Special Problem A Solutions

1. Consider a linear system of 4 equations in 3 variables. We know that such a system can be abbreviated as a single matrix equation in the form $A\vec{x}=\vec{b}$ where $\vec{x}$ is the 3×1 column matrix (or vector) of variables, A is the 4×3 matrix of coefficients and $\vec{b}$ is the 4×1 column matrix of constants. Let's exclude the peculiar cases in which one or more of the variables is absent from each of the equations. This means that we have decided that no column of A consists only of zero entries.

a. List all row-reduced echelon forms $[A|\vec{b}]_{rref}$ of the augmented matrices $[A|\vec{b}]$ described above that have rank 2 or 3, i.e. that have 2 or 3 pivots. Use lower case letters to label real entries that are not necessarily 0 but indicate wherever entries must be 0 or 1. Also, determine the solution sets in each case.

The rrefs and solution sets in the case when the rank is 2 are there are the following rrefs and corresponding solution sets.

i. $\left[\begin{array}{ccc|c} 1 & 0 & a & c \\ 0 & 1 & b & d \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \left\{\begin{bmatrix} c \\ d \\ 0 \end{bmatrix} + r\begin{bmatrix} a \\ b \\ -1 \end{bmatrix} \middle| r \in \mathbf{R}\right\}$

ii. $\left[\begin{array}{ccc|c} 1 & a & 0 & c \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \left\{\begin{bmatrix} c \\ 0 \\ d \end{bmatrix} + r\begin{bmatrix} a \\ -1 \\ 0 \end{bmatrix} \middle| r \in \mathbf{R}\right\}$

iii. $\left[\begin{array}{ccc|c} 1 & a & b & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \varnothing$

In the case where the rank is 3, we have the following rref and solution set pairs.

iv. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & c \\ 0 & 1 & 0 & d \\ 0 & 0 & 1 & e \\ 0 & 0 & 0 & 0 \end{array}\right], \left\{\begin{bmatrix} c \\ d \\ e \end{bmatrix}\right\}$

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$$v. \left[\begin{array}{ccc|c} 1 & a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right], \varnothing$$

b. For each of the forms you listed in part a, give a concrete example of an augmented matrix $[A|\vec{b}]$. Choose your examples so that no entry in $[A|\vec{b}]$ has the value 0.

We choose to set $a = b = c = d = 1$ in our examples. We apply elementary row operations to generate the equations desired. This is rather like reversing the process of Gaussian elimination. Through replacements, we can "eliminate" all zero entries in $[A|\vec{b}]$.

$$i. \ [A|\vec{b}] = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 2 & 2 & 4 & 4 \\ 3 & 3 & 6 & 6 \end{array}\right].$$

$$ii. \ [A|\vec{b}] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 4 \\ 3 & 3 & 3 & 6 \end{array}\right].$$

$$iii. \ [A|\vec{b}] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{array}\right].$$

$$iv. \ [A|\vec{b}] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 1 & 4 \\ 1 & 1 & 2 & 4 \\ 2 & 2 & 3 & 6 \end{array}\right].$$

$$v. \ [A|\vec{b}] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \\ 2 & 2 & 2 & 2 \end{array}\right].$$