Comp. Laboratory #3: Projectile Motion

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1 Introduction

In physics, the concept of projectile motion is often considered without air resistance, as its presence means the equations of motion become difficult or even impossible to solve analytically. However, this is not the case for most experiments in the real world. Thankfully, numerical methods allow for simulating almost any circumstance, including the motion of a particle under different types of air resistance.

In this experiment, different approximations of the resistive force were considered, and the ranges of velocity for which each of them is useful were found. Motion of a particle, as well as the optimal launching angle were calculated (using numerical integration) for two of these approximations.

2 Methodology

2.1 Exercise 1

Firstly, the resistive force $\vec{F} = -f(V)\hat{V}$ was analyzed. For D - diameter of the spherical object, and B and C depending on the medium, one can define: $b = BD$, $c = CD^2$ to write:

$$f(V) = bV + cV^2$$

One can observe, that the linear term in f can be written as $B(DV)$, and the quadratic term as $C(DV)^2$. This is useful for establishing over which ranges of DV are each of these terms alone a good approximation.

2.2 Exercise 2

To calculate the terminal velocity of an object, one has to find the equilibrium condition for the value of vertical velocity. This means that the acceleration (and also force) have to be equal to 0:

$$\vec{F_{total}} = -gm\hat{z} + f(V_T)\hat{z} = 0$$
$$\implies bV_T + cV_T^2 = gm \implies V_T = \frac{-b}{2c} + \sqrt{b^2/4c^2 + gm/c}$$

In the case where the terminal velocity is sufficiently small, one can use the linear approximation

$$\frac{dV_y}{dt} = -g - \frac{b}{m}V_y$$

2.3 Exercise 3

In the case where the linear approximation is sufficient, the equations of motion for the two cartesian coordinates (x,y) become independent of each other:

$$\begin{aligned}\frac{dV_x}{dt} &= -b/m \cdot V_x \\ \frac{dV_y}{dt} &= -g - b/m \cdot V_y\end{aligned} \tag{1}$$

and thus can be solved separately.

2.4 Exercise 4

For the high-velocity regime, the quadratic term becomes sufficient to represent air resistance. Here, the two coordinates are no longer independent, as they both depend on $\sqrt{V_x^2 + V_y^2}$:

$$\frac{dV_x}{dt} = -c/m\sqrt{V_x^2 + V_y^2}\, V_x \qquad \frac{dV_y}{dt} = -g - c/m\sqrt{V_x^2 + V_y^2}\, V_y \tag{2}$$

3 Results

3.1 Exercise 1

All the plots were generated for the following values:

$$B = 1.6 \times 10^{-4} Ns/m^2 \quad C = 0.25 Ns^2/m^4$$

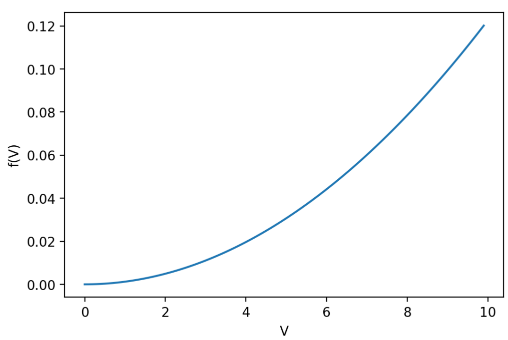


Figure 1: Plot of $f(V)$

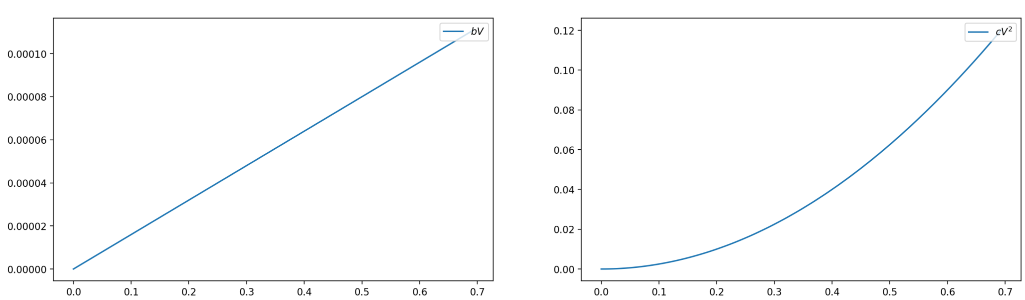


Figure 2: Plots of the linear and quadratic contributions to f(V)

To compare the magnitudes of the linear and quadratic contributions, a simple calculation was done:

$$\frac{cV^2}{bV} = c/b \cdot V = C/B \cdot (D \times V)$$

The y-axis of the generated graph is scaled logarithmically, to better show the difference in orders of magnitude.

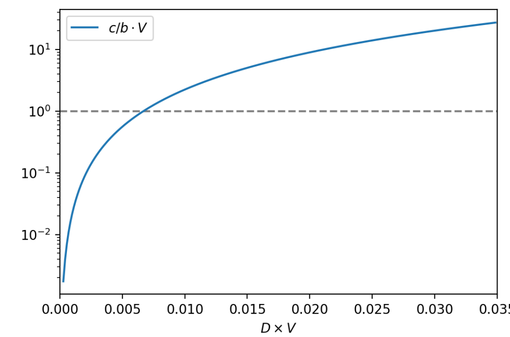


Figure 3: Ratio of the two contributions

It can be observed that for the given conditions, the linear term is an order of magnitude greater than the quadratic when $D \times V$ is below around 2×10^{-3}, and the opposite when $D \times V$ is above around 0.2. Therefore, between these two values, both terms have to be taken into account for a precise result.

Using these values in real world examples:
For a baseball of diameter $D = 7$cm and velocity $V = 5$m/s:

$D \times V = 0.35$, so the linear term is orders of magnitude lower than the quadratic and can be neglected.

For a drop of oil, $D = 1.5 \times 10^{-6}$m, $V = 5 \times 10^{-5}$m/s:

$D \times V = 7.5 \times 10^{-11}$, so the quadratic term is negligible.

For a raindrop, $D = 1$mm, $V = 1$m/s:

$D \times V = 1$, which is in between the two obtained values, meaning both terms have to be included.

3.2 Exercise 2

For a spherical grain of dust with $D = 10^{-4}$m and density 2×10^3kg/m^3, to find out the appropriate approximation, one has to find its terminal velocity. As described before, the terminal velocity takes the form:

$$V_T = \frac{-b}{2c} + \sqrt{b^2/4c^2 + gm/c}$$

where for m, one has to substitute $\pi D^3/6$. For the given conditions, this gives $V_T = 1.605 \times 10^{-8}$m/s, which means the linear approximation will be very accurate over the whole movement.

A simulation was run, assuming the linear (1) form of air resistance, obtaining how the vertical velocity depends on time for particles of different masses (including ones where it wouldn't be a good approximation):

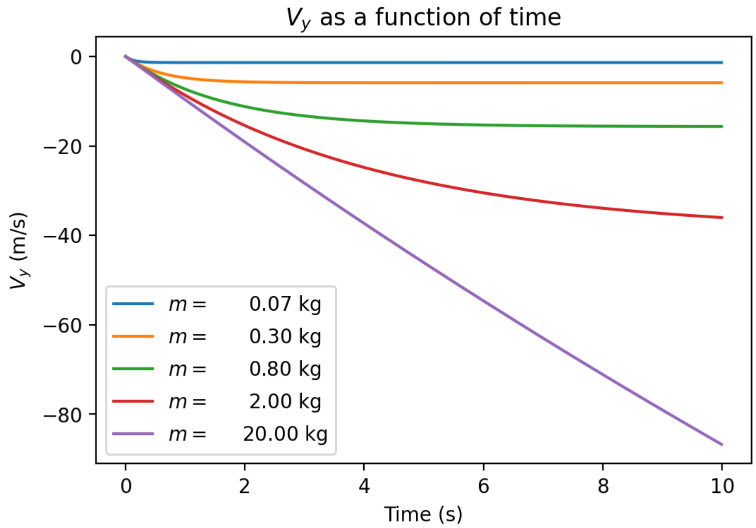


Figure 4: Plot of V_y as a function of time for different masses of particle

It is clear that smaller masses approach their terminal velocity quicker than larger ones. For a mass of 20kg, the vertical velocity seems to just be increasing linearly.

Furthermore, it can be observed that because mass and the coefficient b appear in the equations of motion together, in the form $\frac{b}{m}$, any relative decrease in mass is equivalent to a corresponding relative increase of b.

Comparing the numerical solution to the analytical $V_y = \frac{mg}{b}(e^{-bt/m} - 1)$, one gets the following:

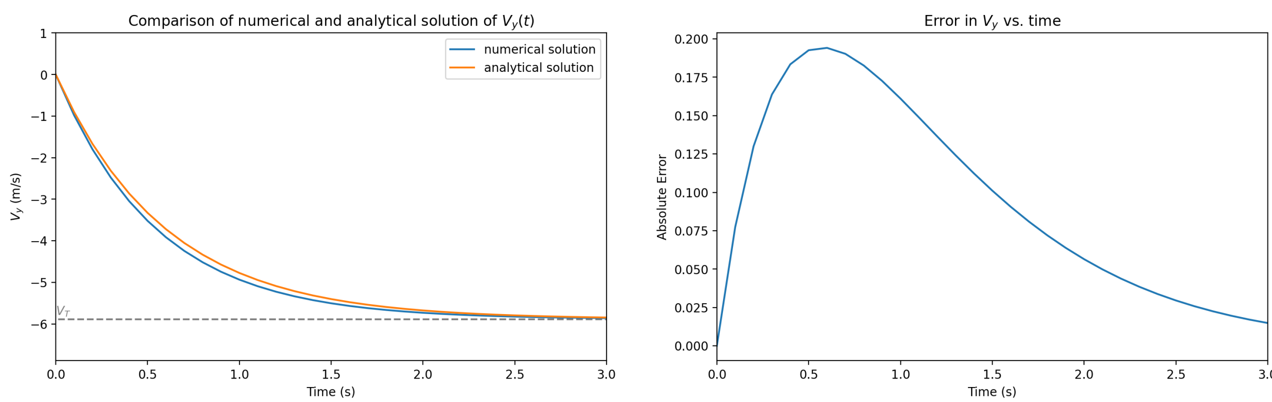


Figure 5: Comparison of numerical/analytical solutions and Error vs time

The error starts at 0, as both the methods are constructed to satisfy the initial conditions, then increases as the numerical solution diverges from the actual values. After some time, the error approaches zero again, as both the methods predict V_y settling at the terminal velocity.

The main way of improving the numerical accuracy is by decreasing the timestep used in the calculations. This, however, directly affect the computation time. Another method could be to use more sophisticated methods of numerical integration, making use of a higher order Taylor expansion.

To generate how the height of the particle changes with time, one may use the fact that:

$$\frac{dy}{dt} = V_y \implies dy = V_y dt \tag{3}$$

This creates a set of two coupled differential equations, that can be simultaneously solved numerically.

The heights of particles of different masses were simulated as a function of time, all starting from $y = 5$m:

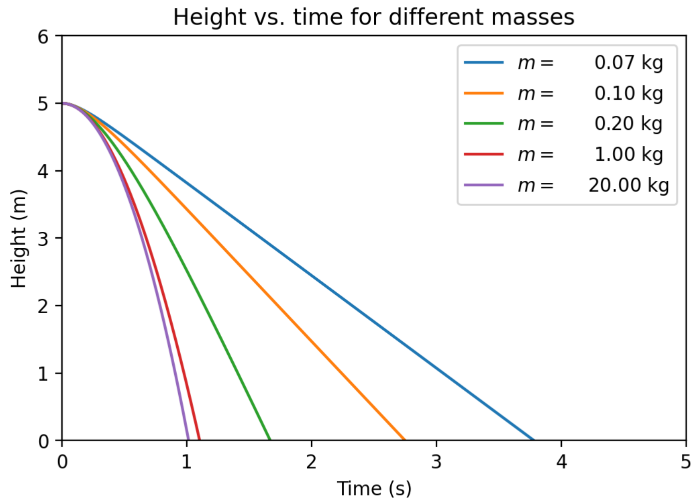


Figure 6: Numerical solutions to $y(t)$ for different masses

It is visible that the particles hit the ground at different times - the lighter ones taking longer. This relation - of time to hit the ground and particle mass was investigated, to check the validity of the statement that 'all objects fall together with the same acceleration regardless of their masses'.

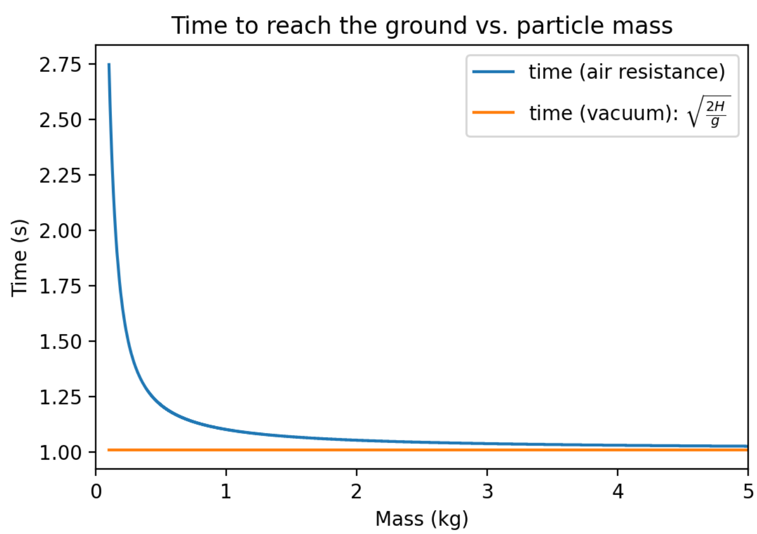


Figure 7: Plot of Time to reach the ground as a function of mass

From the plot can be observed that this statement is far from truth for very small masses or large resistance constants, this agrees with everyday experience - a sheet of paper will fall much slower than a steel ball. However, when the mass considered is sufficiently large, the time to reach the ground approaches the ideal case of motion in vacuum. It must be said, however, that for these large values of m, the used linear approximation of air resistance may not be accurate.

3.3 Exercise 3

To obtain an objects trajectory, the solutions $y(t)$ and $x(t)$ can be first calculated (using (3)) and plotted as $y(x(t)) = y(x)$. This was done in the case of linear air resistance and compared to the analytical solution with no air resistance:

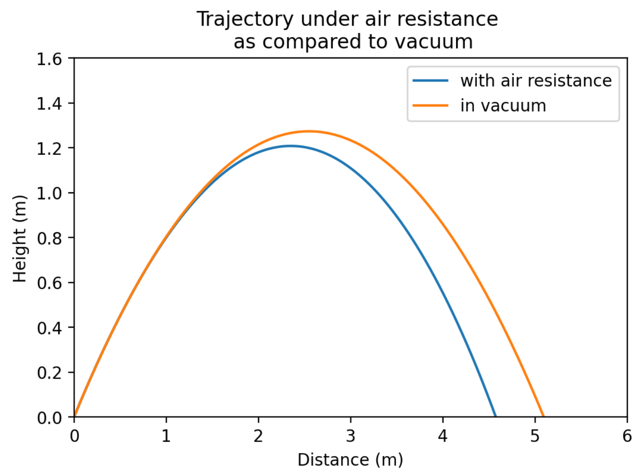


Figure 8: Trajectory under linear air resistance and in vacuum

Another question that was investigated was whether the optimal angle of 45° still holds under air resistance. For this, for different angles, the horizontal distance traveled was compared and the best angle chosen. The procedure was repeated for different values of mass, to produce the following graph:

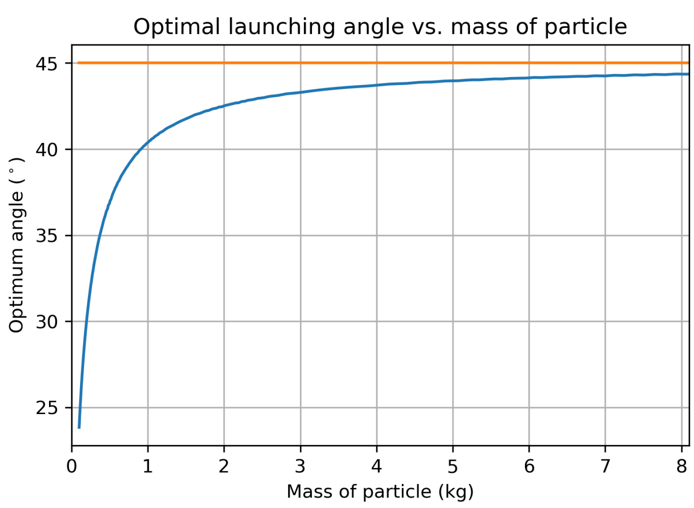


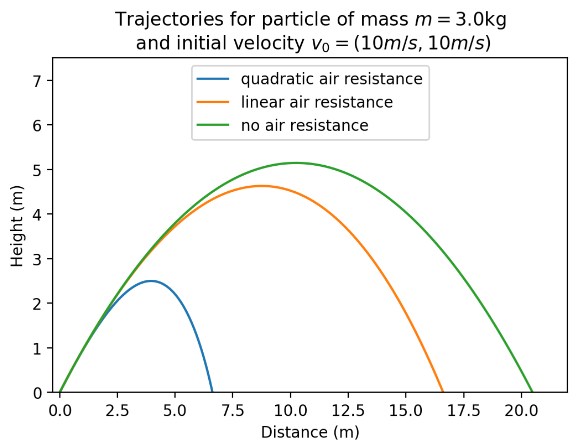
Figure 9: Plot of Optimal launching angle as function of mass

Note on implementation: since the computation of the trajectory for a range of angles with high precision and for many masses requires $\mathcal{O}(\#\ of\ angles \times \#\ of\ masses \times t/tstep)$ time complexity, an optimization method (`scipy.optimize.minimize`) was used to significantly improve time performance of the program.

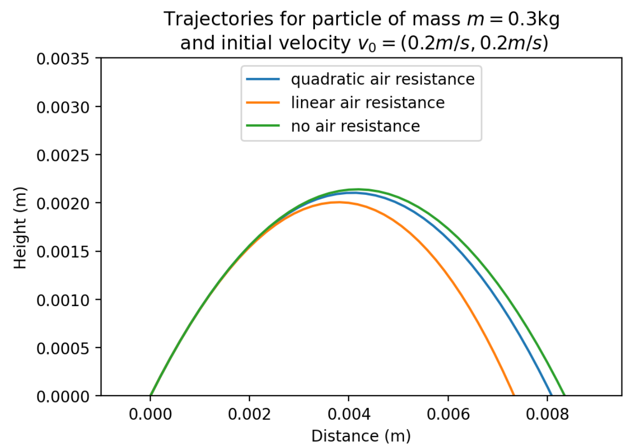
For large enough masses (or weak resistance), the optimal launching angle approaches the expected 45°. However, as was in the case of time to reach ground, for smaller masses the situation looks drastically different with the angle decreasing together with mass.

3.4 Exercise 4

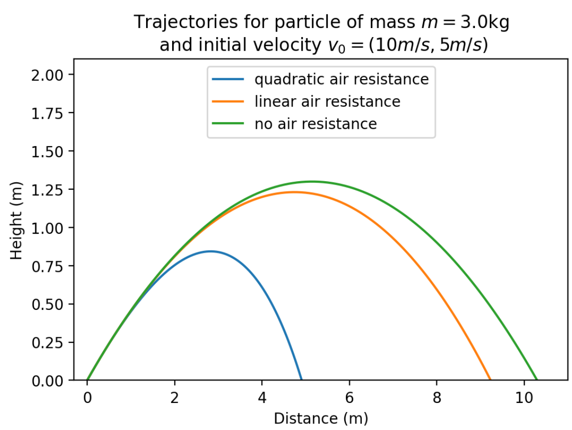
The trajectories of a particle moving with quadratic (linear and no air resistance were plotted and compared. This was done for 4 different sets of parameters:



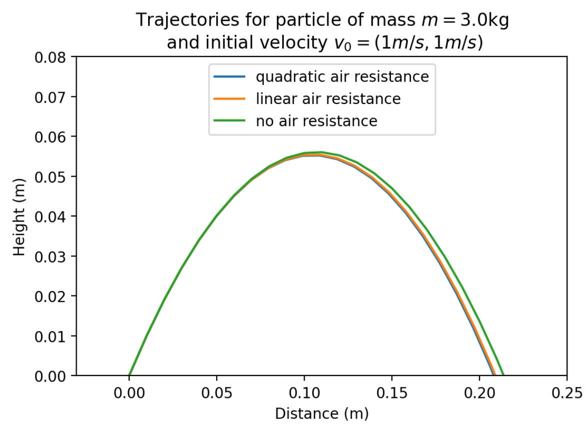
(a) High initial speed



(b) Low initial speed



(c) Lower launching angle



(d) High mass, low speed

Figure 10: Different trajectories under linear, quadratic and no air resistance

Some observations: for high initial speeds (10a), the motion under quadratic air resistance is dampened quicker than that under linear, as expected - and vice versa for low initial speeds (10b). A lower launching angle (10c) leads to a smaller difference between the three trajectories, as does a higher mass/initial speed ratio (10d), but much stronger.

4 Conclusions

As predicted, it was found that there exist different regimes dependent on the diameter of the particle and its velocity, where different approximations for its motion under air resistance are most accurate. It was observed that in air, not all objects fall with the same acceleration and the optimal launching angle is not always 45°, with both being heavily dependent on particle mass. Finally, it was observed that under different initial conditions, the motions in linearly-dependent, quadratic, or no air resistance can be very different or not different at all.