Integer Constraints

<u>Problem</u>. Determine the ways in which 20 bills chosen from the denominations \$1, \$2 and \$5 can have a total value of \$33.

<u>Solution</u>. Let x, y and z represent the number of \$1, \$2 and \$5 bills, respectively. Then, from the information given, we have two linear equations $x+y+z=20$ and $x+2y+5z=33$ in three variables. These equations can be represented by the 2×4 augmented matrix $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 20 \\ 1 & 2 & 5 & 33 \end{array}\right]$. Row-reducing this matrix we obtain, in just two steps, that $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 20 \\ 1 & 2 & 5 & 33 \end{array}\right] \leftrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 20 \\ 0 & 1 & 4 & 13 \end{array}\right] \leftrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -3 & 7 \\ 0 & 1 & 4 & 13 \end{array}\right]$. We first subtracted the top row from the bottom one and then we subtracted the bottom row from the top one. The last matrix is fully row-reduced. From it, we can express the two variables, x and y in terms of z. Converting rows to equations, we have $x=3z+7$ and $y=-4z+13$. We might be tempted to say that z is arbitrary. That is, z is any real number. However, the problem has a condition that was not explicitly stated: x, y and z must be nonnegative integers. This is condition cannot be formulated as a linear equation but it restricts the possible choices of z. Clearly, $x=3z+7$ is a nonnegative integer if z is a nonnegative integer. But, $y=-4z+13$ will be a nonnegative integer provided that z is not too large. In order that $y \geq 0$, we need $z < 4$. So, the only possible solutions arise when $z = 0, 1, 2,$ or 3. We can write the solutions in vector format:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = z\begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix} + \begin{bmatrix} 7 \\ 13 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 13 \\ 0 \end{bmatrix}, \begin{bmatrix} 10 \\ 9 \\ 1 \end{bmatrix}, \begin{bmatrix} 13 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 16 \\ 1 \\ 3 \end{bmatrix}.$$