1. Enter **T** (true) or **F** (false) or nothing next to each statement below. Each correct response earns full credit; each blank response earns no credit; each incorrect response earns negative half credit. The lowest possible total score will be 0.

F	a. If $\vec{u}$, $\vec{v}$ and $\vec{w}$ are non-zero vectors in some vector space and $\vec{u}$ is a linear combination of $\vec{v}$ and $\vec{w}$, then $\vec{w}$ is a linear combination of $\vec{u}$ and $\vec{v}$. Example: $\vec{u} = \vec{v} + 0\vec{w}$.
T	b. If A, B and C are real square matrices of the same size and A is similar to B and B is similar to C, then A is similar to C. Similarity is transitive.
F	c. For real $m \times n$ matrices D and E, rank(D) + rank(E) = rank(D + E). Example: $D = -E = I$.
F	d. For real $m \times n$ matrices D and E, if rank(D) = rank(E), then E may be obtained from D by row operations. Example: two matrices with pivots in different locations.
F	e. If a_{ij} is the ij-th entry of A, then $a_{ij} \mapsto \det(A)$ is a linear function. Example: Doubling the entry in the upper right corner the 2×2 matrix all of whose entries are 1 does not double its determinant.
F	f. If $\vec{a}$ is an eigenvector of A and $\vec{b}$ is an eigenvector of B, $\vec{a} + \vec{b}$ is an eigenvector of $A + B$. Example: $(A+B)(\vec{a}+\vec{b}) = (A+B)\vec{b} + (A+B)\vec{b}$
T	g. If the graph of the characteristic polynomial of A passes through the origin, dim(ker(A)) > 0. If the graph passes through the origin, 0 is a root and so 0 is an eigenvalue. A corresponding eigenvector belongs to ker(A).
F	h. There are real 2013×2013 matrices with no real eigenvalues. The characteristic polynomial is real and of odd degree. It must have a root.
F	i. If A is a real $m \times n$ matrix and rank(A) = m, then $\vec{x} \mapsto A\vec{x}$ is a one-to-one mapping. Example: Any matrix with $m < n$.
T	j. If H is a subspace of $\mathbf{R}^n$, there is an $n \times n$ matrix A such that H = im(A). Let the columns of A be the vectors in any basis for H and add $n - \dim(H)$ zero columns.

2. List 8 <u>distinct</u> statements about a real $n \times n$ matrix A equivalent to the statement ***A* is invertible** (non-singular). Two statements are considered the same (indistinct) if they differ only by the interchange of *row* and *column* or A and A^T.

(1) There is an $n \times n$ matrix B so that $AB = I_n$ (or $BA = I_n$). (2) rref(A) = I_n. (3) det(A) $\neq$ 0. (4) im(A) = $\mathbf{R}^n$. (5) ker(A) = $\{\vec{0}\}$. (6) $0 \notin \mathrm{spec}(A)$. (7) The columns of A span $\mathbf{R}^n$. (8) The columns of A are linearly independent.

3. Two bases $(\vec{v}_1, \ldots, \vec{v}_n)$ and $(\vec{w}_1, \ldots, \vec{w}_n)$ for $\mathbf{R}^n$ are said to be ***dual*** if and only if $\vec{v}_i \cdot \vec{w}_j = \delta_{ij}$ for $1 \le i, j \le n$. For example, here is a pair of dual bases in $\mathbf{R}^2$: $\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$ and $\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix}\right)$. Given a basis $\mathcal{B} = (\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n)$ for $\mathbf{R}^n$, describe specifically, how to find its dual basis $\mathcal{C} = (\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n)$ by row-reduction.

Let the vectors in $\mathcal{B}$ be the corresponding column vectors of the matrix B and let $C = B^{-1}$. Then, the list of row vectors of C comprise the desired basis dual to $\mathcal{B}$. C is found by row-reduction since rref[$B \mid I$] = [$I \mid C$].

4. Suppose that $A, B \in \mathbf{R}^{n \times n}$ and $(\vec{v}_1, \ldots, \vec{v}_n)$ is a real eigenbasis for both A and B. Prove that A and B commute, that is $AB = BA$.

Suppose the list of corresponding eigenvalues of A is $(\lambda_1, \ldots, \lambda_n)$ and the list of corresponding eigenvalues for B is $(\mu_1, \ldots, \mu_n)$. Then, for any k = 1, …, n.

$$(AB)\vec{v}_k = A(B\vec{v}_k) = A(\mu_k \vec{v}_k) = \mu_k (A\vec{v}_k) = \mu_k \lambda_k \vec{v}_k$$

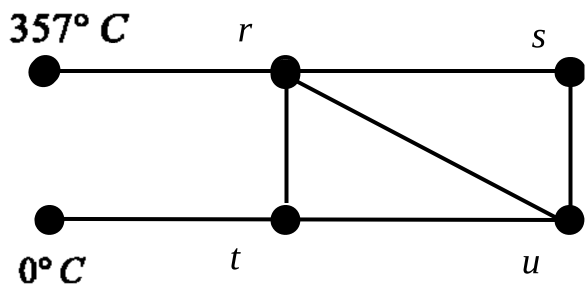
$$(BA)\vec{v}_k = B(A\vec{v}_k) = B(\lambda_k \vec{v}_k) = \lambda_k (B\vec{v}_k) = \lambda_k \mu_k \vec{v}_k$$

Therefore, $(AB)\vec{v}_k = (BA)\vec{v}_k$. Any vector $\vec{x} \in \mathbf{R}^n$ is a linear combination of $\vec{v}_1, \ldots, \vec{v}_n$. So, $(AB)\vec{x} = (AB)(c_1\vec{v}_1 + \ldots + c_n\vec{v}_n) = c_1(AB)\vec{v}_1 + \ldots + c_n(AB)\vec{v}_n = c_1(BA)\vec{v}_1 + \ldots + c_n(BA)\vec{v}_n$ $= (AB)\vec{x} = (BA)\vec{x}$. Therefore, $AB = BA$.

5. A is a real 3×3 matrix whose column vectors are nonzero, coplanar, and no pair of them is collinear. Describe all possible forms for rref(A).

$$\mathrm{rref}(A) = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 0 \end{bmatrix}$$ where a and b are real and neither is zero (i.e. $ab \neq 0$).

6. A grid with six nodes is diagrammed at the right. Two nodes are maintained at the fixed temperatures $0°C$ and $357°C$. Determine the temperatures r, s, t, and u at the other four nodes assuming that all nodes are at thermal equilibrium and, consequently, their temperatures are the averages of the temperatures at all nodes adjacent to them. Two nodes are adjacent if connected by a line segment.



$$\left\{\begin{array}{c} r = \frac{1}{4}(357+s+t+u) \\ s = \frac{1}{2}(t+u) \\ t = \frac{1}{4}(0+s+u) \\ u = \frac{1}{3}(s+t) \end{array}\right\} \Rightarrow \begin{bmatrix} 4 & -1 & -1 & -1 \\ 1 & -2 & 0 & 1 \\ 1 & 0 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{bmatrix}\begin{bmatrix} r \\ s \\ t \\ u \end{bmatrix} = \begin{bmatrix} 357 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} r \\ s \\ t \\ u \end{bmatrix} = \begin{bmatrix} 221 \\ 204 \\ 136 \\ 187 \end{bmatrix}$$

7. The fully row-reduced echelon form of the matrix A is $\begin{bmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Find the following or state that it cannot be determined from the information given.

a. The matrix A.

A cannot be determined from A_{rref}.

b. A basis for $\text{im}(A)$.

A basis for $\text{im}(A)$ cannot be determined from A_{rref}.

c. A basis for $\ker(A)$.

Since the kernel of a matrix is unaffected by row operations, a basis for $\ker(A)$ = $\ker(A_{rref})$ is $([2, -1, 0, 0, 0]^T, [3, 0, 4, 5, -1]^T)$ according to the Solution Algorithm.

d. The determinant of A.

Since A_{rref} has zero rows, A is not invertible and $\det(A) = 0$.

e. An eigenvalue of A.

Since $\det(A) = 0$, 0 is an eigenvalue of A.

f. An eigenvector of A.

Any vector in $\ker(A)$ is an eigenvector with eigenvalue 0.

g. All solutions to $A\vec{x} = [1, 1, 1, 1, 1]^T$ if $A[1, 2, 3, 4, 5]^T = [1, 1, 1, 1, 1]^T$.

$\vec{x} = [1, 2, 3, 4, 5]^T + \vec{k}$ where $\vec{k}$ is any vector in $\ker(A)$.

8. T is rotation by $120°$ about the line through the origin in $\mathbf{R}^3$ parallel to $\begin{bmatrix}1\\1\\1\end{bmatrix}$. It maps the x-axis onto the y-axis, the y-axis onto the z-axis, and the z-axis onto the x-axis. What is the standard matrix for T?

The information tells what happens to each of the standard basis vectors. Specifically, T maps the first standard basis vector to the second, the second to the third and the third to the first. These, then are the corresponding columns of the standard matrix for T which is $\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$.

9. Determine which of the following subsets are subspaces of $\mathbf{R}^3$. Find a basis for each subspace. Explain why any subsets are not subspaces.

$$U = \operatorname{span}\left(\begin{bmatrix}1\\2\\1\end{bmatrix},\begin{bmatrix}2\\4\\2\end{bmatrix},\begin{bmatrix}2\\1\\2\end{bmatrix}\right). \quad V = \left\{\begin{bmatrix}x\\y\\z\end{bmatrix} : x+y+z=0\right\}. \quad W = \left\{\begin{bmatrix}x\\y\\z\end{bmatrix} : xyz=0\right\}.$$

U is a subspace and a basis for it is $\left(\begin{bmatrix}1\\2\\1\end{bmatrix},\begin{bmatrix}2\\1\\2\end{bmatrix}\right)$; V is a subspace and a basis for it is $\left(\begin{bmatrix}1\\-1\\0\end{bmatrix},\begin{bmatrix}1\\0\\-1\end{bmatrix}\right)$; W is not a subspace since $\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix}\in W$ but $\begin{bmatrix}1\\0\\0\end{bmatrix}+\begin{bmatrix}0\\1\\1\end{bmatrix}=\begin{bmatrix}0\\1\\1\end{bmatrix}\notin W$.

10. A subspace H of $\mathbf{P}_2$ is spanned by the polynomials q_1 and q_2 where $q_1(x)=1+x$ and $q_2(x)=x-x^2$. $f\cdot g \equiv \int_0^1 f(x)g(x)\,dx$ defines an inner product on $\mathbf{P}_2$. Relative to this inner product, find an orthogonal basis for H.

An orthogonal basis for H is $(q_1, q_2{}^{\perp})$. Now, $q_2{}^{\perp} = q_2 - \left(\dfrac{q_1\cdot q_2}{q_1\cdot q_1}\right)q_1$. We compute

$$q_1\cdot q_2 = \int_0^1 (1+x)(x-x^2)\,dx = \int_0^1 (1-x^3)\,dx = \tfrac{3}{4}.$$

$$q_1\cdot q_1 = \int_0^1 (1+x)^2 dx = \int_0^1 (1+2x+x^2)\,dx = \tfrac{7}{3}.$$

So, $q_2^{\perp}(x) = q_2(x) - \left(\dfrac{q_1 \cdot q_2}{q_1 \cdot q_1}\right) q_1(x) = (x - x^2) - \tfrac{9}{28}(1+x) = -\tfrac{9}{28} + \tfrac{19}{28}x - x^2$.

11. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 3 & 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}$.

a. Determine $\vec{x}$ so that $\left\| A\vec{x} - \vec{b} \right\|$ is as small as possible.

The vector we seek satisfies the normal equation $A^T A\vec{x} = A^T \vec{b}$. It is given by $\vec{x} = (A^T A)^{-1} A^T \vec{b}$. Calculating, we have $(A^T A)^{-1} A^T = \tfrac{1}{14}\begin{bmatrix} -1 & -4 & 5 \\ 6 & 10 & -2 \end{bmatrix}$. So, $\vec{x} = \begin{bmatrix} \tfrac{3}{2} \\ 0 \end{bmatrix}$.

b. Determine the matrix P for projection onto $\mathrm{im}(A)$.

$$P = A(A^T A)^{-1} A^T = \tfrac{1}{14}\begin{bmatrix} 5 & 6 & 3 \\ 6 & 10 & -2 \\ 3 & -2 & 13 \end{bmatrix}.$$

12. $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Which is diagonalizable and why?

Since both matrices are upper triangular, we see that $\mathrm{spec}(A) = \mathrm{spec}(B) = (1, 1, 2)$. The eigenspaces are

$$E_1(A) = \ker(A - I) = \ker\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathrm{span}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix};\ \dim(E_1(A)) = 1.$$

$$E_1(B) = \ker(B - I) = \ker\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right);\ \dim(E_1(B)) = 2.$$

$$E_2(A) = \ker(A - 2I) = \ker\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \mathrm{span}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix};\ \dim(E_2(A)) = 1.$$

$$E_2(B) = \ker(B - 2I) = \ker\begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \mathrm{span}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix};\ \dim(E_2(B)) = 1.$$

So, A is not diagonalizable since its eigenvectors do not span $\mathbf{R}^3$ whereas B is diagonalizable since the list of eigenvectors $\left(\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix},\begin{bmatrix}1\\1\\0\end{bmatrix}\right)$ is basis for $\mathbf{R}^3$.

13. The equations $\begin{Bmatrix} w+y+z=0 \\ x+y+z=0 \end{Bmatrix}$ describe a subspace V of $\mathbf{R}^4$. Find a basis for $V^\perp$, the subspace of $\mathbf{R}^4$ that is orthogonal to V.

$$V^\perp = \left(\ker\begin{bmatrix}1&0&1&1\\0&1&1&1\end{bmatrix}\right)^T = \operatorname{im}\begin{bmatrix}1&0\\0&1\\1&1\\1&1\end{bmatrix} \Rightarrow \text{a basis for } V^\perp \text{ is } \left(\begin{bmatrix}1\\0\\1\\1\end{bmatrix},\begin{bmatrix}0\\1\\1\\1\end{bmatrix}\right).$$

14. A linear dynamical system with two variables, x_1 and x_2, each dependent on the time evolves according to $\begin{Bmatrix} 4x_1(k+1)=11x_1(k)-9x_2(k) \\ 4x_2(k+1)=3x_1(k)-x_2(k) \end{Bmatrix}$.

a. Determine the matrix A so that these equations may be written $\vec{x}(k+1)=A\vec{x}(k)$ and find the eigenvalues and eigenvectors of A <u>exactly</u> (no decimal approximations).

$$A=\tfrac{1}{4}\begin{bmatrix}11&-9\\3&-1\end{bmatrix},\quad \operatorname{spec}(A)=(\tfrac{1}{2},\,2),$$

$$E_{\frac{1}{2}}(A)=\operatorname{span}\begin{bmatrix}1\\1\end{bmatrix},\quad E_2(A)=\operatorname{span}\begin{bmatrix}3\\1\end{bmatrix}.$$

b. Solve this system for any k if $\vec{x}(0)=\begin{bmatrix}4\\2\end{bmatrix}$ and find $\lim_{k\to\infty}\left(x_1(k)/x_2(k)\right)$.

$$\vec{x}(k)=A^k\vec{x}(0)=A^k\begin{bmatrix}4\\2\end{bmatrix}=A^k\left(\begin{bmatrix}1\\1\end{bmatrix}+\begin{bmatrix}3\\1\end{bmatrix}\right)=\left(\tfrac{1}{2}\right)^k\begin{bmatrix}1\\1\end{bmatrix}+2^k\begin{bmatrix}3\\1\end{bmatrix} \quad \text{and}$$

$$\frac{x_1(k)}{x_2(k)}=\frac{\left(\frac{1}{2}\right)^k+3\cdot 2^k}{\left(\frac{1}{2}\right)^k+2^k}=\frac{\left(\frac{1}{4}\right)^k+3}{\left(\frac{1}{4}\right)^k+1}\to 3,\quad \text{as } k\to\infty.$$

15. $A = \left(\frac{1}{7}\begin{bmatrix} 2 & 6 & 3 \\ 3 & 2 & -6 \\ 6 & -3 & 2 \end{bmatrix} \right) \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\frac{1}{7}\begin{bmatrix} 2 & 6 & 3 \\ 3 & 2 & -6 \\ 6 & -3 & 2 \end{bmatrix} \right)^{-1}$.

a. Determine the eigenvalues and eigenspaces of A.

$\operatorname{spec}(A) = (-1, 0, 1)$, $E_{-1}(A) = \operatorname{span}\begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$, $E_0(A) = \operatorname{span}\begin{bmatrix} 6 \\ 2 \\ -3 \end{bmatrix}$, $E_1(A) = \operatorname{span}\begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix}$.

b. Determine $\left(\frac{1}{7}\begin{bmatrix} 2 & 6 & 3 \\ 3 & 2 & -6 \\ 6 & -3 & 2 \end{bmatrix} \right)^{-1}$ without calculator or row-reduction and explain how you arrived at your result.

Since the column vectors of $M = \frac{1}{7}\begin{bmatrix} 2 & 6 & 3 \\ 3 & 2 & -6 \\ 6 & -3 & 2 \end{bmatrix}$ are orthonormal, M is orthogonal and so, $M^{-1} = M^T = \frac{1}{7}\begin{bmatrix} 2 & 3 & 6 \\ 6 & 2 & -3 \\ 3 & -6 & 2 \end{bmatrix}$.

16. An entry of the matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 2 & 1 & 1 & 1 & \cdots & 1 \\ 3 & 3 & 1 & 1 & \cdots & 1 \\ 4 & 4 & 4 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ n & n & n & n & \cdots & 1 \end{bmatrix}$ is 1 if it is on or above the main diagonal; otherwise, it is the row of the entry. Find $\det(A)$ for any n.

$$\det(A) = \det\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 2 & 1 & 1 & 1 & \cdots & 1 \\ 3 & 3 & 1 & 1 & \cdots & 1 \\ 4 & 4 & 4 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ n & n & n & n & \cdots & 1 \end{bmatrix} = \det\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 2 & 2 & 0 & 0 & \cdots & 0 \\ 3 & 3 & 3 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ n-1 & n-1 & n-1 & n-1 & \cdots & 0 \end{bmatrix}$$

The last result was obtained by subtracting the top row from each of the succeeding ones. Now we can use the Laplace expansion down the last column. $\det(A) = (-1)^{n+1}\det(B)$. B is the matrix obtained from A, by removing its first row and its last column of A. B is lower triangular with the positive integers $1, 2, \ldots, n-1$ on the main diagonal. So, $\det(A) = (-1)^{n+1}(n-1)!$

17. Explain why all matrices representing orthogonal projections onto planes through the origin in $\mathbf{R}^3$ are similar to one another and to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Discuss the eigenvalues and eigenvectors of such projections and the coordinate transformations that connect them.

Let M be any plane through the origin in $\mathbf{R}^3$ and let N be the line through the origin normal to M. M and N are subspaces of dimension 2 and 1, respectively. Let P be the matrix for projection onto M. If $\vec{x} \in M$, $A\vec{x} = \vec{x}$ and if $\vec{x} \in N$, $A\vec{x} = \vec{0}$. This shows that $\mathrm{spec}(A) = (1, 1, 0)$, $E_1(A) = M$ and $E_0(A) = N$. Choose a basis $(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ for $\mathbf{R}^3$ where $\vec{v}_1$ and $\vec{v}_2$ are two non-collinear vectors in M and $\vec{v}_3$ is any non-zero vector in N. Relative to this basis, the matrix for orthogonal projection onto M is evidently $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and P is similar to this matrix with coordinate change matrix $S = [\vec{v}_1, \vec{v}_2, \vec{v}_3]$.