Final Examination
SM261
1330, Tuesday, December 14, 2004

1. [*12 points*] Let L and T be the linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$ described by $L\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ and $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

   a. Are the composites $T \circ L$ and $L \circ T$ the same? Explain.

   b. Find $\vec{v}$ so that $(T \circ L)(\vec{v}) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

   c. Let S be the square with vertices given by $\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Carefully sketch and label the image of S under $T \circ L$.

2. [*12 points*] $T: \mathbb{R}^4 \to \mathbb{R}^3$ is described by $T(\vec{x}) = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 2 & 1 & 3 & 1 \\ 3 & 2 & 5 & 1 \end{bmatrix} \vec{x}$.

a. Find a nonzero vector in ker(T). Explain.

b. Find a nonzero vector in im(T). Explain.

c. What is the dimension of ker(T)? Explain.

d. What is the dimension of im(T)? Explain.

e. Consider solutions $\vec{x}$ of the equation $T(\vec{x}) = \vec{b}$ for fixed $\vec{b}$ in $\mathbb{R}^3$. For which $\vec{b}$, if any, will this equation have: no solutions? a unique solution? infinitely many solutions?

3. [*12 points*] Let H be the set of vectors in $\mathbb{R}^4$ that are orthogonal to $\begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix}$.

a. Show that H is a subspace of $\mathbb{R}^4$.

b. Find a basis for H.

c. What is the dimension of H? Explain.

4. [*13 points*] Give examples for each of the following and explain why your examples satisfy the specified conditions.

a. a transformation M: $\mathbb{R}^2 \to \mathbb{R}^2$ that is not linear.

b. a set of vectors in $\mathbb{R}^4$ that is not linearly independent.

c. a non-empty set of vectors in $\mathbb{R}^3$ that does not span $\mathbb{R}^3$.

d. a set of vectors in $\mathbb{R}^4$ that is not a subspace.

5. [*13 points*] B is the basis for $\mathbb{R}^3$ consisting of the vectors

$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$ and $T: \mathbb{R}^3 \to \mathbb{R}^3$ is given by $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

a. $[\vec{v}]_B = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is the B-coordinate vector for $\vec{v}$. What is $\vec{v}$? I.e., what is the coordinate vector with respect to the standard (natural) basis for $\vec{v}$?

b. Determine the matrix that converts standard (natural) coordinate vectors to B-coordinate vectors.

c. Determine the B-matrix for T. That is, find the matrix for T with respect to the basis B.

6. [*12 points*] The vectors $\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}$ whose components satisfy the equation $x + y - z + t = 0$ comprise a hyperplane W in $\mathbb{R}^4$.

a. Find an orthogonal basis for W.

b. Write the vector $\begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ as a linear combination of your basis vectors using the dot product to find the coefficients.

c. Find the orthogonal projection of $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ onto W.

7. [*13 points*] Compute the volume of the tetrahedron whose vertices are given by $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$, and $\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$. Recall that a tetrahedron is a pyramid with a triangular base and its volume is $\frac{1}{3}$(*base area*)(*height*).

8. [*13 points*] A is a 2×2 matrix. $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of A with eigenvalue 2 and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is another eigenvector of A with eigenvalue –3.

a. Find a diagonal matrix D similar to A.

b. Find the matrix P such that $P^{-1} A P = D$.

c. Find A.