The statistical distribution of magnetotelluric apparent resistivity and phase

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SUMMARY

The marginal distributions for the magnetotelluric (MT) magnitude squared response function (and hence apparent resistivity) and phase are derived from the bivariate complex normal distribution that describes the distribution of response function estimates when the Gauss–Markov theorem is satisfied and the regression random errors are normally distributed. The distribution of the magnitude squared response function is shown to be non-central chi-squared with 2 degrees of freedom, with the non-centrality parameter given by the squared magnitude of the true MT response. The standard estimate for the magnitude squared response function is biased, with the bias proportional to the variance and hence important when the uncertainty is large. The distribution reduces to the exponential when the expected value of the MT response function is zero. The distribution for the phase is also obtained in closed form. It reduces to the uniform distribution when the squared magnitude of the true MT response function is zero or its variance is very large. The phase distribution is symmetric and becomes increasingly concentrated as the variance decreases, although it is shorter-tailed than the Gaussian. The standard estimate for phase is unbiased. Confidence limits are derived from the distributions for magnitude squared response function and phase. Using a data set taken from the 2003 Kaapvaal transect, it is shown that the bias in the apparent resistivity is small and that confidence intervals obtained using the non-parametric delta method are very close to the true values obtained from the distributions. Thus, it appears that the computationally simple delta approximation provides accurate estimates for the confidence intervals, provided that the MT response function is obtained using an estimator that bounds the influence of extreme data.

Key words: electromagnetic induction, geostatistics, magnetotellurics, statistical methods.

1 INTRODUCTION

The fundamental datum in the magnetotelluric (MT) method is the site-specific, frequency-dependent, second rank tensor **Z** relating the horizontal electric and magnetic fields measured at Earth's surface. The MT response tensor must be estimated statistically from data using methods that are ultimately based on least-squares principles. Considerable success has been achieved using robust and/or bounded influence estimators that limit the effect of unusual or extreme electric and magnetic field time-series intervals (e.g. Chave & Thomson 2004), and this class of estimator is now in general use. The ensuing estimates of the MT response function **Z** are generally unbiased, statistically reliable, and consistent with the requirements of the Gauss–Markov theorem, as further discussed in Section 2. In addition, parametric estimates of the uncertainty (i.e. the variance or standard error) $\delta \mathbf{Z}$ on **Z** are also statistically meaningful when the influence of extreme data is bounded, while non-parametric estimators such as the jackknife (Thomson & Chave 1991) are more accurate under these conditions.

However, MT interpretation is often based on an additional statistical entity derived for each tensor element of **Z**, the apparent resistivity whose magnitude is given by

$$\rho_{ij} = \mu_0 \left| Z_{ij} \right|^2 / \omega, \tag{1}$$

where μ_0 is the magnetic permeability of free space, ω is the angular frequency and **Z** is measured in units of electric field **E** divided by magnetic induction **B**. Because the tensor elements Z_{ij} are complex, there is a phase corresponding to the squared response, but this is rarely used in practice. Instead, eq. (1) is combined with the phase ϕ_{ij} of Z_{ij} as substitute data for the real and imaginary parts of Z_{ij}. Use of $\{\rho_{ij}, \phi_{ij}\}$ instead of $\{\text{Re}[Z_{ij}], \text{Im}[Z_{ij}]\}$ is often preferred because the ubiquitous occurrence of galvanic distortion biases ρ_{ij} or $\{\text{Re}[Z_{ij}], \text{Im}[Z_{ij}]\}$, but not ϕ_{ij}.

While the transformation from $\{\text{Re}[Z_{ij}], \text{Im}[Z_{ij}]\}$ to $\{\rho_{ij}, \phi_{ij}\}$ is straightforward, statistical inference about the result is not. The most widely used approach is a first-order Taylor series approximation for $\delta \rho_{ij}$ and $\delta \phi_{ij}$ usually called the delta method (Stuart & Ord 1994, section 10.5)

$$\begin{aligned} \delta \rho_{ij} &= 2\mu_0 |Z_{ij}| \delta Z_{ij} / \omega \\ \delta \phi_{ij} &= \sin^{-1}(\delta Z_{ij} / |Z_{ij}|) \end{aligned}, \tag{2}$$

where the error terms in eq. (2) are $O(1/N)$ and N is the sample size. The arcsine function in the second equation is sometimes omitted at some loss of accuracy. Propagation of error methods are sometimes employed (e.g. Gamble *et al.* 1979; Stodt 1983), but their range of validity remains unquantified, as is also true for the delta approximation. Yet, interpretation of accurate MT parameters combined with inaccurate estimates for their uncertainty will bias the resolution of derived models for Earth.

In this paper, the statistical distributions for $\{\psi_{ij},\ \phi_{ij}\}$, where $\psi_{ij} = |Z_{ij}|^2$ is a substitute statistic for ρ_{ij}, given $\{\text{Re}[Z_{ij}],\ \text{Im}[Z_{ij}]\}$ that satisfy the extended Gauss–Markov conditions are derived from first principles. In Section 2, the general linear model is reviewed to establish a statistical model for $\{\text{Re}[Z_{ij}],\ \text{Im}[Z_{ij}]\}$, and the joint distribution of $\{\psi_{ij},\ \phi_{ij}\}$ is obtained from it. Sections 3 and 4 derive and describe the marginal distributions for ψ_{ij} and ϕ_{ij}, respectively, along with symmetric confidence intervals on them. Section 5 contains further discussion and a data-based comparison of confidence limits computed from the distributions with approximate results from the delta method.

2 JOINT DISTRIBUTION OF ψ AND ϕ

The magnetotelluric response function **Z** is usually estimated using a variant of the Welch overlapped section averaging method (Welch 1967) in which a time-series is broken into segments, each segment is pre-whitened and tapered with a data window, discrete Fourier transforms are taken, and pre-whitening is corrected for. The ensuing Fourier estimates at a given frequency become data in the context of MT data processing. The MT response tensor **Z** follows from a linear regression of the electric on the magnetic data using a robust M-estimator (Egbert & Booker 1986; Chave *et al.* 1987). Additional steps to bound the influence of extreme predictor (i.e. magnetic field) data should also be taken (Chave & Thomson 2003, 2004). Accurate statistical inference about **Z** depends critically on minimizing bias in the Fourier estimates through effective pre-whitening and the use of low bias data windows, on assuring the independence of the Fourier estimates for distinct data sections and frequencies, and on satisfying conditions on the linear regression problem, including those of the Gauss–Markov theorem. The first two of these require careful spectral analysis procedures; the issues are reviewed by Thomson & Chave (1991) and Chave & Thomson (2004).

The standard linear regression model for a row of the full MT response tensor **Z** is

$$\mathbf{e} = \mathbf{b}\,\zeta + \varepsilon, \tag{3}$$

where **e** is the complex response (horizontal electric field) N-vector, **b** is the complex N × 2, rank-2 predictor (horizontal magnetic field) matrix, ζ is the complex solution 2-vector, and ε is a complex N-vector of random errors. The least-squares solution for ζ obtains from minimizing the error power, yielding

$$\mathbf{z} = (\mathbf{b}^{\text{H}}\mathbf{b})^{-1}(\mathbf{b}^{\text{H}}\mathbf{e}), \tag{4}$$

where the superscript H denotes the Hermitian (complex conjugate) transpose. The conditions on the variables in eq. (3) and their moments that yield a least-squares solution eq. (4) that is optimal in a well-defined sense are given by the Gauss–Markov theorem of classical statistics (Stuart *et al.* 1999, chapter 29). The textbook version of the Gauss–Markov theorem applies when the predictor variables in eq. (3) are fixed, but Shaffer (1991) has extended it to cover a wide range of cases where **b** contains random variables. The linear

regression solution eq. (4) is an unbiased estimate with an associated unbiased variance estimate when the random errors ε have zero mean, are mutually uncorrelated and share a common variance independent of any assumptions about their statistical properties except that the variance must exist. In addition, if the random errors are complex Gaussian, then the least-squares result is a maximum likelihood, fully efficient, minimum variance estimate. The regression residuals $\{r_i\}$ are the differences between the measured values of the response variable **e** and those predicted by the linear regression, and serve as an estimate for the random errors ε that can be examined to check the validity of these conditions. It is well known that MT data frequently violate the Gauss–Markov conditions, and that the use of robust or bounded influence estimators that introduce data-dependent weights into eq. (4) can mitigate this problem. Further, the residuals can be tested for consistency with the Gauss–Markov and residual normality conditions, as described by Chave & Thomson (2004).

Assuming the Gauss–Markov theorem is satisfied and the random errors ε in eq. (3) are complex Gaussian, it follows that the response variables **e** are complex normal, independent, and homoscedastic (i.e. share a common variance). Their distribution is

$$\mathbf{e} : \text{CN}_N(\mathbf{b}\,\zeta,\ \sigma^2\mathbf{I}), \tag{5}$$

where CN_N is the N-variate complex normal distribution with expected value $\mathbf{b}\zeta$ and common variance σ^2; **I** is the identity matrix. It can also be shown (e.g. Mardia *et al.* 1979) that the elements of a row of the MT response are bivariate complex normal

$$\mathbf{z} : \text{CN}_2[\zeta,\ \sigma^2(\mathbf{b}^{\text{H}}\mathbf{b})^{-1}]. \tag{6}$$

In addition, $\{|r_i|^2/\sigma^2\}$ are χ^2 distributed with $N-2$ degrees of freedom.

Despite the seeming simplicity of eq. (6), it still contains eight unknown real parameters (the real and imaginary parts of two elements of the MT response, the population variance for each element and the real and imaginary parts of the population covariance). Following Kotz *et al.* (2000 section 45.13), let $z_j = x_j + iy_j$ have expected value μ_j, and assume $\text{var}(x_j) = \text{var}(y_j) = \sigma_j^2$, $\text{cov}(x_j, y_j) = 0$, $\text{cov}(x_i, x_j) = \text{cov}(y_i, y_j) = \alpha_{ij}$, and $\text{cov}(x_j, y_i) = -\text{cov}(x_i, y_j) = \beta_{ij}$. These conditions apply to any signal that may be expressed as a Fourier transform (Wooding 1956), and hence are not restrictive in the present context. The Hermitian covariance matrix Σ follows directly from the definition. Define the correlation coefficient $\Lambda = (\alpha_{12} + i\beta_{12})/\sigma_1\sigma_2$. The joint probability density function (pdf) for z_1 and z_2 is the bivariate complex normal given by

$$\begin{aligned} &f(z_1,\ z_2\,|\,\mu_1,\ \mu_2,\ \Sigma\,) \\ &= \frac{1}{4\pi^2\sigma_1^2\sigma_2^2(1-|\Lambda|^2)} e^{-[|\xi_1|^2 - 2\text{Re}\{\Lambda\xi_1^*\xi_2\} + |\xi_2|^2]/[2(1-|\Lambda|^2)]}, \end{aligned} \tag{7}$$

where $\xi = (z_i - \mu_i)/\sigma_i$. The marginal pdf for a single element z_1 follows by integration over all possible values of z_2, and it is well known that the result is univariate complex Gaussian and hence independent of the correlation coefficient. It is also identical in form for z_1 and z_2.

The marginal pdf is traditionally used for statistical inference, and especially to derive confidence intervals on the elements of **z**. However, there are instances where the more appropriate choice would be the conditional pdf of z_1 given a particular value Ω for z_2

$$\begin{aligned} &f(z_1\,|\,z_2 = \Omega,\ \mu_1,\ \mu_2,\ \Sigma) \\ &= \frac{1}{2\pi\sigma_1^2(1-|\Lambda|^2)} e^{-[|\xi_1|^2 - 2\text{Re}\{\Lambda\xi_1^*\xi_2\} + |\Lambda|^2|\xi_2|^2]/[2(1-|\Lambda|^2)]}, \end{aligned} \tag{8}$$

where ξ_2 is evaluated at $z_2 = \Omega$. For example, when the 2-D approximation is used for inversion so that the diagonal elements of **Z** are ignored, it is more appropriate to estimate confidence intervals on the off-diagonal elements conditional on the diagonal elements being zero. The resulting confidence intervals may differ significantly from those estimated using the marginal distribution for which no constraints on the diagonal elements pertain, depending on the correlation coefficient Λ and the size of $|\Lambda\xi_2|$. Derivation of confidence intervals using eq. (8) is a straightforward extension of the results presented in this paper.

In the interest of simplicity, only the marginal distribution for a single complex element of the MT response tensor will be further considered. This marginal pdf for z_1, which is also the joint pdf for the real and imaginary parts of the MT response, is given by

$$f(z_R, \ z_I \,|\, \mu_R, \ \mu_I, \ \sigma) = \frac{1}{2\pi\sigma^2}e^{-[(z_R-\mu_R)/\sigma]^2/2}e^{-[(z_I-\mu_I)/\sigma]^2/2}. \quad (9)$$

The joint distribution for the magnitude squared response function $\psi = z_R^2 + z_I^2$ and the phase $\phi = \tan^{-1}(Z_I/Z_R)$ follows from standard transformation methods (de Groot & Schervish 2002, section 3.9)

$$g(\psi, \ \phi \,|\, \lambda, \ \nu, \ \beta) = \frac{\beta}{2\pi}e^{-\beta(\lambda+\psi)}e^{2\beta\sqrt{\lambda\psi}\cos(\phi-\nu)}, \quad (10)$$

where $\beta = 1/(2\sigma^2)$ is the square of the precision modulus, $\lambda = \mu_R^2 + \mu_I^2$ and $\nu = \tan^{-1}(\mu_I/\mu_R)$. This result is exact. The parameters λ and ν are anticipated to be the population values of ψ and ϕ, respectively, although the expected values and/or the maximum likelihood estimates may differ. The marginal distributions for ψ and ϕ are obtained by integrating eq. (10) over all possible values of ϕ and ψ, respectively. When $\mu_R = \mu_I = 0$, this is straightforward, reducing to the exponential distribution for ψ and the uniform distribution for ϕ. The distributions are substantially more complicated in the more common instance where the expected value of the MT response function differs from zero.

3 MARGINAL DISTRIBUTION FOR THE MAGNITUDE SQUARED RESPONSE FUNCTION

The marginal pdf for the magnitude squared response function ψ is eq. (10) integrated over the range of ϕ

$$g_1(\psi \,|\, \lambda, \ \nu, \ \beta) = \frac{\beta}{2\pi}e^{-\beta(\lambda+\psi)}\int_{-\pi}^{\pi}e^{2\beta\sqrt{\lambda\psi}\cos(\phi-\nu)}\, d\phi. \quad (11)$$

Using the generating function for modified Bessel functions of the first kind $I_n(t)$, the exponential term in the integrand may be expanded as

$$e^{t\cos\theta} = I_0(t) + 2\sum_{k=1}^{\infty}I_k(t)\cos(k\theta). \quad (12)$$

Performing the integration in eq. (11) yields

$$g_1(\psi \,|\, \lambda, \ \beta) = \beta e^{-\beta(\lambda+\psi)}I_0(2\beta\sqrt{\lambda\psi}) \quad (13)$$

which is the non-central χ^2 distribution with 2 degrees of freedom with non-centrality parameter λ whose properties are described by Johnson *et al.* (1995 chapter 29). As expected, it reduces to the exponential distribution when $\lambda = 0$.

Defining the precision parameter $\kappa = \beta\lambda$ and the non-dimensional magnitude squared response function $\eta = \psi/\lambda$, eq. (13) may be transformed under the requirement that probability is preserved to yield

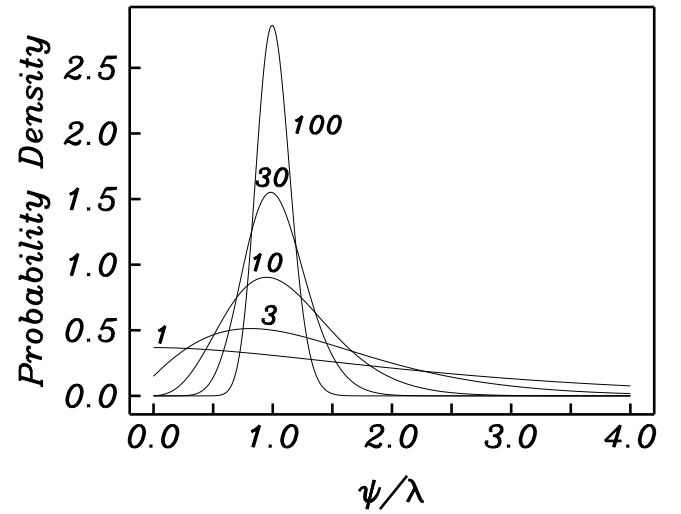$$\hat{g}_1(\eta \,|\, \kappa) = \kappa e^{-\kappa(\eta+1)}I_0(2\kappa\sqrt{\eta}). \quad (14)$$

Figure 1. The probability density function (14) for the dimensionless magnitude squared response function $\eta = \psi/\lambda$ as a function of the precision parameter $\kappa = \beta\lambda = \lambda/(2\sigma^2)$ at values of 1, 3, 10, 30 and 100. Note the increasingly skewed, non-Gaussian shape for $\kappa < 30$, and the lack of an obvious mode for small values of κ.

It can be shown by integrating eq. (14) for its first two moments that the expected value and variance of η are $1 + 1/\kappa$ and $(2\kappa + 1)/\kappa^2$, respectively. Consequently, the sample value of λ is a downward biased estimator for ψ, with the bias given by $1/\beta$. This may be important when the variance of MT response estimates is large. The corresponding expected value and variance of the apparent resistivity are $\mu(\lambda + 2\sigma^2)$ and $2\mu^2\sigma^2(\lambda + \sigma^2)/\omega^2$, respectively.

Fig. 1 shows the pdf (eq. 14) for $\kappa = 1, 3, 10, 30$ and 100. The distribution is symmetric and peaked near $\eta = 1$ for large κ, but is highly skewed and lacking an obvious mode for small values. It takes on an increasingly Gaussian form as κ increases, but differs substantially for $\kappa < 30$.

Confidence intervals are always non-unique and may be central or non-central about a given value, but minimum size is typically achieved in the central case. A central confidence interval about $\eta = 1$ may be derived by solving

$$\int_{\lceil 0, \ 1-c\rceil}^{1+c}\hat{g}_1(\eta \,|\, \kappa)\, d\eta = \gamma \quad (15)$$

for c at an appropriate probability level γ, where $\lceil\;\rceil$ denotes the supremum. The lower bound on the integral reflects the non-negative form of η. The confidence intervals about the expected value of η may be found by replacing 1 with $1 + 1/\kappa$ in the integral bounds. Fig. 2 shows c from eq. (15) evaluated as a function of κ for $\gamma = 0.68, 0.95$ and 0.99, respectively. The confidence interval is approximately linear with κ on a log–log scale, exhibiting slight upward curvature for small values. Large values of κ must be achieved for the confidence limits to become small; for example, 3 per cent on either side of the centre at the 95 per cent level is obtained only for $\kappa > 10\,000$. This corresponds to a relative error $\delta Z/|Z|$ of about 0.7 per cent.

Eq. (13) may easily be transformed to the distribution of the gain factor $\sqrt{\psi}$. The result is the Rice distribution, reducing to the Rayleigh distribution when $\lambda = 0$.
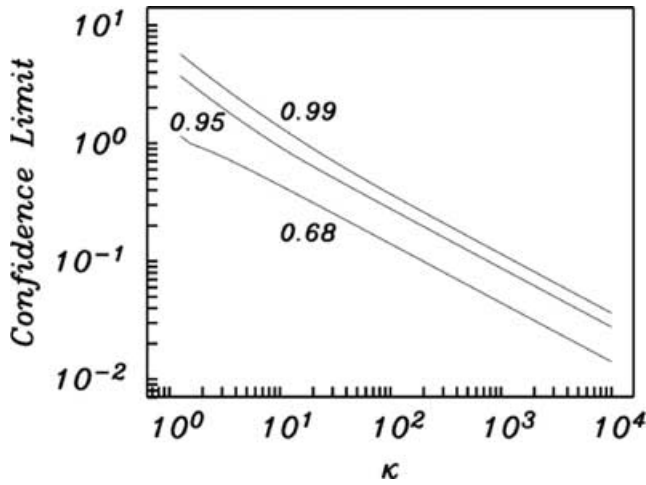
Figure 2. The confidence interval c about 1 for the dimensionless magnitude squared response function $\eta = \psi/\lambda$ as a function of the precision parameter $\kappa = \lambda/(2\sigma^2)$ at probability levels of 0.68, 0.95 and 0.99. The statistics λ and σ^2 are the expected value and variance of ψ, respectively. A confidence interval of 3 per cent on either side of the centre at the 95 per cent level is obtained only for $\kappa > 10\,000$, and corresponds to a relative error $\delta Z/|Z|$ of about 0.7 per cent.

4 MARGINAL DISTRIBUTION FOR PHASE

The marginal pdf for phase is obtained by integrating the joint pdf eq. (10) over the range of ψ. Converting to non-dimensional form as in eq. (14) gives

$$\hat{g}_2(\vartheta \mid \kappa) = \frac{\kappa}{2\pi} e^{-\kappa} \int_0^\infty e^{-\kappa\eta} e^{2\kappa\sqrt{\eta}\cos\vartheta}\,\mathrm{d}\eta, \tag{16}$$

where $\theta = \phi - \nu$. This can be integrated using the Mathematica 5 package (Wolfram 2003) subject to verification by numerical quadrature of eq. (16). The result is

$$\hat{g}_2(\vartheta \mid \kappa) = \frac{e^{-\kappa}}{2\pi}[1 + \sqrt{\pi\kappa}\cos\vartheta\; e^{\kappa\cos^2\vartheta}\mathrm{erfc}(-\sqrt{\kappa}\cos\vartheta)], \tag{17}$$

where erfc(x) is the complementary error function. Eq. (17) reduces to the uniform distribution when $\kappa = 0$. The expected value of θ is zero, so θ is the expected value of ϕ. A closed form solution for the variance cannot be obtained. The distribution eq. (17) is symmetric about $\phi = \nu$ for all values of κ (Fig. 3), becoming increasingly concentrated as κ increases and approaching a point distribution in the limit of large κ. However, the phase distribution is increasingly shorter tailed than the Gaussian as κ increases, asymptotically falling off algebraically rather than exponentially with θ.

Central confidence limits on θ about 0 (and hence on ϕ about ν) follow from solution of

$$\int_{-c}^{c} \hat{g}_2(\vartheta \mid \kappa)\,\mathrm{d}\vartheta = \gamma. \tag{18}$$

Fig. 4 shows c from eq. (18) evaluated for $\gamma = 0.68$, 0.95 and 0.99. As for the confidence limits on the magnitude squared response function, these are approximately linear on a log–log scale, exhibiting a departure from linearity when the influence of the bounds on θ at $\{-\pi, \pi\}$ is evident. Large values of κ must be achieved for the confidence limits to become small; for example, 0.02 radian ($\sim1°$) on either side of the centre at the 95 per cent level is obtained only for $\kappa > 10\,000$, and 0.05 radian ($\sim3°$) is observed for $\kappa = 800$.
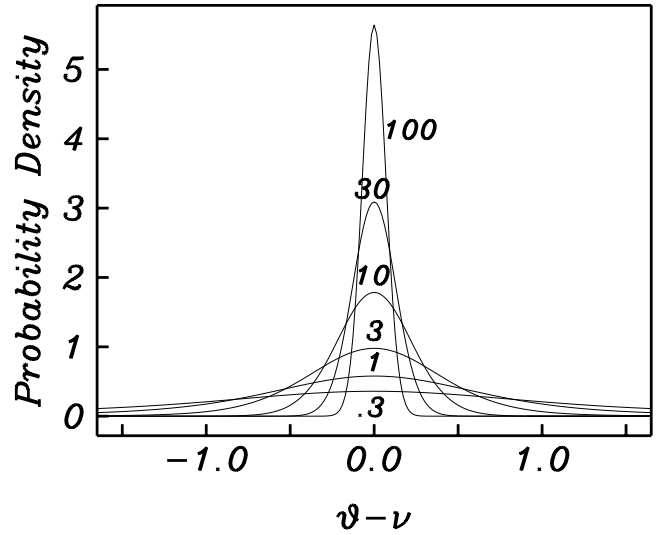

Figure 3. The probability density function (17) for the phase as a function of the precision parameter $\kappa = \lambda/(2\sigma^2)$ at values of 0.3, 1, 3, 10, 30 and 100. Note the increasing concentration and more Gaussian-like behaviour as κ increases, and the approach to a uniform distribution for small κ.
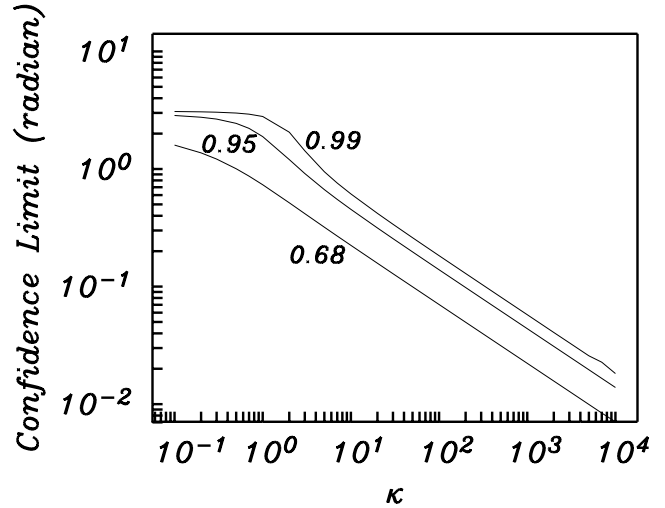

Figure 4. The symmetric confidence interval c about the expected value ν for the phase as a function of the precision parameter $\kappa = \lambda/(2\sigma^2)$ at probability levels of 0.68, 0.95 and 0.99. The total confidence interval is $2c$. A confidence interval of 0.02 radian ($\sim1°$) on either side of the centre at the 95 per cent level is obtained only for $\kappa > 10\,000$, and 0.05 radian ($\sim3°$) is observed for $\kappa > 800$. These values correspond to relative errors $\delta Z/|Z|$ of about 0.7 and 2.5 per cent, respectively.

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5 DISCUSSION

Confidence intervals obtained independently from eqs (15) and (18) tend to underestimate the true value because they are exclusive, placing all of the uncertainty in ψ or ϕ, respectively. Unless there is an *a priori* reason to believe that one parameter is substantially more accurate than the other, it is simultaneous confidence intervals on both the magnitude squared response and the phase at a given probability level that are required for inference purposes. Let $\gamma = 1 - \alpha$ in eqs (15) and (18). Simultaneous Bonferroni confidence

Table 1. Apparent resistivity statistics.

Period (s)	κ	ρ (Ω-m)	B[ρ] (Ω-m)	Exact 95 per cent	Delta 95 per cent	γ
17 067	5.30	3.40	0.641	5.72	4.67	0.949
12 800	1.65	0.384	0.233	1.44	0.948	0.913
8533	10.5	2.15	0.205	2.36	2.10	0.962
6400	6.35	0.564	0.089	0.846	0.709	0.953
4267	18.2	1.23	0.067	0.970	0.912	0.967
3200	32.1	2.24	0.070	1.29	1.25	0.971
2133	47.0	1.83	0.039	0.866	0.847	0.972
1600	45.0	2.36	0.053	1.14	1.12	0.972
1067	81.9	2.47	0.030	0.876	0.865	0.973
800	104	3.60	0.035	1.13	1.12	0.974
533	277	3.76	0.014	0.719	0.717	0.974
400	291	4.36	0.015	0.812	0.809	0.974
267	542	5.68	0.010	0.775	0.773	0.975
200	509	6.85	0.013	0.965	0.962	0.975
133	1234	8.52	0.007	0.770	0.769	0.975
100	1664	9.48	0.006	0.737	0.737	0.975
66.7	4346	11.4	0.003	0.548	0.548	0.975
50.0	6880	13.1	0.002	0.502	0.501	0.975
33.3	14204	13.4	0.001	0.357	0.357	0.975
25.0	14100	16.0	0.001	0.427	0.427	0.975
16.7	6550	13.7	0.002	0.537	0.536	0.975
12.5	317	5.94	0.019	1.06	1.06	0.975

intervals on both parameters may be obtained by replacing α with α/p (Rencher 1998, section 7.5), where $p = 2$ is the number of parameters. Thus, simultaneous confidence intervals on magnitude squared response and phase at the 95 per cent level would utilize the 97.5 per cent level in both eqs (15) and (18). Alternately, critical values from Hotelling's T^2 distribution rather than Student's t may be used. It is recommended that the Bonferroni or Hotelling's T^2 method be more widely adopted, as failure to use simultaneous confidence intervals will tend to underestimate the uncertainty in apparent resistivity and phase.

For illustrative purposes, time-series from site 127 (28°48′S, 23°47′E) of the 2003 Kaapvaal, South Africa, transect are employed. The time-series were sampled every 5 s for about 40 d. The horizontal magnetic field from site 172 (22°38′S, 29°31′E) was used as a remote reference. The time-series were converted to MT responses using the bounded influence estimator described in Chave & Thomson (2004); the Z_{yx} (where y is east and x is north) component of the response function is explored in detail. The electric field at site 127 is strongly polarized to the north, and the Z_{yx} tensor element is concomitantly noisy, making it suitable for evaluating the statistics under less than optimal conditions. The delete-one jackknife given in Thomson & Chave (1991) was used to estimate the standard error s. Z_{yx} and s serve as sample estimates for $\mu_R + i\mu_I$ and σ, respectively, from which sample estimates of λ, ν and β follow directly. Note that jackknife and parametric variance estimates are comparable for data that even approximately meet the Gauss–Markov conditions, although the jackknife is conservative and hence always yields a slightly larger result (Efron & Stein 1981), so that use of a parametric variance estimate on the response function would not substantially alter the conclusions. Bonferroni 95 per cent confidence intervals were obtained using the delta method (eq. 2) by scaling δZ by 2.24 (or the inverse normal distribution at the $1 - \alpha/2p$ level with $\alpha = 0.95$, where the extra factor of 2 follows from symmetry), as well as from eq. (15) expressed as apparent resistivity and eq. (18) for phase, respectively, with $\gamma = 0.975$. Note that the lower integration limit in eq. (15) guarantees that the resulting confidence interval will be at the γ level, but the delta method does not

include the non-negativity constraint. As a result, the delta method will systematically underestimate the size of the apparent resistivity confidence band when its lower limit intersects zero.

Table 1 contains period, the sample estimate of the precision parameter κ, apparent resistivity computed from eq. (1), the bias (i.e. the difference between the expected and sample values) of the apparent resistivity, the 95 per cent confidence limit on apparent resistivity from (15), the 95 per cent confidence limit from the delta method, and the actual probability level achieved in the latter by computing (15) with the delta method value for c. The precision parameter κ varies from 2 to 14 000, primarily reflecting decreasing variance at short periods due to higher coherence and increasing degrees of freedom in the MT response estimates. The apparent resistivity at the longest four periods is not useful, as the confidence band is extremely broad and intersects zero. The bias in the apparent resistivity is small except at long periods, and even then is not significant when compared to the confidence limits. Further, the differences between the confidence limits estimated using the actual distribution and the delta method are also small, and certainly insignificant once κ exceeds ∼100. The delta method systematically underestimates the confidence band, although the difference is not important unless the lower limit intersects zero. Thus, it appears that the delta method does produce accurate confidence intervals for the apparent resistivity, presuming that the MT response function estimates and their standard errors are themselves reliable.

Table 2 shows period, the phase, the 95 per cent confidence limit on phase from eq. (18), and the 95 per cent confidence limit from the delta method. The precision parameter estimates are identical to those in Table 1. Agreement of the exact and delta method confidence intervals is excellent, except at 12 800 s where the delta method yields nothing meaningful. The delta method systematically underestimates the confidence band, although the discrepancy is unimportant.

These are gratifying results given the simplicity of eq. (2), and would not be known without the statistical basis for comparison given in this paper. The Site 127 data set is of low quality due to strong polarization of the electromagnetic field and cultural noise,

Table 2. Phase statistics.

Period (s)	Phase (°)	Exact 95 pre cent (°)	Delta 95 per cent (°)
17067	13.56	43.47	43.48
12800	41.03	101.1	180.0
8533	76.83	29.28	29.26
6400	61.88	38.96	38.94
4267	66.01	21.78	21.77
3200	58.63	16.24	16.23
2133	66.34	13.37	13.36
1600	62.57	13.67	13.66
1067	60.84	10.08	10.08
800	66.29	8.94	8.93
533	66.46	5.47	5.46
400	66.24	5.33	5.32
267	69.56	3.90	3.90
200	65.87	4.03	4.03
133	67.04	2.59	2.58
100	64.56	2.23	2.23
66.7	59.52	1.38	1.38
50.0	58.39	1.09	1.09
33.3	51.86	0.76	0.76
25.0	47.29	0.76	0.76
16.7	44.85	1.12	1.12
12.5	14.98	5.11	5.11

and hence the delta method is probably valid for all save extremely noisy or very short duration data sets, where useful response estimates are difficult to obtain in any case.

The apparent resistivity and phase are derived quantities rather than entities that can be estimated directly from data using a linear least-squares-based procedure. The approach used in this paper is to first compute bounded influence estimates for the elements of the MT tensor $\mathbf{Z}$ along with their standard errors (either parametrically or based on the jackknife), and then transform these to apparent resistivity, phase, and their associated confidence limits. An alternate approach would apply the jackknife directly to the apparent resistivity and phase by deleting data with replacement from estimates of $\mathbf{Z}$. However, the jackknife yields accurate confidence limits only if the underlying distribution is approximately Gaussian, which does not apply to eq. (14) without applying a variance stabilizing transformation (Stuart *et al.* 1999, section 32.38). It is easy to show that $\log(\psi - \sigma^2)$ should be jackknifed instead of ψ. Phase estimates may be jackknifed directly, as the distribution is symmetric and quasi-Gaussian in appearance. An alternate approach would be application of the bootstrap at a significant increase in computational load. However, consistency of the much simpler delta method confidence limits with the full parametric ones suggests that more complicated approaches are not generally required.

On the basis of empirical analyses, Bentley (1973) and Fournier & Febrer (1976) claimed that apparent resistivity is log normally distributed, and this result has been widely cited. It is not difficult to understand this conclusion if it were derived from ordinary least-squares MT response function estimates, as would be standard practice in the 1970s. Such estimates are frequently dominated by a small number of extreme data, so that the apparent resistivity will be very long-tailed and its distribution might be approximated as log normal. However, the correct distribution for the apparent resistivity based on statistical theory is non-central χ^2 with 2 degrees of freedom, which is always shorter tailed than log normal, especially as the non-centrality parameter (or the squared response function) increases. Further, the shapes of the correct distributions for both

apparent resistivity and phase change markedly as the non-centrality parameter increases, in contrast to log normal or normal approximations to each. It is recommended that the correct distributions be used for future inference.

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