Special Problem D Solutions.

These exercises invite you to explore a few algebraic and geometric properties of elementary linear transformations in Euclidean 2-space.

1. Let R_θ denote the 2×2 matrix for rotation in $\mathbf{R}^2$ through the angle θ. We have seen that $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

   a. Verify by direct computation that $R_{\alpha+\beta} = R_\alpha R_\beta = R_\beta R_\alpha$.

$$R_{\alpha+\beta} = \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$$

$$R_\alpha R_\beta = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} (\cos\alpha)(\cos\beta) - (\sin\alpha)(\sin\beta) & -(\cos\alpha)(\sin\beta) - (\sin\alpha)(\cos\beta) \\ (\sin\alpha)(\cos\beta) + (\cos\alpha)(\sin\beta) & (\cos\alpha)(\cos\beta) - (\sin\alpha)(\sin\beta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$$

$$R_\beta R_\alpha = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} (\cos\beta)(\cos\alpha) - (\sin\beta)(\sin\alpha) & -(\cos\beta)(\sin\alpha) - (\sin\beta)(\cos\alpha) \\ (\sin\beta)(\cos\alpha) + (\cos\beta)(\sin\alpha) & (\cos\beta)(\cos\alpha) - (\sin\beta)(\sin\alpha) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\beta+\alpha) & -\sin(\beta+\alpha) \\ \sin(\beta+\alpha) & \cos(\beta+\alpha) \end{bmatrix}$$

In the above computations, we made use of the trigonometric identities for the sine and cosine of a sum of angles. The three matrices are the same.

   b. What are the geometrical interpretations of these formulae?

In the plane, a rotation by α followed by a rotation by β is the same as performing the rotations in the reverse order and is the same as performing a single rotation by the sum, $\alpha + \beta$, of the two angles.

2. Let H_s be the 2×2 matrix for horizontal scaling in $\mathbf{R}^2$ by the factor s. So,

$$H_s = \begin{bmatrix} s & 0 \\ 0 & 1 \end{bmatrix}.$$

   a. Verify that $H_s R_\theta \neq R_\theta H_s$.

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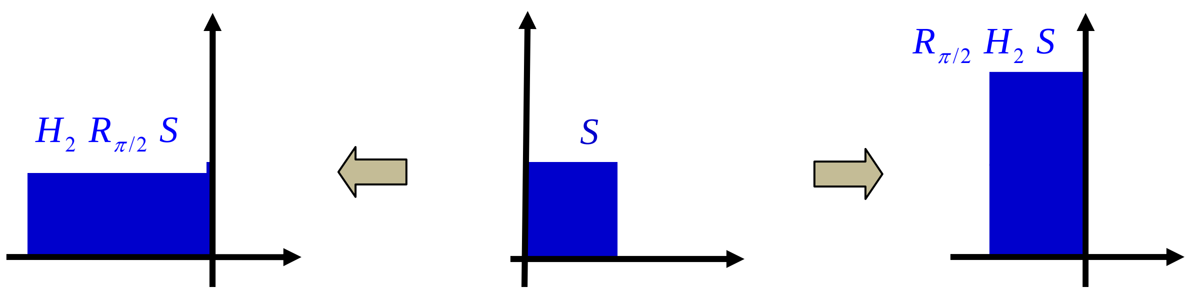
$$H_s\,R_\theta = \begin{bmatrix} s & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} s\cdot\cos\theta & -s\cdot\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \neq$$

$$R_\theta\,H_s = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} s & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} s\cdot\cos\theta & -\sin\theta \\ s\cdot\sin\theta & \cos\theta \end{bmatrix}$$

Clearly the two matrices are different if $s \neq 1$.

b. Convince yourself that, in general, rotation and horizontal scaling do not commute by considering the effect of $H_2\,R_{\pi/2}$ and $R_{\pi/2}\,H_2$ on all the points in the unit square consisting of points (x, y) such that $0 \le x \le 1$ and $0 \le y \le 1$. Draw pictures to illustrate.

Let S be the unit square. Then,



3. Let $F_\theta = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$ be the 2×2 matrix for reflection across the line through the origin in $\mathbf{R}^2$ at the angle θ relative to the horizontal.

a. Verify that two successive reflections across the same line through the origin is the identity matrix which is the linear transformation that leaves all vectors unchanged.

$$F_\theta\,F_\theta = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2(2\theta)+\sin^2(2\theta) & 0 \\ 0 & \sin^2(2\theta)+\cos^2(2\theta) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

b. Now, consider two reflections, one followed by the other, across two lines through the origin at angles α and β with respect to the horizontal. What is the matrix for the composite transformation? Simplify it. This composite of two reflections is, in fact, a simple elementary transformation of another kind. Identify this transformation.

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$$F_\theta F_\phi = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(2\theta)\cos(2\phi) + \sin(2\theta)\sin(2\phi) & \cos(2\theta)\sin(2\phi) - \sin(2\theta)\cos(2\phi) \\ \sin(2\theta)\cos(2\phi) - \cos(2\theta)\sin(2\phi) & \sin(2\theta)\sin(2\phi) + \cos(2\theta)\cos(2\phi) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(2(\theta-\phi)) & -\sin(2(\theta-\phi)) \\ \sin(2(\theta-\phi)) & \cos(2(\theta-\phi)) \end{bmatrix} = R_{2(\theta-\phi)}.$$

So, the composite of two reflections across lines through the origin at angles ϕ and then θ is a rotation by twice the difference between the angles, namely $2(\theta - \phi)$.

4. Define $Horz_a = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$, $Vert_b = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$ as the 2×2 matrices that represent the horizontal and vertical shears of size a and b in $\mathbf{R}^2$.

a. Do two horizontal shears commute? Does a horizontal shear commute with a horizontal scaling? Does a horizontal and a vertical shear commute?

$$Horz_a\, Horz_b = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+b \\ 0 & 1 \end{bmatrix} =$$

$$Horz_b\, Horz_a = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & b+a \\ 0 & 1 \end{bmatrix}$$

$$Horz_a\, H_b = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} b & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} b & a \\ 0 & 1 \end{bmatrix} \neq$$

$$H_b\, Horz_a = \begin{bmatrix} b & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} b & ab \\ 0 & 1 \end{bmatrix}$$

$$Horz_a\, Vert_b = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} = \begin{bmatrix} 1+ab & a \\ b & 1 \end{bmatrix} \neq$$

$$Vert_b\, Horz_a = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a \\ b & 1+ab \end{bmatrix}$$

So, two horizontal shears do commute. But, a horizontal shear and a horizontal scaling don't commute and a horizontal shear and a vertical shear don't commute.

b. Sketch $\{(u,v) \in \mathbf{R}^2 \mid -\frac{1}{2} \le u,v \le \frac{1}{2}\}$ and its image under the $\vec{x} \mapsto Horz_2\, \vec{x}$.