1 Monomials

A compilation of integral lemmas, not rigorously proved.

Lemma 1. Let $w : [0, t_\mathrm{f}] \to \mathbb{R}^m$ be Riemann integrable.

$$\int_0^t w(\tau)\,\mathrm{d}\tau = 0 \quad \forall\, t \in [0, t_\mathrm{f}] \quad \Rightarrow \quad w \equiv 0 \tag{1}$$

Proof. Define $\Delta t = t_\mathrm{f}/N$ sufficient small and N_t such that $N_t \Delta t = t$. Then

$$\lim_{N\to\infty} \Delta t \sum_{k=0}^{N_t} w(k\Delta t) = 0 \quad \forall N_t \tag{2}$$

which means

$$w(0) = 0 \tag{3}$$
$$w(0) + w(\Delta t) = w(\Delta t) = 0 \tag{4}$$
$$w(0) + w(\Delta t) + w(2\Delta t) = w(2\Delta t) = 0 \tag{5}$$
$$\vdots \tag{6}$$

□

Corollary. Assume there is a function w such that

$$\int_0^t w(\tau)\,\mathrm{d}\tau = \int_0^t \tau\, w(\tau)\,\mathrm{d}\tau \quad \forall\, t \tag{7}$$

which is equivalent to

$$\int_0^t w(\tau)(1-\tau)\,\mathrm{d}\tau = 0 \quad \forall\, t \quad . \tag{8}$$

By the preceding lemma we know $w(\tau)(1-\tau) = 0\,\forall\tau$ and hence $w(t) = 0$ a.e. and because w is Riemann integrable $w \equiv 0$. With $\tilde{w}(\tau) = \tau\, w(\tau)$ we get $\tau\, w(\tau)(1-\tau) = 0\,\forall\tau$. Thus if for any N and for all t

$$\int_0^t \tau^N w(\tau)\,\mathrm{d}\tau = 0 \tag{9}$$

then $w \equiv 0$.

2 Hidden Input Observability

Consider the linear system $\mathscr{S}$

$$\dot{x} = Ax + Bu + Dw \tag{$\mathscr{S}1$}$$

$$y = Cx \tag{$\mathscr{S}2$}$$

$$x(0) = x_0 \tag{$\mathscr{S}3$}$$

where $x : [0, t_\mathrm{f}] \to \mathbb{R}^n$, $u : [0, t_\mathrm{f}] \to \mathbb{R}^{m'}$, $w : [0, t_\mathrm{f}] \to \mathbb{R}^m$ and $y : [0, t_\mathrm{f}] \to \mathbb{R}^r$. Furthermore $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m'}$, $D \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{r\times n}$. It is well known that

$$y(t) = C\mathrm{e}^{At}x_0 + C\int_0^t \mathrm{e}^{A(t-\tau)}\left(Bu(\tau) + Dw(\tau)\right)\mathrm{d}\tau \tag{10}$$

is a solution of $\mathscr{S}$.

Definition 1 (Hidden Input Observability)**.** Let w_1 and w_2 be admissible functions and let y_1 and y_2 be the solutions of $\mathscr{S}$ with w_1 and w_2, respectively. The system is called *hidden input observable*, if

$$w_1 \neq w_2 \quad \Rightarrow \quad y_1 \neq y_2 \quad . \tag{11}$$

Theorem 1 (Target Controllability)**.** For the system $\mathscr{S}$ with $B = 0$ define the matrix

$$M_k = [CD, CAD, CA^2D, \ldots, CA^{k-1}D] \quad . \tag{12}$$

By the Cayley-Hamilton theorem we know that this system is target controllable, i.e. any $y_\mathrm{f} \in \mathbb{R}^r$ can be reached within a finite time interval, if and only if

$$\operatorname{rank} M_n = r \quad . \tag{13}$$

Remark. Since M_n is a $r \times nm$ matrix, the rank condition (13) needs $r \leq nm$. Obviously only $m \leq n$ makes sense and hence $r \leq n^2$.

We now search for a sufficient or necessary condition for hidden input observability. Let y_1 and y_2 be solutions of $\mathscr{S}$ with w_1 and w_2, respectively. If we define $y = y_1 - y_2$ and $w = w_1 - w_2$, then

$$y(t) = C\int_0^t \mathrm{e}^{A(t-\tau)}Dw(\tau)\,\mathrm{d}\tau \tag{14}$$

is a solution of $\mathscr{S}$ with $B = 0$ and $x_0 = 0$. We call this simplified system $\mathscr{S}'$. Furthermore $w_1 \neq w_2 \Leftrightarrow w \not\equiv 0$ and for y analogous.

2.1 necessary condition

Assume the system is hidden input observable, i.e. $w \not\equiv 0 \Rightarrow y \not\equiv 0$.

Lemma 2. If $\mathscr{S}$ is hidden input observable, then

$$w \not\equiv 0 \Leftrightarrow y \not\equiv 0 \tag{15}$$

$$\text{or equivalently} \quad w \equiv 0 \Leftrightarrow y \equiv 0 \quad . \tag{16}$$

Proof.

1. $w \not\equiv 0 \Rightarrow y \not\equiv 0$ and equivalently $y \equiv 0 \Rightarrow w \equiv 0$ by definition of hidden input observability and $w \equiv 0 \Rightarrow y \equiv 0$ is obvious.
2. Assume $y \not\equiv 0$ and $w \equiv 0$. The explicit formula (14) results in $y \equiv 0$ which is in conflict with the assumption. Thus $y \not\equiv 0 \Rightarrow w \not\equiv 0$.

□

Expanding (14) in a power series we get

$$y(t) = \sum_{k=0}^{\infty} CA^k D \int_0^t \frac{(t-\tau)^k w(\tau)}{k!} \mathrm{d}\tau \quad , \tag{17}$$

and by the Cayley-Hamilton theorem we know that this is equivalent to

$$y(t) = M_n V(t) \tag{18}$$

with an arbitrary $V : [0, t_\mathrm{f}] \to \mathbb{R}^{nm}$.

Lemma 3. We have

$$\operatorname{rank} M_n = nm. \tag{19}$$

Proof.

1. Let $V(t) \in \operatorname{kernel} M_n \; \forall t$. Then $y \equiv 0 \Rightarrow w \equiv 0 \Rightarrow V \equiv 0$.
2. Now $V \equiv 0$ then $V(t) \in \operatorname{kernel} M_n$ because M_n is linear.

Hence $\dim \operatorname{kernel} M = 0$. By the rank theorem

$$f : X \to Y \text{ linear} \quad \text{then} \quad \dim V = \dim \operatorname{kernel} f + \operatorname{rank} f \tag{20}$$

we see

$$M_n : \mathbb{R}^{nm} \to \mathbb{R}^r \quad \text{and} \quad nm = \operatorname{rank} M_n \quad . \tag{21}$$

□

Remark. The preceding lemma only makes sense if $r \geq nm$ because $\operatorname{rank} M_n \leq \min(nm, r)$. The theorem 1 needs $r \leq nm$, so that we can conclude:

A system $\mathscr{S}$ can be hidden input observable only if $r \geq nm$.

A system $\mathscr{S}'$ that is target controllable can belong to a system $\mathscr{S}$ that is hidden input observable only if $r = nm$.

2.2 sufficient condition