SM261 Final Examination

09 May 2013

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1. Let $\vec{u} = \frac{1}{5}\begin{bmatrix}3\\4\end{bmatrix}$ and $\vec{y} = \begin{bmatrix}5\\10\end{bmatrix}$.

   a. What is the angle between $\vec{u}$ and $\vec{y}$?

$\vec{u}\cdot\vec{y} = \|\vec{u}\|\,\|\vec{y}\|\cos\theta \Rightarrow 11 = 1\cdot 5\sqrt{5}\cos\theta \Rightarrow \cos(\theta) = \frac{11}{5\sqrt{5}}$

$\Rightarrow \theta = \arccos\left(\frac{11}{5\sqrt{5}}\right) \Rightarrow \theta \cong .18$ radians, or about $10°$

   b. What is $proj_L(\vec{y})$, where L is the line through the origin parallel to $\vec{u}$?

$proj_L(\vec{y}) = (\vec{u}\cdot\vec{y})\vec{u} = 11\vec{u} = \frac{11}{5}\begin{bmatrix}3\\4\end{bmatrix}$

   c. What is $ref_L(\vec{y})$? $ref_L(\vec{y}) = 2proj_L(\vec{y}) - \vec{y}$

$$= \frac{22}{5}\begin{bmatrix}3\\4\end{bmatrix} - \begin{bmatrix}5\\10\end{bmatrix}$$

$$= \begin{bmatrix}\frac{66}{5}-\frac{25}{5}\\ \frac{88}{5}-\frac{50}{5}\end{bmatrix} = \begin{bmatrix}41/5\\38/5\end{bmatrix}$$

   d. Find the matrix of the linear transformation T which is a rotation and satisfies $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \vec{u}$.

The matrix is orthogonal. Its first column is $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \vec{u} = \begin{bmatrix}3/5\\4/5\end{bmatrix}$

so its second column will be a unit vector orthogonal to $\vec{u}$.

The matrix is $\begin{bmatrix}3/5 & -4/5\\4/5 & 3/5\end{bmatrix}$

   e. Find the matrix of the linear transformation R which is a reflection and satisfies $R\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \vec{u}$.

8

2. Use Gaussian elimination techniques to find all solutions to the following system.

$$\begin{aligned} x + 2y + 3z &= 1 \\ 3x + 2y + z &= 1 \\ 7x + 2y - 3z &= 1 \end{aligned}.$$

Form the augmented matrix and find its rref:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 3 & 2 & 1 & 1 \\ 7 & 2 & -3 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -4 & -8 & -2 \\ 0 & -12 & -24 & -6 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 1/2 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left.\begin{aligned} x_1 - x_3 &= 0 \\ x_2 + 2x_3 &= \tfrac{1}{2} \end{aligned}\right\} \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -2x_3 + \frac{1}{2} \\ x_3 \end{bmatrix} = s\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/2 \\ 0 \end{bmatrix}$$

3. Define the rank of a matrix.
   a. Suppose A is a 3×5 matrix. What are the possible values of the rank of the matrix?

Answer: 3 is the maximum since each $\underbrace{\text{row}}_{\text{of rref}(A)}$ can have at most leading 1.

Other possibilities are 2, 1, 0.

   b. Suppose A is a 3×5 matrix. What are the possible values of the nullity of the matrix?

By the Rank-Nullity Theorem, rank + nullity = 5, so nullity can be

2, 3, 4, 5

   c. Suppose A is a 3×5 matrix and has rank 3. Must the system $A\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ have a solution?

   Explain.

Yes, rref(A) has 3 leading 1's since it has rank 3. We can set the free variables to 0. The leading one variables will equal a, b, c where rref$\left(\left[A \middle| \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}\right]\right)$ has last column $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$

   d. Give an example of a 3×5 matrix A which has rank 2 but $A\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ has no solution.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} \quad \text{so cannot equal} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

12

4. Let A be the matrix $\begin{bmatrix} 1 & 1 & 1 & -1 & -2 \\ 1 & 1 & 2 & -2 & -2 \\ 2 & 2 & 3 & -2 & -2 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}$ and let $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 2 \end{bmatrix}$. In the following you may use the fact that the augmented matrix $(A|\vec{b})$ has rref equal to $\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$.

a. Find all solutions to the system $A\vec{x} = \vec{b}$.

$x_1 = -x_2 + 2x_5 + 1$

$x_3 = -2x_5 + 1$

$x_4 = -2x_5 + 1$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -x_2 + 2x_5 + 1 \\ x_2 \\ -2x_5 + 1 \\ -2x_5 + 1 \\ x_5 \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ 0 \\ -2 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

b. Find a basis for $\ker(A)$. ker(A) can be read from the answer above.

$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \\ -2 \\ 1 \end{bmatrix} \right\}$ is a basis for ker(A)

c. Let $V = im(A)$. What is $\dim(V^{\perp})$?

V is a subspace of $\mathbb{R}^4$ of dimension equal to rank(A) = 3

So $V^{\perp}$ has dimension 1.

5. Find the area of the parallelogram with vertices (1,4), (2,5), (2,-1) and (3,0).

Two sides of the parallelogram correspond to $\vec{v} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and

$\vec{w} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$

Then $\det \begin{bmatrix} \vec{v} & \vec{w} \end{bmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -5 \end{vmatrix} = -6$ is (minus) the area.

6. Let $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.

a. State the Cauchy-Schwarz Inequality and show that it holds for these vectors.

$|\vec{v} \cdot \vec{w}| \le \|\vec{v}\| \, \|\vec{w}\|$

$3 \le \sqrt{3}\sqrt{5} = \sqrt{15}$

b. State the Triangle Inequality and show that it holds for these vectors.

$\|\vec{v} + \vec{w}\| \le \|\vec{v}\| + \|\vec{w}\|$ or $\left\| \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right\| \le \left\| \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\| + \left\| \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\|$ or

$\sqrt{14} \le \sqrt{3} + \sqrt{5}$

c. Find an orthonormal basis for $V = span(\vec{v}, \vec{w})$.

$\vec{u}_1 = \frac{1}{\|\vec{v}\|} \vec{v} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\vec{v}_2 = \vec{w} - (\vec{w} \cdot \vec{u}_1) \vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \left(\frac{3}{\sqrt{3}}\right) \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$,

$\vec{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.

d. Find an orthonormal basis for $V^{\perp}$.

Take $\vec{u}_1 \times \vec{u}_2 =$

$\frac{1}{\sqrt{2}} \frac{1}{\sqrt{3}} \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 0 & -1 & 1 \end{vmatrix} = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$

8/16

7. Let $A = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$.

a. Use Gaussian elimination techniques to find A^{-1}.

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{3}{2} & 0 & 0 & -\frac{1}{2} & 1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -\frac{1}{3} & \frac{2}{3} \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 0 & 1 & 0 & \frac{2}{3} & -\frac{4}{3} \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 & 0 & -\frac{1}{3} & \frac{2}{3} \end{array}\right] \longrightarrow \left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 0 & 1 & 0 & \frac{2}{3} & -\frac{4}{3} \\ 0 & 1 & 0 & 0 & 0 & 1 & -\frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 & 0 & -\frac{1}{3} & \frac{2}{3} \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -2 & 2 & -2 \\ 0 & 1 & 0 & 0 & 0 & 1 & -\frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & \frac{2}{3} \end{array}\right]$$

b. Find all eigenvalues of A. For each eigenvalue determine its algebraic and geometric multiplicity.

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 & 0 & 2 \\ 0 & 1-\lambda & 1 & 0 \\ 0 & 0 & 2-\lambda & 1 \\ 0 & 0 & 1 & 2-\lambda \end{vmatrix} = (1-\lambda)(1-\lambda)\left((2-\lambda)^2 - 1\right) = (\lambda-1)^2(\lambda^2 - 4\lambda + 3)$$
$$= (\lambda-1)^3(\lambda-3)$$

For eval $\lambda = 1$, find all $\vec{x}$ so that $(A-I)\vec{x} = \vec{0}$

$$\left[\begin{array}{cccc|c} 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \text{only evecs are scalar multiples of } \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

so 1 has alg mult 3, geom 1
3 " " 1, " " 1.

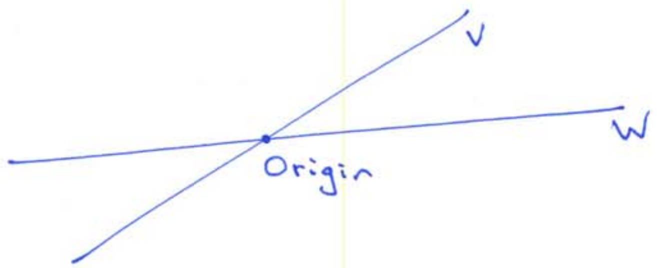
c. Does A have an eigenbasis? Explain.

No because the algebraic multiplicity must match the geometric multiplicity for each e-value

8. Define "V is a subspace of R^n ".

a. Show by example that the union of a pair V, W of subspaces of R^n need not be a subspace.

If V, W are a pair of lines their union is not a subspace



b. Show that the set of all vectors $\vec{v} + \vec{w}$, where $\vec{v}$ is in V and $\vec{w}$ is in W, forms a subspace.

1.) Let $\vec{v}_1 + \vec{w}_1$ and $\vec{v}_2 + \vec{w}_2$ be in the set, then their sum is $(\vec{v}_1 + \vec{v}_2) + (\vec{w}_1 + \vec{w}_2)$ and $\vec{v}_1 + \vec{v}_2$ in V, $\vec{w}_1 + \vec{w}_2$ in W

2.) If $\vec{v} + \vec{w}$ in the set then $c(\vec{v} + \vec{w}) = c\vec{v} + c\vec{w}$ is in the set, since $c\vec{v}$ is in V and $c\vec{w}$ is in W.

9. State the Rank-Nullity Theorem and illustrate for the matrix $A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 1 \\ 1 & 2 & 3 & 0 \end{bmatrix}$.

Rank + Nullity = number of columns

$\text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ so rank of A is 2

Any vector in ker (A) satisfies $x_1 = -2x_2 - 3x_4$, $x_3 = x_4$

so $\vec{x} = s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ so nullity = dim(ker(A)) = 2

and 2 + 2 = 4.

10. Use Cramer's Rule to find the solution $\vec{x}$ to $\begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}\vec{x} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

$$x_1 = \frac{\begin{vmatrix} 3 & 2 \\ 4 & 8 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 3 & 8 \end{vmatrix}} = \frac{16}{2} = 8 \quad \text{and} \quad x_2 = \frac{\begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 3 & 8 \end{vmatrix}} = \frac{-5}{2}$$

11. Consider the vectors $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

a. Show that these vectors form a basis of R^3.

$$\text{rref}\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} = I$$ so the vectors form a basis.

b. Find the coordinates of the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ with respect to this basis.

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 2 \\ 2 & 1 & 1 & 3 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$ so $[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$.

c. Let T be the linear transformation given by $T(\vec{x}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\vec{x}$. Find the matrix of T with respect to the basis. With $S = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$ the matrix is

$$B = S^{-1}AS = \begin{bmatrix} -1 & 0 & -1 \\ 2 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

10/14

12. Determine an invertible matrix S and a diagonal matrix D such that $S^{-1}AS = D$, where $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$. $|A - \lambda I| = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3 = (\lambda-3)(\lambda-1)$

Evecs for $\lambda = 3$ satisfy $(A-3I)\vec{x} = \vec{0} \Rightarrow \vec{x}$ is a scalar mult of $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

" " $\lambda = 1$ " " $(A-I)\vec{x} = \vec{0} \Rightarrow \vec{x}$ " " " " $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Then with $S = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$, $S^{-1}AS = D = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

13. A square matrix A is called a permutation matrix if there is a single 1 in each row and each column, and if all other entries are 0. For example, $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ is a permutation matrix.

a. List all 3×3 symmetric permutation matrices.

I, $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

b. Explain why all $n \times n$ permutation matrices are invertible.

They are orthogonal matrices

c. Must the inverse of a permutation matrix also be a permutation matrix? Explain.

Since each such is orthogonal, its inverse is $A^{-1} = A^T$, but the transpose of a permutation matrix is a permutation matrix.