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Geometric Cognition in a Nested Polytopal Architecture

Abstract

We propose a formal synthesis of **Geometric Information Theory (GIT)** and **Polytopal Projection Processing (PPP)** to define a multi-scale geometric architecture for cognition. At its core is the 4-dimensional *24-cell* polytope (icositetrachoron), which we identify as a cognitive “fulcrum” embedded within a higher-order matrix of the *600-cell* (hexacosichoron). We show that the 600-cell’s 120 vertices decompose into **25 interlinked 24-cells**, with precise overlap structure proven by recent work [1] [2]. The symmetry group of this arrangement (H_4 or 2I, the binary icosahedral group) and its **5×5 Latin-square** structure of 24-cell partitions [3] [4] are derived and analyzed. We further review Conway’s icosian approach to the E_8 lattice, **projecting E_8 onto 4D** to reveal an $H_4 \oplus \phi H_4$ construction: a 600-cell plus a ϕ-scaled 600-cell (i.e. 120-cell) intertwined via the golden ratio φ [5] [6]. This provides a rigorous *projection formalism* from 8D to 4D, linking the 25-fold 24-cell complex to E_8’s geometry.

Using this framework, we formalize **moiré interference patterns** among discrete vertex sets in 4D: overlapping rotations of 24-cells within the 600-cell create emergent periodicities analogous to moiré fringes, yielding *quasi-lattice* patterns at larger scales. Category theory is invoked to construct these polytopal systems: we treat each polytope’s face lattice as a poset category and show that complex compounds (like the 25×24-cell network) can be seen as colimits (glued by shared vertices and facets) in a “polytopal category.” Within GIT, we interpret **convex polytopes as information-theoretic equilibria**: convexity serves as an equilibrium constraint both in thermodynamic potentials and in cognitive state-space. We cite the uniqueness of the Legendre transform as the order-reversing duality on convex functions [7] and show that entropy–energy duality induces convex, self-consistent constraints [8]. By analogy, we adopt a **Convexity Axiom** for conceptual spaces: natural categories correspond to convex regions in the high-dimensional *Orthocognitum* (cognitive state-space) [9]. This aligns with both information geometry (where free-energy landscapes are convex [10]) and cognitive geometry (where concept regions are convex to ensure semantic stability [9]).

Finally, we delineate speculative but testable conjectures stemming from this architecture. We hypothesize a *fractal-like* scaling whereby iterative φ-projections of H_4 (embedding 600-cells within 120-cells recursively) produce self-similar cognitive state-space structure. We propose that new *homological cycles* (topological holes) emerge from the 25-fold polytope overlap, corresponding to emergent memory or concept loops in cognition. We describe how **triadic rotations** of 16-cells inside each 24-cell (an S_3 symmetry of an inscribed 3-16-cell compound [11] [12]) could implement *dialectical transformations* of cognitive states. We discuss how the model could be **falsified or supported** by cross-domain evidence: e.g. detecting 4D lattice interference patterns in neural data (grid cell firing correlating with 24-cell geometry) or achieving combinatorial generalization in AI systems built to compute via 24-cell rotations. The result is a comprehensive formalism positioning the 24-cell and its golden-ratio nested polytopes as the scaffolding of a multi-scale, geometric theory of cognition, unifying information-theoretic optimality with high-dimensional neural coding.

Introduction

Modern approaches to artificial and natural cognition are increasingly turning to geometry and topology to bridge symbolic structure and statistical learning [13] [14]. *Geometric Cognition* hypothesizes that mental representations are not arbitrary symbols or distributed vectors, but points, regions, and transformations in a structured geometric space [15] [16]. Within this paradigm, **Polytopal Projection Processing (PPP)** has emerged as a proposed computational framework in which cognitive states are encoded as *high-dimensional polytopes* whose rotations and projections enact reasoning [17]. In parallel, **Geometric Information Theory (GIT)** seeks a deep unity between information processes (inference, learning, thermodynamic analogues) and geometric structures, positing that fundamental information invariances correspond to convexity, duality, and curvature in an abstract space [18] [8]. This paper aims to formally synthesize and extend the PPP framework with GIT principles, grounding the cognitive geometry hypothesis in rigorous mathematics of *4D polytopes*, lattice theory, and information geometry.

A central claim of PPP is that cognition can be realized in *four-dimensional* state-spaces, notably using regular 4-polytopes (the *Polychora*) as scaffolding [17]. Among these, the **24-cell** (Schläfli symbol {3,4,3}) is unique: a self-dual regular polytope with 24 vertices and 24 octahedral cells, existing only in 4D. Prior work suggests the 24-cell's symmetry and self-duality make it a natural "coordinate system" for combinatorial cognition (it has no lower-dimensional analog) [19]. However, a single 24-cell by itself is static and homogeneous; to capture *contextual, multi-scale structure* (essential for ambiguity, hierarchy, and generalization in cognition), we must embed the 24-cell in a larger geometric architecture. We posit that the **600-cell** {3,3,5} – a 4D polytope with 120 vertices and 600 tetrahedral cells – provides this higher-order structure, with the 24-cell as a repeating subunit. Specifically, the 600-cell contains **25 distinct inscribed 24-cells** linked by a precise overlap graph [2]. Each 24-cell represents a "local" cognitive frame, and the 600-cell's global structure links these frames via shared vertices in a manner akin to a network of overlapping concepts. Our thesis is that *the 24-cell functions as a cognitive fulcrum within the 600-cell*, mediating between local transformations (modeled by 24-cell rotations) and global coherence (enforced by 600-cell symmetries). By analyzing this 25-fold 24-cell complex and its symmetries, we aim to define a **multi-scale cognitive architecture**: one that spans fine-grained, triadic state transitions inside a 24-cell and broad, interference-based patterns across the entire 600-cell.

In what follows, **Section 2** reviews the necessary mathematical background on 4D regular polytopes and lattice projections. We derive the *25-fold 24-cell decomposition of the 600-cell* and present a proof of this structure based on the work of Denney *et al.* (2020) [1]. We introduce the symmetry group actions (Coxeter group H_4 and its subgroups) that organize the 24-cells, explaining how a *5×5 array* of 24-cells (rows and columns corresponding to distinct 5-fold partitions) emerges from the geometry [3]. We also describe the *overlap graph* of the 24-cells: each 24-cell shares certain vertices with others, and in fact any given 600-cell vertex is the intersection of 5 different 24-cells [20]. This yields a highly symmetric overlap network in which 144 closed pentagonal loops interlink all 25 nodes (24-cells) of the network [2]. **Section 3** then connects these 4D structures to the 8-dimensional E_8 lattice. We summarize Conway's *icosian construction*, wherein the **icosian quaternion group** (vertices of a 600-cell) is extended to the full E_8 lattice by adjoining $\sqrt{5}$ (the golden ratio field) [5] [6]. By projecting E_8 onto 4D via the icosian mapping, one obtains two interpenetrating copies of the 120-vertex 600-cell: one scaled by a factor of φ (the golden ratio $\phi=\frac{1+\sqrt5}{2}$) relative to the other [21] [5]. We formally define this $H_4 \oplus \phi H_4$ structure and discuss how the **120-cell** (the dual of the 600-cell, with 600 vertices) naturally arises as the ϕ-scaled companion. This projection formalism not only proves that the 600-cell's symmetry (non-crystallographic H_4) embeds in E_8 (a crystallographic root lattice) [6], but also provides a **multi-scale**

lattice for cognition: the unscaled 600-cell defines a "module" of states, and the scaled copy defines a larger-scale module, whose combination yields a quasicrystalline, **moiré-like superstructure** in 4D.

In **Section 4**, we leverage category theory and information geometry to interpret the above polytopal architecture in cognitive terms. We treat the system of polytopes as a **categorical construct**: each polytope's internal structure (faces, cells, etc.) forms a lattice that can be viewed as a category (with inclusion maps as morphisms). We show that the compound structures (e.g. the compound of 25 24-cells) can be regarded as **colimits** of smaller polytope-categories, glued along shared sub-faces. This provides a high-level, coordinate-free description of how simpler cognitive components might compose into a complex knowledge space. We then address the role of **convexity** as a unifying principle. In information theory and thermodynamics, convexity of certain functions (e.g. entropy $S(U)$ concave in energy vs. free energy $F(T)$ convex in temperature) is not just a mathematical curiosity but a *necessary condition for stable equilibria* [10] [8]. The Legendre duality between entropy and free energy – or generally between any conjugate pair of thermodynamic potentials – guarantees convexity and hence stability (since a violation of convexity signals a phase transition or instability) [22] [23]. **Artstein-Avidan & Milman (2009)** proved that the Legendre transform is essentially the *unique* involutive duality transform on convex functions (up to affine changes), cementing its foundational role [7]. By analogy, we argue that convex geometries (e.g. convex polytopes in state-space) are the *signatures of cognitive equilibrium*. We adopt as a postulate that **meaningful cognitive states correspond to convex regions** in the state-space. This *Convexity Constraint* has empirical support in human cognitive semantics: Gärdenfors (2000) noted that natural concept classes tend to form convex regions in a psychological feature space (e.g. color categories are convex in the Lab color space) [24] [9]. We integrate this constraint with our polytopal model by asserting that the fundamental *concept cells* of the Orthocognitum are convex polytopes – specifically, the 24-cells and their unions. In this way, the GIT notion of convex equilibrium and the PPP notion of geometric concept representation converge.

In **Section 5**, we venture into carefully delineated **conjectures and implications** of the proposed architecture. We label these ideas as speculative, distinguishing them from the rigorous results of Sections 2–4. We explore the possibility of **fractal-like or quasicrystalline scaling** in cognitive representation: since the golden ratio φ is incommensurate (its powers never repeat rationally), a system that includes scales of size 1 and φ (like the 600-cell and 120-cell lattice) can produce *aperiodic tilings* in the 4D state-space. We hypothesize that this could endow the cognitive geometry with self-similar structure across scales – an appealing property for modeling hierarchical or recursive thought patterns. We also discuss **homological emergence**: the 25 interlinked 24-cells might support higher-dimensional cycles (e.g. a closed chain of conceptual transformations that cannot be continuously deformed to a point) – essentially, nontrivial homology groups in the neural state-space. Such topological features could correlate with recurrent or cyclic cognitive processes (for instance, revisiting a concept in a complex analogy might correspond to traversing a loop in this network). We then consider **dynamic rotations and interference**: small rotations of individual 24-cells (implementing local state transitions or "thought moves") will interact via the shared vertices to produce large-scale *interference patterns*, akin to moiré patterns when lattices are overlaid at slight relative angles. We formalize a notion of *4D moiré* arising from multiple 24-cells being rotated isoclinically (in 4D, an isoclinic rotation can rotate two orthogonal planes simultaneously). The resultant pattern may manifest as constructive and destructive interference in the occupancy of certain vertices (states), effectively creating an interference-based mechanism for *context gating* or *conceptual blending*. Intriguingly, this resonates with how grid cells in the brain might form large-scale periodic representations by interference of multiple smaller grids – we note that entorhinal grid fields at different scales can superpose to encode very large spaces via beat patterns (a 2D analogy) [16]. Our model provides a concrete

4D instance of this idea, potentially relevant both to biological cognition and to engineered AI systems. Finally, we articulate how the theory can be **validated or falsified**. We propose specific predictions: for example, if the brain indeed utilizes something like a 24-cell representation, one might find neural populations whose activity relations correspond to the 24-cell's 96 edges or 24 octahedral cells (e.g. forming an F_4 symmetry arrangement). In AI, one could implement a *Chronomorphic Polytopal Engine* (as per PPP) [25] [26] – essentially, a neural network whose latent states are constrained to lie on polytope structures – and test if it exhibits better generalization or interpretability. The falsifiability lies in the rigidity of our geometric predictions: if cognition does **not** rely on these polytopal structures, then forcing such structure in a model should hinder, not help, its performance. We also underscore open problems and limitations: How might noise and imperfect symmetry in real neural systems affect the ideal polytopal model? How might the continuous dynamics of thought map onto piecewise-linear polytope transitions? These questions frame a path for future research.

In summary, this work provides a detailed theoretical foundation for a **nested polytopal model of cognition**, grounded in established geometry and extending into new hypotheses. We proceed now with the mathematical core: the 24-cell within the 600-cell and its golden-ratio lattice context.

2. 25-Fold Partition of the 600-Cell into 24-Cells

2.1 The 600-Cell and Its 24-Cell Constituents: The *600-cell* (also called the hexacosichoron) is one of the six regular convex 4-polytopes, with Schläfli symbol {3,3,5}. It consists of 600 tetrahedral cells, 120 vertices, 720 edges, and 1200 triangular faces [27]. Its symmetry group is the order-14400 Coxeter group H_4, which is isomorphic to the *binary icosahedral group* (denoted 2I) – the double cover of the icosahedron's rotational symmetry [28] [29]. The 600-cell is dual to the 120-cell {5,3,3}, but here we focus on a remarkable property of the 600-cell: **it contains 25 inscribed 24-cells** in a highly symmetric arrangement [2]. Each inscribed 24-cell is a set of 24 vertices (one fifth of the 600-cell's 120 vertices) that themselves form a regular 24-cell polytope. The existence of these 24-cells and their exact enumeration were conjectured over a century ago (by Schoute in 1905) and verified computationally by Roberson (2017). Recently, Denney *et al.* (2020) provided a human-readable proof and a deeper analysis of this structure [30] [1].

We first present a self-contained derivation of this **25-fold partition**. *Partition* here means that the 25 inscribed 24-cells are arranged such that any given vertex of the 600-cell belongs to multiple 24-cells in a structured way, and in fact one can select 5 of them to form a disjoint cover of all 120 vertices. To avoid confusion: there are **only 10 ways** to pick 5 disjoint 24-cells that cover the vertices without overlap [31] [32], but overall *25 distinct 24-cells* exist, each sharing some vertices with others [2]. A helpful analogy is a 5×5 grid or Latin square: there are 5 disjoint 24-cells in each partition, and 10 such partitions, which altogether account for $5\times 5 = 25$ unique 24-cells [3] [1]. We can label these 24-cells as $C_{i,j}$ for $i,j\in\{0,1,2,3,4\}$, where each fixed i (row) corresponds to one partition and each fixed j (column) to another partition [1]. The *row* i contains 5 disjoint 24-cells $C_{i,0},\dots,C_{i,4}$ whose union of vertices is the whole 600-cell. Likewise, each *column* j contains 5 disjoint 24-cells $C_{0,j},\dots,C_{4,j}$ forming another complete partition. The grid of $C_{i,j}$ thus enumerates all 25 inscribed 24-cells, and each appears exactly once in each of two partitions (its row and its column). This structure was in fact described as "something like a Latin square" by Baez [33] [34], reflecting the combinatorial design governing the overlaps.

Proof Sketch: The existence of 5 disjoint 24-cells whose vertices partition the 600-cell can be understood via group theory. The vertices of a 600-cell can be identified with the elements of the *binary icosahedral group* 2I (of order 120) [28] [29] – concretely, the 120 vertices correspond to unit quaternions forming the icosian group [35] [36]. The **binary tetrahedral group** 2T (order 24) is a subgroup of 2I [37] [38]; geometrically, 2T's 24 elements are the vertices of a 24-cell (this is one way to construct the 24-cell, as the quaternion group of unit Hurwitz quaternions) [37]. Now, consider *any* embedding of 2T into 2I (there are many conjugacy classes of subgroups). The cosets of 2T in 2I will be of the form $g\,2T$ for $g\in 2I$, and since $[2I:2T]=120/24=5$, there are 5 distinct cosets [39] [40]. These cosets are pairwise disjoint and each coset $g\,2T$ has 24 elements; identifying each coset's elements as vertices, we obtain 5 disjoint 24-vertex sets, each forming a regular 24-cell inscribed in the 600-cell [41] [40]. This yields one partition of the 600-cell's vertices into 5 disjoint 24-cells. Different choices of the subgroup 2T (i.e. different conjugates $h2T h^{-1}$ inside 2I) will yield different partitions. It turns out all such partitions can be related by the action of the symmetry group 2I itself. In fact, Denney *et al.* prove that exactly 10 distinct partitions exist [30], confirming Schoute's count. The union of all partitions' 24-cells gives 25 unique 24-cells total (since each partition contributes 5, but each 24-cell appears in exactly 2 partitions, consistent with $10\times5/2=25$) [1].

To characterize the **overlap structure**, we note some key facts: (a) Each 600-cell vertex belongs to exactly 5 of the 25 inscribed 24-cells [20] [2]. Indeed, "five 24-cells meet at each 600-cell vertex" [42]. In terms of our 5×5 grid $C_{i,j}$, one can deduce that each vertex lies in one 24-cell from each row and one from each column (making 5 total) – hence the grid structure. (b) Any two distinct 24-cells either share **0, 1, or 12** vertices, depending on their relation. They share 0 if they lie in the same partition (since those 5 are disjoint by construction), they share 12 vertices if they intersect in a symmetric way (half the vertices of one coincide with half of the other – such pairs are those $C_{i,j}$ that are neither same row nor same column), and they share exactly 1 vertex if one is in the same row as the other is in the same column, meaning they intersect at a single vertex on a common pentagonal loop (explained below) [2] [43]. (c) The 24-cells are *interlinked by pentagons*: The 600-cell contains 144 distinct great circle *pentagons* (closed 5-cycles of vertices) [2]. Each such pentagon lies on one vertex in each of 5 different 24-cells, effectively linking those 5 24-cells in a cycle [44]. Conversely, any given 24-cell participates in a number of these pentagonal loops, each loop picking out one vertex from that 24-cell. It can be shown that each 24-cell intersects each pentagon in at most one vertex [44]. The picture emerges of an "overlap graph" of 25 nodes (the 24-cells) where every pentagon in the 600-cell corresponds to a 5-node cycle. Since each 600-cell edge lies in 3 pentagons, and each vertex in 6 pentagons, this overlap graph is highly connected (in fact strongly regular). A particularly important sub-structure is the *compound of five 24-cells*: any of the 10 partitions can be seen as a 5-cell compound inscribed in the 600-cell [39] [30], analogous to the classical compound of five tetrahedra inscribed in a dodecahedron (the 3D case) [45] [46]. This compound has full icosahedral symmetry and is self-dual (the compound of five 24-cells is the 4D analogue of the compound of five tetrahedra being self-dual) [47] [48]. The remaining 20 out of 25 24-cells can be regarded as rotations of this compound through the symmetry group.

For concreteness, we can give **explicit coordinates**. One convenient coordinate system for the 600-cell is obtained from the *icosian* quaternions [49]. Using the representation from Conway and Sloane [50] [21], the 120 vertices of a 600-cell centered at the origin can be taken as all even permutations of: - $\tfrac{1}{2}(\pm 2,0,0,0)$ (8 vertices), - $\tfrac{1}{2}(\pm1,\pm1,\pm1,\pm1)$ (16 vertices), - $\tfrac{1}{2}(0,\pm1,\pm\phi^{-1},\pm\phi)$ (96 vertices), in the 4-dimensional space $\mathbb{R}^4$ (here $\phi=(1+\sqrt5)/2$ is the golden ratio) [50]. These 120 points form the 600-cell; they also form the group 2I under quaternionic multiplication. Now, one choice of a 24-cell inside this is given by the subgroup 2T, which in these coordinates can be identified as the 24 Hurwitz units: - $2T =$ all even permutations of $(\pm1,0,0,0)$ (8 vertices) and $(\pm\tfrac{1}{2},\pm\tfrac{1}{2},\pm\tfrac{1}{2},\pm\tfrac{1}{2})$ with an even number of minus signs (16 vertices) [12]. These 24 points indeed form a 24-cell (the binary tetrahedral group) [11] [12]. For example, one such 24-cell $C_{0,0}$ can be explicitly listed as:

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(±1,0,0,0) and permutations;
(±½,±½,±½,±½) with even number of negatives.
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Now, to get the other 24-cells, we multiply this set on the left by elements of 2I not in 2T. Take an element g corresponding to one vertex of the 600-cell, say $g=(0,1/φ,1/φ,1/φ)$ (one of the 120 vertices). The coset $g\cdot 2T$ yields a second 24-cell $C_{0,1}$ consisting of g times each of the above 24 quaternion-vertices. Doing this for 5 appropriately chosen coset representatives $g_0=1, g_1, g_2, g_3, g_4$ (which can be selected as any set of one from each coset) yields 5 disjoint 24-cells forming one partition. If we choose a different initial 24-cell (a different 2T subgroup, i.e. conjugate), we get another partition. Ultimately, one finds 25 distinct sets of the above form. A detailed listing appears in Denney *et al.* (they provide tables enumerating vertices for each 24-cell) [3]. For our purposes, the existence and symmetry properties are most important.

2.2 Symmetry Group Actions (Weyl groups and S_5 structure): The symmetry group of the 600-cell is H_4, of order 14,400. It acts transitively on the set of 120 vertices, and also on the set of 25 inscribed 24-cells (indeed, by symmetry all 24-cells are equivalent). However, there is a higher combinatorial symmetry relating the 24-cells: as noted, they can be arranged in a 5×5 grid where rows and columns correspond to 5-partitions. This suggests an S_5 permutation symmetry acting on the indices. In fact, the action of the full Coxeter group H_4 on the 25 labeled 24-cells factors through a group of order 120 which is isomorphic to S_5 [1]. Intuitively, the 5 cosets in one partition can be thought of as being permuted by some symmetry (related to the 5-fold rotational symmetry of the icosahedral structure), and similarly the 5 partitions themselves are interrelated. More concretely, Baez describes the 25 24-cells as ${g^i (2T) g^j: i,j\in{0,\dots,4}}$ where g is an element of order 5 in 2I [51] [52]. Here left-multiplying by g^i and right-multiplying by g^j generate the row and column index shifts respectively. This construction makes the 5×5 structure manifest: one can regard i as labeling cosets and j labeling conjugate subgroups, or vice versa. The group of order 25 generated by $g^i 2T g^j$ (with i,j mod 5) is a Sylow-5 subgroup of H_4 which is abelian (in fact isomorphic to $\mathbb{Z}_5 \times \mathbb{Z}_5$) and corresponds to the grid translations. The full symmetry that permutes rows and columns (i.e. any Latin square permutation) yields an S_5. Indeed, as the abstract of Denney *et al.* states, the geometry of 24-cells in the 600-cell "in §7… show[s] how the 600-cell transforms $E_8/2E_8$… into a 4-space over F_4 whose points, lines and planes are labeled by the geometric objects of the 600-cell" [53] – in that correspondence, the 25 24-cells plus 60 vertex pairs (antipodal vertex pairs of 600-cell) amount to 85 objects, which matches the number of 1-dimensional subspaces of a 4D vector space over $\mathbb{F}_4$ (since $(4^4-1)/(4-1)=85$) [54]. In that mapping, the 25 24-cells correspond to 25 specific lines in $\mathbb{F}_4^4$, and the S_5 symmetry arises from the projective linear group $PGL(4,4)$ (order 120) which is isomorphic to S_5. This beautiful connection is beyond the scope of our paper to fully detail, but it reinforces the combinatorial regularity of the 25 24-cells: they can be treated as projective points or lines in a finite geometry, with symmetry group S_5.

For a more intuitive picture, one can examine smaller analogues. In 3 dimensions, the icosahedron (20 vertices) has an inscribed compound of five tetrahedra (each tetrahedron 4 vertices, five of them making 20 total) [45]. The rotational symmetry group A_5 of the icosahedron permutes these five tetrahedra as the alternating group on 5 objects [55] [56]. Each vertex of the icosahedron belongs to exactly two tetrahedra (since 5 tetrahedra ×4 vertices each would count 20×2=40 incidences, each of 20 vertices in exactly 2) – analogously to each 600-cell vertex lying in 5 of the 24-cells (since 25×24 would count each of 120 vertices in exactly 5). The 4D case can thus be seen as an expansion of this 3D compound: from five 4-vertex tetrahedra to twenty-five 24-vertex 24-cells. The overlap graph in 3D was the famous *icosahedral Petersen graph* (each pentagon linking 5 tetrahedra, etc.), and in 4D it becomes a more complex 5-regular graph on 25 nodes (each node connected to others via pentagon circuits). We will not delve further into the graph theory, but emphasize that the proven existence of this 25-24cell configuration [2] is a cornerstone of our cognitive model: it provides a concrete multi-component structure in which local units (24-cells) are globally interwoven.

2.3 The Compound of Three 16-Cells in the 24-Cell: Before moving on to the E_8 lattice, we highlight one more nested structure: *inside each 24-cell, there is a tripartite rotation symmetry between three 16-cell sub-polytopes*. The **16-cell** (hexadecachoron) is another regular 4-polytope, Schläfli symbol {3,3,4}, having 8 vertices (and 16 tetrahedral cells – hence the name). It is the dual of the tesseract (8-cell). A 24-cell of 24 vertices can be *exactly partitioned into 3 disjoint 16-cells* (each with 8 vertices) in a unique way [11] [12]. This fact is known in the literature as the *regular compound of three 16-cells* inside the 24-cell, and is related to the subgroup structure $W(D_4) < W(F_4)$ of the Weyl groups (since the symmetry group of the 24-cell is the Weyl group $W(F_4)$ of order 1152, and that of the 16-cell is $W(B_4)\cong W(C_4)$ of order 384, which is isomorphic to $W(D_4)$ due to the exceptional isomorphism in rank 4; $[1152:384]=3$) [57] [58]. The 3 inscribed 16-cells are analogous to the 3 orthogonal axes or the 3 quaternion units: indeed, if one takes the 24-cell vertices in coordinates as above, one 16-cell can be the 8 coordinate vectors $(\pm1,0,0,0)$ etc., and the remaining 16 vertices (the half-diagonals $(\pm½,\pm½,\pm½,\pm½)$ split by parity of signs) form the other two 16-cells [12]. The symmetry quotient $W(F_4)/W(D_4)\cong S_3$ permutes these three 16-cells cyclically [11]. In quaternionic terms, if the 24 vertices of the 24-cell are the binary tetrahedral group 2T, then left-multiplication by the quaternion $\omega=\frac{-1+i+j+k}{2}$ (of order 3) cycles through the three cosets of the binary cubical group 2O (which would be a 16-cell of order 8? Actually 2O has 24 elements too, hmm). More directly, one can observe that an isoclinic rotation by 120° in a certain invariant plane of 4D space will map each 16-cell to the next. **Why is this relevant for cognition?** Because it provides an internal three-way *dialectic rotation* within each cognitive atom (24-cell). We speculate that these could represent, for example, three fundamental *perspectives* or *modalities* that a concept can take, with a 120° rotation (an element of order 3 in the symmetry) cycling through them. In Section 5 we will link this to the idea of triadic reasoning or thesis–antithesis–synthesis cycles. For now, the key takeaway is: **each 24-cell has an inherent triadic structure (3 inscribed 16-cells) with an S_3 symmetry**, and each 16-cell in turn can be seen as spanned by 4 orthogonal axes (like basis vectors) giving it a bit-structure. This nested symmetry (16-cell inside 24-cell, 24-cell inside 600-cell) yields a hierarchy: 16-cell vertices are like basis concepts, 24-cell vertices are compound concepts, 600-cell vertices are global states.

Having established the polytope-within-polytope relationships and symmetries, we now move outward to the 8-dimensional lattice context that ties everything together via the golden ratio.

3. Projection from E_8 to $H_4 \oplus \phi H_4$ (600-Cell and 120-Cell)

3.1 The E_8 Lattice and the Icosian Construction: The E_8 lattice is a highly symmetric 8-dimensional lattice with 240 minimal vectors (root vectors) and exceptional symmetry (Weyl group of type E_8). Conway and Sloane famously described an **icosian construction** of E_8 using quaternions [5] [6]. The idea is to leverage the fact that the **icosian ring** (Hamilton's quaternions with components in $\mathbb{Z}[\phi]$) can produce an 8D lattice isomorphic to E_8 when one considers the norm form induced by quaternion multiplication [5]. In simpler terms, one takes pairs of *icosians* (elements of the form

$a+b\sqrt5$ in the four quaternionic coordinates) which gives an 8-dimensional vector, and one defines a norm such that the set of all such pairs with certain integrality forms the E_8 lattice [5]. The crucial fact used is that the unit *icosians* (the 120 vertices of the 600-cell) and their $\sqrt5$-scaled counterparts (related to the 120-cell) together fill out the E_8 structure [21] [5]. Specifically, as noted earlier: - The 120 **unit icosians** (norm 1) form the 600-cell's vertices [50]. - The 600 **icosians of norm 2** form the 120-cell's vertices [59] (since the 120-cell has 600 vertices). These are the two shell layers of the E_8 root system when projected appropriately. Indeed, the E_8 roots can be described as $E_8 = {(x,y) \in \mathbb{Z}^8 \cup (\mathbb{Z}+\tfrac{1}{2})^8 : \sum x_i \equiv 0 \mod 2}$, or via quaternions: $E_8 = { q_1 + q_2\sqrt5 : q_1,q_2 \in \mathbb{H}(\mathbb{Z})}$ with the appropriate norm [5]. Under this construction, one finds exactly 240 shortest vectors, which split naturally into two sets of 120: one corresponding to $(1,0,0,0)$-type quaternions (the 600-cell vertices) and one to $\tfrac{1}{\phi}(1, \phi,0,0)$-type (which actually yield the 120-cell after normalizing lengths) [50] [59]. The golden ratio ϕ enters because the quaternion components involve $\frac{1}{2}(0, \pm1, \pm\phi^{-1}, \pm\phi)$ terms as seen earlier. A more group-theoretic statement is: **H_4 (the symmetry of the 600-cell) is a subgroup of E_8's symmetry**, and one can choose an H_4-invariant 4D subspace of $\mathbb{R}^8$ such that the projection of the E_8 roots onto this subspace yields the 600-cell and 120-cell vertices. In fact, one can embed H_4 as a *maximal sublattice* of E_8—this is part of a wider phenomenon where non-crystallographic Coxeter groups in 2,3,4 dimensions correspond to A_4, D_6, E_8 in 4,6,8 dimensions respectively [60] [61]. The icosian ring formalism rigorously establishes that **the golden field extension $\mathbb{Q}(\sqrt5)$ is the key** to going from 4D to 8D: it allows a $\mathbb{Z}$-basis of 8 vectors where 4 are essentially the 600-cell basis and 4 are ϕ times those basis vectors (in a different orientation). The resulting set of all integer combinations is E_8 [5].

Crucially, this shows that one can decompose the E_8 lattice as a direct sum of two *scaled copies of the H_4 lattice* (the H_4 lattice being the set of all integer linear combinations of 600-cell vertices). Symbolically, one might write $E_8 = H_4 \oplus \phi H_4$, though this is a bit of an oversimplification (the union ${v + w : v\in H_4, w\in \phi H_4}$ doesn't produce a lattice unless certain integrality conditions are met, but in the quaternionic formulation it does). This **$H_4 \oplus \phi H_4$ structure** is exactly what is meant by the "φ-scaled 600-cell and 120-cell geometries" in our thesis. The 600-cell and 120-cell are dual polytopes (one's vertices correspond to the other's cells). In coordinates, if the 600-cell has edge length 1, the 120-cell inscribed in the same 3-sphere will have edge length $1/\phi$ (or vice versa, depending on normalization) [62] [63]. In the icosian coordinate given, the 600-cell vertices are length-1 quaternions and the 120-cell vertices end up length-ϕ or 2, consistent with that scaling [59]. The golden ratio being present is intimately connected to the 5-fold symmetry. In summary, *the E_8 lattice provides a unifying 8D structure in which a 4D observer sees both a 600-cell and a scaled 120-cell superimposed*. This can be visualized as a 4D quasicrystal: the two sets of points (600-cell and 120-cell vertices) interpenetrate in a non-periodic fashion (since φ is irrational with respect to 1), giving a highly symmetric but aperiodic array of points in 4D. This, in fact, has been studied in the context of quasicrystals and theoretical physics [60] [61].

3.2 Multi-Scale Implications – Moiré and Quasicrystal Patterns: Why is the $H_4 \oplus \phi H_4$ decomposition important for cognition? It provides a natural way to incorporate *multiple scales* or *frequencies* of representation in the model. In neural terms, one could imagine one set of grid-cell-like representations with a certain "periodicity" (the 600-cell scale) and another with a larger scale (the 120-cell, which has vertices further apart in the 4D space). The overlay of these two can produce interference patterns that have much larger effective periods (like a beat frequency). In 2D, the interference of two grids with incommensurate periods produces a moiré pattern that can have a very large repeat length relative to the base grids. In 4D, the concept is similar: overlaying two lattices (600-cell lattice and a scaled copy) yields a *quasiperiodic* set of points that can encode very large structures (in fact, an infinite aperiodic tiling of 4D space) [64] [65]. We conjecture that cognitive maps might exploit this, just as the brain's spatial navigation system appears to (with multiple modules of grid cells of different spacings). For example, one 600-cell could represent fine distinctions between concepts, while a φ-scaled 600-cell overlay provides a coarser coding of contexts or situational frames. A given cognitive state might be represented by a point that is simultaneously near a vertex of the fine lattice and a vertex of the coarse lattice; as one moves through state-space, the fine representation moves through many small cells while the coarse one moves through fewer large cells – only when both align do we get a full repeat, enabling a combinatorially large space with manageable bases. This is precisely the function of moiré or quasicrystal codes: they extend range or capacity without losing resolution [66] [67].

Mathematically, one can describe the moiré pattern by considering the intersection of the two lattices or their difference set. If Λ is the H_4 lattice (vertices of 600-cell scaled appropriately) and $\Lambda'=\phi \Lambda$ the scaled version, the intersection $\Lambda \cap \Lambda'$ is sparser (in fact trivial aside from 0 if φ is irrational relative to the lattice basis). But one can look at pairs $(v,v') \in \Lambda \times \Lambda'$ that approximately coincide. In the cognitive interpretation, a state might require "resonance" between a fine and coarse code, meaning we look for $v \in \Lambda$ and $w \in \Lambda'$ that are close in $\mathbb{R}^4$. Those nearly coincident points occur in a periodic fashion within a higher-dimensional embedding (the combined system lives effectively in 8D or in 4D with a quasi-periodic modulation). As an analogy, consider a 1D case: if you have points at integers (lattice $\mathbb{Z}$) and points at multiples of π (incommensurate), the "aligned" points are those where $n \approx k\pi$ for integers n,k. These never exactly coincide, but for any large length L you can find an n and k such that $n \approx k\pi$ within $1/L$. This property yields an effective large cycle. In 4D, with φ, the least common multiple in a rational sense doesn't exist, but functionally one can define a tolerable alignment scale. Biological systems might not require exact recurrence, only sufficient similarity.

Returning to rigorous geometry: the combination of the 600-cell and 120-cell (scaled) in E_8 also means the **24-cells are present at multiple scales**. The 120-cell (600 vertices) contains many inscribed 24-cells as well – in fact, by duality and the correspondence under projection, one can expect 75 inscribed 24-cells in the 120-cell by an analogous count (since the roles of 5 and something might swap under duality, but let's not speculate; Coxeter's tables of regular compound facetings indicate possible compounds of 24-cells in the 120-cell too) [68]. So one might even get 25 24-cells at one scale and another 25 (or more) at the other scale. The union could be up to 50 distinct 24-cells, etc. However, many of those will not lie in the same 4-space after projection, so we consider the original 25 as our core set and imagine the scaled ones as a second set.

Finally, the E_8 connection provides some powerful algebraic tools. It implies that one can use the language of **Lie algebras and root systems** to talk about cognitive states: for instance, the 24-cell's structure is tied to the D_4 root system (since 24-cell vertices are the D_4 roots), the 600-cell to the H_4 non-crystallographic root system, and E_8 contains both. One might ask if cognitive transformations correspond to reflections or rotations in these root systems. The binary icosahedral group 2I we discussed is actually isomorphic to the rotational symmetry group of the 600-cell (order 600), whereas the full 120-cell symmetry (including reflections) is H_4 (order 14400). The E_8 Weyl group is much larger (order ~696729600), so obviously not all E_8 symmetries are used by the brain (that would be overkill!). But it provides a scaffolding. One alluring speculation is that something like *error-correcting codes* or *neural codes* might live in E_8 (since E_8 is famously a perfect sphere-packing). It's known for example that a binary code related to the 24-cell (the extended Golay code of length 24) is linked to the

Leech lattice in 24D; here we have an analog: the E_8 lattice is like an error-correcting code that could protect cognitive states against noise (since E_8 has the highest kissing number in 8D, it's optimal for separating points). While we won't pursue coding theory here, this connection underscores that the geometric framework is not just abstractly elegant but could serve functional roles (error tolerance, efficient representation) crucial for cognition.

4. Categorical and Information-Geometric Formulations

Having laid out the multi-polytope geometry, we now shift perspective to describe it in more abstract frameworks: category theory (for structural composition) and information geometry (for equilibrium constraints). The goal is to show that our model is compatible with – and indeed illuminated by – these formalisms, which are pillars of GIT and modern mathematical system theory.

4.1 Polytopes as Categories (Face Lattices and Colimits): Each convex polytope P can be associated with a *face lattice* $L(P)$: an ordered set of all faces of P (including the polytope itself and the empty face) ordered by inclusion. For example, the 24-cell has faces of dimensions 0 (vertices), 1 (edges), 2 (faces, which are triangles for the 24-cell), and 3 (cells, which are octahedra). The face lattice encodes the combinatorial structure of P. This lattice is in fact a graded poset and can be treated as a **category**: objects are faces and there is a morphism $F \to F'$ iff $F \subseteq F'$ (face inclusion). Such categories are thin (at most one arrow between any two objects) and in fact equivalent to their posets. Why consider this trivial category? Because it allows us to use category-theoretic operations to build complex structures from simpler ones. Specifically, the notion of **gluing polytopes along shared faces** can be formalized as taking a *pushout* or *colimit* in the category of polytopes (or their face posets). For regular polytopes, one often considers their *compounds* or *facetings* in classical terms, but a categorical view might describe the 25-24cell configuration as a colimit of 25 copies of a 24-cell identically glued wherever a vertex (0-face) is common. The "overlap graph" of 24-cells we discussed essentially indicates how the colimit is formed from pairwise intersections that are either a single vertex or an entire 12-vertex set. One could sketch a diagram of inclusions: for each vertex v of the 600-cell, we have 5 inclusion maps of v into the 5 distinct 24-cells that meet there. The colimit of this diagram (five 24-cells with their vertex v identified as one) yields a sub-complex, and doing this for all vertices consistently yields the entire 600-cell (plus perhaps its interior structure). In other words, **the 600-cell can be seen as the union of 24-cells identified along shared vertices** [2]. Each pentagon loop of vertices imposes a certain identification cycle of 5 vertices (one from each of 5 24-cells) forming that pentagon. In category terms, those identifications are additional commutativity constraints in the diagram. The result is a complex colimit that yields the full vertex set and incidence structure of the 600-cell. This categorical composition underscores that the 25 24-cells are *like pieces of a puzzle* that fit together to create a consistent 4D structure. It hints at a higher-level **topos** or category of "Polytopal cognition" wherein objects are knowledge states (polytopal complexes) and morphisms are geometric mappings or refinements between them. In that vein, one might consider functors that take a polytope to its face lattice or to its symmetry group, etc., and study natural transformations (for instance, mapping each polytope's face lattice to a Boolean algebra of propositions about that state – linking geometry to logic).

Another categorical aspect is the idea of **colimits as cognitive integration**. If each 24-cell represents a context or domain of knowledge (with its internal structure of concepts as vertices and relations as edges), then the full cognitive space is a colimit of these contexts identified on common elements (i.e. the same concept appearing in multiple contexts). The overlap of 24-cells on a vertex can be seen as the formal analog of *two contexts sharing a concept*. Five contexts sharing one concept (as in the pentagon linking five 24-cells by one common vertex) is a configuration where a particular core concept is interpreted in five different frames or perspectives – yielding a rich connection (one might poetically think of a concept that sits at the intersection of five broad ideas). Category theory assures us that if we specify all pairwise overlaps (like a diagram of contexts and common concepts), there is a well-defined colimit that amalgamates them into a single global structure (the pushout of all pairwise identifications). Our polytopal model thus slots naturally into this framework: the 25 24-cells plus their overlaps is exactly such a diagram. The **Mesoscale Emergence Schema (MES)** – although not explicitly defined earlier, we interpret MES model as perhaps a theoretical construct for emergent structure – could refer to how larger wholes emerge from such categorical gluings of simpler parts. In our case, the 600-cell (plus φ-scaled copy) is the emergent whole from gluing 24-cells. The MES could be formalized as the colimit of an indexed family of polytopal categories representing local state-spaces.

4.2 Convexity, Duality, and Equilibrium in Information Geometry: We now connect the above structural picture to *information-theoretic geometry*. A recurring theme in both thermodynamics and information theory is the interplay of convex and concave functions and their Legendre duals. We recall a few fundamentals: In thermodynamics, if $S(U)$ is entropy as a function of internal energy (with volume, particle number fixed), $S(U)$ is concave (reflecting the second law's maximization of entropy at equilibrium), and the *free energy* $F(T)$ (a function of temperature T) is convex. S and F are Legendre transforms of each other (one switches from extensive U to intensive $T=\partial S/\partial U$ variables) [8] [10]. The Legendre transform $f^*(y) = \sup_{x}(x\cdot y - f(x))$ maps a concave entropy to a convex free energy [8]. This transform is essentially an information-geometric duality as well (between likelihood parameterization and expectation parameterization of statistical models – in exponential families, the log-partition function and the entropy are Legendre duals) [69]. The result, as noted, is that equilibrium conditions (maximizing entropy or minimizing free energy) yield convex functions whose maxima or minima are stable. Artstein-Avidan & Milman's characterization result [7] emphasizes that this convex duality (Legendre transform) is not just one arbitrary choice but the canonical one, under mild axioms (order-reversing involution, etc.).

Translating to cognition: if we view the cognitive state-space as an analog of a thermodynamic manifold, then stable cognitive states (concepts, percepts, belief configurations) should correspond to optima of some potential function. In GIT, one might posit that there is an "entropy of belief" or a "surprisal" function over possible states, and the system seeks a balance that maximizes entropy given constraints (or minimizes a free-energy-like function as in some models of the brain). If so, the *geometric representation of those states should reveal convexity*. For example, if a concept is defined by being the "most uninformative generalization" of some examples (maximum entropy subject to fitting the data), then that concept will lie in a convex region of feature space (since any mixture of two data-fitting distributions also fits, by convexity of the entropy maximization problem). Indeed, as referenced, Gärdenfors' **Convexity Axiom** for concepts [9] states that natural categories are convex regions. Information geometry provides a rationale: **Fisher information metric** on the probability simplex and the associated dually-flat affine connections guarantee that exponential family models lead to convex information projections [70] [71]. In other words, when the brain picks an explanation or a category for stimuli, it likely does so by something analogous to a projection in a convex space (like finding the nearest "mean" on a convex manifold of prototypes).

Our polytopal model enforces convexity at a structural level: every state is within the convex hull of extreme states (vertices). The 24-cell, 600-cell, etc., are all convex polytopes. Moreover, any intermediate state (a point inside these polytopes) can be expressed as a mixture (convex combination) of vertices. This resonates with the idea that if you have two pure concepts (vertices), any blend or ambiguity between them is represented by a point along the edge connecting them – which lies inside the polytope and is thus *still a* *valid state in the same conceptual space*. If concepts were non-convex, a straight interpolation might leave the space of meaningful states, which would imply a strange discontinuity or disallowed partial state. But cognitive blends do seem possible (e.g. a color between red and yellow is orange, still a valid color). Enforcing convexity ensures no “forbidden gaps” between representable mixtures. In our architecture, **convexity is preserved through all levels**: the local 24-cells are convex, their union (the entire state-space) is also convex (since ultimately all vertices of 24-cells are vertices of the 600-cell, which is convex). However, one must be cautious: the union of polytopes is not convex if taken naively (25 separate 24-cells make a non-convex union). But the 600-cell is convex and contains all vertices, so if we consider the whole 600-cell, that’s convex. If cognition only allowed states at the vertices or edges of those 24-cells, it’d be discrete. But presumably continuous trajectories through the interior are allowed, corresponding to gradual transitions. The *Orthocognitum* introduced in PPP [72] [73] is essentially the union of all concept polytopes, but if one imposes the convexity constraint, it might actually be the convex hull of all concept regions, making it an overall convex “knowledge manifold” (likely with piecewise-linear facets corresponding to boundaries between concepts).

From a thermodynamic analogy: each 24-cell might be like an “energy well” or mode, and the 600-cell convex hull provides a basin of attraction for states. The **MES (Monotonic Entropy Structure?)** could imply that as systems reach equilibrium, they form convex hulls (polytopes) naturally. Indeed, one could speculate a principle: *Opposing cognitive forces (like expectations vs. observations, or different drives) find equilibrium in a convex compromise region.* This would align with the idea that conceptual spaces are shaped by balancing forces (perhaps analogous to prediction errors minimization, etc.) resulting in convex regions where those forces cancel out (the Hessian being positive-definite ensures a stable fixed point).

In summary, convexity in our model ensures both **stability** (no weird concave aberrations that would cause multiple local optima in inference) and **compositionality** (mixtures of states are allowed and meaningful). The Legendre duality’s presence hints at possibly a deeper dual structure: there might be a dual polytope (like the 600-cell’s dual is the 120-cell) that represents a dual space of cognitive parameters. Could it be that the 120-cell (with 600 vertices) plays the role of dual potential? Perhaps. If the 600-cell represents “states” (points in $\mathbb{R}^4$ perhaps corresponding to extensive quantities or features), the dual 120-cell might correspond to “constraints” or “contexts” (like intensive parameters). In information geometry, one often has dual coordinate systems (expectation vs. natural parameters). It is intriguing that E_8 gives both at once. Perhaps one could assign the 600-cell vertices to basis *concepts* and 120-cell vertices to basis *constraints*, and the golden ratio coupling means each constraint interacts with concept coordinates in a specific scaled way. While speculative, this is a fertile ground for future work: exploiting the polarity (duality) between polytopes to model the duality between questions and answers, or stimuli and responses, etc.

4.3 Integrating PPP and GIT: Now we clearly articulate how PPP’s constructs map to GIT’s principles in our model: - The **Orthocognitum** (PPP’s global manifold of conceptual states) [72] [73] is, in our formalism, the *total state-space polytope* (or union of polytopes) that the cognitive system can occupy. We propose that this is essentially the 600-cell structure (possibly extended by φ-scaling to incorporate multi-scale). It is “topologically bounded” and tessellated by convex regions [74] [75], consistent with a polytope tiling. - **Geoepistatic Localization** (PPP’s notion that a concept’s expression depends on its location in state-space context) [76] [72] can be understood via the overlaps of 24-cells: if a vertex (concept) lies in multiple 24-cells, its “meaning” or relational role can differ depending on which 24-cell (context frame) is currently active. The geometry literally localizes how that concept is projected along edges to other vertices within each 24-cell. - **Epistaorthognition** (verification of truth-states against topological invariants) [77] might relate to ensuring that a state lies within the convex hull of known consistent states. A “truth” in this model could correspond to a point lying inside a valid concept polytope rather than outside. Topological invariants (like the homology loops we suspect) could serve as records of logical consistency (closing a loop could mean returning to the starting concept after a series of transformations – if the loop is nontrivial, perhaps an inconsistency or a novel concept emerges). - The **Chronomorphic Polytopal Engine (CPE)** [78] [79] is the dynamic realization of PPP. In our terms, it is a process that moves the cognitive state vector through the 4D polytopal space by rotations and reflections corresponding to reasoning steps. The engine would use operations of the symmetry group (H_4 rotations, or flips through ϕ correspondences) to compute. Because these are algebraic (quaternionic) operations, they can be done extremely fast (PPP even mentions photonic implementations) [80] . The polytopal approach might allow certain inferences to be done by simple geometric moves rather than iterative numeric methods (like how one might "query" a nearest concept by literally moving in that direction along an edge).

Crucially, both GIT and PPP emphasize an optimization or equilibrium principle: PPP implicitly aims to solve problems by geometric intersection (which often equates to solving equations or minimizing differences in geometry), and GIT explicitly uses variational principles. In our architecture, one can imagine that a cognitive question (constraint) is introduced as an energy term that deforms the convex polytope or carves out a face, and the answer is found where some potential is minimized on that polytope (like a ball rolling to the lowest corner consistent with the constraint – essentially a linear programming on the polytope). This mechanical metaphor aligns with cognitive processes that settle into an interpretation that best fits constraints (e.g. vision resolving an ambiguous image by settling on one gestalt – a minimum free energy principle as proposed in some brain theories). The convex geometry ensures that this minimum is unique or bounded and the Legendre duality assures we can shift between constraint-space and state-space seamlessly.

5. Conjectures and Implications (Fractals, Homology, Dynamics, Testability)

Having established the formal scaffolding, we now turn to forward-looking **conjectures**. These are hypotheses that extend beyond currently proven results, offered to stimulate cross-disciplinary exploration. We clearly separate these speculative ideas from the established content above.

5.1 Fractal-Like Scaling and Self-Similarity: One conjecture is that the nested golden-ratio structure (H_4 and ϕH_4 in E_8) might imply a **scale invariance or fractal-like repetition** in cognitive state-space. If one could continue the nesting (e.g. E_8 contains H_4 and φH_4; does φH_4 in turn contain a φ^2-scaled H_4 when projected to some subspace of 4D?), one might get an infinite tower of ever larger polytopes, akin to a quasicrystal that is self-similar. In practice, biological or AI systems won't realize infinite scales, but they might use 2-3 scales. Still, the presence of an incommensurate ratio φ means the pattern of one scale relative to the other is aperiodic and lacks a characteristic length – a hallmark of fractals and quasicrystals. **Conjecture:** *The cognitive state-space is quasicrystalline, with no single characteristic scale, allowing concepts to robustly generalize across magnitudes.* This could tie to how humans can reason from small-scale to large-scale analogies (the patterns repeat). A concrete mathematical question: does the projection of the E_8 lattice (and possibly E_8 + next layer of lattices like $\phi^2 H_4$ if one existed) produce a point distribution with fractal boundary or self-similar hole structure? Quasicrystals often have fractal boundary measures (like diffraction patterns showing self-similarity) [81] [65] . It would be interesting if the distribution of points in state-space exhibited fractal dimension between 0 and 4, perhaps reflecting a complexity measure of cognitive repertoire.

5.2 Homological Emergence (Topological Signatures of Thought): The arrangement of overlapping 24-cells likely creates nontrivial cycles in the space of representations. For instance, consider looping through a sequence of 24-cells: starting in 24-cell $C_{0,0}$ at a concept v, then moving to another concept w that lies also in $C_{0,1}$, then to a concept x in $C_{1,1}$, and so on, eventually returning to v after traversing a network of shared vertices. Such a loop might correspond to *conceptual change after a full cycle*, analogous to how transporting a vector around a loop on a curved surface can rotate it (holonomy). **Conjecture:** *There exist higher-dimensional loops (1-cycles, 2-cycles, etc.) in the cognitive polytope complex that correspond to emergent structures (thought loops, conceptual blending paths) not reducible to any single context.* We might find that the first homology group H_1 of the 25-24cell complex is nonzero. If each pentagon loop of 5 cells is like a 1-cycle at the overlap graph level, perhaps there are 144 such cycles (from the 144 pentagons) but many might be homologous or cancel out. If a homological cycle exists, it could carry information (maybe a persistent ambiguity or a creative blend lives “around the cycle”). In topological data analysis of neural activity, one does sometimes find cyclic structure (e.g. in hippocampal place cell ensembles as an animal runs a loop, a topological loop in activity space forms). We speculate similarly that certain cognitive processes could form loops in this high-dimensional space (like thinking in circles yields a loop that might be detected by persistent homology techniques). More exotically, higher homology (2-cycles) might correspond to surfaces of consistent states. The **MES model** we referenced could incorporate such homological features as indicators of multi-scale integration (the phrase “homological emergence” implies that new homology groups appear at higher scales of organization, providing robustness or new capacity at the collective level). Testing this would involve computing the homology of a point-cloud of neural states or AI hidden states and seeing if it matches the predicted structure from our model.

5.3 Dynamic Rotation and Isoclinic Trajectories: In 4D, a special kind of rotation is an **isoclinic rotation**, where two orthogonal planes rotate by the same angle. The quaternion representation makes isoclinic rotations very natural (a quaternion q of unit norm rotates any vector via $v \mapsto qvq^{-1}$; if q is something like $\exp(\theta (i+j)/\sqrt{2})$, that rotates the ij-plane and kl-plane by θ simultaneously). The 120° rotation $\omega = \frac{-1+i+j+k}{2}$ mentioned for cycling the 16-cells is an isoclinic rotation of order 3 in 4D [82] . Our conjecture is that *cognitive processes correspond to sequences of isoclinic rotations in 4D state-space*, rather than arbitrary moves. An isoclinic rotation has the property that it preserves certain subsystems (in the above, one plane’s structure relative to another). This could implement a kind of *parallel processing*: rotating in one plane might correspond to updating two aspects of a concept in sync. The triadic 16-cell cycle is one example; one could have 5-fold isoclinic rotations that cycle through 5 24-cells (maybe related to the pentagon loops). Because 5 and 3 are involved (icosahedral symmetry yields rotations of order 3 and 5), we anticipate some cognitive operations might require a sequence of three 120° steps (like a three-step argument returning to start) or five 72° steps (like considering something from five perspectives returns you to original view, akin to a pentagram of thought). If such rotations are dynamic, we might measure them as oscillatory modes in neural populations (perhaps EEG rhythms corresponding to these symmetries? pure speculation: a gamma oscillation might implement a rapid rotation among 3 states, etc.). The *Chronomorphic engine* plan fits: time is treated as a geometric deformation [83] , meaning as time evolves, the state point traces a path. If that path is along an isocline, then the “shadow” in any 2D projection appears as a regular rotation. Interestingly, a 4D isoclinic rotation projected to 3D can produce a **moiré-like motion** – because two planes rotating in sync, when flattened, might produce beating patterns. We hypothesize the brain might harness such things to create interference patterns in oscillations (some theories propose beats between theta and gamma waves for coding).

5.4 Links to Grid Cells and AI Architectures: Our model can be probed by looking at grid cells in the brain’s entorhinal cortex. Grid cells fire in multiple locations that form a hexagonal grid in 2D spaces. When an animal navigates, different grid scales combine to cover large environments. It has been suggested that grid-cell activity could also map conceptual or task spaces (not just physical) in an abstract hexagonal lattice. If our model is right, then a 4D analog should exist: one might find neurons whose firing fields correspond to vertices of a 4D lattice (like a 600-cell or 24-cell projection). Obviously we can't directly see in 4D, but mathematically the firing pattern might produce multi-modal distributions that match 4D distances. One might test rats or AI agents in tasks that have group-theoretic structure (like needing to combine sequences in a way that fits F4 symmetry) and see if their neural codes reflect something like a 24-cell coordinate system. On the AI side, one could design a latent space of a neural network explicitly to be a toric 4D structure. For instance, represent the latent state as a point on S^3 (3-sphere) with a binary icosahedral group action, effectively constraining it to 120 possibilities or their convex span. Then train the network on relational reasoning tasks. If PPP is correct, such an architecture might learn faster or more systematically than an unconstrained one, because it has the inductive bias of geometric compositionality built in. Conversely, if an unconstrained network spontaneously learns something isomorphic to our structure (e.g. cluster centers aligning with 24-cell vertices), that would be remarkable evidence that the geometry is natural for the problem. In fact, recent work on *neurosymbolic* models or *vector symbolic architectures* often emphasize binding and superposition of vectors – operations which could be elegantly done by quaternion algebra (which underlies our rotations). The binary tetrahedral and icosahedral groups are subgroups of the unit quaternions, which might be used for binding operations (multiplying two quaternions could represent combining two concepts, with phase angles adding). If we find performance advantages in such systems, it would lend credence to the cognitive role of these symmetries.

5.5 Falsifiability and Prediction Constraints: It is important that our framework not degenerate into unfalsifiable hand-waving. We have embedded specific structures (24-cells, 600-cells, golden ratio lattices) that can be searched for in data. If the brain or a cognitive model does not show any evidence of these (for example, no hint of a 5-fold or 3-fold symmetry in representational geometry where it should, or conceptual similarity judgments not matching geodesic distances on these polytopes), then the theory would be weakened. One direct prediction: *Concept combinations will obey the triangle inequality in a 4D metric consistent with F_4 symmetry.* This could be tested by human similarity ratings or ML embedding distances – if people consider concept A to B and B to C as unrelated, then A to C should not be very related either, etc., forming a metric. Our model further predicts some distances take only discrete values (since in 600-cell there are limited distances between vertices: about 8 distinct distances) [84] [85]. If one finds that human conceptual distances cluster around specific ratios (maybe reflecting those distances), that's support. Another test: *neural representations of sequences might reveal pentagon cycles.* For instance, if one records neurons as a subject cycles through a set of tasks logically, the latent state might show a loop after 5 steps (like a pentagon in PCA projection). If it doesn't or if it loops in say 4 steps, maybe a different symmetry is at play (like a cube rather than a dodecahedron analogy).

From an engineering stance, building an AI that explicitly uses these polytopes (e.g. reinforcement learning agent whose state representation is a point in a 4D torus discretized by 600-cell vertices) could produce unique failure modes if wrong. If the agent can't represent certain combinations easily, that's a falsification of the completeness of the polytope basis. But if it excels and is robust to noise (maybe because E_8 packing gives maximal error margins), that's a validation of the design.

Finally, one could even look at physics: some theories (e.g. Garrett Lisi's E_8 Theory of Everything) have toyed with E_8 as fundamental. If the brain leveraged similar math, there might be observable at micro-scales (maybe microtubules? though that's far-fetched). But at least at a metaphorical level, it is amusing that the cognitive model here aligns with the most symmetric structure in mathematics (E_8), hinting at a potential unity of physical and informational elegance.

Conclusion

We have presented a comprehensive theoretical framework that unites geometric and information-theoretic principles in a polytopal model of cognition. The **24-cell**, with its exceptional symmetry and internal triadic decomposition, emerges as a fundamental unit of cognitive representation – a "concept manifold" wherein basic inferences are rotations among 16-cell subspaces. The **600-cell** then provides the next scale up: a global *conceptual space* in which multiple 24-cell frames coexist, overlap, and interfere in a structured way. Mathematically, we substantiated the remarkable **25-fold partition of the 600-cell into 24-cells** [2] [1] and linked it to group actions and combinatorial designs (a 5×5 Latin-square array of 24-cells with S_5 symmetry). We then connected this 4D structure to the **8D E_8 lattice** via Conway's icosian method, showing that the interplay of a 600-cell and a φ-scaled 120-cell reproduces the richest lattice known [5] [6]. This not only lends our model a multi-scale (and perhaps physically optimal) character, but also situates it in a lineage of deep mathematical results bridging Coxeter groups and crystallographic lattices [60] [61].

On the theoretical side, by incorporating **categorical thinking**, we interpreted the polytopal assembly as a colimit of simpler parts, aligning with how complex cognitive structures can be built from simpler schemas. By invoking **convexity and duality**, we anchored our model in the language of equilibria and optimality, echoing the Legendre duality that underlies both thermodynamic and inferential systems [22] [7]. This provides a principled reason why such geometric structures would arise: they are the natural shape of information at balance – the shape of thought at harmony with itself. The **Convexity Constraint** on concept regions is both empirically and theoretically justified, and our polytopes elegantly satisfy it by construction [9].

Our conjectural extensions venture that this framework could explain various phenomena (fractal scaling of knowledge, topologically distinct modes of thought, cyclic re-entrant ideas, etc.) and yield novel predictions. While these remain to be tested, they demonstrate the *fertility* of the geometric approach: instead of treating cognition as an opaque black box or an ad-hoc network, we treat it as a *space with structure*, enabling the import of powerful mathematical machinery. This opens doors to applying homology, representation theory, and metric geometry to cognitive science and AI. It also suggests an avenue for **cross-domain unification**: the same geometric code might be manifest in neuronal firing patterns, abstract conceptual maps, and even engineered computing systems (like neuromorphic photonics) [86].

In conclusion, by synthesizing GIT and PPP, we arrive at a vision of cognition as **polytope computation**: the mind as moving through a landscape of polytopes, where **vertices are ideas, edges are transformations, faces are constraints, and symmetries are analogies**. This white paper has laid the formal groundwork, proving key structural features and highlighting consistency with known mathematics (Denney et al.'s 24-cell arrangement [1], Conway & Sloane's E_8 links [5], Artstein-Avidan & Milman's convex duality [7], etc.). What remains is to empirically verify and refine these ideas in the crucible of experiment and practical implementation. If validated, the payoff would be profound: a *geometric unification* of principles across mind, mathematics, and information – with the 24-cell, glistening at the center of a 600-cell, as a crystal of thought itself.

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