1. List all the combinations of 32 bills chosen from the denominations \$1, \$5, and \$10 that have a total worth of \$100.

Let $s,$ $f,$ and t be the number of singles, fives, and tens, respectively, and let $\vec{x}$ be the vector in $\mathbf{R}^3$ whose components are, in order, $s,$ $f,$ and $t.$ Then, we seek the solutions to $A\,\vec{x} = \vec{b}$ where $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 5 & 10 \end{bmatrix},\ b = \begin{bmatrix} 32 \\ 100 \end{bmatrix},\ \vec{x} = \begin{bmatrix} s \\ f \\ t \end{bmatrix}.$

Employing row-reduction, we have $[A\,|\,\vec{b}\,] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 32 \\ 1 & 5 & 10 & 100 \end{array}\right] \leftrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 32 \\ 0 & 4 & 9 & 68 \end{array}\right]$

$\leftrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 32 \\ 0 & 1 & \frac{9}{4} & 17 \end{array}\right] \leftrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -\frac{5}{4} & 15 \\ 0 & 1 & \frac{9}{4} & 17 \end{array}\right] \Rightarrow \vec{x} = \begin{bmatrix} 15 \\ 17 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -\frac{5}{4} \\ \frac{9}{4} \\ -1 \end{bmatrix}$ for $\alpha \in \mathbf{R}$. Since the number of each bill must be a non-negative integer, we set $\alpha = -4k$, where k is a non-negative integer. So, $\vec{x} = \begin{bmatrix} 15 \\ 17 \\ 0 \end{bmatrix} + k \begin{bmatrix} 5 \\ -9 \\ 4 \end{bmatrix}$. The only values for k that yield non-negative values for $s,$ $f,$ and t are $k = 0$ and 1. Our solutions are, therefore,

$\vec{x} = \begin{bmatrix} s \\ f \\ t \end{bmatrix} = \begin{bmatrix} 15 \\ 17 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 20 \\ 8 \\ 4 \end{bmatrix}$.

2. a. Given a list $L = (\vec{v}_1, \ldots, \vec{v}_p)$ of p vectors in $\mathbf{R}^n$, describe a procedure, involving row-reduction, to unambiguously determine if L is linearly independent.

Let $A = \left[\vec{v}_1 \,|\, \ldots \,|\, \vec{v}_p\right]$. Then, L is linearly independent if and only if every column of A_{rref} is a pivot column.

b. Given a vector $\vec{w}$ in $\mathbf{R}^n$ and the list $L = (\vec{v}_1, \ldots, \vec{v}_p)$ of p vectors in $\mathbf{R}^n$, describe a procedure, involving row-reduction, to unambiguously determine if $\vec{w} \in \operatorname{span}(L)$.

Let $A = \left[\vec{v}_1 \,|\, \ldots \,|\, \vec{v}_p\right]$. Then, $\vec{w} \in \operatorname{span}(L)$ if and only if the last column of $[A\,|\,\vec{b}\,]_{rref}$ is not a pivot column.

c. Given a list $L=(\vec{v}_1,\ldots,\vec{v}_n)$ of n vectors in $\mathbf{R}^n$, describe how to unambiguously determine if L is a basis for $\mathbf{R}^n$.

L is a basis for $\mathbf{R}^n$ if and only if $\det\left[\vec{v}_1|\ldots|\vec{v}_p\right] \neq 0$. Alternatively, L is a basis for $\mathbf{R}^n$ if and only if $\left[\vec{v}_1|\ldots|\vec{v}_p\right]_{rref} = I$.

3. $L=(\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3)=\left(\begin{bmatrix}1\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\1\end{bmatrix},\begin{bmatrix}1\\2\\2\\1\end{bmatrix}\right)$ and $\vec{w}=\begin{bmatrix}1\\2\\1\\1\end{bmatrix}$.

a. Determine an orthonormal basis $M=(\hat{u}_1,\ \hat{u}_2,\ \hat{u}_3)$ for $\operatorname{span}(L)$.

Since the first two vectors in L are orthogonal to each other, we obtain the first two vectors of M by normalization. $\hat{u}_1=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\\0\end{bmatrix}$ and $\hat{u}_2=\frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\0\\1\end{bmatrix}$. Then,

$$\vec{v}_3^{\perp}=\vec{v}_3-(\vec{v}_3\cdot\hat{u}_1)\,\hat{u}_1-(\vec{v}_3\cdot\hat{u}_2)\,\hat{u}_2=\begin{bmatrix}1\\2\\2\\1\end{bmatrix}-\tfrac{3}{2}\begin{bmatrix}1\\0\\1\\0\end{bmatrix}-\tfrac{3}{2}\begin{bmatrix}0\\1\\0\\1\end{bmatrix}=\begin{bmatrix}-\frac{1}{2}\\ \frac{1}{2}\\ \frac{1}{2}\\ -\frac{1}{2}\end{bmatrix}.\quad \hat{u}_3=\frac{\vec{v}_3^{\perp}}{\left\|\vec{v}_3^{\perp}\right\|}=\tfrac{1}{2}\begin{bmatrix}-1\\1\\1\\-1\end{bmatrix}.$$

b. Compute the 4×4 matrix P for projection onto $\operatorname{span}(L)$.

$P=\hat{u}_1\,\hat{u}_1^{T}+\hat{u}_2\,\hat{u}_2^{T}+\hat{u}_3\,\hat{u}_3^{T}$

$$=\tfrac{1}{2}\begin{bmatrix}1&0&1&0\\0&0&0&0\\1&0&1&0\\0&0&0&0\end{bmatrix}+\tfrac{1}{2}\begin{bmatrix}0&0&0&0\\0&1&0&1\\0&0&0&0\\0&1&0&1\end{bmatrix}+\tfrac{1}{4}\begin{bmatrix}1&-1&-1&1\\-1&1&1&-1\\-1&1&1&-1\\1&-1&-1&1\end{bmatrix}=\tfrac{1}{4}\begin{bmatrix}3&-1&1&1\\-1&3&1&1\\1&1&3&-1\\1&1&-1&3\end{bmatrix}.$$

c. Find the vector $\vec{w}^*$ in $\operatorname{span}(L)$ for which $\left\|\vec{w}-\vec{w}^*\right\|$ is smallest.

The projection of any vector $\vec{w}$ onto the subspace $\operatorname{span}(L)$ is the vector in this subspace that is closest to $\vec{w}$. So, $\vec{w}^*=P\,\vec{w}=\frac{1}{4}\begin{bmatrix}3&-1&1&1\\-1&3&1&1\\1&1&3&-1\\1&1&-1&3\end{bmatrix}\begin{bmatrix}1\\2\\1\\1\end{bmatrix}=\frac{1}{4}\begin{bmatrix}3\\7\\5\\5\end{bmatrix}$.

4. Imagine positive and negative electrically charged particles stored in three capacitors C_1, C_2, and C_3. The capacitors are linked by superconducting circuitry that has no resistance to the flow of electricity. The net charge throughout the system remains constant. Every millisecond, the particles are allowed to flow between the capacitors in the following way. 80% of the net charge that was in C_1 remains there, while 10% flows to C_2 and 10% flows to C_3; C_2 retains 60% of its net charge while the remaining 40% flows to C_3; and no net charge flows from C_3 to either of the other capacitors. Let $x_k(t)$ be the total charge in C_k after t milliseconds where $k = 1, 2,$ or 3.

a. Formulate a single vector equation that describes the relationships between the net charges in the capacitors after t milliseconds and at 0 seconds.

$$\vec{x}(t+1) = A\,\vec{x}(t) \text{ where } A = \begin{bmatrix} .8 & 0 & 0 \\ .1 & .6 & 0 \\ .1 & .4 & 1 \end{bmatrix}. \text{ So, } \vec{x}(t) = A^t\,\vec{x}(0.)$$

b. Solve the equation in part a, if the initial charges in the capacitors C_1, C_2, and C_3 are 1, 0, and 0 coulombs, respectively.

Since A is lower triangular, spec(A) = (.8, .6, 1). Correspondingly,

$$E_{.8}(A) = \ker(A - .8I) = \ker\begin{bmatrix} 0 & 0 & 0 \\ .1 & -.2 & 0 \\ .1 & .4 & .2 \end{bmatrix} = \ker\begin{bmatrix} 1 & 0 & \frac{2}{3} \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix} = \operatorname{span}\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}.$$

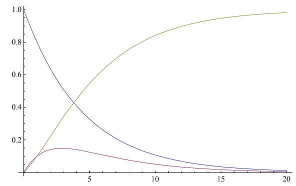
$$E_{.6}(A) = \ker(A - .6I) = \ker\begin{bmatrix} .2 & 0 & 0 \\ .1 & 0 & 0 \\ .1 & .4 & .4 \end{bmatrix} = \ker\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \operatorname{span}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

$$E_{1}(A) = \ker(A - 1I) = \ker\begin{bmatrix} -.2 & 0 & 0 \\ .1 & -.4 & 0 \\ .1 & .4 & 0 \end{bmatrix} = \ker\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \operatorname{span}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore, $\vec{x}(t) = a\,(.8)^t\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} + b\,(.6)^t\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. The coefficients a, b, and c are obtained from the initial conditions by setting $t = 0$.

$$\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = a\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \begin{Bmatrix} a = \frac{1}{2} \\ b = -\frac{1}{2} \\ c = 1 \end{Bmatrix}.$$ So, the complete solution is

$$\vec{x}(t)=\tfrac{1}{2}(.8)^t\begin{bmatrix}2\\1\\-3\end{bmatrix}+\tfrac{1}{2}(.6)^t\begin{bmatrix}0\\-1\\1\end{bmatrix}+\begin{bmatrix}0\\0\\1\end{bmatrix}.$$



c. For which initial charges on the capacitors will their proportions remain the same for all time?

The proportions of the charges on each capacitor will remain constant in time only if $\vec{x}(0)$ belongs to one of the eigenspaces $E_{.6}(A),\ E_{.8}(A),$ or $E_1(A)$. If $\vec{x}(0)$ is a vector in one of the eigenspaces, the magnitude of may $\vec{x}(t)$ change as t changes but $\vec{x}(t)$ will always remain in the same eigenspace.

5. A is a 6×6 matrix such that

$$A_{rref}=\begin{bmatrix}1&0&1&0&1&-2\\0&1&0&0&-1&1\\0&0&0&1&1&1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix},\quad {A^T}_{rref}=\begin{bmatrix}1&0&0&1&1&1\\0&1&0&-1&-4&3\\0&0&1&0&1&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix},\text{ and } A\begin{bmatrix}0\\0\\0\\0\\1\\0\end{bmatrix}=\begin{bmatrix}2\\0\\3\\2\\5\\2\end{bmatrix}$$

Determine a basis for each of the following subspaces of $\mathbf{R}^6$.

a. im(A).

The pivot rows of ${A^T}_{rref}$ provide a basis for im(A) since they are, by construction, linearly independent, and as a result of row-reduction, they must span the rows of A^T which correspond to the columns of A.

$$\operatorname{im}(A)=\operatorname{span}\left(\begin{bmatrix}1\\0\\0\\1\\1\\1\end{bmatrix},\begin{bmatrix}0\\1\\0\\-1\\-4\\3\end{bmatrix},\begin{bmatrix}0\\0\\1\\0\\1\\0\end{bmatrix}\right)$$

b. ker(A).

Applying the Solution Algorithm to A_{rref} provides a basis for ker(A) since ker(A) is the solution subspace for the equation $A\vec{x}=\vec{0}$.

$$\ker(A)=\operatorname{span}\left(\begin{bmatrix}1\\0\\-1\\0\\0\\0\end{bmatrix},\begin{bmatrix}1\\-1\\0\\1\\-1\\0\end{bmatrix},\begin{bmatrix}-2\\1\\0\\1\\0\\-1\end{bmatrix}\right).$$

Find all solutions for each of the following equations.

c. $A\vec{x} = \begin{bmatrix} 2 \\ 0 \\ 3 \\ 2 \\ 5 \\ 2 \end{bmatrix}$. The solution set is $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \ker(A)$. (We used the data above.)

d. $A\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. There is no solution, since $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is clearly not in $\operatorname{im}(A)$.

6. Suppose that $(\vec{v}_1, \ldots, \vec{v}_p)$ is a basis for the subspace V in $\mathbf{R}^n$ and $\vec{w}$ belongs to V. Then, there is list $C = (c_1, \ldots, c_p)$ of scalar coefficients such that $\vec{w} = c_1\vec{v}_1 + \ldots + c_p\vec{v}_p$.

a. Explain why this is so.

Every vector in a subspace is a linear combination of the vectors in any basis for that subspace.

b. Prove that this list C of coefficients is unique.

Suppose, to the contrary, that there was another list of coefficients, call this list $(b_1, \ldots, b_p)$. Now, we have $c_1\vec{v}_1 + \ldots + c_p\vec{v}_p = b_1\vec{v}_1 + \ldots + b_p\vec{v}_p$. Subtracting the right side from the left, we obtain $(b_1 - c_1)\vec{v}_1 + \ldots + (b_p - c_p)\vec{v}_p = \vec{0}$. But, the only linear combination of linearly independent vectors that is $\vec{0}$ is the trivial one, i.e., $b_1 = c_1, \ldots, b_p = c_p$.

c. Obtain a formula for c_k, $k = 1, 2, \ldots, p$, if $(\vec{v}_1, \ldots, \vec{v}_p)$ is an orthogonal list.

If we take the inner product of both sides of the equation $\vec{w} = c_1\vec{v}_1 + \ldots + c_p\vec{v}_p$ with $\vec{v}_k$, we easily obtain $c_k = \vec{w}\cdot\vec{v}_k / \vec{v}_k\cdot\vec{v}_k$.

7. V is the subspace of $\mathbf{R}^3$ described by the single equation $x+y+z=0$ and W is the subspace of $\mathbf{R}^3$ described by the pair of equations $x-y+2z=0$ and $x+y+3z=0$.

a. Determine bases for V and for W.

$$V=\ker\begin{bmatrix}1 & 1 & 1\end{bmatrix}=\operatorname{span}\left(\begin{bmatrix}1\\-1\\0\end{bmatrix},\begin{bmatrix}1\\0\\-1\end{bmatrix}\right)\Rightarrow\left(\begin{bmatrix}1\\-1\\0\end{bmatrix},\begin{bmatrix}1\\0\\-1\end{bmatrix}\right)\text{ is a basis for } V.$$

$$W=\ker\begin{bmatrix}1 & -1 & 2\\1 & 1 & 3\end{bmatrix}=\ker\begin{bmatrix}1 & 0 & \frac{5}{2}\\0 & 1 & \frac{1}{2}\end{bmatrix}=\operatorname{span}\left(\begin{bmatrix}5\\1\\-2\end{bmatrix}\right)\Rightarrow\left(\begin{bmatrix}5\\1\\-2\end{bmatrix}\right)\text{ is a basis for } W.$$

Joining the two lists in part a gives us a basis $\mathfrak{B}=(\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3)$ for $\mathbf{R}^3$.

b. What is the connection between any $\vec{x}\in\mathbf{R}^3$ and its $\mathfrak{B}$-coordinate representative $[\vec{x}]_{\mathfrak{B}}$?

$$\vec{x}=S[\vec{x}]_{\mathfrak{B}}\quad\text{where}\quad S=[\vec{v}_1\,|\,\vec{v}_2\,|\,\vec{v}_3]=\begin{bmatrix}1 & 1 & 5\\-1 & 0 & 1\\0 & -1 & -2\end{bmatrix}.$$

Now, suppose that $T:\mathbf{R}^3\to\mathbf{R}^3$ is the linear transformation that doubles all vectors in V and reverses all vectors in W.

c. Determine B, the matrix for T in $\mathfrak{B}$-coordinates.

$$B=\left[[T(\vec{v}_1)]_{\mathfrak{B}}\,|\,[T(\vec{v}_2)]_{\mathfrak{B}}\,|\,[T(\vec{v}_3)]_{\mathfrak{B}}\right]=\left[[2\vec{v}_1]_{\mathfrak{B}}\,|\,[-\vec{v}_2]_{\mathfrak{B}}\,|\,[-\vec{v}_3]_{\mathfrak{B}}\right]=\begin{bmatrix}2 & 0 & 0\\0 & -1 & 0\\0 & 0 & -1\end{bmatrix}.$$

d. Compute the matrix A for T in standard coordinates.

$$A=SBS^{-1}=\begin{bmatrix}1 & 1 & 5\\-1 & 0 & 1\\0 & -1 & -2\end{bmatrix}\begin{bmatrix}2 & 0 & 0\\0 & -1 & 0\\0 & 0 & -1\end{bmatrix}\begin{bmatrix}1 & 1 & 5\\-1 & 0 & 1\\0 & -1 & -2\end{bmatrix}^{-1}=-\tfrac{1}{4}\begin{bmatrix}1 & 9 & -3\\3 & -5 & 3\\0 & 0 & 4\end{bmatrix}.$$

e. Determine the list of all eigenvalues, spec(A), for A.

spec(A) = $(2,-1,-1)$.

f. What is the eigenspace $E_\lambda(A)$ for each eigenvalue $\lambda\in\operatorname{spec}(A)$?

$E_2(A)=V$ and $E_{-1}(A)=W$.

8. a. The 3×3 matrix S has column vectors $\vec{c}_1,\ \vec{c}_2,\ \hat{c}_3$. That is, $A=\left[\vec{c}_1 \mid \vec{c}_2 \mid \vec{c}_3\right]$. We find that $\det(S) = 7$. What is the value of $\det\left(\left[\vec{c}_1 + 2\vec{c}_3 \mid 3\vec{c}_1 \mid -4\vec{c}_2\right]\right)$?

$$\det\left(\left[\vec{c}_1 + 2\vec{c}_3 \mid 3\vec{c}_1 \mid -4\vec{c}_2\right]\right) = (3)(-4)\det\left(\left[\vec{c}_1 + 2\vec{c}_3 \mid \vec{c}_1 \mid \vec{c}_2\right]\right)$$
$$= (3)(-4)\det\left(\left[2\vec{c}_3 \mid \vec{c}_1 \mid \vec{c}_2\right]\right) = (3)(-4)(2)\det\left(\left[\vec{c}_3 \mid \vec{c}_1 \mid \vec{c}_2\right]\right)$$
$$== (3)(-4)(2)(-1)^3\det\left(\left[\vec{c}_1 \mid \vec{c}_2 \mid \vec{c}_3\right]\right) = (24)(7) = 168.$$

The ijth entry of the 5×5 matrix A is a_{ij}. Recall that the ijth minor of A is the 4×4 matrix obtained from A by deleting its ith row and jth column. Let b_{ij} be the determinant of the ijth minor of A. Given that $\det(A) = 3$, determine the numerical values of the following expressions.

b. $a_{15}b_{13} - a_{25}b_{23} + a_{35}b_{33} - a_{45}b_{43} + a_{55}b_{53}$

This is a "wrong" Laplace expansion that pairs the entries of the fifth column with the minors for a different (the third) column. The result is 0.

c. $a_{41}b_{41} - a_{42}b_{42} + a_{43}b_{43} - a_{44}b_{44} + a_{45}b_{45}$

This is a Laplace expansion that pairs the entries of the fourth row with the minors for that row. This differs from $\det(A)$ only in sign since $(1)^{4+1} = -1$. So, the value of this expression is $-\det(A) = -3$.

9. For any real matrix A, $\ker(A^TA) = \ker(A)$. Assertions that provide a proof of this theorem follow. For each of the six assertions labeled a through f below, provide a brief justification or rationale.

Assume $\vec{x} \in \ker(A)$.

a. It follows that $\vec{x} \in \ker(A^TA)$ and so, $\ker(A) \subseteq \ker(A^TA)$.

$A\vec{x} = \vec{0} \ \Rightarrow\ A^TA\vec{x} = A^T\vec{0} = \vec{0} \ \Rightarrow\ \vec{x} \in \ker(A^TA) \ \Rightarrow\ \ker(A) \subseteq \ker(A^TA)$.

Now, assume $\vec{x} \in \ker(A^TA)$.

b. Then, $A\vec{x} \in \left(\operatorname{im}(A)\right)^{\perp}$.

$(A^TA)\vec{x} = \vec{0} \ \Rightarrow\ A^T(A\vec{x}) = \vec{0} \ \Rightarrow\ A\vec{x} \in \ker(A^T) = \left(\operatorname{im}(A)\right)^{\perp}$.

c. $A\vec{x} \in \operatorname{im}(A) \bigcap \left(\operatorname{im}(A)\right)^{\perp}$.

$A\vec{x} \in \operatorname{im}(A)$ by definition of image and, with part b, we have

$A\vec{x} \in \operatorname{im}(A) \cap \left(\operatorname{im}(A)\right)^{\perp}$,

d. So, $A\vec{x} = \vec{0}$.

The intersection of a subspace and its orthogonal complement is the zero vector.

e. It follows then that $\ker(A^T A) \subseteq \ker(A)$.

From part d, we have shown that vectors belonging to $\ker(A^T A)$ are annihilated by A and so also belong to $\ker(A)$.

Therefore,

f. $\ker(A^T A) = \ker(A)$.

This follows from parts a and e since $X \subseteq Y$ and $Y \subseteq X \Rightarrow X = Y$.

10. Completely solve the following coupled initial-value problem for $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ using eigenvalue-eigenvector methods. $\frac{d}{dt}\vec{x}(t) + \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}\vec{x}(t) = \vec{0}$; $\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Let $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$. Then, $0 = \det(A - \lambda I) = (1-\lambda)(4-\lambda) + 2 = \lambda^2 - 5\lambda + 6 =$ $(\lambda - 2)(\lambda - 3) \Rightarrow \text{spec}(A) = (2, 3)$. The eigenspaces of A are found next.

$$E_2(A) = \ker(A - 2I) = \ker\left(\begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right).$$

$$E_3(A) = \ker(A - 3I) = \ker\left(\begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right).$$

It follows then that $\vec{x}(t) = a e^{-2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + b e^{-3t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. From the initial conditions,

$$\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow a = 2,\ b = -1 \Rightarrow \vec{x}(t) = 2e^{-2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - e^{-3t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

11. Two variables, s and t are related by an equation of the form $s = at + bt^2$ for a pair of constants a and b. Find the best (in the least-squares sense) choices of a and b consistent with the following data.

t	s
1	-1
1	0
2	1
2	2

Extending the table in order to determine how each value of s corresponds to a linear combination of the two variables a and b, we have

s	t	$at+bt^2$
-1	1	$a+b$
0	1	$a+b$
1	2	$2a+4b$
2	2	$2a+4b$

If the tabulated data fit a parabola with equation $s = at + bt^2$ exactly (they do not), the data would be summarized by the vector equation $A\vec{x} = \vec{b}$ where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 2 & 4 \\ 2 & 4 \end{bmatrix},\ \vec{x} = \begin{bmatrix} a \\ b \end{bmatrix},\ \text{and}\ \vec{c} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}.$$ The vector equation clearly has no solution since $\vec{c} \notin \operatorname{im}(A)$. Instead, we seek the solution to $A\vec{x} = \vec{c}\,*$ where $\vec{c}\,*$ is the vector in $\operatorname{im}(A)$ that is closest to $\vec{c}$. This is equivalent to solving $A^T A\vec{x} = A^T\vec{c}$.

We have $$\vec{x} = \begin{bmatrix} a \\ b \end{bmatrix} = (A^T A)^{-1} A^T \vec{c} = \left(\begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 4 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 2 & 4 \\ 2 & 4 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 4 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 18 \\ 18 & 34 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 4 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix} = \tfrac{1}{4} \begin{bmatrix} -7 \\ 5 \end{bmatrix}.$$ The equation for the parabola of best fit is therefore $s = -\tfrac{7}{4}t + \tfrac{5}{4}t^2 = \tfrac{5}{4}t(t - \tfrac{7}{5}) = 1.25t(t - 1.4)$.

12. $\mathbf{R}^{3\times 3}$ is the 9-dimensional vector space consisting of all real 3×3 matrices wherein vector addition is defined to be the ordinary addition of matrices and multiplication by scalars is the familiar multiplication of a matrix by a scalar. Consider the subset V of $\mathbf{R}^{3\times 3}$ consisting of the symmetric 3×3 matrices whose trace is 0.

a. Explain why V is a subspace of $\mathbf{R}^{3\times 3}$.

All linear combinations of symmetric 3×3 matrices are symmetric 3×3 matrices. All linear combinations of 3×3 matrices with zero trace are 3×3 matrices with zero trace. So, V is closed under linear combinations and is a subspace.

b. Find a basis for and the dimension of V.

$$A \in V \;\Rightarrow\; A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & -(a+d) \end{bmatrix}$$ for arbitrary reals a, b, c, d, e. So,

$$A = a\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} + e\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Here, we have expressed an arbitrary matrix in V as a linear combination of 5 clearly linearly independent 3×3 matrices that are symmetric and have zero trace. In other words, the 5 matrices constitute a basis for V and dim(V) = 5. So, a basis for V is $\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right)$.

Let S be the matrix for any rotation in $\mathbf{R}^3$ and consider the transformation $f : \mathbf{R}^{3\times 3} \to \mathbf{R}^{3\times 3}$ where $f(A) = S\,A\,S^T$. It is easy to see that f is linear.

c. Explain why $f(A) \in V$ for any $A \in V$. In other words, show that the image, under f, of any traceless symmetric 3×3 matrix is itself a traceless symmetric 3×3 matrix.

Since a rotation matrix S is orthogonal, $S^T S = I$ or, equivalently $S^T = S^{-1}$. Therefore, $f(A) = S A S^{-1}$ is similar to A. If A belongs to V, tr($f(A)$) = tr(A) = 0 because similar matrices have the same trace. $f(A)$ is symmetric because $\left(f(A)\right)^T = (S\,A\,S^T)^T = S\,A\,S^T = f(A)$. So, $f(A)$ is symmetric and has zero trace. Therefore, $f(A) \in V$ for any $A \in V$.