1. Use the methods of this course to describe all the ways in which 200 coins with a total value of 300¢ can be chosen from among pennies, nickels, and dimes.

Let x_1, x_2, and x_3 be the number of pennies, nickels and dimes, respectively. We have $x_1 + x_2 + x_3 = 200$ as the equation for the total number of coins and $x_1 + 5x_2 + 10x_3 = 300$ as as the equation for their total value. The matrix equation is $A\vec{x} = \vec{b}$ where $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 5 & 10 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 200 \\ 300 \end{bmatrix}$. Row-reduction yields

$$[A\,|\,\vec{b}] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 200 \\ 1 & 5 & 10 & 300 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 200 \\ 0 & 4 & 9 & 100 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 200 \\ 0 & 1 & \frac{9}{4} & 25 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 200 \\ 0 & 1 & \frac{9}{4} & 25 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -\frac{5}{4} & 175 \\ 0 & 1 & \frac{9}{4} & 25 \end{array}\right] = [A\,|\,\vec{b}]_{rref}.$$ From the Solution Algorithm, $\vec{x} = \begin{bmatrix} 175 \\ 25 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -\frac{5}{4} \\ \frac{9}{4} \\ -1 \end{bmatrix}$ for any $\alpha \in \mathbf{R}$. Since the number of each coin must be a nonnegative integer, we replace α by $-4k$ where k is a nonnegative integer. Then, $\vec{x} = \begin{bmatrix} 175 + 5k \\ 25 - 9k \\ 4k \end{bmatrix}$.

Now $x_2 \geq 0$, and so $0 \leq k \leq 2$, from which we find $\vec{x} = \begin{bmatrix} 175 \\ 25 \\ 0 \end{bmatrix}, \begin{bmatrix} 180 \\ 16 \\ 4 \end{bmatrix}$, or $\begin{bmatrix} 185 \\ 7 \\ 8 \end{bmatrix}$.

2. a. What is A_{rref} if A is a matrix for a rotation in $\mathbf{R}^3$ by some angle about a line through the origin?

The column vectors of a rotation matrix are the images of the three standard basis vectors in $\mathbf{R}^3$ under the rotation. Since rotations do not change the angles nor the lengths of vectors, the column vectors of A must be mutually orthogonal unit vectors. So, the column vectors of A are non-coplanar and nonzero. So, the column vectors are linearly independent. Put another way, A is invertible and so,

$$A_{rref} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

b. What is A_{rref} if A is the matrix for projection onto a line through the origin in $\mathbf{R}^3$ not coincident with any of the coordinate axes?

The column vectors of A are projections of the standard basis vectors onto the given line. So, the column vectors of A must be non-zero vectors that are collinear and, therefore, proportional to each other. $A_{rref} = \begin{bmatrix} 1 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ where a, b are nonzero reals.

c. A is an $m \times n$ matrix and B is an $n \times p$ matrix. If the column vectors of AB are linearly independent, prove that the column vectors of B must also be linearly independent.

We know that if the column vectors of B are linearly dependent, the column vectors of AB must also be linearly dependent. Therefore, if the column vectors of AB are linearly independent, the column vectors of B must also be linearly independent. This statement is merely the contra-positive of the first.

Alternatively, note that the column vectors of any $r \times s$ matrix C are linearly independent if and only if when $\vec{x} \in \mathbf{R}^s$, $C\vec{x} = \vec{0}$ always implies that $\vec{x} = \vec{0}$. So, suppose that $B\vec{x} = \vec{0}$ for $x \in \mathbf{R}^p$. Then, $A(B\vec{x}) = A\vec{0} = (AB)\vec{x} = \vec{0}$. Since the column vectors of AB are assumed to be linearly independent, it follows that $\vec{x} = \vec{0}$ and we have shown that the column vectors of B are linearly independent.

3. Suppose that $[\vec{a}_1 \,|\, \vec{a}_2 \,|\, \vec{a}_3 \,|\, \vec{a}_4 \,|\, \vec{b}]_{rref} = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

a. Find all the linear combinations of the vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4$ that sum to $\vec{b}$.

Let the coefficients of the linear combinations be the corresponding components of $\vec{x}$. Then, $[\vec{a}_1 \,|\, \vec{a}_2 \,|\, \vec{a}_3 \,|\, \vec{a}_4]\,\vec{x} = \vec{b}$ and, from the results above, using the Solution Algorithm, we have $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + \alpha \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}$ for any $\alpha \in \mathbf{R}$.

b. Are there vectors $\vec{c}$ in $\mathbf{R}^4$ so $[\vec{a}_1 \,|\, \vec{a}_2 \,|\, \vec{a}_3 \,|\, \vec{a}_4]\vec{x} = \vec{c}$ has no solution? Why?

Yes. There is a $\vec{c}$ in $\mathbf{R}^4$ so that $[\vec{a}_1 \,|\, \vec{a}_2 \,|\, \vec{a}_3 \,|\, \vec{a}_4 \,|\, \vec{c}]_{rref} = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

This implies that there are no solutions to $[\vec{a}_1 \,|\, \vec{a}_2 \,|\, \vec{a}_3 \,|\, \vec{a}_4]\vec{x} = \vec{c}$. This $\vec{c}$ is obtained by applying to $\hat{e}_4$, in reverse order, the inverses of the same elementary row operations that reduce A to A_{rref}.

c. Find all vectors $\vec{x}$ in $\mathbf{R}^4$ so that $[\vec{a}_1 \,|\, \vec{a}_2 \,|\, \vec{a}_3 \,|\, \vec{a}_4]\vec{x} = \vec{0}$.

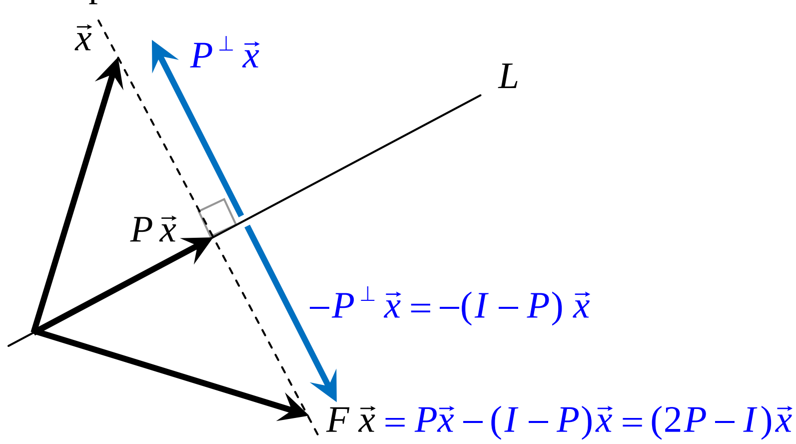
From part a, if we replace $\vec{b}$ by the zero vector, $\vec{x} = \alpha \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}$ for any $\alpha \in \mathbf{R}$.

d. If $\vec{d}$ is one solution of $[\vec{a}_1 \,|\, \vec{a}_2 \,|\, \vec{a}_3 \,|\, \vec{a}_4]\vec{x} = \hat{e}_4$, find all other solutions.

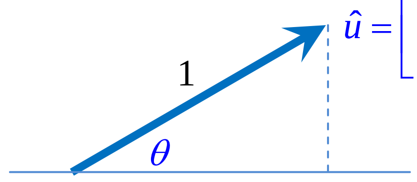
$\vec{x} = \vec{d} + \alpha \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}$ for any $\alpha \in \mathbf{R}$

4. L is a line in $\mathbf{R}^2$ through the origin at an angle θ to the 1-axis. P is the matrix for projection onto L and F is the matrix for reflection across L.

a. Using the following diagram, show that $F\vec{x} = P\vec{x} - (I - P)\vec{x}$ for any vector $\vec{x}$ in the plane.



b. Determine $\hat{u}$, the unit vector parallel to L.



$\hat{u} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

c. From part b, determine P in terms of θ.

$$P = \begin{bmatrix} u_1 u_1 & u_1 u_2 \\ u_2 u_1 & u_2 u_2 \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\theta & \cos\theta\sin\theta \\ \sin\theta\cos\theta & \sin\theta\sin\theta \end{bmatrix}$$

d. From parts a and b, compute F in terms of θ and simplify .

Since $F\vec{x}=(2P-I)\vec{x}$ for all $\vec{x}$, $F=2P-I$. So,

$$F=2P-I=\begin{bmatrix}2\cos^2\theta-1 & 2\sin\theta\cos\theta\\ 2\sin\theta\cos\theta & 2\sin^2\theta-1\end{bmatrix}=\begin{bmatrix}\cos 2\theta & \sin 2\theta\\ \sin 2\theta & -\cos 2\theta\end{bmatrix}.$$

5. a. Solve $A(B+X)^{-1}=B^{-1}$ for X if $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$ and $B=\begin{bmatrix}3 & -2\\ -1 & 2\end{bmatrix}$

We take the inverse of both sides of $A(B+X)^{-1}=B^{-1}$ to obtain $\left(A(B+X)^{-1}\right)^{-1}=(B+X)A^{-1}=\left(B^{-1}\right)^{-1}=B$. So, $(B+X)A^{-1}=B.$ Now, multiply both sides of this equation on the right by A to obtain $B+X=BA$ and so,

$$X=BA-B=B(A-I)=\begin{bmatrix}3 & -2\\ -1 & 2\end{bmatrix}\left(\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}-\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)=\begin{bmatrix}3 & -2\\ -1 & 2\end{bmatrix}\begin{bmatrix}0 & 2\\ 3 & 3\end{bmatrix}=\begin{bmatrix}-6 & 0\\ 6 & 4\end{bmatrix}.$$

b. Determine the matrix A if $A\begin{bmatrix}2\\ -1\end{bmatrix}=\begin{bmatrix}1\\ 1\end{bmatrix}$ and $A\begin{bmatrix}3\\ 1\end{bmatrix}=\begin{bmatrix}2\\ 5\end{bmatrix}$.

The two equations may be combined into one:: $A\begin{bmatrix}2 & 3\\ -1 & 1\end{bmatrix}=\begin{bmatrix}1 & 2\\ 1 & 5\end{bmatrix}$ and so,

because the matrix multiplying A is clearly invertible, we multiply this equation on the right by the inverse of this matrix.

$$A=\begin{bmatrix}1 & 2\\ 1 & 5\end{bmatrix}\begin{bmatrix}2 & 3\\ -1 & 1\end{bmatrix}^{-1}=\begin{bmatrix}1 & 2\\ 1 & 5\end{bmatrix}\cdot\tfrac{1}{5}\begin{bmatrix}1 & -3\\ 1 & 2\end{bmatrix}=\tfrac{1}{5}\begin{bmatrix}3 & 1\\ 6 & 7\end{bmatrix}.$$