1. Let $A = \begin{bmatrix} 1 & 1 & 5 & 0 & 3 \\ 1 & 1 & 5 & 1 & 4 \\ 2 & 0 & 6 & 1 & 5 \\ 1 & 0 & 3 & 0 & 2 \end{bmatrix}$.

a. Find a basis for the image of A and find a basis for the kernel of A.

We find that $rref(A) = \begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. By identifying the pivot columns of A and using the Solution Algorithm we determine that

a basis for $\text{im}(A)$ is $(\vec{v}_1, \vec{v}_2, \vec{v}_3) = \left(\begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right)$, and

a basis for $\ker(A)$ is $(\vec{w}_1, \vec{w}_2) = \left(\begin{bmatrix} 3 \\ 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right)$,

b. Use your results in part a to describe all vectors $\vec{b}$ for which the linear equation $A\vec{x} = \vec{b}$ has a solution.

$\vec{b}$ must be in $\text{im}(A)$, so $\vec{b} = a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 \;\; \forall \;\; a_1, a_2, a_3 \in \mathbf{R}$..

c. If $\vec{x}_1$ and $\vec{x}_2$ are any solutions of the linear equation $A\vec{x} = \vec{b}$, use your results in part a to describe the vector $\vec{x}_1 - \vec{x}_2$.

$\vec{x}_1 - \vec{x}_2$ must belong to $\ker(A)$, so $\vec{x}_1 - \vec{x}_2 = c_1\vec{w}_1 + c_2\vec{w}_2 \;\; \forall \;\; c_1, c_2. \in \mathbf{R}$.

2. Let $P = \frac{1}{9}\begin{bmatrix} 5 & 4 & -2 \\ 4 & 5 & 2 \\ -2 & 2 & 8 \end{bmatrix}$ and let $W = \text{im}(P)$ and let $W^{\perp}$ be the subset of $\mathbf{R}^3$ consisting of all vectors in $\mathbf{R}^3$ orthogonal to all vectors in W.

a. What algebraic properties of P verify that P is a projection matrix?

$P^2 = P^T = P$.

b. Demonstrate that $W^{\perp}$ is a subspace of $\mathbf{R}^4$.

$W^{\perp}$ is the set of all vectors in $\mathbf{R}^4$ that are orthogonal to all the vectors in

W. We have to show that $W^\perp$ is nonempty and it is closed under vector addition and multiplication by scalars. $\vec{0}$ is in $W^\perp$ since $\vec{0}$ is orthogonal to every vector. Next, suppose that $\vec{y}_1$ and $\vec{y}_2$ are both in $W^\perp$. Then each is orthogonal to any $\vec{w}$ in W. Take any linear combination of $\vec{y}_1$ and $\vec{y}_2$, say $a_1\,\vec{y}_1 + a_2\,\vec{y}_2$. We obtain $\vec{w}\cdot(a_1\,\vec{y}_1 + a_2\,\vec{y}_2) = a_1\,\vec{w}\cdot\vec{y}_1 + a_2\,\vec{w}\cdot\vec{y}_2$ $= a_1\,0 + a_2\,0 = 0 + 0 = 0$. So, we have shown that $a_1\,\vec{y}_1 + a_2\,\vec{y}_2$ is also orthogonal to any vector in W.

c. Determine the matrix $P^\perp$ for projection onto the subspace $W^\perp$.

Projection onto $W^\perp$ is what remains after removing projection onto W, i.e.

$$P^\perp = I - P = \tfrac{1}{9}\begin{bmatrix} 4 & -4 & 2 \\ -4 & 4 & -2 \\ 2 & -2 & 1 \end{bmatrix}.$$

d. Calculate the smallest value for the expression $\|\vec{v}-\vec{w}\|$ if $\vec{v} = \begin{bmatrix} 9 \\ 9 \\ 9 \end{bmatrix}$ and $\vec{w}$ is any vector in W.

The expression is smallest when $\vec{w}$ is the projection of $\vec{v}$ onto W, i.e. for

$$\|\vec{v}-\vec{w}\| = \|\vec{v}-P\vec{v}\| = \|(I-P)\,\vec{v}\| = \|P^\perp\,\vec{v}\| = \left\|\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}\right\| = 3.$$

3. Consider the list $\mathcal{B} = (\vec{v}_1, \vec{v}_2, \vec{v}_3) = \left(\begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 11 \\ -1 \\ 5 \end{bmatrix}\right)$ and notice that the first two entries are orthogonal. $\mathcal{B}$ is a basis for $\mathbf{R}^3$.

a. Use the Gram-Schmidt process to determine, from $\mathcal{B}$, an orthonormal basis $\mathcal{C} = (\hat{u}_1, \hat{u}_2, \hat{u}_3)$ for $\mathbf{R}^3$.

The first two vectors in $\mathcal{C}$ are found by normalization and the third by the usual G-S process. We have $\hat{u}_1 = \vec{v}_1/\|\vec{v}_1\| = \tfrac{1}{7}\begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$ and $\hat{u}_2 = \vec{v}_2/\|\vec{v}_2\| = \tfrac{1}{7}\begin{bmatrix} 6 \\ 2 \\ -3 \end{bmatrix}$.

Then, $\vec{v}_3^{\,\perp} = \vec{v}_3 - (\vec{v}_3\cdot\hat{u}_1)\hat{u}_1 - (\vec{v}_3\cdot\hat{u}_2)\hat{u}_2 = \vec{v}_3 - 7\hat{u}_1 - 7\hat{u}_2 = \begin{bmatrix} 11 \\ -1 \\ 5 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} - \begin{bmatrix} 6 \\ 2 \\ -3 \end{bmatrix}$

$$= \begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix}. \text{ So, } \hat{u}_3 = \vec{v}_3^{\,\perp} / \left\| \vec{v}_3^{\,\perp} \right\| = \tfrac{1}{7} \begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix}.$$

b. Determine the coordinate vector $\begin{bmatrix} 7 \\ 7 \\ 7 \end{bmatrix}_{\mathcal{C}}$ for $\begin{bmatrix} 7 \\ 7 \\ 7 \end{bmatrix}$ by using the orthonormality of $\mathcal{C}$.

The coefficients c_1, c_2, and c_3 in $\vec{w} = \begin{bmatrix} 7 \\ 7 \\ 7 \end{bmatrix} = c_1\,\hat{u}_1 + c_2\,\hat{u}_2 + c_3\,\hat{u}_3$ are the components of the coordinate vector. By taking the inner product of both sides of this equation with $\hat{u}_k$ and using the orthonormality of $\mathcal{C}$, we find $c_k = \vec{w} \cdot \hat{u}_k$. So, $\begin{bmatrix} 7 \\ 7 \\ 7 \end{bmatrix}_{\mathcal{C}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 11 \\ 5 \\ -1 \end{bmatrix}$.

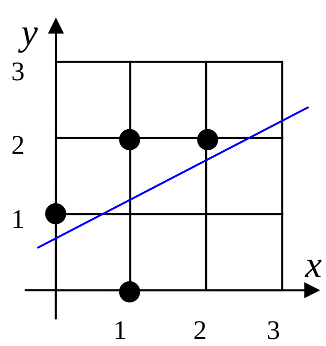
c. Let $Q = [\hat{u}_1 \,|\, \hat{u}_2 \,|\, \hat{u}_3]$. Suppose $\vec{w}_1$ and $\vec{w}_2$ are vectors in $\mathbf{R}^3$ with magnitudes 1 and 2, respectively, and the angle between $\vec{w}_1$ and $\vec{w}_2$ is $\pi/3$, calculate the angle between $Q\,\vec{w}_1$ and $Q\,\vec{w}_2$.

Q is the matrix for an orthogonal transformation and so, it preserves angles between vectors. Therefore, the angle between $Q\,\vec{w}_1$ and $Q\,\vec{w}_2$ is the same as that between $\vec{w}_1$ and $\vec{w}_2$, namely $\pi/3$.

4. Find the coefficients α and β for the equation

$$y = \alpha\, x + \beta$$

of the straight line that best fits, in the least squares sense, the data points at the right.



Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$. Then, we seek the solution to $A^T A\,\vec{x} = A^T\,\vec{b}$. Since the columns of A are linearly independent, we have

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \vec{x} = \left(A^T A\right)^{-1} A^T \vec{b} = \left(\begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 4 \\ 4 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ 5 \end{bmatrix} = \tfrac{1}{8} \begin{bmatrix} 4 & -4 \\ -4 & 6 \end{bmatrix} \begin{bmatrix} 6 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{3}{4} \end{bmatrix}.$$ The least squares line has the equation $y = \frac{1}{2}x + \frac{3}{4}$.

5. For each assertion below, state whether it is ***TRUE*** or ***FALSE***. You are not asked to provide a proof or explanation. However, each correct statement earns full credit and each incorrect statement loses half credit. No credit is assigned to a statement left blank.

a. $\|\vec{v} + \vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2$ for vectors in $\mathbf{R}^n$.

FALSE. The statement (Pythagorean Theorem) is true only when $\vec{v}$ and $\vec{w}$ are orthogonal. As a counterexample, choose nonzero vectors with $\vec{v} = -\vec{w}$.

b. Any list of n spanning vectors in $\mathbf{R}^n$ is linearly independent.

TRUE. n spanning vectors in $\mathbf{R}^n$ constitute a basis.

c. All matrices representing reflections across any lines through the origin in the plane are similar.

TRUE. Rotations change one reflection into any other.

d. The union of any two subspaces of $\mathbf{R}^n$ is always a subspace of $\mathbf{R}^n$.

FALSE. For example, the union of two different lines through the origin in $\mathbf{R}^2$ is not a subspace.

e. The square of an orthogonal matrix is also orthogonal.

TRUE. If Q is orthogonal, $(Q^2)^T(Q^2) = Q^T Q^T Q Q = Q^T I Q = I$.

f. If A and B are symmetric $n\times n$ matrices, AB is also symmetric.

FALSE. $(AB)^T = B^T A^T = BA \neq AB$ unless A and B commute.

g. For any matrix A, $(\ker(A))^\perp = \operatorname{im}(A^T)$.

TRUE. This follows from the identity $(\operatorname{im}(A))^\perp = \ker(A^T)$ by replacing A by A^T and taking the orthogonal complement of both sides.

h. If $\hat{v}$ and $\hat{w}$ are unit vectors in $\mathbf{R}^n$, $\hat{v}\,\hat{v}^T + \hat{w}\,\hat{w}^T$ is a projection matrix.

FALSE. This is true only if $\hat{v}$ and $\hat{w}$ are also orthogonal.