Comp. Laboratory #2: The Pendulum

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1 Introduction

The aim of the lab is to learn about the dynamics of systems with nonlinear forces as well as to solve coupled ODEs numerically, using multiple different methods implemented in python. These methods include the trapezoid rule as well as the Runge-Kutta integration schemes. Phase portraits for several different damped, driven systems are also generated and analyzed.

2 Methodology

For all of the exercises, the motion of a pendulum of length L is analysed. According to Newton's second law, the equation of motion of a such pendulum is

$$mL\frac{d^2\theta}{dt^2} = -mgsin\theta$$

with m being the mass of the moving particle, θ - the angle between the string and vertical and g - the gravitational acceleration.
This equation can be simplified to the following form:

$$\frac{d^2\theta}{dt^2} = -g/L \cdot sin\theta$$

.

2.1 Exercise 1

If it can be assumed that the angle θ is sufficiently small, one can use the linear approximation $sin\theta \approx \theta$, which allows for the equation of motion to be solved analytically, giving:

$$\theta = Bsin(\omega_0 t + \delta)$$

with $\omega_0 = \sqrt{g/L}$ being the natural frequency and B, δ - constants of integration. To help with solving the ODE numerically, one can write it in terms of two coupled, first order equations:

$$\begin{cases} \frac{d\theta}{dt} &= \omega \\ \frac{d\omega}{dt} &= f(\theta, \omega, t) \end{cases}$$

$$f(\theta, \omega, t) = -g/L \cdot \theta$$

In this exercise, the above linear case is solved using the so-called trapezoid rule, which works based on the following Taylor approximations:

$$\theta_{n+1} \approx \theta_n + \frac{\Delta t}{2}\left(\frac{d\theta(t)}{dt} + \frac{d\theta(t+\Delta t)}{dt}\right)$$
$$\theta_{n+1} \approx \theta_n + \frac{\Delta t}{2}\left(\omega_n + (\omega_n + f(\theta_n, \omega_n, t))\Delta t\right)$$

and similarly for ω:

$$\omega_{n+1} \approx \omega_n + \frac{\Delta t}{2}\left(\frac{d\omega(t)}{dt} + \frac{d\omega(t+\Delta t)}{dt}\right)$$
$$\omega_{n+1} \approx \omega_n + \frac{\Delta t}{2}\left(f(\theta_n, \omega_n, t)\right) + f(\theta_{n+1}, \omega_n + f(\theta_n, \omega_n, t)\Delta t, t_{n+1}))$$

These approximations allow for iterative solving of the equation of motion, and their accuracy depends on the choice of Δt.

2.2 exercise 2

In this exercise, the trapezoid method was used again, but this time to solve the nonlinear case of the equation of motion, i.e. the one in which:

$$f(\theta, \omega, t) = -g/L \cdot sin\theta$$

2.3 Exercise 3

In the following exercises, the trapezoid rule was replaced by the fourth-order Runge-Kutta algorithm, which works on the same principles, but uses a fourth-order, instead of linear, expansion in Δt. This provides higher accuracy at a comparable computational cost.

2.4 Exercise 4

Here, the fourth order Runge-Kutta algorithm was used to predict the motion of a damped pendulum. The addition of a damping force $F_d = -k\frac{d\theta}{dt}$ results in the change of the function f:

$$f(\theta, \omega, t) = -g/L \cdot sin\theta - k\omega$$

.

2.5 Exercise 5

In this exercise, the motion of a damped pendulum driven by a periodic external force is analysed. Because the resulting motion is quite complex, it gets represented in the form of a phase portrait - a graph of θ versus ω. After the pendulum reaches a steady state motion, it can either behave periodically, with the period being an integer multiple of the force's period, or it can behave chaotically. This can be clearly observed on a phase diagram as periodic motion results in a closed loop - a limit cycle. Here, $f(\theta, \omega, t) = -g/L \cdot sin\theta - k\omega + Acos(\phi t)$.

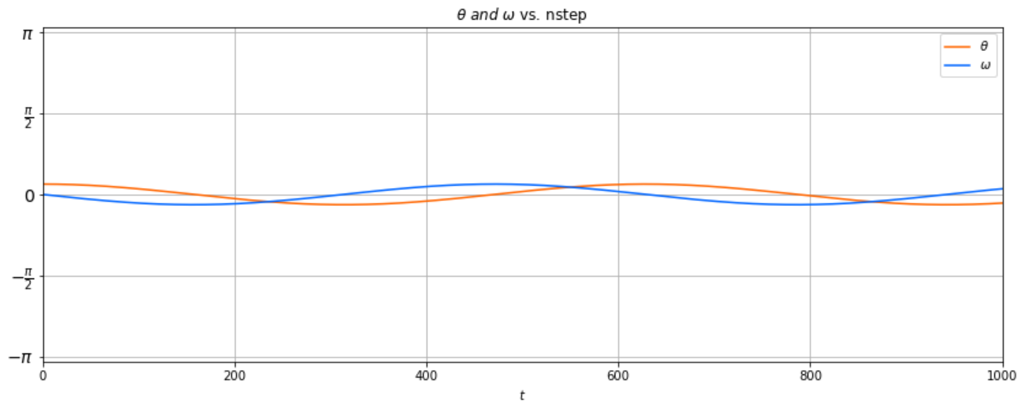

Figure 1: Plot of θ and ω vs. the number of iteration steps (trapezoid method, simple pendulum)

3 Results

For the simple pendulum, to check if the predicted motion of the particle was sinusoidal, as in the analytical solution, the graph of both its angle and angular velocity versus the number of iterations was produced. This graph does in fact seem to closely resemble the two functions (up to a phase shift):

Next, these parameters were plotted starting from different initial conditions:

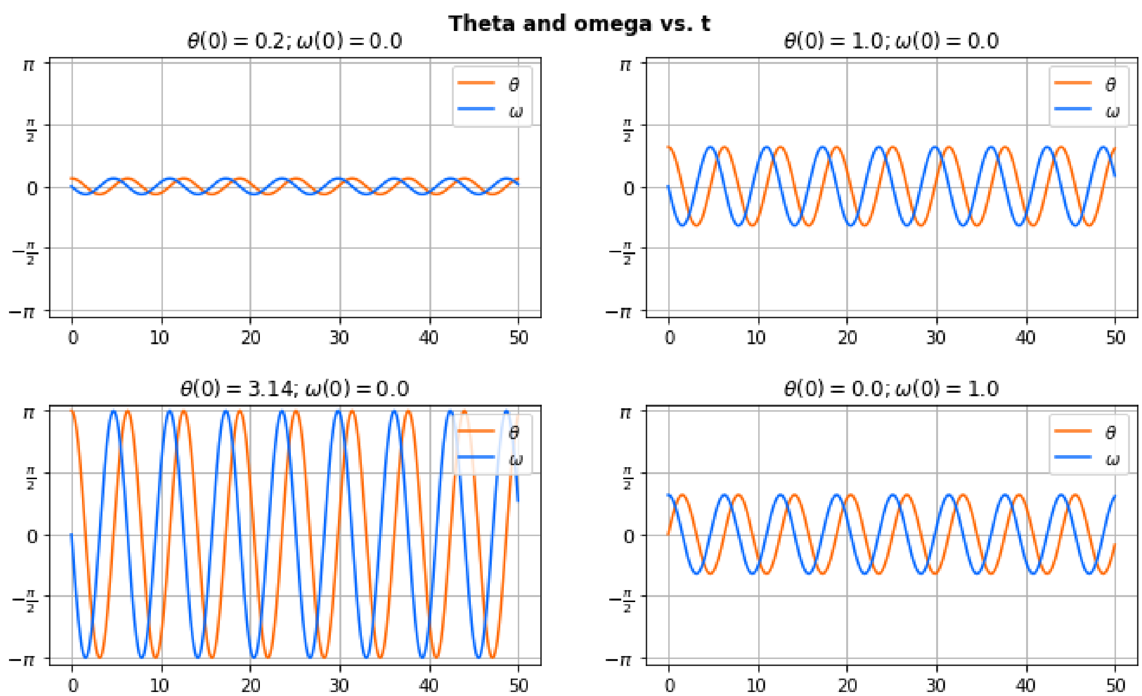

Figure 2: Plots of θ and ω vs. t for different initial conditions (trapezoid method, simple pendulum)

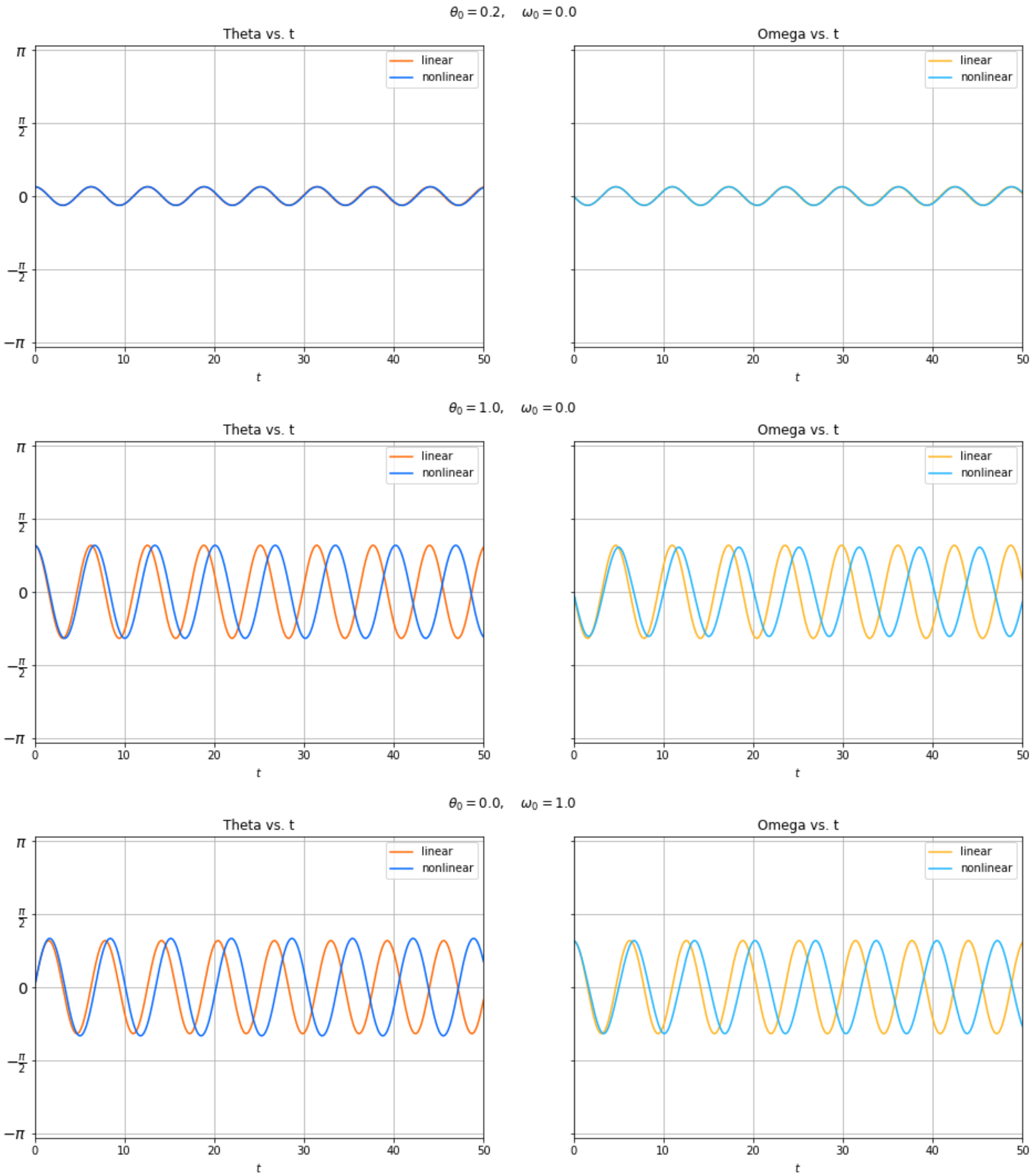

Figure 3: Comparison of θ and ω vs. t for the simple and nonlinear pendulum

The trapezoid rule was used here to produce graphs of both the simple and nonlinear pendulum motion, starting from the same sets of initial conditions. In the first case, $\theta = 0.2$ is the maximum angle that is achieved during that motion. Because it is small, the approximation $sin\theta \approx \theta$ is quite precise, which is apparent in the near-indistinguishable shapes of the linear and nonlinear plots.

The second and third sets of initial conditions was similar to the first one, except either the angle or the angular velocity were much larger. This makes the first order expansion produce significantly different results from the actual function, enough to shift completely out of phase by less than 10 oscillations.

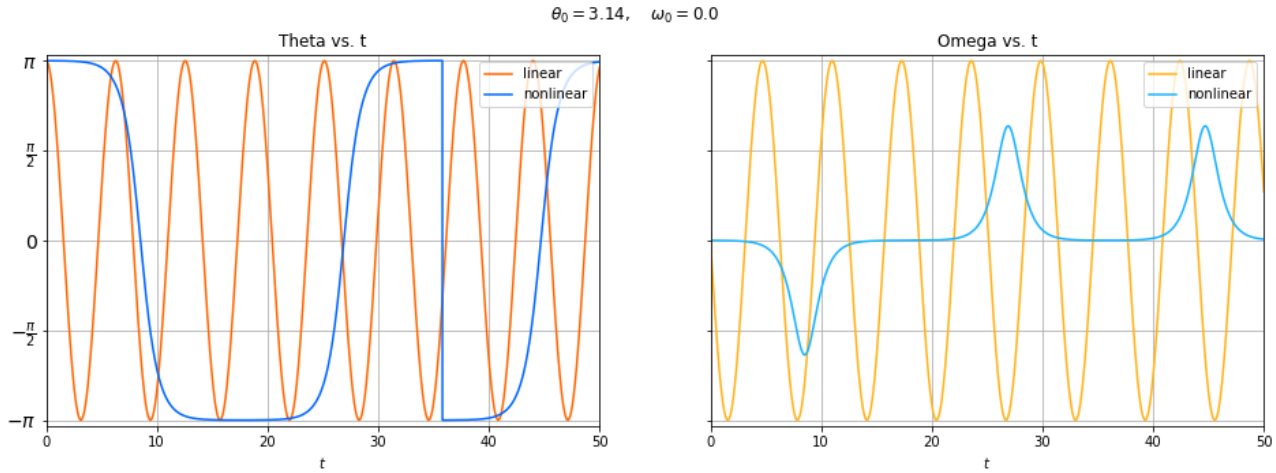

Figure 4: Comparison of θ and ω vs. t for the simple and nonlinear pendulum

The last set of conditions showed the most difference between the two models. The starting point was with the particle almost completely over the pivot point. Whereas the simple algorithm just treats it like any other simple oscillator, the nonlinear one correctly describes the, at first, slow motion of the pendulum. It is also worth noting that after about 35s, this algorithm seems to predict the particle completing a full rotation - signified by the sharp change of the value of θ. This is not physical, as it breaks energy conservation and is most likely due to the limited accuracy of the trapezoid method.

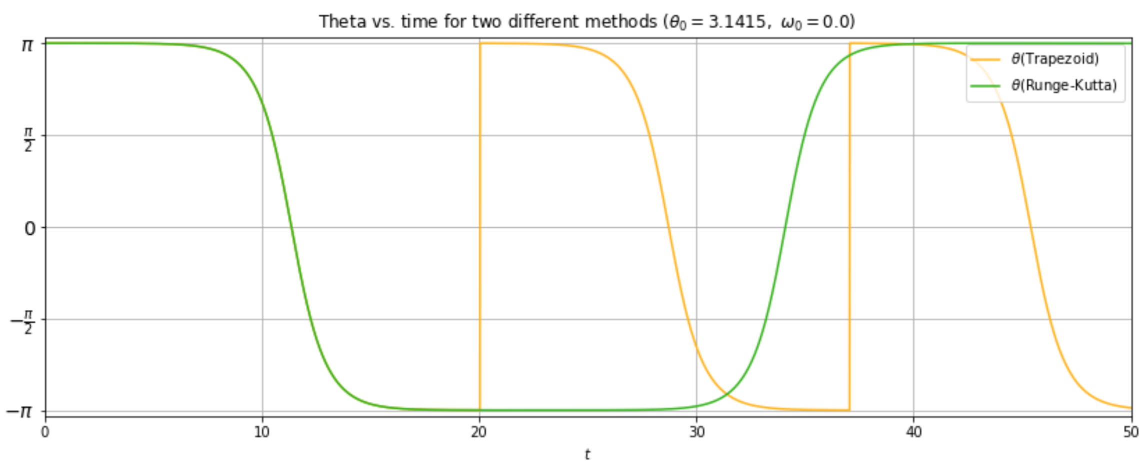

Figure 5: Comparison of the Runge-Kutta method to the trapezoid rule

Here, the fourth order Runge-Kutta method was implemented to achieve greater accuracy of predicted movement of the pendulum for the same Δt. The graph shows the predicted θ for both trapezoid rule as well as the new method, and the starting condition was $\theta_0 = 3.1415$,$\omega_0 = 0.0$. As observed before, the condition where initial angle is close to π seems to be the most challenging to the algorithms - the linear algorithm failed completely, while the trapezoid was not accurate enough to keep the total energy conserved. As is apparent in this graph, however, the Runge-Kutta method seems to be by far the best of the methods, not allowing for complete rotations and keeping the correct shape of the curve. This is due to the fact that at values of θ as high as π, there is a distinguishable difference between the first and fourth order Taylor expansions of the used function.

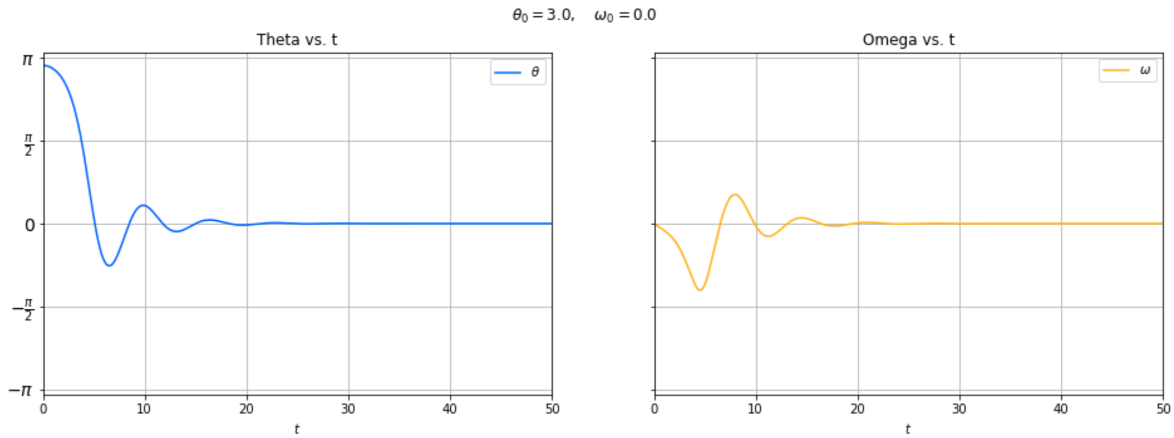

Figure 6: Plot of θ and ω vs. t for a damped ($k = 0.5$) pendulum (nonlinear model, Runge-Kutta method)

When a positive damping coefficient was introduced, the shapes of the graphs changed significantly. In the example shown on the plots, the pendulum starts off at a large initial angle but its movement is quickly - exponentially - dampened. The damping coefficient is small enough to allow for oscillation, meaning the pendulum is underdamped, however the oscillation 'dies down' to a very small level after only about 30 seconds. Therefore, it can be speculated that the shapes of the graphs are compound - comprised of a negative-exponential term multiplying an oscillating term.

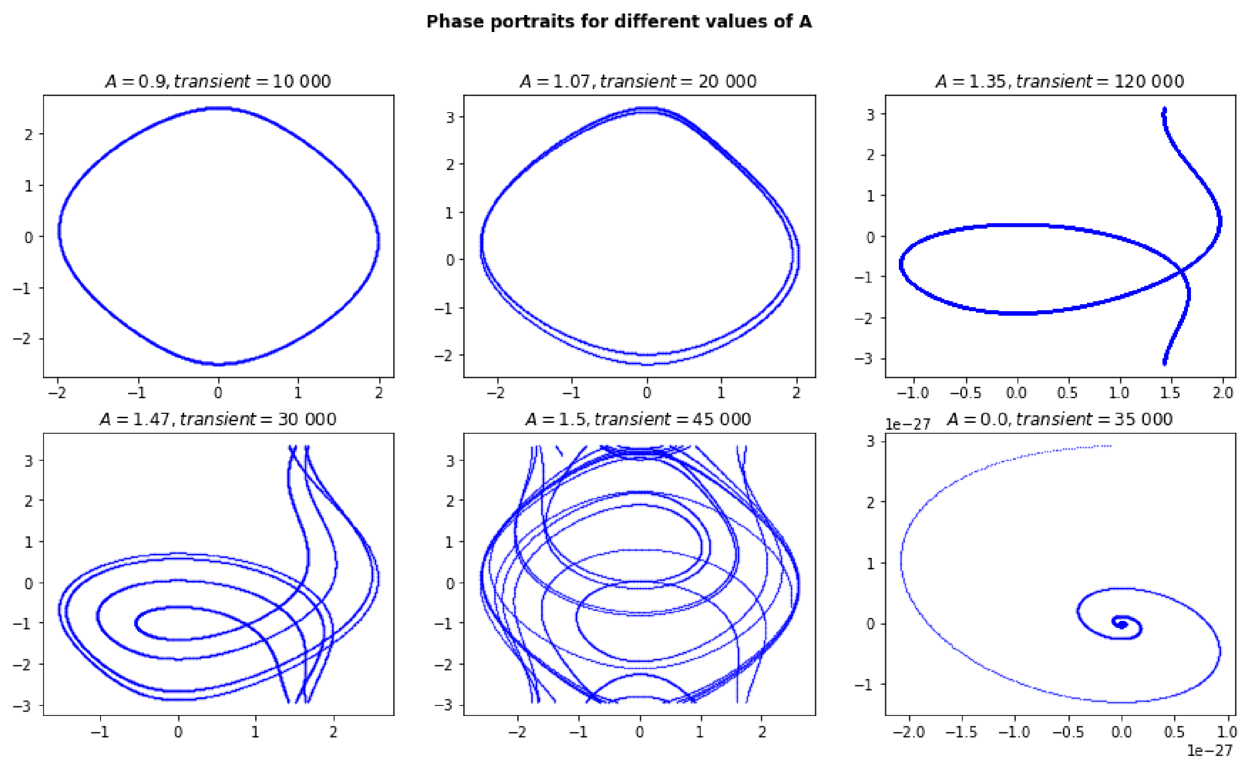

Figure 7: Plot of θ vs. ω for a damped ($k = 0.5$) and driven pendulum (nonlinear model, Runge-Kutta method)

In the last exercise, the phase portraits of damped pendulums driven with different amplitude cosinusoidal forces were generated. To capture only the steady-state behaviour, a variable *transient* was introduced, signifying the amount of steps that would be calculated, but not graphed. After this amount of steps, some additional amount was calculated and the results plotted. Because of the varying complexities of motion for different amplitudes, and to save on computation time, this variable was different for each case.

The first two phase diagrams are closed loops - characteristic of periodic, stable motion. However, none of the other conditions have produced such effects. For $A = 1.35$ and $A = 1.47$, the phase diagram clearly shows some repeating pattern, however one that doesn't produce a loop. When $A = 1.5$, the motion appears very chaotic. An additional example of completely undriven motion was also analysed, and the resulting graph is a spiral, approaching the point $(0, 0)$.

4 Conclusions

In the process of analysing different types of pendulums, the theoretical predictions were confirmed: a simple, linear model is completely sufficient at small enough angles, producing results virtually indistinguishable from more sophisticated methods. However, the larger one of the values of θ or ω gets, the greater the need for higher order approximations. A nonlinear model produces much more realistic results when starting the pendulum close to being above the pivot point.

A comparison of different numerical integration methods was also done, showing that, again the higher order Runge-Kutta method becomes necessary at larger amplitudes in order to avoid errors such as violating energy conservation.

Finally, it was found that for some amplitudes of the driving force, the motion would become periodic, for larger amplidudes the phase diagrams appear to point at chaotic motion.