# Three.II Homomorphisms

*Linear Algebra*

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# Definition

## Homomorphism

1.1 *Definition* A function between vector spaces $h\colon V \to W$ that preserves addition

$$\text{if } \vec{v}_1, \vec{v}_2 \in V \text{ then } h(\vec{v}_1 + \vec{v}_2) = h(\vec{v}_1) + h(\vec{v}_2)$$

and scalar multiplication

$$\text{if } \vec{v} \in V \text{ and } r \in \mathbb{R} \text{ then } h(r \cdot \vec{v}) = r \cdot h(\vec{v})$$

is a *homomorphism* or *linear map*.

*Example* Of these two maps $h, g\colon \mathbb{R}^2 \to \mathbb{R}$, the first is a homomorphism while the second is not.

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The map $h$ respects addition

$$\begin{aligned} h(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}) = h(\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}) &= 2(x_1 + x_2) - 3(y_1 + y_2) \\ &= (2x_1 - 3y_1) + (2x_2 - 3y_2) = h(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}) + h(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}) \end{aligned}$$

and scalar multiplication.

$$r \cdot h(\begin{pmatrix} x \\ y \end{pmatrix}) = r \cdot (2x - 3y) = 2rx - 3ry = (2r)x - (3r)y = h(r \cdot \begin{pmatrix} x \\ y \end{pmatrix})$$

*Example* Of these two maps $h, g\colon \mathbb{R}^2 \to \mathbb{R}$, the first is a homomorphism while the second is not.

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In contrast, $g$ does not respect addition.

$$g(\begin{pmatrix} 1 \\ 4 \end{pmatrix} + \begin{pmatrix} 5 \\ 6 \end{pmatrix}) = -17 \qquad g(\begin{pmatrix} 1 \\ 4 \end{pmatrix}) + g(\begin{pmatrix} 5 \\ 6 \end{pmatrix}) = -16$$

We proved these two while studying isomorphisms.

1.6 *Lemma* A homomorphism sends the zero vector to the zero vector.

1.7 *Lemma* The following are equivalent for any map $f\colon V \to W$ between vector spaces.

(1) $f$ is a homomorphism

(2) $f(c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2) = c_1 \cdot f(\vec{v}_1) + c_2 \cdot f(\vec{v}_2)$ for any $c_1, c_2 \in \mathbb{R}$ and $\vec{v}_1, \vec{v}_2 \in V$

(3) $f(c_1 \cdot \vec{v}_1 + \cdots + c_n \cdot \vec{v}_n) = c_1 \cdot f(\vec{v}_1) + \cdots + c_n \cdot f(\vec{v}_n)$ for any $c_1, \ldots, c_n \in \mathbb{R}$ and $\vec{v}_1, \ldots, \vec{v}_n \in V$

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*Example* Between any two vector spaces the zero map $Z\colon V \to W$ given by $Z(\vec{v}) = \vec{0}_W$ is a linear map. Using (2): $Z(c_1\vec{v}_1 + c_2\vec{v}_2) = \vec{0}_W = \vec{0}_W + \vec{0}_W = c_1 Z(\vec{v}_1) + c_2 Z(\vec{v}_2)$.

*Example* The *inclusion map* $\iota\colon \mathbb{R}^2 \to \mathbb{R}^3$

$$\iota(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

is a homomorphism.

$$\begin{aligned}
\iota(c_1 \cdot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + c_2 \cdot \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}) &= \iota(\begin{pmatrix} c_1x_1 + c_2x_2 \\ c_1y_1 + c_2y_2 \end{pmatrix}) \\
&= \begin{pmatrix} c_1x_1 + c_2x_2 \\ c_1y_1 + c_2y_2 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} c_1x_1 \\ c_1y_1 \\ 0 \end{pmatrix} + \begin{pmatrix} c_2x_2 \\ c_2y_2 \\ 0 \end{pmatrix} \\
&= c_1 \cdot \iota(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}) + c_2 \cdot \iota(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix})
\end{aligned}$$

*Example* The derivative is a transformation on polynomial spaces. For instance, consider $d/dx\colon \mathcal{P}_2 \to \mathcal{P}_1$ given by

$$d/dx\,(ax^2 + bx + c) = 2ax + b$$

(examples are $d/dx\,(3x^2 - 2x + 4) = 6x - 2$ and $d/dx\,(x^2 + 1) = 2x$).

It is a homomorphism.

$$\begin{aligned} d/dx\,\bigl(\,r_1(a_1x^2 + b_1x + c_1) + r_2(a_2x^2 + b_2x + c_2)\,\bigr) \\ &= d/dx\,\bigl(\,(r_1a_1 + r_2a_2)x^2 + (r_1b_1 + r_2b_2)x + (r_1c_1 + r_2c_2)\,\bigr) \\ &= 2(r_1a_1 + r_2a_2)x + (r_1b_1 + r_2b_2) \\ &= (2r_1a_1x + r_1b_1) + (2r_2a_2x + r_2b_2) \\ &= r_1 \cdot d/dx\,(a_1x^2 + b_1x + c_1) + r_2 \cdot d/dx\,(a_2x^2 + b_2x + c_2) \end{aligned}$$

*Example* The *trace* of a square matrix is the sum down the upper-left to lower-right diagonal. Thus $\mathrm{Tr}\colon \mathcal{M}_{2\times 2} \to \mathbb{R}$ is this.

$$\mathrm{Tr}(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = a + d$$

It is linear.

$$\begin{aligned}
&\mathrm{Tr}(\, r_1 \cdot \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + r_2 \cdot \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\,) \\
&\qquad = \mathrm{Tr}(\begin{pmatrix} r_1 a_1 + r_2 a_2 & r_1 b_1 + r_2 b_2 \\ r_1 c_1 + r_2 c_2 & r_1 d_1 + r_2 d_2 \end{pmatrix}) \\
&\qquad = (r_1 a_1 + r_2 a_2) + (r_1 d_1 + r_2 d_2) \\
&\qquad = r_1 (a_1 + d_1) + r_2 (a_2 + d_2) \\
&\qquad = r_1 \cdot \mathrm{Tr}(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}) + r_2 \cdot \mathrm{Tr}(\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix})
\end{aligned}$$

1.9 *Theorem* A homomorphism is determined by its action on a basis: if $V$ is a vector space with basis $\langle\vec{\beta}_1,\dots,\vec{\beta}_n\rangle$, if $W$ is a vector space, and if $\vec{w}_1,\dots,\vec{w}_n\in W$ (these codomain elements need not be distinct) then there exists a homomorphism from $V$ to $W$ sending each $\vec{\beta}_i$ to $\vec{w}_i$, and that homomorphism is unique.

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*Proof* For any input $\vec{v}\in V$ let its expression with respect to the basis be $\vec{v}=c_1\vec{\beta}_1+\dots+c_n\vec{\beta}_n$. Define the associated output by using the same coordinates $h(\vec{v})=c_1\vec{w}_1+\dots+c_n\vec{w}_n$. This is well defined because, with respect to the basis, the representation of each domain vector $\vec{v}$ is unique.

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This map is a homomorphism because it preserves linear combinations: where $\vec{v_1}=c_1\vec{\beta}_1+\cdots+c_n\vec{\beta}_n$ and $\vec{v_2}=d_1\vec{\beta}_1+\cdots+d_n\vec{\beta}_n$, here is the calculation.

$$\begin{aligned}
h(r_1\vec{v}_1+r_2\vec{v}_2)&=h(\,(r_1c_1+r_2d_1)\vec{\beta}_1+\cdots+(r_1c_n+r_2d_n)\vec{\beta}_n\,)\\
&=(r_1c_1+r_2d_1)\vec{w}_1+\cdots+(r_1c_n+r_2d_n)\vec{w}_n\\
&=r_1h(\vec{v}_1)+r_2h(\vec{v}_2)
\end{aligned}$$

This map is unique because if $\hat{h}\colon V \to W$ is another homomorphism satisfying that $\hat{h}(\vec{\beta}_i) = \vec{w}_i$ for each $i$ then $h$ and $\hat{h}$ have the same effect on all of the vectors in the domain.

$$\begin{aligned}\hat{h}(\vec{v}) = \hat{h}(c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n) &= c_1\hat{h}(\vec{\beta}_1) + \cdots + c_n\hat{h}(\vec{\beta}_n) \\ &= c_1\vec{w}_1 + \cdots + c_n\vec{w}_n = h(\vec{v})\end{aligned}$$

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1.10 *Definition* Let $V$ and $W$ be vector spaces and let $B = \langle\vec{\beta}_1, \ldots, \vec{\beta}_n\rangle$ be a basis for $V$. A function defined on that basis $f\colon B \to W$ is *extended linearly* to a function $\hat{f}\colon V \to W$ if for all $\vec{v} \in V$ such that $\vec{v} = c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n$, the action of the map is $\hat{f}(\vec{v}) = c_1 \cdot f(\vec{\beta}_1) + \cdots + c_n \cdot f(\vec{\beta}_n)$.