

Three.VI Projection

Linear Algebra

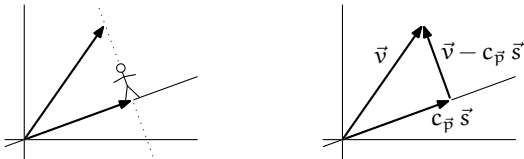
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<http://joshua.smcvt.edu/linearalgebra>

Orthogonal Projection Into a Line

Project a vector into a line

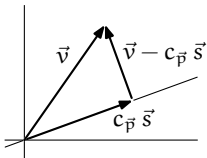
This shows a figure walking out on the line to a point \vec{p} such that the tip of \vec{v} is directly above them, where “above” does not mean parallel to the y-axis but instead means orthogonal to the line.



Since the line is the span of some vector $\ell = \{c \cdot \vec{s} \mid c \in \mathbb{R}\}$, we have a coefficient $c_{\vec{p}}$ with the property that $\vec{v} - c_{\vec{p}} \vec{s}$ is orthogonal to $c_{\vec{p}} \vec{s}$.

To solve for this coefficient, observe that because $\vec{v} - c_{\vec{p}} \vec{s}$ is orthogonal to a scalar multiple of \vec{s} , it must be orthogonal to \vec{s} itself. Then $(\vec{v} - c_{\vec{p}} \vec{s}) \cdot \vec{s} = 0$ gives that $c_{\vec{p}} = \vec{v} \cdot \vec{s} / \vec{s} \cdot \vec{s}$.

We have decomposed \vec{v} into two parts $\vec{v} = (c_{\vec{p}} \vec{s}) + (\vec{v} - c_{\vec{p}} \vec{s})$.



Intuitively, some of \vec{v} lies with the line and that gives the first part $c_{\vec{p}} \vec{s}$. The part of \vec{v} that lies with a line orthogonal to ℓ is $\vec{v} - c_{\vec{p}} \vec{s}$. What's compelling about pairing these two parts is that they don't interact, in that the projection of one into the line spanned by the other is the zero vector.

Note: We have not given a definition of 'angle' in spaces other than \mathbb{R}^n 's, so we will stick here to those spaces. Extending the definitions to other spaces is perfectly possible but we don't need them here.

1.1 *Definition* The *orthogonal projection of \vec{v} into the line spanned by a nonzero \vec{s}* is this vector.

$$\text{proj}_{[\vec{s}]}(\vec{v}) = \frac{\vec{v} \cdot \vec{s}}{\vec{s} \cdot \vec{s}} \cdot \vec{s}$$

Example The projection of this \mathbb{R}^3 vector into the line

$$\vec{v} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad L = \{c \cdot \vec{s} = c \cdot \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \mid c \in \mathbb{R}\}$$

is this vector.

$$\frac{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}} \cdot \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/6 \\ 1/6 \end{pmatrix}$$