Geometric Decompositions of the 24-Cell in Musical Voice Leading: Polytope Sub-Structures as Models for Harmonic Deviation and Voice Independence

1. Introduction: The Geometric Turn in Cognitive Music Theory

The intersection of geometry and music theory has witnessed a paradigm shift over the last two decades, moving from discrete, low-dimensional lattice models toward high-dimensional, continuous representations of musical space. This "geometric turn" is predicated on the insight that musical objects—notes, chords, scales, and voice leadings—can be rigorously modeled as points, vectors, and manifolds within a hyperspace. Central to this inquiry is the 24-cell (or icositetrachoron), a regular convex 4-polytope that has emerged as a "Rosetta Stone" for Western tonal harmony. Its 24 vertices map isomorphically to the 24 major and minor keys, capturing the intricate web of symmetries, dualities, and cyclic relationships that govern the common practice period and beyond.

However, existing geometric models often suffer from a "monolithic" limitation. By treating the 24-cell as a singular, unified manifold, theorists and computational systems implicitly prioritize "archetypal" harmony—the idealized, mathematically perfect chords that align with the polytope's vertices. This approach, while elegant for analyzing functional harmony, struggles to account for the messy reality of musical performance: the independence of individual polyphonic voices, the phenomenon of "imperfect" harmony (such as bitonality or psychoacoustic roughness), and the expressive nuance of microtonal deviation. In a monolithic model, a voice leading is a single trajectory, and any deviation from the grid is treated as error rather than information.

This report investigates a novel theoretical and computational framework: the decomposition of the 24-cell into constituent sub-polytopes. Specifically, we examine two structural decompositions: the partition into **three disjoint 16-cells** (sets of 8+8+8 vertices) and the hierarchical separation into a **Tesseract and a 16-cell** (16+8 vertices). We posit that these decompositions offer a superior mechanism for mapping distinct musical voices and handling non-archetypal sound. By assigning voices to separate sub-polytopes, we can mathematically enforce independence and define "imperfect harmony" not as a failure of the system, but as a measurable geometric relationship between orthogonal or interstitial spaces.

This analysis synthesizes principles from algebraic geometry, group theory, cognitive science (specifically the theory of Conceptual Spaces), and emerging computational paradigms like Polytopal Projection Processing (PPP). It aims to demonstrate that the internal architecture of the 24-cell—its "bones" and "gaps"—provides the necessary scaffolding for a new generation of music-AI systems capable of reasoning about texture, timbre, and intonation with human-like

subtlety.

2. Theoretical Foundations: The Geometry of the 24-Cell

To understand the power of decomposition, one must first grasp the unique properties of the parent structure. The 24-cell is the only regular convex polytope in four dimensions that has no analogue in three dimensions (unlike the Tesseract, which corresponds to the cube, or the 16-cell, which corresponds to the octahedron). It is a self-dual object composed of 24 vertices, 96 edges, 96 triangular faces, and 24 octahedral cells.

2.1 Coordinate Systems and Symmetry Groups

The vertices of a 24-cell centered at the origin with a radius of $\sqrt{2}$ are given by the 24 permutations of the coordinate vector $(\pm 1, \pm 1, 0, 0)$ in $\mathbb{R}^4$. This coordinate definition is non-trivial; it reveals that the 24-cell can be viewed as a "rectified 16-cell" or, conversely, as the convex hull of a Tesseract and its dual 16-cell combined.
The symmetry group of the 24-cell is the Weyl group F_4, which has order 1,152. This massive symmetry allows the polytope to encode a vast array of musical transformations, including transposition, inversion, and retrograde motion, as isometric rotations. Crucially, the F_4 group contains subgroups corresponding to the symmetries of the 16-cell (B_4) and the Tesseract (BC_4 or C_4). It is the interplay between the parent group and these subgroups that makes decomposition possible.

2.2 The Isomorphism of Keys

In the specific implementation of the MusicGeometryDomain framework reviewed for this research, the 24 vertices are mapped to the 24 keys of Western music. This mapping is not arbitrary; it exploits the **self-duality** of the 24-cell to mirror the **Major/Minor duality** of the tonal system.
- **Type A Vertices (12 Vertices):** These correspond to the 12 Major Keys. Geometrically, these are the permutations involving specific axes pairs (e.g., xy, xz, xw) that define the "primary" structure of the tonal space.
- **Type B Vertices (12 Vertices):** These correspond to the 12 Minor Keys. Geometrically, these are the relative duals to the Type A vertices, often defined by the complementary coordinate permutations (e.g., zw, yw, yz).

In a monolithic model, traversing from C Major to A Minor involves moving along an edge of the polytope. However, this movement obscures the fact that C Major and A Minor belong to different *geometric classes* within the structure—one defines the "face" and the other the "vertex" of the underlying dual relationship. By decomposing the polytope, we can make these class distinctions explicit.

2.3 The Problem of the Monolithic Surface

The primary limitation of current geometric models is the "Flattening Problem." When we map a four-part chorale (Soprano, Alto, Tenor, Bass) onto a single 24-cell, we are essentially projecting four distinct data streams onto a single manifold.

1. **Loss of Independence:** If the Soprano moves from C to G, and the Bass moves from E to C, the geometric model sees a single "chord vector" rotating. It does not natively "know" that two independent agents are acting. The voice leading is inferred post-hoc, rather than being a constraint of the geometry.
2. **Ambiguity of Crossing:** If voices cross (e.g., the Tenor sings a higher pitch than the Alto), the geometric projection often folds onto itself, creating ambiguity.
3. **Binary Harmonic State:** A point is either "on" a vertex (perfect key) or "between" vertices (modulation). There is no native geometric distinction between a "transitional" chord and a "detuned" chord. Both are simply off-vertex vectors.

To solve these problems, we must break the 24-cell apart.

3. Decomposition I: The Triadic Partition (Three 16-Cells)

The first decomposition partitions the 24 vertices of the 24-cell into three disjoint sets of eight vertices. Geometrically, each of these sets forms a regular **16-cell** (also known as a hexadecachoron or 4-orthoplex).

3.1 Geometric Derivation

The 24 coordinate permutations of $(\pm 1, \pm 1, 0, 0)$ can be grouped based on the disjoint pairs of axes that contain the non-zero entries. Since we are in 4D space with axes x, y, z, w, the non-zero entries must occupy two positions. The possible pairs of axes are (x,y), (x,z), (x,w), (y,z), (y,w), (z,w).

We can group these six planes into three **orthogonal sets**:
1. **Set α (The Cardinal 16-Cell):** Combining the xy-plane and the zw-plane.
   ○ Vertices: $(\pm 1, \pm 1, 0, 0) \cup (0, 0, \pm 1, \pm 1)$.
   ○ This forms a 16-cell because the xy vectors are orthogonal to the zw vectors. It is the union of two orthogonal squares (or hyper-rectangles).
2. **Set β (The Diagonal 16-Cell A):** Combining the xz-plane and the yw-plane.
   ○ Vertices: $(\pm 1, 0, \pm 1, 0) \cup (0, \pm 1, 0, \pm 1)$.
3. **Set γ (The Diagonal 16-Cell B):** Combining the xw-plane and the yz-plane.
   ○ Vertices: $(\pm 1, 0, 0, \pm 1) \cup (0, \pm 1, \pm 1, 0)$.

These three sets are mutually disjoint (sharing no vertices) and exhaustive (their union is the 24-cell).

3.2 Musical Significance: The Axes of Tonal Space

When we apply the MusicGeometryDomain key mapping to these sets, a profound musical structure emerges. The decomposition essentially partitions the Circle of Fifths into three "Axes of Tonality."
- **16-Cell α (The Natural Axis):** This cell contains the keys C, G, D, A (Major) and their relative minors A, E, B, F#. This corresponds to the "diatonic heart" of Western music—the keys with few sharps or flats.
- **16-Cell β (The Sharp/Enharmonic Axis):** This cell captures the keys further along the sharp side: E, B, F#, C# (Major) and their relatives.
- **16-Cell γ (The Flat Axis):** This cell captures the flat side: F, Bb, Eb, Ab (Major)

and their relatives.
This is not merely a clustering; it is a **structural orthogonalization**. The keys in 16-Cell \alpha are mathematically orthogonal to the keys in 16-Cell \beta within the 4D embedding space.

3.3 Modeling Voice Independence

The primary application of this decomposition is the modeling of **Polyphonic Voice Independence**. In a complex contrapuntal texture, the goal is to maintain the identity of each voice. By assigning different voices to different sub-polytopes, we can enforce this independence geometrically.
The "Lane" Hypothesis: Imagine we assign the Bass voice to navigate the vertices of **16-Cell \alpha**, the Tenor to **16-Cell \beta**, and the Soprano to **16-Cell \gamma**.
- **Collision Avoidance:** Because the vertex sets are disjoint, it is geometrically impossible for the Bass and Soprano to occupy the same "slot" in the state space. Even if they play the same pitch class (e.g., C), the *context* of that C is different. For the Bass, C is a vertex of 16-Cell \alpha. For the Soprano, C must be approximated or projected from 16-Cell \gamma.
- **Orthogonal Trajectories:** The movement of the Bass is a rotation within subspace \alpha. The movement of the Tenor is a rotation within subspace \beta. These rotations can occur simultaneously without interfering with each other's vector magnitude. This allows the system to model **polyrhythmic harmony** or **independent harmonic rhythms**—where the Bass moves slowly (half notes) while the Tenor moves quickly (eighth notes)—without the global state vector becoming "jittery" or unstable.

3.4 Handling "Imperfect Harmony": Polychords and Bitonality

This decomposition excels at modeling **Imperfect Harmony** defined as **Bitonality** or **Polychordalism**.
Consider the famous "Petrushka Chord" used by Stravinsky, which superimposes C Major and F# Major triads.
- **Monolithic View:** In a single 24-cell model, the centroid of C Major (vertices 0, 4, 1) and F# Major (vertices 6, 10, 7) would likely land somewhere in the middle—perhaps near a diminished chord or in a "dead zone" of the polytope. The system would interpret this as high entropy or error.
- **Decomposed View:**
  - C Major resides comfortably in **16-Cell \alpha**.
  - F# Major resides comfortably in **16-Cell \beta**.
  - The system does not average them. Instead, it reports **two simultaneous, perfect geometric states** occurring in orthogonal dimensions.
  - The "imperfection" (dissonance) is quantified not as error, but as the **Cross-Polytope Tensor Tension**. We can calculate the angle between the \alpha-vector and the \beta-vector. Since C and F# are tritone-related (antipodal in the circle of fifths, orthogonal in the 16-cell decomposition), the system recognizes this as a structured, maximal tension—a stable dissonance.

This allows AI systems to generate or analyze music that is "poly-harmonic" rather than just "noisy." It provides a geometric basis for the "Garden of Forking Paths," where multiple tonalities coexist without collapsing into a single decision.

4. Decomposition II: The Hierarchical Partition (Tesseract + 16-Cell)

The second decomposition is hierarchical rather than egalitarian. It partitions the 24 vertices into a set of 16 (forming a Tesseract) and a set of 8 (forming a 16-cell).

4.1 Geometric Derivation

The 24-cell can be constructed by rectifying a 16-cell or, alternatively, by taking the convex hull of a Tesseract and its dual 16-cell.
- **The Tesseract (16 Vertices):** We can form a Tesseract by taking the union of **any two** of the disjoint 16-cells defined in the previous section. For example, $\alpha \cup \beta$. These 16 vertices form the vertices of a hypercube.
- **The 16-Cell (8 Vertices):** The remaining set (e.g., γ) forms the dual 16-cell.

Crucially, the vertices of the 16-cell align with the **cell centers** of the Tesseract. This geometric relationship—Duals nested within Primaries—is the key to modeling hierarchy.

4.2 Musical Mapping: The "Grid" and the "Gap"

This decomposition provides a powerful model for **Tonal Hierarchy** and **Chromaticism**.
- **The Tesseract (The "Grid"):** This polytope represents the **Diatonic** or **Archetypal** framework of a piece. It contains 16 keys—enough to encompass the primary key, its dominant, subdominant, and relative minors. In a stable musical passage, the state vector is confined to the Tesseract.
- **The 16-Cell (The "Residual"):** This polytope represents the **Chromatic**, **Alien**, or **Passing** tones. It contains the 8 keys that are most distant from the Tesseract's center of gravity.

4.3 Modeling Microtonal Deviation and "Rub"

The most significant advantage of this decomposition is its ability to handle **Microtonal Deviation** and the "rub" of imperfect intonation.

Research indicates that there are "measurable 4-dimensional interstices" between the envelope of the inscribed Tesseract and the envelope of the enclosing 24-cell. This interstitial volume is not empty space; it is the **Geometric Home of Microtonality**.

The Vector Space Model of Intonation: In a standard model, a note is a point. In this decomposed model, we can define a microtonal note as a **vector sum** of a Tesseract vertex and a 16-cell vertex.

Let V_{T} be a vertex of the Tesseract (representing a perfectly tuned C). Let V_{16} be a vertex of the 16-cell (representing a distant key or chromatic neighbor).

We can define a microtonal pitch P as:

where λ is a "Deviation Coefficient."
- **When $\lambda = 0$:** The note is perfectly "in tune" (on the Tesseract grid).
- **When $\lambda > 0$:** The note drifts away from the grid towards the 16-cell dual. This drift represents **expressive intonation** (e.g., a "blue note" or a leading tone pushed sharp).
- **"Rub":** When two voices have different λ values, their vectors create friction in the

interstitial space. The decomposed model allows us to quantify this "rub" as the volume of the simplex formed by the Tesseract basis and the deviation vectors.

Why is this better than the Monolithic Model? In a monolithic 24-cell, a microtonal note is simply "off-vertex." It has no semantic anchor. Is it a sharp C or a flat C#? The model cannot tell. In the decomposed model, the deviation is **vectorized**. We know it is "a C pulling towards the dual." The 16-cell vertices act as **attractors** for chromatic deviation. This allows the system to distinguish between "randomly out of tune" (error) and "expressively out of tune" (structural tension).

5. Comparative Analysis: Monolithic vs. Decomposed Architectures

This section explicitly compares the three approaches to synthesize the findings.

Feature	Monolithic 24-Cell	Decomposition I (Three 16-Cells)	Decomposition II (Tesseract + 16-Cell)
Primary Use Case	Global Key Navigation, Modulation	Polyphonic Voice Independence, Bitonality	Hierarchical Harmony, Microtonality, Intonation
Voice Representation	Single Centroid Vector (Average of all notes)	Three Independent Vectors (Orthogonal Subspaces)	Core Vector (Grid) + Deviation Vector (Gap)
Imperfect Harmony	Modeled as **Noise/Entropy**. (Distance from nearest vertex)	Modeled as **Structural Tension**. (Angle between sub-polytopes)	Modeled as **Interstitial Displacement**. (Vector magnitude in gaps)
Microtonality	Ambiguous. Hard to distinguish from modulation.	Limited. Can map phase shifts between axes.	**High Fidelity.** Maps microtones as trajectories between Duals.
Computational Load	Low (Single F_4 lookup).	Medium (Three parallel B_4 lookups).	High (Requires convex hull & interstitial volume calc).
Cognitive Analogy	"Concept" (Static State)	"Parallel Processing" (Independent Streams)	"Figure-Ground" (Focus vs. Periphery)

5.1 Insight: Adaptive Complexity in AI Music Generation

The existence of these multiple decompositions suggests an architecture of **Adaptive Complexity** for AI music generation.

An advanced music AI should not use a single geometry. Instead, it should dynamically switch between decompositions based on the musical texture:

- **Phase 1 (Simple Chorale):** Use the Monolithic 24-cell for efficient, functional harmony.
- **Phase 2 (Fugal/Polyphonic):** Switch to the **Three 16-Cells** model to track independent voice lines and prevent collision.
- **Phase 3 (Expressive Solo/Jazz):** Switch to the **Tesseract + 16-Cell** model to capture the "blue notes," microtonal inflections, and the push-pull against the grid.

This mirrors the biological principle of **recruitment**, where the brain engages additional neural

resources (or grid cell modules) to handle increased complexity.

6. Computational Framework: Polytopal Projection Processing (PPP)

The decompositions described above are not merely abstract constructs; they are the data structures for a new computational paradigm known as **Polytopal Projection Processing (PPP)**. As outlined in the research material , PPP treats reasoning as geometric rotation.

6.1 The "Garden of Forking Paths"

The user query references the "Garden of Forking Paths," a metaphor for multi-future prediction in generative systems. The decomposed 24-cell provides the rigorous geometry for this garden. In a monolithic model, predicting the next chord is a probabilistic guess—collapsing the wave function to a single outcome. In the **Three 16-Cells** model, the "forking" is literal:
- The system can project three simultaneous futures: one for the Bass (in Cell α), one for the Tenor (in Cell β), and one for the Soprano (in Cell γ).
- Because the spaces are orthogonal, these futures do not need to agree immediately. The system can maintain a superposition of harmonies (e.g., a "suspended" state) where one voice resolves while another holds tension.
- This allows for the generation of **long-range dependencies** and delayed resolutions, traits of high-quality human composition that Markov chains and simple RNNs struggle to capture.

6.2 Implementation via Hyperdimensional Computing (HDC)

These geometric operations map directly to **Hyperdimensional Computing (HDC)** vectors.
- **Superposition:** The monolithic state is $S = A + B + C$.
- **Decomposition:** The decomposed state is a tensor product $S = A \otimes B \otimes C$.
- **Rotation:** Voice leading is modeled as multiplying the state vector by a rotor (from Clifford Algebra). In the decomposed model, we apply *independent rotors* to each subspace: $S_{next} = (R_\alpha \cdot A) \otimes (R_\beta \cdot B) \otimes (R_\gamma \cdot C)$.

This algebraic independence is what allows the AI to "reason" about voices separately.

7. Implications for Hardware: The Photonic Advantage

The computational cost of maintaining and rotating multiple high-dimensional polytopes is significant for standard CMOS hardware. However, the report research highlights **Neuromorphic Photonics** as the ideal substrate for this architecture.
- **The 16-Cell as Optical Native:** The 16-cell (Cross-Polytope) is the geometric dual of the Hypercube. In optical computing, signals are naturally represented as phase/amplitude vectors in complex space. The "Binding" operation in HDC (which creates the tensor product) corresponds to simple optical interference or element-wise multiplication, which photonics does at the speed of light.
- **Parallelism:** The "Three 16-Cells" decomposition is perfectly suited for **Wavelength Division Multiplexing (WDM)**. We can encode the α-axis on one wavelength (e.g.,

1550nm), the \beta-axis on another, and the \gamma-axis on a third. A single photonic core can thus process the entire decomposed harmony in parallel, with zero latency penalties for the added complexity.

- **Sparse Activation:** In moments of simple harmony (perfect Tesseract alignment), the system can power down the "16-cell deviation" circuits, saving energy. This mimics the brain's energy-efficient coding.

8. Conclusion

The investigation into the decomposition of the 24-cell polytope reveals that the monolithic model, while structurally beautiful, is functionally insufficient for capturing the nuance of human musical performance. By breaking the 24-cell into sub-structures, we gain the geometric vocabulary to describe "imperfection."

1. **The Triadic Decomposition (Three 16-Cells)** solves the problem of **Voice Independence**. By partitioning the tonal space into orthogonal axes, it allows multiple voices to coexist and move without collapsing into a single data point. It effectively "channelizes" harmony, enabling the modeling of bitonality and complex counterpoint.
2. **The Hierarchical Decomposition (Tesseract + 16-Cell)** solves the problem of **Intonation and Hierarchy**. By establishing a "Grid vs. Gap" topology, it allows us to define imperfect harmony and microtonality not as errors, but as measurable vectors occupying the interstices of the polytope.

Ultimately, this research suggests that "Harmony" is not a static property of a chord, but a dynamic tension between geometric containers. The "perfect" 24-cell is merely the resting state; the music lives in the decompositions. For AI to truly create music that breathes and moves like a human performance, it must learn to navigate not just the vertices of the polytope, but the spaces in between.

Data Tables

Table 1: Comparison of Polytope Decompositions for Musical Application

Feature	Monolithic 24-Cell	Three 16-Cells (8+8+8)	Tesseract + 16-Cell (16+8)
Vertex Count	24	8 + 8 + 8 (Disjoint)	16 (Grid) + 8 (Dual)
Geometric Basis	Permutations of $(\pm 1, \pm 1, 0, 0)$	Orthogonal Coordinate Planes (e.g., xy/zw)	Convex Hull of Hypercube & Orthoplex
Musical Mapping	Global Key Map (Circle of Fifths)	Three "Axes" of Tonality (Natural, Sharp, Flat)	Diatonic Core vs. Chromatic Shell
Voice Independence	**Low.** Voices collapse to centroid.	**High.** Voices occupy orthogonal subspaces.	**Medium.** Focus is on Foreground vs. Background.
Imperfect Harmony	Interpreted as **Noise/Error**.	Interpreted as **Inter-Axis Tension** (Polychords).	Interpreted as **Interstitial Vector** (Microtonality).
Symmetry Group	F_4 (Order 1152)	$B_4 \times 3$ (Order 384×3)	BC_4 (Hypercubic)
AI Application	General Harmonic	Polyphonic Generation	Expressive

Feature	Monolithic 24-Cell	Three 16-Cells (8+8+8)	Tesseract + 16-Cell (16+8)
	Navigation	/ Source Separation	Performance / Intonation Modeling

Table 2: Vertex Mapping to Musical Keys in the Monolithic vs. Decomposed Models

Decomposition Set	Vertex Indices (from MusicGeometryDomain)	Coordinate Characteristics	Musical Keys (Major/Minor)	Functional Role
16-Cell \alpha	0-3, 20-23	xy-plane dominant / zw-plane dominant	C, G, D, A (Maj) / F, C, G, D (Min)	**"Natural Axis"**: Home keys, low chromaticism.
16-Cell \beta	4-7, 16-19	xz-plane dominant / yw-plane dominant	E, B, F#, Db (Maj) / Bb, D#, G#, C# (Min)	**"Sharp Axis"**: Tension, brightness, dominant expansion.
16-Cell \gamma	8-11, 12-15	xw-plane dominant / yz-plane dominant	Ab, Eb, Bb, F (Maj) / A, E, B, F# (Min)	**"Flat Axis"**: Relaxation, darkness, subdominant drift.
Tesseract	Union of any two above (e.g., \alpha \cup \gamma)	e.g., xy, zw, xw, yz planes	All Natural + Flat Keys	**"Diatonic Grid"**: The stable harmonic framework.
Residual 16-Cell	The remaining set (e.g., \beta)	e.g., xz, yw planes	All Sharp Keys	**"Chromatic Dual"**: The keys that destabilize the grid.

Citations

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