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Name

Problem	Value	Score
1	12	
2	10	
3	8	
4	8	
5	14	
6	10	
7	12	
8	10	
9	6	
10	6	
11	12	
12	18	
13	8	
14	10	
15	16	
16	10	
17	30	
Sum	200	

1. Determine all the ways in which 40 bills may be chosen from among bills with denominations of \$1, \$5 and \$10 so their total value is \$120.

2. Consider a 3×4 matrix A. Determine A_{rref} with as much specificity as possible if

a. no pair of the column vectors of A is collinear and all are coplanar.

b. no triplet of the column vectors of A is coplanar.

3. Solve $A\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 7 \\ 0 \end{bmatrix}$ completely if $A\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = A\begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \end{bmatrix} = A\begin{bmatrix} 4 \\ 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 7 \\ 0 \end{bmatrix}$ and rank(A) = 2.

4. a. How is im(A) related to the existence of solutions to the linear system $A\vec{x} = \vec{b}$?

b. How is ker(A) related to the uniqueness of solutions to the system $A\vec{x} = \vec{b}$?

5. a. What effect do elementary row operations have on linear relationships among the column vectors of a matrix?

b. What effect do elementary row operations have on the span of the row vectors of a matrix?

If $A \in \mathbf{R}^{6\times 6}$, $A_{rref} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & -2 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and $A^T{}_{rref} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 & -4 & 3 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$,

determine a basis for each of the following subspaces of $\mathbf{R}^6$.

c. $\operatorname{im}(A)$.

d. $\ker(A)$.

6. a. Determine if $\begin{bmatrix} 3 \\ 4 \\ 3 \\ 5 \end{bmatrix} \in \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 7 \\ 3 \\ 6 \end{bmatrix} \right)$.

b. Find a basis for and the dimension of the subspace of $\mathbf{R}^4$ consisting of all vectors of the form $\begin{bmatrix} 2a-3b \\ a+b \\ -2a \\ a+2b \end{bmatrix}$ where a and b are any reals.

7. Let $W = \text{span}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\1\\3\end{bmatrix}\right)$. Determine:

a. a basis for $W^{\perp}$.

b. an orthogonal basis for W.

c. the matrix P for projection onto W.

8. Let $\vec{s} = [2\ \ 1\ \ 3]^T$ and $\vec{t} = [2\ \ 2\ \ 4]^T$ and let $\vec{x}$ be any vector in $\mathbf{R}^3$. Define the transformation $f(\vec{x}) = \det\left[\vec{s} \mid \vec{x} \mid \vec{t}\,\right]$.

a. Using this definition, explain why f is a <u>*linear*</u> transformation from $\mathbf{R}^3$ to $\mathbf{R}^1$.

b. Determine the standard matrix for f.

9. Suppose $A \in \mathbf{R}^{10\times 10}$ and the diagonal entries of A are odd positive integers and the off-diagonal entries are even positive integers. Explain why A is invertible.

10. $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ are vectors in $\mathbf{R}^3$. Compute $\left|\det[\vec{v}_1 \,|\, \vec{v}_2 \,|\, \vec{v}_3]\right|$, if $\left\|\vec{v}_1\right\| = 3$, $\left\|\vec{v}_2^{\perp}\right\| = 2$ and $\left\|\vec{v}_3^{\perp}\right\| = 5$; where $\vec{v}_2^{\perp}$ is the projection of $\vec{v}_2$ orthogonal to $\vec{v}_1$; and $\vec{v}_3^{\perp}$ is the projection of $\vec{v}_3$ orthogonal to both $\vec{v}_1$ and $\vec{v}_2$.

11. $T : \mathbf{R}^2 \to \mathbf{R}^2$ is a linear transformation that doubles all vectors in $\mathbf{R}^2$ parallel to the vector $[3 \;\; 1]^T$ and reverses all vectors in $\mathbf{R}^2$ parallel to the vector $[1 \;\; 2]^T$.

a. Determine a basis $\mathfrak{B}$ for $\mathbf{R}^2$ relative to which the matrix for T is diagonal.

b. What is the $\mathfrak{B}$–matrix for T?

c. What is the standard matrix for T?

d. What is $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$?

12. Suppose that all the cadets at West Point have ice cream at evening meal every day. Washington Hall serves three flavors: chocolate, vanilla and strawberry. It is found that of cadets who choose chocolate one evening, 10% will choose vanilla and 10% will choose strawberry the next evening; of cadets who choose vanilla one evening, 20% will choose chocolate and 10% will choose strawberry the next evening; and of cadets who choose strawberry one evening, 30% will choose chocolate and 10% will choose vanilla the next evening. Let $x_1(k)$, $x_2(k)$ and $x_3(k)$ be the number of cadets who choose chocolate, vanilla and strawberry, respectively, for the kth evening. Here, $k = 0, 1, 2, ...$

a. Show that the circumstances described above may be summarized by a single vector equation of the form $\vec{x}(k+1) = A\vec{x}(k)$ where $\vec{x}(k) = [x_1(k)\ \ x_2(k)\ \ x_3(k)]^T$. Identify the matrix A.

b. Use the vector equation found in part a to express $\vec{x}(k)$ in terms of A and $\vec{x}(0)$, the initial vector of cadet flavor choices for any k.

c. Explain why 1 must be an eigenvalue of A as a consequence of the fact that the entries in each column of A must sum to 1. [Hint: Think A^T.]

d. Determine spec(A).

e. Find $\vec{x}(k)$ for any k, assuming $\vec{x}(0) = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$ for fixed constants c_1, c_2, and c_3 and fixed eigenvectors $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ corresponding to the distinct eigenvalues λ_1, λ_2 and $\lambda_3 \in \text{spec}(A)$.

f. Determine $E_1(A)$ and find the exact ratios of chocolate to vanilla to strawberry consumed each evening in Washington Hall as the number of days approach infinity.

13. What is the distance between the subspace $\operatorname{span}\left(\begin{bmatrix}1\\1\\-1\\1\end{bmatrix},\begin{bmatrix}1\\-1\\1\\1\end{bmatrix}\right)$ and $\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$?

14. $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d} \in \mathbf{R}^3$. Complete the table below. The entry in each cell is the scalar (or inner or dot) product of the vector to its left and the vector above it.

	$\vec{a}$	$\vec{b}$	$\vec{c}$	$\vec{d}$
$\vec{a}$	1	0		
$\vec{b}$		1	-2	
$\vec{c}$	3			5
$\vec{d}$	0	0		1

15. Let $A=\begin{bmatrix}1&1&0&0\\0&2&a&0\\0&0&1&1\\0&0&0&2\end{bmatrix}$ where a is a real number. Determine:

a. the characteristic polynomial for A.

b. $E_\lambda(A)$, the eigenspace of A for each λ in spec(A). Make clear the dependence on the value of a.

c. the values of a for which A diagonalizable and the diagonal matrices to which A is similar.

16. Two variables, s and t, are related by an equation of the form $s = x_1 t + x_2 t^2$ for a pair of constants x_1 and x_2. Find the best (least-squares) choices of x_1 and x_2 consistent with the following data.

s	t
-1	1
0	1
1	2
2	2

17. For **10** of the assertions below, state whether **True** or **False**. No reason is required. A correct response will earn full credit; no response will earn no credit; and an incorrect response will earn *negative* credit.

___ a. Any list of mutually orthogonal nonzero vectors in $\mathbf{R}^n$ is linearly independent.

___ b. If $\vec{u}$, $\vec{v}$ and $\vec{w} \in \mathbf{R}^5$, and $\vec{u}$ is a linear combination of $\vec{v}$ and $\vec{w}$, then $\vec{w}$ is a linear combination of $\vec{u}$ and $\vec{v}$.

___ c. There is a real 2×2 matrix A other than I so that $A^{13} = I$.

___ d. There is an invertible 10×10 matrix whose entries include 92 ones.

___ e. All rotations through a fixed angle θ around axes through the origin in $\mathbf{R}^3$ are similar.

___ f. The magnitudes of the entries of an orthogonal matrix never exceed 1.

___ g. Every invertible matrix is the product of an orthogonal matrix and an upper triangular matrix.

___ h. For any $m\times n$ matrix A, $\operatorname{im}(A^T) = (\ker(A))^{\perp}$.

___ i. If A is an invertible square matrix, $\ker(A) = \ker(A^{-1})$.

___ j. There are $n\times n$ matrices A and B such that $AB = 0$ but $BA \neq 0$.

___ k. If each of the column vectors of a square matrix A is a unit vector, $|\det(A)| \leq 1$.

___ l. If A is an invertible $n\times n$ matrix, at least one of its submatrices, obtained by deleting one row and one column of A, is also invertible.

___ m. If $\vec{v}$ and $\vec{w}$ are eigenvectors of a square matrix A, then $\vec{v} + \vec{w}$ is also an eigenvector of A.

___ n. If A is an invertible 3×3 matrix and B is a 3×4 matrix, then $\ker(AB) = \{\vec{0}\}$.

___ o. Suppose that T is a linear transformation on $\mathbf{R}^n$ and A is its standard matrix. Then, all n^2 entries of A are completely determined by the vectors $\left(T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_n)\right)$ where $\left(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\right)$ is any basis for $\mathbf{R}^n$.

___ p. If $A \in \mathbf{R}^{m\times n}$ and $B \in \mathbf{R}^{n\times p}$, then $\operatorname{im}(AB)$ is a subspace of $\operatorname{im}(A)$.