1. $C[0,1]$ is the collection of all real-valued functions of a real variable that are defined and continuous on the closed interval [0, 1]. Using the usual rules for addition of two functions and the multiplication of a function by a scalar, it is straightforward to verify that this is a vector space. However, it is not a finite dimensional vector space since it has no finite basis. [For example, $C[0,1]$ contains all the monomials (the functions m_k where $m_k(x) = x^k$ and k is any non-negative integer). But, these functions are infinite in number and comprise a linearly independent collection.]

Let $f \cdot g \equiv \int_0^1 f(x)g(x)\,dx$ for any f and g in $C[0,1]$.

a. Verify that the definition above is a scalar product on $C[0,1]$.

Suppose that f, g and h are any functions that are continuous on the interval [0, 1]. First, it is evident that $f \cdot g$ is a real number. Then, we observe that $f \cdot f = \int_0^1 f(x)\,f(x)\,dx = \int_0^1 (f(x))^2\,dx \geq 0$ since this is the integral of a nonnegative continuous function. Indeed, the only case where it is 0 is if f is zero. $f \cdot g$ is commutative since $f \cdot g = \int_0^1 f(x)\,g(x)\,dx$ $= \int_0^1 g(x)\,f(x)\,dx = g \cdot h$. Too, it is distributive since $f \cdot (\alpha g + \beta h) =$ $\int_0^1 f(x)\big(\alpha g(x) + \beta h(x)\big)\,dx = \alpha \int_0^1 f(x)\,g(x)\,dx + \beta \int_0^1 f(x)\,h(x)\,dx$ $= \alpha f \cdot g + \beta f \cdot h$. This verifies that the definition given here is a scalar product.

Let $s_k(x) \equiv \sqrt{2}\sin(k\pi x)$ for k = 1, 2, ... Then, $s_k \in C[0,1]$ for every k.

b. Verify that the list $(s_1, s_2, s_3, \ldots)$ is orthonormal.

If $j \neq k$, then $s_j \cdot s_k = \int_0^1 s_j(x)\,s_k(x)\,dx = 2\int_0^1 \sin(j\pi x)\sin(k\pi x)\,dx =$

$$2\left(\left.\frac{\sin((j-k)\pi x)}{2(j-k)}\right|_0^1 - \left.\frac{\sin((j+k)\pi x)}{2(j+k)}\right|_0^1 \right) = 2\big((0-0)-(0-0)\big) = 0.$$

$$s_j \cdot s_j = \int_0^1 \big(s_j(x)\big)^2\,dx = 2\int_0^1 \big(\sin(j\pi x)\big)^2\,dx = 2\left.\left(\frac{x}{2} - \frac{\sin(2a\pi x)}{4a\pi}\right)\right|_0^1 =$$

$$2\left(\left(\frac{1}{2}-0\right)-(0-0)\right) = 1.$$ Therefore, $s_j \cdot s_k = \delta_{jk}$.

Let $h(x) \equiv x - x^2$. Then, $h \in C[0,1]$. Let V_N be the subspace of $C[0,1]$ spanned by $(s_1, s_2, \ldots, s_N)$ for any positive integer N. Clearly, $h \notin V_N$. Let $h_{(N)}$ be the orthogonal projection of h onto V_N. $h_{(N)} \in V_N$.

c. Compute $h_{(N)}(x)$ for any positive integer N.

$h_{(N)}(x) = (h \cdot s_1)s_1(x) + (h \cdot s_2)s_2(x) + \ldots + (h \cdot s_N)s_N(x)$. So, we need to compute $h \cdot s_j = \int_0^1 h(x)s_j(x)\,dx = \sqrt{2}\int_0^1 (x - x^2)\sin(j\pi x)\,dx =$

$$\sqrt{2}\left(\frac{j\pi(1-2x)\sin(j\pi x) - \left((j\pi)^2(x-x^2)+2\right)\cos(j\pi x)}{(j\pi)^3}\right)\Bigg|_0^1 =$$

$$\sqrt{2}\left(\frac{0-\left((j\pi)^2(0)+2\right)\cos(j\pi)}{(j\pi)^3} - \frac{0-\left((j\pi)^2(0)+2\right)}{(j\pi)^3}\right) = \begin{cases} \dfrac{4\sqrt{2}}{(j\pi)^3} & \text{if } j \text{ is odd} \\ 0 & \text{if } j \text{ is even} \end{cases}$$

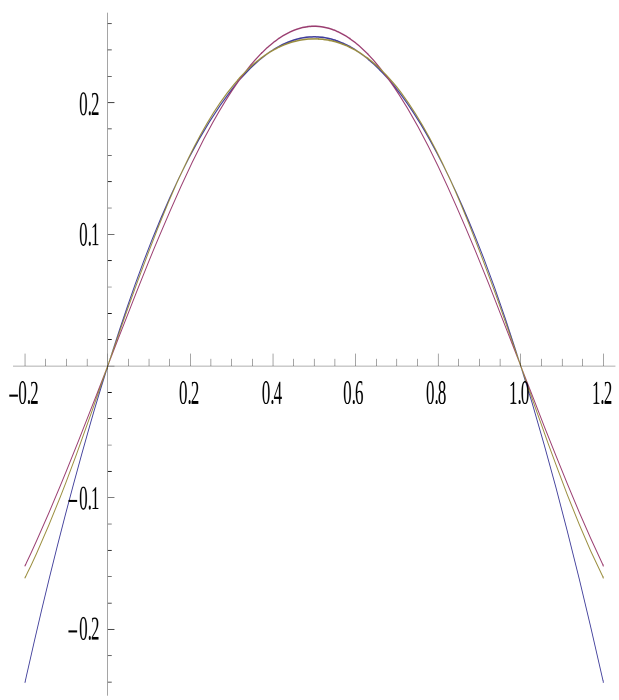
[For integers k, $\sin(k\pi) = 0$, $\cos(k\pi) = 1$ (k even), $\cos(k\pi) = -1$ (k odd).]

$$h_{(1)}(x) = h_{(2)}(x) = \frac{8}{\pi^3}\sin(\pi x), \quad h_{(3)}(x) = h_{(4)}(x) = \frac{8}{\pi^3}\left(\sin(\pi x) + \frac{1}{3^3}\sin(3\pi x)\right),$$

$$h_{(5)}(x) = h_{(6)}(x) = \frac{8}{\pi^3}\left(\sin(\pi x) + \frac{1}{3^3}\sin(3\pi x) + \frac{1}{5^3}\sin(5\pi x)\right), \quad \ldots$$

d. Compare h, $h_{(1)}$, and $h_{(3)}$ graphically [with a computer] or by tabulating their values over the interval $[-0.2, 1.2]$.

x	$h(x)$	$h_1(x)$	$h_3(x)$
-0.2	-0.240	-0.152	-0.161
-0.1	-0.110	-0.080	-0.087
0.0	0.000	0.000	0.000
0.1	0.090	0.080	0.087
0.2	0.160	0.152	0.161
0.3	0.210	0.209	0.212
0.4	0.240	0.245	0.240
0.5	0.250	0.258	0.248
0.6	0.240	0.245	0.240
0.7	0.210	0.209	0.212
0.8	0.160	0.152	0.161
0.9	0.090	0.080	0.087
1.0	0.000	0.000	0.000
1.1	-0.110	-0.080	-0.087
1.2	-0.240	-0.152	-0.161



2. Let P_2 be the collection of all polynomials of degree at most 2. With the usual definition of addition of functions and multiplication of functions by scalars, we know that this is a 3-dimensional vector space. Now let $f \cdot g \equiv \int_{-1}^{1} f(x)g(x)\,dx$ for any f and g in P_2. This defines a scalar product on P_2.

a. Use the Gram-Schmidt process to obtain an orthogonal (it will not be orthonormal) basis (ℓ_0, ℓ_1, ℓ_2) for P_2 such that $\ell_k(1) = 1$ for k = 0, 1, 2. Start with the natural basis (m_0, m_1, m_2) where $m_k(x) = x^k$, for k = 0, 1, 2.

The two polynomials m_0 and m_1 are already orthogonal and have the value 1 at 1 since $m_0 \cdot m_1 = \int_{-1}^{1} m_0(x)\,m_1(x)\,dx = \int_{-1}^{1} x\,dx = 0$ and $m_0(1) = m_1(1) = 1$. So, we choose $\ell_0(x) = m_0(x) = 1$ and $\ell_1(x) = m_1(x) = x$. To obtain the third polynomial in our orthogonal triplet, we use the Gram-Schmidt process, that is, we determine that part of the squaring function that is orthogonal to ℓ_0 and ℓ_1.

$$m_2^{\perp} = m_2 - \left(m_2 \cdot \frac{\ell_0}{\|\ell_0\|}\right)\frac{\ell_0}{\|\ell_0\|} - \left(m_2 \cdot \frac{\ell_1}{\|\ell_1\|}\right)\frac{\ell_1}{\|\ell_1\|} = m_2 - \left(\frac{m_2 \cdot \ell_0}{\|\ell_0\|^2}\right)\ell_0 - \left(\frac{m_2 \cdot \ell_1}{\|\ell_1\|^2}\right)\ell_1.$$

So, we need to calculate some scalar products. $m_2 \cdot \ell_0 = \int_{-1}^{1} x^2 1\,dx = \frac{2}{3}$, $m_2 \cdot \ell_1 = \int_{-1}^{1} x^2\,x\,dx = 0$, $\|\ell_0\|^2 = \ell_0 \cdot \ell_0 = \int_{-1}^{1} (1)(1)\,dx = 2$. Substituting above in the equation for $m_2^{\perp}$, we obtain $m_2^{\perp} = m_2 - \left(\frac{2/3}{2}\right)\ell_0 - (0)\ell_1 = m_2 - \frac{1}{3}\ell_0$.

So, we choose $\ell_2 = m_2^{\perp} / m_2^{\perp}(1) = \frac{3}{2}(m_2 - \frac{1}{3}\ell_0) = \frac{1}{2}(3m_2 - \ell_0)$. To summarize:

$$\ell_0(x) = 1, \quad \ell_1(x) = x, \quad \ell_2(x) = \tfrac{1}{2}(3x^2 - 1).$$

The basis (ℓ_0, ℓ_1, ℓ_2) spans P_2, is an orthogonal list and each polynomial in it has a graph that passes through the point (1, 1).