1. a. Calculate $\det\left(\begin{bmatrix} 1 & 0 & 1 & 2 & 0 \\ 1 & 0 & 4 & 3 & 7 \\ 4 & 5 & 3 & 2 & 1 \\ 1 & 0 & 1 & 4 & 8 \\ 1 & 0 & 1 & 2 & 4 \end{bmatrix}\right)$.

Let A be the 5×5 matrix in the expression above. Then, using the Laplace expansion along the second column of A, we find $\det(A) = 5(-1)^{2+3}\det\left(\begin{bmatrix} 1 & 1 & 2 & 0 \\ 1 & 4 & 3 & 7 \\ 1 & 1 & 4 & 8 \\ 1 & 1 & 2 & 4 \end{bmatrix}\right)$. Next, we subtract the first row of the matrix from all succeeding rows to obtain

$$\det(A) = 5(-1)^{2+3}\det\left(\begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 3 & 1 & 7 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 4 \end{bmatrix}\right) = 5(-1)^{2+3}(1)(3)(2)(4) = -120.$$

In the next to last step, we used the fact that the determinant of an upper (or lower) triangular matrix is the product of its diagonal entries.

b. Find the area of the parallelogram in $\mathbf{R}^4$, two of whose concurrent edges are described by the vectors $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$.

Designating the two vectors above as $\vec{u}$ and $\vec{v}$, respectively, the area desired is

$$\sqrt{\det\left([\vec{u}\mid\vec{v}]^T[\vec{u}\mid\vec{v}]\right)} = \sqrt{\det\left(\begin{bmatrix} 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}\right)} = \sqrt{\det\left(\begin{bmatrix} 3 & 3 \\ 3 & 6 \end{bmatrix}\right)} = \sqrt{9} = 3.$$

2. a. Find the determinant of the $n \times n$ matrix $\begin{bmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 2 & 2 & 3 & 4 & \cdots & n \\ 3 & 3 & 3 & 4 & \cdots & n \\ 4 & 4 & 4 & 4 & \cdots & n \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ n & n & n & n & \cdots & n \end{bmatrix}$. Notice that the ijth entry of this matrix is the maximum of i and j.

Let A be the given matrix. Subtracting the first row A from each of its succeeding rows yields $\begin{bmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 2 & 1 & 0 & 0 & \cdots & 0 \\ 3 & 2 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ n-1 & n-2 & n-3 & n-4 & \cdots & 0 \end{bmatrix}$ whose determinant is the same as $\det(A)$.

Now, using the Laplace expansion along the last column, we have

$\det(A) = n(-1)^{n+1} \det\left(\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 2 & 1 & 0 & \cdots & 0 \\ 3 & 2 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n-1 & n-2 & n-3 & \cdots & 1 \end{bmatrix}\right)$. The last matrix is $(n–1)\times(n–1)$ and lower triangular. Its determinant is 1. So, $\det(A) = (-1)^{n+1}n$.

b. Suppose that $A = \begin{bmatrix} * & 100 & * & * \\ 100 & * & * & * \\ * & * & * & 100 \\ * & * & 100 & * \end{bmatrix}$ where the asterisks are positive integers less than 10 and they are not necessarily the same. Show that A is invertible by considering the size of the terms that sum to its determinant.

Each of the 4! = 24 terms of $\det(A)$ is ± 1 times the product of 4 entries of the matrix, each from a different row and different column. The largest is $(100)^4 = 10^8$. All others have no more than two factors of 100 and two factors of 10.. Clearly, then, $\det(A) > 10^8 - (23)(10^2)(100^2) > 0$. So, A^{-1} exists because its determinant is not 0.

3. Let $A=\begin{bmatrix}-3 & 2 & 1\\ -5 & 4 & 1\\ -5 & 2 & 3\end{bmatrix}$.

a. Calculate $A\begin{bmatrix}1\\1\\1\end{bmatrix}$, $A\begin{bmatrix}1\\2\\1\end{bmatrix}$, and $A\begin{bmatrix}1\\1\\3\end{bmatrix}$.

$$A\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}=0\begin{bmatrix}1\\1\\1\end{bmatrix},\quad A\begin{bmatrix}1\\2\\1\end{bmatrix}=\begin{bmatrix}2\\4\\2\end{bmatrix}=2\begin{bmatrix}1\\2\\1\end{bmatrix},\quad \text{and}\quad A\begin{bmatrix}1\\1\\3\end{bmatrix}=\begin{bmatrix}2\\2\\6\end{bmatrix}=2\begin{bmatrix}1\\1\\3\end{bmatrix}$$

b. From part a, determine the spectrum of A.

From the calculations, 0, 2, and 2 are eigenvalues of A and so spec(A) = (0, 2, 2).

c. From part a, determine an eigenbasis of A for $\mathbf{R}^3$.

The three vectors multiplied by A in part a are eigenvectors of A. Moreover, they are linearly independent and span $\mathbf{R}^3$. So, $(\vec{v}_1,\vec{v}_2,\vec{v}_3)=\left(\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}1\\2\\1\end{bmatrix},\begin{bmatrix}1\\1\\3\end{bmatrix}\right)$ is an eigenbasis of A for $\mathbf{R}^3$.

d. Provide a nonsingular 3×3 matrix S such that $S^{-1}AS=D$ is a diagonal matrix.

$$S=\left[\vec{v}_1\,|\,\vec{v}_2\,|\,\vec{v}_3\right]=\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 1\\ 1 & 1 & 3\end{bmatrix}.$$

e. List all diagonal matrices similar to A.

The order in which the eigenvectors are arranged in S corresponds to the order of the eigenvalues along the diagonal of D. This order is arbitrary and the only possibilities are

$$\begin{bmatrix}0 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix},\begin{bmatrix}2 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 2\end{bmatrix},\text{ and }\begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 0\end{bmatrix}.$$

4. F is the matrix for reflection across a plane M through the origin in $\mathbf{R}^3$.
Being as explicit as possible, describe:

a. the spectrum of F and provide the algebraic multiplicity of each eigenvalue.

Every vector in the subspace M is left unchanged by the reflection. These are the eigenvectors with eigenvalue +1. Every vector in the subspace $M^{\perp}$, the line through the origin normal to the plane is reversed by the reflection. These are the eigenvectors with eigenvalue -1. Since $\dim(M) = 2$, the geometric multiplicity of $+1$ is 2 and the algebraic multiplicity must be at least 2. Since $\dim(M^{\perp}) = 1$, the geometric multiplicity of -1 is 1 and its algebraic multiplicity is at least 1. So, $+1$ has algebraic multiplicity 2 and -1 has algebraic multiplicity 1. $\text{spec}(A) = (+1, +1, -1)$

b. the characteristic polynomial of F.

From part a, $f_A(\lambda) = (\lambda - 1)^2(\lambda + 1) = \lambda^3 + \lambda^2 - \lambda - 1$

c. the eigenspaces of F.

$\mathrm{E}_{+1}(A) = M$ and $\mathrm{E}_{-1}(A) = M^{\perp}$.

5. Consider the linear dynamical system described by the scalar equations

$$x_1(t+1) = .4x_1(t) + .3x_2(t)$$

$$x_2(t+1) = -.2x_1(t) + 1.1x_2(t)$$

where t is a nonnegative integer. Suppose $x_1(0) = 11$ and $x_2(0) = 7$.

a. Determine $x_1(t)$ and $x_2(t)$ for all positive integers t.

Let $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$. The above equations abbreviate to $\vec{x}(t+1) = A\vec{x}(t)$ where $A = \begin{bmatrix} .4 & .3 \\ -.2 & 1.1 \end{bmatrix}$. The solution is $\vec{x}(t) = A^t\,\vec{x}(0)$. The eigens of A are now found.

$0 = \det(A - \lambda I) = (.4 - \lambda)(1.1 - \lambda) + .06 = (\lambda - .5)(\lambda - 1)$ implies $\text{spec}(A) = (.5, 1)$. So,

$$\mathrm{E}_{.5}(A) = \ker(A - .5I) = \ker\begin{bmatrix} -.1 & .3 \\ -.2 & .6 \end{bmatrix} = \ker\begin{bmatrix} -1 & 3 \\ 0 & 0 \end{bmatrix} = \text{span}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$ and

$$\mathrm{E}_{1}(A) = \ker(A - 1I) = \ker\begin{bmatrix} -.6 & .3 \\ -.2 & .1 \end{bmatrix} = \ker\begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} = \text{span}\begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$ An eigenbasis for $\mathbf{R}^2$ is

$\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$ and a diagonalizer for A is $S = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$ with inverse $S^{-1} = \frac{1}{5}\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$.

Since $\vec{x}(0) = \begin{bmatrix} 11 \\ 7 \end{bmatrix} = 3\begin{bmatrix} 3 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ [the coefficients 3 and 2 are the components of $S^{-1}\vec{x}(0)$], we have $\vec{x}(t) = A^t\begin{bmatrix} 11 \\ 7 \end{bmatrix} = 3A^t\begin{bmatrix} 3 \\ 1 \end{bmatrix} + 2A^t\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 3\cdot(.5)^t\begin{bmatrix} 3 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

b. What are $\lim_{t\to\infty} x_1(t)$ and $\lim_{t\to\infty} x_2(t)$?

$$\lim_{t\to\infty}\vec{x}(t) = \lim_{t\to\infty}\left(A^t\begin{bmatrix} 11 \\ 7 \end{bmatrix}\right) = \lim_{t\to\infty}\left(3\cdot 2^{-t}\begin{bmatrix} 3 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} \lim_{t\to\infty} x_1(t) \\ \lim_{t\to\infty} x_2(t) \end{bmatrix}.$$

6. For each of the following statements, A, B, and C are $n \times n$ matrices where $n > 1$. Indicate whether the statement is True (**T**) or False (**F**).

a. $\det(A+B) = \det(A) + \det(B)$.

False. Let $n = 2$, $A = B = I$. Then, $\det(A + B) = 4$ but $\det(A) + \det(B) = 2$.

b. If 0 is an eigenvalue of A, then A is singular (not invertible).

True. 0 is an eigenvalue of A if and only if $0 = \det(A - 0I) = \det(A)$.

c. $\det(A)$ is a linear function of any entry of A.

False. For example, doubling one entry of a square matrix does not double its determinant. The statement would be true if "entry" were replaced by "row" or "column."

d. If A is similar to B and B is similar to C, then A is similar to C.

True. Similarity is a transitive relationship.

e. Similar matrices have the same eigenvectors.

False. The statement is true about eigenvalues.

f. If 1 is the only eigenvalue of A and its algebraic multiplicity is n, then A is the identity matrix.

False. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \neq I$. But A has eigenvalue 1 repeated twice.

g. The characteristic polynomials of A and A^T are the same.

True. $\det(\lambda I - A) = \det\left((\lambda I - A)^T\right) = \det(\lambda I - A^T)$.

h. If $\vec{v}$ is an eigenvector of both A and B, then $\vec{v}$ is also an eigenvector of both $A + B$ and AB.

True. If $A\vec{v} = \alpha\vec{v}$ and $B\vec{v} = \beta\vec{v}$ then,

$$(A+B)\vec{v} = A\vec{v} + B\vec{v} = \alpha\vec{v} + \beta\vec{v} = (\alpha+\beta)\vec{v}$$

$$(AB)\vec{v} = A(B\vec{v}) = A(\beta\vec{v}) = \beta(A\vec{v}) = (\beta\alpha)\vec{v} = (\alpha\beta)\vec{v}$$

i. Every eigenvector of A belongs to $\operatorname{im}(A)$ or to $\ker(A)$.

True. If $\vec{v}$ is an eigenvector with nonzero eigenvalue λ, then $A(1/\lambda)\vec{v} = (\lambda/\lambda)\vec{v} = \vec{v}$ shows that $\vec{v}$ is in the image of A. On the other hand, if $\vec{v}$ is an eigenvector with zero eigenvalue, $A\vec{v} = 0\vec{v} = \vec{0}$ shows that $\vec{v}$ belongs to the kernel of A.