Control theory - part 2, controllability, LQR

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1 Stability

Consider some linear dynamical system. Given some initial condition $x(0)$ we want to simulate what will happen when we rollout time into the future. Let's analyse in which cases the system is going to converge to some fixed state in which it is going to stay. Recall that we use a decomposition $A = TDT^{-1}$, where D is a diagonal matrix.

$$\dot{x} = Ax \tag{1}$$

$$A = TDT^{-1} \tag{2}$$

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \tag{3}$$

In order to analyse a rollout, recall what is the solution to a linear dynamical system.

$$x(t) = e^{At}x(0) = Te^{Dt}T^{-1}x(0) \tag{4}$$

The only thing that is time dependent in the equation above is e^{Dt}, but since D is diagonal we have:

$$e^{Dt} = \begin{pmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{pmatrix} \tag{5}$$

To understand what is the dynamics, we have to inspect the lambdas, in particular their real and imaginary parts.

$$\lambda = a + ib \text{ where } a, b \in \mathbb{R} \tag{6}$$

$$e^{\lambda t} = e^{(a+bi)t} = e^{at}(\cos(bt) + i\sin(bt)) \tag{7}$$

The impact of the imaginary part b drives the oscilation, but is is the real part a that has a decisive impact on convergence or divergence.

Theorem 1. *If all the real parts of the eigenvalues of a linear dynamical systems have real parts smaller than zero, then regardless of the initial state $x(0)$ the system will converge to a fixed point.*

Where the imaginary parts are coming from

Since we assumed that A has only real numbers, why do we have to deal with complex numbers? Because in the decomposition $A = TDT^{-1}$ both T and D can have complex numbers, and even though every $x(t)$ is a vector of real numbers only, the matrix e^{Dt} does potentially contain complex numbers.

2 Controllability

So far we were discussing a system, where no external forces were influencing the system. Given the initial condition, we could simulate what will happen in any moment in time just by following the given system dynamics.

Now, let's introduce control to our system.

$$\dot{x} = Ax + Bu \tag{8}$$

Note that the dimensionality of $u \in \mathbb{R}^d$ does not have to be matching the state space dimensionality $x \in \mathbb{R}^n$. For example you can consider a car driving on a line:

- the state is described by position and velocity, so it is two-dimensional,
- our control is one-dimensional - acceleration.

$$\begin{pmatrix} \dot{\text{pos}} \\ \dot{\text{vel}} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \text{pos} \\ \text{vel} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \text{ where } u \in \mathbb{R} \tag{9}$$

Controllability

Intuitively being able to control the system should mean that we can reach any part of the state space, meaning reaching any desired configuration. For example for a car on a line where we control the acceleration, we want to be able to reach any point on the line with any desired velocity. Even though the control input is single-dimensional, we can reach any state in the two dimensional state space (position+velocity) by choosing the rate of change of velocity over time, i.e., by picking an appropriate acceleration profile. Note that however we should not hope for maintaining a state in time, it is not the case that we can get any trajectory in the state space (e.g., if we have non-zero velocity the position can't stay fixed).

Definition 1. *A linear dynamical system $\dot{x} = Ax + Bu$ is* controllable *iff, for any initial state x_0 and any final state x_f, there exists a control input $u(t)$ that will transfer the state from x_0 to x_f in a finite time T, i.e., if for every $x_0, x_f \in \mathbb{R}^n$ and every $T > 0$, there exists a function $u : [0, T] \to \mathbb{R}^d$ such that the solution $x(t)$ of the system with $x(0) = x_0$ satisfies $x(T) = x_f$.*

Consider the following three systems:

$$\dot{x} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \text{ where } u \in \mathbb{R} \tag{10}$$

$$\dot{x} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} u \text{ where } u \in \mathbb{R}^2 \tag{11}$$

$$\dot{x} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \text{ where } u \in \mathbb{R}^2 \tag{12}$$

Note that for systems (10), (11) we have a clear decoupling of the state dimensions - the dynamics is independent. Since for system (10) we cannot influence the first dimension in the state, the system is not controllable. On the other hand we have independent control knobs (two components of u) in system 11 so it should be easily controllable. What is not obvious, is that system (12) is controllable, which we will verify after brining in an important theorem in section 2.1.

Time Note that due to linearity, scaling control by a constant factor is equivalent to scaling time. This means that by using very large values for control we can reach the desired states in a very short time, which is definitely not grounded in real world assumptions (you can't have arbitrary acceleration when driving a car). Consequently the quantifier *for every* $T > 0$ is equivalent to *exists* $T > 0$ in Definition 1.

2.1 Testing controllability

We present the following theorem without a proof.

Theorem 2. *The following conditions are equivalent:*

- *The system is controllable.*
- *rank*$(C) = n$*, where* $C = [B, AB, A^2B, \ldots, A^{n-1}B]$.
- *For a given set of desired eigenvalues* $\psi = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$*, there exists a feedback matrix* K *such that the eigenvalues of* $A - BK$ *are exactly the eigenvalues in* ψ.

Note that the criterion involving rank(C) is very appealing, because it gives us a tool to verify controllability algorithmically.

The matrix C for systems (10), (11), and (12) are respectively:

$$C = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$$

2.2 Using control to establish desired eigenvalues

Consider the following dynamical system:

$$\dot{x} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

where $x \in \mathbb{R}^2$ is the state vector and $u \in \mathbb{R}$ is the control input.

As an example, imagine that we want to design a state feedback linear controller $u = -Kx$, where $K \in \mathbb{R}^{1\times 2}$ is the feedback gain matrix, so that the eigenvalues of the resulting system $\dot{x} = (A - BK)x$ are $\lambda_1 = -2$, $\lambda_2 = -1$.

The characteristic polynomial of the closed-loop system is

$$\begin{aligned} \det((A - BK) - \lambda I) &= \det \begin{pmatrix} 1-\lambda & 1 \\ -k_1 & 2-k_2-\lambda \end{pmatrix} \\ &= (1-\lambda)(2-k_2-\lambda) + k_1 \\ &= \lambda^2 + (k_2 - 3)\lambda + (2 + k_1 - k_2) \end{aligned} \tag{13}$$

We want this polynomial to be equal to $(\lambda + 1)(\lambda + 2)$, so we need

$$2 + k_1 - k_2 = 2k_2 - 3 = 3 \tag{14}$$

which gives $k_1 = k_2 = 6$.

3 LQR - Linear Quadratic Regulator

Theorem 3. *Consider a linear dynamical system*

$$\dot{x} = Ax + Bu$$

and a cost function

$$J(u) = \int_0^\infty (x^T Q x + u^T R u)\, dt$$

where Q *and* R *are positive semi-definite and positive definite matrices, respectively.*

There exists a controller that minimizes the loss function, which is linear $u^* = -Kx$.

The exact formula for K is out of scope of this course, what is important is that for the defined cost function there is an optimum controller that is linear and we can compute it using standard algebraic operations.

3.1 LQR for cartpole

When expressed as a dynamical system, LQR is not linear. However we can linearize it around a fixed point and apply LQR. By setting different Q and R values we achieve different behaviours of the cart-pole, think of different value of Q and R that lead to:

- Quick (aggresive) stabilization, potentially exerting a lot of energy.
- Energy efficient stabilization.
- Keeping the pole upright at all times.

Additional materials

- Introductory material on ordinary differential equations.
- Control bootcamp - video series by Steve Brunton.
- State-space equations, including LQR - short video series by Brian Douglas.