1 Estimating the source field from the data

Instead of using an explicit description of the railway source based on some technical understanding of the railway installations, an alternative attempt is to determine the source (or the sources) directly from the data. A principal difficulty inherent to this approach is that the observed fields depend on both the unknown source field and the unknown conductivity structure. To deal with this problem, we shall first make an assumption about the conductivity structure in order to find a source field geometry that can account for the observations. A second step would be to adjust the conductivity model to reduce the residuals which can not be explained with our initial estimate of the sources. These steps could be iterated until the data are explained. There is of course no guarantee that this approach converges; however, there is at least the hope, that this approach will help us to improve our understanding of the railway source.

Similar problems exist in the field of global induction where the objective is to determine mantle conductivity from magnetic fields generated by inhomogeneous ionospheric or magnetospheric sources. In theory, a solution to this problem can be obtained because internal and external parts of the magnetic field can be separated from observations with a dense array on an infinitely large or closed surface. Expansion of the external part of the magnetic field into spherical harmonic functions allows then to describe the source geometry as a function of time, and the internal part arises due to induction phenomena within the earth. With the knowledege of both the sources and the fields of internal origin, the conductivity distribution can be determined [2]. In practice, the approach suffers from a sparse distribution of observatories on the earth, which prohibits an exact separation of the observed fields into parts of internal and external origin.

Here, we investigate two approaches to determine the source geometry. The first approach is to find the sources by direct inversion of interstation transfer functions. The second approach adopts multivariate array processing techniques, which have become popular in magnetotellurics over the last years [1], but which have not yet been applied to controlled or un-controlled source electromagnetic problems. Array-processing techniques rely on the estimate of the principal components of observations. We detail this approach below.

1.1 Anna's inversion

1.2 Principal component analysis

Our observations are the windowed and fourier-transformed recordings of the horizontal electric field components at a number of recording stations. The number of total channels corresponds to the number of stations times two. Let us collect these responses $E_{n,i}$ for all channels $n = 1, ..., N$ for a set of time windows $i = 1, ..., I$ in a $N \times I$ data matrix $\mathbf{X}$ as

$$\mathbf{X} = \begin{bmatrix} E_{1,1} & E_{1,2} & \dots & E_{1,I} \\ E_{2,1} & \ddots & & \\ \vdots & & E_{n,i} & \\ E_{N,1} & & & E_{N,I} \end{bmatrix} . \quad (1)$$

Imagine that the observed fields in $\mathbf{X}$ result from linear combinations of P sources. It is clear, that the maximum number of independent linear combinations of observations with our array is P, corresponding to the number of independent sources. It can then be shown, that all observed fields reside in a space that is spanned by P orthogonal coordinate axes. These coordinate axes are the so-called principal components $\mathbf{U}$. In our case, we can denote the principal components as principal electric fields, and we aim at determining these from all observations in a statistically robust way. Each particular observation (time window) results from a particular linear combination and is reflected in the coeffcients of a matrix $\mathbf{A}$.

Therefore, it is useful to denote the elements of $\mathbf{A}$ as polarization parameters. The principal component analysis (PCA) yields exactly this decomposition of a data matrix:

$$\mathbf{X} = \mathbf{U}\mathbf{A} \tag{2}$$

The outcomes of a PCA of the data matrix $\mathbf{X}$ are thus the number P of independent observations, an orthogonal P-dimensional coordinate system $\mathbf{U}$ in which all realizations must reside, and the coordinates $\mathbf{A}$ of these observations in this coordinate system. Other coordinate systems of the same dimension exist also; for instance any rotation is allowed. Therefore the PCA is not unique.

PCA is closely related to the singular value decomposition (SVD) of a data matrix $\mathbf{X}$. Under ideal circumstances (noise-free data and s small number of separable sources) the number of non-vanishing singular values corresponds to the number of independent rows contained in the $\mathbf{X}$. This is then the number of principal components. The principal components $\mathbf{u}_p$ themselves are defined as linear combinations of the data

$$\mathbf{u}_p = \mathbf{X}\mathbf{a}_p^T \tag{3}$$

under the constraint that $\|\mathbf{a}_p\| = 1$ and $\mathbf{u}_p^T\mathbf{u}_q = 0$ for $p \neq q$. Here, $\mathbf{u}_p$ and and $\mathbf{a}_p$ are $N \times 1$ and $1 \times I$ vectors, respectively. In turn, the data can written as linear combinations of the principal components (cf. equation 2). The dimensions of $\mathbf{U} = [\mathbf{u}_1, \ldots, \mathbf{u}_P]$ and $\mathbf{A} = \left[\mathbf{a}_1^T, \ldots, \mathbf{a}_P^T\right]^T$ are $N \times (P \leq N)$ and $P \times I$, respectively. There is a maximum of $P = N$ principal components, but for low-dimensional data, only the first $P \leq N$ components are required (this means that there are less independent linear combinations than channels). The decomposition of $\mathbf{X}$ is not unique, since $\mathbf{U}\mathbf{A}$ can be replaced with $\tilde{\mathbf{U}}\tilde{\mathbf{A}}$, where $\tilde{\mathbf{U}} = \mathbf{U}\mathbf{B}^{-1}$ and $\tilde{\mathbf{A}} = \mathbf{A}\mathbf{B}$ and $\mathbf{B}$ is any non-singular $P \times P$ matrix. Therefore, the SVD

$$\mathbf{X} = \mathbf{U}\mathbf{S}\mathbf{V}^T \tag{4}$$

truncated at the $P-$th singular value yields only possible PCA representation. For this set of principal components, $\mathbf{A} = \mathbf{S}\mathbf{V}^T$.

It is important to understand that the principal components $\mathbf{u}_p$ can be thought of as independent representative observations of fields, which contain as much information as all real observations (i.e. the observations with field polarizations $\mathbf{a}_i$) taken together. Therefore, to explain the entire set of observed data (i.e. all time windows), it is sufficient to estimate the sources which generate the fields $\mathbf{u}_p$. To illustrate this, we consider a simple synthetic example in the sequel.

For simplicity, we consider a non-inductive case. Then, for an electric dipole at position $\mathbf{r}_s$ and with dipole moment $\mathbf{p}_s$, the electric field at coordinate $\mathbf{r}$ is obtained from

$$\mathbf{E}^s(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left(3\frac{\mathbf{p}_s(\mathbf{r}-\mathbf{r}_s)}{r^5}\cdot(\mathbf{r}-\mathbf{r}_s) - \frac{1}{r^3}\mathbf{p}_s\right) \quad , \tag{5}$$

where $r = |\mathbf{r}-\mathbf{r}_s|$. We denote the response due to a unit dipole $\hat{\mathbf{p}}$ as $\hat{\mathbf{E}}(\mathbf{r})$, where $\mathbf{E}^s(\mathbf{r}) = p_s\hat{\mathbf{E}}^s(\mathbf{r})$ and $\mathbf{p} = p\hat{\mathbf{p}}$. Let us distribute a number of S_0 electric dipoles and record the horizontal electric field components with an array of K stations. The total number of recording channels is $N = 2K$.

For the synthetic experiment, we transmit signal with random linear combinations of all sources. Then, at a particular station at position $\mathbf{r}$ the electric field is the superposition of contributions from all source dipoles for a particular random realization i. Hence,

$$\mathbf{E}_i(\mathbf{r}) = \sum_s r_{i,s}\mathbf{E}^s(\mathbf{r}) = \sum_s r_{i,s}p_s\hat{\mathbf{E}}^s(\mathbf{r}) \tag{6}$$

where the random number $r_{i,s}$ determines the strength of source s. As before, we collect the responses $E_{n,i}$ at all channels $n = 1, \ldots, N$ for all realizations $i = 1, \ldots, I$ to construct a synthetic $N \times I$ data matrix

$\mathbf{X}$ as

$$\begin{aligned}
\mathbf{X} &= \begin{bmatrix} E_{1,1} & E_{1,2} & \dots & E_{1,I} \\ E_{2,1} & \ddots & & \\ \vdots & & E_{n,i} & \\ E_{N,1} & & & E_{N,I} \end{bmatrix} + \boldsymbol{\delta} \\
&= \begin{bmatrix} \hat{E}_{1,1} & \hat{E}_{1,2} & \dots & \hat{E}_{1,S} \\ \hat{E}_{2,1} & \ddots & & \\ \vdots & & \hat{E}_{n,s} & \\ \hat{E}_{N,1} & & & \hat{E}_{N,S} \end{bmatrix} \begin{bmatrix} p_1 & & & \mathbf{0} \\ & \ddots & & \\ & & p_s & \\ \mathbf{0} & & & p_S \end{bmatrix} \begin{bmatrix} r_{1,1} & r_{1,2} & \dots & r_{1,I} \\ r_{2,1} & \ddots & & \\ \vdots & & r_{s,i} & \\ r_{S,1} & & & r_{S,I} \end{bmatrix} + \boldsymbol{\delta} \\
&= \hat{\mathbf{E}}\mathbf{pR} + \boldsymbol{\delta} = \hat{\mathbf{E}}\mathbf{S} + \boldsymbol{\delta}
\end{aligned} \tag{7}$$

where $\mathbf{S} = \mathbf{pR}$ and δ is a noise term. We denote $\mathbf{S}$ as the source parameters.

We apply a PCA on $\mathbf{X}$ and keep the P dominant components, i.e.

$$\mathbf{X} = \mathbf{UA} = \hat{\mathbf{E}}\mathbf{S} + \boldsymbol{\delta} \ . \tag{8}$$

Here, we use the robust PCA code `robpca`, which is part of the Matlab-based `LIBRA` library [3]. Recall that $\mathbf{A}$ contains the polarization parameters for each single event. However, to represent the information contained in the observed data, it sufficient to consider P independent realizations. One particular set of P independent realizations, which would be generated with the unknown source parameters $\mathbf{S}^u$ would correspond to $\mathbf{A} = \mathbf{I}_{P\times P}$ being a unity matrix, i.e.

$$\mathbf{U} = \hat{\mathbf{E}}\mathbf{S}^u + \boldsymbol{\delta} \tag{9}$$

It means that this set of P realizations would generate the fields $\mathbf{u}_p$ at the actual receiver positions. We can now estimate for each principal component $p = 1, ..., P$ the source parameters. Let this problem be mixed-determined in the sense that we allow for more unknown sources S than independent components P. Then, the damped least squares estimate for the source parameters is

$$\mathbf{S}^{u,est} = (\hat{\mathbf{E}}^T\hat{\mathbf{E}} + \lambda\mathbf{I})^{-1}\hat{\mathbf{E}^T}\mathbf{U} \tag{10}$$

where λ is a regularization parameter. Each of the columns of $\mathbf{S}^{u,est}$ then corresponds to the moments of each of the S sources, which, when superposed, give rise to the fields given by the columns of $\mathbf{U}$.

Figure 1 illustrates this approach. We use three point dipoles and generate 1000 random realizations of fields to be observed along a receiver line at 1 km offset from the source line (Figure 1a). With noise-free data, the PCA yields exactly $P = 3$ principal components. As outlined above, the principal components can be conceived as independet representative fields for the data set. Red arrows in Figure 1b,c depict these principal fields. To solve for the source parameters, we place point dipoles along the source line and compute the unit dipole responses for each of the receivers. For the example in Figure 1b, we used $S = 5$ sources, three of which placed at the locations of the true sources, and in the example in Figure 1c, we used $S = 30$ sources. The source positions of these hypothetic sources are depicted as crosses. The source strength estimates determined from expression 10 correspond to the size of circles plotted on top of the source line; white and black fillings correspond to negative and positive moments, respectively. Note that in both examples, the estimated sources which contribute to the principal components are close to or exactly at the true source locations; however, each principle component is shwon to be due a particular linear combination of the true sources. This illustrates, that we can not identify each of the true sources individually.

We have repeated the same experiment with a sparse subset of eleven stations. The results are depcited in Figure 2. Even though the principles fields lokk at a first glance spatially inconsistent, the reconstruction of sources resembles the true sources.

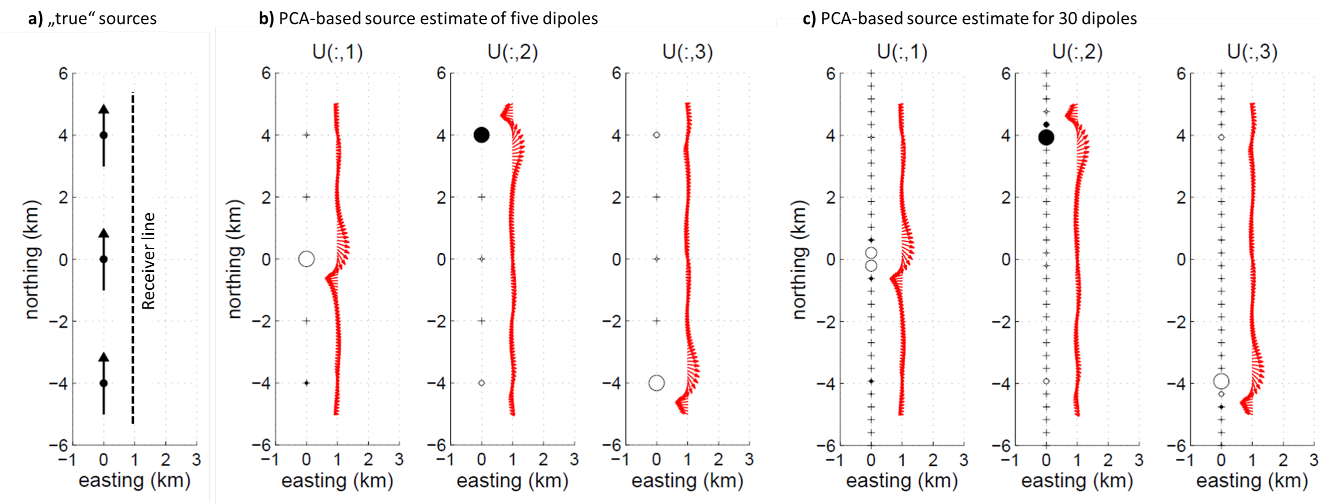

Figure 1: PCA analysis applied to synthetic data. **a)** Black arrows indicate position and orientation of three true source dipoles. Thousand random linear combinations of the fields generated by these sources have been collected along the receiver line (dashed line). **b)** The PCA yields three independent source configurations, which generate the principal electric fields plotted as red arrows (corresponding to the columns of $\mathbf{U}$). The dipole moments of five hypothetical sources (crosses corresspond to hypothetic source positions) were estimated from the principal components. Source strength estimates are depicted as circles with the diameter of each circle corresponding to the estimated dipole strength; white and black colors correspond to negative and positive dipole moments (i.e. the direction) **c)** Same as in panel b), but with 30 hypothetic source locations.

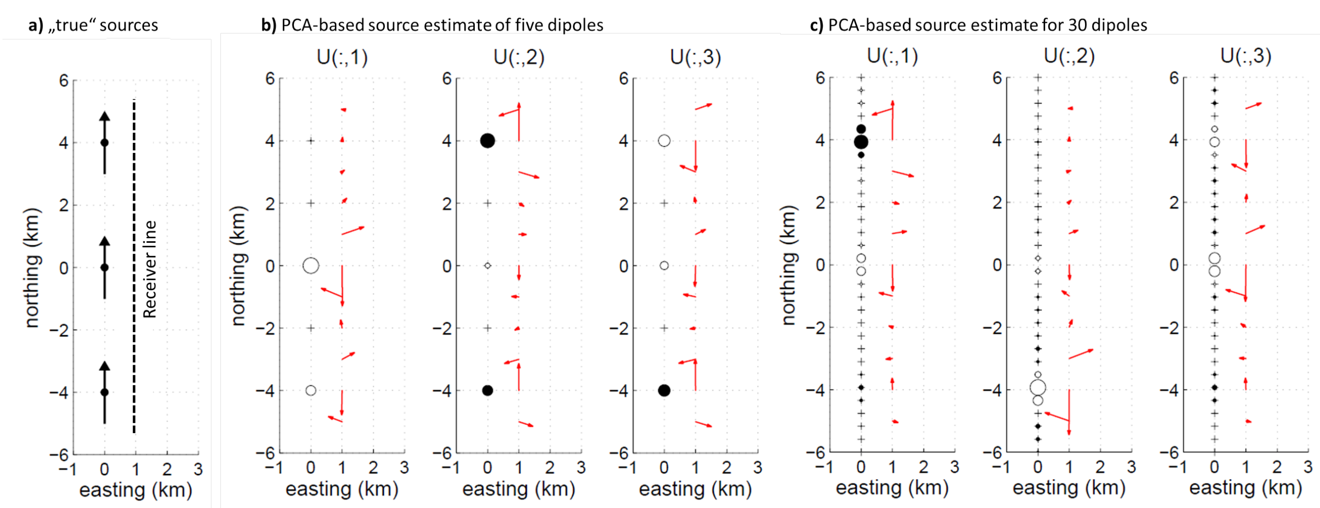

Figure 2: Same as in Figure 1, but only 11 receivers have been used for the PCA.

These synthetic examples suggest that the PCA analysis may be a useful tool to extract information about the railway source from the observed electric field data. We have started to apply this approach to the real data, and we propose to pursue this approach further in this project.

References

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