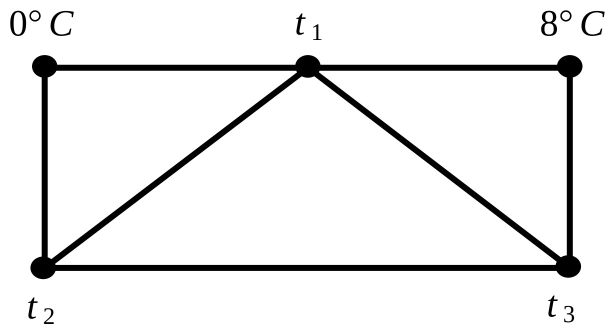
1. Choose either part a (below) *or* part b on the next page.

a. A grid of five nodes and seven connectors is shown in the diagram below. The temperatures at two of the nodes are held fixed. The grid is at thermal equilibrium and so the temperatures at each of the other three nodes is the numerical average of the temperatures at the nodes to which that node is connected. Find the unspecified temperatures at these three nodes. Do so by reformulating the problem so that it has a standard matrix form and solve by row reduction methods.



The temperatures to be determined at the three nodes are given by:

$$\begin{Bmatrix} t_1 = \tfrac{1}{4}(0+t_2+t_3+8) \\ t_2 = \tfrac{1}{3}(0+t_1+t_3) \\ t_3 = \tfrac{1}{3}(t_1+t_2+8) \end{Bmatrix} \Rightarrow A\vec{x}=\vec{b},\ \text{where } A=\begin{bmatrix} 4 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix},\ \vec{x}=\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix},\ \vec{b}=\begin{bmatrix} 8 \\ 0 \\ 8 \end{bmatrix}$$

Row reduction yields $[A|\vec{b}]_{rref} = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{array}\right]$. So, $\vec{x}=\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}=\begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$.

b, Determine if $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$

This problem is equivalent to finding whether $A\vec{x}=\vec{b}$ has any solutions if $A=\begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 5 \\ 1 & 0 & 2 \end{bmatrix}$, $\vec{x}=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\vec{b}=\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. We find $[A|\vec{b}]_{rref} = \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$ and so there are no solutions since the fourth column is a pivot column (equivalently, the fourth row implies that 0 = 1). Therefore, the given single vector is *not* a linear combination of the given triplet of vectors.

2. a. C is a 3×3 matrix whose column vectors are nonzero and coplanar. Find all possible matrices C_{rref}.

If two matrices are related by a sequence of elementary row operations, a column of one matrix will consist entirely of zeros if and only if the same column of the other matrix consists entirely of zeros. So, C_{rref} contains no zero columns. If three vectors in $\mathbf{R}^3$ are coplanar, then C cannot be row-reduced to the identity matrix. Otherwise, every vector in $\mathbf{R}^3$ would be a unique linear combination of the three column vectors of C. This means that $\operatorname{rank}(C) < 3$ and $\operatorname{rref}(C)$ must have at least one row of zeros and its rank must be less than 3. The only possibilities for C_{rref} are $\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & c & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & d & e \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ where $a, b, c, d,$ and e are any real numbers but, to exclude zero columns, $ab \neq 0,\ c \neq 0,\ de \neq 0$.

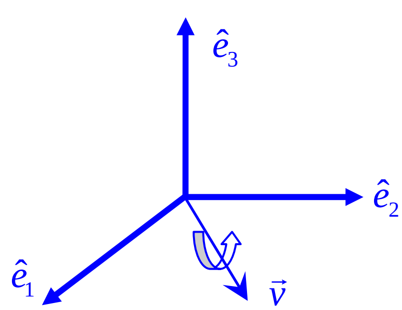
b. Suppose that you are given a fixed 4×4 matrix A and it is determined that linear system $A\vec{x} = \vec{0}$ has infinitely many solutions $\vec{x}$ in $\mathbf{R}^4$. What can you say about the rank of A and the number of solutions $\vec{x}$ in $\mathbf{R}^4$ for the linear systems $A\vec{x} = \vec{b}$ where $\vec{b}$ is some fixed nonzero vector in $\mathbf{R}^4$?

$\operatorname{rref}(A)$ must have at least one non-pivot column and so $\operatorname{rank}(A) < 4$. So, for any nonzero vector $\vec{b}$ in $\mathbf{R}^4$, the equation $A\vec{x} = \vec{b}$ cannot have a unique solution. It may have no solutions or it may have infinitely many.

3. Determine the 3×3 matrix corresponding to each of the following rotations in $\mathbf{R}^3$ where the axis of rotation is a line through the origin parallel to $\vec{v}$ and θ is the angle of rotation. The rotations are counterclockwise when viewed along the positive axis of rotation looking toward the origin

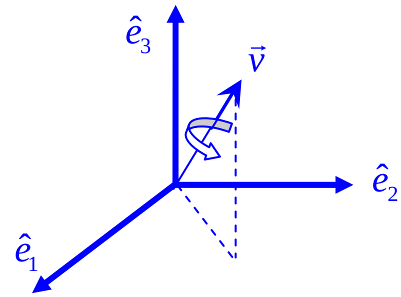
The columns of the matrix are images, under the rotation of $\hat{e}_1$, $\hat{e}_2$, and $\hat{e}_3$. Let r be the rotation and let A be its transformation matrix.

a. $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\theta = \pi$.



Since $r(\hat{e}_1) = \hat{e}_2,\ r(\hat{e}_2) = \hat{e}_1,\ r(\hat{e}_3) = -\hat{e}_2,\ A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

b. $\vec{v}=\begin{bmatrix}1\\1\\1\end{bmatrix}$ and $\theta=2\pi/3$.



Since $r(\hat{e}_1)=\hat{e}_2,\ r(\hat{e}_2)=\hat{e}_3,\ r(\hat{e}_3)=\hat{e}_1,\ A=\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$.

4. Completely solve the equation $A\vec{x}=\vec{b}$ for $\vec{x}$ if

$$[A|\vec{b}]_{rref}=\left[\begin{array}{cccccc|c}1&0&2&0&3&2&3\\0&1&1&0&2&1&2\\0&0&0&1&1&0&1\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{array}\right].$$

Using the Solution Algorithm, $\vec{x}=\begin{bmatrix}3\\2\\0\\1\\0\\0\end{bmatrix}+\alpha\begin{bmatrix}2\\1\\-1\\0\\0\\0\end{bmatrix}+\beta\begin{bmatrix}3\\2\\0\\1\\-1\\0\end{bmatrix}+\gamma\begin{bmatrix}2\\1\\0\\0\\0\\-1\end{bmatrix}$, where α, β, and γ are any real numbers.

5. Suppose $f:\mathbf{R}^2\to\mathbf{R}^3$ is linear and $f\left(\begin{bmatrix}1\\2\end{bmatrix}\right)=\begin{bmatrix}3\\1\\1\end{bmatrix}$ and $f\left(\begin{bmatrix}1\\4\end{bmatrix}\right)=\begin{bmatrix}1\\1\\3\end{bmatrix}$.

a. Use the linearity of f to find $f\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$ and then to find $f\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$.

$f\left(\begin{bmatrix}1\\4\end{bmatrix}\right)-f\left(\begin{bmatrix}1\\2\end{bmatrix}\right)=f\left(\begin{bmatrix}0\\2\end{bmatrix}\right)=\begin{bmatrix}1\\1\\3\end{bmatrix}-\begin{bmatrix}3\\1\\1\end{bmatrix}=\begin{bmatrix}-2\\0\\2\end{bmatrix}$. So, $f\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}-1\\0\\1\end{bmatrix}$.

Now, $f\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=f\left(\begin{bmatrix}1\\2\end{bmatrix}\right)-2f\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}3\\1\\1\end{bmatrix}-\begin{bmatrix}-2\\0\\2\end{bmatrix}=\begin{bmatrix}5\\1\\-1\end{bmatrix}$.

b. Determine the matrix A so that $f(\vec{x})=A\vec{x}$ for any $\vec{x}\in\mathbf{R}^2$.

$$A=\left[f\left(\begin{bmatrix}0\\1\end{bmatrix}\right)\mid f\left(\begin{bmatrix}0\\1\end{bmatrix}\right)\right]=\begin{bmatrix}5&-1\\1&0\\-1&1\end{bmatrix}.$$

6. Solve the equation $\begin{bmatrix}1&2\\3&4\end{bmatrix}X\begin{bmatrix}1&2\\3&4\end{bmatrix}^{-1}=\begin{bmatrix}2&3\\0&1\end{bmatrix}$ for the 2×2 matrix X.

To obtain X alone on the left side, we multiply both sides of this equation on the left by $\begin{bmatrix}1&2\\3&4\end{bmatrix}^{-1}$ and on the right by $\begin{bmatrix}1&2\\3&4\end{bmatrix}$ to obtain

$$X=\begin{bmatrix}1&2\\3&4\end{bmatrix}^{-1}\begin{bmatrix}2&3\\0&1\end{bmatrix}\begin{bmatrix}1&2\\3&4\end{bmatrix}=\frac{1}{1\cdot4-3\cdot2}\begin{bmatrix}4&-2\\-3&1\end{bmatrix}\begin{bmatrix}2&3\\0&1\end{bmatrix}\begin{bmatrix}1&2\\3&4\end{bmatrix}$$

$$=-\tfrac{1}{2}\begin{bmatrix}4&-2\\-3&1\end{bmatrix}\begin{bmatrix}11&16\\3&4\end{bmatrix}=-\tfrac{1}{2}\begin{bmatrix}38&56\\-30&-44\end{bmatrix}=\begin{bmatrix}-19&-28\\15&22\end{bmatrix}.$$

7. Let $\vec{w}=\begin{bmatrix}-1\\2\end{bmatrix}$.

a. Determine the matrices P and Q corresponding to the projection parallel to $\vec{w}$ and the projection orthogonal to $\vec{w}$, respectively.

The unit vector parallel to $\vec{w}$ is $\hat{w}=\frac{1}{\sqrt{5}}\begin{bmatrix}-1\\2\end{bmatrix}=\begin{bmatrix}w_1\\w_2\end{bmatrix}$ and so the projection matrices are $P=\begin{bmatrix}w_1w_1&w_1w_2\\w_2w_1&w_2w_2\end{bmatrix}=\frac{1}{5}\begin{bmatrix}1&-2\\-2&4\end{bmatrix}$ and $Q=I-P=\frac{1}{5}\begin{bmatrix}4&2\\2&1\end{bmatrix}$.

b. Use the results of part a to find the matrix A that corresponds to the linear transformation that doubles all vectors parallel to $\vec{w}$ and reverses the direction of all vectors orthogonal to $\vec{w}$.

Any vector $\vec{x}$ in $\mathbf{R}^2$ may be resolved into its projections parallel and orthogonal to $\vec{w}$. We have $\vec{x}=P\vec{x}+Q\vec{x}$. So, $A\vec{x}=AP\vec{x}+AQ\vec{x}$. But,

$AP\vec{x} = 2P\vec{x}$ and $AQ\vec{x} = -Q\vec{x}$ and so $A\vec{x} = 2P\vec{x} - Q\vec{x} = (2P - Q)\vec{x}$. Since $\vec{x}$ is arbitrary, $A = 2P - Q = 2\,\tfrac{1}{5}\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} - \tfrac{1}{5}\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} = \tfrac{1}{5}\begin{bmatrix} -2 & -6 \\ -6 & 7 \end{bmatrix}$.