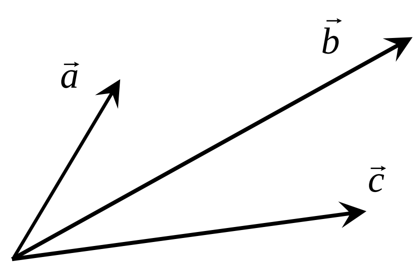
The point value for each of the 19 problems is indicated as follows. [**t**] means there is one part to the problem and its total value is **t**. [**t**: $\mathbf{t}_1$, …, $\mathbf{t}_n$] means that the total value for the problem is **t** and it has *n* parts worth $\mathbf{t}_1$, …, $\mathbf{t}_n$ points, respectively.

1. [4:1 each] For each of the following statements about sets of (distinct) vectors in $\mathbf{R}^4$, indicate whether it is **always** true, **sometimes** true, or **never** true.
   a. $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5\}$ is linearly dependent in $\mathbf{R}^4$.
   b. $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5\}$ spans $\mathbf{R}^4$.
   c. $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent in $\mathbf{R}^4$.
   d. $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ spans $\mathbf{R}^4$.

2. [5:1,4] The matrix $[A\,|\,\vec{b}\,] = \left[\begin{array}{ccccc|c} 0 & \square & * & * & * & 0 \\ 0 & 0 & \square & * & * & 0 \\ 0 & 0 & 0 & 0 & \square & * \end{array}\right]$ is the augmented matrix for a matrix equation $A\,\vec{x} = \vec{b}$ corresponding to a system of three equations. Here and in the next problem, $\square$ stands for a non-zero real number and $*$ stands for a real number (possibly zero).
   a. Is the system consistent?
   b. Find rank(A), dim(Col A), dim(Row A), dim(Nul A).

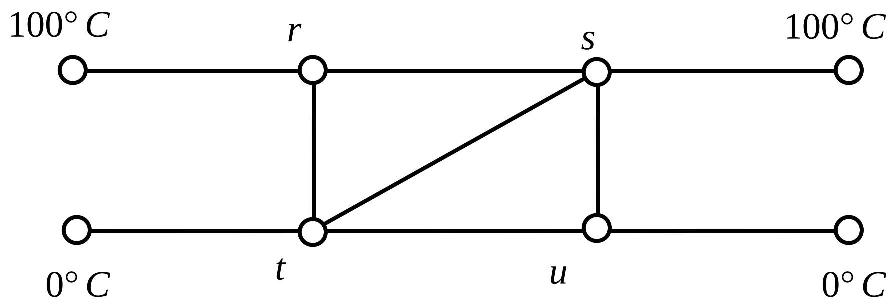
3. [4:2,2] Consider the three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ in $\mathbf{R}^2$ shown at the right. Using the symbols $\square, *, 1$ and 0 only, determine the row-reduced echelon form of the matrices $[\vec{a}\,|\,\vec{b}\,]$ and $[\vec{a}\,|\,\vec{b}\,|\,\vec{c}\,]$.



4. [2] For what values of h is $\vec{y} = \begin{bmatrix} h \\ 2 \\ 10 \end{bmatrix}$ in $\text{span}\left(\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 8 \end{bmatrix}\right)$?

5. [4] A grid with eight nodes is shown below. The four external nodes are maintained at the fixed temperatures shown. The temperatures r, s, t, and

u at the four internal nodes are to be determined from the fact that the internal nodes are at thermal equilibrium and so their temperatures are the averages of the temperatures at the adjacent nodes. Two nodes are adjacent if connected by a line segment in the diagram. Formulate the problem as a linear system and solve for the four temperatures at the internal nodes.



6. [4:1 each] Compute or state "not defined".

a. $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ b. $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ d. $\begin{bmatrix} 2 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 3 & 1 \end{bmatrix}$

7. [7:1,1,1,2,2]. Given $A = \begin{bmatrix} 2 & 3 & 5 \\ -5 & 1 & -4 \\ -3 & -1 & -4 \\ 1 & 0 & 1 \end{bmatrix}$, observe that the third column is the sum of the first two columns.

a. Find a nontrivial solution of $A\vec{x} = \vec{0}$.
b. Do the columns of A form a basis for $\mathbf{R}^4$?
c. Do the rows of A form a basis for $\mathbf{R}^3$?
d. Is the linear transformation $\vec{x} \mapsto A\vec{x}$ one-to-one? Explain.
e. Is the linear transformation $\vec{x} \mapsto A\vec{x}$ onto $\mathbf{R}^4$? Explain.

8. [3:3] Let L be the line in $\mathbf{R}^3$ that consists of all the points whose three coordinates are equal. Suppose that T is a linear transformation that effects a rotation by $120°$ about L, mapping the x-axis onto the y-axis, the y-axis

onto the z-axis, and the z-axis onto the x-axis. What is the matrix for T. That is, what is the matrix A so that $T(\vec{x}) = A\,\vec{x}$?

9. [4:2,2] Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

a. Find an expression for $A\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

b. If det(A) = 7, what is $\det\left(\begin{bmatrix} 3c & 3d \\ 3a & 3b \end{bmatrix}\right)$?

10. [3]. A solution to a linear system of equations is given by Cramer's

Rule: $x = \dfrac{\det\begin{bmatrix} 3 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 0 & -1 \end{bmatrix}}{\det\begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 1 \\ 1 & 0 & -1 \end{bmatrix}}$, $y = \dfrac{\det\begin{bmatrix} 1 & 3 & 2 \\ 1 & 2 & 1 \\ 1 & 1 & -1 \end{bmatrix}}{\det\begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 1 \\ 1 & 0 & -1 \end{bmatrix}}$, $z = \dfrac{\det\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}}{\det\begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 1 \\ 1 & 0 & -1 \end{bmatrix}}$.

Find the system of equations.

11. [6:2,2,2] Determine if the following subsets are subspaces of $\mathbf{R}^3$. If so, find a basis. If not, state a reason.

a. $\operatorname{span}\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}\right)$.

b. $\left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + y + z = 0\right\}$.

c. $\left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix} : x\,y\,z = 0\right\}$.

12. [6:3,3] Given:

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 & 5 \\ 1 & 2 & 1 & 7 & 11 \\ 2 & 4 & -1 & 2 & 4 \\ 1 & 2 & 2 & 11 & 17 \end{bmatrix} \text{ and } A_{rref} = \begin{bmatrix} 1 & 2 & 0 & 3 & 5 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

a. Determine a basis for and the dimension of Nul(A).
b. Determine a basis for and the dimension of Col(A)

13. [6:2,2,2] Two bases for a subspace W of $\mathbf{R}^3$ are $\mathcal{B} = (\vec{b}_1, \vec{b}_2)$ and $\mathcal{C} = (\vec{c}_1, \vec{c}_2)$, where $\vec{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{b}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{c}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and $\vec{c}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

a. Express $\vec{c}_1$ and $\vec{c}_2$ as linear combinations of $\vec{b}_1$ and $\vec{b}_2$.
b. Find the change of basis matrix that changes $\mathcal{C}$ coordinates to $\mathcal{B}$ coordinates.
c. If $\vec{x} = 3\vec{c}_1 - 2\vec{c}_2$, find $[\vec{x}]_{\mathcal{B}}$ and $[\vec{x}]_{\mathcal{C}}$.

14.[5:3,2] Let $A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$. Then, the matrix equation $A\vec{x} = \vec{b}$ is inconsistent.

a. Find a vector $\vec{x}*$ so that $A\vec{x}*$ is closest to $\vec{b}$.
b. Find the parameters c and d so that the equation $s = c\,t + d$ is the line of best fit to the data points (t, s) = (3, 5), (–1,1), and (1,0). [Hint: your work from part a can be used here.]

15 [8:2 each] Let $A = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix}$.

a. Find the eigenvalues of A.
b. For each eigenvalue, find an eigenvector with integer components.
c. Find a matrix P and a diagonal matrix D so that $A = P\,D\,P^{-1}$.
d. Determine the matrix A^n for any positive integer n.

16.[3] List three applications of matrix theory in science or social science not mentioned in this examination.

17.[5] Read the following theorem and its proof. Then, answer the question below.

Theorem. If A is a diagonalizable matrix all of whose eigenvalues are non-negative real numbers, then there is a matrix B with non-negative eigenvalues such that $B^2 = A$.

Proof. Since A is diagonalizable, there is an invertible matrix S such that $L = S^{-1}$ is diagonal. The diagonal entries $\lambda_1, \lambda_2, \ldots, \lambda_n$ of L (which are the eigenvalues of A) are non-negative, by hypothesis. Let $\mu_i = \sqrt{\lambda_i}$, for $i = 1, 2, \ldots, n$, be the non-negative square roots of the eigenvalues and let M be the diagonal matrix with diagonal entries $\mu_1, \mu_2, \ldots, \mu_n$. Clearly, $M^2 = L$. Since similar matrices have the same eigenvalues, the matrix $B = S\,M\,S^{-1}$ has eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$. Moreover,
$B^2 = (S\,M\,S^{-1})^2 = S\,M\,S^{-1}S\,M\,S^{-1} = S\,M^2\,S^{-1} = S\,L\,S^{-1} = A$.

Question. The matrix $A = \begin{bmatrix} 10 & -9 \\ 6 & -5 \end{bmatrix}$ has eigenvalues 1 and 4. Find a matrix B with positive eigenvalues such that $B^2 = A$.

18.[5:1 each]. Indicate which choice of words offered in each case completes the sentence so it will be a true statement.

a. (**Some**, **Every**, **No**) square matrix with two equal columns will have zero determinant.

b. Let A be a square matrix and suppose that $A\vec{x} = \vec{0}$ has a nontrivial solution. Then det(A) (**may**, **cannot**, **must**) equal 0.

c. Let A and B be 5×5 matrices. Then, det($A\,B$) (**could**, **must**, **couldn't**) equal det(A) det(B).

d. Let A and B be 5×5 matrices. Then, det($A + B$) (**could**, **must**, **couldn't**) equal det(A) + det(B).

e. There (**does**, **doesn't**) exist a 5×5 matrix all of whose entries are integers and whose determinant is $\frac{2}{3}$.

19.[16:1 each] For each statement, indicate whether it is **True** or **False**.

a. Every elementary row operation is reversible.

b. If the thirteenth column of a 20×50 matrix is the sum of the first column plus twice the second plus twice the third, the same is true of the row-reduced echelon form of the same matrix.

c. The equation $A\vec{x} = \vec{b}$ is inconsistent if and only if $\vec{b}$ is in the span of the columns of A.

d. Let $\hat{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\hat{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{y}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, and $\vec{y}_2 = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$. Then the linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ such that $T(\hat{e}_1) = \vec{y}_1$ and $\mathrm{T}(\hat{e}_2) = \vec{y}_2$ sends $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ 7 \end{bmatrix}$.

e. If the second column of the matrix B consists entirely of zeroes, then the second column of the product $A\,B$ is all zeroes, too, assuming the product is defined.

f. If the zero vector is in a subset H of $\mathbf{R}^n$, H is a subspace of $\mathbf{R}^n$.

g. The columns of an invertible $n \times n$ matrix form a basis for $\mathbf{R}^n$.

h. Each line through the origin is a 1-dimensional subspace of $\mathbf{R}^n$.

i. $\det(A^T) = (-1)\det(A)$.

j. Assuming A, B, and C are all square matrices of the same size and that B is invertible, the matrix equation $A\,B = B\,C$ implies that $A = C$.

k. Every orthogonal subset of $\mathbf{R}^n$ is linearly independent.

l. A basis is a linearly independent subset of a vector space that is as large as possible.

m. The list $(1, t, t^2, 1 - t + t^2)$ is a basis for $\mathbf{P}_2$, the vector space of all polynomials with real coefficients with degree at most 2.

n. If μ and ω are eigenvalues of the square matrix A, then so is $\mu + \omega$

o. If A is an $m \times n$ matrix and the linear transformation $\vec{x} \mapsto A\vec{x}$ is onto, then $\operatorname{rank}(A) = m$.

p. If A is an 8×6 matrix and $\dim(\mathrm{Nul}(A)) = 2$, then $\dim((\mathrm{Col}(A))^{\perp}) = 2$.