SM261. Final Examination. 5 May 2006.

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1. A linear system of four equations in five variables is equivalent to the single matrix equation $A\vec{x} = \vec{b}$ where $A = \begin{bmatrix} 1 & 2 & 3 & 1 & 3 \\ 0 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 4 \\ 2 & 2 & 2 & 2 & 6 \end{bmatrix}$.

   a. Determine A_{rref}.

   b. Determine a basis for and the dimension of im(A).

   c. Determine a basis for and the dimension of ker(A).

   d. Completely discuss the nature of the solution set of $A\vec{x} = \vec{b}$ using the results of parts b and c.

Show all work.

2. $L = (\vec{v}_1, \ldots, \vec{v}_p)$ is a list of p vectors in $\mathbf{R}^n$ and the $n \times p$ matrix $\left[\vec{v}_1 \mid \cdots \mid \vec{v}_p\right]_{rref}$ has r pivots (leading ones). Describe, in terms of n, p, and r when

a. L is linearly independent.

b. L spans $\mathbf{R}^n$.

3. Solve for the 3×3 matrix X if $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 0 & 1 \end{bmatrix} X = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix}$.

Show all work.

4. Find values for the coefficients a and b so that the line with equation $a\,x + b\,y = 1$ is a best fit, in the least-squares sense, to the following (x,y)-data: (0,1), (1,0), (1,2), and (2,2).

5. The four vectors $\vec{b}$, $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ belong to $\mathbf{R}^3$. Solve the matrix equation $[\vec{v}_1 \mid \vec{v}_2 \mid \vec{v}_3]\,\vec{x} = \vec{b}$ for $\vec{x}$, if $\det\left([\vec{v}_1 \mid \vec{v}_2 \mid \vec{v}_3]\right) = 8$, $\det\left(\left[\vec{b} \mid \vec{v}_2 \mid \vec{v}_3\right]\right) = 19$, $\det\left(\left[\vec{v}_1 \mid \vec{b} \mid \vec{v}_3\right]\right) = -38$, and $\det\left(\left[\vec{v}_1 \mid \vec{v}_2 \mid \vec{b}\right]\right) = 19$.

Show all work.

6. Evaluate $\det\left(\begin{bmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 2 & 2 & 3 & 4 & \cdots & n \\ 3 & 3 & 3 & 4 & \cdots & n \\ 4 & 4 & 4 & 4 & \cdots & n \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ n & n & n & n & \cdots & n \end{bmatrix}\right)$ for any positive integer n.

7. An inner product for $\mathbf{P}_2$, the vector space of polynomials of degree 2 or less, is defined by $f \cdot g = \int_{-1}^{1} f(t)g(t)\,dt$ for any two polynomials f and g in $\mathbf{P}_2$. If $h(t) = 1 + t$ and $k(t) = 1 + t^2$, find $\ell(t)$, where ℓ is the projection of h orthogonal to k.

Show all work.

8. $A=\begin{bmatrix}1&1&1\\0&0&1\\1&0&1\\0&1&0\end{bmatrix}=\begin{bmatrix}\sqrt{2}/2&\sqrt{6}/6&0\\0&0&1\\\sqrt{2}/2&-\sqrt{6}/6&0\\0&-\sqrt{6}/3&0\end{bmatrix}\begin{bmatrix}\sqrt{2}&\sqrt{2}/2&\sqrt{2}\\0&\sqrt{6}/6&0\\0&0&1\end{bmatrix}$ provides the QR-factorization of the 4×3 matrix A into the product of the 4×3 matrix Q and the 3×3 matrix R.

a. How are the column vectors of Q obtained from those of A?

b. What is the geometrical significance of the diagonal entries of R?

9. Determine expressions for each entry of the matrix $\begin{bmatrix}-1&2\\-4&5\end{bmatrix}^n$ for any positive integer n.

Show all work.

10. Let $S = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}\right)$. $T : \mathbf{R}^3 \to \mathbf{R}^3$ is the linear transformation that leaves unchanged each vector in S and doubles each vector in $S^{\perp}$. Let A be the 3×3 matrix so that $T(\vec{x}) = A\,\vec{x}$ for any $\vec{x}$ in $\mathbf{R}^3$.

a. Determine A.

b. What are the eigenvalues of A and their algebraic multiplicities?

c. Describe the eigenspaces of A in terms of S and $S^{\perp}$.

d. Determine all diagonal matrices similar to A.

Show all work.

11. The matrix $Q = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$ represents a rotation in $\mathbf{R}^3$ because it is orthogonal and has determinant +1. Therefore, the linear transformation $\vec{x} \mapsto Q\vec{x}$ preserves the length of any vector, the angle of any pair of vectors, and the orientation of any triplet of vectors. Now, a rotation is completely characterized by its axis and its angle and, conversely, given a rotation, its axis and angle are determined.

a. Determine the rotation axis for Q by finding the vectors parallel to the rotation axis. [Hint: Solve the equation that describes the effect of the rotation on a vector parallel to the axis of rotation.]

b. Determine the rotation angle for Q. [Hint: This is the angle between any vector orthogonal to the rotation axis and that vector's image under the rotation.]

Show all work.

12. List as many distinct statements as you can that are equivalent to

The $n \times n$ matrix A is non-singular.