SM261 Final Examination

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1. (16 pts) Let $\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$.
   a. Find a vector of length 9 parallel to $\vec{v}$.

   b. Find the cosine of the angle between $\vec{v}$ and $\vec{w}$.

   c. Show that the Cauchy-Schwarz Inequality is valid for $\vec{v}$ and $\vec{w}$.

   d. Find $proj_L(\vec{w})$, where L is the line through the origin and parallel to $\vec{v}$.

2. (14 pts) Let $\vec{v}$ be the vector in the previous problem.
   a. Define: "T is a linear transformation from R^m to R^n."

   b. Show that the mapping T on R^3 given by $T(\vec{x}) = proj_L(\vec{x})$ is a linear transformation.

   c. Find the matrix A that satisfies $T(\vec{x}) = A\vec{x}$ for all $\vec{x}$ in R^3.

3. (12 pts) Write the system below as a matrix equation and use Gaussian elimination techniques to find all solutions.

$$\begin{array}{rcrcrcl} x & + & 2y & - & z & = & 2 \\ x & - & y & + & z & = & 1 \\ 3x & + & 0y & + & z & = & 4. \end{array}$$

4. (15 pts) Let $A = \begin{bmatrix} 1 & 2 & 2 & -5 & 6 \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & -8 & 9 \\ 3 & 6 & 1 & 5 & -7 \end{bmatrix}$.

a. Show (by hand) that $rref(A) = \begin{bmatrix} 1 & 2 & 0 & 3 & -4 \\ 0 & 0 & 1 & -4 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

b. Find a basis for the kernel of A.

c. Find a basis for the image of A.

5. (14 pts) Let A be an invertible $n \times n$ matrix. What can you say about each of the following?

   a. The rank of A.

   b. $rref(A)$.

   c. The kernel of A.

   d. The image of A.

   e. The column vectors of A.

   f. The number of solutions to the system $A\vec{x} = \vec{b}$, where $\vec{b}$ is any fixed vector.

   g. $\det(A)$.

6. (12 pts) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{bmatrix}$.

   a. Use row operation techniques to find A^{-1}.

   b. Use your answer in a. to solve the matrix equation $A\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

7. (10 pts) Consider the linear transformation T on R^2 whose matrix is given by the product $\begin{bmatrix} \cos(\frac{\pi}{3}) & -\sin(\frac{\pi}{3}) \\ \sin(\frac{\pi}{3}) & \cos(\frac{\pi}{3}) \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. Describe $T(\vec{x})$ geometrically. Hint: describe what each matrix does separately to a vector.

8. (15 pts) Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$.

   a. Prove that $\mathcal{B}$ is a basis for R^2.

   b. Let $\vec{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. Find $[\vec{v}]_B$.

   c. Let T be the linear transformation on R^2 whose matrix is $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Find the matrix of T with respect to the basis $\mathcal{B}$.

9. (12 pts) Consider a reflection matrix A and a vector $\vec{x}$ in R^2. Define $\vec{v} = \vec{x} + A\vec{x}$ and $\vec{w} = \vec{x} - A\vec{x}$. (Recall that $A(A\vec{x}) = \vec{x}$.)

a. Express $A\vec{v}$ in terms of $\vec{v}$.

b. Express $A\vec{w}$ in terms of $\vec{w}$.

c. If the vectors $\vec{v}$ and $\vec{w}$ are both nonzero, show that they are orthogonal.

d. If the vector $\vec{v}$ is nonzero, what is the relationship between $\vec{v}$ and the line L of reflection?

10. (12 pts) A 2×2 matrix A is called nilpotent if $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

a. Show that there is a nonzero vector in $\ker(A)$. (Hence λ = 0 is an eigenvalue of A.)

b. Show that 0 is the <u>only</u> eigenvalue of A.

11. (12 pts) Let $M = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 \\ 1 & 0 & 3 & 0 \\ 2 & 0 & 2 & 3 \end{bmatrix}$. Use row operation methods to find $\det(M)$.

12. (16 pts) Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 7 & 14 & 7 \\ 1 & 2 & 2 \\ 7 & 14 & 7 \end{bmatrix}$.

a. Find a basis for $im(A)$.

b. Find an orthonormal basis for $im(A)$.

c. Find $proj_{im(A)}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right)$. Hint: use the orthonormal basis from b.

13. (12 pts) Let A be a square matrix.

   a. Define "A is a symmetric matrix".

   b. Define "A is an orthogonal matrix".

   c. Find all symmetric orthogonal 2×2 matrices. Hint: first determine what orthogonal 2×2 matrices look like.

14. (15 pts) Let $A = \begin{bmatrix} -1 & 0 & 1 \\ -3 & 0 & 1 \\ -4 & 0 & 3 \end{bmatrix}$.

   a. Show that 1 and 0 are the eigenvalues of A.

   b. Find a basis of each eigenspace.

   c. Can the matrix A be diagonalized? Explain.

15. (13 pts) (Calculators okay) Use Least Squares techniques to find the coefficients c_0, c_1, and c_2 to the quadratic polynomial $y = c_0 + c_1 t + c_2 t^2$ that best fits the data points in the table below. Hint: first set up the problem as an (inconsistent) system $A\vec{x} = \vec{b}$.

Independent variable t	Dependent variable y
0	1
1	2
2	2
3	4