1. The trace of a square matrix is defined as the sum of it diagonal entries.

If A is $n\times n$ matrix with entries a_{ij}, trace(A) = $a_{11}+a_{22}+\ldots+a_{nn}=\sum_{i=1}^{n} a_{ii}$.

a. For any two $n\times n$ matrices A and B, demonstrate that trace(AB) = trace(BA). Begin with the expression for the *ij*-entry of the product matrix AB.

Let $A=[a_{ij}]$ and $B=[b_{ij}]$. Then, the *ij*-th entry of the product AB is $\sum_{k=1}^{n} a_{ik}b_{kj}$ and so, trace(AB) = $\sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}b_{ki}=\sum_{i=1}^{n}\sum_{k=1}^{n} b_{ki}a_{ik}=\sum_{k=1}^{n}\sum_{i=1}^{n} b_{ki}a_{ik}$ = trace(BA). The second equality interchanges the order of the two factors in the summand. In the next equality, the order of summation (over *i* and over *k*) is swapped to arrive at the result.

b. Show that the traces of two similar square matrices are equal.

A and B are similar if and only if there is an invertible $n\times n$ matrix S so $A=SBS^{-1}$, Thus, $\text{trace}(A)=\text{trace}(SBS^{-1})=\text{trace}(BS^{-1}S)=\text{trace}(B)$. In the next to last step, we used the result from part a.

2. A is an $n\times n$ matrix each of whose diagonal entries is a positive odd integer and each of whose off-diagonal entries is a positive even integer. Show that ker(A) consists of the zero vector only.

ker(A) is trivial if and only if det$A$) is nonzero. If the determinant of A is expanded as the sum of $n!$ terms each containing a product of n entries from distinct columns and rows, there will be one and only one term that is an odd integer. This is the product of all the diagonal entries. All other terms in this expansion are even integers since they will contain at least one positive even integer as a factor. The sum of one positive odd integer and other terms which are all even integers cannot be 0.

3. a. A is an $n\times n$ matrix with real entries. Explain why each eigenvector of A must belong to im(A) or ker(A).

If $\vec{v}$ is any eigenvector for A with nonzero eigenvalue λ, $A\vec{v}=\lambda\vec{v}$ and so, $A(\frac{1}{\lambda}\vec{v})=\vec{v}$. Therefore, $\vec{v}$ belongs to im(A). On the other hand, if $\vec{v}$ is an eigenvector for A with zero eigenvalue, $A\vec{v}=0\vec{v}=\vec{0}$ shows that $\vec{v}$ belongs to ker(A).

Now, suppose that A is the 2008×2008 matrix whose *ij*-th entry is $+1$ if $i+j$ is even and is 0 otherwise.

b. What are the eigenvectors of A that belong to im(A) and what are their eigenvalues?

Row-reducing A, we find the following.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \cdots & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \cdots & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \cdots & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \cdots & 1 \end{bmatrix}_{rref} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \cdots & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

The first two columns of A are linearly independent, span $\mathrm{im}(A)$, and each is an eigenvector of A with eigenvalue 1004.

c. What are the eigenvectors of A that belong to $\ker(A)$?

From the rref of A, above, we see that a basis for $\ker(A)$ is

$$\left(\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ -1 \end{bmatrix} \right)$$

d. What is the characteristic polynomial of A?

From a and b, we see that the spectrum of A consists of two eigenvalues, 0 and 1004. The first has algebraic multiplicity 2006 and the second has algebraic multiplicity 2. So, $f_A(\lambda) = \lambda^{2006}(\lambda - 1004)^2$

4. If A and B are $n\times n$ matrices, $A \sim B$ will be used to indicate that A is similar to B. Show that, for any $n\times n$ matrices, A, B, and C

a. $A \sim A$

Choose $S = I$. Then, $A = SAS^{-1}$. So, A is similar to itself.

b. $A \sim B$ implies $B \sim A$

$A \sim B$ implies that there is an invertible matrix S so that $A = SBS^{-1}$. So, $B = (S^{-1})A(S^{-1})^{-1}$ implies $B \sim A$.

c. If $A \sim B$ and $B \sim C$ then $A \sim C$.

$A \sim B$ and $B \sim C$ implies that there are invertible matrices S and T so that $A = SBS^{-1}$ and $B = TCT^{-1}$. Therefore, $A = STCT^{-1}S^{-1} = (ST)C(ST)^{-1}$. So, $A \sim C$.

5. Since a rotation in $\mathbf{R}^3$ is a proper orthogonal linear transformation, its matrix in standard coordinates is 3×3, orthogonal, and has determinant +1.

a. Verify that the composition of any two rotations on $\mathbf{R}^3$ is also a rotation on $\mathbf{R}^3$.

Suppose that A and B are any two rotation matrices. Then $(AB)^T = B^T A^T = B^{-1}A^{-1} = (AB)^{-1}$ shows that AB is orthogonal. But, also $\det(AB) = \det(A)\det(B) = (+1)(+1) = +1$.

Let's denote by $R(\hat{u},\theta)$ the matrix for rotation in $\mathbf{R}^3$ by the angle θ about the axis through the origin parallel to the unit vector $\hat{u}$.

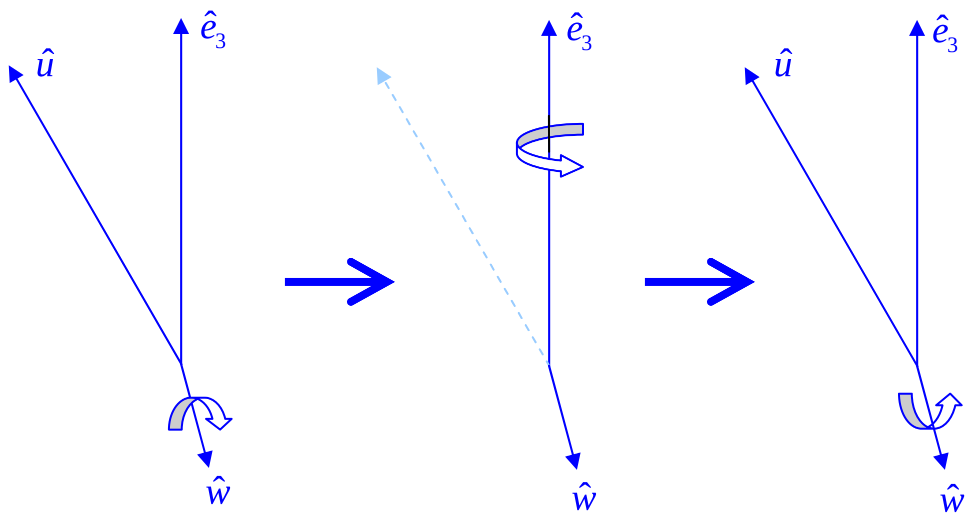
b. Compute $R(\hat{e}_1,\theta)$ and $R(\hat{e}_3,\theta)$.

$$R(\hat{e}_1,\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \text{ and } R(\hat{e}_3,\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Each column for each matrix is the image of the corresponding standard basis vector under the transformation.

c. Show that all rotations in $\mathbf{R}^3$ by the angle θ are similar to each other since they are all similar to $R(\hat{e}_3,\theta)$. See problem 4, above. Proceed by demonstrating that $R(\hat{u},\theta)$ is similar to $R(\hat{e}_3,\theta)$ by finding the unit vector $\hat{w}$ and the angle ϕ so that $R(\hat{u},\theta) = R(\hat{w},\phi)R(\hat{e}_3,\theta)\left(R(\hat{w},\phi)\right)^{-1}$. Express $\hat{w}$ and ϕ in terms of $\hat{u}$ and $\hat{e}_3$.

The angle between $\hat{u}$ and $\hat{e}_3$ is given by $\cos\phi = \hat{u}\cdot\hat{e}_3$. We let $\hat{w} = \hat{e}_3 \times \hat{u} / \sin\phi$. $R(\hat{w},\phi)$ is a rotation through ϕ about the axis normal to both $\hat{u}$ and $\hat{e}_3$. So, rotation about $\hat{u}$ through the angle θ is achieved in three steps as follows. First, we bring the vector $\hat{u}$ to coincidence with the z-axis. This is achieved by the rotation about $\hat{w}$ through the angle $-\phi$. Next, we rotate around the z-axis by the angle θ. Finally, the rotated vector $\hat{u}$ is returned to its original direction by the inverse of the first rotation. The product of the corresponding rotation matrices is the desired is $R(\hat{u},\theta) = R(\hat{w},\phi)R(\hat{e}_3,\theta)\left(R(\hat{w},\phi)\right)^{-1} = R(\hat{w},\phi)R(\hat{e}_3,\theta)R(\hat{w},-\phi)$. So, we see that every rotation by θ is similar to rotation by θ about the z-axis. Hence, all rotations by the same angle are similar. This construction should remind you of our computation of scaling along an arbitrary direction in the plane.



d. What is the trace of a rotation by the angle θ for any rotation axis in $\mathbf{R}^3$?

Since all rotation matrices for the same rotation angle are similar and the traces of any two similar square matrices are the same, we find that the trace of any rotation matrix with rotation angle θ is $\operatorname{trace}\big(R(\hat{e}_1,\theta)\big) = 1 + 2\cos\theta$.

e. Argue that every rotation in $\mathbf{R}^3$ has the eigenvalue $+1$.

If R is a rotation matrix for the rotation angle θ, it is similar $R(\hat{e}_1,\theta)$.

Then, $R - 1I$ is also similar to $R(\hat{e}_1,\theta) - 1I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \cos\theta - 1 & -\sin\theta \\ 0 & \sin\theta & \cos\theta - 1 \end{bmatrix}$.

But, $\det\big(R(\hat{e}_1,\theta) - 1I\big) = 0$. Hence, R has the eigenvalue $+1$. The corresponding eigenspace consists of the vectors parallel to the rotation axis.

f. According to part a above, $R(\hat{e}_1,\frac{\pi}{4})R(\hat{e}_3,\frac{\pi}{4})$ is the composite of two rotations and so it must be a rotation. Use the results above to determine the axis and angle of this composite rotation. That is, find $\hat{u}$ and θ so that $R(\hat{u},\theta) = R(\hat{e}_1,\frac{\pi}{4})R(\hat{e}_3,\frac{\pi}{4})$.

$$R(\hat{u},\theta) = R(\hat{e}_1,\tfrac{\pi}{4})R(\hat{e}_3,\tfrac{\pi}{4}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Since $\operatorname{trace}\big(R(\hat{u},\theta)\big) = 1 + 2\cos\theta = \frac{1}{2} + \sqrt{2}$, $\theta = \arccos\left(\frac{1}{\sqrt{2}} - \frac{1}{4}\right) \approx 1.0906$ $\approx 62.7994°$. The rotation axis belongs to $E_1\big(R(\hat{u},\theta)\big) = \ker\big(R(\hat{u},\theta) - 1I\big)$ =

$$\ker\begin{bmatrix} \frac{1}{\sqrt{2}}-1 & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2}-1 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}}-1 \end{bmatrix} = \ker\begin{bmatrix} \sqrt{2}-1 & 1 & 0 \\ 1 & -1 & -\sqrt{2} \\ 0 & 0 & 0 \end{bmatrix} = \ker\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & \sqrt{2}-1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \operatorname{span}\begin{bmatrix} 1 \\ 1-\sqrt{2} \\ 1 \end{bmatrix}. \text{ Therefore, } \hat{w} = \frac{1}{\sqrt{5-2\sqrt{2}}}\begin{bmatrix} 1 \\ 1-\sqrt{2} \\ 1 \end{bmatrix} \approx \begin{bmatrix} .678598 \\ -.281085 \\ .678598 \end{bmatrix}.$$

6. Let $A = \begin{bmatrix} -9 & 20 \\ -6 & 13 \end{bmatrix}$.

a. Determine the eigenvalues and corresponding eigespaces of A.

$$0 = \det(A-\lambda I) = (-9-\lambda)(13-\lambda) - (-6)(20) = \lambda^2 - 4\lambda + 3 = (\lambda-1)(\lambda-3)$$

So, $\operatorname{spec}(A) = (1, 3)$. $E_1(A) = \ker(A-1I) = \ker\begin{bmatrix} -10 & 20 \\ -6 & 12 \end{bmatrix} = \operatorname{span}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

$E_3(A) = \ker(A-3I) = \ker\begin{bmatrix} -12 & 20 \\ -6 & 9 \end{bmatrix} = \operatorname{span}\begin{bmatrix} 5 \\ 3 \end{bmatrix}$. A diagonalizer for A is

$S = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ with $S^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} = S^{-1}AS$ or $A = SBS^{-1}$.

b. Determine A^n for any nonnegative integer n. Express each entry of this matrix as a function of n.

$$A^n = (SBS^{-1})^n = SB^nS^{-1} = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3^n \end{bmatrix}\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 6-5\cdot 3^n & -10+10\cdot 3^n \\ 3-3\cdot 3^n & -5+6\cdot 3^n \end{bmatrix}$$

c. *Deleted.*

d. If p is any polynomial, what is $p(A)$? Express the entries of this matrix in terms of the values of p.

$$p(A) = S\,p(B)\,S^{-1} = S\,p(B)\,S^{-1} = S\begin{bmatrix} p(1) & 0 \\ 0 & p(3) \end{bmatrix}S^{-1}$$

$$= \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} p(1) & 0 \\ 0 & p(3) \end{bmatrix}\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 3p(1) & -5p(1) \\ -p(3) & 2p(3) \end{bmatrix}$$

$$= \begin{bmatrix} 6p(1)-5p(3) & -10p(1)+10p(3) \\ 3p(1)-3p(3) & -5p(1)+6p(3) \end{bmatrix}.$$

e. Now, calculate $\sin\left(\frac{\pi}{6}\begin{bmatrix} -9 & 20 \\ -6 & 13 \end{bmatrix}\right)$. In so doing, describe any assumptions that you are making.

$$\sin\left(\frac{\pi}{6}\begin{bmatrix}-9 & 20\\ -6 & 13\end{bmatrix}\right)=\sin\left(\frac{\pi}{6}A\right)=S\sin\left(\frac{\pi}{6}B\right)S^{-1}$$
$$=\begin{bmatrix}2 & 5\\ 1 & 3\end{bmatrix}\begin{bmatrix}\sin(\frac{\pi}{6}) & 0\\ 0 & \sin(\frac{\pi}{2})\end{bmatrix}\begin{bmatrix}3 & -5\\ -1 & 2\end{bmatrix}=\begin{bmatrix}2 & 5\\ 1 & 3\end{bmatrix}\begin{bmatrix}\frac{1}{2} & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}3 & -5\\ -1 & 2\end{bmatrix}$$
$$=\begin{bmatrix}2 & 5\\ 1 & 3\end{bmatrix}\begin{bmatrix}\frac{3}{2} & -\frac{5}{2}\\ -1 & 2\end{bmatrix}=\begin{bmatrix}-2 & 5\\ -\frac{3}{2} & \frac{7}{2}\end{bmatrix}.$$

Here we have treated the function $x\mapsto\sin(\frac{\pi}{6}x)$ as though it were a polynomial and could be applied to matrices as we had done in part d with polynomials. While a transcendental function such as sine has a Taylor series that looks like an "infinite" polynomial or power series, this ignores questions of the convergence of power series. However, it can be argued that this particular series does converge and the result can be justified.

7. An idealized ecosystem contains two species: prey and predator. The prey species has an inexhaustible supply of food and, in the absence of the predator species, its population would grow exponentially. The population of the predator species, on the other hand, would decrease exponentially in the absence of the prey on which it depends for its sustenance. Denoting the population (in millions) at any time t as $x_1(t)$ for the prey species and $x_2(t)$ for the predator species, we find the following relationships for the rates of change of the populations of the two species.

$$x_1'(t)=.26x_1(t)-.18x_2(t)$$
$$x_2'(t)=.12x_1(t)-.16x_2(t)$$

a. Use eigen-methods to determine the future populations of the two species, given initial populations of $x_1(0) = 2$, $x_2(0) = 3$.

Let $\vec{x}(t)=\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}$ and $A=\begin{bmatrix}.26 & -.18\\ .12 & -.16\end{bmatrix}$. Then, the equations above may be written as a single first-order linear homogeneous ordinary vector differential equation: $\vec{x}'(t)=A\vec{x}(t)$. The eigens of A are found first.

$$0=\det(A-\lambda I)=(.26-\lambda)(-.16-\lambda)-(.12)(-.18)=\lambda^2-.1\lambda-.02$$
$$=(\lambda-.2)(\lambda+.1).\ \text{So, } \operatorname{spec}(A)=(-.1,.2) \text{ gives us the eigenvalues.}$$

$$\mathrm{E}_{-.1}(A)=\ker(A+.1I)=\ker\begin{bmatrix}.36 & -.18\\ .12 & -.06\end{bmatrix}=\operatorname{span}(\vec{v}_1)\ \text{ where }\ \vec{v}_1=\begin{bmatrix}1\\ 2\end{bmatrix}.$$

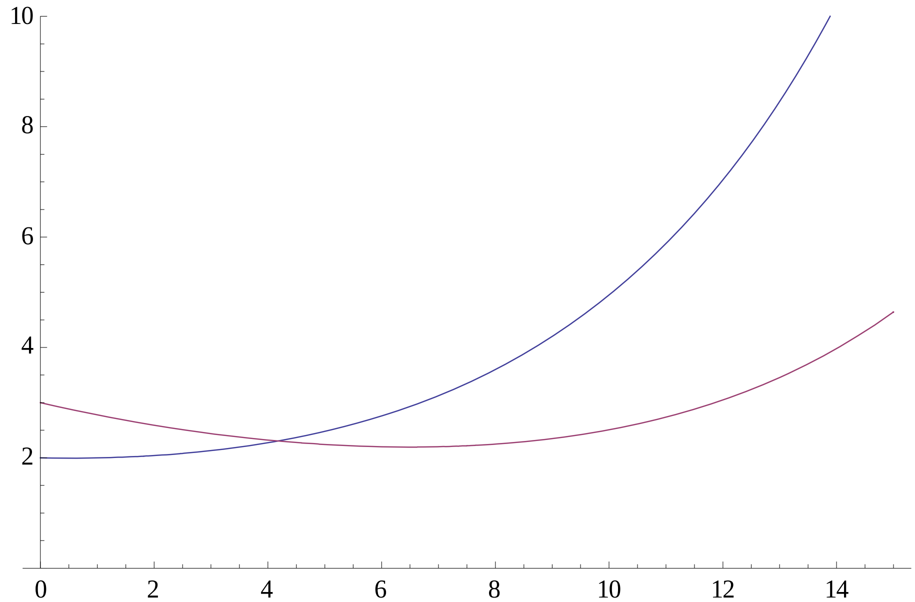
$$\mathrm{E}_{.2}(A)=\ker(A-.2I)=\ker\begin{bmatrix}.06 & -.18\\ .12 & -.36\end{bmatrix}=\operatorname{span}(\vec{v}_2)\ \text{ where }\ \vec{v}_2=\begin{bmatrix}3\\ 1\end{bmatrix}.$$

The solution to the DE above may be written in vector form as

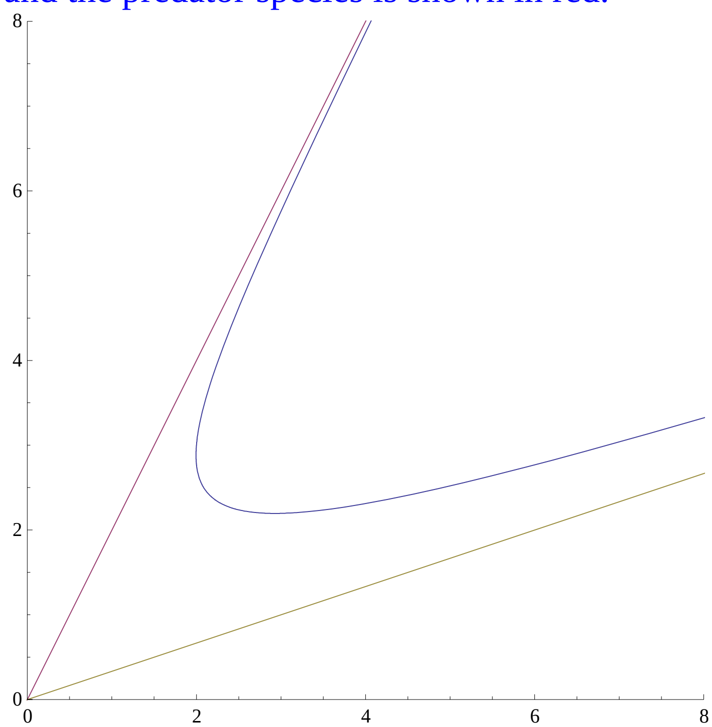
$$\vec{x}(t) = a_1 e^{-.1t}\,\vec{v}_1 + a_2 e^{.2t}\,\vec{v}_2.$$

From the initial conditions, we have $\vec{x}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = a_1\vec{v}_1 + a_2\vec{v}_2 =$ $\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$. Let $S = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$ and $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$. Then, $\vec{a} = S^{-1}\vec{x}(0) = \begin{bmatrix} 1.4 \\ .2 \end{bmatrix}$.

$$\vec{x}(t) = 1.4e^{-.1t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + .2e^{.2t}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.4e^{-.1t} + .6e^{.2t} \\ 2.8e^{-.1t} + .2e^{.2t} \end{bmatrix}.$$



The population, plotted as a function of the time t, of the prey species is shown in blue and the predator species is shown in red.



In the above, $x_2(t)$ is plotted against $x_1(t)$ starting at negative values of t on the upper portion of the curve and moving toward increasing positive values of t on the lower portion. As t increases, the curve is traversed counterclockwise. The straight line asymptotes shown here are, in fact, $E_{-.1}(A)$ and $E_{.2}(A)$.

b. Describe all initial conditions for which the ratio of $x_1(t)$ to $x_2(t)$ remains constant in time.

Only if $\vec{x}(0)$ belongs to one or the other eigenspace will the ratio of the two species remain constant in time. In other words, the ratio will remain constant if, initially, it is in the ratio of the components of an eigenvector, that is either 1:2 or 3:1. To see that this is the case, observe that the solution was found to be $\vec{x}(t) = a_1 e^{-.1t}\vec{v}_1 + a_2 e^{.2t}\vec{v}_2$. Consequently, $\dfrac{x_1(t)}{x_2(t)} = \dfrac{a_1 e^{-.1t}(1) + a_2 e^{.2t}(3)}{a_1 e^{-.1t}(2) + a_2 e^{.2t}(1)}$. But, this expression is constant if and only if either $a_1 = 0$ or $a_2 = 0$. Setting the derivative of the ratio equal to 0 yields the same result, namely $a_1 a_2 = 0$.

c. *Deleted.*

8. In class, on Monday, we will analyze an ideal oscillator consisting of three point masses in a line with adjacent masses connected by Hooke's law springs. Carry out the same analysis with the following modification. Add a third spring, connecting the two end masses, whose stiffness constant is one quarter either of the other two springs and such that this spring is relaxed when the other two masses are relaxed.

The coupled initial value problem to be solved is given by these DEs.

$$x_1'' = -(x_1 - x_2) - \tfrac{1}{4}(x_1 - x_3)$$
$$x_2'' = +(x_1 - x_2) - (x_2 - x_3)$$
$$x_3'' = +(x_2 - x_3) + \tfrac{1}{4}(x_1 - x_3)$$

with the following values specified: $x_1(0), x_2(0), x_3(0), x_1'(0), x_2'(0), x_3'(0)$. These three equations may be rewritten by moving all terms to the left side.

Now, we let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $A = \begin{bmatrix} \frac{5}{4} & -1 & -\frac{1}{4} \\ -1 & 2 & -1 \\ -\frac{1}{4} & -1 & \frac{5}{4} \end{bmatrix}$, whereupon we obtain

$$\vec{x}'' + A\vec{x} = \vec{0} \text{ given } \vec{x}(0) \text{ and } \vec{x}'(0).$$

To solve this homogeneous ODE, we first determine the eigens of A. Solving $0 = \det(A - \lambda I)$, we obtain $\operatorname{spec}(A) = (0, \frac{3}{2}, 3)$ and

$E_0(A) = \text{span}\begin{bmatrix}1\\1\\1\end{bmatrix}$, $E_{\frac{3}{2}}(A) = \text{span}\begin{bmatrix}1\\0\\-1\end{bmatrix}$, $E_3(A) = \text{span}\begin{bmatrix}1\\-2\\1\end{bmatrix}$. These are the same eigenspaces (but with a different middle eigenvalue) as we found in the original problem. Let us abbreviate $\vec{v}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}1\\0\\-1\end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix}1\\-2\\1\end{bmatrix}$ We can write the general solution in the form

$$\begin{aligned}\vec{x}(t) = &\left(a_1 + b_1 t\right)\vec{v}_1\\ &+\left(a_2 \cos\left(\sqrt{\tfrac{3}{2}}\,t\right) + b_2 \sin\left(\sqrt{\tfrac{3}{2}}\,t\right)\right)\vec{v}_2\\ &+\left(a_3 \cos\left(\sqrt{3}\,t\right) + b_3 \sin\left(\sqrt{3}\,t\right)\right)\vec{v}_3\end{aligned}$$

The six constants above are found by using the initial conditions and the orthogonality of the eigenvectors. We obtain

$$a_1 = \tfrac{1}{3}\vec{x}(0)\cdot\vec{v}_1,\ a_2 = \tfrac{1}{2}\vec{x}(0)\cdot\vec{v}_2,\ a_3 = \tfrac{1}{6}\vec{x}(0)\cdot\vec{v}_3,$$
$$b_1 = \tfrac{1}{3}\vec{x}\,'(0)\cdot\vec{v}_1,\ b_2 = \tfrac{1}{\sqrt{6}}\vec{x}\,'(0)\cdot\vec{v}_2,\ a_3 = \tfrac{1}{6}\tfrac{1}{\sqrt{3}}\vec{x}\,'(0)\cdot\vec{v}_3.$$