1. Reformulate each of the following as a problem in finding a solution $\vec{x}$ to the equation $A\vec{x}=\vec{b}$, where A is a matrix and $\vec{x}$ and $\vec{b}$ are vectors. In each case, identify A, $\vec{x}$ and $\vec{b}$. Do not solve.

a. Determine if $\begin{bmatrix} 7 \\ 5 \\ 10 \\ 6 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 3 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 2 \\ 3 \\ 3 \\ 1 \end{bmatrix}$.

A vector $\vec{b}$ is a linear combination of the vectors $\vec{a}_1, \vec{a}_2,$ and $\vec{a}_3$ if and only if there are scalars $x_1, x_2,$ and x_3 so that $x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 = \vec{b}$ if and only if $A\vec{x}=\vec{b}$ where $A = [\vec{a}_1 \,|\, \vec{a}_2 \,|\, \vec{a}_3] = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 3 \\ 2 & 1 & 1 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 7 \\ 5 \\ 10 \\ 6 \end{bmatrix}$.

b. Find all vectors $\vec{v}$ in $\mathbf{R}^3$ whose scalar products with $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$ are, respectively, 3, 4, and 5.

The vectors desired are those which solve $A\vec{x}=\vec{b}$ where

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 1 \\ 4 & 1 & 2 \end{bmatrix}, \quad \vec{x} = \vec{v}, \quad \text{and} \quad \vec{b} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}.$$

c. $F = \frac{1}{3}\begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & 2 \\ -2 & 2 & -1 \end{bmatrix}$ is the matrix for a reflection across a plane through the origin in $\mathbf{R}^3$. Find all vectors $\vec{v}$ normal (perpendicular) to this plane from the fact that the reflected image of a normal vector $\vec{v}$ is $-\vec{v}$.

A normal vector satisfies $F\vec{v} = -\vec{v} = -\vec{v}$ or $F\vec{v} + \vec{v} = \vec{0}$ or $(F+I)\vec{v} = \vec{0}$.

So, normal vectors are solutions to $A\vec{x}=\vec{b}$ where

$$A = F + I = F = \frac{1}{3}\begin{bmatrix} 5 & 1 & -2 \\ 1 & 5 & 2 \\ -2 & 2 & 2 \end{bmatrix}, \quad \vec{x} = \vec{v}, \quad \text{and} \quad \vec{b} = \vec{0}.$$

2. Given $[A\,|\,\vec{b}]_{rref}$ in each case below, solve $A\vec{x}=\vec{b}$ completely.

a. $[A\,|\,\vec{b}]_{rref}=\left[\begin{array}{ccc|c}1&0&0&1\\0&1&0&2\\0&0&1&3\\0&0&0&0\end{array}\right]$. $\vec{x}=\begin{bmatrix}1\\2\\3\end{bmatrix}$.

b. $[A\,|\,\vec{b}]_{rref}=\left[\begin{array}{cccc|c}1&0&0&1&0\\0&1&0&2&0\\0&0&1&3&0\\0&0&0&0&1\end{array}\right]$. This system has no solution.

c. $[A\,|\,\vec{b}]_{rref}=\left[\begin{array}{ccccc|c}1&2&0&0&3&6\\0&0&1&0&4&7\\0&0&0&1&5&8\\0&0&0&0&0&0\end{array}\right]$. $\vec{x}=\begin{bmatrix}6\\0\\7\\8\\0\end{bmatrix}+\alpha\begin{bmatrix}2\\-1\\0\\0\\0\end{bmatrix}+\beta\begin{bmatrix}3\\0\\4\\5\\-1\end{bmatrix}$ where α and β are any reals.

3. Consider the equation $A\vec{x}=\vec{b}$ where A is an $m\times n$ matrix, $\vec{x}$ is in $\mathbf{R}^n$ and $\vec{b}$ is in $\mathbf{R}^m$. Show that this equation cannot have a finite number of solutions greater than 1 by proceeding as follows.

a. First, define what is meant by: $\vec{w}$ is a solution of the equation above.

$\vec{w}$ is a solution if and only if $A\vec{w}=\vec{b}$ is true.

b. Suppose that $\vec{u}$ and $\vec{v}$ are distinct (different) solutions to $A\vec{x}=\vec{b}$. Let $\vec{z}$ be the difference between $\vec{v}$ and $\vec{u}$. Note that $\vec{z}\neq\vec{0}$. What is $A(\alpha\vec{z})$ for any α in $\mathbf{R}$?

$A(\alpha\vec{z})=\alpha A\,\vec{z}=\alpha A(\vec{v}-\vec{u})=\alpha(A\vec{v}-A\vec{u})=\alpha(\vec{b}-\vec{b})=\vec{0}$.

c. Now, show that $\{\vec{u}+\alpha\vec{z}\,|\,\alpha\in\mathbf{R}\}$ is a set with infinitely many (different) vectors each of which is a solution to $A\vec{x}=\vec{b}$.

If $\alpha\neq\beta$, $(\vec{u}+\alpha\vec{z})-(\vec{u}+\beta\vec{z})=(\alpha-\beta)\vec{z}\neq\vec{0}$. Moreover, $A(\vec{u}+\alpha\vec{z})=$ $A\vec{u}+\alpha A\vec{z}=b+\alpha\vec{0}=\vec{b}$ shows that each of the infinitely many different vectors in $\{\vec{u}+\alpha\vec{z}\,|\,\alpha\in\mathbf{R}\}$ is a solution.

4. a. Suppose that $T:\mathbf{R}^2 \to \mathbf{R}^2$ is linear, T triples vectors parallel to $\begin{bmatrix}1\\2\end{bmatrix}$ and doubles vectors perpendicular to $\begin{bmatrix}1\\2\end{bmatrix}$. Find $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)$ for any real x and y.

Let A be the standard matrix for T and let P be the standard matrix for projection onto the vector $\begin{bmatrix}1\\2\end{bmatrix}$. Since $I = P + (I-P)$, $\vec{x} = I\vec{x} = P\vec{x} + (I-P)\vec{x}$. So, $A\vec{x} = A\left(P\vec{x}\right) + A\left((I-P)\vec{x}\right) = 3P\vec{x} + 2(I-P)\vec{x}$. This last equation is true for every $\vec{x}$. So, $Ax = 3P + 2(I-P) = 2I + P$. Now,

$$P = \left(\tfrac{1}{\sqrt{5}}\begin{bmatrix}1\\2\end{bmatrix}\right)\left(\tfrac{1}{\sqrt{5}}\begin{bmatrix}1\\2\end{bmatrix}\right)^T = \tfrac{1}{5}\begin{bmatrix}1 & 2\\2 & 4\end{bmatrix} \text{ and } A = 2\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + \tfrac{1}{5}\begin{bmatrix}1 & 2\\2 & 4\end{bmatrix} = \tfrac{1}{5}\begin{bmatrix}11 & 2\\2 & 14\end{bmatrix}.$$

Consequently, $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \tfrac{1}{5}\begin{bmatrix}11 & 2\\2 & 14\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}\frac{11}{5}x + \frac{2}{5}y\\ \frac{2}{5}x + \frac{14}{5}y\end{bmatrix}$.

b. Solve for the matrix X if $\begin{bmatrix}1 & 1\\1 & 3\end{bmatrix}X^{-1}\begin{bmatrix}1 & 1\\1 & 3\end{bmatrix}^{-1} = \begin{bmatrix}5 & 2\\-3 & 2\end{bmatrix}^{-1}$.

Inverting both sides of the above equation, we obtain

$$\begin{bmatrix}1 & 1\\1 & 3\end{bmatrix}X\begin{bmatrix}1 & 1\\1 & 3\end{bmatrix}^{-1} = \begin{bmatrix}5 & 2\\-3 & 2\end{bmatrix} \text{ and so, } X = \begin{bmatrix}1 & 1\\1 & 3\end{bmatrix}^{-1}\begin{bmatrix}5 & 2\\-3 & 2\end{bmatrix}\begin{bmatrix}1 & 1\\1 & 3\end{bmatrix} =$$

$$\tfrac{1}{2}\begin{bmatrix}3 & -1\\-1 & 1\end{bmatrix}\begin{bmatrix}5 & 2\\-3 & 2\end{bmatrix}\begin{bmatrix}1 & 1\\1 & 3\end{bmatrix} = \tfrac{1}{2}\begin{bmatrix}3 & -1\\-1 & 1\end{bmatrix}\begin{bmatrix}7 & 11\\-1 & 3\end{bmatrix} = \tfrac{1}{2}\begin{bmatrix}22 & 30\\-8 & -8\end{bmatrix} = \begin{bmatrix}11 & 15\\-4 & -4\end{bmatrix}.$$

5. For each assertion below, indicate whether the assertion is true (***T***) or false (***B***) by circling the correct letter. It is not necessary to show any work. However, a correct response earns full credit, an incorrect response earns negative half credit, and no response earns no credit.

T ***F*** a. A linear system of 7 equations in 8 variables always has infinitely many solutions.

False. One of the equations could the inconsistent equation 0 = 1.

T ***F*** b. A 4×4 matrix of rank 4 is always invertible.

True. This is one of many assertions about a square matrix equivalent to its invertibility.

T ***F*** c. Given any three nonzero vectors in $\mathbf{R}^2$, no two of which are collinear, any one of them is a always a linear combination of the other two.
True. Two nonzero noncollinear vectors span $\mathbf{R}^2$

T ***F*** d. The function $T:\mathbf{R}^2 \to \mathbf{R}^2$ defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ 1 \end{bmatrix}$ is linear.

False. $T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

T ***F*** e. There is a 2×2 matrix A, different from the identity matrix, such that A^{2012} is the identity matrix.
True. Perhaps the simplest example is $A = -I$. If the exponent had been an odd integer, this example would not work. Another example is this. Let A be the standard matrix for rotation in the plane by the angle $\frac{2\pi}{2012}$.

T ***F*** f. T and S are two linear transformations defined by $T(\vec{x}) = A\vec{x}$ and $S(\vec{x}) = B\vec{x}$ where A and B are 3×3 matrices. The composite function $T \circ S$ is also a linear transformation and its matrix is AB.
True. This is how matrix multiplication has been defined.

T ***F*** g. A 10×10 matrix with more than 90 ones among its entries is not invertible.

6. T is defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 & 3 & 5 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$.

a. What are the domain and co-domain of T?
$\text{domain}(T) = \mathbf{R}^4$ and $\text{co-domain}(T) = \mathbf{R}^2$.

b. Is T linear? Why?
Yes, it is linear because is multiplication by a constant matrix.

c. Find all vectors $\vec{x}$ so that $T(\vec{x}) = \vec{0}$.

$$\begin{bmatrix} 1 & 1 & 3 & 5 \\ 0 & 1 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix} \Rightarrow \vec{x} = \alpha \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix} \text{ for any } \alpha, \beta \in \mathbf{R}.$$

d. Notice that, $T\left(\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}5\\3\end{bmatrix}$. Use the result of part c to find all vectors $\vec{x}$ for which $T(\vec{x})=\begin{bmatrix}5\\3\end{bmatrix}$ without using row reduction, again.

$$\vec{x}=\begin{bmatrix}0\\0\\0\\1\end{bmatrix}+\alpha\begin{bmatrix}1\\2\\-1\\0\end{bmatrix}+\beta\begin{bmatrix}2\\3\\0\\-1\end{bmatrix} \text{ for any } \alpha,\beta\in\mathbf{R}.$$

7. Suppose that $T:\mathbf{R}^2\to\mathbf{R}^2$ is linear and we are given

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right)=\begin{bmatrix}2\\3\end{bmatrix} \text{ and } T\left(\begin{bmatrix}-2\\3\end{bmatrix}\right)=\begin{bmatrix}1\\5\end{bmatrix}.$$

a. If A is the standard matrix for T, combine the two given equations into a *single* matrix equation for A.

$$A\begin{bmatrix}1&-2\\1&3\end{bmatrix}=\begin{bmatrix}2&1\\3&5\end{bmatrix}.$$

b. Find the matrix A in part a.

$$A=\begin{bmatrix}2&1\\3&5\end{bmatrix}\begin{bmatrix}1&-2\\1&3\end{bmatrix}^{-1}=\begin{bmatrix}2&1\\3&5\end{bmatrix}\tfrac{1}{5}\begin{bmatrix}3&2\\-1&1\end{bmatrix}=\tfrac{1}{5}\begin{bmatrix}5&5\\4&11\end{bmatrix}.$$

c. Determine $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)$ for any reals x and y.

$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=A\begin{bmatrix}x\\y\end{bmatrix}=\tfrac{1}{5}\begin{bmatrix}5&5\\4&11\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x+y\\\tfrac{4}{5}x+\tfrac{11}{5}y\end{bmatrix}.$$