Abschlussarbeit Fallstudien der math. Modellbildung

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1 Whittaker-Shannon Interpolation formula

Sampling and reconstructing a signal from its samples are probably two of the most important properties of modern communication. Even very simple things of our daily life, like making a phone call to our dear friend Massimo, are not possible without digitalizing analog signals, for example our voice.
Whereas digitalizing can be done rather straightforward by sampling, the real art is regaining the original signal from these samples or assessing the information lost in the sampling process.
In the following, we'd like to try to give a small survey of the famous Whittacker-Shannon interpolation formula, ist applications in real-life but also its limitations, keeping in mind the relationship of Fourier transform and the Heisenberg uncertainty principle.

1.1 Preliminary notes and Sampling

Definition 1.1.1 (Fourier transform). *The Fourier transform $\mathcal{F}(f)$ of a d-dimensional, integrable function $f : \mathbb{R}^d \to \mathbb{R}$ is given by*

$$\mathcal{F}f(w) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i w x} \, \mathrm{d}x \tag{1.1}$$

So, the Fourier transform converts a time domain function into a frequency domain function. For example, the Fourier transform of an audio signal identifies the frequency spectrum as peaks in the frequency domain.
If $\mathcal{F}f \in L^1(\mathbb{R}^d)$, then we can define the inverse Fourier transform:

Definition 1.1.2 (Inverse Fourier transform).

$$f(x) = \int_{\mathbb{R}^d} \mathcal{F}f(w) e^{2\pi i w x} \, \mathrm{d}w \tag{1.2}$$

Definition 1.1.3 (bandlimited function). *For $Q = \prod_{i=1}^{d} \omega_i[-1/2, 1/2)$, $\omega \in \mathbb{R}^d$, we define*

$$L^2_Q(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d) : \ supp(\mathcal{F}f) \subset Q\} \tag{1.3}$$

If $f \in L^2_Q(\mathbb{R}^d)$, then it is called ω-bandlimited.

1.1.1 Sampling

To convert a continuous function f into a sequence of discrete values is called sampling. In a mathematical way, sampling can be described as a multiplication of f with a dirac-comb

$$s(t, \Delta T) = \sum_{n \in \mathbb{Z}} \delta(t - n\Delta T),$$

where ΔT is the sampling intervall and δ is the Dirac-function.
The sampled function $\tilde{f}$ of our original f is denoted by

$$\tilde{f}(t) = s(t, \Delta T) f(t) = \sum_{n \in \mathbb{Z}} f(t) \delta(t - n\Delta T) \tag{1.4}$$

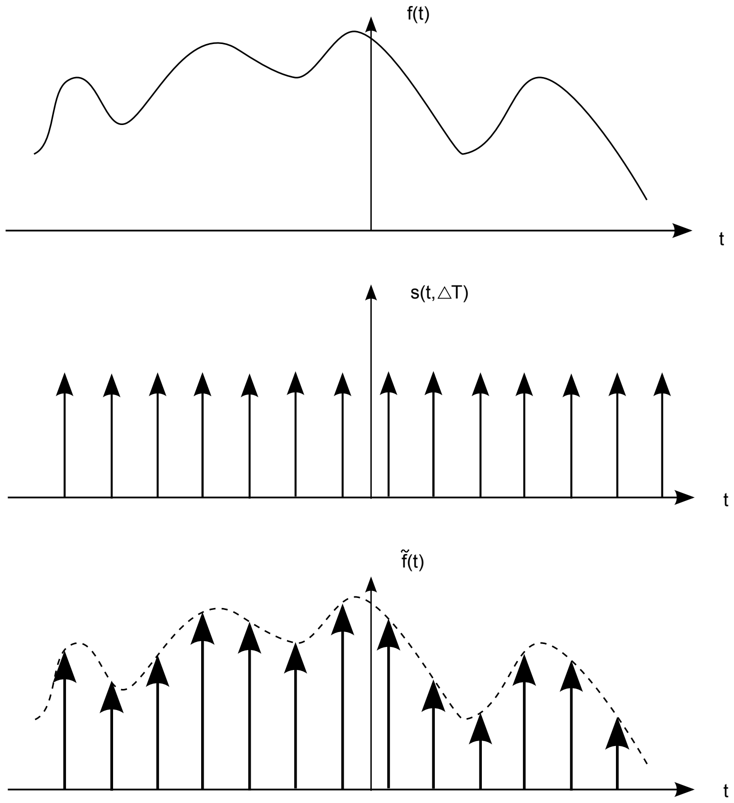

Figure 1.1: (a) continuous function f, (b) the dirac-comb, (c) sampled function as product of (a) and (b)

The following theorem will be essential for our further work:

Theorem 1.1.4 (perturbed sampling in L^2)**.** *Let* $Q = \prod_{i=1}^{d} \omega_i[-1/2, 1/2)$ *and* $f \in C(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ *such that* $f_{|\tau\mathbb{Z}^d} \in l^2$*. We write* $f = \eta + \epsilon$*, where* $\mathcal{F}\eta = \mathcal{F}f$ *on* Q*. Then it holds*

$$f(x) = \sum_{k \in \mathbb{Z}^d} (f^c(\tau k) - \epsilon^c(\tau k)) \prod_{i=1}^{d} sinc(\tau_i^{-1} x_i - k_i) + \epsilon(x), \qquad in\ L^2(\mathbb{R}^d), \tag{1.5}$$

where $sinc(x) = \frac{sin(\pi x)}{x}$.

1.2 The Whittaker-Shannon Interpolation formula

Theorem 1.2.1 (Interpolation formula). *If $f \in L^2(\mathbb{R}^d)$ is a ω-bandlimited function, there exists a $\tau_0 > 0$ such that for all $\tau \in (0, \tau_0]$*

$$f(t) = \sum_{k \in \mathbb{Z}^d} f(\tau k) \prod_{i=1}^{d} sinc\left(\frac{t_i - k_i \tau_i}{\tau_i}\right). \tag{1.6}$$

In other words, every bandlimited L^2 function can be perfectly reconstructed from its samples, if the sampling rate is high enough! Holy Shit!
Of course, this perfect reconstruction is only possible in a theoretical manner, since we would need infinitely many sampling points. But we can interpolate the original signal with arbitrary precision, if we just add enough sampling points.
(Da der Beweis jetzt nicht mehr so umfangreich ist, ist ein eigenes Kapitel so viel denke ich, ich w"'urde ihn einfach direkt nach das Theorem setzen)

Proof. Follows with theorem 1.1.4... □

1.3 Proof of the Theorem

Define $\tau_0 = \frac{1}{\omega} := \left(\frac{1}{\omega_0}, \ldots, \frac{1}{\omega_d}\right)$ and choose $\tau \in (0, \tau_0]$ arbitrarily.
Besides denote $\Omega := \prod_{i=1}^{d} \left[-\frac{1}{2}\omega_i, \frac{1}{2}\omega_i\right]$ and $T := \prod_{i=1}^{d} \left[-\frac{1}{2\tau_i}, \frac{1}{2\tau_i}\right]$.

$$x \in L^2_w(\mathbb{R}^d) \Rightarrow \forall f \notin \Omega : \mathcal{F}(x)(f) = 0 \tag{1.7}$$

The consequence of this condition and of the linearity of $\mathcal{F}(x)$ is:

$$\forall f \in T, k \in \mathbb{Z}^d : \mathcal{F}(x)\left(f + \frac{k}{\tau}\right) = \mathcal{F}(x)(f) + \underbrace{\mathcal{F}(x)\left(\frac{k}{\tau}\right)}_{\notin \Omega} = \mathcal{F}(x)(f)$$

Thus the formula can be rewritten.

$$\forall f \in \Omega : \mathcal{F}(x)(f) = \sum_{k \in \mathbb{Z}^d} \mathcal{F}(x)\left(f + \frac{k}{\tau}\right) \tag{1.8}$$

(1.1) and (1.2) allow us to say:

$$\mathcal{F}(x)(f) = \chi_T(f) \sum_{k \in \mathbb{Z}^d} \mathcal{F}(x)\left(f + \frac{k}{\tau}\right)$$

For using the Poisson summation formula (Theorem 0.3) we have to check the requirements.

i) Since $\mathcal{F}(x)$ has a compact support and is continuous, $\mathcal{F}(x) \in L^1(\mathbb{R}^d)$. Hence the fourier transform is reversible.

$$x(t) = \mathcal{F}^{-1}(\mathcal{F}(x))(t) = \int_{\mathbb{R}^d} \mathcal{F}(x)(f) e^{2\pi i f t} df$$

With that in mind you can't make a secret of:

$$x'(t) = \left(\int_{\mathbb{R}^d} \mathcal{F}(x)(f) e^{2\pi i f t} 2\pi i f_1 df, \ldots, \int_{\mathbb{R}^d} \mathcal{F}(x)(f) e^{2\pi i f t} 2\pi i f_d df \right)$$

You can easily see now that $x \in C^\infty(\mathbb{R}^d)$

ii) Another implication of the compact support is

$$\exists C, \varepsilon > 0 : |\mathcal{F}(x)(f)| \leq C(1 + |f|)^{-d-\varepsilon}$$

iii) Die dritte Bedingung ist schwieriger.

With the aid of the Poisson summation formula (Theorem 0.3) you can conclude:

$$\mathcal{F}(x)(f) = \chi_T(f) \det(\tau) \sum_{n \in \mathbb{Z}^d} x(n\tau) e^{-2\pi i n \tau f} \tag{1.9}$$

Now we want to prove that $\mathcal{F}\left(\prod_{j=1}^d sinc\left(\frac{t_j - n\tau_j}{\tau_j}\right)\right)(f) = \det(\tau)\chi_T(f)e^{-2\pi i n \tau f}$.

$$\begin{aligned}
\mathcal{F}^{-1}\left(\det(\tau)\chi_T(f)e^{-2\pi i n\tau f}\right) &= \int_{\mathbb{R}^d} \det(\tau)\chi_T(f)e^{-2\pi i n \tau f} e^{2\pi i f t} df \\
&= \int_{\mathbb{R}^d} \det(\tau)\chi_T(f) e^{2\pi i f (t - n\tau)} df \\
&= \int_{-\frac{1}{2\tau_1}}^{\frac{1}{2\tau_1}} \cdots \int_{-\frac{1}{2\tau_d}}^{\frac{1}{2\tau_d}} \prod_{j=1}^d \tau_j e^{2\pi i f_j (t_j - n\tau_j)} df_1 \ldots df_d \\
&= \prod_{j=1}^d \int_{-\frac{1}{2\tau_j}}^{\frac{1}{2\tau_j}} \tau_j e^{2\pi i f_j (t_j - n\tau_j)} df_j \\
&= \prod_{j=1}^d \left[\frac{\tau_j}{2\pi i (t_j - n\tau_j)} e^{2\pi i f_j (t_j - n\tau_j)} \right]_{f_j = -\frac{1}{2\tau_1}}^{\frac{1}{2\tau_1}} \\
&= \prod_{j=1}^d \frac{\tau_j}{2\pi i (t_j - n\tau_j)} \left(e^{\pi i \frac{t_j - n\tau_j}{\tau_j}} - e^{-\pi i \frac{t_j - n\tau_j}{\tau_j}} \right) \\
&= \prod_{j=1}^d sinc\left(\frac{t_j - n\tau_j}{\tau_j} \right)
\end{aligned}$$

Hence formula (1.3) is:

$$\mathcal{F}(x)(f) = \sum_{n\in\mathbb{Z}^d} x(n\tau)\mathcal{F}\left(\prod_{j=1}^{d} sinc\left(\frac{t_j - n\tau_j}{\tau_j}\right)\right)(f)$$

Through applying the inverse transform on both sides, the theorem is proved.

$$\begin{aligned} x(t) = \mathcal{F}^{-1}(\mathcal{F}(x))(t) &= \mathcal{F}^{-1}\left(\sum_{n\in\mathbb{Z}^d} x(n\tau)\mathcal{F}\left(\prod_{j=1}^{d} sinc\left(\frac{t_j - n\tau_j}{\tau_j}\right)\right)\right) \\ &= \sum_{n\in\mathbb{Z}^d} x(n\tau)\prod_{j=1}^{d} sinc\left(\frac{t_j - n\tau_j}{\tau_j}\right) \end{aligned}$$

1.4 Meaning, real-life applications and limitations

1.4.1 Meaning

Now we know, that a band-limited L^2 function can be perfectly reconstructed with τ_0 small enough. Indeed, it is possible to determine τ_0 more precisely:

Theorem 1.4.1 (Shannon-Nyquist sampling theorem). *Let $f \in L^2_Q(\mathbb{R}^d)$, λ_{max} the highest frequency of f, $\frac{1}{\tau} \in \mathbb{R}^d$ the sampling rate. If $\frac{1}{\tau_0} \geq 2 \cdot \lambda_{max}$, f can be reconstructed from its samples for all $\tau \leq \tau_0$.*
$\frac{1}{2}\frac{1}{\tau_0}$ is called the Nyquist-rate or Nyquist-frequency.

Proof. See [3]. □

(den Beweis abzuschreiben waere gar zu klaegliche Platzverbraucherei oder? ;))
Remark: The Whittacker-Shannon interpolation formula (Theorem 1.2.1) and the Shannon-Nyquist sampling theorem (Theorem 1.4.1) are often combined as the "Shannon-Nyquist-Whittacker sampling theorem".

1.4.2 Uncertainty Principle

Warum haben wir das nur fuer d=1 gemacht??

Theorem 1.4.2 (Uncertainty principle). *Let $g \in L^2(\mathbb{R})$ and $a, b \in \mathbb{R}$ two arbitrary scalars. Then*

$$\left(\int_{\mathbb{R}} (x-a)^2 |g(x)|^2 \,\mathrm{d}x\right)\left(\int_{\mathbb{R}} (\omega-b)^2 |\mathcal{F}g(\omega)|^2 \,\mathrm{d}\omega\right) \geq \frac{||g||_2^2}{4\pi} \tag{1.10}$$

(die hoch 1/2 im Skript sind falsch denke ich)
Which means, that an analyzing function (window) cannot be arbitrarily concentrated in the time- and frequency-domain at the same time.
If $g \in L^2(\mathbb{R}^d)$ with $||g||_2 = 1$, then, because of the Plancherel equality $||g||_2 = ||\mathcal{F}g||_2$,

both $|g|^2$ and $|\mathcal{F}g|^2$ are probability distributions on $\mathbb{R}^d$. With $a = mean(g)$ and $b = mean(\mathcal{F}g)$, (1.10) can be written as

$$var|g|^2 \cdot var|\mathcal{F}g|^2 \geq \frac{1}{4\pi},$$

the famous Heisenberg uncertainty principle!
Physically, this result says that position distribution and momentum distribution of a quantum particle cannot be sharply peaked at the same time. Mathematically, one can say that position and momentum are Fourier transforms of one another.
If we return to our example of audio signals, (1.10) can be understood in the following way:
If the signal f ist very short, it is not possible to determine the frequencies λ exactly, while on the other hand sound at one exact frequency corresponds to a perfect sine wave with no beginning or ending.

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[3] Claude E. Shannon. Communication in the presence of noise. *Proceedings of the IRE, 37.1*, pages 10–21, 1949.