Special Problem 3 Solutions Due Monday, 28 February 2011

In class, we proved the following.

Theorem A. If $(\vec{u}_1, \dots, \vec{u}_p)$ is a linearly independent list of p vectors in a subspace $\boldsymbol{V}$ of $\mathbf{R}^n$ and $(\vec{w}_1, \dots, \vec{w}_q)$ is a spanning list of q vectors in $\boldsymbol{V}$, then $p \le q$. [Linearly independent lists are never longer than spanning lists.]

We used this theorem to prove that any two bases for a subspace $\boldsymbol{V}$ must have the same number of vectors. We define this number as the dimension of $\boldsymbol{V}$, written dim($\boldsymbol{V}$).

Now prove the following.

Theorem B. If $\boldsymbol{V}$ is a subspace of $\mathbf{R}^n$, $(\vec{v}_1, \dots, \vec{v}_m)$ is a linearly independent list of m vectors in $\boldsymbol{V}$ and $m = \dim(\boldsymbol{V})$, then $(\vec{v}_1, \dots, \vec{v}_m)$ is a basis for $\boldsymbol{V}$.

Your objective here is to show that $(\vec{v}_1, \dots, \vec{v}_m)$ spans $\boldsymbol{V}$. Toward this end, suppose to the contrary that $(\vec{v}_1, \dots, \vec{v}_m)$ does not span $\boldsymbol{V}$. From this, obtain a contradiction. Use basic definitions and Theorem A.

If $(\vec{v}_1, \dots, \vec{v}_m)$ does not span $\boldsymbol{V}$, there must be a vector $\vec{w}$ in $\boldsymbol{V}$ that is not a linear combination of the vectors in $(\vec{v}_1, \dots, \vec{v}_m)$. Therefore, $(\vec{v}_1, \dots, \vec{v}_m, \vec{w})$ is a linearly independent list in $\boldsymbol{V}$ since $\vec{w}$ is not a linear combination of the vectors in this list that precede it. But, this new list has one more vector than in a basis for $\boldsymbol{V}$ which is a spanning list of m vectors. This contradicts Theorem A since we have produced a linearly independent list in $\boldsymbol{V}$ that is longer than a spanning list in $\boldsymbol{V}$. We conclude that $(\vec{v}_1, \dots, \vec{v}_m)$ spans $\boldsymbol{V}$.

There is a companion to Theorem B. Prove it.

Theorem C. If $\boldsymbol{V}$ is a subspace of $\mathbf{R}^n$, $(\vec{w}_1, \dots, \vec{w}_m)$ is a spanning list of m vectors in $\boldsymbol{V}$ and $m = \dim(\boldsymbol{V})$, then $(\vec{w}_1, \dots, \vec{w}_m)$ is a basis for $\boldsymbol{V}$.

Your objective here is to show that $(\vec{w}_1, \dots, \vec{w}_m)$ is linearly independent in $\boldsymbol{V}$. Toward this end, suppose to the contrary that $(\vec{w}_1, \dots, \vec{w}_m)$ is not linearly independent in $\boldsymbol{V}$. From this, obtain a contradiction. Use basic definitions and Theorem A.

If $(\vec{w}_1, \ldots, \vec{w}_m)$ is not linearly independent in $\boldsymbol{V}$, there must be a vector $\vec{w}_k$ in this list that is a linear combination of the others. Its removal from the list leaves a list one vector shorter but a list that still spans $\boldsymbol{V}$. This is a list that is shorter than a basis which linearly independent. This is in contradiction to Theorem A. We conclude that $(\vec{v}_1, \ldots, \vec{v}_m)$ is linearly independent in $\boldsymbol{V}$.