1. $\left\{ \hat{u}_1 = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix},\ \hat{u}_2 = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix},\ \hat{u}_3 = \frac{1}{3}\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \right\}$ is an orthonormal basis for $\mathbf{R}^3$.

a. Find the coefficients $c_1$, $c_2$, and $c_3$ if $\begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix} = c_1\hat{u}_1 + c_2\hat{u}_2 + c_3\hat{u}_3$.

Since $c_k = \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix} \cdot \hat{u}_k$, we find $c_1 = 9$, $c_2 = 0$, and $c_3 = 3$.

b. If $\vec{w} = d_1\hat{u}_1 + d_2\hat{u}_2 + d_3\hat{u}_3$, prove that $\|\vec{w}\|^2 = d_1^{\,2} + d_2^{\,2} + d_3^{\,2}$.

$$\|\vec{w}\|^2 = \vec{w}\cdot\vec{w} = \left(d_1\hat{u}_1 + d_2\hat{u}_2 + d_3\hat{u}_3\right)\cdot\left(d_1\hat{u}_1 + d_2\hat{u}_2 + d_3\hat{u}_3\right)$$
$$= d_1^{\,2}(\hat{u}_1\cdot\hat{u}_1) + d_1d_2(\hat{u}_1\cdot\hat{u}_2) + d_1d_3(\hat{u}_1\cdot\hat{u}_3) + d_2d_1(\hat{u}_2\cdot\hat{u}_1) + d_2^{\,2}(\hat{u}_2\cdot\hat{u}_2) +$$
$$d_2d_3(\hat{u}_2\cdot\hat{u}_3) + d_3d_1(\hat{u}_3\cdot\hat{u}_1) + d_3d_2(\hat{u}_3\cdot\hat{u}_2) + d_3^{\,2}(\hat{u}_3\cdot\hat{u}_3).$$

In view of the fact that $\hat{u}_j\cdot\hat{u}_k = \delta_{jk}$, all the "cross" terms vanish and we are left with $\|\vec{w}\|^2 = d_1^{\,2} + d_2^{\,2} + d_3^{\,2}$.

c. Define the $3\times3$ matrix $Q = \left[\hat{u}_1 \,|\, \hat{u}_2 \,|\, \hat{u}_3\right]$ and compute its inverse $Q^{-1}$.

$Q$ is an orthogonal matrix and it follows immediately that $Q^{-1} = Q^T =$

$$\frac{1}{3}\begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & 2 \\ 2 & -2 & 1 \end{bmatrix}^T = \frac{1}{3}\begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & -2 \\ -2 & 2 & 1 \end{bmatrix}.$$

2. The subspace $V$ of $\mathbf{R}^3$ is spanned by $\vec{v}_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

a. Compute $P_V$, the $3\times3$ matrix that represents projection onto $V$.

Let $A = [\vec{v}_1 \,|\, \vec{v}_2] = \begin{bmatrix} 0 & 1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix}$. Since the columns of $A$ are linearly independent, we have the formula

$$P_V = A\left(A^T A\right)^{-1} A^T = \begin{bmatrix} 0 & 1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix} \left( \begin{bmatrix} 0 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ -2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix} \tfrac{1}{2} \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$= \tfrac{1}{2} \begin{bmatrix} 0 & 1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ 2 & 0 & 0 \end{bmatrix} = \tfrac{1}{2} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

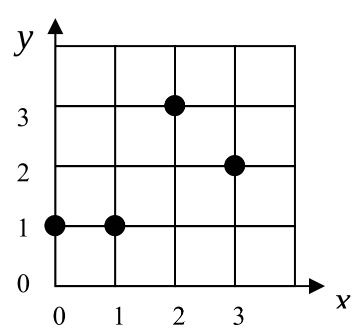
b. Find the vector $\vec{v}$ in $V$ that is closest to $\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$; i.e. find $\vec{v}$ so that $\| \vec{w} - \vec{v} \| \le \| \vec{w} - \vec{x} \|$ for all $\vec{x}$ in $V$.

The desired vector is the projection of $\vec{w}$ onto $V$, i.e. $\vec{v} = P_V \, \vec{w} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

c. Find the area of the parallelogram in $V$ two of whose concurrent edges are $\vec{v}_1$ and $\vec{v}_2$.

The area is $\sqrt{\det\left(A^T A\right)} = \sqrt{\det \begin{bmatrix} 2 & -2 \\ -2 & 3 \end{bmatrix}} = \sqrt{2}$

3. Find the function described by $y = f(x) = a + bx$ whose straight line graph best fits, in the least squares sense, the 4 data points below.



The data pairs are (0,1), (1,1), (2,3), and (3,2). If there were a straight line to fit this data, the matrix equation $\begin{bmatrix}1 & 1\\ 1 & 1\\ 1 & 3\\ 1 & 2\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix}=\begin{bmatrix}0\\ 1\\ 2\\ 3\end{bmatrix}$ would have a solution.

It does not because the vector on the right is not in the image of the 4×2 matrix on the left. However, a least squares approximation is available.

$$\begin{bmatrix}a\\ b\end{bmatrix}=\left(\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & 1 & 3 & 2\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & 1\\ 1 & 3\\ 1 & 2\end{bmatrix}\right)^{-1}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & 1 & 3 & 2\end{bmatrix}\begin{bmatrix}0\\ 1\\ 2\\ 3\end{bmatrix}$$

$$=\begin{bmatrix}4 & 7\\ 7 & 15\end{bmatrix}^{-1}\begin{bmatrix}6\\ 13\end{bmatrix}=\tfrac{1}{11}\begin{bmatrix}15 & -7\\ -7 & 4\end{bmatrix}^{-1}\begin{bmatrix}6\\ 13\end{bmatrix}=\tfrac{1}{11}\begin{bmatrix}-1\\ 10\end{bmatrix}.$$

The straight line that fits the data "best" has the equation $y=-\frac{1}{11}+\frac{10}{11}x$.

4. Let $A=\begin{bmatrix}1 & 0 & 1\\ 1 & 1 & 5\\ 0 & 1 & 1\end{bmatrix}$ and $\vec{b}=\begin{bmatrix}0\\ 1\\ 1\end{bmatrix}$.

a. Use Cramer's Theorem to compute $A^{-1}$, the inverse of $A$.

The cofactor matrix is $C=\begin{bmatrix}+(-4) & -(1) & +(1)\\ -(-1) & +(1) & -(1)\\ +(-1) & -(4) & +(1)\end{bmatrix}=\begin{bmatrix}-4 & -1 & 1\\ 1 & 1 & -1\\ -1 & -4 & 1\end{bmatrix}$ and

$\det(A)=\det\begin{bmatrix}1 & 0 & 1\\ 1 & 1 & 5\\ 0 & 1 & 1\end{bmatrix}=-3$. So, $A^{-1}=C^T/\det(A)=\frac{1}{3}\begin{bmatrix}4 & -1 & 1\\ 1 & -1 & 4\\ -1 & 1 & -1\end{bmatrix}$.

b. Use the result above to solve $A\vec{x}=\vec{b}$.

$$\vec{x}=A^{-1}\vec{b}=\tfrac{1}{3}\begin{bmatrix}4 & -1 & 1\\ 1 & -1 & 4\\ -1 & 1 & -1\end{bmatrix}\begin{bmatrix}0\\ 1\\ 1\end{bmatrix}=\begin{bmatrix}0\\ 3\\ 0\end{bmatrix}.$$

5. Find the eigens for the matrix $A=\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$.

First, we find the eigenvalues: $0=\det(A-\lambda I)=\det\begin{bmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{bmatrix}$

$=(1-\lambda)[(2-\lambda)(1-\lambda)-1]+1[-(1-\lambda)]=(1-\lambda)[(2-\lambda)(1-\lambda)-2]$

$=(1-\lambda)[\lambda^2-3\lambda]=-\lambda(\lambda-1)(\lambda-3) \quad \Rightarrow \quad \text{spec}(A)=(0,1,3).$

The corresponding eigenvectors span $\ker\begin{bmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{bmatrix}$. By row-reduction, we find:

$$\lambda=0: \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad E_0(A)=\text{span}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\lambda=1: \begin{bmatrix} 0 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{bmatrix} \leftrightarrow \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad E_1(A)=\text{span}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\lambda=3: \begin{bmatrix} -2 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -2 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad E_3(A)=\text{span}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

6. For each of the following, state whether the assertion is True or False.

a. $\text{im}(A^T) = \ker(A)$ for any matrix $A$.

False. See part c, below.

b. $(\text{im}(A))^\perp = \ker(A)$ for any matrix $A$.

False. See part c, below.

c. $\text{im}(A^T) = (\ker(A))^\perp$ for any matrix $A$.

True. Since $[\text{im}(A)]^\perp = \ker(A^T)$ is true for any matrix $A$ replace, $A$ by its transpose and take the orthogonal complement of both sides.

d. $\text{im}(A) \cap (\text{im}(A))^\perp = \{\vec{0}\}$ for any matrix $A$.

True. $V \cap V^\perp = \{\vec{0}\}$ for any subspace $V$.

e. $\ker(A^T A) = \ker(A)$ for any matrix $A$.
True. This was a theorem proven in class and in the text.

f. If $P_V$ is the matrix representing orthogonal projection onto the subspace $V$ of $\mathbf{R}^n$ and $\vec{x}$ is any vector in $\mathbf{R}^n$, then $\vec{x} - P_V\,\vec{x}$ belongs to $V^\perp$.
True. $P_V(\vec{x} - P_V\,\vec{x}) = P_V\vec{x} - P_V{}^2\,\vec{x} = P_V\vec{x} - P_V\,\vec{x} = \vec{0}$

g. Orthogonal transformations preserve the scalar product of vectors.
True. It was shown that length preserving transformations, i.e. orthogonal transformations preserve length, angle, and scalar product.

h. If the column vectors of $A$ are linearly dependent, $\dim(\ker(A)) > 0$.
True. There is a nontrivial linear combination of the column vectors that sums to the zero vector. The coefficients of this linear combination are the components of a nonzero vector in $\ker(A)$.

i. If the only vector orthogonal to all the row vectors of $A$ is the zero vector, then the column vectors of $A$ are linearly independent.
True. This asserts that the only linear combination of the column vectors that sum to zero is the trivial linear combination.

j. An orthogonal projection onto a proper subspace of $\mathbf{R}^n$ is represented by an orthogonal matrix.
False. Projections do not preserve length. In fact, they annihilate vectors orthogonal to the subspace onto which they project.

k. If $A$ and $B$ are symmetric $n \times n$ matrices then so is their product $AB$.
False. $(A\,B)^T = B^T A^T = B\,A \neq A\,B$ unless $A$ and $B$ commute.

l. If $\vec{x}$ is orthogonal to each row of a matrix $A$, then $\vec{x}$ is in $\ker(A)$.
True. This asserts that $A\vec{x} = \vec{0}$.

m. If $A$ is a $7 \times 7$ matrix every entry of which is $7$, then $\det(A) = 7^7$.
False. The rows (columns) of $A$ are identical and so $\det(A) = 0$.

n. If $A$ is a $3 \times 3$ matrix with entries $a_{i\,j}$ and corresponding cofactors $c_{i\,j}$, then $a_{11}\,c_{11} + a_{22}\,c_{22} + a_{33}\,c_{33} = \det(\mathrm{A})$.
False. This is a sum along the diagonal.

o. If $A$ is a $3 \times 3$ matrix with entries $a_{i\,j}$ and corresponding cofactors $c_{i\,j}$, then $a_{31}\,c_{21} + a_{32}\,c_{22} + a_{33}\,c_{23} = \det(\mathrm{A})$.
False. This sums entries from the third column with cofactors from the second column. So, the result is always $0$.

p. If $A$ is a square matrix, $\det(AA^T) = [\det(A)]^2$.
True. $\det(AA^T) = \det(A)\det(A^T) = \det(A)\det(A)$.

q. If $A$ and $B$ are both $2009 \times 2009$ matrices, then $\det(3A - 4B) = 3\det(A) - 4\det(B)$
False. Determinants are not linear functions of their arguments; they are linear functions of any fixed row or column.