MINIMUM WEIGHTED VERTEX COVER by Dynamic Programming on Tree Decompositions

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1 Problem description

Given a graph $G = (V, E)$, a tree decomposition $\mathcal{T} = (T, \chi)$ of G of width k and a function $w : V \to \mathbb{R}$, find $X \subseteq V$ such that X is a vertex cover of G and $\sum_{x \in X} w(x)$ is minimal.

2 Dynamic Program description

The problem is solved via a dynamic programming approach on T. As a preprocessing step, the given tree decomposition is converted into a nice tree decompositino. The dynamic programming algorithm then updates a table at each node from the leaves up by the following rules:

Let t be the current node for which to update each $M_t(U)$ and t' and t'' its potential children.

1. Leaf Nodes. Let v be the single vertex in $\chi(t)$:
$$M_t(\{\}) = 0$$
$$M_t(\{v\}) = w(v)$$
2. Introduce Nodes. Let v be the newly introduced vertex, i.e. $\chi(t) \setminus \chi(t') = \{v\}$.
$$\text{if } v \in U\text{: } M_t(U) = M_{t'}(U) + w(v)$$
$$\text{if } v \notin U\text{: } M_t(U) = \begin{cases} M'_t(U) & \text{, if U is a vertex cover of } G_t \\ \infty & \text{, otherwise} \end{cases}$$
3. Forget Nodes. Let v be the newly forgotten vertex, i.e. $\chi(t') \setminus \chi(t) = \{v\}$.
$$M_t(U) = min(M_{t'}(U), M_{t'}(U \cup \{v\}))$$
4. Join Nodes.
$$M_t(U) = M_{t'} + M_{t''} - \sum_{v \in U} w(v)$$