Three.I Isomorphisms

Linear Algebra

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Definition

Example We have the intuition that the vector spaces $\mathbb{R}^2$ and $\mathcal{P}_1$ are "the same," in that they are two-component spaces. For instance

$$\begin{pmatrix}1\\2\end{pmatrix} \quad \text{is just like} \quad 1+2x,$$
$$\text{and} \quad \begin{pmatrix}-3\\1/2\end{pmatrix} \quad \text{is just like} \quad -3+(1/2)x,$$

etc. What makes the spaces alike, not just the sets, is that the association persists through the operations: this illustrates addition

$$\begin{pmatrix}1\\2\end{pmatrix}+\begin{pmatrix}-3\\1/2\end{pmatrix}=\begin{pmatrix}-2\\5/2\end{pmatrix}$$
$$\text{is just like} \quad (1+2x)+(-3+(1/2)x)=-2+(5/2)x$$

and this illustrates scalar multiplication.

$$3\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}3\\6\end{pmatrix} \quad \text{is just like} \quad 3(1+2x)=3+6x$$

Example Similarly, we can link each two-tall vector with a linear polynomial.

$$\begin{pmatrix} a \\ b \end{pmatrix} \quad \longleftrightarrow \quad a + bx$$

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This association holds through the vector space operations of addition

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 \\ b_1 + b_2 \end{pmatrix}$$
$$\longleftrightarrow \quad (a_1 + b_1 x) + (a_2 + b_2 x) = (a_1 + a_2) + (b_1 + b_2) x$$

and scalar multiplication.

$$r \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ra \\ rb \end{pmatrix} \quad \longleftrightarrow \quad r(a + bx) = (ra) + (rb)x$$

We say that the association *preserves the structure* of the spaces.

Example We can think of $\mathcal{M}_{2\times 2}$ as "the same" as $\mathbb{R}^4$ if we associate in this way.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \longleftrightarrow \quad \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

For instance, these are corresponding elements.

$$\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} \quad \longleftrightarrow \quad \begin{pmatrix} 1 \\ -1 \\ 2 \\ -2 \end{pmatrix}$$

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This association persists under addition.

$$\begin{aligned} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} &= \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix} \\ &\longleftrightarrow \quad \begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{pmatrix} + \begin{pmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 \\ b_1 + b_2 \\ c_1 + c_2 \\ d_1 + d_2 \end{pmatrix} \end{aligned}$$

Here is an example of addition being preserved under this association.

$$\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} + \begin{pmatrix} 0 & 4 \\ 3 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 5 & -5 \end{pmatrix}$$

$$\longleftrightarrow \quad \begin{pmatrix} 1 \\ -1 \\ 2 \\ -2 \end{pmatrix} + \begin{pmatrix} 0 \\ 4 \\ 3 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \\ -5 \end{pmatrix}$$

Here is an example of addition being preserved under this association.

$$\begin{pmatrix}1&-1\\2&-2\end{pmatrix}+\begin{pmatrix}0&4\\3&-3\end{pmatrix}=\begin{pmatrix}1&3\\5&-5\end{pmatrix}$$

$$\longleftrightarrow\quad\begin{pmatrix}1\\-1\\2\\-2\end{pmatrix}+\begin{pmatrix}0\\4\\3\\-3\end{pmatrix}=\begin{pmatrix}1\\3\\5\\-5\end{pmatrix}$$

The association also persists through scalar multiplication.

$$r\cdot\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}ra&rb\\rc&rd\end{pmatrix}\quad\longleftrightarrow\quad r\cdot\begin{pmatrix}a\\b\\c\\d\end{pmatrix}=\begin{pmatrix}ra\\rb\\rc\\rd\end{pmatrix}$$

This illustrates.

$$2\cdot\begin{pmatrix}1&-1\\2&-2\end{pmatrix}=\begin{pmatrix}2&-2\\4&-4\end{pmatrix}\quad\longleftrightarrow\quad 2\cdot\begin{pmatrix}1\\-1\\2\\-1\end{pmatrix}=\begin{pmatrix}2\\-2\\4\\-4\end{pmatrix}$$

Isomorphism

1.3 *Definition* An *isomorphism* between two vector spaces V and W is a map $f\colon V \to W$ that

1) is a correspondence: f is one-to-one and onto;
2) *preserves structure:* if $\vec{v}_1, \vec{v}_2 \in V$ then

$$f(\vec{v}_1 + \vec{v}_2) = f(\vec{v}_1) + f(\vec{v}_2)$$

and if $\vec{v} \in V$ and $r \in \mathbb{R}$ then

$$f(r\vec{v}) = rf(\vec{v})$$

(we write $V \cong W$, read "V is isomorphic to W", when such a map exists).

How-to

To verify that $f\colon V\to W$ is an isomorphism, do these four.

- To show that f is one-to-one, assume that $\vec{v}_1,\vec{v}_2\in V$ are such that $f(\vec{v}_1)=f(\vec{v}_2)$ and derive that $\vec{v}_1=\vec{v}_2$.
- To show that f is onto, let $\vec{w}$ be an element of W and find a $\vec{v}\in V$ such that $f(\vec{v})=\vec{w}$.
- To show that f preserves addition, check that for all $\vec{v}_1,\vec{v}_2\in V$ we have $f(\vec{v}_1+\vec{v}_2)=f(\vec{v}_1)+f(\vec{v}_2)$.
- To show that f preserves scalar multiplication, check that for all $\vec{v}\in V$ and $r\in\mathbb{R}$ we have $f(r\cdot\vec{v})=r\cdot f(\vec{v})$.

The first two cover condition (1), that the spaces correspond, that for each member of W there exactly one assocated member of V. The latter two cover (2), that the map preserves structure. For these two, the intuition is in the discussion above. (Later section cover these two at length.)

Example The space of quadratic polynomials $\mathcal{P}_2$ is isomorphic to $\mathbb{R}^3$ under this map.

$$f(a_0 + a_1 x + a_2 x^2) = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$$

Here are two examples of the action of f.

$$f(1 + 2x + 3x^2) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad f(3 + 4x^2) = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$$

To verify that f is an isomorphism we must check condition (1), that f is a correspondence, and condition (2), that f preserves structure.

The first part of the definition's clause (1) is that f is one-to-one. We suppose $f(\vec{v}_1) = f(\vec{v}_2)$, that the function yields the same output on two inputs $f(a_0 + a_1x + a_2x^2) = f(b_0 + b_1x + b_2x^2)$. From that, we must derive that the two inputs are equal. The definition of f gives

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix}$$

and two column vectors are equal only if their entries are equal $a_0 = b_0$, $a_1 = b_1$, and $a_2 = b_2$. Thus the original inputs are equal $\vec{v}_1 = a_0 + a_1x + a_2x^2 = b_0 + b_1x + b_2x^2 = \vec{v}_2$. So f is one-to-one.

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The second part of (1) is that f is onto. We consider an element of the codomain

$$\vec{w} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} \in \mathbb{R}^3$$

and produce an element of the domain that maps to it. Observe that $\vec{w}$ is the image under f of the member $\vec{v} = a_0 + a_1x + a_2x^2$ of the domain. Thus f is onto.

The definition's clause (2) also has two halves. First we show that f preserves addition. Consider f acting on the sum of two elements of the domain.

$$\begin{aligned} f(\,(a_0 + a_1x + a_2x^2) + (b_0 + b_1x + b_2x^2)\,) \\ = f(\,(a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2\,) \end{aligned}$$

The definition of f gives

$$= \begin{pmatrix} a_0 + b_0 \\ a_1 + b_1 \\ a_2 + b_2 \end{pmatrix}$$

and that equals

$$= \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix}$$

which gives

$$= f(\,a_0 + a_1x + a_2x^2) + f(b_0 + b_1x + b_2x^2\,)$$

as required.

We finish by checking that f preserves scalar multiplication. This is similar to the check for addition.

$$\begin{aligned} f(r\cdot(a_0+a_1x+a_2x^2)) &= f(\,(ra_0)+(ra_1)x+(ra_2)x^2\,) \\ &= \begin{pmatrix} ra_0 \\ ra_1 \\ ra_2 \end{pmatrix} \\ &= r\cdot\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} \\ &= r\cdot f(\,a_0+a_1x+a_2x^2\,) \end{aligned}$$

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So the function f is an isomorphism. Because there is an isomorphism, the two spaces are isomorphic $\mathcal{P}_2 \cong \mathbb{R}^3$.

Example Consider these two vector spaces (under the natural operations)

$$V = \{ \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \mid a, b, c \in \mathbb{R} \} \qquad W = \{ (x \;\; y \;\; z) \mid x, y, z \in \mathbb{R} \}$$

and consider this function.

$$\begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \stackrel{f}{\longmapsto} (b \;\; 2a \;\; a + c)$$

Here is an example of the map's action.

$$\begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix} \stackrel{f}{\longmapsto} (2 \;\; 6 \;\; 4)$$

We will verify that f is an isomorphism.

To show that f is one-to-one, suppose that

$$f(\begin{pmatrix} a_1 & b_1 \\ c_1 & 0 \end{pmatrix}) = f(\begin{pmatrix} a_2 & b_2 \\ c_2 & 0 \end{pmatrix})$$

Then $(b_1 \;\; 2a_1 \;\; a_1 + c_1) = (b_2 \;\; 2a_2 \;\; a_2 + c_2)$. The first entries give that $b_1 = b_2$, the second entries that $a_1 = a_2$, and with that the third entries give that $c_1 = c_2$.

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & 0 \end{pmatrix} = \begin{pmatrix} a_2 & b_2 \\ c_2 & 0 \end{pmatrix}$$

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$$\begin{pmatrix} a_1 & b_1 \\ c_1 & 0 \end{pmatrix} = \begin{pmatrix} a_2 & b_2 \\ c_2 & 0 \end{pmatrix}$$

To show that f is onto, consider a member of W.

$$\vec{w} = (x \;\; y \;\; z)$$

We must find a $\vec{v}$ so that $f(\vec{v}) = \vec{w}$. The map sends the upper right entry of the input to the first entry of the output, so the upper right of $\vec{v}$ is x. Similarly, the upper left of $\vec{v}$ is $(1/2)y$. With that, the lower left is $z - (1/2)y$.

$$(x \;\; y \;\; z) = f(\begin{pmatrix} y/2 & x \\ z - y/2 & 0 \end{pmatrix})$$

To show that f preserves addition, assume

$$f(\begin{pmatrix} a_1 & b_1 \\ c_1 & 0 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ c_2 & 0 \end{pmatrix}) = f(\begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & 0 \end{pmatrix})$$

which equals $(b_1 + b_2 \;\; 2(a_1 + a_2) \;\; (a_1 + a_2) + (c_1 + c_2))$. In turn, that equals this.

$$(b_1 \;\; 2a_1 \;\; a_1 + c_1) + (b_2 \;\; 2a_2 \;\; a_2 + c_2) = f(\begin{pmatrix} a_1 & b_1 \\ c_1 & 0 \end{pmatrix}) + f(\begin{pmatrix} a_2 & b_2 \\ c_2 & 0 \end{pmatrix})$$

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which equals $(b_1+b_2\ \ 2(a_1+a_2)\ \ (a_1+a_2)+(c_1+c_2))$. In turn, that equals this.

$$(b_1\ \ 2a_1\ \ a_1+c_1)+(b_2\ \ 2a_2\ \ a_2+c_2)=f(\begin{pmatrix}a_1 & b_1\\ c_1 & 0\end{pmatrix})+f(\begin{pmatrix}a_2 & b_2\\ c_2 & 0\end{pmatrix})$$

Preservation of scalar multiplication is similar.

$$\begin{aligned}
f(r\cdot\begin{pmatrix}a & b\\ c & 0\end{pmatrix})&=f(\begin{pmatrix}ra & rb\\ rc & 0\end{pmatrix})\\
&=(rb\ \ 2ra\ \ ra+rc)\\
&=r\cdot(b\ \ 2a\ \ a+c)\\
&=r\cdot f(\begin{pmatrix}a & b\\ c & 0\end{pmatrix})
\end{aligned}$$

Preservation is special

Many functions do not preserve addition and scalar multiplication. For instance, $f\colon \mathbb{R}^2 \to \mathbb{R}^2$

$$f(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} x^2 \\ y^2 \end{pmatrix}$$

does not preserve addition since the sum done one way

$$f(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix}) = f(\begin{pmatrix} 3 \\ 0 \end{pmatrix}) = \begin{pmatrix} 9 \\ 0 \end{pmatrix}$$

gives a different result than the sum done the other way.

$$f(\begin{pmatrix} 1 \\ 0 \end{pmatrix}) + f(\begin{pmatrix} 2 \\ 0 \end{pmatrix}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

Special case: Automorphisms

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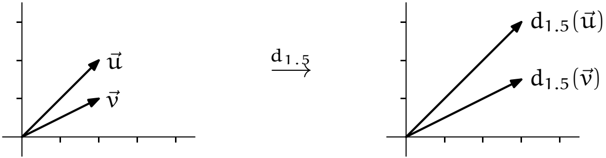

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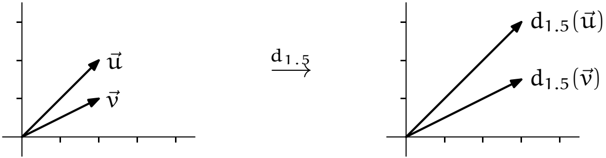
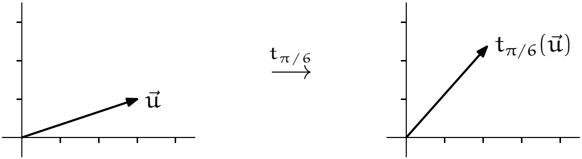
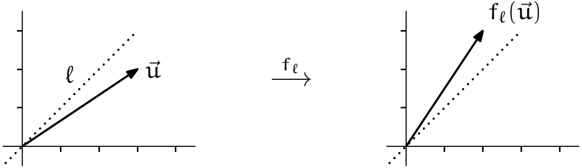


Another automorphism is a *rotation* or *turning map*, $t_\theta\colon \mathbb{R}^2 \to \mathbb{R}^2$ that rotates all vectors through an angle θ.



A third type of automorphism of $\mathbb{R}^2$ is a map $f_\ell\colon \mathbb{R}^2 \to \mathbb{R}^2$ that *flips* or *reflects* all vectors over a line ℓ through the origin.



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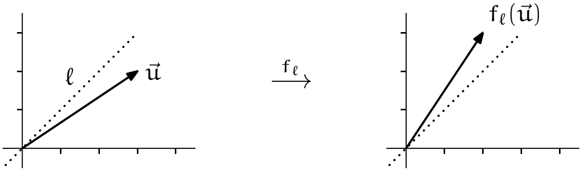

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Why study automorphisms? Isn't it trivial that the plane is just like itself?

Consider the family of automorphisms t_Θ rotating all vectors counterclockwise. They makes precise the intuition that the plane is uniform — that space near the x-axis is just like space near the y-axis.

So one lesson is that we can use maps to describe relationships between spaces. If the maps are isomorphisms then this relation makes precise the intuition “just like”.

A second lesson is that while there is an obvious automorphism of $\mathbb{R}^2$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix}$$

there are reasons to consider maps other than the obvious one.

1.10 *Lemma* An isomorphism maps a zero vector to a zero vector.

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Proof Where $f\colon V \to W$ is an isomorphism, fix some $\vec{v} \in V$. Then $f(\vec{0}_V) = f(0 \cdot \vec{v}) = 0 \cdot f(\vec{v}) = \vec{0}_W$. QED

1.11 *Lemma* For any map $f\colon V \to W$ between vector spaces these statements are equivalent.

(1) f preserves structure

$$f(\vec{v}_1 + \vec{v}_2) = f(\vec{v}_1) + f(\vec{v}_2) \quad \text{and} \quad f(c\vec{v}) = c\, f(\vec{v})$$

(2) f preserves linear combinations of two vectors

$$f(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1 f(\vec{v}_1) + c_2 f(\vec{v}_2)$$

(3) f preserves linear combinations of any finite number of vectors

$$f(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = c_1 f(\vec{v}_1) + \cdots + c_n f(\vec{v}_n)$$

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Proof Since the implications (3) $\implies$ (2) and (2) $\implies$ (1) are clear, we need only show that (1) $\implies$ (3). So assume statement (1). We will prove (3) by induction on the number of summands n.

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The one-summand base case, that $f(c\vec{v}_1) = c\,f(\vec{v}_1)$, is covered by the second clause of statement (1).

For the inductive step assume that statement (3) holds whenever there are k or fewer summands. Consider the $k+1$-summand case. Use the first half of (1) to break the sum along the final '$+$'.

$$f(c_1\vec{v}_1 + \cdots + c_k\vec{v}_k + c_{k+1}\vec{v}_{k+1}) = f(c_1\vec{v}_1 + \cdots + c_k\vec{v}_k) + f(c_{k+1}\vec{v}_{k+1})$$

Use the inductive hypothesis to break up the k-term sum on the left.

$$= f(c_1\vec{v}_1) + \cdots + f(c_k\vec{v}_k) + f(c_{k+1}\vec{v}_{k+1})$$

Now the second half of (1) gives

$$= c_1\, f(\vec{v}_1) + \cdots + c_k\, f(\vec{v}_k) + c_{k+1}\, f(\vec{v}_{k+1})$$

when applied $k+1$ times. QED