SM261 Final Examination

02 May 2014 / 0755

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1. (16 pts) Let $\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$.

   a. Find a vector of length 9 parallel to $\vec{v}$.

$\|\vec{v}\| = \sqrt{2^2+1^2+2^2} = 3$ so $3\vec{v} = \begin{bmatrix} 6 \\ 3 \\ 6 \end{bmatrix}$ has length 9

   b. Find the cosine of the angle between $\vec{v}$ and $\vec{w}$.

$\vec{v}\cdot\vec{w} = (\cos\theta)\|\vec{v}\|\|\vec{w}\| \Rightarrow \cos\theta = \dfrac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|} = \dfrac{4}{3\cdot 3} = \dfrac{4}{9}$

   c. Show that the Cauchy-Schwarz Inequality is valid for $\vec{v}$ and $\vec{w}$.

Cauchy-Schwarz Inequality: $|\vec{v}\cdot\vec{w}| \le \|\vec{v}\|\,\|\vec{w}\|$

Here $\vec{v}\cdot\vec{w} = 4$ and $\|\vec{v}\| = 3 = \|\vec{w}\|$, and $4 \le 3\cdot 3$

   d. Find $proj_L(\vec{w})$, where L is the line through the origin and parallel to $\vec{v}$.

$\text{proj}_L\vec{w} = (\vec{w}\cdot\vec{u})\vec{u}$, where $\vec{u}$ is the unit vector parallel to $\vec{v}$, i.e., $\vec{u} = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$

So $\text{proj}_L\vec{w} = \left(\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}\cdot\begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}\right)\begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$

$= \frac{4}{3}\begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 8/9 \\ 4/9 \\ 8/9 \end{bmatrix}$

2. (14 pts) Let $\vec{v}$ be the vector in the previous problem.

a. Define: "T is a linear transformation from R^m to R^n."

There is an $n \times m$ matrix A such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x}$ in $\mathbb{R}^m$.

b. Show that the mapping T on R^3 given by $T(\vec{x}) = proj_L(\vec{x})$ is a linear transformation.

A mapping T is linear if and only if 1) $T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$ and 2) $T(a\vec{x}) = aT(\vec{x})$ for all $a \in \mathbb{R}$. But $T(\vec{x}) = (\vec{u}\cdot\vec{x})\vec{u}$, so 1) $T(\vec{x}+\vec{y}) = (\vec{u}\cdot(\vec{x}+\vec{y}))\vec{u} = (\vec{u}\cdot\vec{x})\vec{u} + (\vec{u}\cdot\vec{y})\vec{u} = T(\vec{x}) + T(\vec{y})$ and 2) $T(a\vec{x}) = (\vec{u}\cdot a\vec{x})\vec{u} = a(\vec{u}\cdot\vec{x})\vec{u} = aT(\vec{x})$.

c. Find the matrix A that satisfies $T(\vec{x}) = A\vec{x}$ for all $\vec{x}$ in R^3.

$$\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \vec{u} = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}; \ proj_L(\vec{x}) = (\vec{x}\cdot\vec{u})\vec{u} = \left(\tfrac{2}{3}x_1 + \tfrac{1}{3}x_2 + \tfrac{2}{3}x_3\right)\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \begin{matrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{matrix}$$

$$= \begin{bmatrix} \frac{4}{9}x_1 + \frac{2}{9}x_2 + \frac{4}{9}x_3 \\ \frac{2}{9}x_1 + \frac{1}{9}x_2 + \frac{4}{9}x_3 \\ \frac{4}{9}x_1 + \frac{2}{9}x_2 + \frac{4}{9}x_3 \end{bmatrix} = \begin{bmatrix} 4/9 & 2/9 & 4/9 \\ 2/9 & 1/9 & 4/9 \\ 4/9 & 2/9 & 4/9 \end{bmatrix}\vec{x}$$

3. (12 pts) Write the system below as a matrix equation and use Gaussian elimination techniques to find all solutions.

$$\begin{aligned} x + 2y - z &= 2 \\ x - y + z &= 1 \\ 3x + 0y + z &= 4. \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 1 & -1 & 1 & 1 \\ 3 & 0 & 1 & 4 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -3 & 2 & -1 \\ 0 & -6 & 4 & -2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & -2/3 & 1/3 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{3} & \frac{4}{3} \\ 0 & 1 & -\frac{2}{3} & 1/3 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{aligned} x_1 + \tfrac{1}{3}x_3 &= \tfrac{4}{3} \\ x_2 - \tfrac{2}{3}x_3 &= 1/3 \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4/3 - \frac{1}{3}s \\ 1/3 + \frac{2}{3}s \\ s \end{bmatrix},$$

4. (15 pts) Let $A = \begin{bmatrix} 1 & 2 & 2 & -5 & 6 \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & -8 & 9 \\ 3 & 6 & 1 & 5 & -7 \end{bmatrix}$. $\rightarrow \begin{bmatrix} 1 & 2 & 2 & -5 & 6 \\ 0 & 0 & 1 & -4 & 5 \\ 0 & 0 & -3 & 12 & -15 \\ 0 & 0 & -5 & 20 & -25 \end{bmatrix}$

a. Show (by hand) that $rref(A) = \begin{bmatrix} 1 & 2 & 0 & 3 & -4 \\ 0 & 0 & 1 & -4 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. $\leftarrow$

$$x_1 + 2\overset{s}{x_2} \quad + 3\overset{t}{x_4} - 4\overset{u}{x_5} = 0$$
$$x_3 - 4x_4 + 5x_5 = 0$$

b. Find a basis for the kernel of A.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s - 3t + 4u \\ s \\ 4t - 5u \\ t \\ u \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix} + u\begin{bmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix}. \quad \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right\}$$

c. Find a basis for the image of A.

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 5 \\ 1 \end{bmatrix} \right\}$$

5. (14 pts) Let A be an invertible $n \times n$ matrix. What can you say about each of the following?

   a. The rank of A.

   The rank is n

   b. $rref(A)$. is I, the identity matrix

   c. The kernel of A. is $\{\vec{0}\}$

   d. The image of A. is $\mathbb{R}^n$

   e. The column vectors of A. form a basis for $\mathbb{R}^n$.

   f. The number of solutions to the system $A\vec{x} = \vec{b}$, where $\vec{b}$ is any fixed vector. There is one and only one solution.

   g. $\det(A)$. is not zero.

6. (12 pts) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{bmatrix}$.

   a. Use row operation techniques to find A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ 3 & 8 & 2 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 5 & -1 & -3 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 3 & -1 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & -1 & 7 & -5 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 10 & -6 & 1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -7 & 5 & -1 \end{array}\right]$$

   b. Use your answer in a. to solve the matrix equation $A\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

$$\vec{x} = A^{-1}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 & -6 & 1 \\ -2 & 1 & 0 \\ -7 & 5 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}.$$

7. (10 pts) Consider the linear transformation T on R^2 whose matrix is given by the product $\begin{bmatrix} \cos(\frac{\pi}{3}) & -\sin(\frac{\pi}{3}) \\ \sin(\frac{\pi}{3}) & \cos(\frac{\pi}{3}) \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. Describe $T(\vec{x})$ geometrically. Hint: describe what each matrix does separately to a vector.

The transformation doubles the length of the vector, then flips (reflects) the vector about the line $y=x$, then rotates counterclockwise by an angle of $\pi/3$.

8. (15 pts) Let $\mathscr{B} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$.

a. Prove that $\mathscr{B}$ is a basis for R^2.

$$\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

b. Let $\vec{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. Find $[\vec{v}]_{\mathscr{B}}$.

$$\left[\begin{array}{cc|c} 1 & -2 & 3 \\ 2 & 1 & 4 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & 5 & -2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & 1 & -\frac{2}{5} \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 11/5 \\ 0 & 1 & -2/5 \end{array}\right]$$

$$[\vec{v}]_{\mathscr{B}} = \begin{bmatrix} 11/5 \\ -2/5 \end{bmatrix}$$

c. Let T be the linear transformation on R^2 whose matrix is $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Find the matrix of T with respect to the basis $\mathscr{B}$.

$$S = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}, \quad S^{-1} = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

$$B = S^{-1} A S = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} -2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 0 & 5 \\ -5 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

9. (12 pts) Consider a reflection matrix A and a vector $\vec{x}$ in R^2. Define $\vec{v} = \vec{x} + A\vec{x}$ and $\vec{w} = \vec{x} - A\vec{x}$. (Recall that $A(A\vec{x}) = \vec{x}$.)

a. Express $A\vec{v}$ in terms of $\vec{v}$.

$$A\vec{v} = A\vec{x} + A^2\vec{x} = A\vec{x} + \vec{x} = \vec{x} + A\vec{x} = \vec{v}$$

b. Express $A\vec{w}$ in terms of $\vec{w}$.

$$A\vec{w} = A\vec{x} - A^2\vec{x} = A\vec{x} - \vec{x} = -\vec{w}$$

c. If the vectors $\vec{v}$ and $\vec{w}$ are both nonzero, show that they are orthogonal.

$$(\vec{x} + A\vec{x})\cdot(\vec{x} - A\vec{x}) = \vec{x}\cdot\vec{x} + \cancel{A\vec{x}\cdot\vec{x}} - A\vec{x}\cdot A\vec{x} - \cancel{\vec{x}\cdot A\vec{x}}$$

$$= \vec{x}\cdot\vec{x} - A\vec{x}\cdot A\vec{x} = \vec{x}\cdot\vec{x} - \vec{x}\cdot\vec{x}$$ b/c reflections preserve length

d. If the vector $\vec{v}$ is nonzero, what is the relationship between $\vec{v}$ and the line L of reflection?

$A\vec{v} = \vec{v} \Rightarrow \vec{v}$ on L

10. (12 pts) A 2×2 matrix A is called nilpotent if $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

a. Show that there is a nonzero vector in $\ker(A)$. (Hence $\lambda = 0$ is an eigenvalue of A.)

i) $A \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, then $\exists\, \vec{x} \ni A\vec{x} \neq \vec{0}$, then $A(A\vec{x}) = \vec{0} \Rightarrow A\vec{x} \in \ker A$

ii) $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ then $A\vec{x} = 0\vec{x} \quad \forall \vec{x}$

b. Show that 0 is the <u>only</u> eigenvalue of A.

If $A\vec{x} = \lambda\vec{x} \Rightarrow A^2\vec{x} = \lambda A\vec{x} = \lambda^2\vec{x}$

$\Rightarrow 0\vec{x} = \lambda^2\vec{x} \Rightarrow \lambda = 0.$

11. (12 pts) Let $M = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 \\ 1 & 0 & 3 & 0 \\ 2 & 0 & 2 & 3 \end{bmatrix}$. Use row operation methods to find $\det(M)$.

$$\det \begin{bmatrix} 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 \\ 1 & 0 & 3 & 0 \\ 2 & 0 & 2 & 3 \end{bmatrix} = - \begin{vmatrix} 1 & 0 & 3 & 0 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 2 & 0 & 2 & 3 \end{vmatrix} = - \begin{vmatrix} 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 4 \\ 0 & 3 & -2 & 5 \\ 0 & 0 & -4 & 3 \end{vmatrix} = -2 \begin{vmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 3 & -2 & 5 \\ 0 & 0 & -4 & 3 \end{vmatrix} =$$

$$-2 \begin{vmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & -4 & 3 \end{vmatrix} = -2 \begin{vmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & 5 \end{vmatrix} = -2 \cdot 1 \cdot 1 \cdot -2 \cdot 5 = 20$$

12. (16 pts) Let $A = \begin{bmatrix} 1 & \cancel{2} & 0 \\ 7 & \cancel{14} & 7 \\ 1 & \cancel{2} & 2 \\ 7 & \cancel{14} & 7 \end{bmatrix}$.

a. Find a basis for $im(A)$.

$$\underset{\vec{v}_1}{\begin{bmatrix} 1 \\ 7 \\ 1 \\ 7 \end{bmatrix}} \underset{\vec{v}_2}{\begin{bmatrix} 0 \\ 7 \\ 2 \\ 7 \end{bmatrix}}$$

b. Find an orthonormal basis for $im(A)$.

$$\vec{u}_1 = \frac{1}{\|\vec{v}_1\|} \vec{v}_1 = \begin{bmatrix} 1/10 \\ 7/10 \\ 1/10 \\ 7/10 \end{bmatrix}$$

$$\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1 = \begin{bmatrix} 0 \\ 7 \\ 2 \\ 7 \end{bmatrix} - 10\vec{u}_1 = \begin{bmatrix} 7 \\ 2 \\ 7 \end{bmatrix} - \begin{bmatrix} 1 \\ 7 \\ 1 \\ 7 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

c. Find $proj_{im(A)}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right)$. Hint: use the orthonormal basis from b.

$$= (\vec{v} \cdot \vec{u}_1)\vec{u}_1 + (\vec{v} \cdot \vec{u}_2)\vec{u}_2$$

13. (12 pts) Let A be a square matrix.

a. Define "A is a symmetric matrix". $A^T = A$

b. Define "A is an orthogonal matrix". A is the matrix corresponding to an orthogonal linear transformation, so $\|A\vec{x}\| = \|\vec{x}\|$ for all vectors, $\vec{x}$.

c. Find all symmetric orthogonal 2×2 matrices. Hint: first determine what orthogonal 2×2 matrices look like.

$$A^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = A \Rightarrow \sin\theta = 0. \quad \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}^T = A : \text{all are symmetric.}$$

14. (15 pts) Let $A = \begin{bmatrix} -1 & 0 & 1 \\ -3 & 0 & 1 \\ -4 & 0 & 3 \end{bmatrix}$.

4 a. Show that 1 and 0 are the eigenvalues of A.

$$|A - \lambda I| = \begin{vmatrix} -1-\lambda & 0 & 1 \\ -3 & -\lambda & 1 \\ -4 & 0 & 3-\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -1-\lambda & 1 \\ -4 & 3-\lambda \end{vmatrix} = -\lambda\left((\lambda-3)(\lambda+1)+4\right) = -\lambda\left(\lambda^2 - 2\lambda + 1\right) = -\lambda(\lambda-1)^2$$

So $\lambda = 0$ has algebraic multiplicity $\underline{1}$, and $\underline{1}$ has algebraic multiplicity 2.

8 b. Find a basis of each eigenspace.

$\lambda = 0$: $s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for E_0.

$$\lambda = 1: \quad A - I = \begin{bmatrix} -2 & 0 & 1 \\ -3 & -1 & 1 \\ -4 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ -3 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & -1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 - \tfrac{1}{2}x_3 &= 0 \\ x_2 + \tfrac{1}{2}x_3 &= 0 \end{aligned} \quad \begin{bmatrix} \frac{1}{2}t \\ -\frac{1}{2}t \\ t \end{bmatrix} \Rightarrow \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\} \text{ is a basis.}$$

c. Can the matrix A be diagonalized? Explain.

No. The geometric and algebraic multiplicity of the eigenvalue $\lambda = 1$ do not agree.

0
1