algebraic multiplicities we have $\chi_F(\lambda) = (\lambda-1)^2(\lambda+1)$.

d. Describe the eigenspaces of F and their relationships to M and N.

As indicated above, $E_1(F) = M$ and $E_{-1}(F) = N$

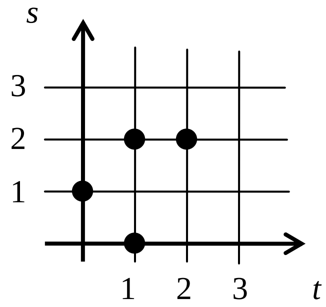
e. Find all diagonal matrices to which F is similar.

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

f. What is $\det(F)$?

F is similar to matrices with determinant -1, so $\det(F) = -1$.

7.



Find the coefficients a and b for which a line with the equation

$$s = a + ct$$

best fits, in the least squares sense, the data points shown at the left.

We seek a and c so that

$$\begin{Bmatrix} 1 = a + 0c \\ 0 = a + 1c \\ 2 = a + 1c \\ 2 = a + 2b \end{Bmatrix} \quad \text{or} \quad A\vec{x} = \vec{b} \quad \text{where } A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, \ \vec{x} = \begin{bmatrix} a \\ c \end{bmatrix}, \ \vec{b} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 2 \end{bmatrix}. \text{ In fact,}$$

it is clear from the graph above as well as the fact that $\vec{b} \notin \mathrm{Col}(A)$ that no solution exists in the conventional sense. We turn to the normalized equation instead for a least squares solution to $A^T A\,\vec{x}^* = A^T\vec{b}$. Since $\mathrm{Nul}(A)$ is trivial, we have $\vec{x}^* = (A^T A)^{-1} A^T \vec{b} =$

$$(A^T A)^{-1} A^T \vec{b} = \left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 4 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$= \tfrac{1}{8} \begin{bmatrix} 6 & -4 \\ -4 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \end{bmatrix}.$$ The least squares line has equation $s = \frac{3}{4} + \frac{1}{2}t$.

8. Determine a matrix $A \in \mathbf{R}^{3\times 3}$ and its characteristic polynomial so that $\mathrm{spec}(A) = (0, 0, -1, 1)$ and

$$E_0(A) = \text{span}\left(\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}\right),\ E_{-1}(A) = \text{span}\left(\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}\right),\ \text{and}\ E_1(A) = \text{span}\left(\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}\right).$$

Since the eigenvalues of A are $0, -1$, and 1 and their multiplicities (both geometric and algebraic) are, respectively, 2, 1, and 1, the characteristic polynomial for A is $\chi_A(\lambda) = \lambda^2(\lambda+1)(\lambda-1) = \lambda^4 - \lambda^2$.

Normalizing the four vectors that appear above yields an orthonormal eigenbasis $(\hat{u}_1, \hat{u}_2, \hat{u}_3, \hat{u}_4) = \left(\frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \frac{1}{2}\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}, \frac{1}{2}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}, \frac{1}{2}\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}\right)$ for $\mathbf{R}^4$. Therefore, the matrix $S = [\hat{u}_1 \,|\, \hat{u}_2 \,|\, \hat{u}_3 \,|\, \hat{u}_4]$ is an orthogonal diagonalizer for A and so,

$$A = S^{-1}DS = S^TDS = \tfrac{1}{4}\begin{bmatrix}1&1&1&1\\1&1&-1&-1\\1&-1&-1&1\\1&-1&1&-1\end{bmatrix}\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&-1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&1&1&1\\1&1&-1&-1\\1&-1&-1&1\\1&-1&1&-1\end{bmatrix}$$

$$= \tfrac{1}{4}\begin{bmatrix}1&1&1&1\\1&1&-1&-1\\1&-1&-1&1\\1&-1&1&-1\end{bmatrix}\begin{bmatrix}0&0&0&0\\0&0&0&0\\-1&1&1&-1\\1&-1&1&-1\end{bmatrix} = \tfrac{1}{2}\begin{bmatrix}0&0&1&-1\\0&0&-1&1\\1&-1&0&0\\-1&1&0&0\end{bmatrix}.$$