Discretization.

Radial Navier-Stokes (Cylindrical, Axisymmetric, No z-resistance)

$$u_r\frac{\partial u_r}{\partial r}+u_z\frac{\partial u_r}{\partial z}-\frac{u_\theta^2}{r}=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu[\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r}\frac{\partial u_r}{\partial r}-\frac{u_r}{r^2}]$$

Tangential Navier-Stokes (Cylindrical, Axisymmetric, No z-resistance)

$$u_r\frac{\partial u_\theta}{\partial r}+u_z\frac{\partial u_\theta}{\partial z}+\frac{u_r u_\theta}{r}=\nu[\frac{\partial^2 u_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r^2}]$$

Note: The above equations assume axisymmetry ($\partial/\partial\theta = 0$) and have removed the z-derivatives in the viscous terms ($\partial^2/\partial z^2 = 0$) to eliminate flow resistance in the z-direction.

Continuity for Pressure

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right)=-\rho\left[\frac{\partial}{\partial r}\left(u_r\frac{\partial u_r}{\partial r}\right)-\frac{u_\theta^2}{r^2}\right]$$

General Case

Second-Order Accurate Discretization for Radial Navier-Stokes

For the radial Navier-Stokes equation:

$$u_r\frac{\partial u_r}{\partial r}+u_z\frac{\partial u_r}{\partial z}-\frac{u_\theta^2}{r}=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu[\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r}\frac{\partial u_r}{\partial r}-\frac{u_r}{r^2}]$$

Here is the second-order discretization of the radial Navier-Stokes equation:

$$u_{r,i}\frac{u_{r,i+1}-u_{r,i-1}}{2\Delta r}+u_{z,i}\frac{u_{r,i+1/2}-u_{r,i-1/2}}{2\Delta z}-\frac{(u_{\theta,i})^2}{r_i}=-\frac{1}{\rho}\frac{p_{i+1}-p_{i-1}}{2\Delta r}+\nu\left[\frac{u_{r,i+1}-2u_{r,i}+u_{r,i-1}}{(\Delta r)^2}+\frac{1}{r_i}\frac{u_{r,i+1}-u_{r,i-}}{2\Delta r}\right.$$

Now, solving for $u_{r,i}$ by rearranging terms:

$$u_{r,i}=\frac{1}{2+\frac{(\Delta r)^2}{r_i^2}}\left[u_{r,i+1}\left(\frac{\nu\Delta t}{(\Delta r)^2}+\frac{\nu\Delta t}{2r_i\Delta r}-\frac{u_{r,i}\Delta t}{2\Delta r}\right)+u_{r,i-1}\left(\frac{\nu\Delta t}{(\Delta r)^2}-\frac{\nu\Delta t}{2r_i\Delta r}+\frac{u_{r,i}\Delta t}{2\Delta r}\right)+u_{r,i}^{old}-\frac{\Delta t}{\rho}\frac{p_{i+1}-p_{i-1}}{2\Delta r}\right.$$

This form assumes an implicit time discretization. For an explicit scheme, you would use the known values from the previous time step for all velocity terms on the right-hand side.

For the explicit formulation of the radial Navier-Stokes equation, all velocity terms would use values from the previous time step:

$$u_{r,i}^{new}=u_{r,i}^{old}+\Delta t\left[-u_{r,i}^{old}\frac{u_{r,i+1}^{old}-u_{r,i-1}^{old}}{2\Delta r}-u_{z,i}^{old}\frac{u_{r,i+1/2}^{old}-u_{r,i-1/2}^{old}}{2\Delta z}+\frac{(u_{\theta,i}^{old})^2}{r_i}-\frac{1}{\rho}\frac{p_{i+1}-p_{i-1}}{2\Delta r}+\nu\left(\frac{u_{r,i+1}^{old}-2u_{r,i}^{old}+u_{r,}^{ol}}{(\Delta r)^2}\right.\right.$$

This explicit scheme is straightforward to implement but has stability restrictions. The time step must satisfy the CFL condition:

$$\Delta t \leq \min\left(\frac{\Delta r}{|u_r|}, \frac{\Delta z}{|u_z|}, \frac{(\Delta r)^2}{2\nu}\right)$$

For improved stability in the viscous terms, you could use a semi-implicit approach where only the viscous terms use the new time step values while keeping convective terms explicit.

Second-Order Accurate Discretization for Tangential Navier-Stokes

For the tangential Navier-Stokes equation:

$$u_r\frac{\partial u_\theta}{\partial r} + u_z\frac{\partial u_\theta}{\partial z} + \frac{u_r u_\theta}{r} = \nu[\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2}]$$

Here is the second-order discretization of the tangential Navier-Stokes equation:

$$u_{r,i}\frac{u_{\theta,i+1} - u_{\theta,i-1}}{2\Delta r} + u_{z,i}\frac{u_{\theta,i+1/2} - u_{\theta,i-1/2}}{2\Delta z} + \frac{u_{r,i}u_{\theta,i}}{r_i} = \nu\left[\frac{u_{\theta,i+1} - 2u_{\theta,i} + u_{\theta,i-1}}{(\Delta r)^2} + \frac{1}{r_i}\frac{u_{\theta,i+1} - u_{\theta,i-1}}{2\Delta r} - \frac{u_{\theta,i}}{r_i^2}\right]$$

Now, solving for $u_{\theta,i}$ by rearranging terms for implicit time marching:

$$u_{\theta,i} = \frac{1}{2 + \frac{(\Delta r)^2}{r_i^2} + \frac{u_{r,i}\Delta t}{r_i}}\left[u_{\theta,i+1}\left(\frac{\nu\Delta t}{(\Delta r)^2} + \frac{\nu\Delta t}{2r_i\Delta r} - \frac{u_{r,i}\Delta t}{2\Delta r}\right) + u_{\theta,i-1}\left(\frac{\nu\Delta t}{(\Delta r)^2} - \frac{\nu\Delta t}{2r_i\Delta r} + \frac{u_{r,i}\Delta t}{2\Delta r}\right) + u_{\theta,i}^{old} - u_{z,i}\Delta t\right.$$

This form implements an implicit time discretization scheme. For an explicit scheme, you would use the known values from the previous time step for all velocity terms on the right-hand side.

For the explicit formulation of the tangential Navier-Stokes equation, all velocity terms would use values from the previous time step:

$$u_{\theta,i}^{new} = u_{\theta,i}^{old} + \Delta t\left[-u_{r,i}^{old}\frac{u_{\theta,i+1}^{old} - u_{\theta,i-1}^{old}}{2\Delta r} - u_{z,i}^{old}\frac{u_{\theta,i+1/2}^{old} - u_{\theta,i-1/2}^{old}}{2\Delta z} - \frac{u_{r,i}^{old}u_{\theta,i}^{old}}{r_i} + \nu\left(\frac{u_{\theta,i+1}^{old} - 2u_{\theta,i}^{old} + u_{\theta,i-1}^{old}}{(\Delta r)^2} + \frac{1}{r_i}\frac{u_{\theta,i+1}^{old}}{2\Delta}\right.\right.$$

This explicit scheme requires the same CFL condition as the radial equation to maintain numerical stability:

$$\Delta t \leq \min\left(\frac{\Delta r}{|u_r|}, \frac{\Delta z}{|u_z|}, \frac{(\Delta r)^2}{2\nu}\right)$$

Second-Order Accurate Spatial Discretization for Continuity Equation

For the continuity equation:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) = -\rho\left[\frac{\partial}{\partial r}\left(u_r\frac{\partial u_r}{\partial r}\right) - \frac{u_\theta^2}{r^2}\right]$$

Here is the second-order discretization of the continuity equation:

$$\frac{1}{r_i}\left[\frac{r_{i+1/2}(p_{i+1} - p_i) - r_{i-1/2}(p_i - p_{i-1})}{(\Delta r)^2}\right] = -\rho\left[\frac{(u_{r,i+1/2})\left(\frac{u_{r,i+1} - u_{r,i}}{\Delta r}\right) - (u_{r,i-1/2})\left(\frac{u_{r,i} - u_{r,i-1}}{\Delta r}\right)}{\Delta r} - \frac{(u_{\theta,i})^2}{r_i^2}\right]$$

For enhanced precision at the mid-points, we can use averages:

$$r_{i+1/2} = \frac{r_i + r_{i+1}}{2}$$

$$r_{i-1/2} = \frac{r_{i-1} + r_i}{2}$$

$$u_{r,i+1/2} = \frac{u_{r,i} + u_{r,i+1}}{2}$$

$$u_{r,i-1/2} = \frac{u_{r,i-1} + u_{r,i}}{2}$$

Rearranging to solve for the pressure at point i:

$$p_i = \frac{r_{i+1/2}p_{i+1} + r_{i-1/2}p_{i-1}}{r_{i+1/2} + r_{i-1/2}} - \frac{(\Delta r)^2 \rho}{r_{i+1/2} + r_{i-1/2}} \left[\frac{(u_{r,i+1/2}) \left(\frac{u_{r,i+1} - u_{r,i}}{\Delta r} \right) - (u_{r,i-1/2}) \left(\frac{u_{r,i} - u_{r,i-1}}{\Delta r} \right)}{\Delta r} - \frac{(u_{\theta,i})^2}{r_i^2} \right]$$

This discretization can be used in a pressure correction or projection method to enforce the divergence-free constraint on the velocity field. It forms a system of linear equations that can be solved using iterative methods such as SOR (Successive Over-Relaxation) or conjugate gradient methods.

fourth order too much