This extends ideas associated with the assigned exercises 1.2: 6-8. The assignment's focus was on the geometry. Here it is on the algebra.

Recall this about multiplication of vectors by matrices. If the column vectors of the $m\times n$ matrix A are $\vec{c}_1,\ldots,\vec{c}_n$ in $\mathbf{R}^m$ (i.e. $A=[\vec{c}_1|\cdots|\vec{c}_n]$), then $x_1\vec{c}_1+\ldots+x_n\vec{c}_n=A\vec{x}$ where $\vec{x}=\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}$. For example, we have

$$x_1\begin{bmatrix}1\\3\end{bmatrix}+x_2\begin{bmatrix}-4\\2\end{bmatrix}=\begin{bmatrix}1&-4\\3&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}x_1-4x_2\\3x_1+2x_2\end{bmatrix}.$$

The Situation. Let $\vec{r}=\begin{bmatrix}2\\1\end{bmatrix}$, $\vec{s}=\begin{bmatrix}3\\2\end{bmatrix}$, $\vec{t}=\begin{bmatrix}5\\4\end{bmatrix}$, $\vec{u}=\begin{bmatrix}4\\1\end{bmatrix}$. No pair of these four vectors is collinear; every pair spans all of $\mathbf{R}^2$. Any one of these vectors is a linear combination of any pair of them. Moreover, there is a non-trivial linear combination of any triplet whose value is the zero vector. Your objective is to demonstrate two of these assertions.

Problem 1. $\vec{t}$ is a linear combination of $\vec{r}$ and $\vec{s}$. This means that there are scalars α and β so that $\alpha\vec{r}+\beta\vec{s}=\vec{t}$. Show that this vector equation can be written as an equivalent matrix equation in the form $A\vec{x}=\vec{b}$. Identify A, $\vec{x}$, and $\vec{b}$ and, by row-reduction, solve for α and β. How many choices are there for these two scalars?

The equation $\alpha\vec{r}+\beta\vec{s}=\vec{t}$ is equivalent to $A\vec{x}=\vec{b}$ where $A=[\vec{r}\,|\,\vec{s}]=\begin{bmatrix}2&3\\1&2\end{bmatrix}$, $\vec{x}=\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$, and $\vec{b}=\vec{t}=\begin{bmatrix}5\\4\end{bmatrix}$. We find, by row-reduction, that $[A|\vec{b}]=\left[\begin{array}{cc:c}2&3&5\\1&2&4\end{array}\right]\leftrightarrow\left[\begin{array}{cc:c}1&0&-2\\0&1&3\end{array}\right]=[A|\vec{b}]_{rref}$. Hence, there is exactly one solution, namely $\begin{bmatrix}\alpha\\ \beta\end{bmatrix}=\begin{bmatrix}-2\\3\end{bmatrix}$.

Problem 2. Proceeding as you did above, find all non-trivial (not all zero) scalars σ, ρ and λ so that $\sigma\vec{s} + \rho\vec{t} + \lambda\vec{u} = \vec{0}$. How many different choices are there for these three scalars?

Now, we seek the solutions of $A\vec{x} = \vec{b}$ where $A = [\vec{s} \mid \vec{t} \mid \vec{u}] = \begin{bmatrix} 3 & 5 & 4 \\ 2 & 4 & 1 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} \sigma \\ \rho \\ \lambda \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Using row-reduction, we find $[A \mid \vec{b}] = \left[\begin{array}{ccc:c} 3 & 5 & 4 & 0 \\ 2 & 4 & 1 & 0 \end{array}\right] \leftrightarrow \left[\begin{array}{ccc:c} 1 & 0 & \frac{11}{2} & 0 \\ 0 & 1 & -\frac{5}{2} & 0 \end{array}\right] = [A \mid \vec{b}]_{rref}$. Therefore, by the Solution Algorithm, $\vec{x} = \begin{bmatrix} \sigma \\ \rho \\ \lambda \end{bmatrix} = \alpha \begin{bmatrix} 11 \\ -5 \\ -2 \end{bmatrix}$ where $\alpha \in \mathbf{R}$ and so, there are infinitely many non-trivial solutions, one for each choice of $\alpha \neq 0$.