AAMAS: 05-12-17

Distributed Constraint Optimization Problem (DCOP):

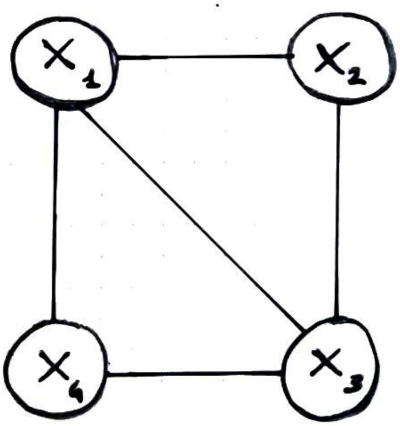
2 algorithms:

1. DPOP (exact solution)
2. MAX-SUM (approximated solution)

Obs: msgs in DPOP can be exponentially large.

Variables x_i
Domains $x_i \in D(x_i)$
Values $F_i(x_i, x_j)$

Ex1: DPOP



X_1	X_2	F_{12}
0	0	1
0	1	2
1	0	0
1	1	1

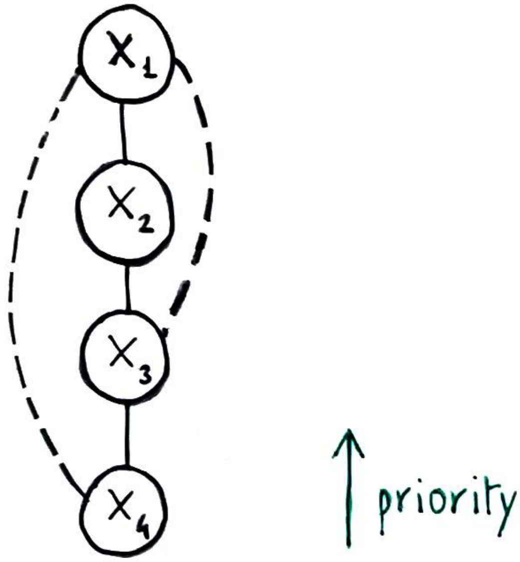
X_3	X_4	F_{34}
0	0	1
0	1	2
1	0	0
1	1	1

X_1	X_4	F_{14}
0	0	2
0	1	1
1	0	0
1	1	0

X_2	X_3	F_{23}
0	0	2
0	1	1
1	0	0
1	1	0

X_1	X_3	F_{13}
0	0	0
0	1	2
1	0	2
1	1	0

- 1st STEP: Tree generation



- 2nd STEP: bottom-up propagation

X_1	X_2	X_3	$F_{12} \otimes F_{23}$
0	0	0	$1 + 2 = 3$
0	0	1	$1 + 1 = 2$
0	1	0	$2 + 0 = 2$
0	1	1	$2 + 0 = 2$
1	0	0	$0 + 2 = 2$
1	0	1	$0 + 1 = 1$
1	1	0	$1 + 0 = 1$
1	1	1	$1 + 0 = 1$

$\Rightarrow$

X_1	X_3	$\overbrace{\max(F_{12} \otimes F_{23})}^{M_2}$
0	0	3
0	1	2
1	0	2
1	1	1

X_1	X_3	X_4	$F_{13} \otimes F_{34} \otimes M_2$
0	0	0	$0 + 1 + 3 = 4$
0	0	1	$0 + 2 + 3 = 5$
0	1	0	$2 + 0 + 2 = 4$
0	1	1	$2 + 1 + 2 = 5$
1	0	0	$2 + 1 + 2 = 5$
1	0	1	$2 + 2 + 2 = 6$
1	1	0	$0 + 0 + 1 = 1$
1	1	1	$0 + 1 + 1 = 2$

$\Rightarrow$

X_1	X_4	$\overbrace{\max(F_{13} \otimes F_{34} \otimes M_2)}^{M_3}$
0	0	4
0	1	5
1	0	5
1	1	6

X_1	X_4	$F_{14} \otimes M_3$
0	0	$2 + 4 = 6$
0	1	$1 + 5 = 6$
1	0	$0 + 5 = 5$
1	1	$0 + 6 = 6$

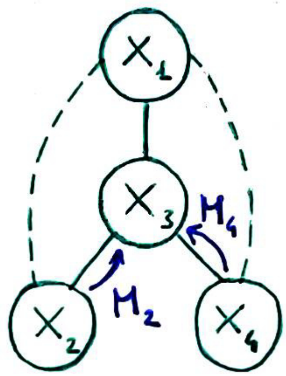
$\Rightarrow$

X_1	$\overbrace{\max(F_{14} \otimes M_3)}^{M_4}$
0	6
1	6

- 3rd STEP: value propagation

$$X_1 = 0,\ X_4 = 1,\ X_3 = 0,\ X_2 = 0$$

Obs: what if we build the tree in a different way?



X_1	X_2	X_3	$F_{12} \otimes F_{23}$
0	0	0	$1 + 2 = 3$
0	0	1	$1 + 1 = 2$
0	1	0	$2 + 0 = 2$
0	1	1	$2 + 0 = 2$
1	0	0	$0 + 2 = 2$
1	0	1	$0 + 1 = 1$
1	1	0	$1 + 0 = 1$
1	1	1	$1 + 0 = 1$

$\Rightarrow$

X_1	X_3	$\overbrace{\max(F_{12} \otimes F_{23})}^{M_2}$
0	0	3
0	1	2
1	0	2
1	1	1

x_1	x_3	x_4	$F_{14} \otimes F_{34}$
0	0	0	$2+1=3$
0	0	1	$1+2=3$
0	1	0	$2+0=2$
0	1	1	$1+1=2$
1	0	0	$0+1=1$
1	0	1	$0+2=2$
1	1	0	$0+0=0$
1	1	1	$0+1=1$

$\Rightarrow$

x_1	x_3	$\overbrace{\max(F_{14} \otimes F_{34})}^{M_4}$
0	0	3
0	1	2
1	0	2
1	1	1

When we have many children of the same node, the computation becomes exponential.

Ex2: MAX-SUM

Assume we have the same constraint graph and tables.

- 1ST ITERATION:

- Considering agent 1:

$$m_{1\to 2}(x_2) = \alpha_{12} + \max_{x_1}\Big(F_{12}(x_1, x_2) + \underbrace{\sum_{i \neq 2} m_{i\to 1}(x_1)}_{0 \text{ because it is the 1st iteration}}\Big) =$$

	x_2
-0,5	0
0,5	1

x_2	F_{12}
0	1
1	2

We sum α_{12} to every possible row

$\Rightarrow (\alpha_{12} + 1) + (\alpha_{12} + 2) = 0$

$\alpha_{12} = -1,5$

$$m_{1\to 3}(x_3) = \alpha_{13} + \max_{x_1}\left(F_{13}(x_1, x_3)\right) =$$

	x_3
0	0
0	1

x_3	F_{13}
0	2
1	2

$\Rightarrow (\alpha_{13} + 2) + (\alpha_{13} + 2) = 0$

$\alpha_{13} = -2$

$$m_{1\to4}(x_4) = \alpha_{14} + \max_{x_1}(F_{14}(x_1, x_4)) = \begin{array}{c|c} & x_4 \\ \hline 0{,}5 & 0 \\ -0{,}5 & 1 \end{array}$$

x_4	F_{14}
0	2
1	1

$$\Rightarrow (\alpha_{14} + 2) + (\alpha_{14} + 1) = 0$$
$$\alpha_{14} = -1{,}5$$

Answers to agent 1:

$$m_{2\to1}(x_1) = \alpha_{21} + \max_{x_2}(F_{12}(x_1, x_2)) = \begin{array}{c|c} & x_1 \\ \hline -1{,}5 & 0 \\ -0{,}5 & 1 \end{array}$$

x_1	F_{12}
0	2
1	1

$$\Rightarrow (\alpha_{21} + 2) + (\alpha_{21} + 1) = 0$$
$$\alpha_{21} = -1{,}5$$

$$m_{3\to1}(x_1) = \alpha_{31} + \max_{x_3}(F_{13}(x_1, x_3)) = \begin{array}{c|c} & x_1 \\ \hline 0 & 0 \\ 0 & 1 \end{array}$$

x_1	F_{13}
0	2
1	2

$$\Rightarrow (\alpha_{31} + 2) + (\alpha_{31} + 2) = 0$$
$$\alpha_{31} = -2$$

$$m_{4\to1}(x_1) = \alpha_{41} + \max_{x_4}(F_{14}(x_1, x_4)) = \begin{array}{c|c} & x_1 \\ \hline 1 & 0 \\ -1 & 1 \end{array}$$

x_1	F_{14}
0	2
1	0

$$\Rightarrow (\alpha_{41} + 2) + (\alpha_{41} + 0) = 0$$
$$\alpha_{41} = -1$$

Then:

$$x_1 = \operatorname{argmax}_{x_1}(m_{2\to1} + m_{3\to1} + m_{4\to1}) = 0$$

x_1	$m_{2\to1} + m_{3\to1} + m_{4\to1}$
0	$-1{,}5 + 0 + 1 = -0{,}5$
1	$-0{,}5 + 0 + (-1) = -1{,}5$

- Considering agent 2:

$m_{2\to1}$ has already been computed

$$m_{2\to3}(x_3) = \alpha_{23} + \max_{x_2}\left(F_{23}(x_2, x_3)\right) = \begin{array}{c|c} & x_3 \\ \hline 0{,}5 & 0 \\ -0{,}5 & 1 \end{array}$$

x_3	F_{23}
0	2
1	1

$$\Rightarrow (\alpha_{23}+2) + (\alpha_{23}+1) = 0$$

$$\alpha_{23} = -1{,}5$$

Answers to agent 2:

$m_{1\to2}$ has already been computed.

$$m_{3\to2}(x_2) = \alpha_{32} + \max_{x_3}\left(F_{23}(x_2, x_3)\right) = \begin{array}{c|c} & x_2 \\ \hline 1 & 0 \\ -1 & 1 \end{array}$$

x_2	F_{23}
0	2
1	0

$$\Rightarrow (\alpha_{32}+2) + (\alpha_{32}+0) = 0$$

$$\alpha_{32} = -1$$

Then:

$$x_2 = \operatorname*{argmax}_{x_2}\left(m_{1\to2} + m_{3\to2}\right) = 0$$

x_2	$m_{1\to2}$		$+ m_{3\to2}$	
0	$-0{,}5$	$+$	1	$= 0{,}5$
1	$0{,}5$	$+$	(-1)	$= -0{,}5$

- Considering agent 3:

$m_{3\to1}$ and $m_{3\to2}$ have already been computed.

$$m_{3\to4}(x_4) = \alpha_{34} + \max_{x_3}\left(F_{34}(x_3, x_4)\right) = \begin{array}{c|c} & x_4 \\ \hline -0{,}5 & 0 \\ 0{,}5 & 1 \end{array}$$

x_4	F_{34}
0	1
1	2

$$\Rightarrow (\alpha_{34}+1)+(\alpha_{34}+2)=0$$
$$\alpha_{34}=-1,5$$

Answers to agent 3:

$m_{2\to3}$ has already been computed.

$$m_{4\to3}(x_3)=\alpha_{43}+\max_{x_4}(F_{34}(x_3,x_4))=$$

	x_3
0,5	0
-0,5	1

x_3	F_{34}
0	2
1	1

$$\Rightarrow (\alpha_{34}+2)+(\alpha_{34}+1)=0$$
$$\alpha_{34}=-1,5$$

Then:

$$x_3=\operatorname*{argmax}_{x_3}(m_{2\to3}+m_{4\to3})=0$$

x_3	$m_{2\to3}+m_{4\to3}$
0	$0,5 + 0,5 = 1$
1	$(-0,5)+(-0,5)=-1$

- Considering agent 4:

$m_{4\to3}$ and $m_{4\to1}$ have already been computed.

Answers to agent 4:

$m_{1\to4}$ and $m_{3\to4}$ have already been computed.

Then:

$$x_4=\operatorname*{argmax}_{x_4}(m_{1\to4}+m_{3\to4})=0$$

↑ it's a tie

x_4	$m_{1\to4}+m_{3\to4}$:
0	$0,5 + (-0,5) = 0$
1	$(-0,5) + 0,5 = 0$

Therefore, after the 1st iteration MAX-SUM returns the following solution:

$$x_1 = x_2 = x_3 = x_4 = 0$$

- **2ND ITERATION:**

· Considering agent 1:

$$m_{1\to 2}(x_2) = \alpha_{12} + \max_{x_1}\left(F_{12}(x_1, x_2) + m_{3\to 1}(x_1) + m_{4\to 1}(x_1)\right) =$$

$$= \begin{array}{c|c} & x_2 \\ \hline 0,5 & 0 \\ 1,5 & 1 \end{array}$$

computed in the previous iteration!

x_1	x_2	x_3	F_{12}		$+ m_{3\to 1}$		$+ m_{4\to 1}$	
0	0	0	1	+	0	+	1	= 2
0	0	1	1	+	0	+	1	= 2
0	1	0	2	+	0	+	1	= 3
0	1	1	2	+	0	+	1	= 3
1	0	0	1	+	0	+	(-1)	= 0
1	0	1	1	+	0	+	(-1)	= 0
1	1	0	2	+	0	+	(-1)	= 1
1	1	1	2	+	0	+	(-1)	= 1

$\Rightarrow \alpha_{12} = -1,5$

Answers to agent 1:

$$m_{2\to 1}(x_1) = \alpha_{21} + \max_{x_2}\left(F_{12}(x_1, x_2) + m_{3\to 2}(x_2)\right) =$$

$$= \begin{array}{c|c} & x_1 \\ \hline 1,5 & 0 \\ 0,5 & 1 \end{array}$$

x_1	x_2	F_{12}		$+ m_{3\to 2}$	
0	0	2	+	1	= 3
0	1	2	+	(-1)	= 1
1	0	1	+	1	= 2
1	1	1	+	(-1)	= 0

$\Rightarrow \alpha_{21} = -1,5$

$$m_{3\to 1}(x_1) = \alpha_{31} + \max_{x_3}\left(F_{13}(x_1, x_3) + m_{2\to 3}(x_3) + m_{4\to 3}(x_3)\right) =$$

$$= \begin{array}{c|c} & x_1 \\ \hline 1 & 0 \\ 1 & 1 \end{array}$$

x_1	x_3	$F_{13} + m_{2\to3} + m_{4\to3}$
0	0	$2 + 0{,}5 + 0{,}5 = 3$
0	1	$2 + (-0{,}5) + (-0{,}5) = 1$
1	0	$2 + 0{,}5 + 0{,}5 = 3$
1	1	$2 + (-0{,}5) + (-0{,}5) = 1$

$\Rightarrow \alpha_{31} = -2$

$$m_{4\to 1}(x_1) = \alpha_{41} + \max_{x_4}\left(F_{14}(x_1, x_4) + m_{3\to 4}(x_4)\right) = \begin{array}{c|c} & x_1 \\ \hline 1{,}5 & 0 \\ -0{,}5 & 1 \end{array}$$

x_1	x_4	$F_{14} + m_{3\to4}$
0	0	$2 + (-0{,}5) = 1{,}5$
0	1	$2 + 0{,}5 = 2{,}5$
1	0	$0 + (-0{,}5) = -0{,}5$
1	1	$0 + 0{,}5 = 0{,}5$

$\Rightarrow \alpha_{41} = -1$

Then:

$$x_1 = \operatorname*{argmax}_{x_1}\left(m_{2\to1} + m_{3\to1} + m_{4\to1}\right) = 0$$

x_1	$m_{2\to1} + m_{3\to1} + m_{4\to1}$
0	$1{,}5 + 1 + 1{,}5 = 4$
1	$0{,}5 + 1 + (-0{,}5) = 1$

The same thing has to be done for agent 2, 3 and 4.