Suppose that $\hat{a}$ and $\hat{b}$ are two unit vectors in $\mathbf{R}^n$. We have seen that $A = \hat{a}\,\hat{a}^T$ and $B = \hat{b}\,\hat{b}^T$ are the $n \times n$ matrices for projection onto the one-dimensional subspaces spanned by $\hat{a}$ and $\hat{b}$, respectively. Since A and B are projection matrices, they must satisfy two algebraic properties of all projection matrices; they must be symmetric and idempotent, that is, each satifies $C = C^T$ and $C^2 = C$. For example, $A^T = \left(\hat{a}\,\hat{a}^T\right)^T = \hat{a}^{TT}\,\hat{a}^T$ $= \hat{a}\,\hat{a}^T = A$ and $A^2 = (\hat{a}\,\hat{a}^T)(\hat{a}\,\hat{a}^T) = \hat{a}(\hat{a}^T\hat{a})\,\hat{a}^T = \hat{a}(1)\,\hat{a}^T = \hat{a}\,\hat{a}^T = A.$ Here, we used these facts: the transpose of a product of two matrices is the product of the transposes but in reverse order, $\hat{a}^{TT} = \hat{a}$ and $\hat{a}^T\hat{a} = 1.$ Identical arguments demonstrate that B is symmetric and idempotent.

A natural question arises as to whether $A + B$ is the matrix for projection onto the subspace spanned by $\hat{a}$ and $\hat{b}$. Even more fundamental is the question whether $A + B$ is a projection matrix. It is easy to verify that it is symmetric since the sum of symmetric matrices is symmetric.

a. Show that, if $\hat{a}$ and $\hat{b}$ are orthogonal, $A + B$ is idempotent, i.e $(A+B)^2 = A + B.$

$(A+B)^2 = \left(\hat{a}\,\hat{a}^T + \hat{b}\,\hat{b}^T\right)^2 = \hat{a}\,\hat{a}^T\hat{a}\,\hat{a}^T + \hat{a}\,\hat{a}^T\hat{b}\,\hat{b}^T + \hat{b}\,\hat{b}^T\hat{a}\,\hat{a}^T + \hat{b}\,\hat{b}^T\hat{b}\,\hat{b}^T$ $= \hat{a}\,(1)\,\hat{a}^T + \hat{a}\,(0)\,\hat{b}^T + \hat{b}\,(0)\,\hat{a}^T + \hat{b}\,(1)\,\hat{b}^T = \hat{a}\,\hat{a}^T + \hat{b}\,\hat{b}^T = A + B.$ In making these calculations, we used the associativity of matrix multiplication and the basic facts: $\hat{a}\cdot\hat{b} = \hat{b}\cdot\hat{a} = \hat{a}^T\,\hat{b} = \hat{b}^T\,\hat{a} = 0$ and $\hat{a}\cdot\hat{a} = \hat{b}\cdot\hat{b} = \hat{a}^T\,\hat{a} = \hat{b}^T\,\hat{b} = 1.$

b. Show that, for orthogonal $\hat{a}$ and $\hat{b}$, $(A+B)(\alpha\,\hat{a} + \beta\,\hat{b}) = \alpha\,\hat{a} + \beta\,\hat{b}$ for any scalars α and β. This demonstrates that vectors in the subspace spanned by $\hat{a}$ and $\hat{b}$ are unaffected by the projection $A + B.$

$(A+B)(\alpha\,\hat{a} + \beta\,\hat{b}) = (\hat{a}\,\hat{a}^T + \hat{b}\,\hat{b}^T)(\alpha\,\hat{a} + \beta\,\hat{b}) = \alpha\,(\hat{a}\,\hat{a}^T + \hat{b}\,\hat{b}^T)\,\hat{a} + \beta\,(\hat{a}\,\hat{a}^T + \hat{b}\,\hat{b}^T)\,\hat{b}$
$= \alpha\,(\hat{a}\,\hat{a}^T\hat{a} + \hat{b}\,\hat{b}^T\hat{a}) + \beta\,(\hat{a}\,\hat{a}^T\hat{b} + \hat{b}\,\hat{b}^T\hat{b}) = \alpha\,(\hat{a}\,(1) + \hat{b}\,(0)) + \beta\,(\hat{a}\,(0) + \hat{b}\,(1))$
$= \alpha\,\hat{a} + \beta\,\hat{b}.$

c. Give a specific example of non-orthogonal vectors $\hat{a}$ and $\hat{b}$ to demonstrate that $A+B$ is not a projection matrix.

Let $\hat{a}=\frac{1}{5}\begin{bmatrix}3\\4\end{bmatrix}$ and $\hat{b}=\frac{1}{5}\begin{bmatrix}4\\3\end{bmatrix}$, a pair of unit vectors that is not orthogonal but does span $\mathbf{R}^2$. The matrices for projection onto their one-dimensional subspaces are $A=\frac{1}{25}\begin{bmatrix}9 & 12\\12 & 16\end{bmatrix}$ and $B=\frac{1}{25}\begin{bmatrix}16 & 12\\12 & 9\end{bmatrix}$. Their sum is $A+B=\frac{1}{25}\begin{bmatrix}25 & 24\\24 & 25\end{bmatrix}$. This is not a projection matrix because it is not idempotent as is easily checked. Moreover, the sum of these two projection matrices is not the matrix for projection onto the span of $\hat{a}$ and $\hat{b}$. After all, since $\operatorname{span}(\hat{a},\ \hat{b})=\mathbf{R}^2$, the matrix for projection onto this subspace is $I=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$.