Errors Bars for Transfer Function Elements in Z–files

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1 File Format

The Z–files contain all information needed to compute standard transfer functions (e.g., impedances), with error bars in any coordinate system. Here is an overview of the format, followed by an artificial example, with some annotation off to the side. The basic idea is that there are *NCH* channels, with the first two used as the "local reference"i.e., these are the input or predictor channels (classically the local horizontal magnetics), and the remaining *NCH-2* are the output or predicted channels (the electrics, and/or vertical magnetics). Note that there might also have been another pair of channels (or a whole array) used as a remote reference. These possible other channels are not referred to explicitly in this file (but they were used to compute the contents of the file). A file of this same format can in principle be produced from single station, standard remote reference, or the multiple station program. This effects how the contents of this file was created, but not any subsequent calculations using this file.

In overview, the file is ASCII, with a short header block which identifies the *NCH* channels. There are then a series of *NBANDS* blocks, one for each period for which an estimate has been computed. Each period block contains three complex arrays:

1) **Z** : the transfer function (TF) array. This array is *NCH-2* rows by 2 columns. For $NCH = 4$, with two reference channels H_x and H_y, and two predicted channels E_x and E_y, **Z** is just the impedance tensor.

2) **S** : the "inverse signal covariance" array. This is a 2×2 Hermitian matrix. Only the 3 elements corresponding to the part on and below the diagonal are actually in the file. These elements are given in the order S_{11}, S_{21}, S_{22}. The missing element satisfies $S_{12} = S_{21}{}^*$, where the superscript asterisk denotes the complex conjugate. Note that in the case of a single station impedance estimate **S** is just the inverse of the **H** cross power matrix. The exact form is slightly different for the case of remote reference or array results. This matrix is needed for the error calculation. (Actually only the diagonal elements, which are real, are needed unless you rotate the coordinate systems).

3) **N** the residual covariance matrix. This is an $(NCH - 2) \times (NCH - 2)$ Hermitian matrix, output in the same symmetric form as **S**. This gives the covariance of the residuals for all predicted channels. Again, only the diagonals (also real) of this matrix are needed

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for error calculations in the "default" coordinate system, but other parts of the matrix will be used for a correct treatment of coordinate changes/rotations.

(Mike: Actually there might be a slight difference in the header block format in the version you have; this is for the most recent version, and I'm not sure exactly what Clark is using).

```
 TRANSFER FUNCTIONS IN MEASUREMENT COORDINATES    <====  line 1 of file
 ********** WITH FULL ERROR COVARAINCE*********

S2                                             <===== some sort of "station" id
coordinate    49.28   102.91 declination    0.00    <=== station coordinates
number of channels   5   number of frequencies  26  <===  NCH, NBANDS
 orientations and tilts of each channel
    5     0.00     0.00 S2H  Hx     <===  for each channel in this "station":
    6    90.00     0.00 S2H  Hy     (1) channel number (this came from multmtrn
    7     0.00     0.00 S2H  Hz         and so the #s aren't 1,2 ... ;
    8     0.00     0.00 S2E  Ex      (2) orientation (deg. E of N);  (3) tilt ;
    9    90.00     0.00 S2E  Ey      (4) Data loger ID  (5) channel type

period :      4.65455    decimation level   1    freq. band from   25 to   30
number of data point   2496 sampling freq.   1.000 Hz   <=== info about 1 band
 Transfer Functions
  0.2498E+00 -0.2049E-03 -0.9341E-04  0.2517E+00
 -0.6246E-02 -0.5245E-01 -0.7291E+01 -0.7318E+01       <=== Z
  0.7292E+01  0.7346E+01 -0.3806E-01  0.5754E-02
 Inverse Coherent Signal Power Matrix
  0.2947E-07  0.5753E-16                               <=== S
 -0.1575E-09  0.1391E-09  0.2895E-07  0.2386E-15
 Residual Covariance
  0.3198E+02  0.0000E+00
  0.2252E+03 -0.2185E+03  0.2660E+05  0.0000E+00        <=== N
  0.2424E+03  0.2418E+03  0.4577E+03  0.3710E+03  0.2781E+05  0.0000E+00
```

The block given above for one period should be pretty much self explanatory. Note that the last two rows of the TF matrix are the local impedance tensor.

2 Error Calculation

First I just give the formula for calculating errors in the TF given in the files (i.e., in the "default" measurement coordinate system). Next, I'll give formulas for transforming the matrices **Z, S, N** into a different coordinate system (not necessarily by rotation). The initial formulas for TF error in the measurement coordinate system can then be applied to the transformed matrices. Finally, linear error propagation is applied to give the standard error estimates for ρ_a and ϕ computed from the off-diagonal elements of the impedance.

2.1 Errors In Transfer Functions

The error covariance for the elements of the transfer function matrix $\mathbf{Z}$ is given by:

$$\mathbf{Cov}[Z_{ij}Z_{i'j'}] = N_{ii'}S_{jj'} \quad j,j' = 1,2 \quad , \; i,i' = 1, NCH-2. \tag{1}$$

You will normally only care about the variances (i.e., the case where $i = i'$ and $j = j'$). In this case you would use only the diagonal elements of $\mathbf{S}$ and $\mathbf{N}$. In the following I refer to these variances as

$$\sigma_{ij}{}^2 = \mathbf{Var}[Z_{ij}] = \mathbf{Cov}[Z_{ij}Z_{ij}] = N_{ii}S_{jj}. \tag{2}$$

For example, the impedance element $Z_{xy}(= E_x/H_y)$ in the above example is element (2,2) (row = i = 2, column = j = 2) in the TF matrix $\mathbf{Z}$. The error variance is obtained from the product of the second diagonal element of the inverse coherent signal power matrix S_{22}, and the second diagonal element of the residual covariance N_{22}. The other off-diagonal impedance element Z_{yx} corresponds to $i = 3$ and $j = 1$, and the variance is $\sigma_{31} = N_{33}S_{11}$. Note that this gives the variances of the complex transfer functions; variances of real and imaginary parts separately are each one half of the complex variance given by (1).

2.2 Transformation Of Transfer Functions and Errors

The transformation of error covariance can be computed for any linear transformation of the predicted and predictor channels. Here I just give expressions for the most standard rotations. Denote by $\theta_1, \theta_2, ...\theta_{NCH}$ the channel orientations (these are given in the header block of the Z_ file). Let θ be the desired rotation of the $x-$axis, relative to the same reference direction used to define the channel orientations (e.g., geographic or geomagnetic north). Note that the sort of coordinate changes we focus on here implicitly involve pairs of channels (the two reference magnetics; a pair of electric channels). Vertical magnetics are not rotated (well ... we *could* get into allowing for tilt ...), and when there are multiple electrics, it will be necessary to identify pairs of channels to transform together. I thus describe transformation of one pair of channels at a time, say channels l, m. Form the matrix

$$\mathbf{U_{lm}} = \begin{bmatrix} \cos(\theta_l - \theta) & \sin(\theta_l - \theta) \\ \cos(\theta_l - \theta) & \sin(\theta_m - \theta) \end{bmatrix}^{-1} \tag{3}$$

Note that if you form the 2-vector $\mathbf{x}$ from the (l,m) pair of measured data channels, then $\mathbf{U_{lm}x}$ gives the vector expressed in the new right-handed orthogonal coordinate system (with $x-$axis pointing in the direction θ degrees E of the reference direction). Note that in the "usual" MT case where there is one pair of reference channels H_x, H_y and one pair of predicted channels E_x, E_y, and both are expressed in the same orthogonal coordinate system, then the same matrix $\mathbf{U_{lm}}$ would be used for coordinate transformation of both pairs, and we would also have $\theta_m = \theta_l + 90$. In this case $\mathbf{U_{lm}}$ would reduce to the more familiar form for the impedance tensor rotation matrix. The formulas given here work for any orientations, including the case of non-orthogonal measurment component pairs.

First consider transformation of the predicting channels $l = 1$, $m = 2$ (normally these would be H_x, H_y). $\mathbf{Z}$ and $\mathbf{S}$ are effected by this part of the transformation. In the new coordinate system the matrices are:

$$\mathbf{Z}' = \mathbf{Z}\mathbf{U_{12}}^T \qquad \mathbf{S}' = \mathbf{U_{12}}\mathbf{S}\mathbf{U_{12}}^T \tag{4}$$

The output residual covariance $\mathbf{N}$ of course remains unchanged by a transformation of only the input channels.

Next consider transformation of two of the output channels, $3 \leq l, m \leq NCH$. (Note the numbering convention: output channels start with 3, and go to NCH, for a total of $NCH - 2$). The simplest way to express the result in general is to define an $(NCH - 2) \times (NCH - 2)$ transformation matrix $\mathbf{V_{lm}}$ which rotates only channels l and m. For the example file above, where $NCH = 5$, the matrix for rotating the coordinate system for the pair of electric field channels (i.e., $l = 4, m = 5$), $\mathbf{V_{45}}$ would take the form

$$\mathbf{V_{45}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_4 - \theta) & \cos(\theta_5 - \theta) \\ 0 & \sin(\theta_4 - \theta) & \sin(\theta_5 - \theta) \end{bmatrix}. \tag{5}$$

More generally the following pseudo-code defines $\mathbf{V_{lm}}$, assuming $l < m$:

$\mathbf{V_{lm}} = (NCH - 2) \times (NCH - 2)$ identity matrix

$\mathbf{V_{lm}}(l - 2, l - 2) = \cos(\theta_l - \theta)$

$\mathbf{V_{lm}}(m - 2, l - 2) = \sin(\theta_l - \theta)$

$\mathbf{V_{lm}}(l - 2, m - 2) = \cos(\theta_m - \theta)$

$\mathbf{V_{lm}}(m - 2, m - 2) = \sin(\theta_m - \theta)$

With $\mathbf{V_{lm}}$ thus defined the transformations of $\mathbf{Z}$ and $\mathbf{N}$ are:

$$\mathbf{Z}' = \mathbf{V_{lm}}\mathbf{Z} \qquad \mathbf{N}' = \mathbf{V_{lm}}\mathbf{N}\mathbf{V_{lm}}^T \tag{6}$$

In general both input and output channels will be rotated, so both (4) and (6) will be used. In the 5 channel example given above the full transfromation of all arrays is thus:

$$\mathbf{Z}' = \mathbf{V_{45}}\mathbf{Z}\mathbf{U_{12}}^T \qquad \mathbf{S}' = \mathbf{U_{12}}\mathbf{S}\mathbf{U_{12}}^T \qquad \mathbf{N}' = \mathbf{V_{45}}\mathbf{N}\mathbf{V_{45}}^T. \tag{7}$$

More generally there may be a series of electric field pairs, requiring that (6) be applied for each pair. Note that in this case a single matrix $\mathbf{V}$ can be derived which transforms all channel pairs, by starting with the $(NCH - 2) \times (NCH - 2)$ identity matrix and modifying the appropriate four elements of $\mathbf{V}$ for each pair l, m. Error (co)variances for the transformed impedance elements are then as given in (2) and (1), with $\mathbf{N}'$ and $\mathbf{S}'$ replacing $\mathbf{N}$ and $\mathbf{S}$.

3 Apparent Resistivities and Phases

After transforming all three arrays, apparent resistivities (ρ_a), phases (ϕ) and error bars ($\sigma_\rho; \sigma_\phi$) can be computed from the appropriate off-diagonal impedance elements (say Z_{ij}), the period T, and the associated error variance σ_{ij} given above. For completness here are the expressions derived from linear propogation of errors, under the assumption that errors are small compared to the impedance.

$$\rho_a = \frac{T|Z_{ij}|^2}{5}$$

$$\sigma_\rho = \left[\frac{(2T\rho\sigma_{ij}{}^2}{5}\right]^{1/2}$$

$$\phi = \frac{180}{\pi}\arctan[\Im(Z_{ij})/\Re(Z_{ij})]$$

$$\sigma_\phi = \frac{180}{\pi|Z_{ij}|}\left[\frac{\sigma_{ij}{}^2}{2}\right]^{1/2}$$