This test is due, in my office, not later than 1200 on Thursday, 12 April. You must work in the three- person teams assigned. All members of a team will earn the same grade. You may use any published text, the notes of members of your team, the web, and any calculator or computer program. You may not receive any assistance whatsoever from any person aside from me and your team members. -fid

1. Biography. Provide a brief biography of the mathematician for whom your team is named. It should be not longer than a single page. Describe this person's contributions to linear algebra in terms understandable to all of your classmates. Give proper citations for all your sources.

2. Polynomials as vectors. This problem illustrates a linear space or vector space distinct from the Euclidean spaces we have studied so far.

Let $\mathbf{P}_2$ denote the collection of all polynomials of degree 2 or less. So, f belongs to $\mathbf{P}_2$ if and only if, for any real number t, $f(t) = c_0 + c_1 t + c_2 t^2$ where the coefficients $c_0, c_1,$ and c_2 are any fixed real numbers. For example, here are the values of some members of $\mathbf{P}_2$ at any point t: $3,\ 2+6t,\ 3-t^2,\ 2010-2009t+2007t^2$. These polynomials are of degree 0, 1, 2, and 2, respectively. With the usual rules for adding functions and multiplying them by scalars, $\mathbf{P}_2$ is a vector space. Unlike Euclidean spaces, the vectors in $\mathbf{P}_2$ are not objects that we ordinarily represent as arrows. Nevertheless, all the rules by which Euclidean space vectors combine have direct analogs for the polynomials in $\mathbf{P}_2$.

Notions of linear combination, span, linear dependence, basis and dimension carry over immediately. For example, the span of the pair (f, g) where $f(t) = 1 + 3t^2$ and $g(t) = 2t - t^2$ is the set of all polynomials whose value at any t is $a + 2bt + (3a - b)t^2$ for reals a and b. For another example, suppose $f(t) = 1 - t,\ g(t) = t - t^2,\ h(t) = t^2 - 1$, then (f, g, h) is a linearly dependent list in $\mathbf{P}_2$ since $1f + 1g + 1h$ is the zero polynomial.

Let $m_0(t) = 1,\ m_1(t) = t,\ m_2(t) = t^2$. These are the monomials in $\mathbf{P}_2$. Clearly, $\mathfrak{B} = (m_0, m_1, m_2)$ is a basis for $\mathbf{P}_2$. Consequently, $\dim(\mathbf{P}_2) = 3$.

Here is an inner (scalar, dot) product for $\mathbf{P}_2$: for any f and g in $\mathbf{P}_2$, define $f \cdot g = \int_{-1}^{1} f(t)g(t)\,dt$. This scalar product has all the properties of the familiar scalar product for vectors in Euclidean spaces. As in the case of vectors in Euclidean spaces, we also define magnitude by $\|f\| = +\sqrt{f \cdot f}$.

a. Use the Gram-Schmidt process to generate an orthonormal basis for $\mathbf{P}_2$ from the basis $\mathfrak{B}$.

b. Explain why ordinary differentiation D (where $Df = f'$) is a linear transformation on $\mathbf{P}_2$.

Let $f(t) = c_0 + c_1 t + c_2 t^2$ for all t. Then f belongs to $\mathbf{P}_2$ and we can associate the vector $\langle f \rangle = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}$ in $\mathbf{R}^3$ to f. It should be clear that this association is one-to-one and onto, connecting polynomials in $\mathbf{P}_2$ uniquely with 3-component vectors in $\mathbf{R}^3$.

d. Let $\langle\langle D \rangle\rangle$ be the $3{\times}3$ matrix for the transformation $\langle f \rangle \mapsto \langle Df \rangle$. Then, $\langle Df \rangle = \langle\langle D \rangle\rangle \langle f \rangle$. Determine $\langle\langle D \rangle\rangle$ and explain $\langle\langle D^n \rangle\rangle = \langle\langle D \rangle\rangle^n$. Calculate the matrix $\langle\langle D \rangle\rangle^3$, and explain the result.

3. Least squares. Find the parameters a, b, and c in the equation $z = a\,x + b\,y + c$ for the plane in $\mathbf{R}^3$ that fits the five points (1,0,0), (0,1,0), (1,1,0), (0,0,1), and (1,0,1) best in the least-squares sense.

4. Projection. Let P be the matrix for the orthogonal projection onto the subspace V of $\mathbf{R}^4$ spanned by the vectors $\begin{bmatrix} 2 \\ 3 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} -6 \\ 0 \\ 3 \\ 2 \end{bmatrix}$, and $\begin{bmatrix} -6 \\ 15 \\ 8 \\ 19 \end{bmatrix}$.

a. Compute P in three distinctly different ways.

b. Find the point in V that is closest to (1,1,1,1).

5. Rotations in $\mathbf{R}^3$. Rotations are orthogonal transformations that preserve the relative orientation of a list of vectors. As a consequence, the matrix for a rotation in Euclidean n-space has determinant +1 and its inverse is its transpose.

In this problem, we develop a means for identifying matrices that represent rotations.

A rotation is characterized by its axis and angle of rotation. Fix any rotation and call its matrix Q. Let $\vec{v}$ be any vector parallel to its axis and let θ be its angle of rotation.

a. Show the following.

i. The rotation leaves $\vec{v}$ unchanged if and only if $\vec{v}$ belongs to $\ker(Q - I)$.

ii. The angle between a vector $\vec{x}$ and its rotated image $Q\vec{x}$ is not, in general, θ. Illustrate with a diagram.

iii. The angle between a nonzero vector in $(\ker(Q - I))^{\perp}$ and its rotated image is θ.

b. Using the results above, demonstrate that $Q = \frac{1}{7}\begin{bmatrix} 2 & 6 & -3 \\ 3 & 2 & 6 \\ 6 & -3 & -2 \end{bmatrix}$ is the matrix for a rotation in $\mathbf{R}^3$ and find its axis and angle.

6. Rotations in $\mathbf{R}^3$. Our objective is to construct the matrix Q for a rotation in $\mathbf{R}^3$ with rotation axis described by the unit vector $\hat{u}$ and rotation angle θ. We proceed by analyzing the effect of the rotation on vectors parallel and orthogonal to $\hat{u}$.

Let P be the matrix for projection onto the 1-dimensional subspace of $\mathbf{R}^3$ spanned by $\hat{u}$, i.e. $P\vec{x} = (\hat{u}\cdot\vec{x})\hat{u}$ and let K be the matrix that represents the cross product with $\hat{u}$, i.e. $K\vec{x} = \hat{u}\times\vec{x}$.

a. Using these definitions, prove the following algebraic properties for the matrices P and K. $P^2 = P$, $K^2 = P - I$, $KP = PK = 0$. [Recall that $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})b - (\vec{a}\cdot\vec{b})\vec{c}$.]

b. If A and B are two $n\times n$ matrices and $\vec{x}\cdot A\vec{y} = \vec{x}\cdot B\vec{y}$ for every vector $\vec{x}$ and $\vec{y}$ in $\mathbf{R}^3$, then $A = B$. Use this and the definitions of P and K to show that $P = P^T$ and $K^T = -K$. [Recall that $\vec{v}\cdot\vec{w} = \vec{v}^T\vec{w}$ and $\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$.]

Clearly, for any vector $\vec{v}$ parallel to the rotation axis, $Q\vec{v} = \vec{v}$. Now, suppose that $\vec{w}$ is any vector orthogonal to the rotation axis.

c. Explain why $Q\vec{w} = \cos\theta\,\vec{w} + \sin\theta\,K\vec{w}$. [A diagram should help.]

d. Now, use the fact that $\vec{x} = P\vec{x} + (I - P)\vec{x}$ for any vector $\vec{x}$, to prove that $Q = \cos\theta\, I + (1-\cos\theta)P + \sin\theta\, K$.

e. Demonstrate that Q is orthogonal by using the properties of P and K derived above.

f. Determine the matrices P, K, and Q if $\hat{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$.

g. Calculate the matrix for rotation by $\pi/4$ about the unit vector $\frac{1}{3}\begin{bmatrix}1\\2\\2\end{bmatrix}$.

7. Determinants. Calculate the determinant of $\begin{bmatrix}a_{11} & \cdots & a_{1n}\\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn}\end{bmatrix}$ if

a. $a_{jk} = \max(j, k)$.
b. $a_{jk} = \min(j, k)$.

8. True/False. Prove each statement below that is true. For each statement below that is false, provide a specific counterexample.

a. For any matrix A, $\ker(A) = \left(im(A^T)\right)^{\perp}$.

b. If A is the matrix of an orthogonal projection onto a plane in $\mathbf{R}^3$, then there is an orthogonal matrix Q such that $Q^T A Q$ is diagonal.

c. If $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4$ are vectors in $\mathbf{R}^4$ and $A = [\vec{a}_1 \,|\, \vec{a}_2 \,|\, \vec{a}_3 \,|\, \vec{a}_4]$, then $|\det(A)| \le \|\vec{a}_1\| \cdot \|\vec{a}_2\| \cdot \|\vec{a}_3\| \cdot \|\vec{a}_4\|$.

d. If all the entries of an $n \times n$ matrix A are positive integers such that the diagonal entries are odd and the off-diagonal entries are even, then A is invertible.