MIT18.S096: Topics in Mathematics with Applications in Finance **2013**

Problem Set 8: Stochastic Calculus

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Problem A-1.

(a) True.

(b) False.

(c) True. $\mathbb{E}[X_{t+1}|X_t] = X_t + \mathbb{E}[Y_t - \frac{1}{\lambda}] = X_t$

(d) False. $\mathbb{E}[X_{t+1}|X_t] = X_t + \mathbb{E}[Z_t Z_{t-1} \dots Z_0] = X_t + \mathbb{E}[e^{W_t}] = X_t + (t+1)\mu + \frac{t+1}{2}\sigma^2$, where $W_t \sim \mathcal{N}((t+1)\mu, (t+1)\sigma^2)$

(e) True. $\mathbb{E}[X_{t+1}|X_t] = \mathbb{E}[B_{t+1}^{(1)} B_{t+1}^{(2)} | B_t^{(1)} B_t^{(2)}] = \mathbb{E}[(B_t^{(1)} + Z_1)(B_t^{(2)} + Z_2)] = B_t^{(1)} B_t^{(2)}$, where $Z_1 \sim \mathcal{N}(0,1)$, $Z_2 \sim \mathcal{N}(0,1)$, and $Z_1 \perp Z_2$.

Problem A-2.

(a) $\mathbb{E}[B(t)|B(s)] = B(s)$ (Martingale). $\mathrm{Var}[B(t)|B(s)] = \mathrm{Var}[B(s) + Z] = t - s$.

(b) $\mathbb{E}[X(t)|X(s)] = \mu(t-s) + B(s)$. $\mathrm{Var}[X(t)|X(s)] = t - s$.

(c) $\mathbb{E}[e^{\sigma(B(t)-B(s))}] = \mathbb{E}[e^{\sigma Z}] = \frac{t-s}{2}\sigma^2$, where $Z \sim \mathcal{N}(0, t-s)$.

Problem A-3.

(a) False. Y_t depends on the future of X_t.

(b) False. $Y_t = \max_{0 \le s \le 2t} X_s$ also depends on the future of X_t.

(c) True.

Problem A-4.

(a) $df = 3B_t dt + 3B_t^2 dB_t$

(b) $df = -\frac{1}{2}\sin B_t dt + \cos B_t dB_t$

(c) $df = -[(1+3t^2)\sin(t^2 + B_t^2) + 2\cos(t^3 + B_t^2)B_t^2]dt - 2\cos(t^3 + B_t^2)B_t dB_t$

(d) $df = (1 + 2B_t^2)e^{B_t^2} dt + 2B_t e^{B_t^2} dB_t$

(e) $df = B_t^2 dB_t$

(f) $df = B_t dt$

(g) $df = (2\mu X_t + \sigma^2)dt + 2\sigma X_t dB_t$

Problem A-5.

Proof: Let $p(x)$ be the pdf given by B_t, $\tilde{p}(x)$ be the pdf given by B_t^2. For all $X \subseteq (-\infty, 0)$, we have $p(X) > 0$ while $\tilde{p}(X) = 0$. Therefore, $B(t)$ and $B(t)^2$ are not equivalent.

Problem B-1.

(i) For a fixed value t, we have $X_1(t) = Z_1 \sim \mathcal{N}(0,t)$ and $X_2(t) = Z_2 \sim \mathcal{N}(0,t)$, and $Z_1 \perp Z_2$.

$$p_Y(z_1, z_2) = p_{Z_1}(z_1) p_{Z_2}(z_2) = \frac{1}{2\pi t} e^{-\frac{z_1^2 + z_2^2}{2t}} \tag{8.1}$$

(ii) $P(Y(t) \in D_\rho) = \int_{-\pi}^{\pi} \int_0^\rho r \cdot \frac{1}{2\pi t} e^{-\frac{r^2}{2t}} dr d\theta = \int_0^\rho \frac{r}{t} e^{-\frac{r^2}{2t}} dr = 1 - e^{-\frac{\rho^2}{2t}}$

Problem B-2.

Let $\Delta(t) = B_t$, which is definitely adapted to B_t, then by using Ito isometry, we have

$$\mathbb{E}[(\int_0^t B_s dB_s)^2] = \mathbb{E}[\int_0^t B_s^2 ds] = \int_0^t \mathbb{E}[B_s^2] ds = \frac{1}{2} t^2. \tag{8.2}$$

Or we can directly calculate X_t by solving the SDE associated with the Ito integral according to the definition, which is

$$X_t = \frac{1}{2} B_t^2 - \frac{t}{2}. \tag{8.3}$$

Then we have

$$\mathbb{E}[X_t^2] = \mathbb{E}[(\frac{1}{2} B_t^2 - \frac{t}{2})^2] = \mathbb{E}[\frac{1}{4} B_t^4 - \frac{1}{2} t B_t^2 + \frac{t^2}{4}] = \frac{3t^2}{4} - \frac{t^2}{2} + \frac{t^2}{4} = \frac{t^2}{2}. \tag{8.4}$$

Problem B-3.

(i) $\int_0^t h(s) dB_s = h(s) B_s |_0^t - \int_0^t B_s h'(s) ds = h(t) B_t - \int_0^t B_s h'(s) ds$

(ii) Let $h(t) = t$. By using part (i), we have

$$\int_0^T s dB_s = T B_T - \int_0^T B_s ds. \tag{8.5}$$

Then we can obtain $\int_0^T B_s ds = T B_T - \int_0^T s dB_s = \int_0^T T dB_s - \int_0^T s dB_s = \int_0^T (T - s) dBs$, which can be seen as the sum of independent normal random variables. Therefore, $\int_0^T B_s ds$ is also normal distributed.