1. Consider the linear system $\left\{\begin{array}{rcrcrcrcr} x & + & 2y & & & - & w & = & 0 \\ 2x & + & 6y & - & 3z & - & 3w & = & 3 \\ 3x & + & 10y & + & kz & - & 5w & = & 2 \end{array}\right\}$.

For which real values of k does the system have

a. a unique solution?
b. infinitely many solutions?
c. no solutions?

2. Consider the matrix $A = \begin{bmatrix} 1 & 1 & -1 \\ 4 & 5 & 0 \\ 5 & 8 & 7 \\ -1 & -2 & -3 \end{bmatrix}$. Given that $\operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$,

a. find a basis for im(A). What is the dimension of im(A)?
b. find a basis for ker(A). What is the dimension of ker(A)?

3. State whether each of the following statements is true or false. Justify your answer.

a. If A is a 3×5 matrix, then there must exist at least two linearly independent vectors in ker(A).

b. If B is a 4×3 matrix and the system $B\vec{x}=\vec{0}$ has a unique solution, then for every vector $\vec{b}$, the system $B\vec{x}=\vec{b}$ also has a unique solution.

c. If C is a 4×3 matrix and, for some vector $\vec{c}$, the system $C\vec{x}=\vec{c}$ has a unique solution, then the system $C\vec{x}=\vec{0}$ also has a unique solution.

4. Let $T:\mathbf{R}^2\to\mathbf{R}^3$ be a linear transformation such that

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}1\\2\\0\end{bmatrix} \text{ and } T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}1\\-1\\1\end{bmatrix}.$$

Consider also the linear transformation $S:\mathbf{R}^3\to\mathbf{R}^2$ given by

$$S\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)=\begin{bmatrix}2x-y-4z\\x-z\end{bmatrix}.$$

a. Find the matrix A of T and the matrix B of S.

b. Show that the map $Q:\mathbf{R}^2 \to \mathbf{R}^2$ given by $Q(\vec{x}) = S(T(\vec{x}))$ is a rotation in the plane and determine the angle of the rotation.

5. Consider the vectors $\vec{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ in $\mathbf{R}^2$.

a. Explain why the set $\mathfrak{B} = \{\vec{v}_1, \vec{v}_2\}$ is a basis for $\mathbf{R}^2$.

b. Find the vector $\vec{y}$ in $\mathbf{R}^2$ whose coordinate vector with respect to $\mathfrak{B}$ is $[\vec{y}]_{\mathfrak{B}} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

c. Let $\vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Find $[\vec{x}]_{\mathfrak{B}}$.

d. Suppose that $T:\mathbf{R}^2 \to \mathbf{R}^2$ is a linear transformation such that

$$T(\vec{v}_1) = \vec{v}_1 - 2\vec{v}_2 \text{ and } T(\vec{v}_2) = \vec{v}_1.$$

Find:

i. The matrix B of T with respect to the basis $\mathfrak{B}$.

ii. The standard matrix A of T.

6. Let V be the subspace of $\mathbf{R}^4$ spanned by the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix}$.

a. Find an orthonormal basis for V.

b. Find the matrix P of the orthogonal projection onto V.

c. Find the orthogonal projection of the vector $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \end{bmatrix}$ onto V.

d. Find the matrix of the orthogonal projection onto $V^{\perp}$.

7. Let W be the subspace of $\mathbf{R}^3$ consisting of all vectors perpendicular to $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

a. Find a basis for W.

b. Find a basis for $W^{\perp}$.

c. Suppose that $T : \mathbf{R}^3 \to \mathbf{R}^3$ is the reflection across W and let A be the matrix for T. Find all the eigenvalues and a basis for each eigenspace of A.

d. Find an invertible matrix S and a diagonal matrix D such that $A = S\,D\,S^{-1}$.

e. Explain why A is invertible and compute A^{-1}.

8. Determine whether each of the following statements is true or false. Justify your answers.

a. The function $T : \mathbf{R}^2 \to \mathbf{R}^2$ defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ 1-x \end{bmatrix}$ is a linear transformation.

b. If the non-zero vectors $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ satisfy the relation $\vec{v}_1 + 2\vec{v}_2 + 3\vec{v}_3 = 4\vec{v}_1 + 5\vec{v}_2 + 6\vec{v}_3$ then they must be linearly dependent.

c. If A is a 4×4 matrix and deta. = 4, then the rank of A must be 4.

d. If A is a 2×2 matrix with eigenvalues 1 and 0, we must have A^2 = A.

9. Suppose that $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ are the rows of a 3×3 matrix A; i.e. $A = \begin{bmatrix} \vec{v}_1 \\ \hline \vec{v}_2 \\ \hline \vec{v}_3 \end{bmatrix}$.

Suppose also that det(A) = 2 and that B is a 3×3 matrix with det(B) = –3. Compute the following, giving reasons for your answers.

a. $\det\left(\begin{bmatrix} \vec{v}_3 \\ \hline \vec{v}_2 \\ \hline \vec{v}_1 \end{bmatrix}\right)$.

b. $\det\left(\begin{bmatrix} \vec{v}_2 - 2\vec{v}_1 \\ \hline \vec{v}_2 \\ \hline \vec{v}_1 - \vec{v}_3 \end{bmatrix}\right)$.

c. det(3A).

d. $\det(B\,A\,B^T)$.
e. $\operatorname{rank}(A)$.

10. Find the equation of the straight line $y = a\,x + b$ that best fits (in the least squares sense) the points $(1, 1)$, $(2, -1)$ and $(3, 2)$.

11. Consider the system of equations $\left\{\begin{matrix} a\,x & - & 4\,y & = & 1 \\ 9\,x & + & a\,y & = & 3 \end{matrix}\right\}$ where a is a parameter.

a. Prove that, for each value of a, this system has a unique solution.
b. Use Cramer's Rule to solve the system for each value of a.

12. Let $A = \begin{bmatrix} .50 & .25 \\ .50 & .75 \end{bmatrix}$.

a. Find an invertible matrix S and a diagonal matrix D such that $A = S\,D\,S^{-1}$.
b. Compute the matrix A^n for any positive integer n.
c. Compute $\lim\limits_{n \to \infty} A^n$.
d. Find a matrix B such that $B^2 = A$.