Name: Solutions	

1. Suppose that $L = (\vec{v}_1, \ldots, \vec{v}_p)$ is a list of p vectors in $\mathbf{R}^n$.

a. Assuming that $0 < p < n$, describe completely but succinctly a specific procedure, employing standard techniques of this course, to unambiguously determine if L is linearly independent.

Let A be the $n \times p$ matrix whose columns are the corresponding vectors in L, i.e. $A = [\vec{v}_1 | \cdots | \vec{v}_p]$. Find the rref of A. Then, L is linearly independent if and only if every column of A_{rref} contains a leading 1 [every column of A is a pivot column or rank(A) = p].

b. Assuming that $p > 0$ and $n > 0$, describe completely but succinctly a specific procedure, employing standard techniques of this course, to unambiguously determine if a given vector $\vec{w}$ in $\mathbf{R}^n$ belongs to span(L).

This is equivalent to determining whether a solution exists to the equation $A\vec{x} = \vec{w}$ where the column vectors of A are the corresponding vectors in L. So, find the rref of $[A | \vec{w}]$. Then, $\vec{w}$ belongs to span(L) if and only if the last column of $[A | \vec{w}]_{rref}$ does not contain a leading 1 [the last column of A is not a pivot column].

2. a. Suppose that $n > 0$, and $\vec{a}$ is a nonzero vector in $\mathbf{R}^n$, and S is the subset of all vectors in $\mathbf{R}^n$ that are orthogonal to $\vec{a}$. Prove that S is a subspace of $\mathbf{R}^n$ and determine dim(S).

If $\vec{v}_1$ and $\vec{v}_2$ both belong to S and α_1 and α_2 are any reals, then $a \cdot (\beta_1 \vec{v}_1 + \beta_2 \vec{v}_2) = \beta_1 a \cdot \vec{v}_1 + \beta_2 a \cdot \vec{v}_2 = 0 + 0 = 0$. That is, all linear combinations of vectors in S are also in S. Hence, S is closed under vector addition and multiplication by scalars and is therefore a subspace. Another way of seeing that this is a subspace is observe that it is the kernel of the $1 \times n$ matrix $\begin{bmatrix} a_1 & a_2 & \ldots & a_n \end{bmatrix}$ and its dimension is clearly $n - 1$ since there is only one pivot row and so only one pivot column for such a matrix.

b. Prove that the subset T of all vectors $\vec{x}$ in $\mathbf{R}^4$ that satisfy $\|\vec{x}\| \le 1$ is <u>not</u> a subspace of $\mathbf{R}^4$.

The first standard basis vector $\hat{e}_1$ clearly belongs to T since $\|\hat{e}_1\| = 1$. However, $2\hat{e}_1$ does not belong to T since $\|2\hat{e}_1\| = 2$. So, T is not closed under multiplication by scalars and is therefore not a subspace.

4. $A = \begin{bmatrix} 1 & 2 & 1 & 3 & 4 \\ 2 & 0 & 1 & 0 & 3 \\ 0 & 1 & 2 & 5 & 3 \\ 1 & 3 & 3 & 8 & 7 \end{bmatrix}$ and $A_{rref} = \begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

a. Determine a basis $\mathscr{K}$ for $\ker(A)$ and find $\dim(\ker(A))$.

From the Solution Algorithm, $\mathscr{K} = \left(\begin{bmatrix} 1 \\ -1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right)$ and $\dim(\ker(A)) = 2$.

b. Determine a basis $\mathscr{M}$ for $\operatorname{im}(A)$ and find $\dim(\operatorname{im}(A))$.

Examination of A_{rref} reveals that the fourth and fifth columns of A are linear combinations of the first three columns of A which comprise a basis for $\operatorname{im}(A)$. Hence, $\mathscr{M} = \left(\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix} \right)$ and $\dim(\operatorname{im}(A)) = 3$.

c. For which vectors $\vec{b}$ does $A\vec{x} = \vec{b}$ have solutions? Explain.

$A\vec{x} = \vec{b}$ has solutions if and only if $\vec{b}$ belongs to $\operatorname{im}(A)$.

d. If $\vec{b}$ is a vector for which $A\vec{x} = \vec{b}$ has solutions, when is the solution unique? Explain.

If $\vec{b}$ belongs to $\operatorname{im}(A)$, $A\vec{x} = \vec{b}$ will always have infinitely many solutions since $\ker(A)$ consists of infinitely many distinct vectors. Any nonzero vector in $\ker(A)$ added to a solution results in a different solution.

e. Labeling the columns of A as $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5$, respectively, find the coordinate vectors $[\vec{v}_2]_{\mathscr{M}}$ and $[\vec{v}_4]_{\mathscr{M}}$.

Since $\mathscr{M}$ is a basis for $\operatorname{im}(A)$, every vector in $\operatorname{im}(A)$ is a unique linear combination of the vectors in $\mathscr{M}$. The coefficients of these linear combinations provide the $\mathscr{M}$-coordinate vectors of any vector in $\operatorname{im}(A)$. But, A_{rref} reveals how each column vector of A is a specific linear combination of the vectors in $\mathfrak{M}$ (the first three column vectors of im(A)).

Specifically, we see that $[\vec{v}_2]_{\mathfrak{M}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $[\vec{v}_4]_{\mathfrak{M}} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$

5. M is the line in $\mathbf{R}^2$ whose equation is $2x - y = 0$. Suppose that $T : \mathbf{R}^2 \to \mathbf{R}^2$ is the linear transformation that triples all vectors in $\mathbf{R}^2$ that are parallel to M and leaves unchanged all vectors in $\mathbf{R}^2$ that are perpendicular to M.

a. Choose a convenient basis $\mathfrak{B} = (\vec{v}_1, \vec{v}_2)$ for $\mathbf{R}^2$ so that T transforms the basis vectors in a simple way. Identify $\vec{v}_1$ and $\vec{v}_2$. Also express $T(\vec{v}_1)$ and $T(\vec{v}_2)$ in terms of $\vec{v}_1$ and $\vec{v}_2$.

A vector parallel to M is $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and a vector perpendicular to M is

$\vec{v}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Clearly, $T(\vec{v}_1) = 3\vec{v}$, and $T(\vec{v}_2) = \vec{v}_2$.

b. What is the matrix S such that $\vec{x} = S[\vec{x}]_{\mathfrak{B}}$ for any vector $\vec{x}$ in $\mathbf{R}^2$ and what is its inverse?

$$S = \left[\vec{v}_1 \mid \vec{v}_2\right] = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \text{ and so, } S^{-1} = \frac{1}{-5}\begin{bmatrix} -1 & -2 \\ -2 & 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}.$$

c. What is the matrix B for T in $\mathfrak{B}$-coordinates? That is, what is the matrix B such that $[T(\vec{x})]_{\mathfrak{B}} = B[\vec{x}]_{\mathfrak{B}}$ for any $\vec{x}$ in $\mathbf{R}^2$?

$$B = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}.$$

d. Determine the standard matrix A for T. That is, find A so that $T(\vec{x}) = A\vec{x}$ for any vector $\vec{x}$ in $\mathbf{R}^2$.

$$A = SBS^{-1} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 3 & 6 \\ 2 & -1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 7 & 4 \\ 4 & 13 \end{bmatrix}.$$

e. Compute $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$.

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 7 & 4 \\ 4 & 13 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 11/5 \\ 13/5 \end{bmatrix}.$$