1 HIO using Volterra-operators

Let $w : [0, T] \to \mathbb{R}^m$, $y : [0, T] \to \mathbb{R}^p$, $A \in \mathbb{R}^{n\times n}$, $D \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$ such that

$$y(t) = \int_0^t C \exp\{A(t-\tau)\} D w(\tau)\,\mathrm{d}\tau \quad . \tag{1}$$

Due to Cayley-Hamilton

$$A^k = \sum_{l=0}^{n-1} c_{k,l} A^l \tag{2}$$

with coefficients $c_{k,l}$ that in general are not unique. By choosing $N \le n$ the smallest number such that

$$A^N \in \operatorname{span}\left(A^0, A^1, \dots, A^{N-1}\right) \quad , \tag{3}$$

the coefficients $c_{k,l}$ count $l = 0, 1, \dots, N-1$ and are unique. Expanding the exponential function to its power series we get

$$y(t) = \sum_{l=0}^{N-1} C A^l D \Phi_l[w](t) \quad . \tag{4}$$

where

$$\Phi_l[w](t) := \int_0^t \sum_{k=0}^{\infty} \frac{(t-\tau)^k}{k!} c_{k,l} w(\tau)\,\mathrm{d}\tau \quad . \tag{5}$$

Proposition 1 (Without proof)**.** Each operator Φ_l is injective, i.e.

$$\Phi_l[w] \equiv 0 \qquad \Rightarrow \qquad w \equiv 0 \quad . \tag{6}$$

Here "$\equiv$" denotes equality in function space and Φ_l operates component-wise on $(w_1, w_2, \dots, w_m)^{\mathrm{T}} : [0, T] \to \mathbb{R}^m$.

1.1 Linearly independent HI

Consider the simple case $D = \mathbb{1}$ ($m = n$). Then for the μ-th column of CA^k we find

$$\left(CA^k\right)_\mu = \sum_{\omega=1}^{n} A^k_{\omega\mu} C_\omega \tag{7}$$

where $A^k_{\omega\mu}$ is the ($\omega\mu$) component of A^k and C_ω is the ω-th column of C. Therefore each column of any CA^k is a linear combination of column vectors C_ω and thus

$$\operatorname{rank}[C, CA, CA^2, \dots, CA^{N-1}] = \operatorname{rank} C \quad . \tag{8}$$

Furthermore we assume the hidden inputs to be linearly independent, i.e.

$$\sum_{\mu=1}^{n} d_\mu w_\mu \equiv 0 \quad \Leftrightarrow \quad d_\mu w_\mu \equiv 0 \quad \forall \mu \quad . \tag{9}$$

1.1.1 Nilpotent dynamics

Let A be a nilpotent matrix, i.e. there is a regular $n \times n$ matrix P such that

$$A = P^{-1} A_\triangle P \tag{10}$$

with $A_{\triangle\omega\mu} = 0$ when $\omega \le \mu$. As a graphical condition this means, that A can be represented by a directed acyclic graph. Inserting in (4) yields

$$y(t) = \sum_{l=0}^{N-1} \underbrace{CP^{-1}}_{\operatorname{rank} CP^{-1} = \operatorname{rank} C} A^l_\triangle \Phi_l [\underbrace{Pw}_{\text{lin.indep.}}](t) \quad . \tag{11}$$

Thus without loss of generality can assume that A is strictly lower triangular and $N = n$. Furthermore we see that (5) reduces to

$$\Phi_l[w_\mu](t) = \int_0^t \frac{(t-\tau)^l}{l!} w_\mu(\tau)\, \mathrm{d}\tau \tag{12}$$

with the properties

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_l[w_\mu](t) = \Phi_{l-1}[w_\mu](t) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\Phi_0[w_\mu](t) = w_\mu(t) \quad , \tag{13}$$

and equation (4) becomes

$$y(t) = \sum_{\omega=1}^{n} \underbrace{\sum_{l=0}^{\omega-1} \sum_{\mu=1}^{\omega-l} \Phi_l \left[A^l_{\omega\mu} w_\mu \right](t)\, C_\omega}_{:=\varphi_\omega(t)} \quad . \tag{14}$$

Case: C is regular Assume the columns of C, $\{C_1, C_2, \dots, C_n\}$ form a linearly independent set of $\mathbb{R}^p$ vectors. Setting $y \equiv 0$ and equating coefficients we get

$$\propto C_1 : \qquad \varphi_1 = \Phi_0[\underbrace{A^0_{11}}_{=1} w_1] \equiv 0 \qquad \Rightarrow \qquad w_1 \equiv 0 \tag{15}$$

$$\propto C_2 : \qquad \varphi_2 = \Phi_0[\underbrace{A^0_{22}}_{=1} w_2] + \Phi_1[A^1_{21} \underbrace{w_1}_{\equiv 0}] \equiv 0 \qquad \Rightarrow \qquad w_2 \equiv 0 \tag{16}$$

etc.

which shows, that such a system is hidden input observable.

Example 1. Consider the system

$$A = \begin{pmatrix} 2 & -1 & -1 & 0 \\ 2 & -1 & -1 & 1/2 \\ 1 & 0 & -1/2 & -3/4 \\ -2 & 2 & 1 & -1/2 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix} \quad . \tag{17}$$

With the matrix

$$P = \begin{pmatrix} 1 & -1/2 & -1/2 & 1/4 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/2 \\ 0 & 0 & 1/2 & -1/4 \end{pmatrix} \tag{18}$$

we transform the system to

$$A_\triangle = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{pmatrix} \quad \hat{C} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 3 & 1 & 3 \\ 0 & 2 & 3 & 2 \\ 0 & 2 & 1 & 2 \\ 1 & 1 & 3 & 3 \end{pmatrix} \quad , \tag{19}$$

to see that this system is hidden input observable.

Remark.
- To have a set $\{C_1, \dots, C_n\}$ of linear independent vectors, its crucial to have $n \leq p$.
- A nilpotent matrix has only one Eigenvalue 0. Thus also $\operatorname{tr} A = 0$ and $\det A = 0$ are necessary conditions.

C_Z is a linear combination