Hidden Input Observability

We consider a mapping $L^2([0,T])^{\otimes m} \to L^2([0,T])^{\otimes p}$ defined by

$$y(t) := C\int_0^t \mathrm{e}^{A(t-\tau)} D w(\tau)\,\mathrm{d}\tau \tag{1}$$

where $w:[0,T]\to\mathbb{R}^m$, $y:[0,T]\to\mathbb{R}^p$, $A\in\mathbb{R}^{n\times n}$, $D\in\mathbb{R}^{n\times m}$ and $C\in\mathbb{R}^{p\times n}$.

We firstly rewrite the problem by expanding (1),

$$y(t) = \sum_{k=0}^{\infty} CA^k D \int_0^t \frac{(t-\tau)^k}{k!} w(\tau)\,\mathrm{d}\tau \quad . \tag{2}$$

Writing $(CA^kD)_{.\mu}$ means the μ-th column and w_μ the ν-th component. We get

$$y(t) = \sum_{k=0}^{\infty}\sum_{\mu=1}^{m} \int_0^t \frac{(t-\tau)^k}{k!} w_\mu(\tau)\,\mathrm{d}\tau (CA^kD)_{.\mu} \tag{3}$$

and if we only consider the ν-th component of y we get

$$y_\nu(t) = \sum_{k=0}^{\infty}\sum_{\mu=1}^{m} \int_0^t \frac{(t-\tau)^k}{k!} w_\mu(\tau)\,\mathrm{d}\tau (CA^kD)_{\nu\mu} \quad . \tag{4}$$

We introduce the coefficients $c_k^{\mu\nu} := (CA^kD)_{\nu\mu}$ to get

$$y_\nu(t) = \sum_{\mu=1}^{m}\sum_{k=0}^{\infty} c_k^{\mu\nu} \int_0^t \frac{(t-\tau)^k}{k!} w_\mu(\tau)\,\mathrm{d}\tau \quad \forall \nu = 1,2,\ldots,p \quad . \tag{5}$$

Here, it is legit to commute the sums, since the infinite sum over k is absolutely convergent.

It is now convenient to introduce some operators.

Definition 1

Let $v \in L^2$ be a function and for simplicity we write $c = (c_k)_{k\in\mathbb{N}_0}$.

1. $\phi_k[v](t) := \int_0^t \frac{(t-\tau)^k}{k!} v(\tau)\,\mathrm{d}\tau$
2. $\Phi_c[v] := \sum_{k=0}^{\infty} c_k \phi_k[v]$

We always exclude the sequence $c_k = 0 \forall k$, since this obviously produces the operator that maps everything to the zero function and thus cannot be injective.

We rewrite (5) as

$$y_\nu(t) = \sum_{\mu=1}^{m} \Phi_{c^{\mu\nu}}[w_\mu](t) \quad \forall \nu = 1,2,\dots,p \tag{6}$$

and discuss some properties.

Lemma 1: Properties of ϕ_k

1. $\phi_k[v](0) = 0$
2. $\frac{\mathrm{d}}{\mathrm{d}t}\phi_0[v](t) = v(t)$ and $\frac{\mathrm{d}}{\mathrm{d}t}\phi_k[v](t) = \phi_{k-1}[v](t)$

Proof. The first property is trivial.
Computing $\phi_k[v](t+\Delta t)$ yields

$$\phi_k[v](t+\Delta t) = \int\limits_0^t \frac{(t-\tau+\Delta t)^k}{k!} v(\tau)\,\mathrm{d}\tau + \int\limits_t^{t+\Delta t} \frac{(t-\tau+\Delta t)^k}{k!} v(\tau)\,\mathrm{d}\tau \tag{7}$$

and using $(t+\Delta t)^k = t^k + k t^{k-1}\Delta t + \mathcal{O}(\Delta t^2)$ for $k > 0$ and $(t+\Delta t)^0 = 1$ and a small Δt leads to

$$\phi_k[v](t+\Delta t) \simeq \int\limits_0^t \frac{(t-\tau)^k}{k!} v(t)\,\mathrm{d}\tau + \Delta t \int\limits_0^t \frac{(t-\tau)^{k-1}}{(k-1)!} v(t)\,\mathrm{d}\tau \tag{8}$$

if $k > 0$ and to

$$\int\limits_0^t \frac{(t-\tau)^0}{k!} v(t)\,\mathrm{d}\tau + \Delta t\, v(t) \tag{9}$$

if $k=0$. Comparing this with the definitions of the operators we get $\phi_k[v](t+\Delta t) \simeq \phi_k(t)+\Delta t \phi_{k-1}[v](t)$ and $\phi_0[v](t+\Delta t) \simeq \phi_0[v](t)+\Delta t\, v(t)$. Taking the limit $\Delta t \to 0$ ends the proof of the derivation rules.

□

Proposition 1

Let v be an integrable function and element of $C^\infty(0^-, 0^+)$ and let $\Phi_c[v] \equiv 0$. Then

$$\sum_{l=0}^{q} c_l\, v^{(q-l)}(0) = 0 \quad \forall q \in \mathbb{N}_0 \tag{$\star$}$$

where $v^{(q)}(0)$ denotes the q-th derivative of v at $t=0$.

Proof. Consider

$$\Phi_c[v] = c_0\phi_0[v] + c_1\phi_1[v] + \ldots + c_{q-1}\phi_{q-1}[v] + c_q\phi_q[v] + c_{q+1}\phi_{q+1}[v] + \ldots \tag{10}$$

and the $(q+1)$-th derivative

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{q+1} \Phi_c[v] = c_0\, v^{(q)} + c_1\, v^{(q-1)} + \ldots + c_{q-1}\, v^{(1)} + c_q\, v^{(0)} + c_{q+1}\phi_0[v] + \ldots \tag{11}$$

for short

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{q+1} \Phi_c[v] = \sum_{l=0}^{q} c_l\, v^{(q-l)} + \sum_{l=0}^{\infty} c_{q+1+l}\phi_l[v] \quad . \tag{12}$$

Evaluating the latter expression at $t=0$ completes the proof. □

To illustrate the idea of the following lemma and theorem we write down some instances of ($\star$):

$$\begin{array}{llllllllll}
q=0 & c_0\, v^{(0)} \\
q=1 & c_0\, v^{(1)} & + & c_1\, v^{(0)} \\
q=2 & c_0\, v^{(2)} & + & c_1\, v^{(1)} & + & c_2\, v^{(0)} \\
q=3 & c_0\, v^{(3)} & + & c_1\, v^{(2)} & + & c_2\, v^{(1)} & + & c_3\, v^{(0)} \\
q=4 & c_0\, v^{(4)} & + & c_1\, v^{(3)} & + & c_2\, v^{(2)} & + & c_3\, v^{(1)} & + & c_4\, v^{(0)} \\
\vdots & & \ddots & & \ddots & & \ddots & & \ddots & & \ddots
\end{array}$$

For a better readability each $v^{(q)}$ is understood to be evaluated at $t=0$. The structure of this triangle remains the same if $c_0 = 0$, i.e. the first column vanishes, and if $v^{(0)} = 0$, i.e. the diagonal at the top vanishes.

Lemma 2: Induction Step

Assume proposition 1 holds and let c_K be the first nonzero coefficient. If there is a $r \in \mathbb{N}_0$ with $v^{(0)}(0) = v^{(1)}(0) = \ldots = v^{(r-1)}(0) = 0$ then $v^{(r)}(0) = 0$.

Proof. Using $(\star)$ with $q = r + K$ yields

$$\sum_{l=0}^{r+K} c_l v^{(r+K-l)}(0) = c_K v^{(r)}(0) = 0 \tag{13}$$

since all other terms of the sum vanish due to $c_l = 0$ or $v^{(l)} = 0$. This shows that also $v^{(r)}(0) = 0$. □

Theorem 1

Let $v \in L^2([0,T]) \cap C^\infty(0^-, 0^+)$ and c a sequence. Then

$$\Phi_c[v] \equiv 0 \quad \Rightarrow \quad v^{(q)}(0) = 0 \quad \forall q \in \mathbb{N}_0 \quad . \tag{14}$$

Proof. Let c_K be the first nonzero coefficient. Using $(\star)$ with $q = K$ yields

$$c_K v^{(0)} = 0 \tag{15}$$

which shows that $v^{(0)} = 0$. Using lemma 2 completes the inductive proof. □

Corollary 1

If v can be represented by its Taylor-expansion, then $\Phi_c[v] \equiv 0$ implies $v \equiv 0$.
If there is a disjoint union $[0^-, T^+] = I_1 \dot{\cap} I_2 \dot{\cap} \ldots$ such that v has a valid Taylor-expansion on each interval $I_j = (t_{j-1}, t_j)$, we can argue that $v \equiv 0$ on I_1. This leads to $\phi_k[v](t_1) = 0$ which allows us to get a modification of $(\star)$, written out

$$\sum_{l=0}^{q} c_l v^{(q-l)}(t_1) = 0 \quad \forall q \in \mathbb{N}_0 \quad . \tag{16}$$

Formally this can again be handled as an inductive proof to show, that v must vanish on each interval.

As equation (6) shows, we usually do not have a simple $y(t) = \Phi_c[v](t)$ relation but a sum with different sequences $c^{\mu\nu}$ and functions w_μ. Since summation and differentiation are linear operations, we directly get the following extension of proposition 1:

Proposition 2

Let $w_\mu \in L^2([0,T]) \cap C^\infty(0^-, 0^+)$ for $\mu = 1,2,\ldots,m$ and let c^μ be sequences. For each $\nu = 1,2,\ldots,p$

$$\sum_{\mu=1}^{m} \Phi_{c^{\mu\nu}}[w_\mu] \equiv 0 \quad \Rightarrow \quad \sum_{\mu=1}^{m} \sum_{l=0}^{q} c_l^{\mu\nu} w_\mu^{(q-l)}(0) = 0 \quad \forall q \in \mathbb{N}_0 \tag{$\star\star$}$$

Proof. The proof works analogous to that of proposition 1. □

Whereas theorem 1 holds for any sequence c and function v, proposition 2 allows cancellation of different functions w_μ. We shortly demonstrate, that a fully observed system will always be hidden input observable. Considering a fully observed system with possible hidden inputs on each state, i.e. $C = D = \mathbb{1}$ and $p = n = m$, we directly get $c_0^{\mu\nu} = \delta_{\mu\nu}$. Inserting this into ($\star\star$) with $q = 0$ yields

$$w_\nu^{(0)} = 0 \quad \text{for} \quad \nu = 1,2,\ldots,n \tag{17}$$

Following the idea of lemma 2 we proceed with an induction step to get

$$w_\nu^{(q)} = 0 \quad \forall q \in \mathbb{N}_0 \quad \text{for} \quad \nu = 1,2,\ldots,n \quad . \tag{18}$$

As argued in corollary 1, if we assume that each w_μ can be represented by a Taylor series, we know, that this system is hidden input observable.