The Convex Geometry of Blind Deconvolution

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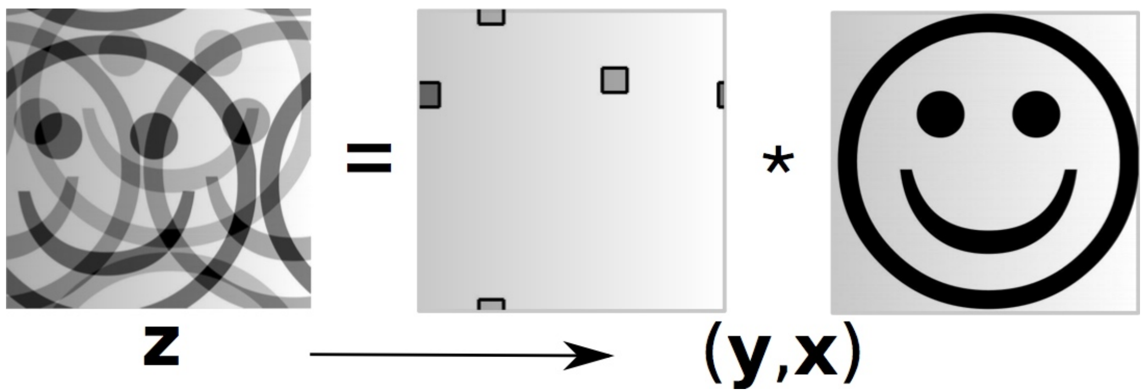
July 12, 2019

Joint work Felix Krahmer (TUM),

Funded by the DFG in the context of SPP 1798 CoSIP

Blind deconvolution in imaging

- *Blind deconvolution ubiquituous in many applications:*
  - Imaging: $\boldsymbol{x}$ signal, $\boldsymbol{y}$ blur



- *(Circular) convolution of $\boldsymbol{w}, \boldsymbol{x} \in \mathbb{C}^L$:* $(\boldsymbol{w} * \boldsymbol{x})_k := \sum_{\ell=1}^{L} \boldsymbol{w}_k \boldsymbol{x}_{(\ell-k) \mod L}$.

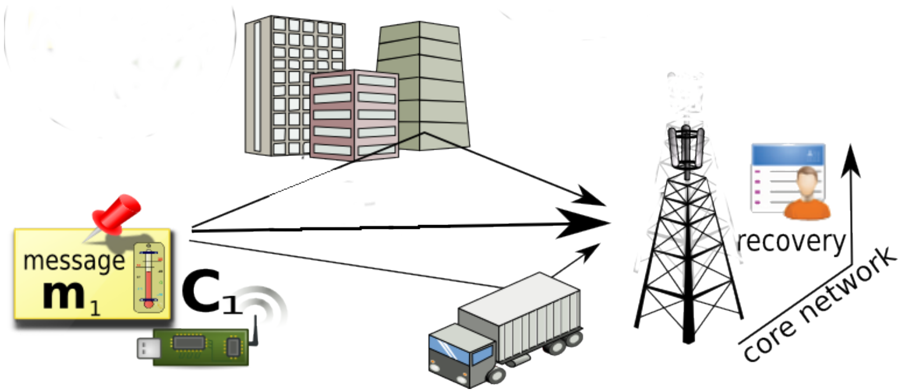
Blind deconvolution in wireless communications

- *Task:* deliver message $\boldsymbol{m} \in \mathbb{C}^N$ via unknown channel.
  Proposed approach: introduce redundancy before transmission.

- *Linear encoding:*
  $\boldsymbol{x} = \boldsymbol{C}\overline{\boldsymbol{m}}$ with $\boldsymbol{C} \in \mathbb{C}^{L \times N}$
  the signal x is transmitted
- *Channel model:*
  only most direct paths are active
  $\boldsymbol{w} = \boldsymbol{B}\boldsymbol{h}$, where $\boldsymbol{B} \in \mathbb{C}^{L \times K}$
- *Received signal:* $\boldsymbol{e}$ noise

$$\boldsymbol{y} = \boldsymbol{w} * \boldsymbol{x} + \boldsymbol{e} \in \mathbb{C}^L$$

- Introduced by Ahmed, Recht, Romberg (IEEE IT ’14)



Goal: recover $\boldsymbol{m}$ from $\boldsymbol{y}$

Lifting

- *Observation:* $\boldsymbol{w} * \boldsymbol{x} = \boldsymbol{B}\boldsymbol{h} * \boldsymbol{C}\overline{\boldsymbol{m}}$ is bilinear in $\boldsymbol{h}$ and $\overline{\boldsymbol{m}}$
  $\Rightarrow$ There is a unique linear map $\mathcal{A} : \mathbb{C}^{K \times N} \to \mathbb{C}^{L}$ such that
  $$\boldsymbol{B}\boldsymbol{h} * \boldsymbol{C}\overline{\boldsymbol{m}} = \mathcal{A}(\boldsymbol{h}\boldsymbol{m}^*)$$
  for arbitrary $\boldsymbol{h}$ and $\boldsymbol{m}$

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- Thus, the rank 1 matrix $\boldsymbol{X}_0 = \boldsymbol{h}\boldsymbol{m}^*$ satisfies
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- Finding $\boldsymbol{X}_0$ is a **low rank matrix recovery problem**

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- Finding $\boldsymbol{X}_0$ is a **low rank matrix recovery problem**
- Ideally find
  $$\operatorname{argmin} \ \operatorname{rank} \boldsymbol{X} \quad \text{subject to } \|\mathcal{A}(\boldsymbol{X}) - \boldsymbol{y}\|_2 \leq \eta$$
- Such problems are NP-hard in general
  $\rightarrow$ try convex relaxation

A convex approach

SDP relaxation (Ahmed, Recht, Romberg '14)

Solve the semidefinite program (SDP)

$$\widetilde{\boldsymbol{X}} = \operatorname{argmin} \|\boldsymbol{X}\|_* \qquad \text{subject to } \|\mathcal{A}(\boldsymbol{X}) - \boldsymbol{y}\|_2 \leq \eta. \tag{SDP}$$

The *nuclear norm* $\|\boldsymbol{X}\|_* := \sum_{j=1}^{\operatorname{rank}(\boldsymbol{X})} \sigma_j(\boldsymbol{X})$ is the sum of all singular values.

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Model assumptions:

- $\boldsymbol{y} = \boldsymbol{Bh} * \boldsymbol{C\bar{m}} + \boldsymbol{e}$
- **Adversarial noise:** $\|\boldsymbol{e}\|_2 \leq \eta$
- $\boldsymbol{C} \in \mathbb{C}^{L \times N}$ has i.i.d. standard Gaussian entries
- $\boldsymbol{B} \in \mathbb{C}^{L \times K}$ satisfies $\boldsymbol{B}^* \boldsymbol{B} = \boldsymbol{Id}$ and is such that $\boldsymbol{FB}$ (for $\boldsymbol{F}$ the DFT) has rows of equal norm.

Recovery guarantees

Theorem (Ahmed, Recht, Romberg '14)

Assume

$$\frac{L}{\log^3 L} \geq C\left(K + N\mu_{\boldsymbol{h}}^2\right).$$

Then with high probability every minimizer $\widetilde{\boldsymbol{X}}$ of (SDP) satisfies

$$\|\widetilde{\boldsymbol{X}} - \boldsymbol{h}\boldsymbol{m}^*\|_F \lesssim \sqrt{K+N}\,\eta.$$

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    $\rightarrow$ Exact recovery with a near optimal-amount of measurements

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- **Consequences:**
  - **No noise**, i.e., $\eta = 0$:
    $\rightarrow$ Exact recovery with a near optimal-amount of measurements
  - **Noisy scenario**, i.e., $\eta > 0$:
    $\rightarrow$ dimension factor $\sqrt{K+N}$ appears in the noise
    Does not explain empirical success of (SDP)

Noise robustness in low-rank matrix recovery

- Gaussian measurement matrices (implies RIP) ✓
- phase retrieval ✓
- *blind deconvolution* (this presentation) **?**
- matrix completion **?**
- Robust PCA **?**
- many more... **?**

Noise robustness in low-rank matrix recovery

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Despite the popularity of convex relaxations for low-rank matrix recovery in the literature, their **noise robustness is not well-understood**.

What is the problem?

- **Proof technique for these models:**

  - *Idea:* Show existence of (approximate) dual certificate w.h.p.
  - *Golfing scheme* originally developed by D. Gross.

D. Gross

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- **Proof technique for these models:**

  – *Idea:* Show existence of (approximate) dual certificate w.h.p.
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D. Gross

- Works well in the noiseless case, where $\boldsymbol{X}_0$ is expected to be the minimizer
- **Problem:** In noisy models we do not know the minimizer

Are the dimension factors necessary?

Recall: We are interested in the scenario $L \ll KN$ and we optimize

$$\widetilde{\boldsymbol{X}} = \operatorname{argmin} \|\boldsymbol{X}\|_* \qquad \text{subject to } \|\mathcal{A}(\boldsymbol{X}) - \boldsymbol{y}\|_2 \le \eta. \tag{SDP}$$

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Theorem (Krahmer, DS '19)

There exists an admissible $\boldsymbol{B}$ *such that:*
With high probability there is $\tau_0 > 0$ *such that for all* $\tau \le \tau_0$ *there exists an adversarial noise vector* $\boldsymbol{e} \in \mathbb{C}^L$ *with* $\|\boldsymbol{e}\|_2 \le \tau$ *that admits an alternative solution* $\widetilde{\boldsymbol{X}}$ *with the following properties.*

- $\widetilde{\boldsymbol{X}}$ *is feasible, i.e.,* $\|\mathcal{A}\left(\widetilde{\boldsymbol{X}}\right) - \boldsymbol{y}\|_2 = \tau$
- $\widetilde{\boldsymbol{X}}$ *is preferred to* $\boldsymbol{hm}^*$ *by (SDP) i.e.,* $\|\widetilde{\boldsymbol{X}}\|_* \le \|\boldsymbol{hm}^*\|_*$*, but*
- $\widetilde{\boldsymbol{X}}$ *is far from the true solution in Frobenius norm, i.e.,*

$$\|\widetilde{\boldsymbol{X}} - \boldsymbol{hm}^*\|_F \ge \frac{\tau}{C_3}\sqrt{\frac{KN}{L}}.$$

What does this mean?

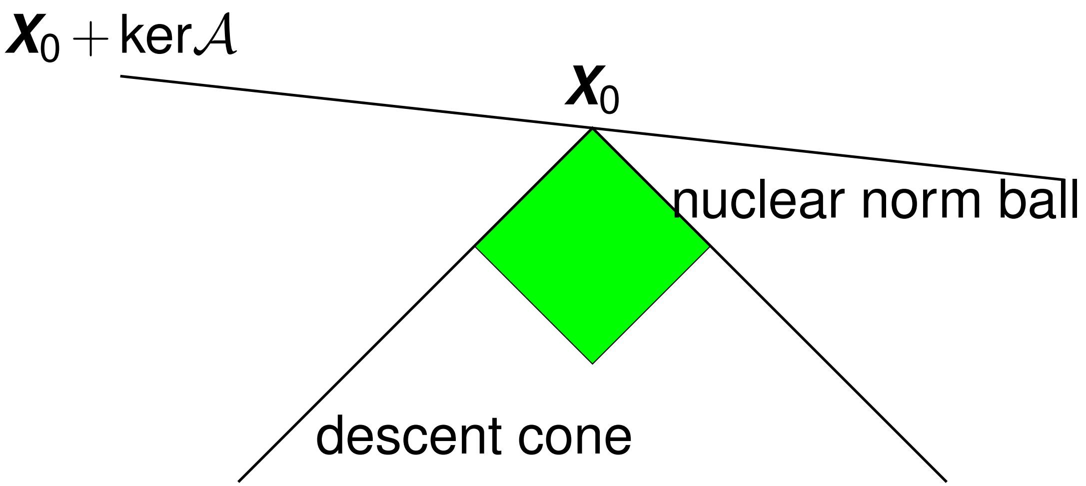
- Assume $K = N$ and $L \approx CK$ up to log-factors

$$\Rightarrow \quad \|\widetilde{\boldsymbol{X}} - \boldsymbol{hm}^*\|_F \gtrsim \tau \sqrt{\frac{KN}{L}} \approx \tau \sqrt{K+N}.$$

  up to log-factors
  $\rightarrow$ The factor $\sqrt{K+N}$ is not a pure proof artifact.
- *Caution:* $\widetilde{\boldsymbol{X}}$ might not be the minimizer of (SDP)!

- Analogous result can be shown for matrix completion.

Ideas of the analysis I



- *Crucial geometric object:* Descent cone for $\boldsymbol{X}_0 \in \mathbb{C}^{K \times N}$

$$\mathcal{K}_* (\boldsymbol{X}_0) = \left\{ \boldsymbol{Z} \in \mathbb{C}^{K \times N} : \|\boldsymbol{X}_0 + \varepsilon \boldsymbol{Z}\|_* \leq \|\boldsymbol{X}_0\|_* \text{ for some small } \varepsilon > 0 \right\}$$

Ideas of the analysis II

- **Minimum conic singular value**:

$$\lambda_{\min}(\mathcal{A}, \mathcal{K}_*(\boldsymbol{X}_0)) := \min_{\boldsymbol{Z} \in \mathcal{K}_*(\boldsymbol{X}_0)} \frac{\|\mathcal{A}(\boldsymbol{Z})\|_2}{\|\boldsymbol{Z}\|_F}$$

- **Noiseless scenario, i.e., $\eta = 0$:**
  Exact recovery $\Longleftrightarrow \lambda_{\min}(\mathcal{A}, \mathcal{K}_*(\boldsymbol{X}_0)) > 0$

- **Noisy scenario:** Conic singular value controls stability [Chandrasekaran et al. '12]:

$$\|\widetilde{\boldsymbol{X}} - \boldsymbol{X}_0\|_F \leq \frac{2\eta}{\lambda_{\min}(\mathcal{A}, \mathcal{K}_*(\boldsymbol{X}_0))}$$

  (As $\mathcal{A}$ is Gaussian, $\lambda_{\min}(\mathcal{A}, \mathcal{K}_*(\boldsymbol{hm}^*)) \asymp 1$ w.h.p., whenever $L \gtrsim K + N$)

Ideas of the analysis III

Lemma (Krahmer, DS '19)

There exists $\mathbf{B} \in \mathbb{C}^{L \times K}$ satisfying $\mathbf{B}^* \mathbf{B} = Id_K$ and $\mu_{\max}^2 = 1$, whose corresponding measurement operator $\mathcal{A}$ satisfies the following:
Let $\mathbf{m} \in \mathbb{C}^N \setminus \{0\}$ and let $\mathbf{h} \in \mathbb{C}^K \setminus \{0\}$ be incoherent. Then with high probability it holds that

$$\lambda_{\min}\left(\mathcal{A}, \mathcal{K}_*\left(\mathbf{h}\mathbf{m}^*\right)\right) \leq C_3 \sqrt{\frac{L}{KN}}.$$

- Lemma can be used to prove the previous theorem.
- (Analogous result holds for matrix completion.)

All hope is lost???

Recovery for high noise levels

Theorem (Krahmer, DS '19)

Let $\alpha > 0$. Assume that

$$L \geq C_1 \frac{\mu^2}{\alpha^2}(K+N)\ \log^2 L.$$

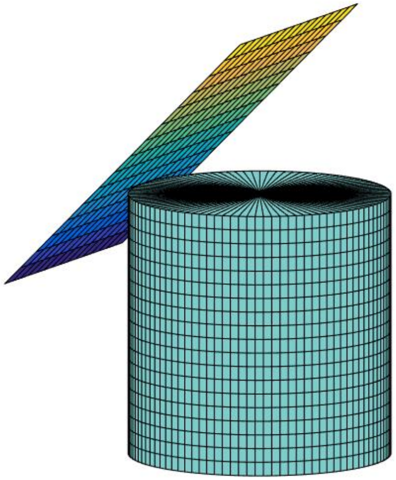
Then with high probability the following statement holds for all $\boldsymbol{h} \in S^{K-1}$ with $\mu_{\boldsymbol{h}} \leq \mu$, all $\boldsymbol{m} \in S^{N-1}$, all $\tau > 0$, and all $\boldsymbol{e} \in \mathbb{C}^L$ with $\|\boldsymbol{e}\|_2 \leq \tau$:
Any minimizer $\widetilde{\boldsymbol{X}}$ of (SDP) satisfies

$$\|\widetilde{\boldsymbol{X}} - \boldsymbol{h}\boldsymbol{m}^*\|_F \leq \frac{C_3 \mu^{2/3} \log^{2/3} L}{\alpha^{2/3}} \max\{\tau; \alpha\}.$$

$\rightarrow$ **Near-optimal recovery guarantees** for high noise-levels.

Proof sketch I

- Descent cone **local** approximation to descent set near $\boldsymbol{hm}^*$.
- *Geometric Intuition:* Close to $\ker \mathcal{A}$, the descent set is not pointy.



- Consider the partition $\mathcal{K}_*(\boldsymbol{hm}^*) = \mathcal{K}_1 \cup \mathcal{K}_2$, where
  - $\mathcal{K}_1$ contains all elements in $\mathcal{K}_*(\boldsymbol{hm}^*)$, which are near-orthogonal to $\boldsymbol{hm}^*$
  - $\mathcal{K}_2 := \mathcal{K}_*(\boldsymbol{hm}^*) \setminus \mathcal{K}_1$

Proof sketch II

Geometric intution: No large error can occur in directions belonging to $\mathcal{K}_1$ due to the curved nature of the nuclear norm ball

- $\lambda_{\min}(\mathcal{A}, \mathcal{K}_2)$ can be bounded from below using *Mendelson's small-ball method*
- $\rightarrow$ No large error can occur in these directions

S. Mendelson

Combining these two ideas yields the result.

Outlook and open questions

- What can we say about the actual minimizer in the scenario of small noise?
- Stability of matrix completion?

Thank you for your attention!