Contents

1 Hidden Input Observability

We consider a mapping $L^2([0,T])^{\otimes m} \to L^2([0,T])^{\otimes p}$ defined by

$$y(t) := C \int_0^t \mathrm{e}^{A(t-\tau)} D w(\tau)\,\mathrm{d}\tau \tag{1}$$

where $w : [0,T] \to \mathbb{R}^m$, $y : [0,T] \to \mathbb{R}^p$, $A \in \mathbb{R}^{n\times n}$, $D \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$.

We assume that in an biological system the functions w and y are defined on a natural interval of time that covers $[0,T]$. This means that for a small ϵ the model could be extended to an interval $(0-\epsilon, T+\epsilon)$ and thus differentiation of w and y at $t=0$ and $t=T$ makes sense. To denote this idea without introducing ϵ we use the notation $[0^-, T^+]$.

We firstly rewrite the problem by expanding (1),

$$y(t) = \sum_{k=0}^{\infty} CA^k D \int_0^t \frac{(t-\tau)^k}{k!} w(\tau)\,\mathrm{d}\tau \quad . \tag{2}$$

Writing $(CA^kD)_{.\mu}$ means the μ-th column and w_μ the μ-th component. We get

$$y(t) = \sum_{k=0}^{\infty} \sum_{\mu=1}^{m} \int_0^t \frac{(t-\tau)^k}{k!} w_\mu(\tau)\,\mathrm{d}\tau (CA^kD)_{.\mu} \tag{3}$$

and if we only consider the ν-th component of y we get

$$y_\nu(t) = \sum_{k=0}^{\infty} \sum_{\mu=1}^{m} \int_0^t \frac{(t-\tau)^k}{k!} w_\mu(\tau)\,\mathrm{d}\tau (CA^kD)_{\nu\mu} \quad . \tag{4}$$

We introduce the coefficients $c_k^{\mu\nu} := (CA^kD)_{\nu\mu}$ to get

$$y_\nu(t) = \sum_{\mu=1}^{m} \sum_{k=0}^{\infty} c_k^{\mu\nu} \int_0^t \frac{(t-\tau)^k}{k!} w_\mu(\tau)\,\mathrm{d}\tau \quad \forall \nu = 1,2,\ldots,p \quad . \tag{5}$$

Here, it is legit to commute the sums, since the infinite sum over k is absolutely convergent.

1.1 Basic Properties

It is now convenient to introduce some operators.

Definition 1

Let $v \in L^2$ be a function and $(c_k)_{k\in\mathbb{N}_0}$. For the sake of readability we write $c = (c_k)_{k\in\mathbb{N}_0}$ for sequences.

1. $\phi_k[v](t) := \int_0^t \frac{(t-\tau)^k}{k!} v(\tau)\,\mathrm{d}\tau$
2. $\Phi_c[v] := \sum_{k=0}^{\infty} c_k \phi_k[v]$

We always exclude the sequence $c_k = 0 \forall k$, since this obviously produces the operator that maps everything to the zero function and thus cannot be injective.

We rewrite (5) as

$$y_\nu(t) = \sum_{\mu=1}^{m} \Phi_{c^{\mu\nu}}[w_\mu](t) \quad \forall \nu = 1, 2, \ldots, p \tag{6}$$

and discuss some properties.

Lemma 1: Properties of ϕ_k

1. $\phi_k[v](0) = 0$
2. $\frac{\mathrm{d}}{\mathrm{d}t}\phi_0[v](t) = v(t)$ and $\frac{\mathrm{d}}{\mathrm{d}t}\phi_k[v](t) = \phi_{k-1}[v](t)$

Proof. The first property is trivial.
Computing $\phi_k[v](t+\Delta t)$ yields

$$\phi_k[v](t+\Delta t) = \int_0^t \frac{(t-\tau+\Delta t)^k}{k!} v(\tau)\,\mathrm{d}\tau + \int_t^{t+\Delta t} \frac{(t-\tau+\Delta t)^k}{k!} v(\tau)\,\mathrm{d}\tau \tag{7}$$

and using $(t+\Delta t)^k = t^k + k t^{k-1}\Delta t + \mathcal{O}(\Delta t^2)$ for $k > 0$ and $(t+\Delta t)^0 = 1$ and a small Δt leads to

$$\phi_k[v](t+\Delta t) \simeq \int_0^t \frac{(t-\tau)^k}{k!} v(t)\,\mathrm{d}\tau + \Delta t \int_0^t \frac{(t-\tau)^{k-1}}{(k-1)!} v(t)\,\mathrm{d}\tau \tag{8}$$

if $k > 0$ and to

$$\int_0^t \frac{(t-\tau)^0}{k!} v(t)\,\mathrm{d}\tau + \Delta t\, v(t) \tag{9}$$

if $k = 0$. Comparing this with the definitions of the operators we get $\phi_k[v](t+\Delta t) \simeq \phi_k(t)+\Delta t\phi_{k-1}[v](t)$ and $\phi_0[v](t+\Delta t) \simeq \phi_0[v](t)+\Delta t v(t)$. Taking the limit $\Delta t \to 0$ ends the proof of the derivation rules.

□

1.2 SISO-System

We start with a *Single Input Single Output (SISO)*-System. That means (6) simplifies to

$$y(t) = \Phi_c[v](t) \tag{10}$$

with a real-valued y and v and a scalar $c_k = (CA^k D)$.

Proposition 1

Let v be an integrable function on $[0^-, T^+]$ and let $\Phi_c[v] \equiv 0$. Then

$$\sum_{l=0}^{q} c_l v^{(q-l)}(0) = 0 \quad \forall q \in \mathbb{N}_0 \tag{$\star$}$$

where $v^{(q)}(0)$ denotes the q-th derivative of v at $t = 0$.

Proof. Consider

$$\Phi_c[v] = c_0\phi_0[v] + c_1\phi_1[v] + \ldots + c_{q-1}\phi_{q-1}[v] + c_q\phi_q[v] + c_{q+1}\phi_{q+1}[v] + \ldots \tag{11}$$

and the $(q+1)$-th derivative

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{q+1} \Phi_c[v] = c_0 v^{(q)} + c_1 v^{(q-1)} + \ldots + c_{q-1} v^{(1)} + c_q v^{(0)} + c_{q+1}\phi_0[v] + \ldots \tag{12}$$

for short

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{q+1} \Phi_c[v] = \sum_{l=0}^{q} c_l v^{(q-l)} + \sum_{l=0}^{\infty} c_{q+1+l}\phi_l[v] \quad . \tag{13}$$

Evaluating the latter expression at $t = 0$ completes the proof. □

To illustrate the idea of the following lemma and theorem we write down some instances of $(\star)$:

$$
\begin{array}{lccccccccc}
q=0 & c_0 v^{(0)} & & & & & & & & \\
q=1 & c_0 v^{(1)} & + & c_1 v^{(0)} & & & & & & \\
q=2 & c_0 v^{(2)} & + & c_1 v^{(1)} & + & c_2 v^{(0)} & & & & \\
q=3 & c_0 v^{(3)} & + & c_1 v^{(2)} & + & c_2 v^{(1)} & + & c_3 v^{(0)} & & \\
q=4 & c_0 v^{(4)} & + & c_1 v^{(3)} & + & c_2 v^{(2)} & + & c_3 v^{(1)} & + & c_4 v^{(0)} \\
\vdots & & \ddots & & \ddots & & \ddots & & \ddots & & \ddots
\end{array}
$$

For a better readability each $v^{(q)}$ is understood to be evaluated at $t=0$. The structure of this triangle remains the same if $c_0 = 0$, i.e. the first column vanishes, and if $v^{(0)} = 0$, i.e. the diagonal at the top vanishes.

Lemma 2: Induction Step

Assume proposition 1 holds and let c_K be the first nonzero coefficient. If there is a $r \in \mathbb{N}_0$ with $v^{(0)}(0) = v^{(1)}(0) = \ldots = v^{(r-1)}(0) = 0$ then $v^{(r)}(0) = 0$.

Proof. Using $(\star)$ with $q = r + K$ yields

$$\sum_{l=0}^{r+K} c_l v^{(r+K-l)}(0) = c_K v^{(r)}(0) = 0 \tag{14}$$

since all other terms of the sum vanish due to $c_l = 0$ or $v^{(l)} = 0$. This shows that also $v^{(r)}(0) = 0$. □

Theorem 1

Let $v \in L^2([0,T]) \cap C^\infty(0^-, 0^+)$ and c a sequence. Then

$$\Phi_c[v] \equiv 0 \quad \Rightarrow \quad v^{(q)}(0) = 0 \quad \forall q \in \mathbb{N}_0 \quad . \tag{15}$$

Proof. Let c_K be the first nonzero coefficient. Using $(\star)$ with $q = K$ yields

$$c_K v^{(0)} = 0 \tag{16}$$

which shows that $v^{(0)} = 0$. Using lemma 2 completes the inductive proof. □

Corollary 1

If v can be represented by its Taylor-expansion, then $\Phi_c[v] \equiv 0$ implies $v \equiv 0$.

If there is a disjoint union $[0^-, T^+] = I_1 \dot\cap I_2 \dot\cap \dots$ such that v has a valid Taylor-expansion on each interval $I_j = (t_{j-1}^-, t_j^+)$, we can argue that $v \equiv 0$ on I_1. This leads to $\phi_k[v](t_1) = 0$ which allows us to get a modification of $(\star)$, written out

$$\sum_{l=0}^{q} c_l v^{(q-l)}(t_1) = 0 \quad \forall q \in \mathbb{N}_0 \quad . \tag{17}$$

Formally this can again be handled as an inductive proof to show, that v must vanish on each interval.

A SISO-system is HIO if and only if there is an $k \in \mathbb{N}_0$ such that $CA^kD \neq 0$.

1.3 MIMO-Systems

As equation (6) shows, we usually do not have a simple $y(t) = \Phi_c[v](t)$ relation but a sum with different sequences $c^{\mu\nu}$ and functions w_μ. Since summation and differentiation are linear operations, we directly get the following extension of proposition 1.

Proposition 2

Let w_μ integrable on $[0^-, T^+]$ for $\mu = 1, 2, \dots, m$ and let $c^{\mu\nu}$ be sequences. For each $\nu = 1, 2, \dots, p$

$$\sum_{\mu=1}^{m} \Phi_{c^{\mu\nu}}[w_\mu] \equiv 0 \quad \Rightarrow \quad \sum_{\mu=1}^{m} \sum_{l=0}^{q} c_l^{\mu\nu} w_\mu^{(q-l)}(0) = 0 \quad \forall q \in \mathbb{N}_0 \tag{$\star\star$}$$

Proof. The proof works analogous to that of proposition 1. □

Whereas theorem 1 holds for any sequence c and function v, proposition 2 allows cancellation of different functions w_μ. We shortly demonstrate, that a fully observed system will always be hidden input observable. Considering a fully observed system with possible hidden inputs on each state, i.e. $C = D = \mathbb{1}$ and $p = n = m$, we directly get $c_0^{\mu\nu} = \delta_{\mu\nu}$. Inserting this into $(\star\star)$ with $q = 0$ yields

$$w_\nu^{(0)} = 0 \quad \text{for} \quad \nu = 1, 2, \dots, n \tag{18}$$

Following the idea of lemma 2 we proceed with an induction step to get

$$w_\nu^{(q)} = 0 \quad \forall q \in \mathbb{N}_0 \quad \text{for} \quad \nu = 1, 2, \ldots, n \quad . \tag{19}$$

As argued in corollary 1, if we assume that each w_μ can be represented by a Taylor series, we know, that this system is hidden input observable.

To generalize the idea of an fully observed system we find the following lemma in analogy to lemma 2.

Lemma 3: Induction Step

Assume proposition 2 holds and assume for an integer K that $CD = CAD = \ldots = CA^{K-1}D = 0$ and $CA^K D$ has rank m. If there is an $r \in \mathbb{N}$ such that $w_\mu^{(0)}(0) = \ldots = w_\mu^{(r-1)}(0) = 0$ for all $\mu = 1, 2, \ldots, m$, then $w_\mu^{(r)}(0) = 0$ for all μ.

Proof. Using $(\star\star)$ with $q = r + K$ yields

$$\sum_{\mu=1}^{m} c_K^{\mu\nu} w_\mu^{(r)}(0) = 0 \tag{20}$$

which is by definition of $c_K^{\mu\nu}$ equivalent to the linear equation

$$CA^K D w^{(r)}(0) = 0 \tag{21}$$

where $w^{(r)}(0)$ is the vector $(w_1^{(r)}(0), \ldots, w_m^{(r)}(0))$. If and only if $\operatorname{rank} CA^K D = m$ this implies $w^{(r)}(0) = 0 \in \mathbb{R}^m$. □

Theorem 2: Observed Hidden Inputs

Let $w \in L^2([0,T])^{\otimes m} \cap C^\infty(0^-, 0^+)^{\otimes m}$ and (A, C, D) the matrices of (1) and assume that $CA^K D$ is the first nonvanishing coefficient matrix.
If $\operatorname{rank} CA^K D = m$, then

$$y \equiv 0 \quad \Rightarrow \quad w_\mu^{(q)}(0) = 0 \quad \forall q \in \mathbb{N}_0 \quad . \tag{22}$$

Proof. Due to the assumptions we can use proposition 2. Equation $(\star\star)$ with $q = K$ yields

$$\sum_{\mu=1}^{m} c_K^{\mu\nu} w_\mu^{(0)}(0) = \sum_{\mu=1}^{m} \left(CA^K D\right)_{\nu\mu} w_\mu^{(0)}(0) = 0 \quad \forall \nu \in \{1, 2, \ldots, p\} \tag{23}$$

which implies $w_\mu^{(0)}(0) = 0 \forall \mu$, since $(CA^K D) : \mathbb{R}^m \to \mathbb{R}^p$ is injective. This allows us to use lemma 3 as an induction step to complete the proof. □

We complete this section with the last corollary in analogy to corollary 1 and some examples.

Corollary 2

If there are a disjoint unions $[0^-, T^+] = I_1^\mu \dot{\cap} I_2^\mu \dot{\cap} \ldots$ such that each w_μ such that w_μ can be represented by its Taylor-expansion on each interval $I_j^\mu = ((t_{j-1}^\mu)^-, (t_j^\mu)^)$ and if the first nonvanishing CA^kD has rank m, then the MIMO-system is HIO.

Example 1

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad D = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix} \tag{24}$$

A SISO-system and $CAD = 1$ thus it is HIO.

Example 2

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad D = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix} \tag{25}$$

A SISO-system and $CA^kD = 0$ for any integer k thus it is not HIO.

Example 3

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{26}$$

A MIMO-system and CD is the 2×2 unity matrix. Thus it is HIO.

2 Zero Dynamics

We follow the ideas of [**\IeC {\textbullet }**]. The zero dynamics of a linear system (A, B, C) is a set of triplets $(x, u, y) : [0, T] \to \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p$ that solve the system equations and $y \equiv 0$. Assuming we have found such an triplet $(x^*, u^*, 0)$, due to linearity we can perturb any other triplet (x, u, y) that solves the system equations since $(x + x^*, u + u^*, y)$ again solves the systems equations and produces the same output y as the unperturbed system.

Definition 2

A subset V of $\mathbb{R}^n$ is called (A, B) invariant if

$$AV \subseteq V + \operatorname{Im} B \quad . \tag{27}$$

For the largest (A, B) invariant V we write

$$\mathfrak{V} = V \cap \operatorname{kernel} C \quad . \tag{28}$$

Proposition 3

A trajectory $x : [0, T] \to \mathfrak{V}$ that solves the system equations is part of the zero dynamics of the corresponding system.

Proof. Assume $x(0) \in \mathfrak{V}$. We define recursively

$$\begin{aligned} Ax(0) &= \gamma_0 + \lambda_0 \\ A\gamma_i &= \gamma_{i+1} + \lambda_{i+1} \end{aligned}$$

where each $\gamma_i \in \mathfrak{V}$ and each $\lambda_i \in \operatorname{Im} B$. By the definition of $\mathfrak{V}$ we know that this is possible. We do a short inductive proof to show

$$A^k(\gamma_0 + \lambda_0) = \gamma_k + \sum_{l=0}^{k} A^l \lambda_{k-l} \quad . \tag{29}$$

1. $k = 0$ is trivial.

2. Assume the statement is correct.

$$\begin{aligned}A^{k+1}(\gamma_0+\lambda_0) &= A A^k(\gamma_0+\lambda_0) = A\left(\gamma_k+\sum_{l=0}^{k} A^l\lambda_{k-l}\right)\\ &= \gamma_{k+1}+\lambda_{k+1}+\sum_{l=0}^{k} A^{l+1}\lambda_{k-l} = \gamma_{k+1}+\lambda_{k+1}+\sum_{l=1}^{k+1} A^l\lambda_{k-(l-1)}\\ &= \gamma_{k+1}+\sum_{l=0}^{k+1} A^l\lambda_{k+1-l} \quad \text{q.e.d.}\end{aligned}$$

Furthermore we know that

$$x(t) = \mathrm{e}^{At}x(0)+\int_0^t \mathrm{e}^{A(t-\tau)}Bw(\tau)\,\mathrm{d}\tau \quad . \tag{30}$$

Expanding the exponentials and shifting the index in the $x(0)$-term yields

$$x(t) = x(0)+\sum_{k=0}^{\infty} A^k\left(\frac{t^{k+1}}{(k+1)!}(\gamma_0+\lambda_0)+\int_0^t \frac{(t-\tau)^k}{k!}Bw(\tau)\,\mathrm{d}\tau\right) \tag{31}$$

and using the letter statement

$$x(t) = x(0)+\sum_{k=0}^{\infty}\left\{\frac{t^{k+1}}{(k+1)!}\left(\gamma_k+\sum_{l=0}^{k} A^l\lambda_{k-l}\right)+A^k\int_0^t \frac{(t-\tau)^k}{k!}Bw(\tau)\,\mathrm{d}\tau\right\} \quad . \tag{32}$$

Applying C leads to

$$y(t) = C\sum_{k=0}^{\infty}\left\{\frac{t^{k+1}}{(k+1)!}\sum_{l=0}^{k} A^l\lambda_{k-l}+A^k\int_0^t \frac{(t-\tau)^k}{k!}Bw(\tau)\,\mathrm{d}\tau\right\} \tag{33}$$

At order $CA^{k'}$ we have

$$\sum_{r=0}^{\infty}\frac{t^{k'+1+r}}{(k'+1+r)!}\lambda_r+\int_0^t \frac{(t-\tau)^{k'}}{k'!}Bw(\tau)\,\mathrm{d}\tau \quad . \tag{34}$$

Since each λ_r is element of $\operatorname{Im} B$ we may find a w that suppresses each term in the summation separately. Without proof we assume that this is always possible.
As a special case we consider $\operatorname{Im} B \subseteq \operatorname{kernel} C$, which makes the problem trivial. □

2.1 Zero Dynamics vs. HIO

In the following example we see, that HIO and zero dynamics are closely related but not equivalent.

Example 4

Consider a dynamic system with the matrices

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad . \tag{35}$$

For $\mathfrak{V} := \operatorname{span}\{(0,0,1)^T\}$ we find $A\mathfrak{V} = \operatorname{span}\{(1,0,0)^T\}$ is a subset of $\operatorname{Im} B$ and thus $\mathfrak{V}$ is (A,B)-invariant.
For instance the set

$$x^*(0) = \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix} \quad \text{and} \quad w^*(t) = \begin{pmatrix} -a \\ 0 \end{pmatrix} \tag{36}$$

leads to a constant

$$x^*(t) = \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix} \tag{37}$$

which produces zero output $y = Cx^*$ with nonvanishing hidden inputs w^*. This shows that the zero dynamics of the system is not trivial.
At the same time we find that CB is injective which means the system is HIO and thus zero output should imply zero hidden intput.

As illustrated by the example, HIO does not imply trivial zero dynamics. The reason for that is, HIO is independent of the initial value $x^*(0)$, whereas the hidden input w^* of the zero dynamics may only work for specific initial values $x^*(0)$. If, for instance, in the above example $x^*(0)$ would not have zeroes in the first two components, it is not possible to find a nontrivial hidden input.
Consider the following example

Example 5

A dynamic system with the matrices

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad . \tag{38}$$

We find

$$\operatorname{Im} B = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \quad \text{and} \quad \operatorname{kernel} C = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\} \quad . \tag{39}$$

We notice

$$A \operatorname{kernel} C \subseteq \operatorname{Im} B \tag{40}$$

and therefore clearly see that $\operatorname{kernel} C$ is (A, B)-invariant.
The system is not HIO for CB has rank 1.

As one can see in the above example, as soon as the image of A restricted to the kernel of C is a subset of the image of B is nonempty, the zero dynamics are nontrivial. In the following proposition we show that for not HIO systems this will always be the case.

Proposition 4: HIO is necessary for Trivial Zero Dynamics

For a linear system of (A, B, C)

$$\text{trivial zero dynamics} \quad \Rightarrow \quad \text{HIO} \tag{41}$$

or equivalently

$$\mathfrak{V} = \{0\} \quad \Rightarrow \quad CA^k B \quad \text{injective} \tag{42}$$

for the first nonvanishing $CA^k B$.

Proof. We show that ¬HIO leads to nontrivial zero dynamics.
Assume for simplicity that $CB \neq 0$ and B injective.

1. B cannot be 0. Thus $\operatorname{Im} B \supset \{0\}$ while $\operatorname{kernel} B = \{0\}$.

2. Since $\operatorname{kernel} CB \supset \{0\}$ we know that $\operatorname{Im} B \cap \operatorname{kernel} C \supset \{0\}$.
   Therefore $\operatorname{kernel} C \supset \{0\}$.

3. If A has full rank, then $\dim \operatorname{kernel} CA \;=\; \dim \operatorname{kernel} C \;> 0$.
   If A has rank less than n, A is not injective hence CA is not injective.
   In either case $\operatorname{kernel} CA \supset \{0\}$.

If we now choose

$$i \tag{43}$$

□