2. Notice that $\int_{-1}^{1} h(t)\,dt$ is 0 if h is an odd polynomial. Therefore, m_0 and m_1 are orthogonal. $\|m_0\| = \sqrt{\int_{-1}^{1} 1\,dt} = \sqrt{2}$ and $\|m_1\| = \sqrt{\int_{-1}^{1} t^2\,dt} = \sqrt{2/3}$. Let $u_0 = m_0/\sqrt{2}$ and $u_1 = m_1/\sqrt{2/3} = \sqrt{3/2}\,m_1$. In other words, $u_0(t) = 1/\sqrt{2}$ and $u_1(t) = \sqrt{3/2}\,t$. Continuing with the orthogonalization, we let $m_2^{\perp} = m_2 - (m_2 \cdot u_0)u_0 - (m_2 \cdot u_1)u_1$. Now, $m_2 \cdot u_0 = \int_{-1}^{1} t^2 \frac{1}{\sqrt{2}}\,dt = \frac{\sqrt{2}}{3}$ and $m_2 \cdot u_1 = 0$. So, $m_2^{\perp} = m_2 - \frac{\sqrt{2}}{3}u_0$. That is, $m_2^{\perp}(t) = t^2 - \frac{1}{3}$. Now let $u_2 = \frac{m_2^{\perp}}{\|m_2^{\perp}\|}$. Since $= \|m_2^{\perp}\| = \sqrt{\int_{-1}^{1}\left(t^4 - \frac{2}{3}t^2 + \frac{1}{9}\right)dt} = \sqrt{\frac{8}{45}}$, we now have $u_2 = \frac{m_2 - \frac{\sqrt{2}}{3}u_0}{\sqrt{8/45}}$. So, $u_0(t) = \frac{1}{\sqrt{2}}$, $u_1(t) = \sqrt{\frac{3}{2}}\,t$, $u_2(t) = \sqrt{\frac{45}{8}}\left(t^2 - \frac{1}{3}\right)$. and our new orthonormal basis for $\mathbf{P}_2$ is $\mathcal{C} = (u_0, u_1, u_2)$.

b. The derivative of a polynomial of degree 2 or less is a polynomial of degree 1 or less. Hence, if f is in $\mathbf{P}_2$, then Df is also in $\mathbf{P}_2$. Moreover, as is well-known, for any polynomials (or any differentiable functions) f and g and any scalars a and b, $D(af + bg) = aDf + bDg$. Therefore, D is a linear transformation on $\mathbf{P}_2$.

c. $Df(t) = c_1 + 2c_2 t$. Therefore $\langle Df \rangle = \begin{bmatrix} c_1 \\ 2c_2 \\ 0 \end{bmatrix}$. The matrix that maps $\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}$ to $\begin{bmatrix} c_1 \\ 2c_2 \\ 0 \end{bmatrix}$, for any c_0, c_1, and c_2, is therefore $\langle\langle D \rangle\rangle = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

$\langle\langle D \rangle\rangle^n$ is the product of the matrix $\langle\langle D \rangle\rangle$ with itself n times. This corresponds to the application of differentiation to a polynomial n times. In other words, the n-fold matrix product is the same thing as the matrix corresponding to the n-fold composite of differentiation with itself. This is the matrix corresponding to the nth derivative. Now,

$$\begin{bmatrix}0&1&0\\0&0&2\\0&0&0\end{bmatrix}^3=\begin{bmatrix}0&1&0\\0&0&2\\0&0&0\end{bmatrix}\begin{bmatrix}0&0&2\\0&0&0\\0&0&0\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}.$$

But, this result is not surprising in view of the fact that the third derivative of any polynomial of degree 2 or less is the zero polynomial.

3. If the data were coplanar, we would seek the solution to $A\vec{x}=\vec{b}$ where

$$A=\begin{bmatrix}1&0&1\\0&1&1\\1&1&1\\0&0&1\\1&0&1\end{bmatrix},\quad \vec{x}=\begin{bmatrix}a\\b\\c\end{bmatrix},\quad \vec{b}=\begin{bmatrix}0\\0\\0\\1\\1\end{bmatrix}.$$

But, $\vec{b}$ does not belong to the image of A, so we attempt to solve $A^TA\vec{x}=A^T\vec{b}$, instead. Since the column vectors of A are linearly independent, we have $\vec{x}=\left(A^TA\right)^{-1}A^T\vec{b}=$

$$\left(\begin{bmatrix}1&0&1&0&1\\0&1&1&0&0\\1&1&1&1&1\end{bmatrix}\begin{bmatrix}1&0&1\\0&1&1\\1&1&1\\0&0&1\\1&0&1\end{bmatrix}\right)^{-1}\begin{bmatrix}1&0&1&0&1\\0&1&1&0&0\\1&1&1&1&1\end{bmatrix}\begin{bmatrix}0\\0\\0\\1\\1\end{bmatrix}=$$

$$\begin{bmatrix}3&1&3\\1&2&2\\3&2&5\end{bmatrix}^{-1}\begin{bmatrix}1\\0\\2\end{bmatrix}=\frac{1}{7}\begin{bmatrix}6&1&-4\\1&6&-3\\-4&-3&5\end{bmatrix}\begin{bmatrix}1\\0\\2\end{bmatrix}=\begin{bmatrix}-2/7\\-5/7\\6/7\end{bmatrix}$$

4. a. i. The three vectors are linearly independent. So, V has dimension 3 and its orthogonal complement $V^{\perp}$ has dimension 1. Our strategy in this first approach is to find the projection $P^{\perp}$ onto $V^{\perp}$. Then, $P = I-P^{\perp}$. Now, $V^{\perp} = \ker(A^T)$, where A is the 4×3 matrix whose columns are the three vectors given here, i.e. $A=\begin{bmatrix}2&-6&-6\\3&0&15\\0&3&8\\6&2&19\end{bmatrix}$. We have $V^{\perp} = \ker(A^T)$

$$= \ker\begin{bmatrix} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \operatorname{span}\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{3} \\ 1 \\ 0 \end{bmatrix} = \operatorname{span}\left(\frac{1}{7}\begin{bmatrix} 3 \\ -2 \\ 6 \\ 0 \end{bmatrix}\right) = \operatorname{span}(\hat{u})$$. Therefore, $P^{\perp}$

$$= \hat{u}\,\hat{u}^T = \frac{1}{49}\begin{bmatrix} 9 & -6 & 18 & 0 \\ -6 & 4 & -12 & 0 \\ 18 & -12 & 36 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } P = \frac{1}{49}\begin{bmatrix} 40 & 6 & -18 & 0 \\ 6 & 45 & 12 & 0 \\ -18 & 12 & 13 & 0 \\ 0 & 0 & 0 & 49 \end{bmatrix}.$$

ii. The three vectors given above comprise a basis for V. Let's call them $\vec{v}_1, \vec{v}_2,$ and $\vec{v}_3$. Moreover, the first two vectors are orthogonal, so we can easily find an orthonormal basis by application of the Gram-Schmidt process. We have $\hat{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{7}\begin{bmatrix} 2 \\ 3 \\ 0 \\ 6 \end{bmatrix}$, $\hat{u}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|} = \frac{1}{7}\begin{bmatrix} -6 \\ 0 \\ 3 \\ 2 \end{bmatrix}$. The third vector in our orthonormal triplet is found next. $\vec{v}_3^{\perp} = \vec{v}_3 - (\hat{u}_1 \cdot \vec{v}_3)\hat{u}_1 - (\hat{u}_2 \cdot \vec{v}_3)\hat{u}_2 =$

$$\begin{bmatrix} -6 \\ 15 \\ 8 \\ 19 \end{bmatrix} - \left(\frac{1}{7}\begin{bmatrix} 2 \\ 3 \\ 0 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} -6 \\ 15 \\ 8 \\ 19 \end{bmatrix}\right)\frac{1}{7}\begin{bmatrix} 2 \\ 3 \\ 0 \\ 6 \end{bmatrix} - \left(\frac{1}{7}\begin{bmatrix} -6 \\ 0 \\ 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -6 \\ 15 \\ 8 \\ 19 \end{bmatrix}\right)\frac{1}{7}\begin{bmatrix} -6 \\ 0 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 2 \\ -3 \end{bmatrix}.$$ Therefore,

$\hat{u}_3 = \frac{\vec{v}_3^{\perp}}{\|\vec{v}_3^{\perp}\|} = \frac{1}{7}\begin{bmatrix} 0 \\ 6 \\ 2 \\ -3 \end{bmatrix}$. Since $(\hat{u}_1, \hat{u}_2, \hat{u}_3)$ is an orthonormal basis for V, the projection onto V is $P = P_1 + P_2 + P_3$, where $P_k = \hat{u}_k\,\hat{u}_k^{\,T}$, the projection onto the 1-dimensional subspace spanned by $\hat{u}_k$, for $k = 1, 2, 3$. Therefore,

$$P = \frac{1}{49}\begin{bmatrix} 4 & 6 & 0 & 12 \\ 6 & 9 & 0 & 18 \\ 0 & 0 & 0 & 0 \\ 12 & 18 & 0 & 36 \end{bmatrix} + \frac{1}{49}\begin{bmatrix} 36 & 0 & -18 & -12 \\ 0 & 0 & 0 & 0 \\ -18 & 0 & 9 & 6 \\ -12 & 0 & 6 & 4 \end{bmatrix} + \frac{1}{49}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 36 & 12 & -18 \\ 0 & 12 & 4 & -6 \\ 0 & -18 & -6 & 9 \end{bmatrix}$$

$$=\frac{1}{49}\begin{bmatrix} 40 & 6 & -18 & 0 \\ 6 & 45 & 12 & 0 \\ -18 & 12 & 13 & 0 \\ 0 & 0 & 0 & 49 \end{bmatrix}.$$

iii. Finally, since $(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ is linearly independent, $\ker(A) = \{\vec{0}\}$, and P, the projection onto $V = \operatorname{im}(A)$ is given by $P = A\left(A^T A\right)^{-1} A^T$

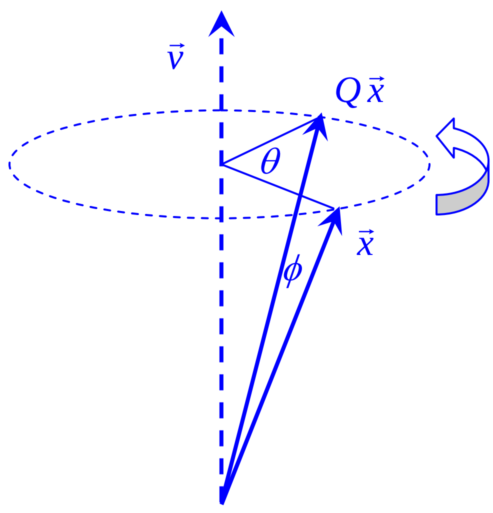
$$=\begin{bmatrix} 2 & -6 & -6 \\ 3 & 0 & 15 \\ 0 & 3 & 8 \\ 6 & 2 & 19 \end{bmatrix}\left(\begin{bmatrix} 2 & 3 & 0 & 6 \\ -6 & 0 & 3 & 2 \\ -6 & 15 & 8 & 19 \end{bmatrix}\begin{bmatrix} 2 & -6 & -6 \\ 3 & 0 & 15 \\ 0 & 3 & 8 \\ 6 & 2 & 19 \end{bmatrix}\right)^{-1}\begin{bmatrix} 2 & 3 & 0 & 6 \\ -6 & 0 & 3 & 2 \\ -6 & 15 & 8 & 19 \end{bmatrix}$$

$$\frac{1}{49}\begin{bmatrix} 40 & 6 & -18 & 0 \\ 6 & 45 & 12 & 0 \\ -18 & 12 & 13 & 0 \\ 0 & 0 & 0 & 49 \end{bmatrix}.$$

b. The displacement vector to this closest point is $P\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4/7 \\ 6/7 \\ 1/7 \\ 1 \end{bmatrix}$.

5. a. i. $Q\vec{v} = \vec{v} \Rightarrow Q\vec{v} - \vec{v} = (Q - I)\vec{v} = \vec{0} \Rightarrow \vec{v} \in \ker(Q - I)$

ii.



The closer the direction of $\vec{x}$ is to the rotation axis, the smaller will be the angle ϕ between $\vec{x}$ and its rotated image $Q\vec{x}$, regardless of the angle θ.

iii. The angle between a vector orthogonal to the rotation axis and its rotated image is exactly the rotation angle. See the diagram above.

b. Straightforward calculation shows that $\det(Q) = +1$. According to the results above, the axis of rotation belongs to $\ker(Q-I)$. So, we use row reduction to find it. $\ker(Q-I) = \ker\frac{1}{7}\begin{bmatrix} -5 & 6 & -3 \\ 3 & -5 & 6 \\ 6 & -3 & -9 \end{bmatrix} = \ker\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$ $= \operatorname{span}\begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$. So, the rotation axis is parallel to $\begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$. We choose $\vec{x} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ as a vector orthogonal to the rotation axis. The rotation angle is given by $\cos\theta = (\vec{x}\cdot Q\vec{x})/\|\vec{x}\|^2 = -5/14$ and we find $\theta = \arccos(-5/14) \approx 1.94$.

6. a. $\forall\, \vec{x} \in \mathbf{R}^3$

$$P^2\vec{x} = PP\vec{x} = (\hat{u}\cdot[(\hat{u}\cdot\vec{x})\hat{u}])\hat{u} = [(\hat{u}\cdot\vec{x})(\hat{u}\cdot\hat{u})]\hat{u} = (\hat{u}\cdot\vec{x})\hat{u} = P\vec{x} \Rightarrow P^2 = P$$

$$K^2\vec{x} = KK\vec{x} = \hat{u}\times(\hat{u}\times\vec{x}) = (\hat{u}\cdot\vec{x})\hat{u} - (\hat{u}\cdot\hat{u})\vec{x} = (P-I)\vec{x} \Rightarrow K^2 = P - I$$

$$PK\vec{x} = (\hat{u}\cdot[\hat{u}\times\vec{x}])[\hat{u}\times\vec{x}] = 0[\hat{u}\times\vec{x}] = \vec{0} \Rightarrow PK = 0$$

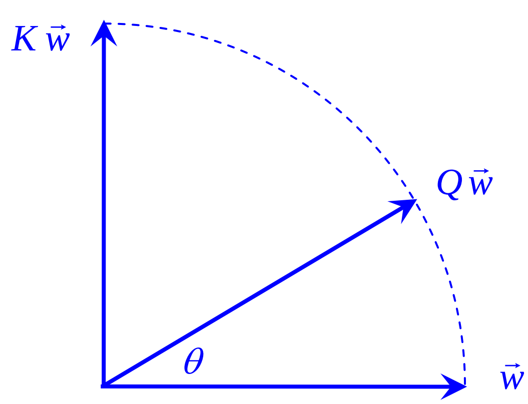
$$KP\vec{x} = \hat{u}\times[(\hat{u}\cdot\vec{x})\hat{u}] = (\hat{u}\cdot\vec{x})[\hat{u}\times\hat{u}] = (\hat{u}\cdot\vec{x})\vec{0} = \vec{0} \Rightarrow KP = 0$$

b. $\forall\, \vec{x},\vec{y} \in \mathbf{R}^3$

$$\vec{x}\cdot P\vec{y} = \vec{x}\cdot(\hat{u}\cdot\vec{y})\hat{u} = (\hat{u}\cdot\vec{y})(\hat{u}\cdot\vec{x}) = [(\hat{u}\cdot\vec{x})\hat{u}]\cdot\vec{y} = P\vec{x}\cdot\vec{y} = (P\vec{x})^T\vec{y}$$
$$= \vec{x}^T P^T\vec{y} = \vec{x}\cdot P^T\vec{y} \Rightarrow P = P^T$$

$$\vec{x}\cdot K\vec{y} = \vec{x}\cdot\hat{u}\times\vec{y} = \vec{x}\times\hat{u}\cdot\vec{y} = -\hat{u}\times\vec{x}\cdot\vec{y} = -K\vec{x}\cdot\vec{y} = -(K\vec{x})^T\vec{y}$$
$$= \vec{x}^T(-K^T)\vec{y} = \vec{x}\cdot(-K^T)\vec{y} \Rightarrow K^T = -K$$

c.



d.
$$
\begin{aligned}
Q\vec{x} &= Q\left(P\vec{x} + (I-P)x\right) = QP\vec{x} + Q(I-P)\vec{x} \\
&= P\vec{x} + \cos\theta(I-P)\vec{x} + \sin\theta K(I-P)\vec{x} \\
&= P\vec{x} + \cos\theta(I-P)\vec{x} + \sin\theta K\vec{x} \\
&= \cos\theta I\vec{x} + (1-\cos\theta)P\vec{x} + \sin\theta K\vec{x} \\
&= [\cos\theta I + (1-\cos\theta)P + \sin\theta K]\vec{x} \\
\Rightarrow Q &= \cos\theta I + (1-\cos\theta)P + \sin\theta K
\end{aligned}
$$

e.
$$
\begin{aligned}
Q^TQ &= \left(\cos\theta I + (1-\cos\theta)P + \sin\theta K\right)^T\left(\cos\theta I + (1-\cos\theta)P + \sin\theta K\right) \\
&= \left(\cos\theta I^T + (1-\cos\theta)P^T + \sin\theta K^T\right)\left(\cos\theta I + (1-\cos\theta)P + \sin\theta K\right) \\
&= \left(\cos\theta I + (1-\cos\theta)P - \sin\theta K\right)\left(\cos\theta I + (1-\cos\theta)P + \sin\theta K\right) \\
&= \cos^2\theta I^2 + (1-\cos\theta)^2P^2 - \sin^2\theta K^2 + 2\cos\theta(1-\cos\theta)P \\
&= \cos^2\theta I + (1-\cos\theta)^2P - \sin^2\theta(P-I) + 2\cos\theta(1-\cos\theta)P \\
&= (\cos^2\theta + \sin^2\theta)I + [(1-\cos\theta)^2 - \sin^2\theta + 2\cos\theta(1-\cos\theta)]P \\
&= I + [1 - 2\cos\theta + \cos^2\theta - \sin^2\theta + 2\cos\theta - 2\cos^2\theta]P \\
&= I
\end{aligned}
$$

f. This was assigned earlier. The kth column of P and K are, respectively, $P\hat{e}_j$ and $K\hat{e}_j$, the images of the standard basis vectors under the transformations corresponding to P and K. We find

$$
P = \begin{bmatrix} u_1u_1 & u_1u_2 & u_1u_3 \\ u_2u_1 & u_2u_2 & u_2u_3 \\ u_3u_1 & u_3u_2 & u_3u_3 \end{bmatrix} \text{ and } K = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}
$$

g. $P = \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix}$ and $K = \frac{1}{3}\begin{bmatrix} 0 & -2 & 2 \\ 2 & 0 & -1 \\ -2 & 1 & 0 \end{bmatrix}$ and so,

$$
\begin{aligned}
Q &= \tfrac{\sqrt{2}}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + (1-\tfrac{\sqrt{2}}{2})\tfrac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix} + \tfrac{\sqrt{2}}{2}\tfrac{1}{3}\begin{bmatrix} 0 & -2 & 2 \\ 2 & 0 & -1 \\ -2 & 1 & 0 \end{bmatrix} \\
&= \tfrac{1}{9}\begin{bmatrix} 1+8r & 2-8r & 2+4r \\ 2+4r & 4+5r & 4-7r \\ 2-8r & 4-r & 4+5r \end{bmatrix}, \text{ where } r = \tfrac{\sqrt{2}}{2}.
\end{aligned}
$$

7. a. For the first matrix, we subtract the first row from each of its successors to obtain

$$\det\begin{bmatrix}1&2&3&4&\cdots&n\\2&2&3&4&\cdots&n\\3&3&3&4&\cdots&n\\4&4&4&4&\cdots&n\\\vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\n&n&n&n&\cdots&n\end{bmatrix}=\det\begin{bmatrix}1&2&3&4&\cdots&n\\1&0&0&0&\cdots&0\\2&1&0&0&\cdots&0\\3&2&1&0&\cdots&0\\\vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\n-1&n-2&n-3&n-4&\cdots&0\end{bmatrix}$$

Using a Laplace expansion by the last column, we obtain $(-1)^{n+1}n$ times the determinant of the $1,n$ minor. But this is a lower triangular matrix all of whose diagonal entries are 1. So, the determinant has the value $(-1)^{n+1}$.

b. Let the given matrix be $A_{(n)}$. Again we subtract the first row from its successors to obtain

$$\det A_{(n)}=\det\begin{bmatrix}1&1&1&1&\cdots&1\\1&2&2&2&\cdots&2\\1&2&3&3&\cdots&3\\1&2&3&4&\cdots&4\\\vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\1&2&3&4&\cdots&n\end{bmatrix}=\det\begin{bmatrix}1&1&1&1&\cdots&1\\0&1&1&1&\cdots&1\\0&1&2&2&\cdots&2\\0&1&2&3&\cdots&3\\\vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\0&1&2&3&\cdots&n-1\end{bmatrix}$$

Using a Laplace expansion along the first column, we see that $\det(A_{(n)})$ = $\det(A_{(n-1)})$. Applying this argument repeatedly, we conclude that $\det(A_{(n)})$ $=\det(A_{(1)})$ = det[1] = 1.

8. a. **True**. We have already established that $\ker(A^T)=\left(im(A)\right)^{\perp}$ is true for any matrix A. So, substituting A^T for A, we arrive at the desired result.

b. **True**. All reflections across a plane in $\mathbf{R}^3$ are the same apart from the orientation of the plane. That is, they change the sign of vectors normal to the plane and leave unchanged vectors parallel to the plane. So, all that is needed to change any reflection to reflection across the xy-plane is to rotate the normal of the given reflection to the z-axis. Reflection across the xy-plane is given by the diagonal matrix $\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}$. If the rotation is given by the orthogonal matrix Q, its inverse is Q^{-1} = Q^T. Hence, Q^TAQ is diagonal.

c. **True**. We have seen that, geometrically, $|\det(A)|$ is the 4-volume of the 4-dimensional parallelepiped whose concurrent edges are described by its columns. This is $\|\vec{a}_1\| \cdot \|\vec{a}_2{}^\perp\| \cdot \|\vec{a}_3{}^\perp\| \cdot \|\vec{a}_4{}^\perp\|$, where $\vec{a}_k{}^\perp$ is the projection of $\vec{a}_k$ orthogonal to the subspace spanned by $\vec{a}_1, \ldots, \vec{a}_{k-1}$. Of course, $\|\vec{a}_k{}^\perp\| \le \|\vec{a}_k\|$. So, we have $|\det A| \le \|\vec{a}_1\| \cdot \|\vec{a}_2\| \cdot \|\vec{a}_3\| \cdot \|\vec{a}_4\|$.

d. **True**. The determinant of A is a sum of $n!$ terms. Each term is $+1$ or -1 times the product of n entries of the matrix with each entry coming from a different row and different column. Exactly one of these terms is a product of the diagonal entries only and its value, the product of odd integers, is odd. All other terms in this expansion have at least one off-diagonal entry as a factor and so they must be even. The sum of one odd integer and even integers is odd. Therefore, the determinant of such a matrix must be an odd number and so cannot be 0. Hence, A is invertible.