FALL 2003 FINAL EXAM FOR SM261
0755 DECEMBER 11, 2003

SHOW ALL WORK

PART 1. On this part you may **NOT** use a calculator. Show all your work.

1. Complete the following definitions.

- **a.** A subset W of $\mathbb{R}^n$ is called a subspace of $\mathbb{R}^n$ if...
- **b.** If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ are vectors in $\mathbb{R}^n$, then span$(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n)$=...
- **c.** If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then the image of T is...
- **d.** A set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$ in a subspace V of $\mathbb{R}^n$ forms a basis for V if...
- **e.** If V is a subspace of $\mathbb{R}^n$, then the dimension of V is...

2. Consider the linear system of equations

$$\begin{aligned} x + y + 5z &= 9 \\ -x + 2y + z &= 6 \\ 3x - y + 7z &= 7 \end{aligned}$$

- **a.** If we write the system in the matrix form $A\vec{v} = \vec{b}$, what are A, $\vec{v}$ and $\vec{b}$?
- **b.** Find the reduced row echelon form of the augmented matrix $[A|\,\vec{b}\,]$ for this system. Show all steps.
- **c.** Solve the system.
- **d.** Is the solution set a subspace of $\mathbb{R}^3$? Explain. If so, what is its dimension?

e. Is the solution set of the corresponding homogenous system $Av = 0$ a subspace of $\mathbb{R}^3$? Explain. If so, what is its dimension?

3. Find the determinant of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 3 & 1 & 2 & 2 \\ 4 & 0 & 0 & 2 \\ 3 & 4 & 5 & 0 \end{bmatrix}.$$

4. Find the inverse of the matrix

$$\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}.$$

5. Find the product AB, where

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 & 1 \\ 2 & -1 \\ 3 & 4 \end{bmatrix}.$$

This is the end of Part 1. Turn in your answers and proceed to Part 2.

PART 2. You may use a calculator on this part. Show all your work.

6. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation whose matrix is

$$\begin{bmatrix} 1 & 3 & 0 \\ 4 & 0 & 1 \end{bmatrix}$$

and $S : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation whose matrix is

$$\begin{bmatrix} 2 & 0 \\ 4 & 1 \\ 0 & 3 \end{bmatrix}.$$

Find the matrix of the linear transformation $S \circ T$. (Recall that $S \circ T(v) = S(T(v))$.)

7. $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation such that $T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $T \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$. Find the matrix of T.

8. Let A be the matrix

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 1 & -2 \\ 0 & 1 & -2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

a. Find a basis for $\ker(A)$.
b. Find $\dim(\ker(A))$.
c. Find a basis for $\operatorname{im}(A)$.
d. Find $\operatorname{rank}(A)$.

9. Let $V = \operatorname{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$, where $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -2 \\ -3 \\ -1 \\ 4 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 2 \end{bmatrix}$, and $\vec{v}_4 = \begin{bmatrix} 4 \\ 5 \\ 3 \\ -6 \end{bmatrix}$.

a. Find a basis for V.

b. Is the vector $\begin{bmatrix} 5 \\ 0 \\ 1 \\ 2 \end{bmatrix}$ in V?

10. Let V be the subspace of $\mathbb{R}^4$ which consists of all multiples of the vector $\begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}$. Find a basis for $V^{\perp}$.

11. Let V be the subspace of $\mathbb{R}^3$ with basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$. Find the $\mathcal{B}$-coordinate vector $[\vec{v}]_{\mathcal{B}}$ for $\vec{v} = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$.

12. Use the Gram-Schmidt process to find an orthonormal basis for the subspace V of $\mathbb{R}^3$ spanned by the vectors $\vec{v}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$.

13. Let V be the subspace of $\mathbb{R}^3$ spanned by $\begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

a. Find the matrix of the orthogonal projection of $\mathbb{R}^3$ onto V.
b. Find the orthogonal projection of the vector $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ onto V.

14. Find the linear function $a + bt$ which best fits the data points $(0, 4)$, $(1, 5)$ and $(2, 0)$ using least squares.

15. Suppose that $T : \mathbb{R}^5 \mapsto \mathbb{R}^8$ is a linear transformation and $\dim(\ker(T)) = 2$. What are the possible values of $\dim(\mathrm{im}(T))$?

In Problems 16-19, do all calculations by hand and show all steps. You may check your work with a calculator.

16. Let A be the matrix

$$\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}.$$

a. Find the characteristic polynomial $f_A(\lambda)$ of A.
b. Find the eigenvalues of A. Find the algebraic multiplicity of each eigenvalue.
c. For each eigenvalue λ, find a basis for the eigenspace E_λ.
d. Is A diagonalizable? If so, find an eigenbasis for A and a diagonal matrix D which is similar to A. If not, give your reasons and evidence.

17. Repeat Problem 16 for

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

18. Repeat Problem 16 for

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

19. a. Find the 4-volume of the parallelepiped in $\mathbb{R}^4$ formed from the vectors

$$\begin{bmatrix} 2 \\ 2 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -4 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 1 \\ 7 \end{bmatrix}.$$

b. Use Cramer's Rule to solve the linear algebraic system below. No credit for other methods.

$$\begin{aligned} 2x_1 + 3x_3 + 4x_4 &= 1 \\ 2x_1 + x_3 + 2x_4 &= 0 \\ 4x_1 - 4x_2 + x_3 + x_4 &= 0 \\ 2x_1 + 5x_3 + 7x_4 &= 0. \end{aligned}$$

c. Let $T\vec{x} = A\vec{x}$ be the linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^4$ where A is the matrix whose columns are the 4 vectors shown above in part **(a)**. If we apply T to a cube whose 4-volume is 6, what will be the volume of the image?

20. True or False. If true, give a reason. If false, give a reason or a counterexample.

a. If A is a square matrix and $A^4 = I$, then A is invertible.

b. Any set of vectors which spans $\mathbb{R}^n$ is a subset of some basis of $\mathbb{R}^n$.

c. Every square matrix has at least one real eigenvalue.

d. If V is the set of all vectors $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ in $\mathbb{R}^2$ with $|x| = |y|$, then V is a subspace of $\mathbb{R}^2$.

e. If $T\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} x+y+1 \\ x+y-1 \end{smallmatrix}\right]$, then T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$.

END OF TEST