Special Problem 2 Due 18 February 2011

Theorem. If A and B are two row-equivalent matrices, i.e. each is obtainable from the other by a sequence of elementary row operations, then

- linear relations among the columns of A are the same among the corresponding columns of B
- the span of the rows of A is the span of the rows of B

Prove these two statements in the case of 3×3 matrices for three row operations: (1) interchange of the first and third rows, (2) multiplication of the second row by $s \neq 0$, and (3) replacement of the third row by t times the second row plus the third row. Altogether, you are proving six results. While not a proof of the general statement, all the ingredients for a more general proof are present in this restricted example. Denote the rows and columns of the two matrices as follows.

$$A = \begin{bmatrix} \vec{s}_1 \\ \hline \vec{s}_2 \\ \hline \vec{s}_3 \end{bmatrix} = \begin{bmatrix} \vec{t}_1 \mid \vec{t}_2 \mid \vec{t}_3 \end{bmatrix} \text{ and } B = \begin{bmatrix} \vec{u}_1 \\ \hline \vec{u}_2 \\ \hline \vec{u}_3 \end{bmatrix} = \begin{bmatrix} \vec{v}_1 \mid \vec{v}_2 \mid \vec{v}_3 \end{bmatrix}.$$

Assume that the columns of A have the following linear relationship.

$$c_1 \vec{t}_1 + c_2 \vec{t}_2 + c_3 \vec{t}_3 = \vec{0} \quad \text{where } c_1,\ c_2 \text{ and } c_3 \text{ are reals.}$$

Note that the jth row or the kth column of either matrix is a vector.
Six proofs are required.

Provide two specific examples to show each of the following two statements is false. If A and B are two row-equivalent matrices, i.e. each is obtainable from the other by a sequence of elementary row operations, then

- linear relations among the rows of A are the same among the corresponding rows of B
- the span of the columns of A is the span of the columns of B