SM261 Matrix Theory Final Exam Fall 2010

Name: ______________________________________

Calculators may be used in any capacity. No other aids are allowed. You are to answer all questions on the exam paper in the space provided. The exam totals 100 points. Points are indicated next to each question, in brackets (e.g. [4:1,2,1] means that the question has parts worth 1,2, and 1 point, totaling 4 points).

Problem	Grading Scheme	Points
1	2	
2	5:1,2,2	
3	4:3,1	
4	8:2,2,2,2	
5	2	
6	3:1,1,1	
7	2	
8	2	
9	4	
10	9:2,4,1,1,1	
11	5:2,3	
12	3	
13	6:1,3,2	
14	3	
15	4	
16	7:5,2	
17	5:1 each	
18	4:1 each	
19	9:3,3,3	
20	13:1 each	

Total Score: ______________

1. [2] Using matrix methods, determine the point of intersection of the lines $x - y = 4$ and $2x + y = -1$.

2. [5:1,2,2] (a) Write a system of equations that is equivalent to the vector equation

$$x_1 \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 8 \\ 5 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -6 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

(b) Is the set $\mathcal{B} = \left\{ \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -6 \\ 0 \end{bmatrix} \right\}$ linearly independent? If so, explain why. If not, find a linear dependence relation among the three vectors.

(c) Does the set $\mathcal{B}$ span all of $\mathbb{R}^3$? Explain.

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3. [4:3,1] The linear transformation T: $\mathbb{R}^3 \to \mathbb{R}^3$ rotates 3-dimensional space 90 degrees about the z-axis (sending the positive x-axis to the positive y-axis, the positive y-axis to the negative x-axis, and leaving the z-axis fixed).
(a) Find the 3x3 matrix A so that $T(\mathbf{v}) = A\mathbf{v}$. for all $\mathbf{v} \in \mathbb{R}^3$.

(b) Find T$\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right)$.

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4. [8:2,2,2,2] (a) Find the matrix A that represents the linear transformation T: $\mathbb{R}^4 \to \mathbb{R}^4$ such that $T(x_1, x_2, x_3, x_4) = (0, x_1 + x_2, x_2 + x_3, x_3 + x_4)$.

(b) Rank $A =$ _______ dim Nul $A =$ _______

(c) Is T onto? Explain.

(d) Is T 1-1? Explain.

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5. [2] Let A, B, C, D and X be $n \times n$ matrices and assume that A and B are invertible. Solve the following matrix equation for X:

$$AXB + C = D.$$

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6. [3:1,1,1] Suppose that $\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 5.$

(a) $\det \begin{bmatrix} d & e & f \\ g & h & i \\ a & b & c \end{bmatrix} =$

(b) $\det \begin{bmatrix} b & a & c \\ e & d & f \\ h & g & i \end{bmatrix} =$

(c) $\det \begin{bmatrix} 2a & 2b & 2c \\ d & e & f \\ h-a & g-b & i-c \end{bmatrix} =$

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7. [2] Use determinants to find the area of the parallelogram $ABCD$ with vertices $A(-1,2)$, $B(0,5)$, $C(3,6)$ and $D(2,3)$.

8. [2] Cramer's rule says that the solution to a system of linear equations in two variables is

$$x_1 = \frac{\det \begin{bmatrix} 6 & 1 \\ 7 & 2 \end{bmatrix}}{\det \begin{bmatrix} 4 & 1 \\ 5 & 2 \end{bmatrix}}$$

and

$$x_2 = \frac{\det \begin{bmatrix} 4 & 6 \\ 5 & 7 \end{bmatrix}}{\det \begin{bmatrix} 4 & 1 \\ 5 & 2 \end{bmatrix}}.$$

Find the system of original system of equations.

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9. [4] Each of the four transformations labeled 1, 2, 3, and 4 corresponds to exactly one of the matrices A, B, C, D, E, F, G, H, or J. Match them up.

1. scaling 2. rotation 3. projection 4. reflection

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} -.6 & .8 \\ -.8 & -.6 \end{bmatrix}$$

$$D = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \quad F = \begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix}$$

$$G = \begin{bmatrix} .6 & .6 \\ .8 & .8 \end{bmatrix} \quad H = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \quad J = \begin{bmatrix} .8 & -.6 \\ .6 & -.8 \end{bmatrix}.$$

$1 =$ ______ $2 =$ ______ $3 =$ ______ $4 =$ ______

10. [9: 2,4,1,1,1] The set $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$ forms a basis for a 3-dimensional subspace W of $\mathbb{R}^4$.

(a) Find the $\mathcal{B}$-coordinates for the vector $\mathbf{x} = \begin{bmatrix} 5 \\ -3 \\ 1 \\ -3 \end{bmatrix}$. Also, express $\mathbf{x}$ as a linear combination of the vectors in $\mathcal{B}$.

(b) Find an orthonormal basis for the subspace W.

(c) Find a basis for the subspace $W^{\perp} = \{\mathbf{v} \in \mathbb{R}^4 : \ \mathbf{w}^T\mathbf{v} = 0 \text{ for all } \mathbf{w} \in W\}$.

(d) Find the vector in W that is closest to the vector $\mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

(e) Express the vector $\mathbf{y}$ from part (d) in the form $\mathbf{w} + \mathbf{v}$ with $\mathbf{w} \in W$ and $\mathbf{v} \in W^{\perp}$.

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11. [5:2,3] Let

$$A = \begin{bmatrix} 1 & 2 & -5 & 11 & -3 \\ 2 & 4 & -5 & 15 & 2 \\ 1 & 2 & 0 & 4 & 5 \\ 3 & 6 & -5 & 19 & -2 \end{bmatrix}.$$

(a) Find a basis for Col(A).

(b) Find a basis for Col$(A)^{\perp}$.

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12. [3] If the null space of an 8x6 matrix A is 2 dimensional, then the dimension of the row space of $A =$ _______ and the dimension of Col A $=$ _______ and the dimension of (Col $A)^{\perp}$ $=$ _______.

13. [6:1,3,2] Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \text{ and } \mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 8 \\ 2 \end{bmatrix}.$$

(a) Without any calculation, explain how you can tell that the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent.

(b) Find a least-squares solution $\hat{\mathbf{x}}$ of $A\mathbf{x} = \mathbf{b}$.

(c) Calculate $\|A\hat{\mathbf{x}} - \mathbf{b}\|$.

14. [3] Describe three applications of matrix theory that are not mentioned in this exam. One sentence about each is sufficient.

15. [4] Given that the distinct eigenvalues of

$$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

are $\lambda = 7$ and $\lambda = -2$, find an invertible matrix P with only integer entries and a diagonal matrix D such that $A = PDP^{-1}$.

16. [7:5,2] (a) Let A be an $n \times n$ matrix with real entries. Place a check mark next to statements that are equivalent to the statement "A is an invertible matrix". Place an X next to statements that are not equivalent to the statement "A is an invertible matrix".

__________ A is row equivalent to the $n \times n$ identity matrix.
__________ Exactly half of the columns of A contain a pivot.
__________ The equation $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions.
__________ The columns of A form a linearly independent set.
__________ The columns of A form an orthogonal set.
__________ 0 is an eigenvalue of A.
__________ Rank A = dim Row A.
__________ The rows of A span $\mathbb{R}^n$.
__________ $\text{Nul}(A^T) = \{\mathbf{0}\}$.
__________ Rank A + dim Nul $A = n$.

(b) Write another condition that is equivalent to the statement "A is an invertible matrix". Do not use the transpose of a condition above.

17. [5:1 each] Let

$$A = \begin{bmatrix} 3 & 0 & 4 \\ -2 & 6 & 2 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 3 & 1 \\ -1 & 6 \\ 2 & 1 \end{bmatrix}.$$

Compute the following or write "not defined", as appropriate:

(a) AB

(b) BA

(c) $A + B^T$

(d) $(AB)^{-1}$

(e) $(BA)^{-1}$

18. [4:1 each] Give an example of each of the following or write “no such example exists”, as appropriate:

(a) A matrix representing a linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^2$ that is onto its codomain but is not 1-1.

(b) A diagonal matrix that is not invertible.

(c) A 2x5 matrix of rank 1.

(d) A matrix A with $A^T = A$.

19. [9:3,3,3] Determine if the following subsets of $\mathbb{R}^3$ are subspaces. If so, find their dimension and compute a basis. If not, provide an explicit example to show that one of the subspace requirements fails.

(a) $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + 2y - z = 0 \right\}$.

(b) $X = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : xyz = 0 \right\}$.

(c) $Y = \left\{ \begin{bmatrix} 2r - s + t \\ s - t \\ r + s - t \end{bmatrix} : r, s, t \in \mathbb{R} \right\}$.

20. [13:1 each] TRUE or FALSE: Circle the most correct answer.

i. TRUE FALSE The 2x2 matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is similar to a diagonal matrix.

ii. TRUE FALSE If A is an invertible matrix then $A^T A^{-1}$ is invertible.

iii. TRUE FALSE If $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for a subspace W of $\mathbb{R}^3$, then the projection of $\mathbf{v}$ in $\mathbb{R}^3$ onto W is given by the formula

$$\text{proj}_W \mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{v} \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2.$$

iv. TRUE FALSE If you remove a vector from a basis of a vector space then the resulting set is linearly dependent.

v. TRUE FALSE Let A, B, C and D be 3x3 matrices; let I denote the 3x3 identity matrix and let 0 denote the 3x3 matrix filled with zeros. If

$$\begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \begin{bmatrix} I & A \\ 0 & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix},$$

then D is invertible and $A = I$.

vi. TRUE FALSE

$$\det \begin{bmatrix} 2 & 0 & 1 & 1 & 2 \\ 0 & 1 & -1 & 2 & 1 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = 4.$$

vii. TRUE FALSE If $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ are nonzero coplanar vectors in $\mathbb{R}^3$ so that no two are parallel then rank $[\mathbf{v}_1|\mathbf{v}_2|\mathbf{v}_3] = 1$.

viii. TRUE FALSE If A is a 20x13 matrix with rank $A = 13$ then the rows of A are linearly independent.

ix. TRUE FALSE If A is an nxn invertible matrix then $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b} \in \mathbb{R}^n$.

x. TRUE FALSE Let $T : \mathbb{R}^{2013} \to \mathbb{R}^{2013}$ be a linear transformation. If T is 1-1 then T is also onto.

xi. TRUE FALSE If the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n = \mathbf{0}$$

has the trivial solution then the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent.

xii. TRUE FALSE If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation with $T(\mathbf{e}_1) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $T\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. Then $T(\mathbf{e}_2) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

xiii. TRUE FALSE Suppose that A is a 2x2 matrix with eigenvalues $\lambda_1 = 0.1$ and $\lambda_2 = 1$, and associated eigenvectors $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Suppose further that $\mathbf{x}_0 = \mathbf{v}_1 + 2\mathbf{v}_2$. If $\mathbf{x}_k = A^k\mathbf{x}_0$ then $\lim_{k\to\infty} x_k = 3\mathbf{e}_2$.