Final Examination

SM261
Matrix Theory

5 May 2009

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name

Problem	Score
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Choose any <u>10</u> of the following 12 problems.

1. List all the combinations of 32 bills chosen from the denominations \$1, \$5, and \$10 that have a total worth of \$100.

2. a. Given a list $L = (\vec{v}_1, \ldots, \vec{v}_p)$ of p vectors in $\mathbf{R}^n$, describe a procedure, involving row-reduction, to unambiguously determine if L is linearly independent.

b. Given a vector $\vec{w}$ in $\mathbf{R}^n$ and the list $L = (\vec{v}_1, \ldots, \vec{v}_p)$ of p vectors in $\mathbf{R}^n$, describe a procedure, involving row-reduction, to unambiguously determine if $\vec{w} \in \operatorname{span}(L)$.

c. Given a list $L = (\vec{v}_1, \ldots, \vec{v}_n)$ of n vectors in $\mathbf{R}^n$, describe how to unambiguously determine if L is a basis for $\mathbf{R}^n$.

3. $L = (\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3) = \left(\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} \right)$ and $\vec{w} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$.

a. Determine an orthonormal basis $M = (\hat{u}_1,\ \hat{u}_2,\ \hat{u}_3)$ for span(L).

b. Compute the 4×4 matrix P for projection onto span(L).

c. Find the vector $\vec{w}*$ in span(L) for which $\| \vec{w} - \vec{w}* \|$ is smallest.

4. Imagine positive and negative electrically charged particles stored in three capacitors C_1, C_2, and C_3. The capacitors are linked by superconducting circuitry that has no resistance to the flow of electricity. The net charge throughout the system remains constant. Every millisecond, the particles are allowed to flow between the capacitors in the following way. 80% of the net charge that was in C_1 remains there, while 10% flows to C_2 and 10% flows to C_3; C_2 retains 60% of its net charge while the remaining 40% flows to C_3; and no net charge flows from C_3 to either of the other capacitors. Let $x_k(t)$ be the total charge in C_k after t milliseconds where $k = 1, 2, \text{or } 3$.

a. Formulate a single vector equation that describes the relationships between the net charges in the capacitors after t milliseconds and at 0 seconds.

b. Solve the equation in part a, if the initial charges in the capacitors C_1, C_2, and C_3 are 1, 0, and 0 coulombs, respectively.

c. For which initial charges on the capacitors will their proportions remain the same for all time?

5. A is a 6×6 matrix such that

$$A_{rref}=\begin{bmatrix}1&0&1&0&1&-2\\0&1&0&0&-1&1\\0&0&0&1&1&1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix},\quad A^{T}{}_{rref}=\begin{bmatrix}1&0&0&1&1&1\\0&1&0&-1&-4&3\\0&0&1&0&1&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix},\ \text{and}\ A\begin{bmatrix}0\\0\\0\\0\\1\\0\end{bmatrix}=\begin{bmatrix}2\\0\\3\\2\\5\\2\end{bmatrix}.$$

Determine a basis for each of the following subspaces of $\mathbf{R}^6$.

a. $\mathrm{im}(A)$.

b. $\ker(A)$.

Find all solutions for each of the following equations.

c. $A\vec{x}=\begin{bmatrix}2\\0\\3\\2\\5\\2\end{bmatrix}$.

d. $A\vec{x}=\begin{bmatrix}1\\1\\1\\0\\0\\0\end{bmatrix}$

6. Suppose that $(\vec{v}_1, \ldots, \vec{v}_p)$ is a basis for the subspace V in $\mathbf{R}^n$ and $\vec{w}$ belongs to V. Then, there is list $C = (c_1, \ldots, c_p)$ of scalar coefficients such that $\vec{w} = c_1\vec{v}_1 + \ldots + c_p\vec{v}_p$.

a. Explain why this is so.

b. Prove that this list C of coefficients is unique.

c. Now suppose that L is orthogonal, but the vectors are not necessarily normalized. Obtain a formula for c_k, $k = 1, 2, \ldots, p$.

7. V is the subspace of $\mathbf{R}^3$ described by the single equation $x+y+z=0$ and W is the subspace of $\mathbf{R}^3$ described by the pair of equations $x-y+2z=0$ and $x+y+3z=0$.

a. Determine bases for V and for W.

Joining the two lists in part a gives us a basis $\mathfrak{B}=(\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3)$ for $\mathbf{R}^3$.

b. What is the connection between any $\vec{x}\in\mathbf{R}^3$ and its $\mathfrak{B}$-coordinate representative $[\vec{x}]_{\mathfrak{B}}$?

Now, suppose that $T:\mathbf{R}^3\to\mathbf{R}^3$ is the linear transformation that doubles all vectors in V and reverses all vectors in W.

c. Determine B, the matrix for T in $\mathfrak{B}$-coordinates.

d. Compute the matrix A for T in standard coordinates.

e. Determine the list of all eigenvalues, $\operatorname{spec}(A)$, for A.

f. What is the eigenspace $E_\lambda(A)$ for each eigenvalue $\lambda\in\operatorname{spec}(A)$?

8. a. The 3×3 matrix S has column vectors $\vec{c}_1,\ \vec{c}_2,\ \hat{c}_3$. That is, $A=\left[\vec{c}_1 \mid \vec{c}_2 \mid \vec{c}_3\right]$. We find that $\det(S) = 7$. What is the value of $\det\left(\left[\vec{c}_1 + 2\vec{c}_3 \mid 3\vec{c}_1 \mid -4\vec{c}_2\right]\right)$?

The ijth entry of the 5×5 matrix A is a_{ij}. Recall that the ijth minor of A is the 4×4 matrix obtained from A by deleting its ith row and jth column. Let b_{ij} be the determinant of the ijth minor of A. Given that $\det(A) = 3$, determine the numerical values of the following expressions.

b. $a_{15}b_{13} - a_{25}b_{23} + a_{35}b_{33} - a_{45}b_{43} + a_{55}b_{53}$

c. $a_{41}b_{41} - a_{42}b_{42} + a_{43}b_{43} - a_{44}b_{44} + a_{45}b_{45}$

9. For any real matrix A, $\ker(A^T A) = \ker(A)$. Assertions that provide a proof of this theorem follow. For each of the six assertions labeled a–f below, provide a brief justification or rationale.

Assume $\vec{x} \in \ker(A)$.

a. It follows that $\vec{x} \in \ker(A^T A)$ and so, $\ker(A) \subseteq \ker(A^T A)$.

Now, assume $\vec{x} \in \ker(A^T A)$.

b. Then, $A\,\vec{x} \in \left(\operatorname{im}(A)\right)^{\perp}$.

c. $A\,\vec{x} \in \operatorname{im}(A) \cap \left(\operatorname{im}(A)\right)^{\perp}$.

d. So, $A\,\vec{x} = \vec{0}$.

e. It follows then that $\ker(A^T A) \subseteq \ker(A)$.

Therefore,

f. $\ker(A^T A) = \ker(A)$.

10. Completely solve the following coupled initial-value problem for $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ using eigenvalue-eigenvector methods. $\frac{d}{dt}\vec{x}(t) + \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}\vec{x}(t) = \vec{0}$; $\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

11. Two variables, s and t are to related by an equation of the form $s = a\,t + b\,t^2$ for a pair of constants a and b. Find the best (in the least-squares sense) choices of a and b consistent with the following data.

s	t
-1	1
0	1
1	2
2	2

12. $\mathbf{R}^{3\times 3}$ is the 9-dimensional vector space consisting of all real 3×3 matrices wherein vector addition is defined to be the ordinary addition of matrices and multiplication by scalars is the familiar multiplication of a matrix by a scalar. Consider the subset V of $\mathbf{R}^{3\times 3}$ consisting of the symmetric 3×3 matrices whose trace is 0.

a. Explain why V is a subspace of $\mathbf{R}^{3\times 3}$.

b. Find a basis for and the dimension of V.

Let S be the matrix for any rotation in $\mathbf{R}^3$ and consider the transformation $f:\mathbf{R}^{3\times 3}\to\mathbf{R}^{3\times 3}$ where $f(A)=S\,A\,S^T$. It is easy to see that f is linear.

c. Explain why $f(A)\in V$ for any $A\in V$. In other words, show that the image, under f, of any traceless symmetric 3×3 matrix is itself a traceless symmetric 3×3 matrix.