Name: Solutions		

1. Determine the coefficients c_1, c_2, and c_3 for the cubic polynomial p defined by the equation $p(x) = c_1x + c_2x^2 + c_3x^3$ if the graph of p passes through the points $(-1,4)$, $(1,0)$, and $(2,6)$. Proceed by obtaining a linear system of equations for the coefficients, reformulating this system as a single matrix equation, and solving the matrix equation by row-reduction.

Using the elementary fact that a point (a, b) lies on the graph of a function p if and only if $p(a) = b$ and the information above, we obtain 3 linear equations in the 3 variables c_1, c_2, and c_3.

$$\left\{\begin{array}{c} p(-1) = -c_1 + c_2 - c_3 = 4 \\ p(1) = c_1 + c_2 + c_3 = 0 \\ p(2) = 2c_1 + 4c_2 + 8c_3 = 6 \end{array}\right\} \text{ which is equivalent to}$$

$$A\vec{c} = \vec{b} \text{ where } A = \begin{bmatrix} -1 & 1 & -1 \\ 1 & 1 & 1 \\ 2 & 4 & 8 \end{bmatrix}, \ \vec{b} = \begin{bmatrix} 4 \\ 0 \\ 6 \end{bmatrix}, \text{ and } \vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

To solve, we row-reduce the augmented matrix,

$$[A|\vec{b}] = \left[\begin{array}{ccc|c} -1 & 1 & -1 & 4 \\ 1 & 1 & 1 & 0 \\ 2 & 4 & 8 & 6 \end{array}\right] \leftrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ -1 & 1 & -1 & 4 \\ 1 & 2 & 4 & 3 \end{array}\right] \leftrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 4 \\ 0 & 1 & 3 & 3 \end{array}\right]$$

$$\leftrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 3 & 3 \end{array}\right] \leftrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 3 & 1 \end{array}\right] \leftrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & \frac{1}{3} \end{array}\right]$$

$$\leftrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{7}{3} \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & \frac{1}{3} \end{array}\right] = [A|\vec{b}]_{rref}. \text{ Therefore, } \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -\frac{7}{3} \\ 2 \\ \frac{1}{3} \end{bmatrix}.$$

2. A linear system of equations is equivalent to the single matrix equation $A\vec{x} = \vec{b}$. Suppose that $[A|\vec{b}]_{rref} = \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 3 & 6 \\ 0 & 0 & 1 & 0 & 4 & 7 \\ 0 & 0 & 0 & 1 & 5 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$.

a. Determine all solutions of the linear system in vector form.

From the Solution Algorithm, we have

$$\vec{x}=\begin{bmatrix}6\\0\\7\\8\\0\end{bmatrix}+\alpha\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix}+\beta\begin{bmatrix}-3\\0\\-4\\-5\\1\end{bmatrix} \text{ where } \alpha,\beta\in\mathbf{R}.$$

b. Find all solutions of the equation $A\vec{x}=\vec{0}$.

Note that if $\vec{b}=\vec{0}$ if and only if the last column of $[A|\vec{b}]_{rref}$ is also $\vec{0}$. So,

$$\vec{x}=\alpha\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix}+\beta\begin{bmatrix}-3\\0\\-4\\-5\\1\end{bmatrix} \text{ where } \alpha,\beta\in\mathbf{R}.$$

3. Suppose that $f:\mathbf{R}^2\to\mathbf{R}^2$ is linear, $f\left(\begin{bmatrix}1\\2\end{bmatrix}\right)=\begin{bmatrix}3\\1\end{bmatrix}$, and $f\left(\begin{bmatrix}1\\4\end{bmatrix}\right)=\begin{bmatrix}1\\3\end{bmatrix}$. Determine the 2×2 matrix A corresponding to f in two <u>different</u> ways.

a. Use linearity to find $f\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$ and $f\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$ and from this find A.

$$f\left(\begin{bmatrix}1\\2\end{bmatrix}\right)=f\left(\begin{bmatrix}1\\0\end{bmatrix}\right)+2f\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}3\\1\end{bmatrix}$$

$$f\left(\begin{bmatrix}1\\4\end{bmatrix}\right)=f\left(\begin{bmatrix}1\\0\end{bmatrix}\right)+4f\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}1\\3\end{bmatrix}$$

Now, subtract the second equation from twice the first equation and subtract half the first equation from half the second equation to obtain

$$f\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=2\begin{bmatrix}3\\1\end{bmatrix}-\begin{bmatrix}1\\3\end{bmatrix}=\begin{bmatrix}5\\-1\end{bmatrix}$$

$$f\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\frac{1}{2}\begin{bmatrix}1\\3\end{bmatrix}-\frac{1}{2}\begin{bmatrix}3\\1\end{bmatrix}=\begin{bmatrix}-1\\1\end{bmatrix}$$

So, $A=\left[f\left(\begin{bmatrix}1\\0\end{bmatrix}\right)\,\middle|\,f\left(\begin{bmatrix}0\\1\end{bmatrix}\right)\right]=\begin{bmatrix}5&-1\\-1&1\end{bmatrix}$.

b. Convert the pair of given vector equations to a single matrix equation for A and solve it.

Since $f(\vec{x}) = A\vec{x}$, $A\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}3\\1\end{bmatrix}$ and $A\begin{bmatrix}1\\4\end{bmatrix} = \begin{bmatrix}1\\3\end{bmatrix}$.

Therefore, $A\begin{bmatrix}1 & 1\\2 & 4\end{bmatrix} = \begin{bmatrix}3 & 1\\1 & 3\end{bmatrix}$ and $A = \begin{bmatrix}3 & 1\\1 & 3\end{bmatrix}\begin{bmatrix}1 & 1\\2 & 4\end{bmatrix}^{-1}$

$= \begin{bmatrix}3 & 1\\1 & 3\end{bmatrix}\left(\dfrac{1}{4-2}\right)\begin{bmatrix}4 & -1\\-2 & 1\end{bmatrix} = \dfrac{1}{2}\begin{bmatrix}3 & 1\\1 & 3\end{bmatrix}\begin{bmatrix}4 & -1\\-2 & 1\end{bmatrix}$

$= \dfrac{1}{2}\begin{bmatrix}10 & -2\\-2 & 2\end{bmatrix} = \begin{bmatrix}5 & -1\\-1 & 1\end{bmatrix}$, as above.

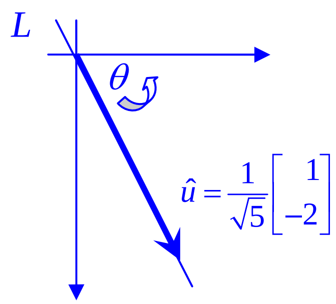
4. The equation $2x + y = 0$ describes a line L through the origin in $\mathbf{R}^2$. Find the matrix that corresponds to reflection across L and use it to find the reflected image of an arbitrary point (r, s).

Two different methods will be used. In the first, we apply the reflection separately to the projections of any vector parallel to and perpendicular to L and add the results. Let f denote the reflection and let F be its corresponding matrix. Then $f(\vec{x}) = F\vec{x}$ for any vector $\vec{x}$. Suppose $\vec{z}$ is a vector parallel to L, then $f(\vec{z}) = \vec{z}$; the reflection leaves $\vec{z}$ unchanged. On the other hand, if $\vec{z}$ is perpendicular to L, then $f(\vec{z}) = -\vec{z}$ because f changes the direction of $\vec{z}$. So, it makes sense to resolve a vector into its projections parallel and perpendicular to L. Let P be the matrix corresponding to projection along L. Then, for any vector $\vec{x}$, we have $\vec{x} = P\vec{x} + (I-P)\vec{x}$. This is an identity. $P\vec{x}$ is the projection of $\vec{x}$ along L and $(I-P)\vec{x}$ is the projection of $\vec{x}$ perpendicular to L. Moreover, linearity allows us to write $f(\vec{x}) = f(P\vec{x}) + f((I-P)\vec{x})$. Now, according to our observations above $f(\vec{x}) = P\vec{x} - (I-P)\vec{x} = (2P-I)\vec{x}$. Since $\vec{x}$ is arbitrary, $F = 2P - I$. So, it remains to find the matrix P. A vector parallel to L is $\begin{bmatrix}1\\-2\end{bmatrix}$ and so, a unit vector parallel to L is $\hat{u} = \begin{bmatrix}u_1\\u_2\end{bmatrix} =$ $\dfrac{1}{\sqrt{5}}\begin{bmatrix}1\\-2\end{bmatrix}$. We have seen that $P = \begin{bmatrix}u_1u_1 & u_1u_2\\u_2u_1 & u_2u_2\end{bmatrix} = \dfrac{1}{5}\begin{bmatrix}1 & -2\\-2 & 4\end{bmatrix}$. So,

$F = \dfrac{2}{5}\begin{bmatrix}1 & -2\\-2 & 4\end{bmatrix} - \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \dfrac{1}{5}\begin{bmatrix}-3 & -4\\-4 & 3\end{bmatrix}$.

To find the image of (r, s) under this reflection, we simply multiply the vector displacement from the origin to (r, s) by F. We have $F\begin{bmatrix} r \\ s \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -3 & -4 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} -\frac{3}{5}r - \frac{4}{5}s \\ -\frac{4}{5}r + \frac{3}{5}s \end{bmatrix}$. The image is $(-\frac{3}{5}r - \frac{4}{5}s, -\frac{4}{5}r + \frac{3}{5}s)$.

For a second approach, we compose f from 3 elementary linear transformations. The matrix F corresponding to f is then the product of the matrices for the three transformation. The first of these rotates the line L counterclockwise to the horizontal; the second is a reflection across the horizontal; and the third is the inverse of the first rotation. Let the matrices for these transformations be, respectively, R (ccw rotation from L to horizontal), X (reflection across the horizontal), and R^{-1} (cw rotation from horizontal to L). Clearly, $X = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. The cw rotation matrix is $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ where the entries of this matrix are determined from the line L or the unit vector parallel to L found above.



$$\sin\theta = \frac{2}{\sqrt{5}}$$

$$\cos\theta = \frac{1}{\sqrt{5}}$$

$R = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ and $R^{-1} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$. Finally, then,

$$F = R^{-1}\,X\,R = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -2 & -1 \end{bmatrix} =$$

$$\frac{1}{5}\begin{bmatrix} -3 & -4 \\ -4 & 3 \end{bmatrix},$$ as before.

5. Explain each of the following assertions.

a. Suppose that $\vec{u}$, $\vec{v}$, and $\vec{w}$ are 3 vectors in $\mathbf{R}^2$ and no pair of them is collinear. There are always infinitely many different linear combinations of these 3 vectors that sum to the zero vector.

Since $\vec{u}$ and $\vec{v}$ are non-collinear, every vector, $\vec{w}$ included, is a linear combination of these two vectors. That is, $\vec{w} = \alpha\vec{u} + \beta\vec{v}$ for some nonzero scalars α and β. Therefore, $\alpha\vec{u} + \beta\vec{v} - \vec{w} = \vec{0}$. This means that there is a nontrivial linear combination of these three vectors that sums to the zero vector. Of course, a trivial linear combination that sums to zero is the following: $0\vec{u} + 0\vec{v} + 0\vec{w} = \vec{0}$. But, the previous equation can be multiplied by <u>any</u> scalar γ to get $\gamma\alpha\vec{u} + \gamma\beta\vec{v} - \gamma\vec{w} = \vec{0}$ and this shows that there are an infinite number of linear combinations (a different linear combination for each choice of γ) of the three vectors that sum to the zero vector.

b. Projection onto a line in $\mathbf{R}^3$ is not an invertible transformation.
A function is not invertible if it maps two different vectors to the same vector. Projection onto a line L maps all the vectors perpendicular to L to the zero vector. Also, any two vectors that differ by a vector that is perpendicular to L will be mapped to the same vector parallel to L. So, a projection is not an invertible transformation because it maps many vectors to one. Notice that this argument is valid for any $\mathbf{R}^n$.

Infinitely many vectors have the same projection onto L.