MIT18.S096: Topics in Mathematics with Applications in Finance **2013**

Problem Set 9: Stochastic Differential Equations

Name: Jinnian Zhang *University of Wisconsin-Madison*

Problem A-1.

(a) According to Ito calculus, we have $dX_t = \frac{1}{2}e^{B_t}dt + e^{B_t}dB_t = \frac{1}{2}X_t dt + X_t dB_t$

(b) $dX_t = -\frac{B_t}{(1+t)^2}dt + \frac{1}{1+t}dB_t = -\frac{1}{1+t}X_t dt + \frac{1}{1+t}dB_t$

(c) $dX_t = -\frac{1}{2}\sin B_t dt + \cos B_t dB_t = -\frac{1}{2}X_t dt + \cos B_t dB_t \neq -\frac{1}{2}X_t dt + \sqrt{1-X_t^2}dB_t$

Problem A-2.

Proof: Assume that $X_t = f(t, B_t)$, then according to Ito calculus, we have

$$dX_t = (\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial B^2})dt + \frac{\partial f}{\partial B}dB_t = \frac{1}{3}X_t^{1/3}dt + X_t^{2/3}dB_t \tag{9.1}$$

$$\begin{cases} \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial B^2} = \frac{1}{3}X_t^{1/3}dt \\ \frac{\partial f}{\partial B}dB_t = X_t^{2/3}dB_t \Rightarrow f = (\frac{1}{3}B_t + g(t))^3 \end{cases} \tag{9.2}$$

Then we will obtain

$$3(\frac{1}{3}B_t + g(t))^2 \cdot g'(t) + \frac{1}{3}f^{\frac{1}{3}} = \frac{1}{3}f^{\frac{1}{3}} \Rightarrow 3(\frac{1}{3}B_t + g(t))^2 \cdot g'(t) = 0 \tag{9.3}$$

We can see that only when $g(t) = C$ will satisfy this equation, where C is a constant. When $t = 0$, we have $C^3 = a \Rightarrow C = a^{\frac{1}{3}}$. Therefore, the solution of the SDE given the initial condition $X_0 = a$ is $X_t = (a^{1/3} + \frac{1}{3}B_t)^3$.

Problem A-3.

Proof: According to Ito calculus, the derivative of $R(t)$ (Vasicek interest rate model) is

$$\begin{aligned} dR(t) &= -\beta R(0)e^{-\beta t}dt + \alpha e^{-\beta t}dt - (\beta\sigma e^{-\beta t}\int_0^t e^{\beta s}dB_s)dt + \sigma dB_t && (9.4) \\ &= [-\beta(R(0)e^{-\beta t} - \frac{\alpha}{\beta}e^{-\beta t} + \frac{\alpha}{\beta} + \sigma e^{-\beta t}\int_0^t e^{\beta s}dB_s) + \alpha]dt + \sigma dB_t && (9.5) \\ &= (\alpha - \beta R(t))dt + \sigma dB_t. && (9.6) \end{aligned}$$

We can see that the derivative of $R(t)$ is equal to the SDE. Besides, when $t = 0$, we have $R(0) = R(0)$, which means that it also satisfies the initial condition. Therefore, $R(t)$ is the solution for the SDE.