1. $A \in \mathbf{R}^{4\times 5}$, $A_{rref} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $\left(A^T\right)_{rref} = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$,

and $A[1\ 0\ 0\ 0\ 0]^T = [1\ 2\ 3\ 4]^T$. [Notes: (1) The information provided here is insufficient to determine A. (2) Recall that elementary row operations may change the rows of a matrix but do not change their span.]. Determine each of the following.

a. a basis for $\ker(A)$

From the Solution Algorithm applied to A_{rref}, we have the following basis for $\ker(A)$: $\left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \right)$.

b. a basis for $\operatorname{im}(A)$

According to Note (2), the row vectors of A^T, which are, of course, the column vectors of A, have the same span as the row vectors of $\left(A^T\right)_{rref}$. But, the nonzero rows of $\left(A^T\right)_{rref}$ are linearly independent by observation and construction. So, they comprise a basis for the row vectors of A^T. Therefore, the transposes of the nonzero rows of $\left(A^T\right)_{rref}$ provide a basis for $\operatorname{im}(A)$: $\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -5 \end{bmatrix} \right)$.

c. the solution set for $A\vec{x} = [1\ 2\ 3\ 4]^T$

This is the kernel of A shifted by the particular solution provided above, that is $[1\ 0\ 0\ 0\ 0]^T + \ker(A)$.

d. the solution set for $A\vec{x} = [1\ 0\ 0\ 1]^T$

Since $[1\ 0\ 0\ 1]^T$ is clearly not in the span of our basis for $\operatorname{im}(A)$, this set is empty.

2. Suppose that $A \in \mathbf{R}^{4\times 2}$ and $P = \frac{1}{3}\begin{bmatrix} 2 & 1 & 0 & -1 \\ 1 & 1 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 1 \end{bmatrix}$ is the matrix for projection onto $\text{im}(A)$.

a. How are $\text{im}(A)$ and $\text{im}(P)$ related?

They are the same.

b. Determine $\text{im}(A)$.

Since the first two columns of P comprise a linearly independent pair and A has only two columns, $\text{rank}(A) = 2$. A basis for $\text{im}(A)$ is, $\left(\begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}\right)$.

c. Determine $\ker(A)$.

$\dim(\text{im}(A)) = 2$ and the Rank-Nullity Theorem assures us that $\ker(A) = \left\{\vec{0}\right\}$.

d. Find the projections of $\vec{b} = [3\ \ 0\ \ 0\ \ 0]^T$ and $\vec{c} = [2\ \ 1\ \ 0\ \ -1]^T$ onto $\text{im}(A)$.

Direct calculation reveals that $P\vec{b} = \vec{c}$ and $P\vec{c} = \vec{c}$.

e. How many solutions do each of $A\vec{x} = \vec{b}$ and $A\vec{x} = \vec{c}$ have? Explain.

$\vec{b}$ does not belong to $\text{im}(A)$, so $A\vec{x} = \vec{b}$ has no solutions. $\vec{c}$ belongs to $\text{im}(A)$ and $\ker(A)$ is trivial, so $A\vec{x} = \vec{c}$ has exactly one solution.

f. How is solving $Ax = \vec{b}$ related to solving $Ax = \vec{c}$?

The least squares solution to $Ax = \vec{b}$ is the (unique) solution to $Ax = \vec{c} = P\vec{b}$.

g. Provide a formula for P in terms of A.

$P = A(A^T A)^{-1} A^T$.

3. a. $S = \left\{\begin{bmatrix} a \\ b \end{bmatrix} \mid a, b \in \mathbf{R};\ ab = 0\right\}$ is a subset of $\mathbf{R}^2$ but it is not a subspace. Show that, although S is nonempty and closed under multiplication by scalars, S is not closed under vector addition.

S consists of all points on the coordinate axes. $\vec{x}=\begin{bmatrix}a\\b\end{bmatrix}\in S$ iff $ab=0$.

Then, $t\,\vec{x}=t\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}t\,a\\t\,b\end{bmatrix}$ and $(t\,a)(t\,b)=t^2(a\,b)=0$ and so, $t\,\vec{x}\in S$ for any real t. So, S is nonempty and is closed under multiplication by scalars. However, $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ both belong to S but their sum, $\begin{bmatrix}1\\1\end{bmatrix}$ does not. Therefore, S is not closed under vector addition.

b. $V=\left\{\begin{bmatrix}a & b\\c & d\end{bmatrix} \mid a,b,c,d\in\mathbf{R};\ a+d=0\right\}$ is a subspace of $\mathbf{R}^{2\times 2}$. Find a basis for V.

$$V=\left\{a\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}+b\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}+c\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} \mid a,b,c\in\mathbf{R}\right\}=\text{span}(L)$$

where $L=\left(\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix},\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix},\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}\right)$ is clearly linearly independent and spanning in V, and so is a basis for V.

4. A basis for the subspace V of $\mathbf{R}^4$ is $(\vec{v}_1,\vec{v}_2,\vec{v}_3)=\left(\begin{bmatrix}1\\0\\2\\2\end{bmatrix},\begin{bmatrix}2\\2\\-1\\0\end{bmatrix},\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right)$.

a. Use the Gram-Schmidt Process to find an orthonormal basis for V.

The first two vectors are orthogonal and need only be normalized. We have $\hat{u}_1=\frac{1}{3}\vec{v}_1$ and $\hat{u}_2=\frac{1}{3}\vec{v}_2$. The third vector in our desired basis is obtained next. $\vec{v}_3^{\perp}=\vec{v}_3-(\vec{v}_3\cdot\hat{u}_1)\,\hat{u}_1-(\vec{v}_3\cdot\hat{u}_2)\,\hat{u}_2=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}-\frac{5}{9}\begin{bmatrix}1\\0\\2\\2\end{bmatrix}-\frac{3}{9}\begin{bmatrix}2\\2\\-1\\0\end{bmatrix}=\frac{1}{9}\begin{bmatrix}-2\\3\\2\\-1\end{bmatrix}$ and so,

$\hat{u}_3=\vec{v}_3^{\perp}/\left\|\vec{v}_3^{\perp}\right\|=\frac{1}{2\sqrt{3}}\begin{bmatrix}-2\\3\\2\\-1\end{bmatrix}$. So, an orthonormal basis for V is

$$(\hat{u}_1, \hat{u}_2, \hat{u}_3) = \left(\tfrac{1}{3}\begin{bmatrix}1\\0\\2\\2\end{bmatrix}, \tfrac{1}{3}\begin{bmatrix}2\\2\\-1\\0\end{bmatrix}, \tfrac{1}{2\sqrt{3}}\begin{bmatrix}-2\\3\\2\\-1\end{bmatrix} \right).$$

b. From part a, present an expression for the matrix P for projection onto V. It is not necessary to evaluate the expression.

$P = \hat{u}_1 \hat{u}_1{}^T + \hat{u}_2 \hat{u}_2{}^T + \hat{u}_3 \hat{u}_3{}^T$.

c. Determine an orthonormal basis for $V^\perp$.

$V^\perp$ is the subspace of $\mathbf{R}^4$ consisting of all vectors orthogonal to the given basis vectors. This is the same as the kernel of the matrix whose rows are the basis vectors (either basis) of V. So, we row-reduce to find

$$V^\perp = \ker\left(\left[\vec{v}_1 \,|\, \vec{v}_2 \,|\, \vec{v}_3\right]^T\right) = \ker\begin{bmatrix}1 & 0 & 2 & 2\\2 & 2 & -1 & 0\\1 & 1 & 1 & 1\end{bmatrix} = \ker\begin{bmatrix}1 & 0 & 0 & \tfrac{2}{3}\\0 & 1 & 0 & -\tfrac{1}{3}\\0 & 0 & 0 & \tfrac{2}{3}\end{bmatrix}$$

$= \operatorname{span}\begin{bmatrix}2\\-1\\2\\-3\end{bmatrix}$. Consequently, an orthonormal basis for $V^\perp$ is $(\hat{u}) = \left(\tfrac{1}{\sqrt{18}}\begin{bmatrix}2\\-1\\2\\-3\end{bmatrix}\right)$.

d. Determine the matrix P using the results of part c.

The matrix for projection onto V is $P = I - \hat{u}\,\hat{u}^T =$

$$I - \tfrac{1}{18}\begin{bmatrix}4 & -2 & 4 & -6\\-2 & 1 & -2 & 3\\4 & -2 & 4 & -6\\-6 & 3 & -6 & 9\end{bmatrix} = \tfrac{1}{18}\begin{bmatrix}14 & 2 & -4 & 6\\2 & 17 & 2 & -3\\-4 & 2 & 14 & 6\\6 & -3 & 6 & 9\end{bmatrix}.$$

5. M is the plane in $\mathbf{R}^3$ described by the equation $x + 2y + 3z = 0$. T is the linear transformation on $\mathbf{R}^3$ that reverses all vectors in $M^\perp$ and doubles all vectors in M. Our objective is to find A, the (standard) matrix for T so that $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbf{R}^3$

a. Find a basis $\mathcal{B} = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$ for $\mathbf{R}^3$ consisting of $\vec{v}_1$, a vector belonging to $M^\perp$, plus two vectors $\vec{v}_2$ and $\vec{v}_3$, belonging to M.

Choose $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in M^T$. Choose $\vec{v}_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$, two non-collinear vectors in M. These are obtained either by inspection or using the fact that a basis for M is a basis for $(M^{\perp})^{\perp} = \ker(\vec{v}_1^{\,T}) = \ker[\,1\;\;2\;\;3\,]$.

b. What is the relationship between $\vec{x}$ and $[\vec{x}]_{\mathfrak{B}}$ for any $\vec{x} \in \mathbf{R}^3$?

The coordinate transformation matrix is $S = [\vec{v}_1 \,|\, \vec{v}_2 \,|\, \vec{v}_3] = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ 3 & 0 & -1 \end{bmatrix}$ and so, $\vec{x} = S\,[\vec{x}]_{\mathfrak{B}}$.

c. What is the matrix B that represents T relative to the basis $\mathfrak{B}$, i.e., what is B so that $\left[T(\vec{x})\right]_{\mathfrak{B}} = B[\vec{x}]_{\mathfrak{B}}$ for all $\vec{x} \in \mathbf{R}^3$?

Since $T(\vec{v}_1) = -\vec{v}_1$, $T(\vec{v}_2) = 2\vec{v}_1$, $T(\vec{v}_3) = 2\vec{v}_3$, $B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

d. Determine A.

$$A = S\,B\,S^{-1} = \tfrac{1}{14}\begin{bmatrix} 25 & -6 & -9 \\ -6 & 16 & -18 \\ -9 & -18 & 1 \end{bmatrix}.$$