# Assignment 26 Solutions

1. Suppose that $\mathcal{B}$ is a basis for $\mathbf{R}^2$. If the $\mathcal{B}-$coordinate vector for $\begin{bmatrix}1\\2\end{bmatrix}$ is $\begin{bmatrix}3\\4\end{bmatrix}$ and the $\mathcal{B}-$coordinate vector for $\begin{bmatrix}1\\1\end{bmatrix}$ is $\begin{bmatrix}2\\1\end{bmatrix}$, what is the $\mathcal{B}-$coordinate vector for $\begin{bmatrix}3\\1\end{bmatrix}$?

Let $\mathcal{B}=(\vec{v}_1,\vec{v}_2)$ and $S=[\vec{v}_1\,|\,\vec{v}_2]$. We want to determine $\begin{bmatrix}3\\1\end{bmatrix}_{\mathcal{B}}=S^{-1}\begin{bmatrix}3\\1\end{bmatrix}$. We are given $\begin{bmatrix}1\\2\end{bmatrix}_{\mathcal{B}}=\begin{bmatrix}3\\4\end{bmatrix}$ or $S\begin{bmatrix}3\\4\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}1\\1\end{bmatrix}_{\mathcal{B}}=\begin{bmatrix}2\\1\end{bmatrix}$ or $\begin{bmatrix}1\\1\end{bmatrix}=S\begin{bmatrix}2\\1\end{bmatrix}$. This data allows us to find $S$. Combining the vector equations into one matrix equation,

$$\begin{bmatrix}1&1\\2&1\end{bmatrix}=S\begin{bmatrix}3&2\\4&1\end{bmatrix} \text{ or } S=\begin{bmatrix}1&1\\2&1\end{bmatrix}\begin{bmatrix}3&2\\4&1\end{bmatrix}^{-1}=\begin{bmatrix}1&1\\2&1\end{bmatrix}\left(-\tfrac{1}{5}\right)\begin{bmatrix}1&-2\\-4&3\end{bmatrix}=$$

$$\left(-\tfrac{1}{5}\right)\begin{bmatrix}-3&1\\-2&-1\end{bmatrix}=\tfrac{1}{5}\begin{bmatrix}3&-1\\2&1\end{bmatrix}. \text{ So } S^{-1}\begin{bmatrix}3\\1\end{bmatrix}=\left(\tfrac{1}{5}\begin{bmatrix}3&-1\\2&1\end{bmatrix}\right)^{-1}\begin{bmatrix}3\\1\end{bmatrix}=\begin{bmatrix}1&1\\-2&3\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix}=\begin{bmatrix}4\\-3\end{bmatrix}.$$

2. If $A$ and $B$ are two $n\times n$ matrices, $A$ is said to be *similar* to $B$ if and only if there is an invertible $n\times n$ matrix $S$ so that $A=SBS^{-1}$. Now, suppose that $A$, $B$ and $C$ are $n\times n$ matrices.

   a. Show: $A$ is similar to itself.

$I$ is an $n\times n$ invertible matrix and $A=(I)A(I)^{-1}$.

   b. Show: if $A$ is similar to $B$, then $B$ is similar to $A$.

There is an $n\times n$ invertible matrix $S$ so that $A=SBS^{-1}$. But $S^{-1}$ is an $n\times n$ invertible matrix such that $B=(S^{-1})A(S^{-1})^{-1}$.

   c. Show: if $A$ is similar to $B$ and $B$ is similar to $C$, then $A$ is similar to $C$.

There are $n\times n$ invertible matrices $S$ and $T$ so that $A=SBS^{-1}$ and $B=TCT^{-1}$.

$ST$ is an invertible matrix and $A=SBS^{-1}=S(TCT^{-1})S^{-1}=(ST)C(ST)^{-1}$.

3. Consider the matrix $A=\begin{bmatrix}0 & 2\\ 1 & 1\end{bmatrix}$ and the vectors $\vec{v}_1=\begin{bmatrix}1\\ 1\end{bmatrix}$ and $\vec{v}_2=\begin{bmatrix}2\\ -1\end{bmatrix}$. Let $\mathcal{B}=(\vec{v}_1,\vec{v}_2)$. Notice that $A\vec{v}_1=2\vec{v}_1$ and $A\vec{v}_2=-\vec{v}_2$. Find a diagonal matrix $D$ and invertible matrix $S$ so that $SDS^{-1}=A$. This result shows that $A$ is similar to a diagonal matrix.

If we let $S=[\vec{v}_1\,|\,\vec{v}_2]=\begin{bmatrix}1 & 2\\ 1 & -1\end{bmatrix}$ and $D=\begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix}$, then $SDS^{-1}=A$.

4. $\mathcal{B}=(\vec{v}_1,\vec{v}_2)=\left(\begin{bmatrix}3\\ 4\end{bmatrix},\begin{bmatrix}-4\\ 3\end{bmatrix}\right)$ is a basis for $\mathbf{R}^2$. Define the linear transformation $T:\mathbf{R}^2\to\mathbf{R}^2$ by $[T(\vec{v}_1)]_{\mathcal{B}}=\begin{bmatrix}7\\ 0\end{bmatrix}$ and $[T(\vec{v}_2)]_{\mathcal{B}}=\begin{bmatrix}0\\ 1\end{bmatrix}$.

a. Describe $T$ in geometric terms.

The transformation $T$ is a stretch by the factor 7 along the direction parallel to $\begin{bmatrix}3\\ 4\end{bmatrix}$ in the plane.

b. Determine the matrix $A$ such that $T(\vec{x})=A\vec{x}$ for any $\vec{x}$ in $\mathbf{R}^2$.

$A=SDS^{-1}$ where $S=[\vec{v}_1\,|\,\vec{v}_2]=\begin{bmatrix}3 & -4\\ 4 & 3\end{bmatrix}$ and $D=\left[[T(\vec{v}_1)]_{\mathcal{B}}\,|\,[T(\vec{v}_2)]_{\mathcal{B}}\right]=\begin{bmatrix}7 & 0\\ 0 & 1\end{bmatrix}$.

So,

$$A=\begin{bmatrix}3 & -4\\ 4 & 3\end{bmatrix}\begin{bmatrix}7 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}3 & -4\\ 4 & 3\end{bmatrix}^{-1}=\begin{bmatrix}3 & -4\\ 4 & 3\end{bmatrix}\begin{bmatrix}7 & 0\\ 0 & 1\end{bmatrix}\left(\tfrac{1}{25}\right)\begin{bmatrix}3 & 4\\ -4 & 3\end{bmatrix}=\tfrac{1}{25}\begin{bmatrix}3 & -4\\ 4 & 3\end{bmatrix}\begin{bmatrix}21 & 28\\ -4 & 3\end{bmatrix}$$

$$=\tfrac{1}{25}\begin{bmatrix}79 & 72\\ 72 & 121\end{bmatrix}=\begin{bmatrix}3.16 & 2.84\\ 2.84 & 4.84\end{bmatrix}.$$

5. $P=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$ is clearly the matrix for projection onto the *xy*-plane in $\mathbf{R}^3$. It is easy to verify that this matrix has the usual properties that a projection matrix

should have: $P = P^2 = P^T$. By an appropriate change of coordinates, we can even show that the matrix for projection onto *any* plane through the origin in $\mathbf{R}^3$ is similar to $P$. This says, roughly, that all projections onto planes in $\mathbf{R}^3$ are the same, apart from the observer's orientation or viewpoint.

a. Let $M$ be any plane through the origin in $\mathbf{R}^3$. [We note that $M$ is a subspace of $\mathbf{R}^3$ and $\dim(M) = 2$.] Now choose a basis for $\mathbf{R}^3$ as follows. Let $\mathcal{B} = (\hat{u}_1, \hat{u}_2, \hat{u}_3)$ where the vectors in the basis are orthonormal (i.e. each has unit magnitude and they are pairwise orthogonal), $\hat{u}_1$ and $\hat{u}_2$ are in $M$ and $\hat{u}_3$ is normal to $M$. [$(\hat{u}_1, \hat{u}_2)$ is a basis for $M$.] Now define $S = [\hat{u}_1 \,|\, \hat{u}_2 \,|\, \hat{u}_3]$.

a. Show that $S^T S = I$, i.e. $S^T = S^{-1}$.

First note that $\hat{u}_j^T \hat{u}_k = \delta_{jk}$. This says, quite simply, that the scalar (dot or inner) product of two basis vectors is 1 if they are the same and 0 otherwise. But, the $jk$th entry of the product matrix $S^T S$ is precisely the scalar product of the $j$th row vector of $S^T$ (this is $\hat{u}_j^T$) and the $k$th column vector of $S$ (this is $\hat{u}_k$). Of course, $\delta_{jk}$ is the $jk$th entry of the identity matrix.

b. Show that the matrix $Q = S\,P\,S^{-1}$ which is, by definition, similar to $P$, has the following expected properties.

i. $Q = Q^2 = Q^T$.

$$Q^2 = (S\,P\,S^{-1})(S\,P\,S^{-1}) = S\,P\,(S^{-1}S)\,P\,S^{-1} = S\,P\,P\,S^{-1} = S\,P\,S^{-1} = Q.$$

$$Q^T = (S\,P\,S^{-1})^T = (S\,P\,S^T)^T = S^{TT} P\,S^T = S\,P\,S^{-1} = Q.$$

ii. $Q\,\hat{u}_1 = \hat{u}_1$, $Q\,\hat{u}_2 = \hat{u}_2$, and $Q\,\hat{u}_3 = \vec{0}$.

$$Q\hat{u}_1 = S\,P\,S^{-1}\hat{u}_1 = S\,P\,S^T\hat{u}_1 = [\hat{u}_1 \,|\, \hat{u}_2 \,|\, \hat{u}_3]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \hat{u}_1^T \\ \hline \hat{u}_2^T \\ \hline \hat{u}_3^T \end{bmatrix}\hat{u}_1 = [\hat{u}_1 \,|\, \hat{u}_2 \,|\, \hat{u}_3]\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \hat{u}_1$$

$$Q\hat{u}_2 = S\,P\,S^{-1}\hat{u}_2 = S\,P\,S^T\hat{u}_2 = [\hat{u}_1 \,|\, \hat{u}_2 \,|\, \hat{u}_3]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \hat{u}_1^T \\ \hline \hat{u}_2^T \\ \hline \hat{u}_3^T \end{bmatrix}\hat{u}_2 = [\hat{u}_1 \,|\, \hat{u}_2 \,|\, \hat{u}_3]\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \hat{u}_2$$

$$Q\hat{u}_3 = S\,P\,S^{-1}\hat{u}_3 = S\,P\,S^T\hat{u}_3 = [\hat{u}_1 \,|\, \hat{u}_2 \,|\, \hat{u}_3]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \hat{u}_1^T \\ \hline \hat{u}_2^T \\ \hline \hat{u}_3^T \end{bmatrix}\hat{u}_3 = [\hat{u}_1 \,|\, \hat{u}_2 \,|\, \hat{u}_3]\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \vec{0}$$

6. a. Using the results of problem 4, calculate the matrix $Q$ for projection onto the plane $M$ through the origin in $\mathbf{R}^3$ whose equation is $x+2y+2z=0$.

First, we determine an orthonormal basis for $\mathbf{R}^3$ whose third member is chosen to be a unit normal to $M$. So, we set $\hat{u}_3 = \frac{1}{3}\begin{bmatrix}1\\2\\2\end{bmatrix}$. Now any vector orthogonal to this vector lies in $M$. There are infinitely many possibilities for picking a pair of unit vectors in $M$ that are also orthogonal to each other. Any such pair will do. We choose $\hat{u}_1 = \frac{1}{3}\begin{bmatrix}2\\-2\\1\end{bmatrix}$ and $\hat{u}_2 = \frac{1}{3}\begin{bmatrix}2\\1\\-2\end{bmatrix}$. So, $\mathcal{B} = (\hat{u}_1, \hat{u}_2, \hat{u}_3)$ is our orthonormal basis whose first two vectors lie in $M$ and whose third vector is normal to $M$. We have

$$Q = SPS^{-1} = SPS^{T} = \frac{1}{3}\begin{bmatrix}2 & 2 & 1\\-2 & 1 & 2\\1 & -2 & 2\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}\frac{1}{3}\begin{bmatrix}2 & -2 & 1\\2 & 1 & -2\\1 & 2 & 2\end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix}2 & 2 & 1\\-2 & 1 & 2\\1 & -2 & 2\end{bmatrix}\begin{bmatrix}2 & -2 & 1\\2 & 1 & -2\\0 & 0 & 0\end{bmatrix} = \frac{1}{9}\begin{bmatrix}8 & -2 & -2\\-2 & 5 & -4\\-2 & -4 & 5\end{bmatrix}.$$

b. Using a result obtained earlier, calculate the matrix $N$ for projection onto the line through the origin in $\mathbf{R}^3$ that is normal to the plane $M$ in part a.

We determine the matrix for projection onto the line through the origin and parallel to $\hat{u}_3$. Now, the matrix for projection onto a line parallel to a unit vector $\hat{v} = \begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}$ is $\begin{bmatrix}v_1v_1 & v_1v_2 & \cdots & v_1v_n\\v_2v_1 & v_2v_2 & \cdots & v_2v_v\\\vdots & \vdots & \ddots & \vdots\\v_nv_1 & v_nv_2 & \cdots & v_nv_n\end{bmatrix}$. We find $N = \hat{u}_3\hat{u}_3^{T} = \frac{1}{3}\begin{bmatrix}1\\2\\2\end{bmatrix}\frac{1}{3}\begin{bmatrix}1 & 2 & 2\end{bmatrix} = \frac{1}{9}\begin{bmatrix}1 & 2 & 2\\2 & 4 & 4\\2 & 4 & 4\end{bmatrix}$.

c. What is the relationship between $N$ and $Q$? [This should suggest an alternate means to determine $Q$.]

$N + Q = I$. Projection onto the normal line plus projection onto the plane is the identity. Multiplying both sides of this equation on the right by any vector $\vec{x} \in \mathbf{R}^3$ gives us $\vec{x} = N\vec{x} + Q\vec{x}$ which is just another way of saying that any vector can be resolved into its projection normal to a plane and parallel to that plane. Since computing $N$ is simpler than computing $Q$, this suggests that one compute $N$ and then obtain $Q = I - N$ by subtraction from the identity. There is another relationship involving these projection matrices. Since these are projections onto mutually orthogonal subspaces of $\mathbf{R}^3$, we should expect that $N\,Q = Q\,N = 0$. Indeed, straightforward matrix multiplication shows that this is true.