Blind Demixing and Deconvolution at Near-Optimal Rate

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Joint work with Peter Jung (TU Berlin), Felix Krahmer (TU München)

supported by DFG priority program "Compressed Sensing in Information Processing" (CoSIP)

Overview

General Framework

Recovery and guarantees

Proof sketch

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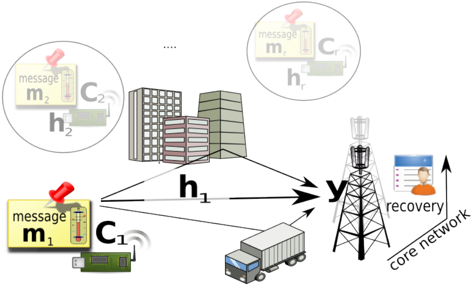
Proof sketch

A problem in Wireless Communication

- r different devices
- each device wants to deliver a message $m_i \in \mathbb{C}^N$
- **Channel model:**
  Only few active paths $w_i = Bh_i$, where $B \in \mathbb{C}^{L \times K}$
- **Linear encoding:**
  $x_i = C_i m_i$ with $C_i \in \mathbb{C}^{L \times N}$
  Device i transmits x_i
- **Received signal:**



$$y = \sum_{i=1}^{r} w_i * x_i \in \mathbb{C}^L$$

Goal: recover **all** m_i from y

Assumptions on B_i and C_i

- Assume w_i is concentrated on the first few entries (most direct paths)
- B: First K columns of the $L \times L$ identity $\Rightarrow$ extends h_i by zeros
- (More general models for B are possible.)
- Choice of C_i arbitrary $\Rightarrow$ randomize
- Choose C_i to have i.i.d. standard complex normal entries, i.e., $(C_i)_{jk} \sim \mathcal{CN}(0,1)$

Lifting

- $w_i * x_i = Bh_i * C_i m_i$ bilinear in h_i and m_i
- $\Rightarrow$ There is a unique linear map $\mathcal{A}_i : \mathbb{C}^{K\times N} \to \mathbb{C}^L$ such that $Bh_i * C_i m_i = \mathcal{A}_i \left(h_i m_i^* \right)$ for arbitrary h_i and m_i
- $$y = \sum_{i=1}^{r} \mathcal{A}_i \left(h_i m_i^* \right) = \mathcal{A}\left(X^0 \right),$$
  where
  $$X^0 = \left(h_1 m_1^*, \ldots, h_r m_r^* \right)$$
- **Low rank matrix recovery problem**
- Corresponding combinatorial problem NP-hard in general
  $\rightarrow$ convex relaxation

Overview

A convex approach for recovery

[Ling, Strohmer 2017]

- Solve the semi-definite program

$$\text{minimize} \sum_{i=1}^{r} \|Y_i\|_* \qquad \text{subject to} \sum_{i=1}^{r} \mathcal{A}_i(Y_i) = y. \qquad \text{(SDP)}$$

- $\|\cdot\|_*$: nuclear norm, i.e., the sum of the singular values
- Recovery is guaranteed with high probability, if

$$L \geq C r^2 \left(K + \mu_h^2 N\right) \log^3 L \log r$$

- μ_h coherence parameter, ranges between $1 \leq \mu_h^2 \leq K$
- (Near-)optimal dependence on K, N, suboptimal dependence on r.
- Previously established for $r = 1$ in [Ahmed, Recht, Romberg 2015]

Main result

Theorem (Jung, Krahmer, S., 2017)

Let $\omega \geq 1$. Assume that

$$L \geq C_\omega r \left(K \log K + N \mu_h^2 \right) \log^3 L, \tag{1}$$

where C_ω is a universal constant only depending on ω. Then with probability $1 - \mathcal{O}\left(L^{-\omega}\right)$ the recovery program (SDP) is successful, i.e., X^0 is its unique minimizer.

- (Near) optimal dependence on K, N, and r

Overview

Proof overview

Our proof follows the same strategy as [Ling, Strohmer 2016].
It consists of the following two main steps:

- Establishing sufficient conditions for recovery
  $\Rightarrow$ approximate dual certificate
- Constructing the dual certificate ("Golfing Scheme")

Proof overview

Our proof follows the same strategy as [Ling, Strohmer 2016].
It consists of the following two main steps:

- Establishing sufficient conditions for recovery
  $\Rightarrow$ approximate dual certificate
- Constructing the dual certificate ("Golfing Scheme")

The following subspace is important for both steps of the proof:

$$\mathcal{T} = \Big\{ (u_1 m_1^* + h_1 v_1^*, \ldots, u_r m_r^* + h_r v_r^*) : \\ u_1, \ldots, u_r \in \mathbb{C}^K, v_1, \ldots, v_r \in \mathbb{C}^N \Big\}$$

$\mathcal{T}_i$ is defined by

$$\mathcal{T}_i = \left\{ u m_i^* + h_i v^* : \; u \in \mathbb{C}^K, \; v \in \mathbb{C}^N \right\}.$$

Local Isometry Property

- Crucial ingredient for the proof:

Definition

We say that $\mathcal{A}$ fulfills the δ-local isometry property, if

$$(1-\delta)\sum_{i=1}^{r}\|X_i\|_F^2 \le \left\| \sum_{i=1}^{r} \mathcal{A}_i\left(X_i\right) \right\|_{\ell_2}^2 \le (1+\delta)\sum_{i=1}^{r}\|X_i\|_F^2$$

for all $X = (X_1, \ldots, X_r) \in \mathcal{T}$.

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for all $X = (X_1, \dots, X_r) \in \mathcal{T}$.

- Our goal: Show that $\mathcal{A}$ fulfills the local isometry property, if L scales linearly with r

Local isometry property

- Define $\hat{\mathcal{T}} = \left\{X = (X_1, \dots, X_r) \in \mathcal{T} : \sum_{i=1}^r \|X_i\|_F^2 = 1\right\}$
- δ-local isometry property is equivalent to

$$\begin{aligned}
\delta &\geq \sup_{X \in \hat{\mathcal{T}}} \left| \|\sum_{i=1}^r \mathcal{A}_i(X_i)\|_{\ell_2}^2 - \sum_{i=1}^r \|X_i\|_F^2 \right| \\
&= \sup_{X \in \hat{\mathcal{T}}} \left| \|\sum_{i=1}^r \mathcal{A}_i(X_i)\|_{\ell_2}^2 - \mathbb{E}\left[\|\sum_{i=1}^r \mathcal{A}_i(X_i)\|_{\ell_2}^2 \right] \right| \\
&= \sup_{X \in \hat{\mathcal{T}}} \left| \|V_X \operatorname{vec}([C_1, \dots, C_r])\|_{\ell_2}^2 - \mathbb{E}\left[\|V_X \operatorname{vec}([C_1, \dots, C_r])\|_{\ell_2}^2 \right] \right|,
\end{aligned}$$

where for $X = (u_1 m_1^* + h_1 v_1^*, \dots, u_r m_r^* + h_r v_r^*) \in \mathcal{T}$

$$V_X\left(\operatorname{vec}([C_1, \dots, C_r])\right) = \sum_{i=1}^r \mathcal{A}_i(X_i)$$

Suprema of Chaos Processes

Theorem (Krahmer, Mendelson, Rauhut 2014)

Let $\mathcal{X}$ be a symmetric set of matrices, i.e., $\mathcal{X} = -\mathcal{X}$, and let ξ be a random vector whose entries ξ_i are independent and have distribution $\mathcal{CN}(0,1)$. Then, for $t > 0$,

$$P\left(\sup_{A\in\mathcal{X}}\left|\|A\xi\|_{\ell_2}^2 - \mathbb{E}\|A\xi\|_{\ell_2}^2\right| \geq c_1 E + t\right) \leq 2\exp\left(-c_2 \min\left(\frac{t^2}{V^2}, \frac{t}{U}\right)\right)$$

where, setting $\mathcal{D}(\mathcal{X}) = \int_0^{+\infty}\sqrt{\log \mathcal{N}(\mathcal{X}, \|\cdot\|_{2\to2}, t)}dt$ the quantities E, V, and U are defined as

$$\begin{aligned}
E &= \mathcal{D}(\mathcal{X})\left(\mathcal{D}(\mathcal{X}) + d_F(\mathcal{X})\right)\\
V &= d_{2\to2}(\mathcal{X})\left(\mathcal{D}(\mathcal{X}) + d_F(\mathcal{X})\right)\\
U &= d_{2\to2}^2(\mathcal{X}).
\end{aligned}$$

The next steps

- Apply Theorem for $\mathcal{X} = \{V_X : X \in T;\ \sum_{i=1}^{r} \|X_i\|_F^2 = 1\}$.
- Bound covering numbers

$\Longrightarrow$ δ-local isometry property holds with high probability