1 Example Systems

Circle

Consider the dynamic system depicted in figure 1

$$\begin{cases} \dot{x}_1 = -a_1 x_1 + b_3 x_3 \\ \dot{x}_2 = -a_2 x_2 + b_1 x_1 \\ \dot{x}_2 = -a_3 x_3 + b_2 x_2 \end{cases}, \quad x_i(0) = x_{i,0} \quad i = 1,2,3 \tag{1}$$

with positive parameters $a_1, \ldots, b_3$ and initial values $x_{i,0}$. Depending on the

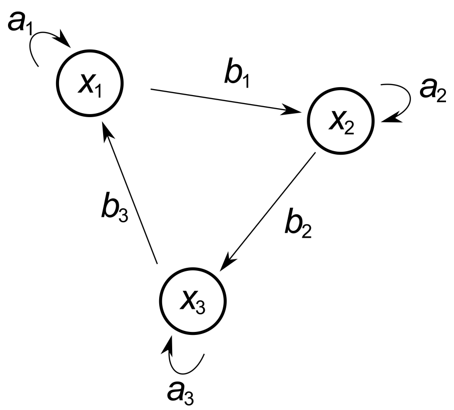

Figure 1: Network Graph of a 3-states circle.

magnitude of the parameters, the system could show different qualitative behaviours, it could be

1. stable $|x_i(t) - x_i^*| \to 0$ as $t \to \infty$ with a fix point x_i^*,
2. periodic $x_i(t) = x_i(t+T)$ with a constant T,
3. unstable $x_i(t) \to \infty$ as $t \to \infty$ or
4. chaotic.

If we want to to use this system as a simplified version of a real system, e.g. the JAK-STAT pathway [**DEN**], we should adjust the parameters in such a way, that the model (1) shows a realistic behaviour before we simulate data.

Classical Equilibration The first step in finding a stable behaviour is to compute the fixed points of the system, i.e. x_i^* such that $\dot{x}_i = 0$. From (1) we deduce the homogeneous linear equation system

$$0 = Ax \quad \text{where} \quad \begin{pmatrix} -a_1 & 0 & b_3 \\ b_1 & -a_2 & 0 \\ 0 & b_2 & -a_3 \end{pmatrix} \quad , \tag{2}$$

and we know that $\dim\ker A = 0$ if and only if $\det A = 0$. So we compute

$$\det A = -a_1 a_2 a_3 + b_1 b_2 b_3 \tag{3}$$

to get: System (1) has only the trivial fixed point $x_i^* = 0$, if

$$a_1 a_2 a_3 \neq b_1 b_2 b_3 \quad . \tag{4}$$

In the case $a_1 a_2 a_3 = b_1 b_2 b_3$ we get an 1-dimensional equilibrium manifold

$$\ker A = \operatorname{span}\left\{ \begin{pmatrix} b_2 \\ b_1 b_2 \\ a_1 a_2 \end{pmatrix} \right\} \quad . \tag{5}$$

Tropical Equilibration Now we try to adjust the parameters such that we obtain a approximately stable system using tropical methods [**Trop**].

Choose a small $1 >> \epsilon > 0$ and define normalized parameters (the notation is always understood as $i = 1,2,3$) $\tilde{a}_i$ and $\tilde{b}_i$ via

$$\begin{array}{llll} a_1 = \tilde{a}_1 \epsilon^{\gamma_1} \quad , & a_2 = \tilde{a}_2 \epsilon^{\gamma_2} \quad , & a_3 = \tilde{a}_3 \epsilon^{\gamma_3} \quad , \\ b_1 = \tilde{b}_1 \epsilon^{\gamma_4} \quad , & b_2 = \tilde{b}_2 \epsilon^{\gamma_5} \quad , & b_3 = \tilde{b}_3 \epsilon^{\gamma_6} \end{array} \tag{6}$$

and normalized state variables $\tilde{x}_i$

$$x_1 = \tilde{x}_1 \epsilon^{\delta_1} \quad , \quad x_2 = \tilde{x}_2 \epsilon^{\delta_2} \quad , \quad x_3 = \tilde{x}_3 \epsilon^{\delta_3} \quad , \tag{7}$$

such that

1. the normalized parameters and variables are of order one $\mathcal{O}(\tilde{a}_i) = \mathcal{O}(\tilde{b}_i) = \mathcal{O}(\tilde{x}_i) = \mathcal{O}(1)$ and
2. the powers are integers $\gamma_i, \delta_i \in \mathbb{Z}$.

Note, that in theory we can perform the limit $\epsilon \to 0$ to get

$$\lim_{\epsilon\to 0} \tilde{a}_i = \lim_{\epsilon\to 0} \tilde{b}_i = \lim_{\epsilon\to 0} \tilde{x}_i = 1 \quad . \tag{8}$$

We now want to express (1) in terms of the normalized parameters and variables. We see that ($i = 2, 3$ analogous)

$$\left.\begin{aligned} \dot{x}_1 &= \dot{\tilde{x}}_1 \epsilon^{\delta_1} \\ -a_1 x_1 + b_3 x_3 &= -\tilde{a}_1 \tilde{x}_1 \epsilon^{\gamma_1+\delta_1} + \tilde{b}_3 \tilde{x}_3 \epsilon^{\gamma_6-\delta_3} \end{aligned}\right\} \quad \Rightarrow \quad \dot{\tilde{x}}_1 = -\tilde{a}_1 \tilde{x}_1 \epsilon^{\gamma_1} + \tilde{b}_3 \tilde{x}_3 \epsilon^{\gamma_6+\delta_3-\delta_1} \tag{9}$$

and we note that since $0 < \epsilon << 1$ the leading summands of (9) are those with the smallest power of ϵ. Now tropical equilibration means, we have to find a parameters, such that

1. the minimal power of ϵ appears in two summands and
2. these two leading summands have opposite signs.

Now by definition of tropical algebra, 1. is just finding the simultaneous tropical roots of

$$\begin{aligned} &\gamma_1 \oplus \gamma_6 \otimes \delta_3 \otimes (-\delta_1) = 0 \\ &\gamma_2 \oplus \gamma_4 \otimes \delta_1 \otimes (-\delta_2) = 0 \\ &\gamma_3 \oplus \gamma_5 \otimes \delta_2 \otimes (-\delta_3) = 0 \end{aligned} \tag{10}$$

or in other words with $f_1(\gamma_1, \ldots, \delta_3) := \gamma_1 \oplus \gamma_6 \otimes \delta_3 \otimes (-\delta_1)$, f_2 and f_3 analogous, we have to compute the tropical variety

$$V(f_1, f_2, f_3) \quad . \tag{11}$$

In this specific case the situation is quiet easy because in (9) we only have two summands with opposite signs. Thus the whole tropical variety also fulfils 2. Addition of the three equations of (10) shows

$$V(f_1, f_2, f_3) \subseteq \left\{\gamma_1, \ldots, \delta_3 \in \mathbb{Z} \middle| \gamma_1 + \gamma_2 + \gamma_3 = \gamma_4 + \gamma_5 + \gamma_6 \right\} \quad . \tag{12}$$

To interpret these results in the nontropical world we apply the exponential function and multiply with $\tilde{a}_i$ and $\tilde{b}_i$ (which are all approximately 1). This yields

$$\begin{aligned} \tilde{a}_1 \tilde{a}_2 \tilde{a}_3 \epsilon^{\gamma_1+\gamma_2+\gamma_3} &= \tilde{b}_1 \tilde{b}_2 \tilde{b}_3 \epsilon^{\gamma_4+\gamma_5+\gamma_6} \\ \tilde{a}_1 \epsilon^{\gamma_1} \tilde{a}_2 \epsilon^{\gamma_2} \tilde{a}_3 \epsilon^{\gamma_3} &= \tilde{b}_1 \epsilon^{\gamma_4} \tilde{b}_2 \epsilon^{\gamma_5} \tilde{b}_3 \epsilon^{\gamma_6} \\ a_1 a_2 a_3 &= b_1 b_2 b_3 \quad . \end{aligned} \tag{13}$$

We found, that in this example, tropical equilibration leads exactly to those parameter sets, that have a nontrivial classical equilibrium manifold.