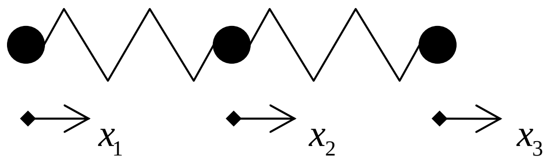
Special Problem J Solution

Three objects, each with a mass of 1 kg, are connected in a line to one another by a pair of ideal springs, each with spring constant 1 N/m and the system is set into motion along the line joining them. The displacements of the objects from a configuration in which both springs are relaxed are $x_1(t),\ x_2(t),$ and $x_3(t)$.



From Newton's Second Law of Motion, one obtains the following three coupled linear, second-order , ordinary, homogeneous differential equations that completely describe the dynamics of the system.

$$x_1'' + x_1 - x_2 = 0$$
$$x_2'' - x_1 + 2x_2 - x_3 = 0$$
$$x_3'' - x_2 + x_3 = 0$$

a. Let $\vec{x}(t)$ be the vector in $\mathbf{R}^3$ whose components are $x_1(t),\ x_2(t),$ and $x_3(t)$. Now, rewrite the equations above as a single vector differential equation of the form

$$\vec{x}'' + A\,\vec{x} = \vec{0}$$

Identify the matrix A and find its eigenvalues and eigenvectors.

$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$. Setting $\det(A - \lambda I) = 0,$ we find $\operatorname{spec}(A) = (0, 1, 3).$

$E_0(A) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right), \quad E_1(A) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right), \quad E_3(A) = \operatorname{span}\left(\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right).$

b. What is the general solution to the equation in part a? To answer this question, you must first determine the general solution to the equation $x'' + \omega^2\, x = 0$ for the scalar function $x(t)$ when $\omega = 0$ and when $\omega > 0$.

The solution to the scalar HODE $x'' + \omega^2\, x = 0$ is $x(t) = a\cos(\omega\, t) + b\sin(\omega\, t)$ when $\omega > 0$ and $x(t) = a + bt$ when $\omega = 0$. Therefore, the general solution to the vector equation above is the following.

$$\vec{x}(t) = \left(a_0 + b_0\, t\right)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \left(a_1\cos(t) + b_1\sin(t)\right)\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \left(a_3\cos(\sqrt{3}\,t) + b_3\sin(\sqrt{3}\,t)\right)\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

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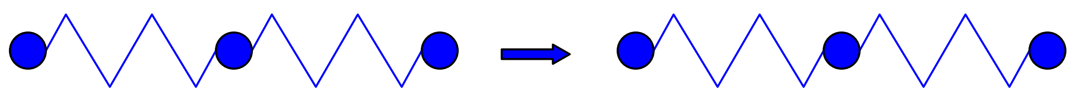
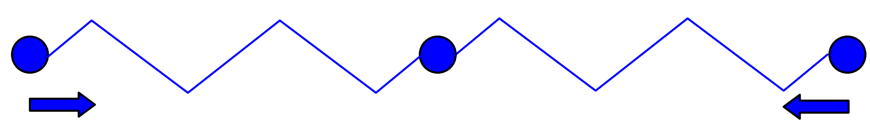
There are three terms above. Each corresponds to a different eigenvalue of A and is the product of the general solution for the scalar HODE $x''+\omega^2 x=0$ and an eigenvector of A for the eigenvalue ω^2. This result was established in class but we will present it again at the <u>end</u> of the discussion of these notes. It is straightforward, but a bit tedious, to verify that the result above is a solution. To do so, one simply substitutes the expression above into the equation $\vec{x}''+A\vec{x}=\vec{0}$ to obtain an identity.

c. Discuss how the solution in part b corresponds to three distinct and independent motions, each associated to an eigenvalue and eigenvector of A. Qualitatively describe each of these fundamental motions or "modes". You should find that two of these modes are oscillatory and one is merely a translation at constant speed.

The first term of the three in part b contains a scalar affine function in the time multiplying the eigenvector $\begin{bmatrix}1\\1\\1\end{bmatrix}$ for the eigenvalue zero. This describes constant speed motion of all three objects in tandem. This is not an oscillatory motion.



The second term is the product of a scalar function in t describing simple harmonic oscillation with angular frequency 1 and the eigenvector $\begin{bmatrix}1\\0\\-1\end{bmatrix}$. This describes an oscillation in which the middle object is stationary while the two outside objects move in opposite directions.



The third term is the product of a simple harmonic scalar function with an angular frequency $\sqrt{3}$ and the eigenvector $\begin{bmatrix}1\\-2\\1\end{bmatrix}$ that corresponds to the eigenvalue 3. This describes an oscillation in which the two outside objects move in tandem while the middle object moves in the opposite direction with an amplitude twice as great. The angular frequency is $\sqrt{3}$.

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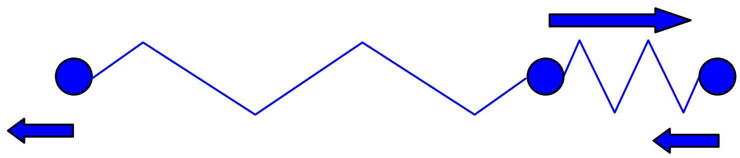

d. Solve the system, given the initial conditions $\vec{x}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and $\vec{x}'(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

This corresponds to the following. At time $t = 0$, the two springs are relaxed and the objects each have zero displacement from their equilibrium positions. At this time, the middle and right objects are at rest but the left object is struck and imparted with a speed of 1 m/s to the right.

The initial conditions are these.

$$\vec{x}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = a_0 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + a_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + a_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

$$\vec{x}'(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = b_0 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + b_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + b_3\sqrt{3} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

The six coefficients may be found by ordinary row reduction, or much more efficiently, by using the orthogonality of the eigenvectors. In each of the two vector equations above, simply take the inner product of both sides with each of the three eigenvectors. Note that they are orthogonal to one another.

$a_0 = a_1 = a_3 = 0$ and $b_0 = \frac{1}{3}$, $b_1 = \frac{1}{2}$, $b_3 = \frac{1}{6\sqrt{3}}$.

So, $\vec{x}(t) = \frac{1}{3}\, t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{2} \sin(t) \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \frac{1}{6\sqrt{3}} \sin(\sqrt{3}t) \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$

This is the superposition (linear combination) of a constant speed motion and two different oscillations with different frequencies.

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As promised in part b, above, we will now develop the solution from first principles. Let $\vec{v}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. These are eigenvectors of A

corresponding to the three eigenvalues 0, 1, and 3, respectively. They comprise a basis for $\mathbf{R}^3$ and so, the matrix $S=[\vec{v}_0\,|\,\vec{v}_1\,|\,\vec{v}_3]$ is invertible. Moreover,

$$S^{-1}A\,S=D=\begin{bmatrix}0&0&0\\0&1&0\\0&0&3\end{bmatrix}.$$

So, we manipulate the equation $\vec{x}\,''+A\,\vec{x}=\vec{0}$ as follows. $S^{-1}\vec{x}\,''+S^{-1}A\,(S\,S^{-1})\vec{x}=\vec{0}$ or $(S^{-1}\vec{x})''+(S^{-1}A\,S)\,(S^{-1}\vec{x})=\vec{0}$. Now, we introduce a change of variables. Let $\vec{y}=S^{-1}\vec{x}$ or $\vec{x}=S\,\vec{y}$. Substituting above, we obtain $\vec{y}\,''+D\,\vec{y}=\vec{0}$. This vector differential equation is uncoupled. If we write out its scalar components, (let's call them f, g, and h, respectively) we have $f\,''=0$, $g\,''+g=0$, $h''+3h=0$. The solutions are $f(t)=a_0+b_0\,t$, $g(t)=a_1\cos(t)+b_1\sin(t)$, and $h(t)=a_3\cos(\sqrt{3}t)+b_3\sin(\sqrt{3}t)$ So,

$$\vec{x}(t)=S\;\vec{y}(t)=[\vec{v}_0\,|\,\vec{v}_1\,|\,\vec{v}_3]\begin{bmatrix}f(t)\\g(t)\\h(t)\end{bmatrix}$$

$$=\left(a_0+b_0\,t\right)\begin{bmatrix}1\\1\\1\end{bmatrix}+\left(a_1\cos(t)+b_1\sin(t)\right)\begin{bmatrix}1\\0\\-1\end{bmatrix}+\left(a_3\cos(\sqrt{3}\,t)+b_3\sin(\sqrt{3}\,t)\right)\begin{bmatrix}1\\-2\\1\end{bmatrix}.$$

as above.