1. a. A is a 2009×2009 matrix each of whose 2009 entries along the cross diagonal (bottom left to top right) are positive integers none of which is evenly divisible by 3. All other entries in the matrix are positive integer multiples of 3. Prove that A is an invertible matrix.

We will show that A is invertible because its determinant cannot be zero. In the expansion of $\det(A)$ by patterns and inversions, exactly one pattern includes only the cross-diagonal entries and the product of these entries is not divisible by 3. On the other hand all other patterns include at least one (to be more accurate, at least two) non-cross-diagonal entries. Therefore, the 2009! – 1 other terms in the determinant expansion are each divisible by 3 and sum to an integer multiple of 3. The sum of these terms plus the cross-diagonal term, yields $\det(A) \neq 0$.

b. Find the 3-volume of the paralleliped in $\mathbf{R}^4$ three of whose concurrent edges are given by the vectors $\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$, $\begin{bmatrix}1\\0\\0\\1\end{bmatrix}$, and $\begin{bmatrix}1\\1\\0\\1\end{bmatrix}$.

The volume is given by the formula $V = \sqrt{\det(A^T A)}$ where A is the 4×3 matrix whose columns are the vectors above. We have $A^T A = \begin{bmatrix}1&1&1&1\\1&0&0&1\\1&1&0&1\end{bmatrix}\begin{bmatrix}1&1&1\\1&0&1\\1&0&0\\1&1&1\end{bmatrix} = \begin{bmatrix}4&2&3\\2&2&2\\3&2&3\end{bmatrix}$.

So, $V = \sqrt{\det\begin{bmatrix}4&2&3\\2&2&2\\3&2&3\end{bmatrix}} = \sqrt{2\det\begin{bmatrix}4&2&3\\1&1&1\\3&2&3\end{bmatrix}} = \sqrt{2\det\begin{bmatrix}2&0&1\\1&1&1\\1&0&1\end{bmatrix}} = \sqrt{2}$.

2. $A \in \mathbf{R}^{4\times 4}$ and $A\begin{bmatrix}0\\1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\2\\2\\2\end{bmatrix}$, $A\begin{bmatrix}1\\0\\1\\1\end{bmatrix} = \begin{bmatrix}3\\0\\3\\3\end{bmatrix}$, $A\begin{bmatrix}1\\1\\0\\1\end{bmatrix} = \begin{bmatrix}2\\2\\0\\2\end{bmatrix}$, $A\begin{bmatrix}1\\1\\1\\0\end{bmatrix} = \begin{bmatrix}1\\1\\1\\0\end{bmatrix}$.

a. What is the characteristic polynomial of A?

Label the vectors on the left side of the four equations $\vec{r}, \vec{s}, \vec{t}, \vec{u}$, respectively. This is a linearly independent quartet. We are given that $A\vec{r} = 2\vec{r}, A\vec{s} = 3\vec{s}, A\vec{t} = 2\vec{t}, A\vec{u} = \vec{u}$. Consequently, $\vec{r}, \vec{s}, \vec{t}, \vec{u}$ are eigenvectors and spec(A) = (2, 3, 2, 1). So, the characteristic polynomial is f_A where $f_A(\lambda) = (\lambda-1)(\lambda-2)^2(\lambda-3)$.

b. For each real $\lambda \in \operatorname{spec}(A)$, provide the corresponding eigenspace $E_\lambda(A)$.

By inspection of the information provided,

$$E_1(A) = \text{span}(\vec{u}) = \text{span}\left(\begin{bmatrix}1\\1\\1\\0\end{bmatrix}\right), \quad E_2(A) = \text{span}(\vec{r}, \vec{t}) = \text{span}\left(\begin{bmatrix}0\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\\1\end{bmatrix}\right),$$

$$E_3(A) = \text{span}(\vec{s}) = \text{span}\left(\begin{bmatrix}1\\0\\1\\1\end{bmatrix}\right).$$

c. Provide Q and $F \in \mathbf{R}^{4\times4}$ so that Q is invertible, F is diagonal, and $F = Q^{-1}AQ$.

$F = \begin{bmatrix}1&0&0&0\\0&2&0&0\\0&0&2&0\\0&0&0&3\end{bmatrix}$ and $Q = \left[\vec{u} \,|\, \vec{r} \,|\, \vec{t} \,|\, \vec{s}\right] = \begin{bmatrix}1&0&1&1\\1&1&1&0\\1&1&0&1\\1&1&1&1\end{bmatrix}$. There are other possibilities that result from reordering the eigenvalues along the diagonal of F and reordering their corresponding eigenvectors that are the columns of Q.

3. Assuming $A \in \mathbf{R}^{n\times n}$ has only real eigenvalues and A is diagonalizable, explain why:

a. the sum of the eigenvalues of A is $\text{tr}(A)$. Each eigenvalue appears as many times in the sum as its algebraic multiplicity.

Since A is diagonalizable, it is similar to a diagonal matrix whose diagonal entries are its eigenvalues repeated according to their algebraic multiplicity. In this case, the sum of the algebraic multiplicities is n. The trace of A and the trace of this diagonal matrix are the same since the trace is the same for all similar matrices.

b. the product of the eigenvalues of A is $\det(A)$. Each eigenvalue appears as many times in the product as its algebraic multiplicity.

Since A is diagonalizable, it is similar to a diagonal matrix whose diagonal entries are its eigenvalues repeated according to their algebraic multiplicity. In this case, the sum of the algebraic multiplicities is n. The determinant of A and the determinant of this diagonal matrix are the same since the determinant is the same for all similar matrices.

4. A is a real square matrix.

a. Carefully explain why every eigenvector of A must belong to $\text{im}(A)$ or $\ker(A)$. Recall that eigenvectors are, by definition, nonzero.

If λ is a nonzero eigenvalue for A with eigenvector $\vec{v}$, then $A\vec{v} = \lambda\vec{v}$ and $A(\frac{1}{\lambda}\vec{v}) = \vec{v}$ shows that $\vec{v}$ belongs to the image of A. On the other hand if 0 is a eigenvalue of A with eigenvector $\vec{v}$, then $A\vec{v} = 0\vec{v} = \vec{0}$ shows that $\vec{v}$ belongs to the kernel of A

b. Of course, one may not infer from part a that every vector in $\operatorname{im}(A)$ is an eigenvector. Provide a specific example of a real 2×2 matrix such that ***no*** vector in $\operatorname{im}(A)$ is an eigenvector of A.

$A = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ represents rotation by $\pi/4$. Its image is $\mathbf{R}^2$ but it has no eigenvectors. .

Now consider the following 8×8 matrix B and its row-reduced echelon form.

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}. \qquad B_{rref} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

c. Employing the results above, determine all the eigenvalues and their corresponding eigenspaces and the characteristic polynomial of B.

From B_{rref}, we deduce that the first and third columns of B comprise a basis for $\operatorname{im}(B)$. Moreover, these are each eigenvectors with eigenvalue 4 as matrix multiplication reveals. So, this pair of vectors is also a basis for $E_4(A)$. The nonpivot columns of B_{rref} and the Solution Algorithm reveal a basis for $\ker(A) = E_0(A)$. In summary, we have the following.

$\operatorname{spec}(A) = (0,0,0,0,0,0,4,4)$; $\quad f_A(\lambda) = \lambda^6(\lambda-4)^2$;

$$E_0(A) = \operatorname{span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}\right); \quad E_4(A) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}\right).$$

5. Find an expression for each of the four entries of the matrix $\begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}^n$ in terms of the positive integer n.

Let A be the matrix whose nth power we are computing. The eigenvalues are given by the characteristic equation $0 = \det(A - \lambda I) = (1-\lambda)(4-\lambda) - (-1)(2) = \lambda^2 - 5\lambda + 5 = (\lambda-2)(\lambda-3)$. So, $\operatorname{spec}(A) = (2, 3)$. The corresponding eigenspaces are

$E_2(A) = \ker(A - 2I) = \ker\begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} = \operatorname{span}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$;

$E_4(A) = \ker(A - 3I) = \ker\begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix} = \operatorname{span}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Now, let $S = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$. Then, $S^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$. Also, let $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. Then, $D = S^{-1} A S$ or $A = S D S^{-1}$. Consequently,

$$A^n = (S D S^{-1})^n = S D^n S^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 2^n & 0 \\ 0 & 3^n \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 2^{n+1} & 2^n \\ -3^n & -3^n \end{bmatrix}$$

$$= \begin{bmatrix} 2^{n+1} - 3^n & 2^n - 3^n \\ -2^{n+1} + 2\cdot 3^n & -2^n + 2\cdot 3^n \end{bmatrix}.$$

6. Let $A = \frac{1}{7}\begin{bmatrix} 6 & 2 & -3 \\ 2 & 3 & 6 \\ 3 & -6 & 2 \end{bmatrix}$. This matrix represents a rotation in $\mathbf{R}^3$ as one can easily check by verifying that $A^T A = I$ and $\det(A) = +1$. Determine the axis and the angle of for this rotation.

The rotation angle is $\theta = \arccos\left(\frac{1}{2}(\operatorname{tr}(A) - 1)\right) = \arccos(\frac{1}{2}(\frac{11}{7} - 1)) = \arccos(\frac{2}{7}) \doteq 1.281$ radians $\doteq 73.40°$. Vectors lying along the rotation axis are left unchanged by the rotation and so are eigenvectors with eigenvalue 1. That is, vectors along the rotation axis belong to $E_1(A)$

$$= \ker(A - I) = \ker\tfrac{1}{7}\begin{bmatrix} -1 & 2 & -3 \\ 2 & -4 & 6 \\ 3 & -6 & -5 \end{bmatrix} = \ker\begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \operatorname{span}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$