1 Hidden Input Observability

Considering a dynamic system $\mathscr{S}$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = Ax(t) + Bu(t) + Dw(t) \tag{1}$$

$$y(t) = Cx(t) \tag{2}$$

$$x(0) = x_0 \tag{3}$$

where x, u, w and y map $[0,T]$ onto $\mathbb{R}^n$, $\mathbb{R}^{m'}$, $\mathbb{R}^m$ and $\mathbb{R}^p$, respectively, and A, B, C, D are matrices of suitable dimensions. The closed form solution for y is

$$y_w(t) = C\int_0^t \exp(A(t-\tau))(Bu(\tau)+Dw(\tau))\,\mathrm{d}\tau \quad . \tag{4}$$

Definition 1. If for a system $\mathscr{S}$ the implication

$$y_w(t) = y_{\hat{w}}(t) \quad \forall\, t\in[0,T] \quad \Rightarrow \quad w=\hat{w} \quad \text{a.e.} \tag{5}$$

holds, $\mathscr{S}$ is called *hidden input observable.*

Due to linearity a system is hidden input observable if and only if

$$\Delta y_w(t) := C\int_0^t \exp(A(t-\tau))Dw(\tau)\,\mathrm{d}\tau = 0 \quad \forall\, t\in[0,T] \quad \Rightarrow \quad w=0 \quad \text{a.e.} \tag{6}$$

For the sake of simplicity we will consider only $p=1$. With the definition of the operators

$$I^t : L^1([0,T]) \to l^2 \quad , \quad w \mapsto \left(\frac{1}{k!}\int_0^t (t-\tau)^k w(\tau)\,\mathrm{d}\tau\right)_{k\in\mathbb{N}_0} \tag{7}$$

and

$$\Sigma : l^2 \to \mathbb{R}^p \quad , \quad (s_k)_{k\in\mathbb{N}_0} \mapsto \sum_{k=0}^{\infty} CA^k D s_k \tag{8}$$

we see that

$$\Delta y_w(t) = \Sigma \circ I^t w \tag{9}$$

and thus

Corollary. A system is hidden input observable if and only if

$$\ker \Sigma \circ I^t = \{0\} \tag{10}$$

where we identify the null function as $w=0$ a.e.

1.1 Properties of the integral operator

By setting $\hat{\tau} = (t - \tau)$ and $f(\hat{\tau}) = w(t - \hat{\tau})$ the integrals in I^t are simplified to

$$I^t(f) = \left(\frac{1}{k!} \int_0^t \tau^k f(\tau)\, \mathrm{d}\tau \right)_{k \in \mathbb{N}_0} \quad . \tag{11}$$

Proposition 1. The operator I^t is bounded and continuous.

Proof. Estimation of the integrals yield

$$\begin{aligned} \|I^t(f)\|_{l^2}^2 &= \sum_{k=0}^{\infty} \left| \frac{1}{k!} \int_0^t \tau^k f(\tau)\, \mathrm{d}\tau \right|^2 \le \sum_0^{\infty} \left(\frac{1}{k!} \int_0^t \left| \tau^k f(\tau) \right| \mathrm{d}\tau \right)^2 \\ &\le \sum_0^{\infty} \left(\frac{t^k}{k!} \int_0^t \left| f(\tau) \right| \mathrm{d}\tau \right)^2 \le \left(\sum_0^{\infty} \frac{t^k}{k!} \int_0^t \left| f(\tau) \right| \mathrm{d}\tau \right)^2 = \left(\mathrm{e}^t \|f\|_{L^1} \right)^2 \quad . \end{aligned} \tag{12}$$

Thus $\|I^t(f)\|_{l^2} \le \mathrm{e}^t \|f\|_{L^1}$ which means I^t is bounded. Each bounded linear operator is continuous. □

1.1.1 Fixed point of time

In the following, $t > 0$ is a fixed point of time. Considering t as a variable will lead to different conditions, as shown in 1.1.2. To prove that I^t is injective we define the n-th integral function F_n by

$$F_{n+1}(\tau) = \int_0^{\tau} F_n(\tau')\, \mathrm{d}\tau' \quad , \quad F_0(\tau) = f(\tau) \tag{13}$$

$$F_n(0) = 0 \quad , \quad n = 1, 2, 3, \dots \tag{14}$$

and use the following lemma.

Lemma 1. [Extreme Value Theorem with marginal conditions]
Let $g : [a, b] \to \mathbb{R}$ be a continuous function, G such that $G'(\tau) = g(\tau)$ and $g(a) = g(b) = G(a) = G(b) = 0$. Let $\{\tau_1, \tau_2, \dots, \tau_N\}$ be the set of roots of g in (a, b). Then G has at most $N - 1$ roots in (a, b).

Proof. The roots of g form a finite set hence there is no interval where g is zero. Then G has an local extremum or an saddle point at each τ_i, and thus at most one root in (τ_i, τ_{i+1}). Since G cannot be zero in (a, τ_1) and (τ_N, b), G has at most $N - 1$ roots. □

In the proof of injectivity, F_n will fit the assumptions of lemma 1. To see that, define the integral operators

$$I^t_{k,n}(f) = \frac{1}{(k-n)!}\int_0^t \tau^{k-n} F_n(\tau)\,\mathrm{d}\tau \quad . \tag{15}$$

We deduce a recursive formula

$$I^t_{k,n}(f) = \frac{t^{k-n}}{(k-1)!}F_{n+1}(t) - I^t_{(k,n+1)}(f) \tag{16}$$

and identify

$$\left(I^t_{k,0}(f)\right)_{k\in\mathbb{N}_0} = I^t(f) \quad . \tag{17}$$

Combining (16) and (17) leads to the closed form expression

$$I^t(f) = \left(\sum_{n=0}^{k}(-1)^n\frac{t^{k-n}}{(k-n)!}F_{n+1}(t)\right)_{k\in\mathbb{N}_0} \quad . \tag{18}$$

Proposition 2. The operator I^t is injective.

Proof. Let $I^t(f) = 0 \in l^2$. With (18) it follows that

$$F_n(t) = 0 \quad , \quad n = 1,2,3,\dots \quad . \tag{19}$$

Let $\mathscr{R}$ be the countable set where $f(\tau) = 0$ for all $\tau \in \mathscr{R}$, let $\mathscr{S}$ be the set where f is not continuous an changes sign and let $\mathscr{I}$ be the set of intervals where f is zero. F_1 cannot have a local extremum on any interval of $\mathscr{I}$. Thus

$$\tau^* \quad \text{is an local extremum} \quad \Rightarrow \quad \tau^* \in \mathscr{R}\cup\mathscr{S} \tag{20}$$

1. If $\mathscr{R}\cup\mathscr{S}$ is finite, then $F_{|\mathscr{R}|+|\mathscr{S}|+1}$ surely has no local extremum and due to (19), $F_{|\mathscr{R}|+|\mathscr{S}|+1}$ has to be the null function. Because of $F'_n(\tau) = F_{n-1}(\tau)$, each F_n is the null function and $f = 0$ a.e.

2. If $\mathscr{R}\cup\mathscr{S}$ is infinite then either f is not in $L^1([0,t])$ or $\mathscr{R}\cup\mathscr{S}$ can be split into a finite part and a null set.

□

1.1.2 Variable time

Now we consider the full information of (10), i.e. I^t maps a function onto a trajectory in l^2.

Proposition 3. The operator I^t is injective.

Proof. Let f such that $I^t = 0 \in l^2$ for each $t \in [0, T]$.

- Setting $t = T$ and using 2 proves that $f = 0$ a.e.
- In general consider a function f that is not zero a.e. but possibly $f(t) = 0$ on $[0, t_1]$. Chose $t_2 > t_1$ to be the first point where f changes sign. Then each component of $I_2^t(f)$ is non zero. Thus f has to be zero on $[t_1, t_2]$ a.e. Repeat this argument until $f = 0$ on $[0, T]$ a.e.

□

The two ways to prove the preceding proposition show, that a variable time puts more much stronger constrains to the hidden input f.

2 Properties of the matrix series

A key statement is the Cayley-Hamilton theorem. A proof can be found in [1].

Theorem 1 (Cayley-Hamilton)**.** Let A be an $n \times n$ matrix and $\sum_{k=0}^{n} \tilde{c}_k \lambda^k$ the characteristic polynomial. Then

$$\sum_{k=0}^{n} \tilde{c}_k A^k = 0 \quad . \tag{21}$$

Corollary.

1. Each A^N, $N \geq n$, can be written as $\sum_{k=0}^{n-1} c_k A^k$ with some coefficients $\{c_k\}$ that can be calculated from $\{\tilde{c}_k\}$.
2. Multiplying with C and D also shows

$$CA^N D = \sum_{k=0}^{n-1} c_k C A^k D \tag{22}$$

   for any $N \geq n$.

3. For each operator $\Sigma_N (s_k)_{k\in\mathbb{N}_0} = \sum_{k=0}^{N-1} CA^k D (s_k)_{k\in\mathbb{N}_0}$ with $N \geq 0$

$$\operatorname{rank} \Sigma_N = \operatorname{rank} \Sigma_n \quad . \tag{23}$$

Lemma 2. The operator series Σ_N converges in norm and $\Sigma_N \longrightarrow \Sigma$.

Proof. The operator norm is defined by

$$||\Sigma_N|| := \sup_{||(s_k)||_{l^2}=1} ||\Sigma_N(s_k)|| \quad . \tag{24}$$

□

Again, considering $t > 0$ as a fixed point in time and assuming I^t is surjective. We use the shorter notation (s_k) for a series $(s_k)_{k\in\mathbb{N}_0} \in l^2$.

Theorem 2. The kernel of Σ is infinite dimensional.

Proof. Define a l^2 series by $s_k = \delta_{k,N}$ for a $N \geq n$. Then

$$\Sigma(s_k) = CA^n D = \sum_{k=0}^{n-1} c_k CA^k D \quad . \tag{25}$$

The l^2 series $\hat{s}_k = c_k$ for $k < n$ and $\hat{s}_k = 0$ for $k \geq n$ leads to the same result, thus

$$\mathrm{span}\,(\hat{s}_k - s_k) \subset \mathrm{kernel}\,\Sigma \quad . \tag{26}$$

Repeating this argument for each $N \geq n$ it is possible to get arbitrarily many linearly independent vectors $(\hat{s}_k - s_k)$. □

Now let $(s_k)(t)$ be a trajectory in l^2.

References

[1] Werner H. *Linear Algebra.* 1963.