1 Gradient Method - Preliminaries

Let $x : [0,T] \to \mathbb{R}^n$, $w : [0,T] \to \mathbb{R}^n$, $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $h : \mathbb{R}^n \to \mathbb{R}^p$ and

$$\dot{x}(t) = f(x(t)) + w(t) \tag{1}$$
$$y(t) = h(x) \tag{2}$$
$$x(0) = x_0 \tag{3}$$

a dynamic system of the DEN. Let $y : [0,T] \to \mathbb{R}^p$ a data set with continuous time. The cost function which is to be minimized is

$$J[x,w,t] = \int_0^T \left\{ \sum_{\mu=1}^{p} \left| y_\mu(t) - h_\mu(x(t)) \right|^2 + \sum_{\nu=1}^{n} \frac{\alpha_2}{2} |w_\nu(t)|^2 \right\} \mathrm{d}t \quad . \tag{4}$$

We can solve this problem using the Hamilton formalism with the Hamiltonian

$$H(x,w,\lambda,t) = \sum_{\mu=1}^{p} \left| y_\mu(t) - h_\mu(x) \right|^2 + \sum_{\nu=1}^{n} \frac{\alpha_2}{2} |w_\nu|^2 + \sum_{\rho=1}^{n} \lambda_\rho (f_\rho(x) + w_\rho) \tag{5}$$

and the canonical equations

$$\nabla_x H|_{x(t),w(t),\lambda(t)}(t) = -\dot{\lambda}(t) \tag{6}$$
$$\nabla_\lambda H|_{x(t),w(t),\lambda(t)}(t) = \dot{x}(t) \tag{7}$$
$$\nabla_w H|_{x(t),w(t),\lambda(t)}(t) = 0 \tag{8}$$

where (6) and (7) ensure the right dynamics and (8) ensures that the solution is extremal. The derivatives are always evaluated at a point $x(t), w(t), \lambda(t)$ so we drop the subscripts. The ∇-derivation maps $\nabla_x : h_\mu(x) \mapsto (\partial/\partial x_1 h_\mu, \ldots, \partial/\partial x_n h_\mu(x))^T$. When deriving the canonical equation you also get $\lambda_\rho(T) = 0$ if there is no terminal cost and no boundary condition on $x(T)$.

Remark. Note that the Hamiltonian is explicitly time dependent via $y(t)$, i.e.

$$\frac{\mathrm{d}H}{\mathrm{d}t}(t) = \frac{\partial H}{\partial t}(t) = \dot{y}(t) \quad . \tag{9}$$

We calculate some derivatives

$$\frac{\partial (w_\nu)^2}{\partial w_\kappa} = \delta_{\nu\kappa} \alpha_2 w_\kappa \quad \text{thus} \quad \nabla_w H(t) = \alpha_2 w(t) + \lambda(t) \tag{10}$$

where $\delta_{\nu\kappa}$ is the Kronecker-delta. Furthermore

$$\frac{\partial (h_\mu(t))^2}{\partial x_\kappa} = 2 \frac{\partial h_\mu(t)}{\partial x_\kappa} h_\mu(x) \quad , \quad \nabla_x (h_\mu(x))^2 = 2 h_\mu(x) \nabla_x h_\mu(x) \tag{11}$$

and the define the Jacobian

$$\mathrm{d}h_x = \begin{pmatrix} \nabla_x h_1(x)^T \\ \vdots \\ \nabla_x h_p(x)^T \end{pmatrix} \quad \text{to get} \quad \nabla_x \sum_{\mu=1}^{p} (h_\mu(x))^2 = 2\mathrm{d}h_x^T h(x) \tag{12}$$

Combining (10) and (8) yields

$$\alpha_2 w(t) + \lambda(t) = 0 \quad \forall t \in [0, T] \tag{13}$$

and since $\lambda(T) = 0$ we get $w(T) = 0$. This may be the best solution in the context of optimal control but seems to be very restrictive if we want to identify the hidden inputs w with model uncertainties.

One possible reason could be that Hamilton formalism of this problem yields a singular problem, i.e. without the α_2 regularisation it would not be possible to solve the problem at all. By adding a convex function $\alpha_2/2|w|^2$ the cost function becomes locally convex, in a vicinity of $w = 0$.

However it would be highly desirable to consider hidden inputs with $w(T) \neq 0$.

1.1 Terminal Cost

One way to get $\lambda(T) \neq 0$ is to introduce a terminal cost

$$V(x) = \Lambda \left| y(T) - h(x) \right|^2 \tag{14}$$

with a new regularisation parameter $\Lambda > 0$. The cost function becomes

$$\bar{J}[x, w, t] = J[x, w, t] + V(x(T)) \tag{15}$$

and the boundary value

$$\lambda(T) = \nabla_x V(T) \quad \text{i.e.} \quad \lambda(T) = -2\Lambda \mathrm{d}h_x^T (y(T) - h(x(T))) \quad . \tag{16}$$

Unfortunately, if we assume $|y(T) - h\left(x^{[i]}\right)(T)| \to 0$ with increasing iterations i, again $\lambda^{[i]}(T) \to 0$.

1.2 Linearisation at T

Though we define x, y as mappings from $[0, T]$ to some vector spaces, the true (real world) system will exist over a larger interval of time, so $\dot{x}(T), \dot{y}(T)$ make sense and even $\lim_{t\to T} \dot{x}(t) = \dot{x}(T)$.
Consider a small $\epsilon > 0$

$$x(T-\epsilon) \cong x(T) - \underbrace{\dot{x}(T)\epsilon}_{=:\delta x} \quad \text{and} \quad y(T-\epsilon) \cong y(T) - \underbrace{\dot{y}(T)\epsilon}_{=:\delta y} \tag{17}$$

Here $\cong$ denotes equality when $\epsilon \to 0$. We linearise h by

$$y(T-\epsilon) \cong h(x(T) - \delta x) \cong \underbrace{h(x(T))}_{=y(T)} - \mathrm{d}h_{x(T)}\delta x \quad . \tag{18}$$

and by comparison we find

$$\mathrm{d}h_x \delta x \cong \delta y \quad \Rightarrow \quad \mathrm{d}h_{x(T)}\dot{x}(T) \cong \dot{y}(T) \tag{19}$$

Using the systems equations and (13) we get

$$\lambda(T) = \alpha_2 \left\{ f(x(T)) - \dot{x}(T) \right\} \tag{20}$$

which would again yield $\lambda^{[0]}(T) = 0$ if we initialize it with the nominal model, since $\dot{x}^{[0]}(t) = f(x^{[0]}(t))$. If we have a pseudo inverse $\mathrm{d}h_x^\dagger$ we write

$$\hat{\dot{x}}(T) = \mathrm{d}h_{x(T)}^\dagger \dot{y}(T) \quad . \tag{21}$$

We could then define

$$\lambda^{[i]}(T) = \alpha_2 \left\{ f\left(x^{[i]}(T)\right) - \mathrm{d}h_{x(T)}^\dagger \dot{y}(T) \right\} \tag{22}$$

or equivalently

$$w^{[i]}(T) = \mathrm{d}h_{x(T)}^\dagger \dot{y}(T) - f\left(x^{[i]}(T)\right) \quad . \tag{23}$$

To calculate (23) we simply have to find an appropriate pseudo inverse $\mathrm{d}h_x^\dagger$, e.g. the Moore-Penrose inverse $\mathrm{d}h_x^\dagger = \left(\mathrm{d}h_x^T \mathrm{d}h_x\right)^{-1} \mathrm{d}h_x^T$, and get the derivative $\dot{y}(T)$, e.g. via (9) as $\dot{y}(T) = {}^{\mathrm{d}}/_{\mathrm{d}t} H(T)$ or simply by numerical differentiation.

1.3 Augmented States

Consider an augmented system with states $\underline{x} = (x, w)^T : [0, T] \to \mathbb{R}^{2n}$, inputs $v : [0, T] \to \mathbb{R}^n$ and dynamics

$$\dot{\underline{x}} = F\left(\underline{x}, v\right) = \begin{pmatrix} f(P_x\underline{x}) + P_w\underline{x} \\ v \end{pmatrix} \tag{24}$$

where we introduced linear projectors $P_x : \underline{x} \mapsto x$ and $P_w : \underline{x} \mapsto w$ with

$$\Pi_x := \frac{\partial P_x\underline{x}}{\partial \underline{x}} = (\mathbb{1}|0) \in \mathbb{R}^{n\times(n+n)} \quad \text{and} \quad \Pi_w := \frac{\partial P_w\underline{x}}{\partial \underline{x}} = (0|\mathbb{1}) \in \mathbb{R}^{n\times(n+n)} \quad . \tag{25}$$

Calculate some derivatives

$$\left.\frac{\partial f}{\partial x}\right|_x = \mathrm{d}f_x \quad \text{thus} \quad \frac{\partial f(P_x\underline{x})}{\partial \underline{x}} = \mathrm{d}f_{P_x\underline{x}}\Pi_x \tag{26}$$

and

$$\frac{\partial F}{\partial \underline{x}} = \begin{pmatrix} \mathrm{d}f_{P_x\underline{x}}\Pi_x + \Pi_w \\ 0 \end{pmatrix} \in \mathbb{R}^{2n\times 2n} \quad \text{and} \quad \frac{\partial F}{\partial v} = \begin{pmatrix} 0 \\ \mathbb{1} \end{pmatrix} \in \mathbb{R}^{2n\times n} \quad . \tag{27}$$

Introduce an augmented Hamiltonian with costates $\lambda : [0, T] \to \mathbb{R}^{2n}$

$$\underline{H}\left(\underline{x}, v, \lambda, t\right) = ||y(t) - h(P_x\underline{x})||^2 + \frac{\alpha_2}{2}||P_w\underline{x}||^2 + \frac{\beta}{2}||v||^2 + \sum_{\mu=1}^{2n}\lambda_\mu F_\mu\left(\underline{x}, v\right) \tag{28}$$

and using the Hamilton equations we get

$$\dot{\lambda}(t) = 2\Pi_x^T\mathrm{d}h_{P_x\underline{x}}^T(y(t) - h(P_x\underline{x})) - \Pi_w^T P_w\underline{x}(t) - \left(\Pi_x^T\mathrm{d}f_{P_x\underline{x}} + \Pi_w^T\right)\Pi_x\lambda(t) \tag{29}$$

multiplying with Π_x and Π_w yields

$$\Pi_x\dot{\lambda}(t) = 2\mathrm{d}h_{P_x\underline{x}}^T(y(t) - h(P_x\underline{x}(t))) - \mathrm{d}f_{P_x\underline{x}}^T\Pi_x\lambda(t) \tag{30}$$

$$\Pi_w\dot{\lambda}(t) = -P_w\underline{x}(t) - \Pi_x\lambda(t) \quad . \tag{31}$$

These equations reproduce the dynamic of the original system as you can see by inserting $P_x\underline{x} = x$, and $\Pi_x\lambda(t)$ are the costates of the original system. The augmented Hamiltonian gives a penalty to the AUCs of w and $\dot{w}$. Now, the constrain $\lambda(T) = 0$ leads to

$$v(T) = 0 \tag{32}$$

while $\underline{x}(T)$ is free. At the same time we need to know $\underline{x}(0) = (x_0, w_0)^T$ which means that, given x_0 as usual, we need a way to estimate w_0.

One way to determine w_0 could again be linearisation at $t = 0$, i.e.

$$w_0 \cong \mathrm{d}h_{x_0}^\dagger\dot{y}(0) - f(x_0) \quad . \tag{33}$$