1. For each of the following, indicate whether the statement is True or False by circling the T or F preceding the statement.

T | F a. If the kernel of a matrix consists of the zero vector only, the matrix is invertible.

T | F b. The determinant of a diagonalizable square matrix is the product of its eigenvalues.

T | F c. An $n \times n$ matrix with n real eigenvalues is always diagonalizable.

T | F d. If the 3×3 matrix A satisfies the equation $3A^2 - A = 2I$, A is invertible.

T | F e. If the list $(\vec{u}, \vec{v}, \vec{w})$ of vectors in $\mathbf{R}^3$ is linearly dependent, then $\vec{u}$ must be a linear combination of $\vec{v}$ and $\vec{w}$.

T | F f. Linear combinations of eigenvectors of a square matrix are also eigen-vectors of that matrix.

2. Suppose that $\vec{r}, \vec{s}, \vec{t}$, and $\vec{u}$ are vectors in $\mathbf{R}^3$. Solve the matrix equation $\left[\vec{r} \mid \vec{s} \mid \vec{t}\right]\vec{x} = \vec{u}$ for $\vec{x}$ if $\det\left[\vec{r} \mid \vec{s} \mid \vec{t}\right] = 4$, $\det\left[\vec{s} \mid \vec{t} \mid \vec{u}\right] = 3$, $\det\left[\vec{t} \mid \vec{u} \mid \vec{r}\right] = 2$, and $\det\left[\vec{u} \mid \vec{r} \mid \vec{s}\right] = 1$.

3. Suppose that $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ and $f : \mathbf{R}^3 \to \mathbf{R}^3$ is the linear transformation that doubles all vectors parallel to $\vec{v}$ and triples all vectors orthogonal to $\vec{v}$. Find the matrix for f.

4. Consider the matrix $A = \dfrac{1}{7}\begin{bmatrix} 2 & 3 & 6 \\ 6 & 2 & -3 \\ -3 & 6 & -2 \end{bmatrix}$.

a. Verify that A is the matrix for a rotation.

b. Find the axis and the angle of this rotation.

5. a. Let $\vec{v} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$. Find the matrix for the linear transformation $f : \mathbf{R}^3 \to \mathbf{R}^3$ defined by $f(\vec{w}) = \vec{v} \times \vec{w}$ for any $\vec{w}$ in $\mathbf{R}^3$.

b. If $\vec{v}$ is any nonzero vector in $\mathbf{R}^n$ and the transformation $g : \mathbf{R}^n \to \mathbf{R}^n$ is defined by $g(\vec{w}) = (\vec{v} \cdot \vec{w})\,\vec{w}$ for any $\vec{w}$ in $\mathbf{R}^n$, show that g is <u>non</u>linear.

6. a. Demonstrate that the subset of all vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbf{R}^2$ for which $xy \geq 0$ is <u>not</u> a subspace of $\mathbf{R}^2$.

b. Prove that $\ker(T)$ is a subspace of the domain of T for any linear transformation $T:\mathbf{R}^n \to \mathbf{R}^m$.

7. Data is collected to determine the two parameters α and β in the relationship $s = \alpha + \beta t$. The following (t, s) data pairs are found: (1, 1), (2, 1), (2, 2). Find values of α and β that fit the data best in a least squares sense.

8. Let $(\vec{v}_1, \vec{v}_2) = \left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 5 \\ 1 \end{bmatrix} \right)$ and define the subspace $V = \operatorname{span}(\vec{v}_1, \vec{v}_2)$ of $\mathbf{R}^4$.

a. Determine the matrix P for the projection onto V.

b. If $\vec{w} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix}$, for which $\vec{v}$ in V is $\|\vec{w} - \vec{v}\|$ least?

9. If $A = \begin{bmatrix} 3 & 2 & 1 & 2 & 2 \\ 1 & 0 & 1 & 1 & 2 \\ 2 & 2 & 0 & 1 & 0 \\ -1 & 2 & -1 & 0 & -2 \end{bmatrix}$ then $A_{rref} = \begin{bmatrix} 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & -1 & -\frac{1}{2} & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

a. Determine a basis for both $\operatorname{im}(A)$ and $\ker(A)$.

b. For which $\vec{b}$ does the equation $A\vec{x} = \vec{b}$ have solutions?

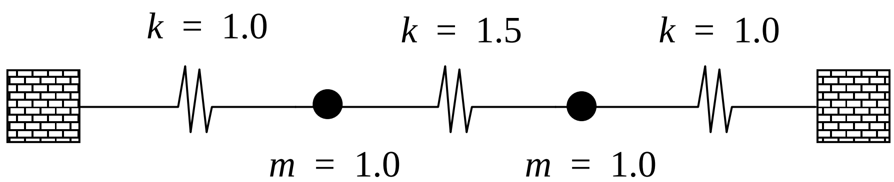
c. Notice that $A \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \\ 5 \\ -2 \end{bmatrix}$. Find all solutions to $A\vec{x} = \begin{bmatrix} 10 \\ 5 \\ 5 \\ -2 \end{bmatrix}$.

10. Describe a step-by-step process and a criterion to unambiguously determine:

a. if a given vector $\vec{w}$ in $\mathbf{R}^n$ belongs to the span of a given list $(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p)$ of vectors in $\mathbf{R}^n$.

b. if a given list $(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p)$ of vectors in $\mathbf{R}^n$ is linearly independent.

11. Below is a diagram of a mechanical oscillator consisting of two particles and three springs. The masses of the two particles are each 1.0 kg. The springs are ideal and obey Hooke's Law. The middle spring constant is 1.5 kg/sec^2 and the two outside spring constants are each 1.0 kg/sec^2. Assume the oscillations occur on a frictionless table along the line between the particles and the fixed points.



$x_1(t)$ and $x_2(t)$ are the displacements of the left and right particles, respectively, relative to a configuration in which all three springs are relaxed. Newton's Second Law gives us a pair of homogeneous, second order, linear, ordinary differential equations. Using $\vec{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$, these equations may be written as one matrix differential equation $\vec{x}\,'' + A\vec{x} = \vec{0}$ where $A = \frac{1}{2}\begin{bmatrix} 5 & -3 \\ -3 & 5 \end{bmatrix}$. Use eigenvalue-eigenvector methods to find the general solution for $\vec{x}(t)$. The solution involves 4 arbitrary constants. Show how the constants are determined in terms of the initial positions and velocities of the particles.

12. Let $A = \frac{1}{2}\begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -2 \\ 1 & -1 & 1 \end{bmatrix}$. Then, $A\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

a. Determine spec(A), the list of eigenvalues of A.

b. Determine the eigenspaces $E_\lambda(A)$ for each of the eigenvalues λ.

c. Determine a matrix Q and its inverse Q^{-1} so that $Q^{-1}AQ$ is diagonal.

d. Compute (all the entries of) A^n for any positive integer n.