Show all work.

1. A linear system of four equations in five variables is equivalent to the single matrix equation $A\vec{x}=\vec{b}$ where $A=\begin{bmatrix}1&2&3&1&3\\0&1&2&1&1\\1&1&1&2&4\\2&2&2&2&6\end{bmatrix}$.

a. Determine A_{rref}.

$$A=\begin{bmatrix}1&2&3&1&3\\0&1&2&1&1\\1&1&1&2&4\\2&2&2&2&6\end{bmatrix}\leftrightarrow\begin{bmatrix}1&1&1&2&4\\0&1&2&1&1\\1&2&3&1&3\\2&2&2&2&6\end{bmatrix}\leftrightarrow$$

$$\begin{bmatrix}1&1&1&2&4\\0&1&2&1&1\\0&1&2&-1&-1\\0&0&0&-2&-2\end{bmatrix}\leftrightarrow\begin{bmatrix}1&0&-1&0&2\\0&1&2&0&0\\0&0&0&1&1\\0&0&0&0&0\end{bmatrix}=A_{rref}$$

b. Determine a basis for and the dimension of $\operatorname{im}(A)$.

From A_{rref}, the first, second and fourth columns of A are linearly independent. So, a basis for $\operatorname{im}(A)$ is $\left(\begin{bmatrix}1\\0\\1\\2\end{bmatrix},\begin{bmatrix}2\\1\\1\\2\end{bmatrix},\begin{bmatrix}1\\1\\2\\2\end{bmatrix}\right)$ and $\dim(\operatorname{im}(A))=3$.

c. Determine a basis for and the dimension of $\ker(A)$.

A_{rref} and the Solution Algorithm yield the basis $\left(\begin{bmatrix}1\\-2\\1\\0\\0\end{bmatrix},\begin{bmatrix}-2\\0\\0\\-1\\1\end{bmatrix}\right)$ for $\ker(A)$ and $\dim(\ker(A))=2$.

d. Completely discuss the nature of the solution set of $A\vec{x}=\vec{b}$ using the results of parts b and c.

Show all work.

There are no solutions if $\vec{b}$ is not in $\mathrm{im}(A)$. If $\vec{b}$ belongs to $\mathrm{im}(A)$ then there is a (doubly-) infinite set of solutions differing from one another by the vectors in $\ker(A)$. If $A\vec{x}_p = \vec{b}$ and $\vec{k} \in \ker(A)$, $\vec{x}_p + \vec{k}$ is also a solution.

2. $L = (\vec{v}_1, \dots, \vec{v}_p)$ is a list of p vectors in $\mathbf{R}^n$ and the $n \times p$ matrix $\left[\vec{v}_1 \mid \cdots \mid \vec{v}_p\right]_{rref}$ has r pivots (leading ones). Describe, in terms of n, p, and r when

a. L is linearly independent.

L is linearly independent iff each column is a pivot column, i.e. iff $r = p$.

b. L spans $\mathbf{R}^n$.

L spans $\mathbf{R}^n$ iff each row is a pivot row, i.e. iff $r = n$.

3. Solve for the 3×3 matrix X if $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 0 & 1 \end{bmatrix} X = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix}$.

This matrix equation has the form $A\,X = B$. By row-reduction, we find

$$[A \mid B] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 3 & 0 \\ 1 & 2 & 1 & 3 & 1 & 1 \\ 2 & 0 & 1 & 0 & 1 & 3 \end{array}\right] \leftrightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 3 & 0 \\ 0 & 1 & 0 & 2 & -2 & 1 \\ 0 & -2 & -1 & -2 & -5 & 3 \end{array}\right] \leftrightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & -1 & 5 & -1 \\ 0 & 1 & 0 & 2 & -2 & 1 \\ 0 & 0 & 1 & -2 & 9 & -5 \end{array}\right] \leftrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -4 & 4 \\ 0 & 1 & 0 & 2 & -2 & 1 \\ 0 & 0 & 1 & -2 & 9 & -5 \end{array}\right] = \left[I \mid A^{-1}B\right].$$

So, $X = \begin{bmatrix} 1 & -4 & 4 \\ 2 & -2 & 1 \\ -2 & 9 & -5 \end{bmatrix}$

4. Find values for the coefficients a and b so that the line with equation $a\,x + b\,y = 1$ is a best fit, in the least-squares sense, to the following (x,y)-data: (0,1), (1,0), (1,2), and (2,2).

Show all work.

We let $A=\begin{bmatrix}0&1\\1&0\\1&2\\2&2\end{bmatrix}$, $\vec{x}=\begin{bmatrix}a\\b\end{bmatrix}$, $\vec{b}=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$ and our problem amounts to finding the least squares solution to $A\vec{x}=\vec{b}$. This means that we seek the solution to the normalized equation $A^T A\vec{x}^*=A^T\vec{b}$ which is $\vec{x}^*=(A^TA)^{-1}A^T\vec{b}$ =

$$\begin{bmatrix}a^*\\b^*\end{bmatrix}=\left(\begin{bmatrix}0&1&1&2\\1&0&2&2\end{bmatrix}\begin{bmatrix}0&1\\1&0\\1&2\\2&2\end{bmatrix}\right)^{-1}\begin{bmatrix}0&1&1&2\\1&0&2&2\end{bmatrix}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

$$=\begin{bmatrix}6&6\\6&9\end{bmatrix}^{-1}\begin{bmatrix}4\\5\end{bmatrix}=\frac{1}{18}\begin{bmatrix}9&-6\\-6&6\end{bmatrix}\begin{bmatrix}4\\5\end{bmatrix}=\begin{bmatrix}5/3\\1/3\end{bmatrix}.$$

5. The four vectors $\vec{b}$, $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ belong to $\mathbf{R}^3$. Solve the matrix equation $[\vec{v}_1 \mid \vec{v}_2 \mid \vec{v}_3]\,\vec{x} = \vec{b}$ for $\vec{x}$, if $\det\left([\vec{v}_1 \mid \vec{v}_2 \mid \vec{v}_3]\right)=8$, $\det\left(\left[\vec{b} \mid \vec{v}_2 \mid \vec{v}_3\right]\right)=19$, $\det\left(\left[\vec{v}_1 \mid \vec{b} \mid \vec{v}_3\right]\right)=-38$, and $\det\left(\left[\vec{v}_1 \mid \vec{v}_2 \mid \vec{b}\right]\right)=19$.

By a straight-forward application of Cramer's Rule, we have

$$\vec{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}\det\left(\left[\vec{b} \mid \vec{v}_2 \mid \vec{v}_3\right]\right)\Big/\det\left([\vec{v}_1 \mid \vec{v}_2 \mid \vec{v}_3]\right)\\ \det\left(\left[\vec{v}_1 \mid \vec{b} \mid \vec{v}_3\right]\right)\Big/\det\left([\vec{v}_1 \mid \vec{v}_2 \mid \vec{v}_3]\right)\\ \det\left(\left[\vec{v}_1 \mid \vec{v}_2 \mid \vec{b}\right]\right)\Big/\det\left([\vec{v}_1 \mid \vec{v}_2 \mid \vec{v}_3]\right)\end{bmatrix}=\begin{bmatrix}19/8\\-38/8\\19/8\end{bmatrix}=\frac{19}{8}\begin{bmatrix}1\\-2\\1\end{bmatrix}.$$

6. Evaluate $\det\left(\begin{bmatrix}1&2&3&4&\cdots&n\\2&2&3&4&\cdots&n\\3&3&3&4&\cdots&n\\4&4&4&4&\cdots&n\\\vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\n&n&n&n&\cdots&n\end{bmatrix}\right)$ for any positive integer n.

Subtracting the first row from each of the succeeding rows, we obtain

Show all work.

$$\det\left(\begin{bmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 2 & 2 & 3 & 4 & \cdots & n \\ 3 & 3 & 3 & 4 & \cdots & n \\ 4 & 4 & 4 & 4 & \cdots & n \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ n & n & n & n & \cdots & n \end{bmatrix}\right) = \det\left(\left[\begin{array}{ccccc:c} 1 & 2 & 3 & 4 & \cdots & n \\ \hdashline 1 & 0 & 0 & 0 & \cdots & 0 \\ 2 & 1 & 0 & 0 & \cdots & 0 \\ 3 & 2 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ n-1 & n-2 & n-3 & n-4 & \cdots & 0 \end{array}\right]\right)$$

In the Laplace expansion along the last column, only one term is nonzero and its value is $(-1)^{1+n} n \det(L)$ where the $(n-1)\times(n-1)$ matrix L is lower triangular and its diagonal entries are all 1's. Therefore $\det(L) = 1$. The value of our determinant is, therefore, $(-1)^{1+n} n$.

7. An inner product for $\mathbf{P}_2$, the vector space of polynomials of degree 2 or less, is defined by $f \cdot g = \int_{-1}^{1} f(t)g(t)\,dt$ for any two polynomials f and g in $\mathbf{P}_2$. If $h(t) = 1+t$ and $k(t) = 1+t^2$, find $\ell(t)$, where ℓ is the projection of h orthogonal to k.

We require $\ell(t) = h(t) - \left(h \cdot \frac{k}{\|k\|}\right)\frac{k(t)}{\|k\|} = h(t) - \left(\frac{h\cdot k}{\|k\|^2}\right)k(t)$ and so, we must calculate $h\cdot k = \int_{-1}^{1} h(t)k(t)dt = \int_{-1}^{1}(1+t)(1+t^2)dt = \int_{-1}^{1}(1+t+t^2+t^3)dt = \frac{8}{3}$ and $\|k\|^2 = k\cdot k = \int_{-1}^{1}(k(t))^2\,dt = \int_{-1}^{1}(1+t^2)^2dt = \int_{-1}^{1}(1+2t^2+t^4)dt = \frac{56}{15}$. So,

$$\ell(t) = (1+t) - \left(\frac{8/3}{56/15}\right)(1+t^2) = (1+t) - \frac{5}{7}(1+t^2) = \frac{2}{7} + t - \frac{5}{7}t^2.$$

8. $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & \sqrt{6}/6 & 0 \\ 0 & 0 & 1 \\ \sqrt{2}/2 & -\sqrt{6}/6 & 0 \\ 0 & -\sqrt{6}/3 & 0 \end{bmatrix}\begin{bmatrix} \sqrt{2} & \sqrt{2}/2 & \sqrt{2} \\ 0 & \sqrt{6}/6 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ provides the QR-factorization of the 4×3 matrix A into the product of the 4×3 matrix Q and the 3×3 matrix R.

a. How are the column vectors of Q obtained from those of A?

Show all work.

The column vectors of Q are obtained by applying the Gram-Schmidt orthogonalization process to the column vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ of A. These three vectors are linearly independent and so, non-coplanar.

b. What is the geometrical significance of the diagonal entries of R?

In order, these entries are $\|\vec{v}_1\|, \|\vec{v}_2{}^{\perp}\|, \|\vec{v}_3{}^{\perp}\|$. $\vec{v}_2{}^{\perp}$ is the projection of $\vec{v}_2$ orthogonal to the subspace spanned by $\vec{v}_1$ and $\vec{v}_2$ and $\vec{v}_3{}^{\perp}$ is the projection of $\vec{v}_3$ orthogonal to the subspace spanned by $\vec{v}_1$ and $\vec{v}_2$. The magnitudes are the length, width, and height, respectively, of the 3-parallelopiped in $\mathbf{R}^4$ whose concurrent edges are $\vec{v}_1, \vec{v}_2, \vec{v}_3$. Their product, $\det(R)$, gives the 3-volume of the 3-parallelopiped.

9. Determine expressions for each entry of the matrix $\begin{bmatrix} -1 & 2 \\ -4 & 5 \end{bmatrix}^n$ for any positive integer n.

We find the eigens of the matrix $A = \begin{bmatrix} -1 & 2 \\ -4 & 5 \end{bmatrix}$. The characteristic equation is $0 = \det(A - \lambda I) = (-1-\lambda)(5-\lambda) - (-4)(2) = \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3)$. Hence $\text{spec}(A) = \{1, 3\}$. The eigenvectors are found by solving determining the kernels of $A - \lambda I$ for each eigenvalue. We obtain the eigenspaces $E_1(A) = \ker(A - I) = \ker\left(\begin{bmatrix} -2 & 2 \\ -4 & 4 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ and $E_3(A) = \ker(A - 3I) = \ker\left(\begin{bmatrix} -4 & 2 \\ -4 & 2 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$. Therefore, $Q = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ is a diagonalizer for A and we have $Q^{-1} A Q = D$ where $D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$. So,

$$\begin{bmatrix} -1 & 2 \\ -4 & 5 \end{bmatrix}^n = Q D^n Q^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}^n \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3^n \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -3^n & 3^n \end{bmatrix} = \begin{bmatrix} 2 - 3^n & -1 + 3^n \\ 2 - 2 \cdot 3^n & -1 + 2 \cdot 3^n \end{bmatrix}.$$

Show all work.

10. Let $S = \operatorname{span}\left(\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}\right)$. $T : \mathbf{R}^3 \to \mathbf{R}^3$ is the linear transformation that leaves unchanged each vector in S and doubles each vector in $S^\perp$. Let A be the 3×3 matrix so that $T(\vec{x}) = A\,\vec{x}$ for any $\vec{x}$ in $\mathbf{R}^3$.

a. Determine A.

Let $\hat{u} = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$ be a unit vector in S. The matrix for orthogonal projection onto S is $P_S = \hat{u}\,\hat{u}^T = \frac{1}{9}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 4 & -4 \\ -2 & -4 & 4 \end{bmatrix}$ and the projection onto $S^\perp$ is $P_{S^\perp} = I - P_S$. So, $A = P_S + 2P_{S^\perp} = P_S + 2(I - P_S) = 2I - P_S = \frac{1}{9}\begin{bmatrix} 17 & -2 & 2 \\ -2 & 14 & 4 \\ 2 & 4 & 14 \end{bmatrix}$.

b. What are the eigenvalues of A and their algebraic multiplicities?

The eigenvalues of A are 1 with algebraic multiplicity 1 and 2 with algebraic multiplicity 2. Recall that algebraic multiplicity is at least as large as the geometric multiplicity. See part c.

c. Describe the eigenspaces of A in terms of S and $S^\perp$.

$E_1(A) = S$ and $E_2(A) = S^\perp$. Clearly, $\dim(S) = 1$ and $\dim(S^\perp) = 2$. So, the geometric multiplicity is 1 for the eigenvalue 1 and 2 for the eigenvalue 2.

d. Determine all diagonal matrices similar to A.

The matrices similar to A are $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

11. The matrix $Q = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$ represents a rotation in $\mathbf{R}^3$ because it is orthogonal and has determinant +1. Therefore, the linear transformation $\vec{x} \mapsto Q\,\vec{x}$ preserves the length of any vector, the angle of any pair of vectors, and the orientation of any triplet of vectors. Now, a rotation is completely

characterized by its axis and its angle and, conversely, given a rotation, its axis and angle are determined.

a. Determine the rotation axis for Q by finding the vectors parallel to the rotation axis. [Hint: Solve the equation that describes the effect of the rotation on a vector parallel to the axis of rotation.]

If $\vec{x}$ is parallel to the rotation axis, $Q\vec{x} = \vec{x}$, that is, the vector is unaffected by the rotation. This means the $\vec{x}$ belongs to $\ker(Q - 1I)$ =

$$\ker\left(\frac{1}{3}\begin{bmatrix} -2 & 2 & -2 \\ 2 & -2 & 2 \\ 2 & -2 & -4 \end{bmatrix}\right) = \ker\left(\begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}\right) = \ker\left(\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right)$$

b. Determine the rotation angle for Q. [Hint: This is the angle between any vector orthogonal to the rotation axis and that vector's image under the rotation.]

A unit vector orthogonal to the rotation axis is, evidently, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Of course, there are infinitely many others, but this one will do. The rotation angle is, therefore, $\arccos\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot Q\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \arccos\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \frac{1}{3}\begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}\right) = \arccos\left(-\frac{1}{3}\right)$.

12. List as many distinct statements as you can that are equivalent to

The $n \times n$ matrix A is non-singular.

A is invertible. There is an $n \times n$ matrix A^{-1} so that $A A^{-1} = A^{-1} A = I_n$.

The matrix equation $A\vec{x} = \vec{b}$ has a unique solution for any $\vec{b}$ in $\mathbf{R}^n$.

A is row (column) equivalent to the identity.

The rows (columns) of A are linearly independent in $\mathbf{R}^n$.

The rows (columns) of A span $\mathbf{R}^n$.

The rows (columns) of A comprise a basis for $\mathbf{R}^n$.

$\det(A) \neq 0$.

0 is not an eigenvalue of A.

$\operatorname{rank}(A) = n$.

$\operatorname{im}(A) = \mathbf{R}^n$.

$\operatorname{row}(A) = \mathbf{R}^n$.

$\ker(A) = \{\vec{0}\}$.