1. The matrix A has rank 2, $A\begin{bmatrix}1\\2\\3\end{bmatrix}=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$, and $A\begin{bmatrix}3\\2\\1\end{bmatrix}=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$.

a. How many rows and columns does A have?

A maps vectors in $\mathbf{R}^3$ to vectors in $\mathbf{R}^4$. Therefore, it is a 4×3 matrix with 4 rows and 3 columns.

b. What is the dimension of the image of A?

$\dim(\text{im}(A)) = \text{rank}(A) = 2$.

c. What is the dimension of the kernel of A?

Rank-Nullity Theorem: $\dim(\ker(A)) = 3 - \dim(\text{im}(A)) = 3 - 2 = 1$.

d. Determine the kernel of A. [Hint: consider the difference of the two equations above.]

Taking the difference of the two equations, we have

$$A\left(\begin{bmatrix}3\\2\\1\end{bmatrix}-\begin{bmatrix}1\\2\\3\end{bmatrix}\right)=A\begin{bmatrix}2\\0\\-2\end{bmatrix}=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}-\begin{bmatrix}1\\1\\1\\1\end{bmatrix}=\vec{0}.$$ This means that $\begin{bmatrix}2\\0\\-2\end{bmatrix}$ is in $\ker(A)$

and since $\dim(\ker(A)) = 1$, $\ker(A)=\text{span}\left(\begin{bmatrix}1\\0\\-1\end{bmatrix}\right)$.

e. Find <u>all</u> solutions to the equation $A\vec{x}=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$.

The set of all solutions is $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}+\alpha\begin{bmatrix}1\\0\\-1\end{bmatrix} \mid \alpha\in\mathbf{R}\right\}=\left\{\begin{bmatrix}1+\alpha\\2\\3-\alpha\end{bmatrix} \mid \alpha\in\mathbf{R}\right\}$. Every solution is any particular solution plus a vector in the kernel.

f. Is the solution set found in part e a subspace? Explain.

No, it is not a subspace. It is easy to check that it is neither closed under vector addition nor multiplication by scalars. But it is even easier to observe that it does not contain $\begin{bmatrix}0\\0\\0\end{bmatrix}$.

2. For each assertion below, state whether it is True or False. It is not necessary to provide any justification.

a. Elementary row operations do not change the linear independence or dependence of the list of row vectors of a matrix.

True. A list of vectors remains linearly independent or dependent if any two are swapped, any one is multiplied by a nonzero constant or any multiple of one is added to another.

b. Elementary row operations do not change the span of the list of column vectors of a matrix.

False. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, then $A_{rref} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $\mathrm{im}(A) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$, and $\mathrm{im}(A_{rref}) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$.

c. Any list of more than 5 vectors in $\mathbf{R}^5$ is linearly dependent.

True. Since $\dim(\mathbf{R}^5) = 5$ any list of more than 5 vectors cannot be linearly independent.

d. Any list of fewer than 5 vectors in $\mathbf{R}^5$ does not span $\mathbf{R}^5$.

True. Since $\dim(\mathbf{R}^5) = 5$ any list of fewer than 5 vectors cannot be spanning.

e. The image of a 5×4 matrix is never $\mathbf{R}^5$.

True. The rank of a 5×4 matrix cannot be greater than 4 = min(5, 4).

f. All $n \times n$ invertible matrices are similar to I_n.

False. The identity matrix is the only matrix similar to itself.

$S^{-1} I S = S^{-1} S = I$.

g. If a subspace of $\mathbf{R}^n$ includes none of the standard basis vectors, the subspace is merely $\{\vec{0}\}$.

False. Counterexample: $\mathrm{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ contains neither $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ nor $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

h. If $\vec{u}, \vec{v}, \vec{w}, \vec{x}, \vec{y},$ and $\vec{z}$ are vectors in $\mathbf{R}^5$, then one of these vectors must be a linear combination of the others.

True. 6 vectors in a 5-dimensional vector space must be linear dependent.

i. The kernel of the product matrix AB contains the kernel of B.

True. $\vec{x} \in \ker(B) \Rightarrow B\vec{x} = \vec{0} \Rightarrow AB\vec{x} = \vec{0} \Rightarrow \vec{x} \in \ker(AB)$.

j. The image of the product matrix AB contains the image of A.

False. The reverse is true; i.e. the image of the product matrix AB is contained in the image of A. After all, the image of B may not be all of the codomain of A. As an example among 2×2 matrices, let $A = I$ and $B = 0$.

3. $A=\begin{bmatrix}1&2&2&3\\1&3&1&4\\2&3&5&5\\3&4&8&7\end{bmatrix}\leftrightarrow\begin{bmatrix}1&2&2&3\\0&1&-1&1\\0&-1&1&-1\\0&-2&2&-2\end{bmatrix}\leftrightarrow\begin{bmatrix}1&0&4&1\\0&1&-1&1\\0&0&0&0\\0&0&0&0\end{bmatrix}=A_{rref}$.

a. Determine a basis for and the dimension of $\ker(A)$.

From A_{rref} above and the Solution Algorithm, a basis for $\ker(A)$ is

$\left(\begin{bmatrix}4\\-1\\-1\\0\end{bmatrix},\begin{bmatrix}1\\1\\0\\-1\end{bmatrix}\right)$ and so, $\dim(\ker(A)) = 2$.

b. Determine a basis for and the dimension of $\operatorname{im}(A)$.

From the observation that the third and fourth columns of A_{rref} are linear combinations of the first two columns of A_{rref} while the first two columns of A_{rref} are a linearly independent pair, we deduce that a basis for $\operatorname{im}(A)$ is

$\left(\begin{bmatrix}1\\1\\2\\3\end{bmatrix},\begin{bmatrix}2\\3\\3\\4\end{bmatrix}\right)$ and so, $\dim(\operatorname{im}(A)) = 2$.

c. For which $\vec{b}$ in $\mathbf{R}^4$ does the equation $A\vec{x}=\vec{b}$ have:

i. no solution?

The equation has no solution for any $\vec{b}$ not in $\operatorname{im}(A)$.

ii. exactly one solution?

This equation never has exactly one solution for any $\vec{b}$ in $\mathbf{R}^4$.

iii. infinitely many solutions?

For any $\vec{b}$ in $\operatorname{im}(A)$, the equation has infinitely many solutions.

4. Choose $\mathfrak{B}=\left(\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}2\\3\end{bmatrix}\right)$ as a basis for $\mathbf{R}^2$.

a. Compute $\begin{bmatrix}6\\7\end{bmatrix}_{\mathfrak{B}}$, the $\mathfrak{B}$-coordinate vector for $\begin{bmatrix}6\\7\end{bmatrix}$.

The coordinate change matrix is $S=\begin{bmatrix}1&2\\2&3\end{bmatrix}$ and, for any $\vec{x}\in\mathbf{R}^2$,

$\vec{x}=S[\vec{x}]_{\mathfrak{B}}$. So, $[\vec{x}]_{\mathfrak{B}}=S^{-1}\vec{x}$. Therefore, $\begin{bmatrix}6\\7\end{bmatrix}_{\mathfrak{B}}=\begin{bmatrix}1&2\\2&3\end{bmatrix}^{-1}\begin{bmatrix}6\\7\end{bmatrix}=$

$\frac{1}{3-4}\begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 7 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 6 \\ 7 \end{bmatrix} = \begin{bmatrix} -4 \\ 5 \end{bmatrix}$. Note, $S^{-1} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$.

Now consider the linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ for which $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

b. If $T(\vec{x}) = A\vec{x}$ and $[T(\vec{x})]_{\mathfrak{B}} = B[\vec{x}]_{\mathfrak{B}}$ for any $\vec{x}$ in $\mathbf{R}^2$, determine the matrices A and B.

Clearly, $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = S^{-1}AS = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} =$
$\begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 5 & 8 \\ 11 & 18 \end{bmatrix} = \begin{bmatrix} 7 & 12 \\ -1 & -2 \end{bmatrix}$.