1 HIO using Volterra-operators and linearly independent hidden inputs

Let $w : [0,T] \to \mathbb{R}^m$, $y : [0,T] \to \mathbb{R}^p$, $A \in \mathbb{R}^{n\times n}$, $D \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$ such that

$$y(t) = \int_0^t C \exp\{A(t-\tau)\} D w(\tau)\,\mathrm{d}\tau \quad . \tag{1}$$

Due to Cayley-Hamilton

$$A^k = \sum_{l=0}^{n-1} c_{k,l} A^l \tag{2}$$

with coefficients $c_{k,l}$ that in general are not unique. By choosing $N \leq n$ the smallest number such that

$$A^N \in \operatorname{span}\left(A^0, A^1, \ldots, A^{N-1}\right) \quad , \tag{3}$$

the coefficients $c_{k,l}$ count $l = 0, 1, \ldots, N-1$ and are unique. Expanding the exponential function to its power series we get

$$y(t) = \sum_{l=0}^{N-1} C A^l D \Phi_l[w](t) \quad . \tag{4}$$

where

$$\Phi_l[w](t) := \int_0^t \sum_{k=0}^{\infty} \frac{(t-\tau)^k}{k!} c_{k,l} w(\tau)\,\mathrm{d}\tau \quad . \tag{5}$$

Proposition 1 (Without proof)**.** Each operator Φ_l is injective, i.e.

$$\Phi_l[w] \equiv 0 \qquad \Rightarrow \qquad w \equiv 0 \tag{6}$$

and Φ_0 is surjective. Here "$\equiv$" denotes equality to the zero function and Φ_l operates component- wise on $(w_1, w_2, \ldots, w_m)^{\mathrm{T}} : [0,T] \to \mathbb{R}^m$.

Consider the simple case $D = \mathbb{1}$ ($m = n$). Then for the μ-th column of CA^k we find

$$\left(CA^k\right)_\mu = \sum_{\omega=1}^{n} A^k_{\omega\mu} C_\omega \tag{7}$$

where $A^k_{\omega\mu}$ is the $(\omega\mu)$ component of A^k and C_ω is the ω-th column of C. Therefore each column of any CA^k is a linear combination of column vectors C_ω and thus

$$\operatorname{rank}\left[C, CA, CA^2, \ldots, CA^{N-1}\right] = \operatorname{rank} C \quad . \tag{8}$$

Furthermore we assume the hidden inputs to be linearly independent, i.e. for any coefficients d_μ

$$\sum_{\mu=1}^{n} d_\mu w_\mu \equiv 0 \quad \Longleftrightarrow \quad d_\mu w_\mu \equiv 0 \quad \forall \mu \quad . \tag{9}$$

1.1 Nilpotent dynamics

Let A be a nilpotent matrix, i.e. there is a regular $n \times n$ matrix P such that

$$A = P^{-1} A_\triangle P \tag{10}$$

with $A_{\triangle\omega\mu} = 0$ when $\omega \le \mu$. As a graphical condition this means, that A can be represented by a directed acyclic graph. Inserting in (4) yields

$$y(t) = \sum_{l=0}^{N-1} \underbrace{CP^{-1}}_{\operatorname{rank} CP^{-1} = \operatorname{rank} C} A_\triangle^l \Phi_l [\underbrace{Pw}_{\text{lin.indep.}}](t) \quad . \tag{11}$$

Thus without loss of generality we can assume that A is strictly lower triangular and $N = n$. Furthermore we see that (5) reduces to

$$\Phi_l[w_\mu](t) = \int_0^t \frac{(t-\tau)^l}{l!} w_\mu(\tau)\,\mathrm{d}\tau \tag{12}$$

with the properties

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_l[w_\mu](t) = \Phi_{l-1}[w_\mu](t) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\Phi_0[w_\mu](t) = w_\mu(t) \quad , \tag{13}$$

and equation (4) becomes

$$y(t) = \sum_{\omega=1}^{n} \sum_{l=0}^{\omega-1} \sum_{\mu=1}^{\omega-l} \Phi_l \left[A_{\omega\mu}^l w_\mu \right](t) C_\omega =: \sum_{\omega=1}^{n} \varphi_\omega(t) C_\omega \quad . \tag{14}$$

From (13) one can also deduce

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^q \varphi_\omega = \sum_{l=q}^{\omega-1} \sum_{\mu=1}^{\omega-l} \Phi_{l-q}\left[A_{\omega\mu}^l w_\mu\right] + \sum_{l=0}^{q-1} \sum_{\mu=1}^{\omega-l} A_{\omega\mu}^l w_\mu^{(q-l-1)} \tag{15}$$

where $w_\mu^{(q)}$ denotes the q-th derivative of w_μ. Useful derivatives are

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_\omega = \sum_{l=1}^{\omega-1} \sum_{\mu=1}^{\omega-l} \Phi_{l-1}\left[A_{\omega\mu}^l w_\mu\right] + w_\omega \quad , \quad \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^\omega \varphi_\omega = \sum_{l=0}^{\omega-1} \sum_{\mu=1}^{\omega-l} A_{\omega\mu}^l w_\mu^{(\omega-l-1)} \quad . \tag{16}$$

Corollary (Without proof)**.** From (15) with $q = 1$ we also conclude that if there is a M such that $w_\mu \equiv 0$ for all $\mu = 1, 2, \ldots, M-1$, then ${}^{\mathrm{d}}/_{\mathrm{d}t}\,\varphi_M = w_M$.

1.1.1 C has rank $p < n$

Following the idea of **??**, let C be any $p \times n$ matrix and let $\mathscr{I} \subset \{1,2,\ldots,n\}$ be an index set such that $\{C_i | i \in \mathscr{I}\}$ are linearly independent and for any $H \notin \mathscr{I}$ there are unique coefficients Λ_i^H such that $C_H = \sum_{i \in \mathscr{I}} \Lambda_i^H C_i$. Furthermore introduce the index sets $\mathscr{H}_i$ such that $H \in \mathscr{H}_i \Leftrightarrow \Lambda_i^H = 0$ and $\mathfrak{H}_i = \bigcap_{\iota \le i} \mathscr{H}_\iota$. Set $i_{\min}$ the smallest i such that $\mathfrak{H}_i = \emptyset$. The notation $\mathfrak{H}_{j-1}$ means the $\mathfrak{H}_i$ with the biggest $i < j$ in $\mathscr{I}$. We get

$$y(t) = \sum_{i \in \mathscr{I}} \left(\varphi_i(t) + \sum_{H \notin \mathscr{I}} \Lambda_i^H \varphi_H(t) \right) C_i \tag{17}$$

setting $y \equiv 0$, equating coefficients and differentiation with respect to t yields for all $i \in \mathscr{I}$

$$\sum_{l=1}^{i-1} \sum_{\mu=1}^{i-l} \Phi_{l-1} \left[A_{i\mu}^l w_\mu \right] + \sum_{H \notin \mathscr{I}} \sum_{l=1}^{H-1} \sum_{\mu=1}^{H-l} \Lambda_i^H \Phi_{l-1} \left[A_{H\mu}^l w_\mu \right] + w_i + \sum_{H \notin \mathscr{I}} \Lambda_i^H w_H \equiv 0 \quad . \tag{18}$$

Algorithmic Approach If $1 \in \mathscr{I}$, choose $i = 1$ and (18) reduces to

$$\sum_{H \notin \mathscr{I}} \sum_{l=1}^{H-1} \sum_{\mu=1}^{H-l} \Lambda_1^H \Phi_{l-1} \left[A_{H\mu}^l w_\mu \right] + w_1 + \sum_{H \notin \mathscr{I}} \Lambda_1^H w_H \equiv 0 \quad . \tag{19}$$

If for all $H \notin \mathscr{I}$ we find $\Lambda_1^H A_{H\mu}^1 = 0 \forall \mu$, then $w_1 \equiv 0$ and $w_H \equiv 0 \forall H \notin \mathfrak{H}_1$. Assume this condition holds.

1. If $2 \in \mathscr{I}$, (18) with $i = 2$ yields

$$\sum_{H \notin \mathscr{I}} \sum_{l=1}^{H-1} \sum_{\substack{\mu=2 \\ \mu \in \mathfrak{H}_1}}^{H-l} \Lambda_2^H \Phi_{l-1} \left[A_{H\mu}^l w_\mu \right] + w_2 + \sum_{H \in \mathfrak{H}_1} \Lambda_2^H w_H \equiv 0 \tag{20}$$

   that is, if for all $H \notin \mathscr{I}$ we find $\Lambda_2^H A_{H\mu}^1 = 0 \forall 2 \le \mu \in \mathfrak{H}_1$, then $w_2 \equiv 0$ and $w_H \equiv 0 \forall H \notin \mathfrak{H}_2$.

2. If $2 \notin \mathscr{I}$ then

   (a) If $2 \notin \mathfrak{H}_1$ then $w_2 \equiv 0$.

   (b) If $2 \in \mathfrak{H}_1$ then we have no information about w_2.

If we can conclude $w_2 \equiv 0$ we increase i and proceed in a similar manner to get $w_1 = \ldots = w_{j-1} \equiv 0$ up to an j with $j \in \mathfrak{H}_{j-1}$ (and consequently $j \notin \mathscr{I}$).

Now assume there is such an j and $j < n$.

1. If $j+1 \in \mathscr{I}$ choose $i = j+1$ to get

$$\Phi_0 \left[A^1_{(j+1)j} w_j \right] + \sum_{H \notin \mathscr{I}} \sum_{l=1}^{H-1} \sum_{\substack{\mu=j \\ \mu \in \mathfrak{H}_j}}^{H-l} \Lambda^H_{j+1} \Phi_{l-1} \left[A^l_{H\mu} w_\mu \right] + w_{j+1} + \sum_{H \in \mathfrak{H}_j} \Lambda^H_{j+1} w_H \equiv 0 \quad . \tag{21}$$

   To get conditions for HIO we can

   (a) If $A^1_{(j+1)j} = 0$ proceed as before assuming $w_j \not\equiv 0$. In the next step we will need $A^1_{(j+2)j} = 0$ etc.

   (b) If $A^1_{(j+1)j} \neq 0$ force $w_j \equiv 0$ and proceed as before.

2. If $j+1 \notin \mathscr{I}$ everything gets worse...

Strict Algorithm

1. If A nilpotent: Transform to strict lower triangular Matrix. Else: break.
2. Get rank C , choose $\mathscr{I}$, compute Λ^H_i and $\mathfrak{H}_i$.
3. $\mathscr{N} = \emptyset$
4. For $i = 1, i \leq n$:

   If $i \in \mathscr{I}$:

      If $\forall H \notin \mathscr{H}_i$ we find $\Lambda^H_i A_{H\mu} = 0$ for $\mu \in \{i, i+1, \ldots, n\} \cap \mathfrak{H}_{i-1}$:

         $i++$

      Else:

         break

   Else:

      If $i \in \mathfrak{H}_{i-1}$:

         Force $w_i \equiv 0$

         $\mathscr{N} = \mathscr{N} \cup \{i\}$

         $i++$

      Else:

         $i++$

5. If the iteration was successful:
   Under the restriction $w_i \equiv 0 \forall i \in \mathscr{N}$, the system is HIO.