1 HIO using Volterra-operators

Let $w : [0, T] \to \mathbb{R}^m$, $y : [0, T] \to \mathbb{R}^p$, $A \in \mathbb{R}^{n\times n}$, $D \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$ such that

$$y(t) = \int_0^t C \exp\{A(t-\tau)\} D w(\tau)\,\mathrm{d}\tau \quad . \tag{1}$$

Due to Cayley-Hamilton

$$A^k = \sum_{l=0}^{n-1} c_{k,l} A^l \tag{2}$$

with coefficients $c_{k,l}$ that in general are not unique. By choosing N the smallest number such that

$$A^N \in \operatorname{span}\left(A^0, A^1, \ldots, A^{N-1}\right) \quad , \tag{3}$$

the coefficients $c_{k,l}$ count $l = 0, 1, \ldots, N-1$ and are unique. For each $k \leq N-1$ we find the Kronecker-delta

$$c_{k,l} = \delta_{k,l} \quad . \tag{4}$$

Expanding the exponential function to its power series we get

$$\begin{aligned} \exp\{A(t-\tau)\} &= \sum_{k=0}^{\infty} \frac{(t-\tau)^k}{k!} A^k \\ &= \sum_{l=0}^{N-1} A^l \underbrace{\sum_{k=0}^{\infty} \frac{(t-\tau)^k}{k!} c_{k,l}}_{\phi_l(t,\tau)} \quad . \end{aligned} \tag{5}$$

Both sums should converge absolutely since they are either finite or suppressed exponentially. Using (5) we can write (1) as

$$y(t) = \sum_{l=0}^{n-1} C A^l D \int_0^t \phi_l(t,\tau)\, w(\tau)\,\mathrm{d}\tau \quad . \tag{6}$$

From Volterra- to matrix-equation

Definition 1. According to [1] or [2] the equation

$$\int_0^t K(t,s) f(s)\,\mathrm{d}s = g(t) \tag{7}$$

where $f, g : [0, T] \to \mathbb{R}$ and $K : \{(t,s) \in [0,T] \times [0,T] | s < t\} \to \mathbb{R}$ is called *Volterra integral equation of the first kind (V1)* and K is called the *kernel*.

Comparing (6) and (7) we see that, if w_i denotes the i-th component of w, we get m V1 equations

$$\int\limits_0^t \phi_l(t,\tau)\, w_i(\tau)\,\mathrm{d}\tau = z_{i,l}(t) \quad . \tag{8}$$

Let $z_l(t)$ be any desired trajectory in $\mathbb{R}^m$ and we want to solve (8).

Definition 2. Again according to [1] and [2]

$$f(s) - \int\limits_0^t K(t,s) f(s)\,\mathrm{d}s = g(t) \tag{9}$$

is called *Volterra integral equation of the second kind (V2).*

Theorem 1 (Proof in [1] and [2])**.** For any admissable function g, V2 has a unique solution f that is given by the Neumann series.

Theorem 2. Let K be a continuous kernel, $K(t,t) \neq 0$ and $\frac{\partial K}{\partial t}$ continuous. Then, given an differentiable g, V1 has a unique solution f.

Proof. We follow the proof in [2].
Differentiation of V1 with respect to t yields

$$K(t,t) f(t) + \int\limits_0^t \frac{\partial K}{\partial t}(t,s) f(s)\,\mathrm{d}s = \frac{\mathrm{d}g}{\mathrm{d}t}(t) \tag{10}$$

Only if $K(t,t) \neq 0$ we can divide the whole equation by $K(t,t)$ and redefine the kernel and right hand side to end up with a V2, which has a unique solution. $\square$

Now turning back to (8), we want to apply theorem 2 to get information about the solution. But we find that the kernels ϕ_l do not fit to the assumption since

$$\phi_l(t,t) = \sum_{k=0}^{\infty} \frac{(t-t)^k}{k!} c_{k,l} = c_{0,l} \tag{11}$$

and comparing to (4)

$$\phi_0(t,t) = 1 \quad , \quad \phi_l(t,t) = 0 \quad \text{for} \quad l = 1,2,\ldots,N-1 \quad . \tag{12}$$

Example 1. To show that (12) will cause problems especially for the hidden input observability, consider

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad , \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad , \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad . \tag{13}$$

This yields

$$\phi_0(t,\tau) = 1 \quad , \quad \phi_1(t,\tau) = t-\tau \quad \text{and} \quad \phi_2(t,\tau) = \frac{(t-\tau)^2}{2} \quad . \tag{14}$$

Choosing $w(\tau) = 2\forall\tau$ we get

$$\int_0^t \phi_1(t,\tau)\, w(\tau)\, \mathrm{d}\tau = 0 \quad \forall t \quad . \tag{15}$$

Assume, we find a set $\left\{A^i\right\}$ with i from an index set $\mathscr{I}$, such that

$$A^k = \sum_{i\in\mathscr{I}} c_{k,i} A^i \quad \text{and} \quad c_{0,i} \neq 0 \forall i \in \mathscr{I} \tag{16}$$

which means $\operatorname{span}\left\{A^i | i \in \mathscr{I}\right\} = \operatorname{span}\left\{A^0, A^1, \ldots, A^{n-1}\right\}$ and $|\mathscr{I}| > N$.

Example 2. Consider $n = 3$

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad , \quad C = D = \mathbb{1} \quad . \tag{17}$$

It is not possible to solve (16). Though $[CD, CAD, CA^2D]$ has rank n.

Example 3. The Matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{18}$$

and the set $\left\{A^1, A^2, A^3\right\}$ or equivalently $\mathscr{I} = \{1,2,3\}$ yields (16) if we choose $c_{0,1} = 1$, $c_{0,2} = 1$ and $c_{0,3} = -1$.

Under an appropriate set $\left\{A^i\right\}$ equation (8) has a unique solution and (6) becomes

$$y(t) = \sum_{i\in\mathscr{I}} CA^i D z_i(t) \quad . \tag{19}$$

Properties of the matrix-equation

Assume that for a given system the preceding conditions hold. Does $y(t) = 0\forall t$ imply $w(t) = 0\forall t$? Writing (19) as

$$\underbrace{\left[CA^{i_1}D, CA^{i_2}D, \ldots, CA^{i_{|\mathscr{I}|}}\right]}_{M:\mathbb{R}^{m|\mathscr{I}|}\to\mathbb{R}^p \text{ linear with rank } r\leq p} \begin{bmatrix} z_{i_1}(t) \\ z_{i_2}(t) \\ \vdots \\ z_{i_{|\mathscr{I}|}}(t) \end{bmatrix} \tag{20}$$

we use the dimension formula

$$
\begin{aligned}
& & m|\mathscr{J}| &= \dim \operatorname{kernel} M \quad + r && (21)\\
\Rightarrow & & mN &< \dim \operatorname{kernel} M \quad + r && (22)\\
\Rightarrow & & mN &\leq \dim \operatorname{kernel} M \quad + p && (23)\\
\Rightarrow & & mN - p &\leq \dim \operatorname{kernel} M \qquad . && (24)
\end{aligned}
$$

Since $N \geq 1$

References

[1] Harro Heuser. *Funktionalanalysis*. 1992.

[2] Andreas Kirsch. *Lineare Integralgleichungen*. 2010.