1 Hidden Input Observability

Consider a dynamic system $\mathscr{S}$

$$\frac{\mathrm{d}x_w}{\mathrm{d}t} = Ax_w(t) + Bu(t) + Dw(t) \tag{1}$$

$$y_w(t) = Cx_w(t) \tag{2}$$

$$x_w(0) = x_0 \tag{3}$$

where x_w, u, w and y_w map $[0,T]$ onto $\mathbb{R}^n$, $\mathbb{R}^{\hat{m}}$, $\mathbb{R}^m$ and $\mathbb{R}^p$, respectively, and A, B, C, D are matrices of suitable dimensions. We assume the function u is known and called the *known input*, the function w is unknown called *hidden input*. The closed form solution for y_w is

$$y_w(t) = C\int_0^t \exp(A(t-\tau))(Bu(\tau) + Dw(\tau))\,\mathrm{d}\tau \quad . \tag{4}$$

Definition 1. If for $\mathscr{S}$ the implication

$$y_w(t) = y_{\hat{w}}(t) \quad \forall\, t \in [0,T] \quad \Rightarrow \quad w = \hat{w} \quad \text{a.e.} \tag{5}$$

holds, $\mathscr{S}$ is called *hidden input observable (HIO)*. If this implication holds only for $m - m'$ components of w, $\mathscr{S}$ is called *limited hidden input observable by m'*.

Our aim is to find necessary or sufficient conditions for the hidden input observability of linear systems.

Due to linearity, $\mathscr{S}$ is HIO if and only if

$$y(t) := C\int_0^t \exp(A(t-\tau))Dw(\tau)\,\mathrm{d}\tau = 0 \quad \forall\, t \in [0,T] \quad \Rightarrow \quad w = 0 \quad \text{a.e.} \tag{6}$$

Rearranging the equation

By the Cayley-Hamilton theorem, for any $k \in \mathbb{N}_0$ there are coefficients $c_{k,l}$ such that

$$A^k = \sum_{l=0}^{n-1} c_{k,l} A^l \quad . \tag{7}$$

Defying

$$\Phi_l[w](t) := \int_0^t \sum_{k=0}^{\infty} c_{k,l} \frac{(t-\tau)^k}{k!} w(\tau)\,\mathrm{d}\tau \tag{8}$$

equation (6) can be written as

$$y(t) = \sum_{l=0}^{n-1} CA^l D \Phi_l[w](t) \quad . \tag{9}$$

1.1 Sufficient Condition

In most cases $D = \mathbb{1}$ is an appropriate choice, thus $m = n$ and the μ-th column of CA^k can be written as

$$\left(CA^k\right)_\mu = \sum_{\omega=1}^{n} A^k_{\omega\mu} C_\omega \tag{10}$$

where $A^k_{\omega\mu}$ is the $(\omega\mu)$ component of A^k and C_ω is the ω-th column of C. Now we can write

$$y(t) = \sum_{\omega=1}^{n} \varphi_\omega(t) C_\omega \tag{11}$$

where

$$\varphi_\omega(t) := \sum_{l=0}^{n-1} \sum_{\mu=1}^{n} \Phi_l \left[A^l_{\omega\mu} w_\mu\right](t) \quad . \tag{12}$$

Now choose an index set $\mathscr{I} \subset \{1,2,\dots,n\}$ such that $\{C_i | i \in \mathscr{I}\}$ are linearly independent and for any $H \in \mathscr{I}^c := \{1,2,\dots,n\} \setminus \mathscr{I}$ there are unique coefficients Λ^H_i such that $C_H = \sum_{i \in \mathscr{I}} \Lambda^H_i C_i$. Furthermore introduce the index sets $\mathscr{H}_i$ such that $H \in \mathscr{H}_i \Leftrightarrow \Lambda^H_i = 0$ and $\mathscr{H}^c_i := \mathscr{I}^c \setminus \mathscr{H}_i$. With this, (11) becomes

$$y(t) = \sum_{i \in \mathscr{I}} \left(\varphi_i(t) + \sum_{H \in \mathscr{H}^c_i} \Lambda^H_i \varphi_H(t) \right) C_i \quad . \tag{13}$$

To get a condition for HIO, let us set $y \equiv 0$ and by equation coefficients

$$\varphi_i + \sum_{H \in \mathscr{H}^c_i} \varphi_H \equiv 0 \quad \forall i \in \mathscr{I} \quad . \tag{14}$$

Proposition 1 (Without proof)**.** Each operator Φ_l is injective, i.e.

$$\Phi_l[w] \equiv 0 \qquad \Rightarrow \qquad w \equiv 0 \tag{15}$$

and Φ_0 is surjective. Here "≡" denotes equality to the zero function and Φ_l operates component-wise on $(w_1, w_2, \dots, w_m)^{\mathrm{T}} : [0,T] \to \mathbb{R}^m$.

Definition 2. Let $\mathscr{L}$ be an index set. A set

$$\left\{ \quad \Phi_l : L^2([0,T]) \to L^2([0,T]) \quad | \quad l \in \mathscr{L} \quad \right\} \tag{16}$$

of linear operators is called *injective set*, if for any functions $\{v_l \in L^2([0,T]) | l \in \mathscr{L}\}$ the implication

$$\sum_{l\in\mathscr{L}} \Phi_l\,[v_l] \equiv 0 \quad \Rightarrow \quad v_l \equiv 0 \forall\, l \in \mathscr{L} \tag{17}$$

holds.

Proposition 2. If $\{\Phi_l | l \in \{0,1,\ldots,n-1\}\}$ defined by (8) is a injective set and if the functions $\{w_\mu\}$ are linearly independent, then:

$$\begin{aligned}
&\text{If for a} \quad \mu \in \{1,2,\ldots,n\} \quad \exists \quad (i,l) \in \mathscr{I} \times \{0,1,\ldots,n-1\} \\
&\text{such that} \quad A^l_{i\mu} + \sum_{H\in\mathscr{H}^c_i} \Lambda^H_i A^l_{H\mu} \neq 0 \\
&\text{then} \quad w_\mu \equiv 0 \quad .
\end{aligned} \tag{18}$$

Proof. Starting with (14) we have for all $i \in \mathscr{I}$

$$\sum_{l=0}^{n-1} \sum_{\mu=1}^{n} \Phi_l \left[\left(A^l_{i\mu} + \sum_{H\in\mathscr{H}^c_i} \Lambda^H_i A^l_{H\mu} \right) w_\mu \right] \equiv 0 \tag{19}$$

and by the definition of an injective set, we get for all $i \in \mathscr{I}$

$$\sum_{\mu=1}^{n} \left(A^l_{i\mu} + \sum_{H\in\mathscr{H}^c_i} \Lambda^H_i A^l_{H\mu} \right) w_\mu \equiv 0 \tag{20}$$

and since $\{w_\mu\}$ is a linearly independent set we can treat each μ separately, hence each function w_μ must vanish at all times if

$$A^l_{i\mu} + \sum_{H\in\mathscr{H}^c_i} \Lambda^H_i A^l_{H\mu} \neq 0 \quad . \tag{21}$$

Therefore it is sufficient to find one pair $(i,l) \in \mathscr{I} \times \{0,1,\ldots,n-1\}$ for which this coefficient is not zero to argue, that w_μ must be zero at all times. □

Theorem 1. Let $\{\Phi_l | l \in \{0,1,\ldots,n-1\}\}$ defined by (8) be an injective set and $\{w_\mu | \mu \in \{1,2,\ldots,n\}\}$ linearly independent functions. If

$$\forall \mu \in \mathscr{M} \exists (i,l) \in \mathscr{I} \times \{0,1,\ldots,n-1\} \big|\, A^l_{i\mu} + \sum_{H\in\mathscr{H}^c_i} \Lambda^H_i A^l_{H\mu} \neq 0 \tag{22}$$

then the system is limited HIO by $n - |\mathscr{M}|$. If $|\mathscr{M}| = n$ then the system is HIO.

Proof. Using the preceding proposition the proof is trivial. □

1.1.1 Nilpotent Dynamics

Let A be a nilpotent matrix, i.e. there is a regular $n \times n$ matrix P such that

$$A = P^{-1} A_{\triangle} P \tag{23}$$

with $A_{\triangle\omega\mu} = 0$ when $\omega \leq \mu$. As a graphical condition this means, that A can be represented by a directed acyclic graph. This yields

$$y(t) = \sum_{l=0}^{n-1} \underbrace{CP^{-1}}_{\operatorname{rank} CP^{-1} = \operatorname{rank} C} A_{\triangle}^{l} \Phi_l[\underbrace{Pw}_{\text{bijection}}](t) \quad . \tag{24}$$

Thus without loss of generality we can assume that A is strictly lower triangular. Furthermore we see that (8) reduces to

$$\Phi_l[w_\mu](t) = \int_0^t \frac{(t-\tau)^l}{l!} w_\mu(\tau)\,\mathrm{d}\tau \quad . \tag{25}$$

Lemma 1 (Without proof)**.** The operators defined by (25) have the properties

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_l[w_\mu](t) = \Phi_{l-1}[w_\mu](t) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\Phi_0[w_\mu](t) = w_\mu(t) \quad . \tag{26}$$

Proposition 3. The operators $\{\Phi_l | l \in \{0, 1, \ldots, n-1\}\}$ from a nilpotent matrix form an injective set.

Proof. Let $\{v_l\}$ be a set of functions with $l \in \mathscr{L} = \{0, 1, \ldots, n-1\}$. Set

$$\sum_{l\in\mathscr{L}} \Phi_l[v_l] \equiv 0 \quad . \tag{27}$$

Writing this as integral equation

$$\int_0^t \sum_{l\in\mathscr{L}} \frac{(t-\tau)^l}{l!} v_l(\tau)\,\mathrm{d}\tau = 0 \quad \forall t \in [0, T] \tag{28}$$

which means

$$\sum_{l\in\mathscr{L}} \frac{(t-\tau)^l}{l!} v_l(\tau) = 0 \quad \forall (t, \tau) \in [0, T] \times [0, t] \quad . \tag{29}$$

Now let $l_{\min}$ be the smallest l in $\mathscr{L}$. This leads to

$$\frac{1}{l_{\min}!} v_l(\tau) = -\sum_{l_{\min} < l \in \mathscr{L}} \frac{(t-\tau)^{l-l_{\min}}}{l!} v_l(\tau) \quad . \tag{30}$$

Since the left hand side of this equation is independent from t, so must the right hand side. Evaluating the derivatives with respect to t leads to

$$v_l(\tau) = 0 \quad \forall \tau \in [0, T] \tag{31}$$

for all $l \in \mathscr{L}$ separately. This means $\{I_l\}$ is an injective set. □

References

[1] J. et al. Gao. “Target control of complex networks”. In: *Nat. Commun.* (2014).

[2] David G. Luenberg. *Introduction to Dynamic Systems*. 1979.