The action of plane transformations

Linear Algebra
Jim Hefferon

http://joshua.smcvt.edu/linearalgebra

Transformations of $\mathbb{R}^2$

Lines go to lines

In a real space $\mathbb{R}^n$ a line through the origin is a set $\{r\cdot\vec{v}\mid r\in\mathbb{R}\}$ of multiples of a nonzero vector.

Consider a transformation $t\colon\mathbb{R}^n\to\mathbb{R}^n$. It is linear and so t's action

$$r\cdot\vec{v}\overset{t}{\longmapsto}r\cdot t(\vec{v})$$

sends members of the line $\{r\cdot\vec{v}\mid r\in\mathbb{R}\}$ in the domain to members of the line $\{s\cdot t(\vec{v})\mid s\in\mathbb{R}\}$ in the codomain.

Thus, under a transformation, lines through the origin map to lines through the origin. Further, the action of t is determined by its effect $t(\vec{v})$ on any nonzero element of the domain line.

Example Consider the line $y = 2x$ in the plane

$$\{r \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} \mid r \in \mathbb{R}\}$$

and this transformation.

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + 3y \\ 2x + 4y \end{pmatrix}$$

The map's effect on any vector in the line is easy to compute.

$$\vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \overset{t}{\longmapsto} \begin{pmatrix} 7 \\ 10 \end{pmatrix}$$

The linear map property $t(r \cdot \vec{v}) = r \cdot t(\vec{v})$ imposes a uniformity on t's action: t has twice the effect on $2\vec{v}$, three times the effect on $3\vec{v}$, etc.

$$\begin{pmatrix} 2 \\ 4 \end{pmatrix} \overset{t}{\longmapsto} \begin{pmatrix} 14 \\ 20 \end{pmatrix} \qquad \begin{pmatrix} -3 \\ -6 \end{pmatrix} \overset{t}{\longmapsto} \begin{pmatrix} -21 \\ -30 \end{pmatrix} \qquad \begin{pmatrix} r \\ 2r \end{pmatrix} \overset{t}{\longmapsto} \begin{pmatrix} 7r \\ 10r \end{pmatrix}$$

In short: the action of t on any nonzero $\vec{v}$ determines its action on any other vector $r\vec{v}$ in the line $[\vec{v}]$.