Three.VI Projection

Linear Algebra
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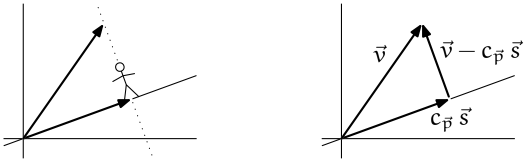
http://joshua.smcvt.edu/linearalgebra

Orthogonal Projection Into a Line

Project a vector into a line

This shows a figure walking out on the line to a point $\vec{p}$ such that the tip of $\vec{v}$ is directly above them, where "above" does not mean parallel to the y-axis but instead means orthogonal to the line.



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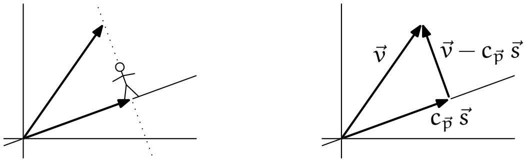

Since the line is the span of some vector $\ell = \{c \cdot \vec{s} \mid c \in \mathbb{R}\}$, we have a coefficient $c_{\vec{p}}$ with the property that $\vec{v} - c_{\vec{p}}\vec{s}$ is orthogonal to $c_{\vec{p}}\vec{s}$.

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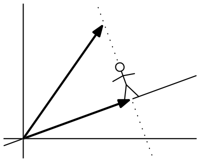

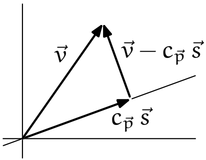

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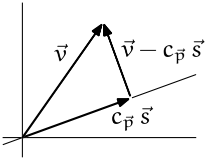
To solve for this coefficient, observe that because $\vec{v} - c_{\vec{p}}\vec{s}$ is orthogonal to a scalar multiple of $\vec{s}$, it must be orthogonal to $\vec{s}$ itself. Then $(\vec{v} - c_{\vec{p}}\vec{s}) \bullet \vec{s} = 0$ gives that $c_{\vec{p}} = \vec{v} \bullet \vec{s} / \vec{s} \bullet \vec{s}$.

We have decomposed $\vec{v}$ into two parts $\vec{v} = (c_{\vec{p}}\vec{s}) + (v - c_{\vec{p}}\vec{s})$.



Intuitively, some of $\vec{v}$ lies with the line and that gives the first part $c_{\vec{p}}\vec{s}$. The part of $\vec{v}$ that lies with a line orthogonal to ℓ is $\vec{v} - c_{\vec{p}}\vec{s}$. What's compelling about pairing these two parts is that they don't interact, in that the projection of one into the line spanned by the other is the zero vector.

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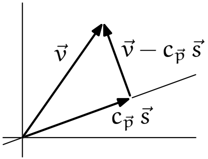

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Note: We have not given a definition of 'angle' in spaces other than $\mathbb{R}^n$'s, so we will stick here to those spaces. Extending the definitions to other spaces is perfectly possible but we don't need them here.

1.1 *Definition* The *orthogonal projection of $\vec{v}$ into the line spanned by a nonzero $\vec{s}$* is this vector.

$$\mathrm{proj}_{[\vec{s}\,]}(\vec{v}) = \frac{\vec{v} \bullet \vec{s}}{\vec{s} \bullet \vec{s}} \cdot \vec{s}$$

Example The projection of this $\mathbb{R}^3$ vector into the line

$$\vec{v} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \qquad L = \{c \cdot \vec{s} = c \cdot \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \mid c \in \mathbb{R}\}$$

is this vector.

$$\frac{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}} \cdot \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/6 \\ 1/6 \end{pmatrix}$$