1. Consider the matrix system $A\vec{x}=\vec{b}$ where $A=\begin{bmatrix}1 & 1 & -1 & -1 & 1\\ 1 & 0 & -1 & 0 & 0\\ 0 & 2 & 0 & -2 & 2\end{bmatrix}$,

$\vec{x}=\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\\ x_5\end{bmatrix}$, and $\vec{b}=\begin{bmatrix}10\\ 8\\ 4\end{bmatrix}$.

a. Find all solutions $\vec{x}$ to the system. Write your answer in column vector form.

$$[A|\vec{b}]=\left[\begin{array}{ccccc|c}1 & 1 & -1 & -1 & 1 & 10\\ 1 & 0 & -1 & 0 & 0 & 8\\ 0 & 2 & 0 & -2 & 2 & 4\end{array}\right]\leftrightarrow\left[\begin{array}{ccccc|c}1 & 1 & -1 & -1 & 1 & 10\\ 0 & -1 & 0 & 1 & -1 & -2\\ 0 & 2 & 0 & -2 & 2 & 4\end{array}\right]$$

$$\leftrightarrow\left[\begin{array}{ccccc|c}1 & 0 & -1 & 0 & 0 & 8\\ 0 & 1 & 0 & -1 & 1 & 2\\ 0 & 0 & 0 & 0 & 0 & 0\end{array}\right]=[A|\vec{b}]_{rref}\quad\Rightarrow\quad\vec{x}=\begin{bmatrix}8\\ 2\\ 0\\ 0\\ 0\end{bmatrix}+\alpha\begin{bmatrix}1\\ 0\\ 1\\ 0\\ 0\end{bmatrix}+\beta\begin{bmatrix}0\\ 1\\ 0\\ 1\\ 0\end{bmatrix}+\gamma\begin{bmatrix}0\\ 1\\ 0\\ 0\\ -1\end{bmatrix}$$

where α, β, and γ are arbitrary reals.

b. What is the rank of A? Explain.

rank(A) = 2, the number of pivot columns.

c. What is the nullity of A? Explain.

nullity(A) = 3, the number of non-pivot columns.

2. Find the cubic polynomial $y(t)=c_0+c_1t+c_2t^2+c_3t^3$ that passes through the points (1, 3), (2, 15), (–1, 3) and (–2, 3).

The given (t, y) pairs correspond to a linear system of four scalar equations which may be written as the single matrix equation $A\vec{c}=\vec{b}$

where $A=\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & 2 & 4 & 8\\ 1 & -1 & 1 & -1\\ 1 & -2 & 4 & -8\end{bmatrix}$, $\vec{c}=\begin{bmatrix}c_0\\ c_1\\ c_2\\ c_3\end{bmatrix}$, and $\vec{b}=\begin{bmatrix}2\\ 15\\ 3\\ 3\end{bmatrix}$. Row-reduction yields

$$[A|\vec{b}]_{rref} = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$ and so the cubic is $y(t) = 1 - 2t + t^2 + t^3$.

3. Consider the system $\left\{\begin{array}{ccccccc} x & + & 2y & + & 3z & = & 4 \\ x & + & ky & + & 4z & = & 6 \\ x & + & 2y & + & (k+2)z & = & 6 \end{array}\right\}$.

This system has the form $A\vec{x} = \vec{b}$ where $[A|\vec{b}] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 1 & k & 4 & 6 \\ 1 & 2 & k+2 & 6 \end{array}\right]$ and $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. After subtracting the first from the second and third rows in $[A|\vec{b}]$, we obtain $[A|\vec{b}] \leftrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & k-2 & 1 & 2 \\ 0 & 0 & k-1 & 2 \end{array}\right]$.

a. For which values of k is there a unique solution to the system?

If $k = 1$ or $k = 2$, there will be fewer than 3 pivots and so there cannot be a unique solution. There is a unique solution so long as $k \neq 1$ and $k \neq 2$.

b. For which values of k is there no solution?

There are no solutions when $k = 1$ since, in this case, the third row corresponds to the equation $0 = 2$.

c. For which values of k are there infinitely many solutions?

There are infinitely many solutions when $k = 2$ since, in this case, the second and third rows are identical and there will be two pivots.

4. Use row operations to find the inverse of the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 5 & 8 \end{bmatrix}$. Show all your work.

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 3 & 5 & 8 & 0 & 0 & 1 \end{array}\right] \leftrightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 2 & 5 & -3 & 0 & 1 \end{array}\right]$$

$$\leftrightarrow\left[\begin{array}{ccc|ccc}1&0&-1&2&-1&0\\0&1&2&-1&1&0\\0&0&1&-1&-2&1\end{array}\right]\leftrightarrow\left[\begin{array}{ccc|ccc}1&0&0&1&-3&1\\0&1&0&1&5&-2\\0&0&1&-1&-2&1\end{array}\right]=[A|\vec{b}]_{rref}=[I\,|\,A^{-1}]$$

Therefore, $A^{-1}=\begin{bmatrix}1&-3&1\\1&5&-2\\-1&-2&1\end{bmatrix}$.

5. Let T and S be the linear transformations from $\mathbf{R}^2$ to $\mathbf{R}^2$ satisfying $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}x+y\\x\end{bmatrix}$ and $S\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}x-y\\y-x\end{bmatrix}$, respectively.

a. Show that T is invertible and find its inverse T^{-1}.

$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}x+y\\x\end{bmatrix}=A\begin{bmatrix}x\\y\end{bmatrix}$ where $A=\begin{bmatrix}1&1\\1&0\end{bmatrix}$. Since $A^{-1}=\begin{bmatrix}0&1\\1&-1\end{bmatrix}$, $T^{-1}\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=A^{-1}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0&1\\1&-1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}y\\x-y\end{bmatrix}$.

b. Show that S is not invertible.

S is not invertible since, for example, $S\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}=\vec{0}$. Too, the matrix for S is $B=\begin{bmatrix}1&-1\\-1&1\end{bmatrix}$ is not invertible since its determinant is 0.

c. Find the matrix of the composite transformation $T\circ S$.

The matrix for the composite is $AB=\begin{bmatrix}1&1\\1&0\end{bmatrix}\begin{bmatrix}1&-1\\-1&1\end{bmatrix}=\begin{bmatrix}0&0\\1&-1\end{bmatrix}$.

d. Explain why there is no linear transformation U from $\mathbf{R}^2$ to $\mathbf{R}^2$ such that $U\circ S$ is invertible.

Since S maps a nonzero vector to the zero vector, the composite transformation $U\circ S$ will do the same for any linear transformation from $\mathbf{R}^2$ to $\mathbf{R}^2$. After all, $(U\circ S)\left(\begin{bmatrix}1\\1\end{bmatrix}\right)=U\left(S\left(\begin{bmatrix}1\\1\end{bmatrix}\right)\right)=U\left(\vec{0}\right)=\vec{0}$.

6. Find all symmetric 2×2 matrices A such that $A^2=I$ and describe geometrically how A acts as a linear transformation from $\mathbf{R}^2$ to $\mathbf{R}^2$.

Since A is symmetric, $A = A^T$. Then, $A^2 = A^T A = I$ tells us that A is also orthogonal. Therefore, it is either a reflection or a rotation or the composite of the two. A reflection across any line through the origin satisfies the given condition that applying it twice in succession results in no change whatever. For a rotation with angle ϕ, repeating the rotation a second time must result in the identity which is a rotation by an integer multiple of 2π. Therefore, the only rotations possible are by integer multiples of π. So, A must be one of these: $\pm\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$ for any real θ.

7. Find all 2×2 matrices that commute with the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

We require $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} a & 2b \\ c & 2d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 2c & 2d \end{bmatrix}$. So, $b = 2b$ and $c = 2c$ or $b = c = 0$. Hence, the only 2×2 matrices that commute with $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ must be diagonal, i.e. $\begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix}$.

8. Let T be the linear transformation from $\mathbf{R}^3$ to $\mathbf{R}^3$ given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ x_2 \end{bmatrix}.$$

a. Find the matrix of T with respect to the standard basis.

The standard matrix for T is $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ whose columns are the corresponding images of the standard basis vectors.

b. Find the matrix of T with respect to the basis $\mathcal{B} = \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right)$.

The transformation matrix for this basis is $S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$.

and its inverse is $S^{-1}=\begin{bmatrix}1&0&0\\0&\frac{1}{2}&\frac{1}{2}\\0&\frac{1}{2}&-\frac{1}{2}\end{bmatrix}$. Therefore, the $\mathfrak{B}$ matrix for T is

$$B=S^{-1}AS=\begin{bmatrix}1&0&0\\0&\frac{1}{2}&\frac{1}{2}\\0&\frac{1}{2}&-\frac{1}{2}\end{bmatrix}\begin{bmatrix}0&0&0\\0&0&0\\0&1&0\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&1\\0&1&-1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&\frac{1}{2}&\frac{1}{2}\\0&\frac{1}{2}&-\frac{1}{2}\end{bmatrix}\begin{bmatrix}0&0&0\\0&0&0\\0&1&1\end{bmatrix}$$

$$=\begin{bmatrix}0&0&0\\0&\frac{1}{2}&\frac{1}{2}\\0&-\frac{1}{2}&-\frac{1}{2}\end{bmatrix}=\frac{1}{2}\begin{bmatrix}0&0&0\\0&1&1\\0&-1&-1\end{bmatrix}.$$

9. Let $\mathbf{R}^2$ be the line in $\mathbf{R}^2$ consisting of all scalar multiples of $\begin{bmatrix}1\\1\end{bmatrix}$.

a. Find the 2×2 matrix corresponding to the projection $proj_L$.

A unit vector parallel to L is $\hat{u}=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$ and so the matrix for $proj_L$ is $P=\hat{u}\,\hat{u}^T=\frac{1}{2}\begin{bmatrix}1&1\\1&1\end{bmatrix}$.

b. Find the 2×2 matrix corresponding to the reflection ref_L.

The matrix for ref_L is $F=P-(I-P)=2P-I=\begin{bmatrix}0&1\\1&0\end{bmatrix}$.

c. Show that the every eigenvector of $proj_L$ is an eigenvector of ref_L and vice versa.

spec$(P)=(0,1)$; spec$(F)=(1,-1)$. $\vec{v}\in E_0(P)$ iff $P\vec{v}=\vec{0}$ iff $F\hat{v}=(2P-I)\vec{v}=-\vec{v}$ iff $\vec{v}\in E_{-1}(F)$. $\vec{v}\in E_1(P)$ iff $P\vec{v}=\vec{v}$ iff $F\hat{v}=(2P-I)\vec{v}$ $\vec{v}$ iff $\vec{v}\in E_1(F)$

10. Let V be a subspace of $\mathbf{R}^n$.

a. State the definition: the list of vectors $(\vec{v}_1,...,\vec{v}_m)$ is linearly independent in V.

$c_1\vec{v}_1+...+c_m\vec{v}_m=\vec{0}$ implies that $c_1=0,...,c_m=0$.

b. State the definition: the list of vectors $(\vec{v}_1,...,\vec{v}_m)$ is a basis for V.

$(\vec{v}_1,...,\vec{v}_m)$ is linearly independent and spans V.

c. Find a basis for im(A) where A is the matrix $A=\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & 2 & -1 & 0\\ 1 & 1 & 1 & 1\\ 1 & 2 & -1 & 0\end{bmatrix}$.

Labelling the column vectors of A as $\vec{v}_1,\vec{v}_2,\vec{v}_3,$ and $\vec{v}_4$, it is evident that $\vec{v}_3=3\vec{v}_1-2\vec{v}_2$ and $\vec{v}_4=2\vec{v}_1-\vec{v}_2$ while $\vec{v}_1$ and $\vec{v}_2$ comprise a linearly independent pair. This is also evident from A_{rref}. Therefore, a basis for im(A) is $(\vec{v}_1,\vec{v}_2)$.

11. Suppose V and W are any subspaces of $\mathbf{R}^n$. For each of the following subsets of $\mathbf{R}^n$, either show that the subset is always a subspace or give a specific example in which the subset is not a subspace.

a. $V\cap W$, the intersection of V and W.

The intersection of any two subspaces of n-space is another subspace since the intersection is close under linear combinations. Let $\vec{a},\vec{b}\in V\cap W$. Then, $\vec{a},\vec{b}\in V$ and $\vec{a},\vec{b}\in W$. Moreover, since linear combinations of vectors belonging to a subspace also belong to that subspace, $\alpha\vec{a}+\beta\vec{b}\in V$ and $\alpha\vec{a}+\beta\vec{b}\in W$. Therefore, $\alpha\vec{a}+\beta\vec{b}\in V\cap W$.

b. $V\cup W$, the union of V and W.

This is not true, in general. For example, let $n=2$ and choose $V=\operatorname{span}(\hat{e}_1)$, $W=\operatorname{span}(\hat{e}_2)$, the standard axes. But, $\hat{e}_1+\hat{e}_2$ does not belong to $V\cup W$. So, $V\cup W$ is not closed under vector addition.

12. Consider the vectors $\vec{v}_1=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$, $\vec{v}_2=\begin{bmatrix}-1\\1\\-1\\1\end{bmatrix}$ and $\vec{v}_3=\begin{bmatrix}0\\2\\0\\6\end{bmatrix}$ in $\mathbf{R}^4$.

a. Find an orthonormal basis for the subspace V spanned by these 3 vectors.

$\vec{v}_1$ and $\vec{v}_2$ are already orthogonal. So, it suffices to simply normalize them to get the first two vectors of the desired basis. That is, we choose $\hat{u}_1=\frac{1}{2}\vec{v}_1$ and $\hat{u}_2=\frac{1}{2}\vec{v}_2$. The Gram-Schmidt process will yield the third basis vector. We have $\vec{v}_3^{\perp}=\vec{v}_3-(\vec{v}_3\cdot\hat{u}_1)\hat{u}_1-(\vec{v}_3\cdot\hat{u}_2)\hat{u}_2=$

$$\vec{v}_3 - \tfrac{1}{4}(\vec{v}_3 \cdot \vec{v}_1)\vec{v}_1 - \tfrac{1}{4}(\vec{v}_3 \cdot \vec{v}_2)\vec{v}_2 = \vec{v}_3 - 2\vec{v}_1 - 2\vec{v}_2 = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 2 \end{bmatrix}.$$ Normalizing this vector gives us the third basis vector. The orthonormal basis we have found is

$$(u_1, u_2, u_3) = \left(\tfrac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \tfrac{1}{2}\begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \tfrac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right).$$

b. Use your answer in a to find the orthogonal projection of the vector $\vec{w} = \begin{bmatrix} 6 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ onto V.

The projection is $(\vec{w} \cdot \hat{u}_1)\hat{u}_1 + (\vec{w} \cdot \hat{u}_2)\hat{u}_2 + (\vec{w} \cdot \hat{u}_3)\hat{u}_3 = 3\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.

13. Let A be a square matrix. Suppose $\vec{v}$, $\vec{w}$, and $\vec{z}$ are eigenvectors of A corresponding to the eigenvalues $\lambda = 0$, 1 and 2, respectively. Show that the three vectors are linearly independent.

None of the three vectors is the zero vector, by definition. Assume that $\alpha\vec{v} + \beta\vec{w} + \gamma\vec{z} = \vec{0}$. Multiplying both sides of this equality on the left by A yields $\beta\vec{w} + 2\gamma\vec{z} = \vec{0}$. This implies that $\vec{w}$ and $\vec{z}$ are proportional or that β and γ are both 0. The former alternative is not possible since scalar multiples of an eigenvector are also eigenvectors for the <u>same</u> eigenvalue. But, if $\beta = \gamma = 0$, the first equation tells us that $\alpha = 0$, too. So, we have shown that the only linear combination of these three eigenvectors that is the zero vector is the trivial linear combination with all coefficients zero.

14. Use row operations to find $\det\left(\begin{bmatrix}1&2&3&4\\2&2&2&2\\1&3&6&10\\1&4&10&20\end{bmatrix}\right)$. Show all your work.

$$\det\left(\begin{bmatrix}1&2&3&4\\2&2&2&2\\1&3&6&10\\1&4&10&20\end{bmatrix}\right)=\det\left(\begin{bmatrix}1&2&3&4\\0&-2&-4&-6\\0&1&3&6\\0&2&7&16\end{bmatrix}\right)=(-2)\det\left(\begin{bmatrix}1&2&3&4\\0&1&2&3\\0&1&3&6\\0&2&7&16\end{bmatrix}\right)$$

$$=(-2)\det\left(\begin{bmatrix}1&0&-1&-2\\0&1&2&3\\0&0&1&3\\0&0&3&10\end{bmatrix}\right)=(-2)\det\left(\begin{bmatrix}1&0&-1&-2\\0&1&2&3\\0&0&1&3\\0&0&0&0\end{bmatrix}\right)=-2.$$

15. Consider the matrix $A=\begin{bmatrix}0.8&0.2\\0.2&0.8\end{bmatrix}$.

a. Find the eigenvalues of A.

$$0=\det(A-\lambda I)=\det\left(\begin{bmatrix}0.8-\lambda&0.2\\0.2&0.8-\lambda\end{bmatrix}\right)=(0.8-\lambda)^2-(0.2)^2$$

$$=\lambda^2-1.6\lambda^2+0.64-0.04=\lambda^2-1.6\lambda^2+0.6=(\lambda-0.6)(\lambda-1).$$

$\operatorname{spec}(A)=(0.6,1)$.

b. Find an eigenbasis of $\mathbf{R}^2$ for the matrix A.

$$E_{0.6}(A)=\ker\left(\begin{bmatrix}0.2&0.2\\0.2&0.2\end{bmatrix}\right)=\ker\left(\begin{bmatrix}1&1\\0&0\end{bmatrix}\right)=\operatorname{span}\left(\begin{bmatrix}1\\-1\end{bmatrix}\right).$$

$E_1(A)=\ker\left(\begin{bmatrix}-0.2&0.2\\0.2&-0.2\end{bmatrix}\right)=\ker\left(\begin{bmatrix}1&-1\\0&0\end{bmatrix}\right)=\operatorname{span}\left(\begin{bmatrix}1\\1\end{bmatrix}\right)$. So, an eigenbasis is $\left(\begin{bmatrix}1\\-1\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix}\right)$.

c. Write the vector $\vec{v}=\begin{bmatrix}1000\\0\end{bmatrix}$ as a linear combination of the vectors in the eigenbasis.

$$\vec{v} = \begin{bmatrix} 1000 \\ 0 \end{bmatrix} = 500\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right).$$

d. Use your answer in c to discuss what happens to $A^t \vec{v}$ for very large positive integers t.

$$A^t \vec{v} = A^t 500\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = 500\left((0.6)^t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \to 500 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ as } t \to \infty.$$

e. Show that A is a diagonalizable matrix by exhibiting a diagonal matrix D and an invertible matrix S so that $D = S^{-1} A S$.

$$D = \begin{bmatrix} 0.6 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } S = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$