Statistics with SpaRRows II

Many models, matrices, and magic

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Learning aims

- Understand mixed models
- Know when and why to use LMMs
- Know when to use random and fixed effects
- Know how to interpret random and fixed effects

Linear models

$$y_i = \boldsymbol{b_0} + \boldsymbol{b_1} x_i + \varepsilon_i$$

- Find solution: these parameter estimates (scalars) that minimise the left-over error residuals (vector)

Linear mixed models have 2 parts

- Fixed part
- Random part

Linear mixed models have 2 parts

- Fixed part
- How mean change with predictors

- Random part
- Partitions variances between groups

Linear mixed models have 2 parts

- Fixed part
- How mean change with predictors
- Mean

- Random part
- Partitions variances between groups
- Variance

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- How mean change with predictors

- Random part
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$$y_{i,j} = b_0 + b_1 x_{i,j} + \alpha_j + \varepsilon_{i,j}$$

- Mean

- Variance

Linear mixed models have 2 parts

- Fixed part
- How mean change with predictors

- Random part
- Partitions variances between groups

$$y_{i,j} = b_0 + b_1 x_{i,j} + \alpha_j + \varepsilon_{i,j}$$

- Mean

- Variance

Linear mixed models have 2 parts

- Fixed part
- How mean change with predictors

- Random part
- Partitions variances between groups

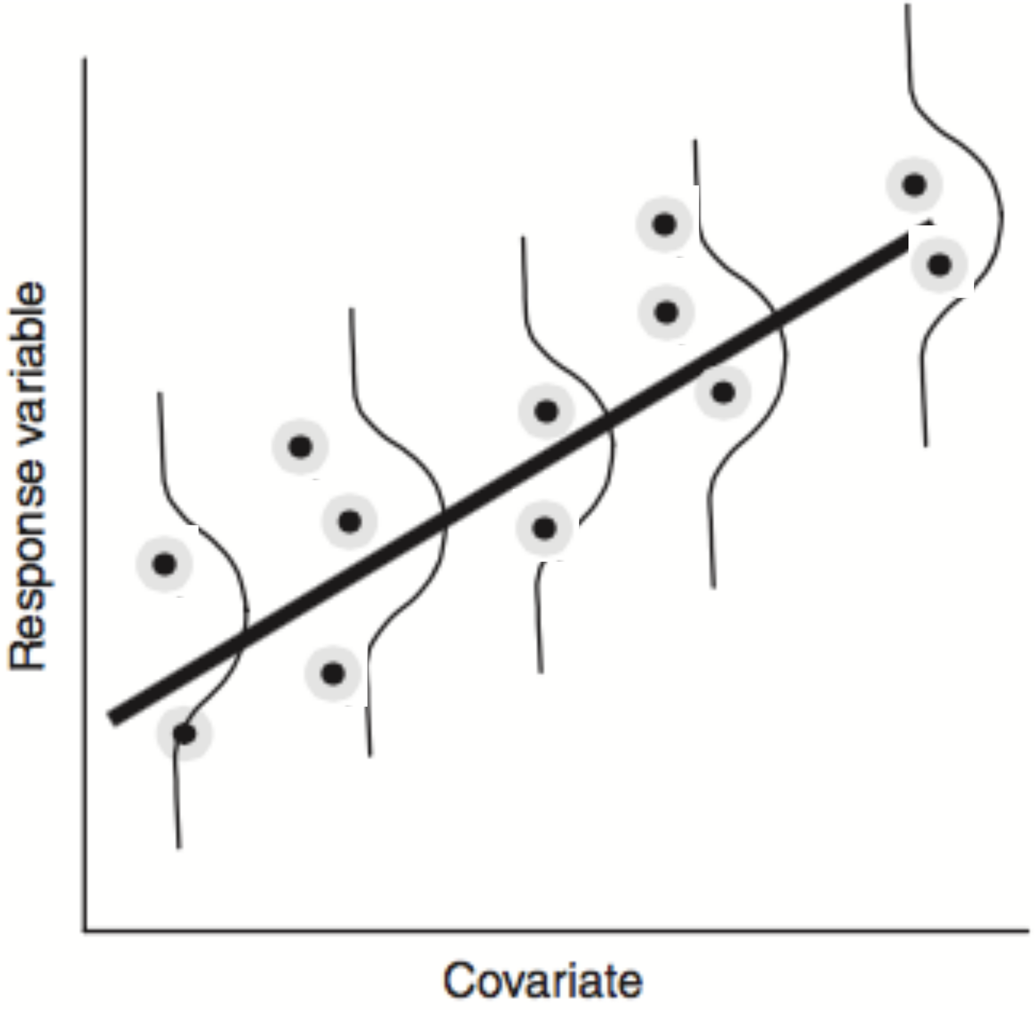
$$y_{i,j} = b_0 + b_1 x_{i,j} + \alpha_j + \varepsilon_{i,j}$$

- Mean

- Variance

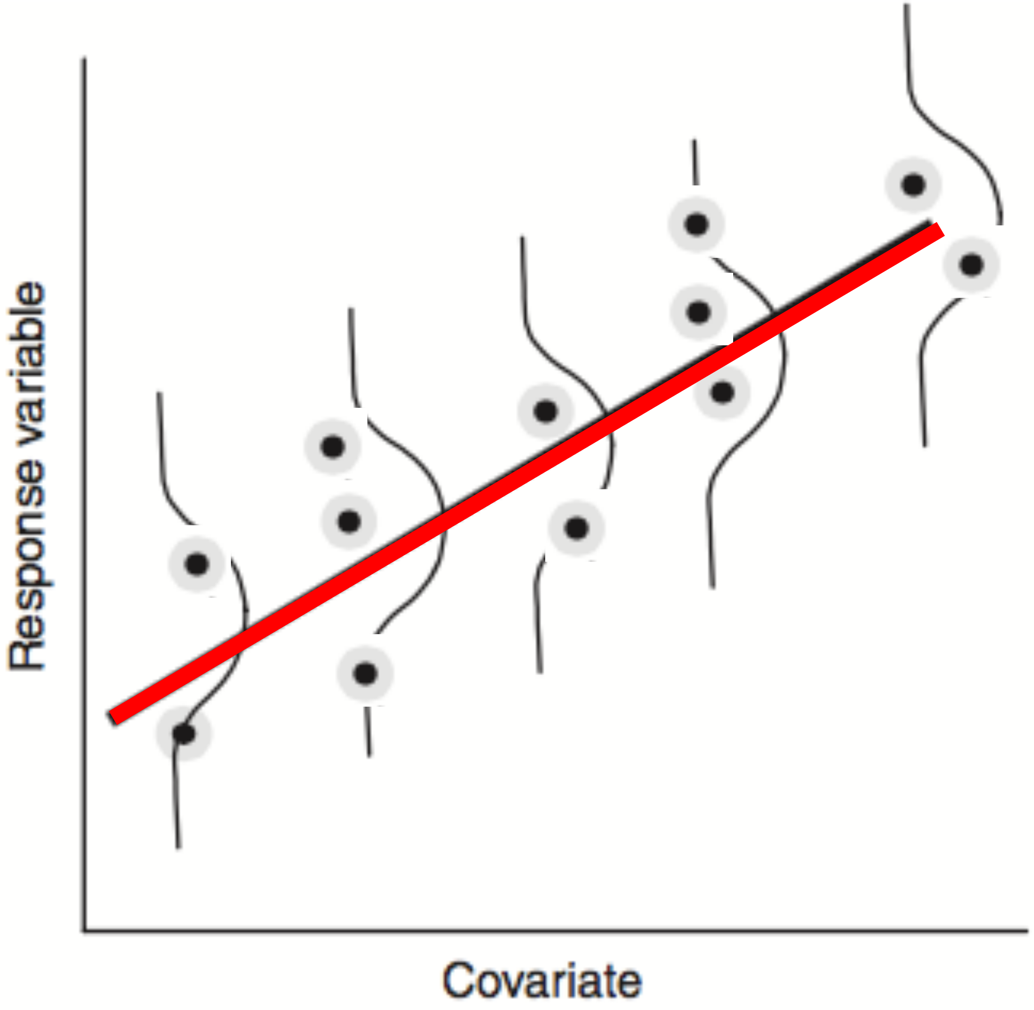
LMMs

- Estimate variance components and fixed parameter estimates simultaneously



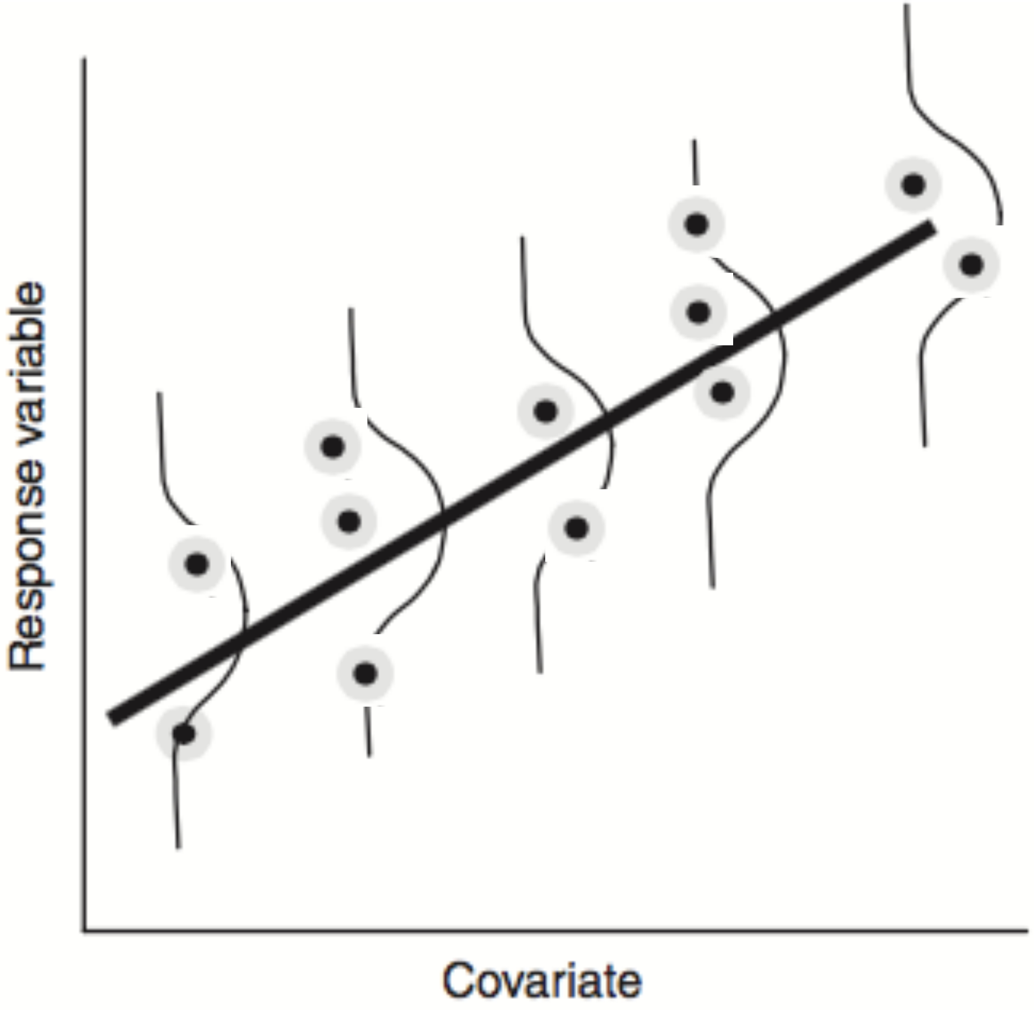
LMMs

- Estimate variance components and fixed parameter estimates simultaneously
- Fixed part: description of red line



LMMs

- Estimate variance components and fixed parameter estimates simultaneously
- Fixed part: description of red line
- Random part: variance within, and between groups



LMMs

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- Fixed part: description of red line
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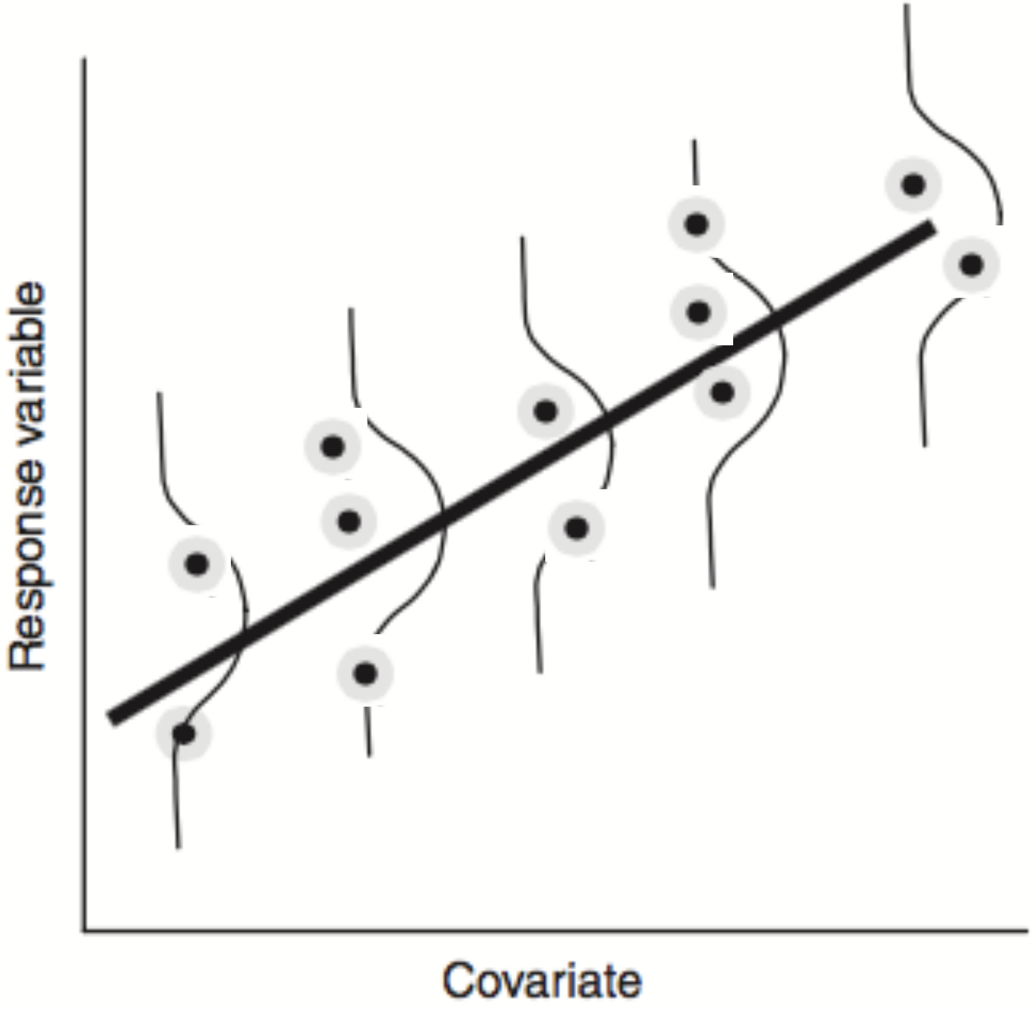
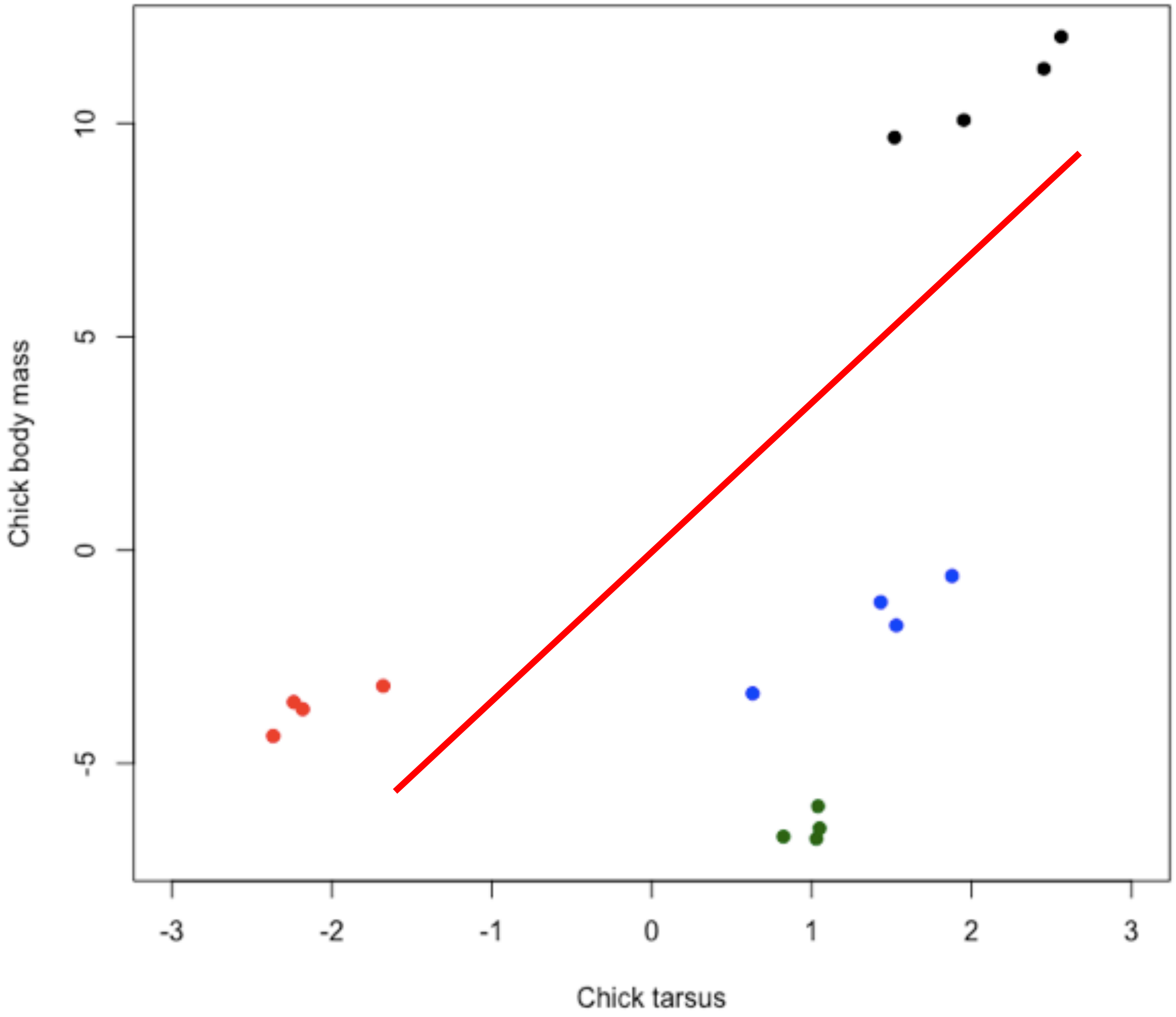


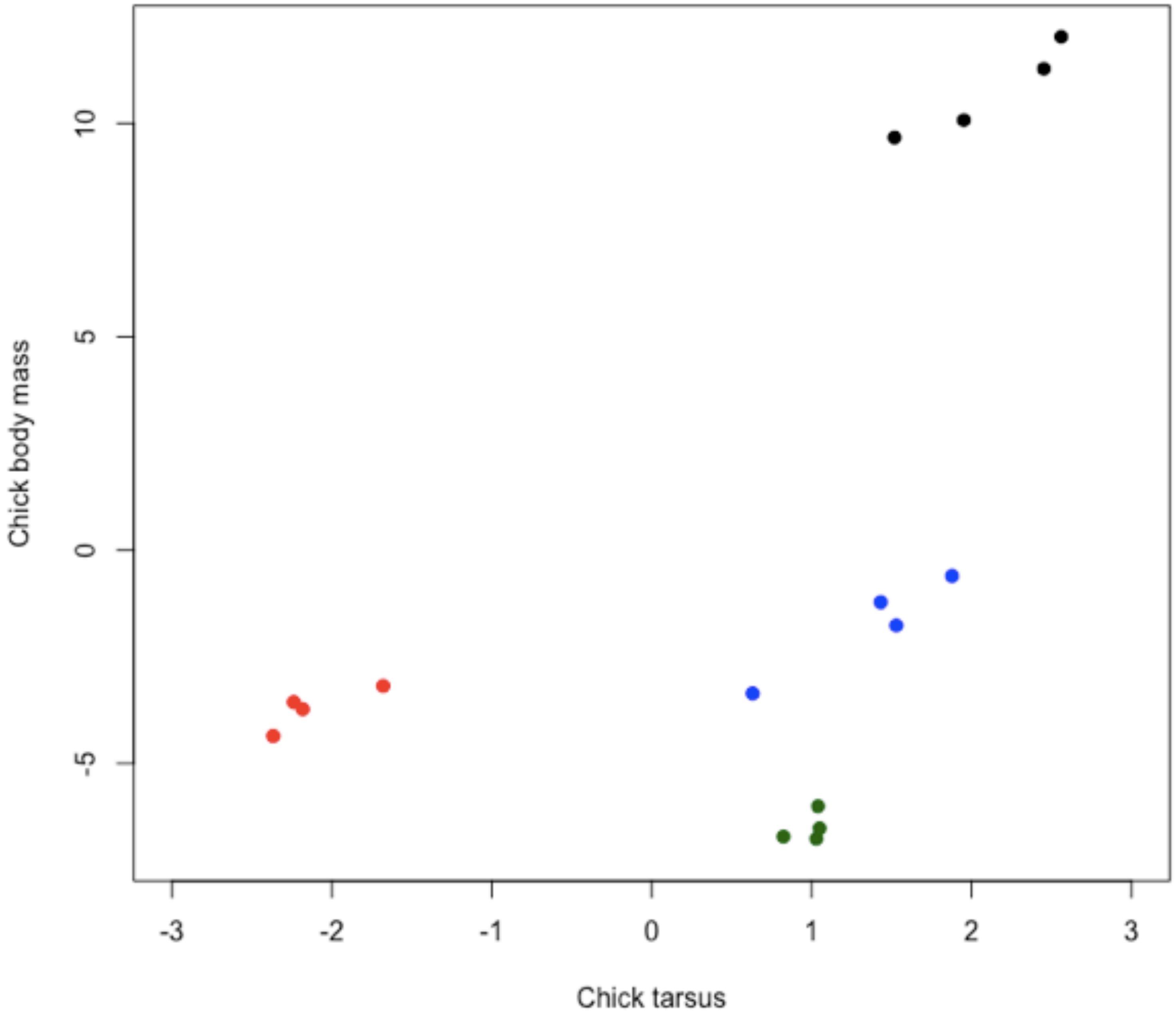
Mixed models

Fixed part



Mixed models

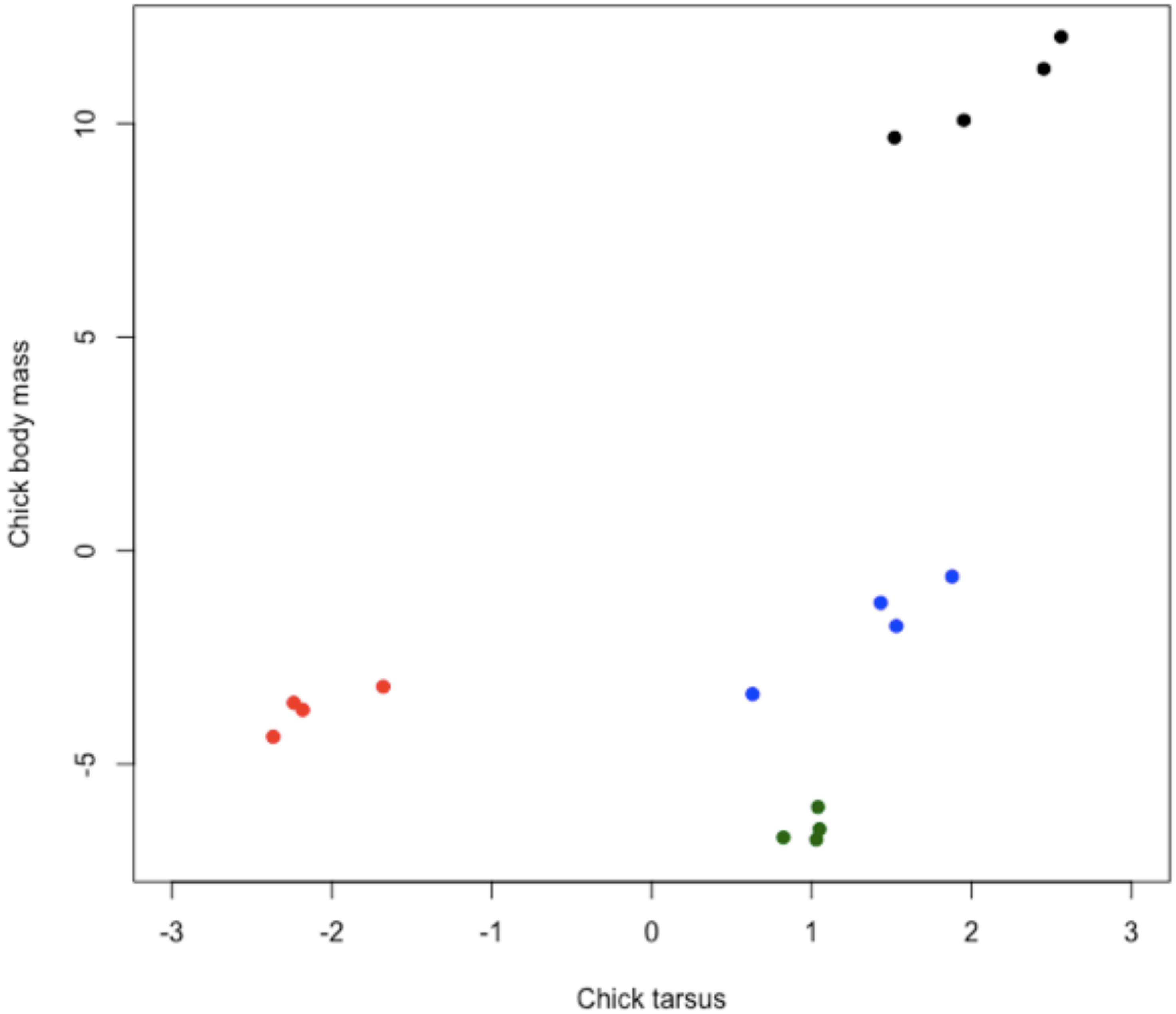
Random part
Groups?



Mixed models

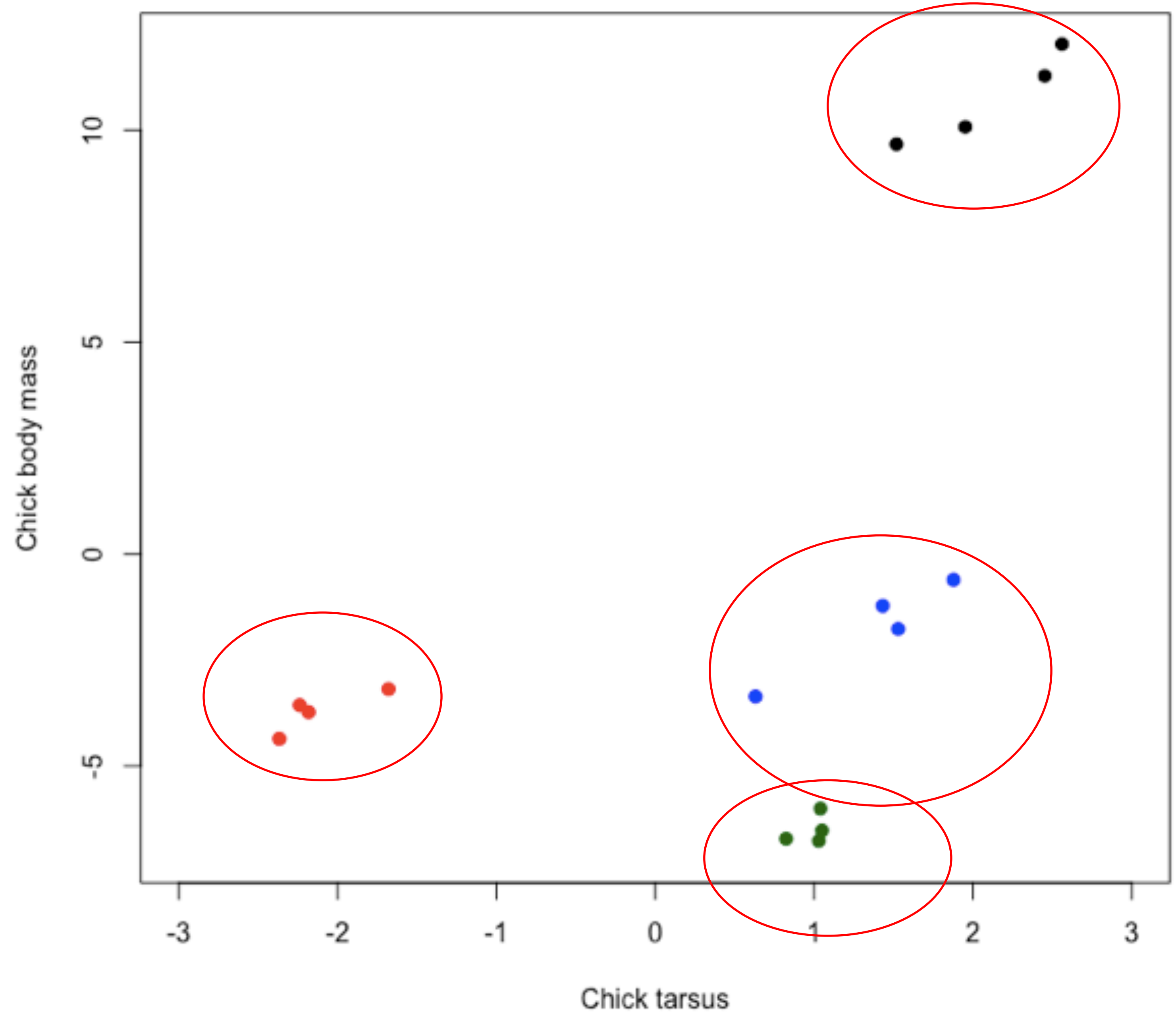
Random part

Groups = broods!



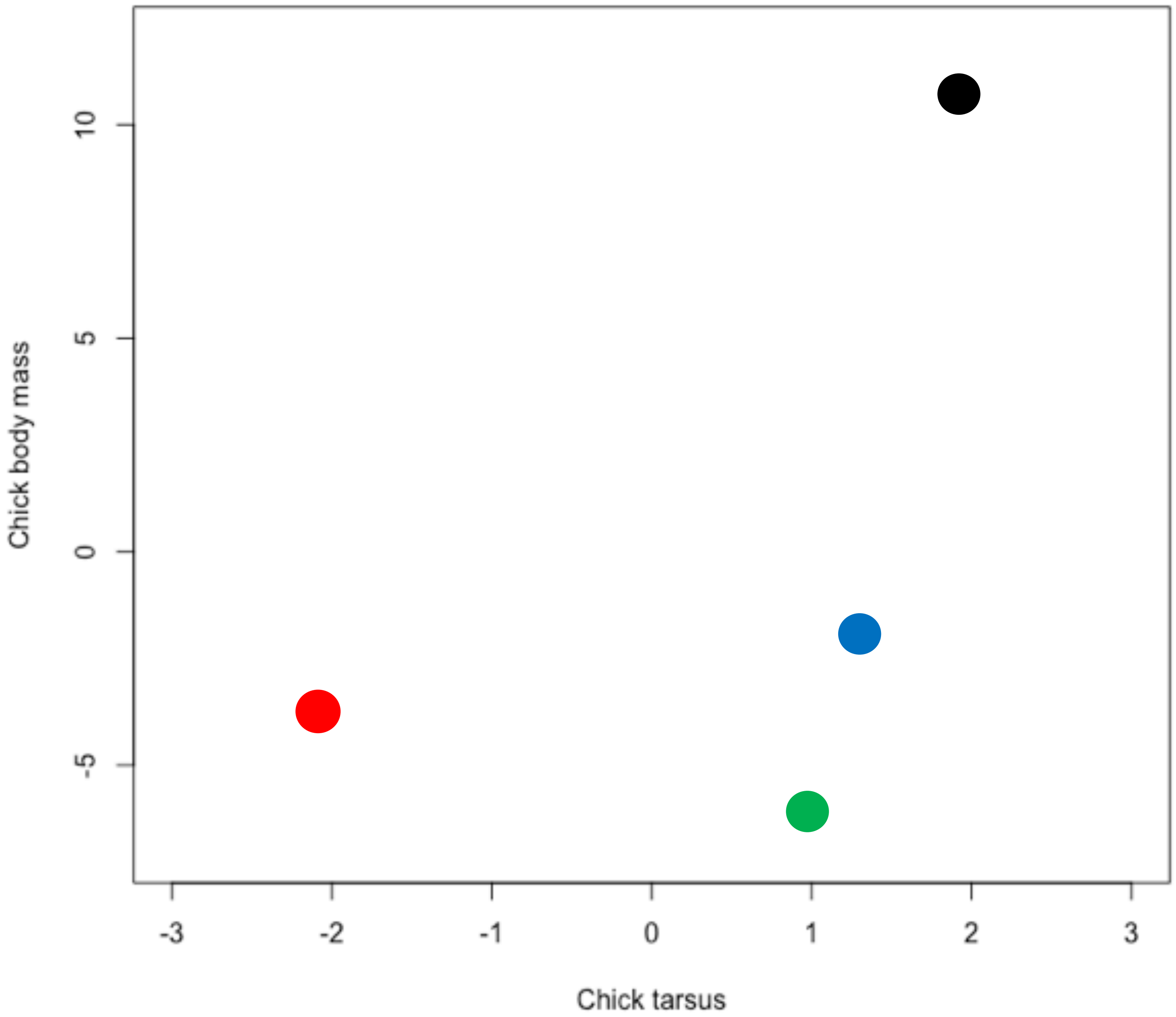
Mixed models

Estimate variances between groups



Mixed models

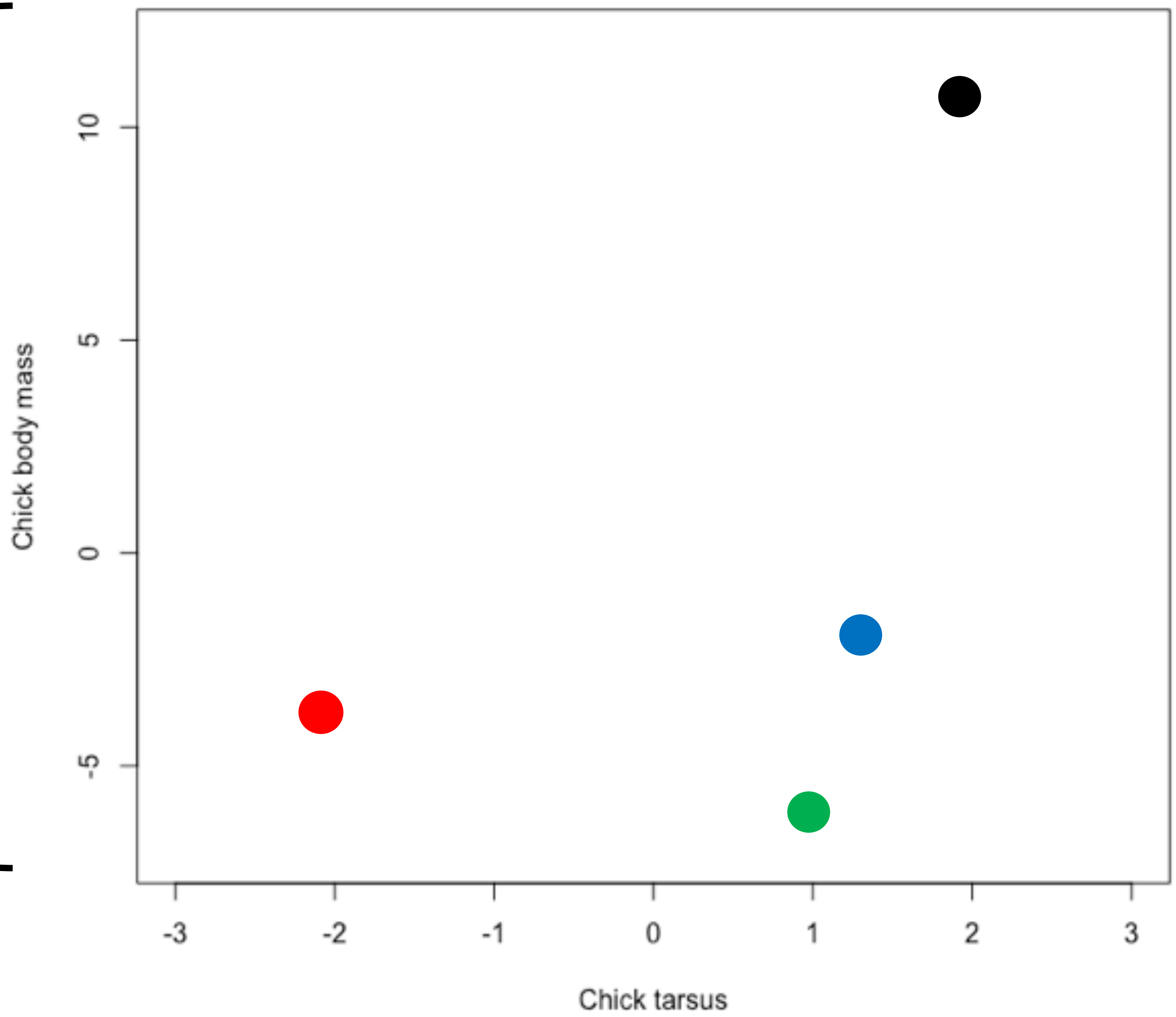
Estimate variances between groups



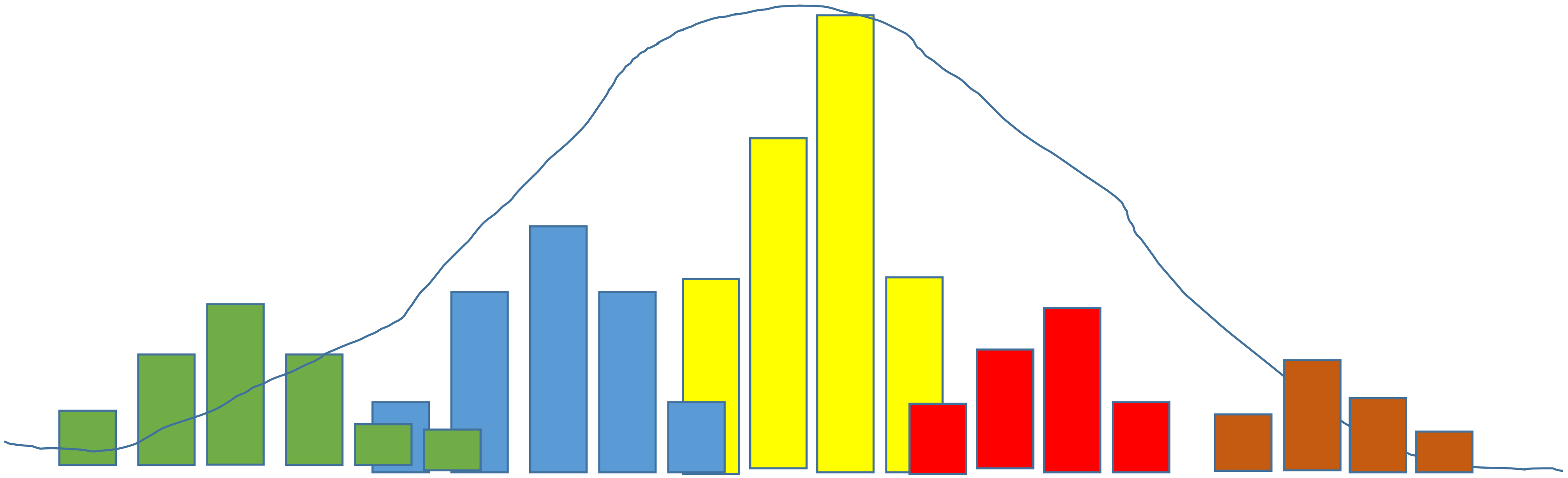
Mixed models

Estimate variances between groups

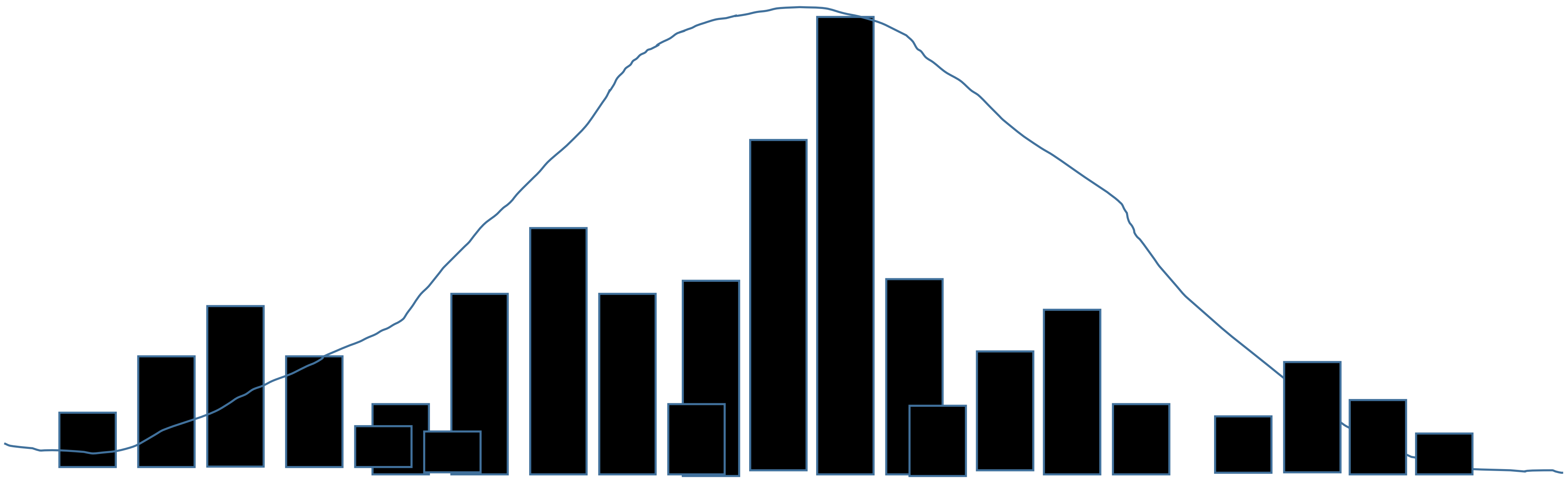
α_i



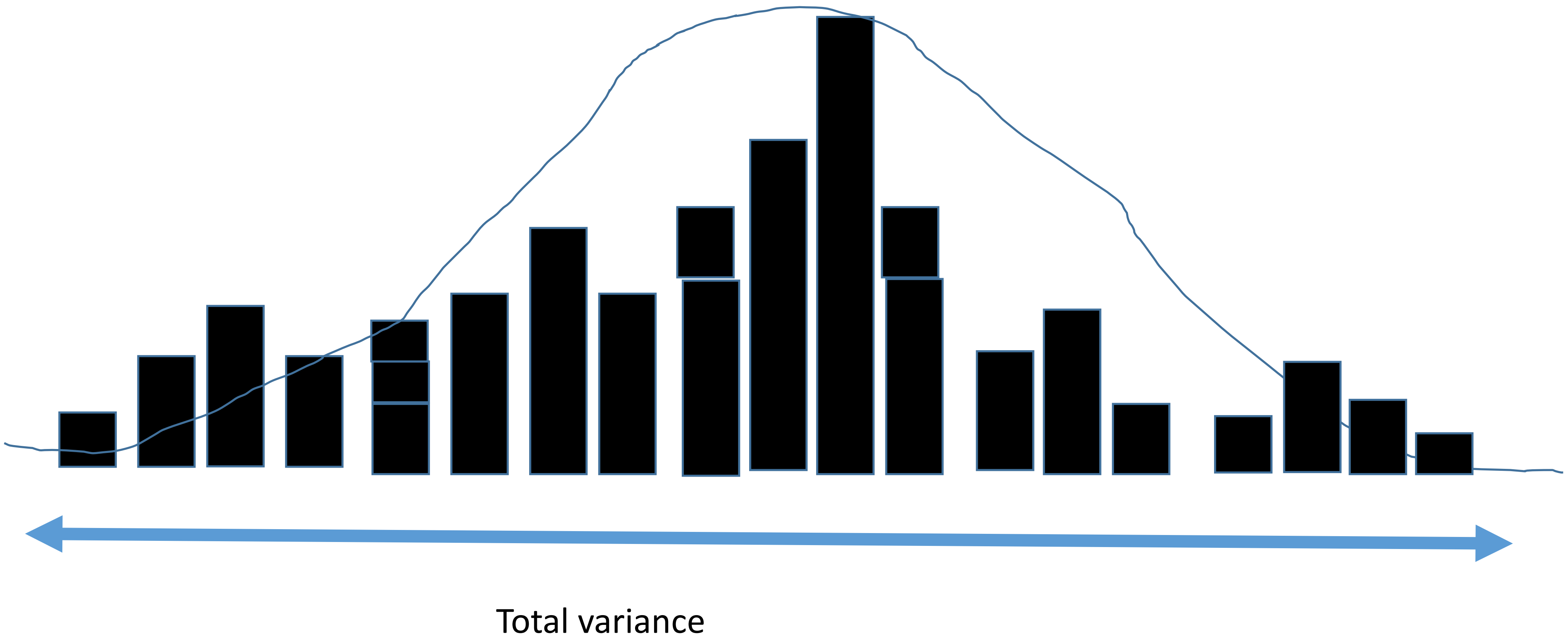
Variances and groups

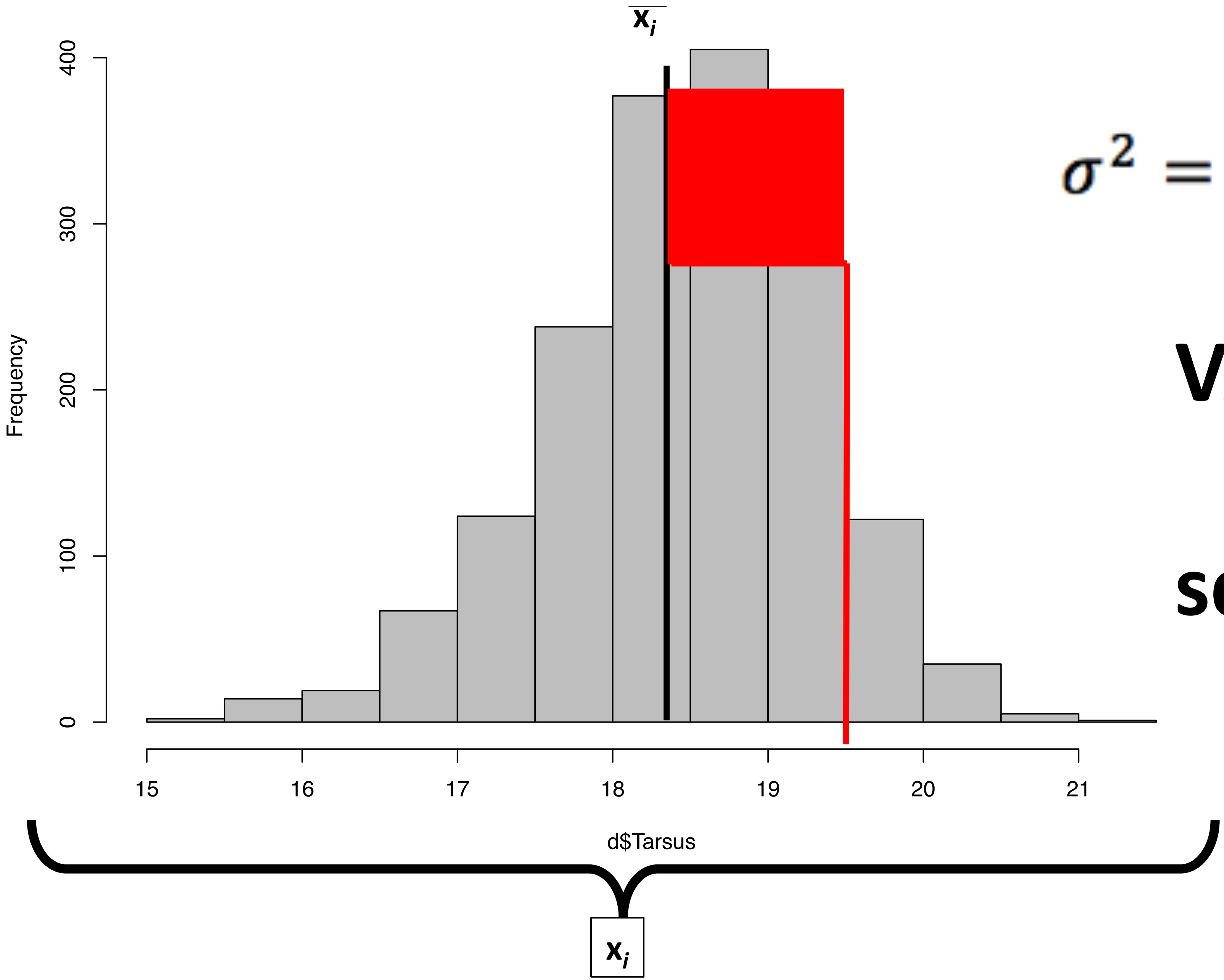


Variances and groups



Variances and groups





$$\sigma^2 = \frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n-1}$$

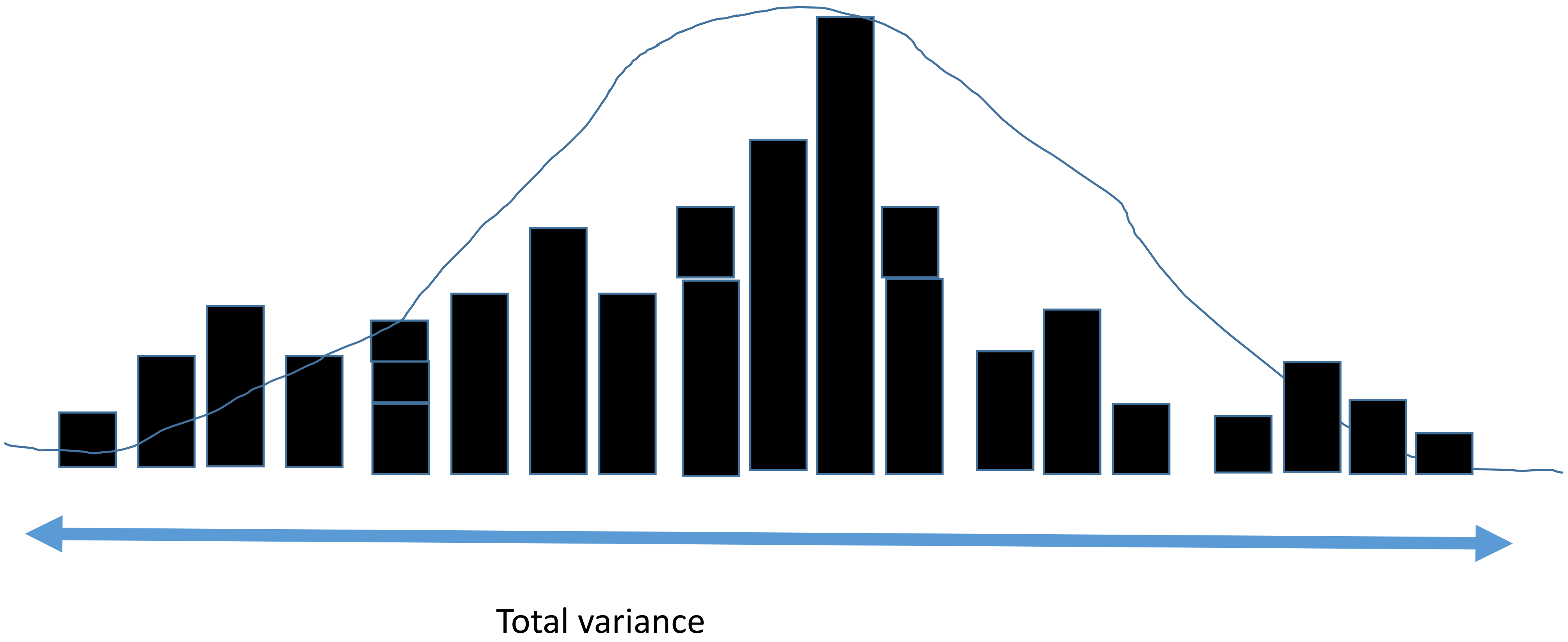
VARIANCE

sd^2

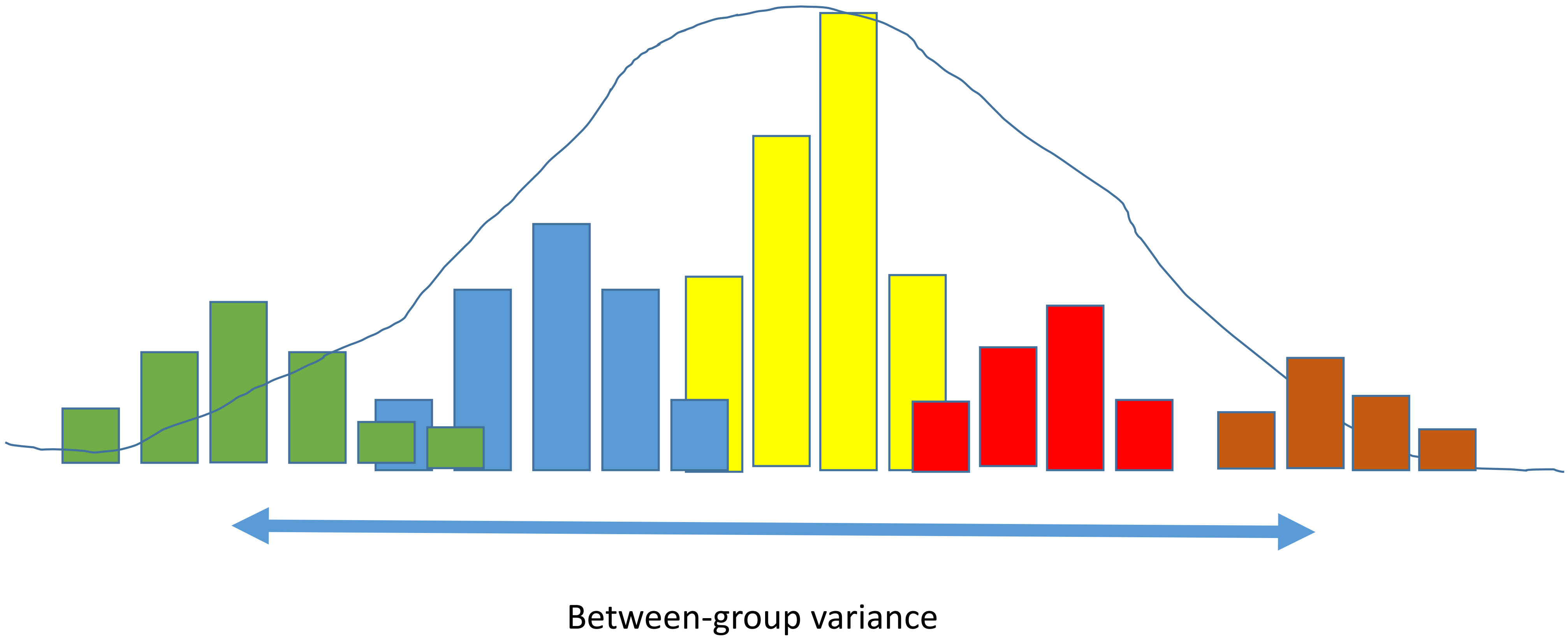
Standard deviation:

stdev video

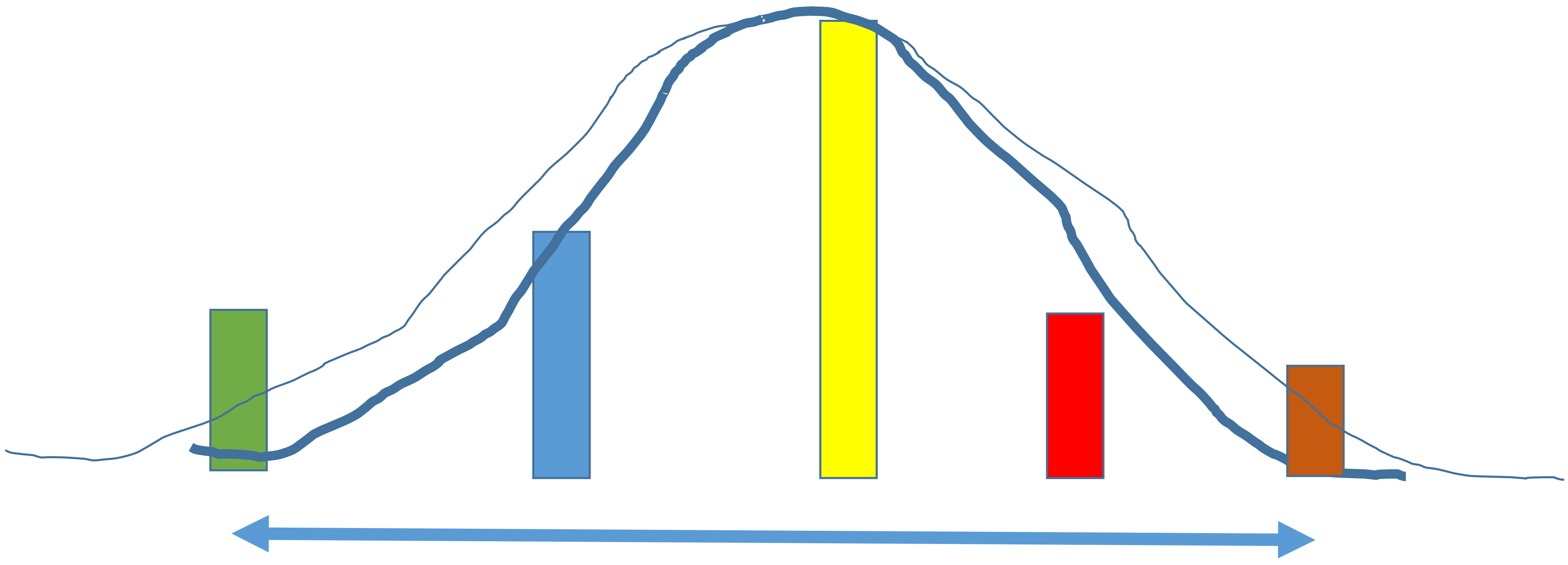
Variances and groups



Variances and groups

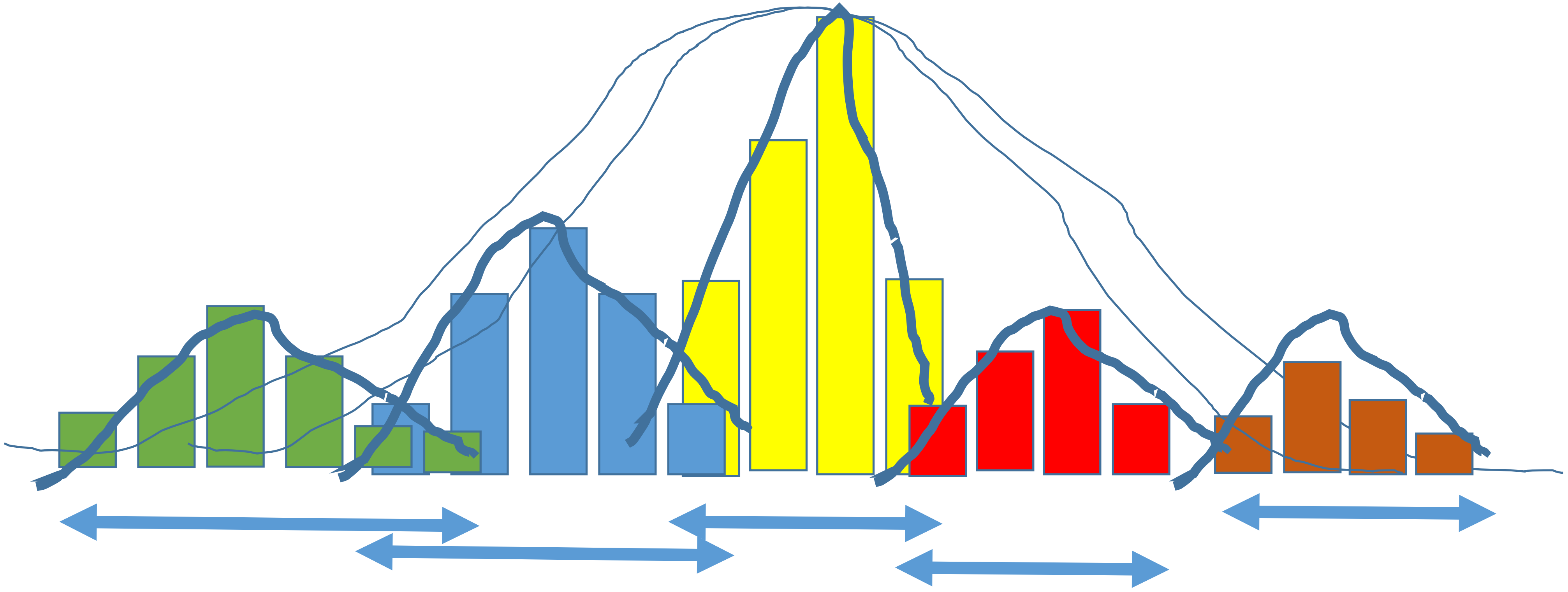


Variances and groups



Between-group variance

Variances and groups



Within-group variance

Variances and groups

1,1,2,1,1,1 5,5,4,5,5,1 3,3,4,3,3,3

Variances and groups

k = number of groups
n = sample size in group
N = total sample size
i = row
j = column
$\bar{x}$ = group mean
$\bar{\bar{x}}$ = grand total mean

1,1,2,1,1,1 5,5,4,5,5,1 3,3,4,3,3,3

Variances and groups

3 =	k = number of groups
6 =	n = sample size in group
18 =	N = total sample size
1:18	i = individual counter
1:3	j = group counter
	$\bar{x}$ = group mean
51	$\bar{\bar{x}}$ = grand total mean

1,1,2,1,1,1 5,5,4,5,5,1 3,3,4,3,3,3

Variances and groups

3 =	k = number of groups
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18 =	N = total sample size
1:18	i = individual counter
1:3	j = group counter
	$\bar{x}$ = group mean
51	$\bar{\bar{x}}$ = grand total mean

j: 1,1,1,1,1,1,2,2,2,2,2,2,3,3,3,3,3,3

1,1,2,1,1,1 5,5,4,5,5,1 3,3,4,3,3,3

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1:18	i = individual counter
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j: 1,1,1,1,1,1,2,2,2,2,2,2,3,3,3,3,3,3

i: 1,2,3,4,5,6,1,2,3, ...5,6

1,1,2,1,1,1 5,5,4,5,5,1 3,3,4,3,3,3

Variances and groups

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j: 1,1,1,1,1,1,2,2,2,2,2,2,3,3,3,3,3,3

i: 1,2,3,4,5,6,1,2,3, ...5,6

1,1,2,1,1,1 5,5,4,5,5,1 3,3,4,3,3,3

$$Y_{3,2} = 4$$

Variances and groups

3 =	k = number of groups
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18 =	N = total sample size
1:18	i = individual counter
1:3	j = group counter
	$\bar{x}$ = group mean
51	$\bar{\bar{x}}$ = grand total mean

j: 1,1,1,1,1,1,2,2,$\boxed{2}$,2,2,2,3,3,3,3,3,3

i: 1,2,3,4,5,6,1,2,$\boxed{3}$, ……………………5,6

1,1,2,1,1,1 5,5,$\boxed{4}$,5,5,1 3,3,4,3,3,3

$$Y_{3,2} = 4$$

3 =	k = number of groups
6 =	n_j = sample size in group
18 =	N = total sample size
1:6	i = individual counter
1:3	j = group counter
	$\bar{x}_j$ = group mean
51	$\bar{\bar{x}}_{i,j}$ = grand total mean

Variances and groups

j: 1,1,1,1,1,1,2,2,2,2,2,2,3,3,3,3,3,3

i: 1,2,3,4,5,6,1,2,3, ……………………5,6

$Y_{i,j}$: 1,1,2,1,1,1 5,5,4,5,5,1 3,3,4,3,3,3

$$Between\ group\ variance = \frac{SSG}{k-1} =$$

Variances and groups

j: 1,1,1,1,1,1,2,2,2,2,2,2,3,3,3,3,3,3

i: 1,2,3,4,5,6,1,2,3, ……………………5,6

$Y_{i,j}$: 1,1,2,1,1,1 5,5,4,5,5,1 3,3,4,3,3,3

$$Between\ group\ variance = \frac{SSG}{k-1} = \frac{\sum_{j=1}^{k} n_j(\overline{x_j} - \overline{\overline{x_{i,j}}})^2}{k-1}$$

3 =	k = number of groups
6 =	n_j = sample size in group
18 =	N = total sample size
1:6	i = individual counter
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	$\bar{x}_j$ = group mean
51	$\bar{\bar{x}}_{i,j}$ = grand total mean

Variances and groups

j: 1,1,1,1,1,1,2,2,2,2,2,2,3,3,3,3,3,3

i: 1,2,3,4,5,6,1,2,3, ………………….5,6

$Y_{i,j}$: 1,1,2,1,1,1 5,5,4,5,5,1 3,3,4,3,3,3

$$Within\ group\ variance = \frac{SSE}{N-k} =$$

3 =	k = number of groups
6 =	n_j = sample size in group
18 =	N = total sample size
1:6	i = individual counter
1:3	j = group counter
	$\bar{x}_j$ = group mean
51	$\bar{\bar{x}}_{i,j}$ = grand total mean

Variances and groups

j: 1,1,1,1,1,1,2,2,2,2,2,2,3,3,3,3,3,3

i: 1,2,3,4,5,6,1,2,3, ………………….5,6

$Y_{i,j}$: 1,1,2,1,1,1 5,5,4,5,5,1 3,3,4,3,3,3

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	$\bar{x}_j$ = group mean
51	$\bar{\bar{x}}_{i,j}$ = grand total mean

$$Within\ group\ variance = \frac{SSE}{N-k} = \frac{\sum_{j=1}^{k}\sum_{j=1}^{n_j}(x_{i,j}-x_j)^2}{N-k}$$

Variances and groups

j: 1,1,1,1,1,1,2,2,2,2,2,2,3,3,3,3,3,3

i: 1,2,3,4,5,6,1,2,3, ...5,6

$Y_{i,j}$: 1,1,2,1,1,1 5,5,4,5,5,1 3,3,4,3,3,3

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$$Within\ group\ variance = \frac{SSE}{N-k} = \frac{\sum_{j=1}^{k}\sum_{j=1}^{n_j}(x_{i,j}-x_j)^2}{N-k}$$

$$Between\ group\ variance = \frac{SSG}{k-1} = \frac{\sum_{j=1}^{k} n_j(\overline{x_j}-\overline{\overline{x_{i,j}}})^2}{k-1}$$

$$Total\ variance = \frac{SST}{n-1}$$

Variances and groups

j: 1,1,1,1,1,1,2,2,2,2,2,2,3,3,3,3,3,3

i: 1,2,3,4,5,6,1,2,3, ……………………5,6

$Y_{i,j}$: 1,1,2,1,1,1 5,5,4,5,5,1 3,3,4,3,3,3

3 =	k = number of groups
6 =	n_j = sample size in group
18 =	N = total sample size
1:6	i = individual counter
1:3	j = group counter
	$\bar{x}_j$ = group mean
51	$\bar{\bar{x}}_{i,j}$ = grand total mean

$$Within\ group\ variance = \frac{SSE}{N-k} = \frac{\sum_{j=1}^{k}\sum_{j=1}^{n_j}(x_{i,j} - x_j)^2}{N-k}$$

$$Between\ group\ variance = \frac{SSG}{k-1} = \frac{\sum_{j=1}^{k} n_j(\bar{x_j} - \overline{\overline{x_{i,j}}})^2}{k-1}$$

$$Total\ variance = \frac{SST}{n-1}$$

$$SST = SSG + SSE$$

LMMs

- So what does a mixed model look like?

LMMs

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LMMs

- So what does a mixed model look like?

$$y_{i,j} = b_0 + b_1 x_{i,j} + \alpha_j + \varepsilon_{i,j}$$

Exam

Chick ID	Chick Mass	Tarsus	Nest				
A	5	3	L1				
B	3	1	L1				
C	6	4	L1				
D	10	8	S5				
E	4	2	S5				

LMMs

- Or estimate both. Let's look at the components of a LMM just like we did with the lm:

$$y_{i,j} = b_0 + b_1 x_{i,j} + \alpha_j + \varepsilon_{i,j}$$

y	i
5	1 A
3	2 B
6	3 C
10	4 D
4	5 D

b_0 = ?

b_1 = ?

x	i
3	1 A
1	2 B
4	3 C
8	4 D
2	5 D

ε	i
?	1 A
?	2 B
?	3 C
?	4 D
?	5 D

LMMs

- First things first. What's with the j?

$$y_{i,j} = b_0 + b_1 x_{i,j} + \alpha_j + \varepsilon_{i,j}$$

y	i	j
5	1 A	1 L1
3	2 B	1 L1
6	3 C	1 L1
10	4 D	2 S5
4	5 D	2 S5

b_0 = ?

b_1 = ?

x	i	j
3	1	1 L1
1	2	1 L1
4	3	1 L1
8	4	2 S5
2	5	2 S5

ε	i	j
?	1	1 L1
?	2	1 L1
?	3	1 L1
?	4	2 S5
?	5	2 S5

LMMs

- First things first. What's with the j?
- J is a grouping factor. BirdID.

$$y_{i,j} = b_0 + b_1 x_{i,j} + \alpha_j + \varepsilon_{i,j}$$

y	i	j
5	1	1
3	2	1
6	3	1
10	4	2
4	5	2

b_0 = ?

b_1 = ?

x	i	j
3	1	1
1	2	1
4	3	1
8	4	2
2	5	2

ε	i	j
?	1	1
?	2	1
?	3	1
?	4	2
?	5	2

LMMs

- First things first. What's with the j?
- J is a grouping factor. BirdID. Observer.

$$y_{i,j} = b_0 + b_1 x_{i,j} + \alpha_j + \varepsilon_{i,j}$$

y	i	j
5	1	1
3	2	1
6	3	1
10	4	2
4	5	2

b_0 = ?

b_1 = ?

x	i	j
3	1	1
1	2	1
4	3	1
8	4	2
2	5	2

ε	i	j
?	1	1
?	2	1
?	3	1
?	4	2
?	5	2

LMMs

- First things first. What's with the j?
- J is a grouping factor. BirdID. Observer. NestID.

$$y_{i,j} = b_0 + b_1 x_{i,j} + \alpha_j + \varepsilon_{i,j}$$

y	i	j
5	1	1
3	2	1
6	3	1
10	4	2
4	5	2

b_0 = ?

b_1 = ?

x	i	j
3	1	1
1	2	1
4	3	1
8	4	2
2	5	2

ε	i	j
?	1	1
?	2	1
?	3	1
?	4	2
?	5	2

LMMs

- First things first. What's with the j?
- J is a grouping factor. BirdID. Observer. NestID. It's a factor in your data. Year.

$$y_{i,j} = b_0 + b_1 x_{i,j} + \alpha_j + \varepsilon_{i,j}$$

y	i	j
5	1	1
3	2	1
6	3	1
10	4	2
4	5	2

$b_0 = ?$

$b_1 = ?$

x	i	j
3	1	1
1	2	1
4	3	1
8	4	2
2	5	2

ε	i	j
?	1	1
?	2	1
?	3	1
?	4	2
?	5	2

LMMs

- First things first. What's with the j?
- J is a grouping factor. BirdID. Observer. NestID. It's a factor in your data. Year.
- It's **categorical**

$$y_{i,j} = b_0 + b_1 x_{i,j} + \alpha_j + \varepsilon_{i,j}$$

y	i	j
5	1	1
3	2	1
6	3	1
10	4	2
4	5	2

$b_0 = ?$

$b_1 = ?$

x	i	j
3	1	1
1	2	1
4	3	1
8	4	2
2	5	2

ε	i	j
?	1	1
?	2	1
?	3	1
?	4	2
?	5	2

LMMs

- Ok. That only leaves alpha-j. What's that?

$$y_{i,j} = b_0 + b_1 x_{i,j} + \alpha_j + \varepsilon_{i,j}$$

y	i	j
5	1	1
3	2	1
6	3	1
10	4	2
4	5	2

b_0 = ?

b_1 = ?

x	i	j
3	1	1
1	2	1
4	3	1
8	4	2
2	5	2

ε	i	j
?	1	1
?	2	1
?	3	1
?	4	2
?	5	2

LMMs

- Ok. That only leaves alpha-j. What's that?

$$y_{i,j} = b_0 + b_1 x_{i,j} + \alpha_j + \varepsilon_{i,j}$$

y	i	j
5	1	1
3	2	1
6	3	1
10	4	2
4	5	2

b_0 = ?

b_1 = ?

x	i	j
3	1	1
1	2	1
4	3	1
8	4	2
2	5	2

Variance of data grouped by j

ε	i	j
?	1	1
?	2	1
?	3	1
?	4	2
?	5	2

LMMs

- Ok. That only leaves alpha-j.
- So it's one number, with a measure of precision, like the bs!

$$y_{i,j} = b_0 + b_1 x_{i,j} + \alpha_j + \varepsilon_{i,j}$$

y	i	j
5	1	1
3	2	1
6	3	1
10	4	2
4	5	2

b_0 = ?

b_1 = ?

x	i	j
3	1	1
1	2	1
4	3	1
8	4	2
2	5	2

Variance of data grouped by j

ε	i	j
?	1	1
?	2	1
?	3	1
?	4	2
?	5	2

LMMs

- Ok. That only leaves alpha-j.
- So it’s one number, with a measure of precision , like the bs!
- And we color it red because we want to estimate it!

$$y_{i,j} = b_0 + b_1 x_{i,j} + \alpha_j + \varepsilon_{i,j}$$

y	i	j
5	1	1
3	2	1
6	3	1
10	4	2
4	5	2

b_0 = ?

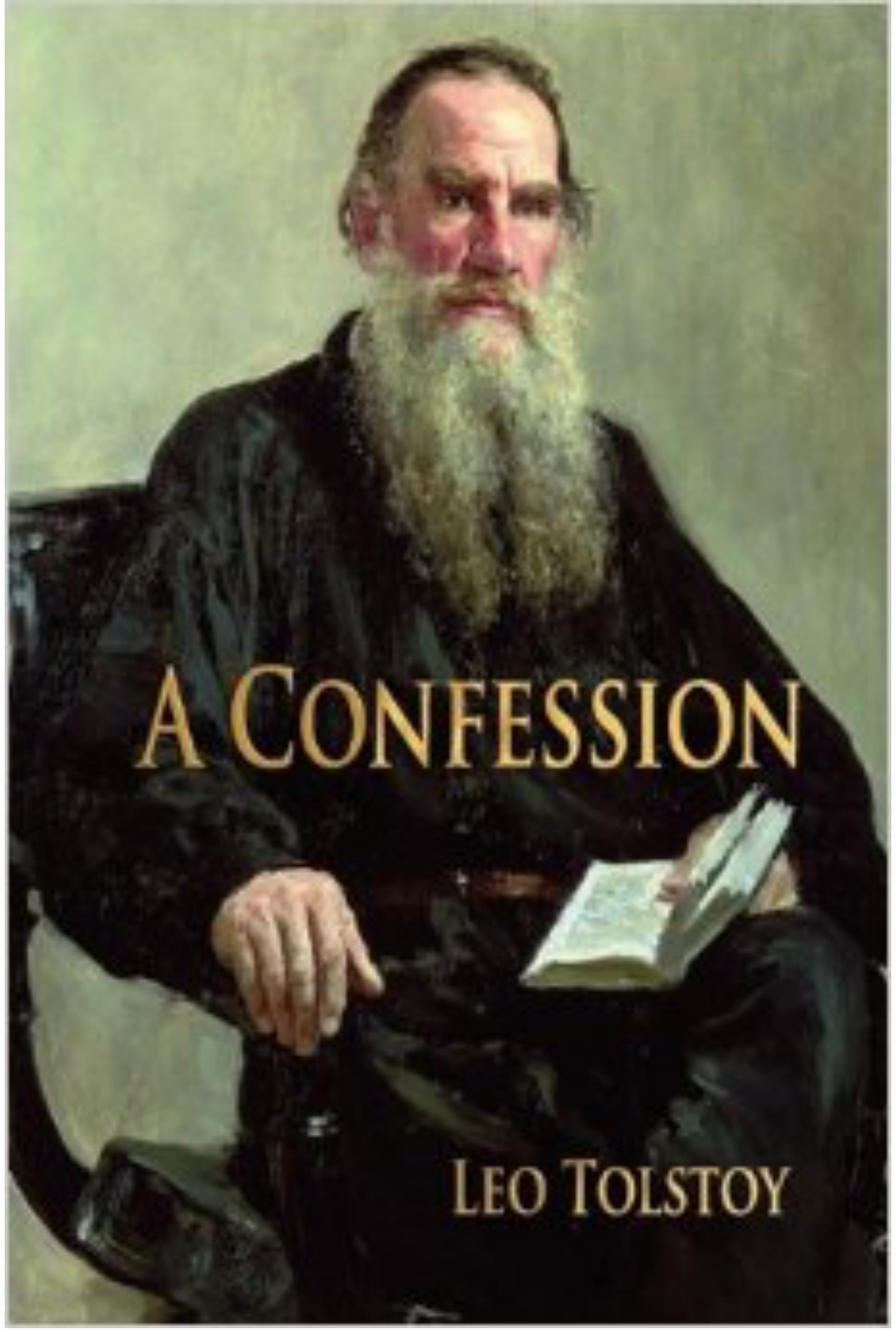
b_1 = ?

x	i	j
3	1	1
1	2	1
4	3	1
8	4	2
2	5	2

y	j
5	1
3	1
6	1
10	2
4	2

Group variance of data y grouped by j

ε	i	j
?	1	1
?	2	1
?	3	1
?	4	2
?	5	2


A CONFESSION
LEO TOLSTOY

LMMs – a confession

- In this equation, e-ij is also a variance. The variance of the residuals. We've somewhat ignored this so far. But really, it's a number.

$$y_{i,j} = b_0 + b_1 x_{i,j} + \alpha_j + \varepsilon_{i,j}$$

y	i	j
5	1	1
3	2	1
6	3	1
10	4	2
4	5	2

b_0 = ?

b_1 = ?

x	i	j
3	1	1
1	2	1
4	3	1
8	4	2
2	5	2

y	j
5	1
3	1
6	1
10	2
4	2

Group variance of data y grouped by j

ε	i	j
?	1	1
?	2	1
?	3	1
?	4	2
?	5	2

Variance of residuals

LMMs – a confession

- Ok, cool. Let's revisit that group variance thing.

$$y_{i,j} = b_0 + b_1 x_{i,j} + \alpha_j + \varepsilon_{i,j}$$

y	i	j
5	1	1
3	2	1
6	3	1
10	4	2
4	5	2

b_0 = ?

b_1 = ?

x	i	j
3	1	1
1	2	1
4	3	1
8	4	2
2	5	2

y	j
5	1
3	1
6	1
10	2
4	2

Group variance of data y grouped by j

ε	i	j
?	1	1
?	2	1
?	3	1
?	4	2
?	5	2

Variance of residuals

Between-group variance

$$y_{i,j} = b_0 + b_1 x_{i,j} + \alpha_j + \varepsilon_{i,j}$$

y	i	j
5	1	1
3	2	1
6	3	1
10	4	2
4	5	2

b_0 = ?

b_1 = ?

x	i	j
3	1	1
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Variance of residuals

Between-group variance

Within-group variance

$$y_{i,j} = b_0 + b_1 x_{i,j} + \alpha_j + \varepsilon_{i,j}$$

y	i	j
5	1	1
3	2	1
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y	j
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6	1
10	2
4	2

Group variance of data y grouped by j

ε	i	j
?	1	1
?	2	1
?	3	1
?	4	2
?	5	2

Variance of residuals

Between-group variance

Within-group variance
Residual variance

$$y_{i,j} = b_0 + b_1 x_{i,j} + \alpha_j + \varepsilon_{i,j}$$

y	i	j
5	1	1
3	2	1
6	3	1
10	4	2
4	5	2

b_0 = ?

b_1 = ?

x	i	j
3	1	1
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8	4	2
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y	j
5	1
3	1
6	1
10	2
4	2

Group variance of data y grouped by j

ε	i	j
?	1	1
?	2	1
?	3	1
?	4	2
?	5	2

Variance of residuals

Between-group variance

Within-group variance

Residual variance

Error variance

$$y_{i,j} = b_0 + b_1 x_{i,j} + \alpha_j + \varepsilon_{i,j}$$

y	i	j
5	1	1
3	2	1
6	3	1
10	4	2
4	5	2

b_0 = ?

b_1 = ?

x	i	j
3	1	1
1	2	1
4	3	1
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y	j
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Group variance of data y grouped by j

ε	i	j
?	1	1
?	2	1
?	3	1
?	4	2
?	5	2

Variance of residuals

Between-group variance

Within-group variance

Residual variance

Error variance

$$Total\ variance = \color{red}{\alpha_j} + \varepsilon_{i,j}$$

$$y_{i,j} = b_0 + b_1 x_{i,j} + \alpha_j + \varepsilon_{i,j}$$

y	i	j
5	1	1
3	2	1
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10	2
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Group variance of data y grouped by j

Between-group variance

ε	i	j
?	1	1
?	2	1
?	3	1
?	4	2
?	5	2

Variance of residuals

Residual variance

$$\underbrace{}_{}\quad y_{i,j} = \overbrace{b_0 + b_1 x_{i,j}}^{\text{Fixed}} + \overbrace{\alpha_j}^{\text{Random}} + \overbrace{\varepsilon_{i,j}}^{\text{Error}}$$

y	i	j
5	1	1
3	2	1
6	3	1
10	4	2
4	5	2

b_0 = ?

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x	i	j
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1	2	1
4	3	1
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Group variance of data y grouped by j

Between-group variance

ε	i	j
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?	2	1
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?	5	2

Variance of residuals

Residual variance

LMMs

- You can choose to only estimate one part and set the other one to fixed:

LMMs

- You can choose to only estimate one part

$$y_{i,j} = b_0 + b_1 x_{i,j} + \alpha_j + \varepsilon_{i,j}$$

- You could set the random part to zero (always keep the error!)

LMMs

- You can choose to only estimate one part

$$y_{i,j} = 1 + \alpha_j + \varepsilon_{i,j}$$

- You could set the fixed part to one (always have an intercept is a good idea)

Mixed models

- Estimate variance components simultaneously to fixed terms

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- Allow to account for nested structure in data

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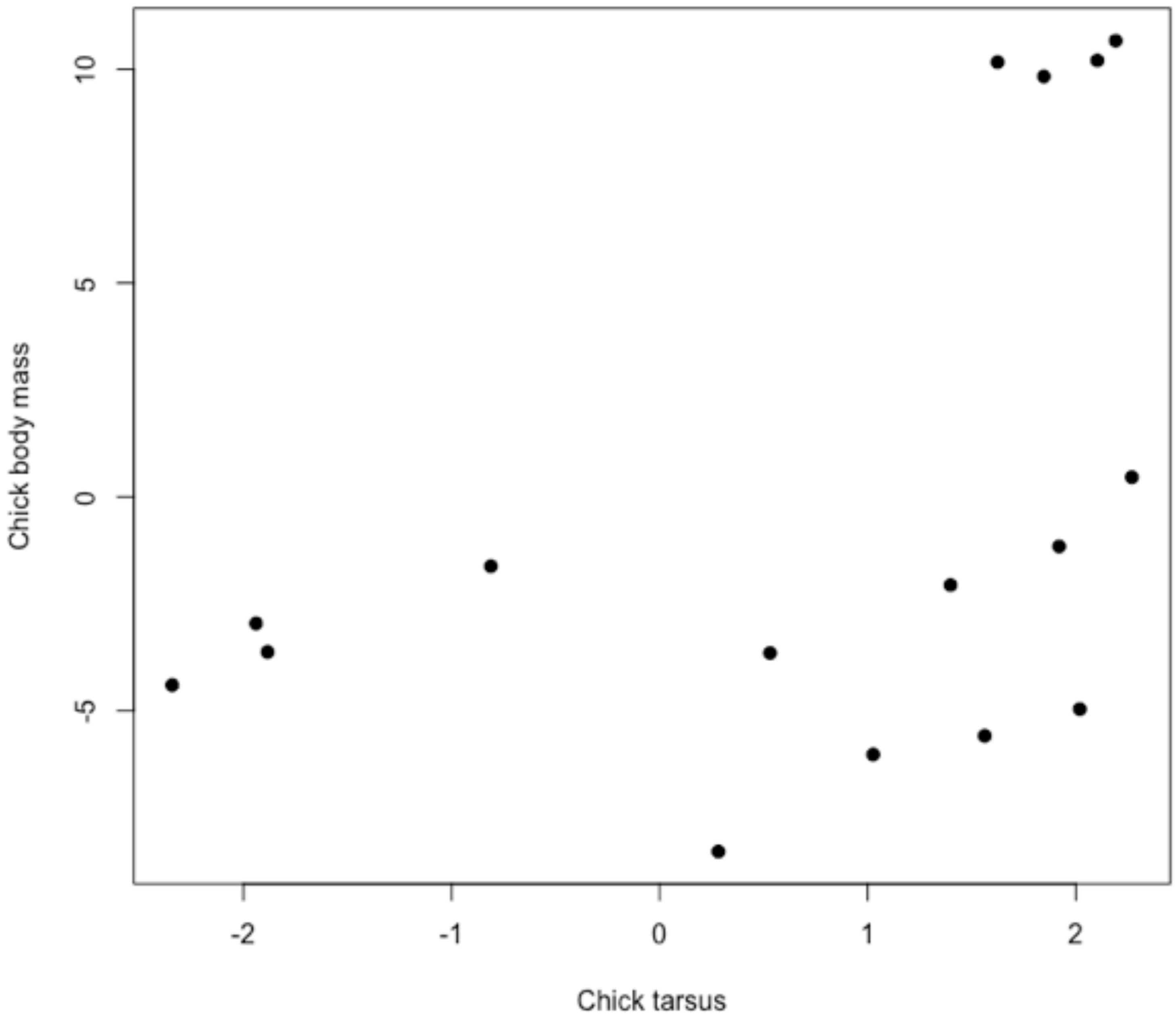


Mixed models

```
> m<-(lm(bm~tarsus))
> summary(m)

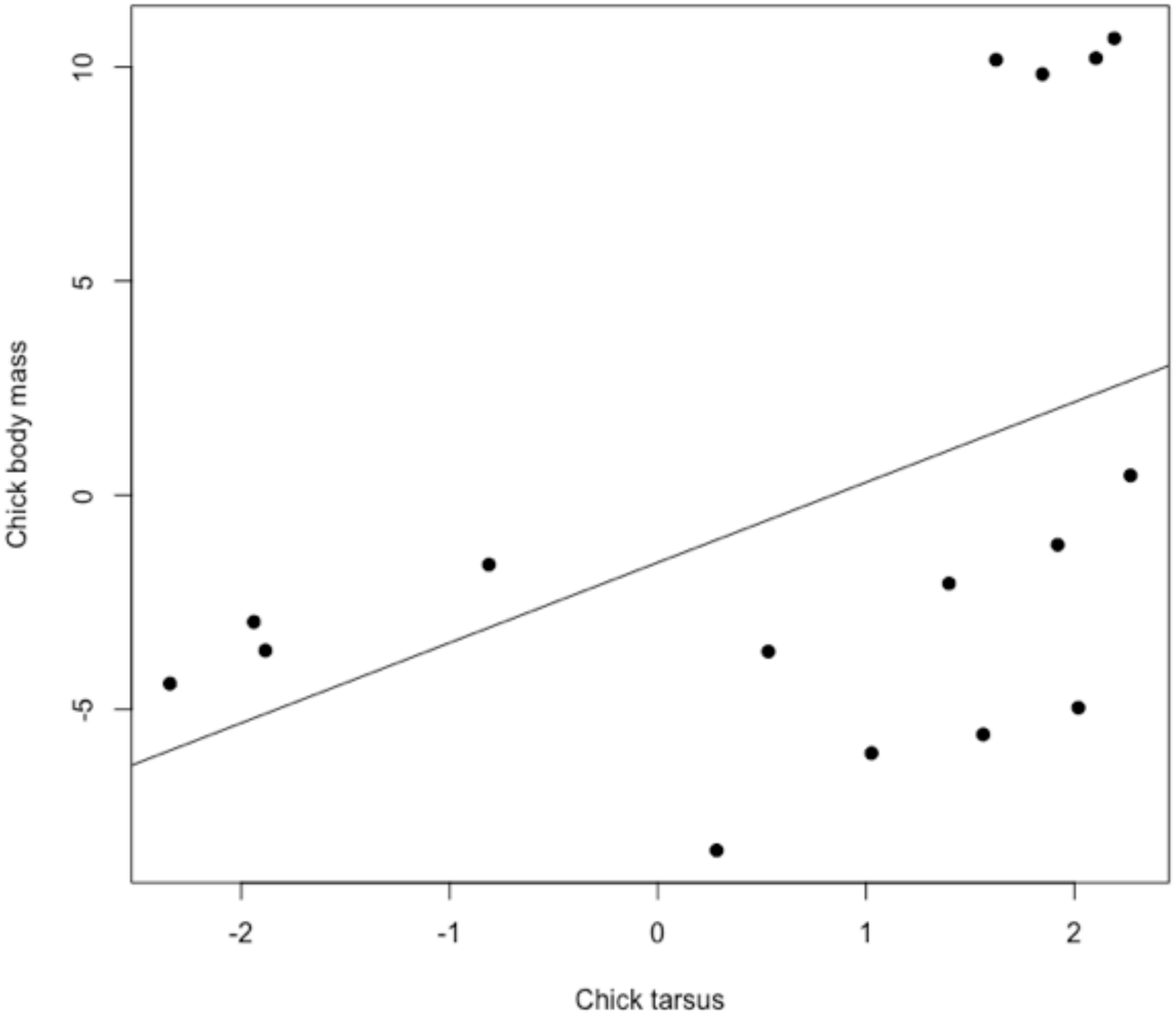
Call:
lm(formula = bm ~ tarsus)

Residuals:
    Min      1Q  Median      3Q     Max 
-7.2535 -3.9865 -0.3775  3.6447  8.6954 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)  
(Intercept)   -1.570      1.662  -0.944   0.3610  
tarsus         1.875      0.965   1.943   0.0725 .
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘

Residual standard error: 6.008 on 14 degrees of freedom
Multiple R-squared:  0.2123,	Adjusted R-squared:  0.1561 
F-statistic: 3.774 on 1 and 14 DF,  p-value: 0.07246

> |
```



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> |
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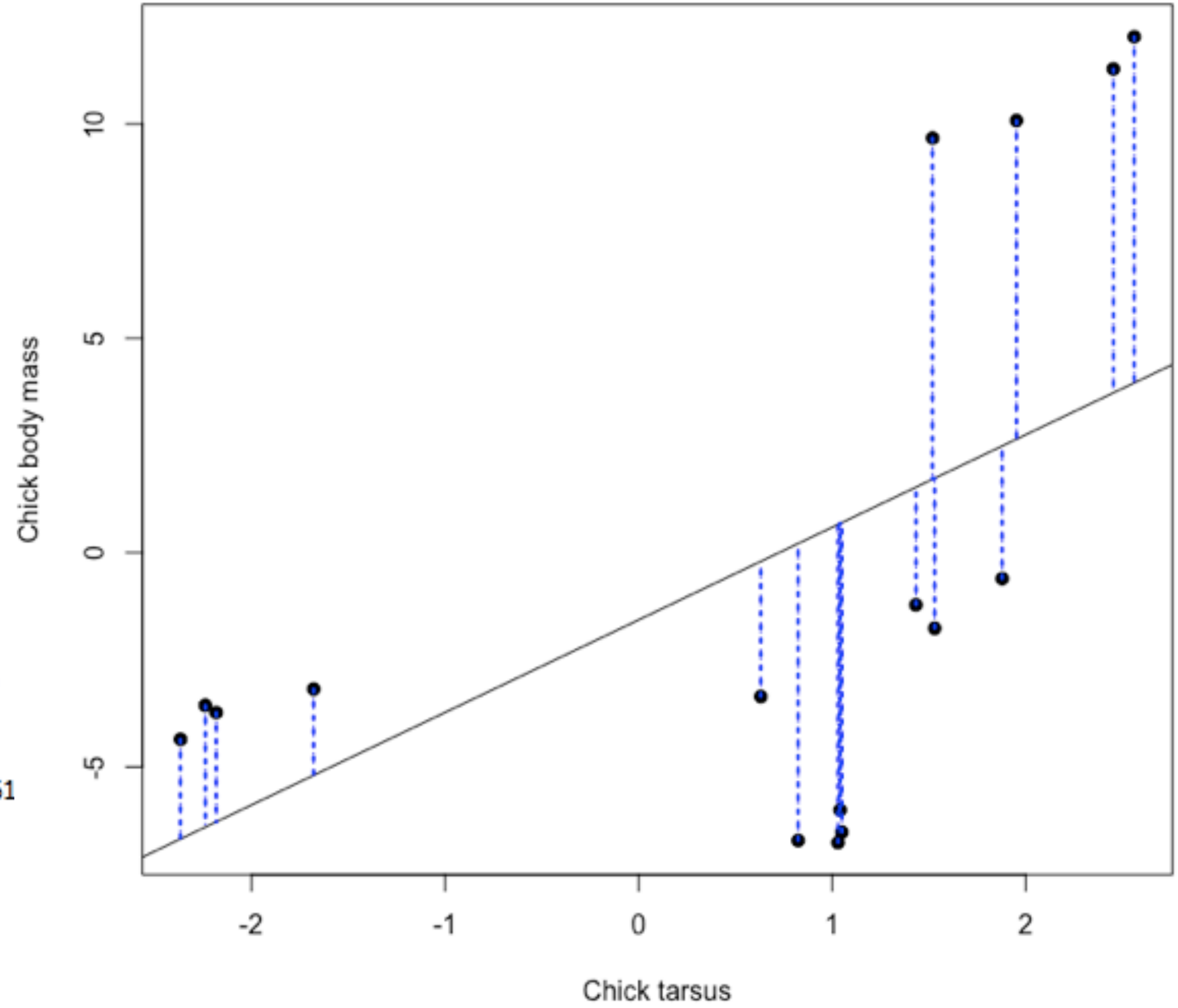

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> |
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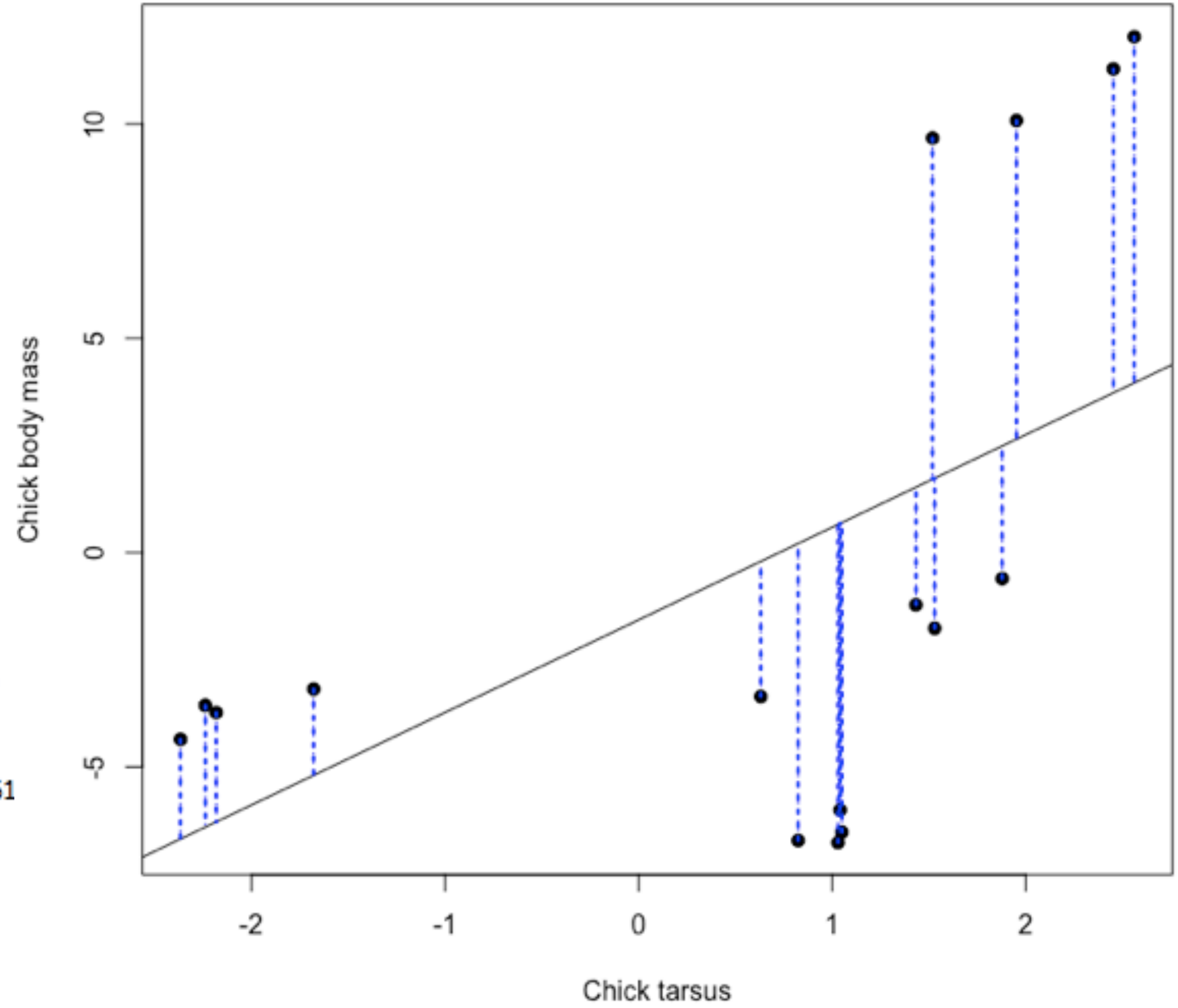
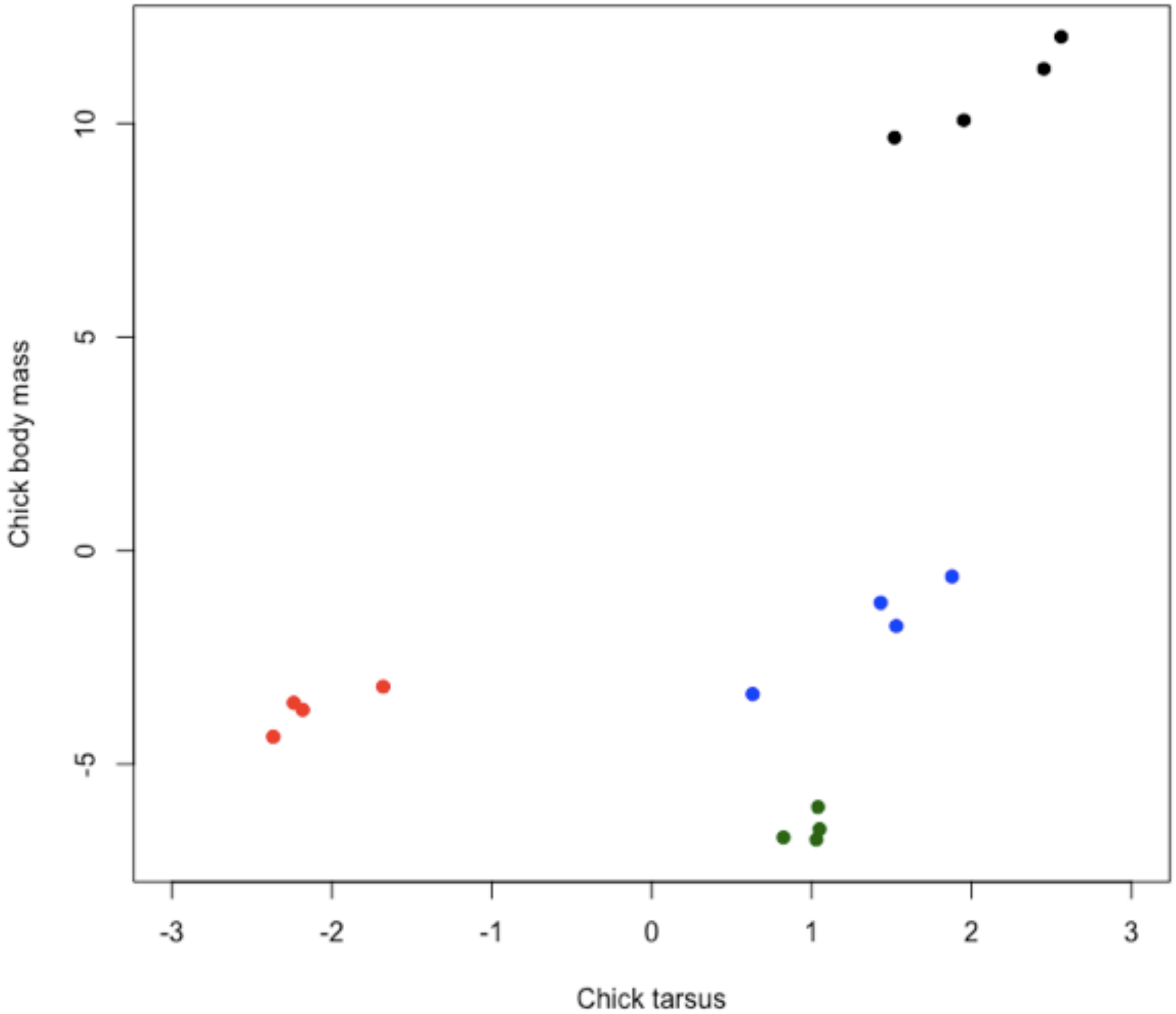
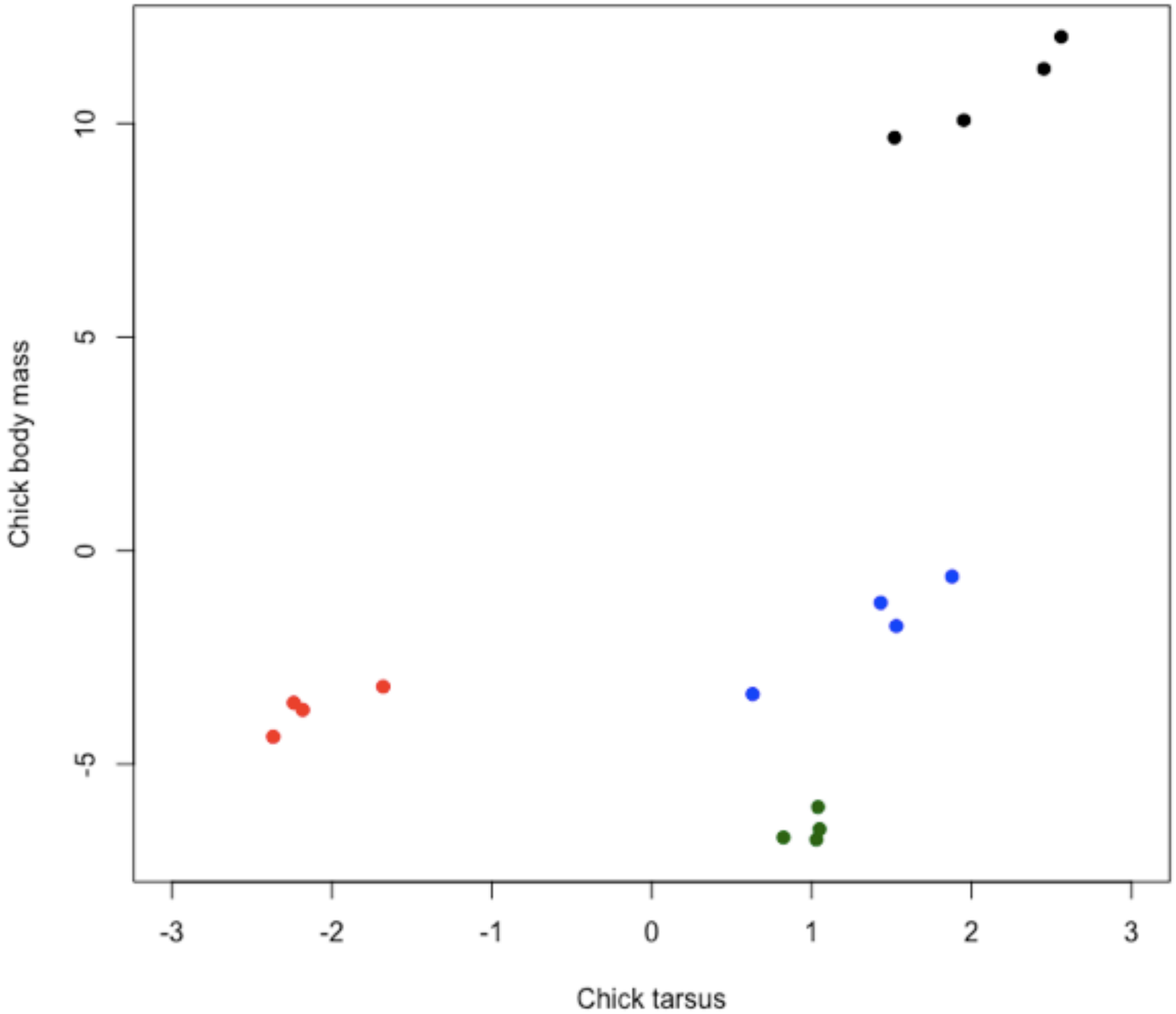


Mixed models



Mixed models

```
> require(lme4)
> mm<-(lmer(bm~tarsus+(1|nest)))
```



Mixed models

```
> require(lme4)
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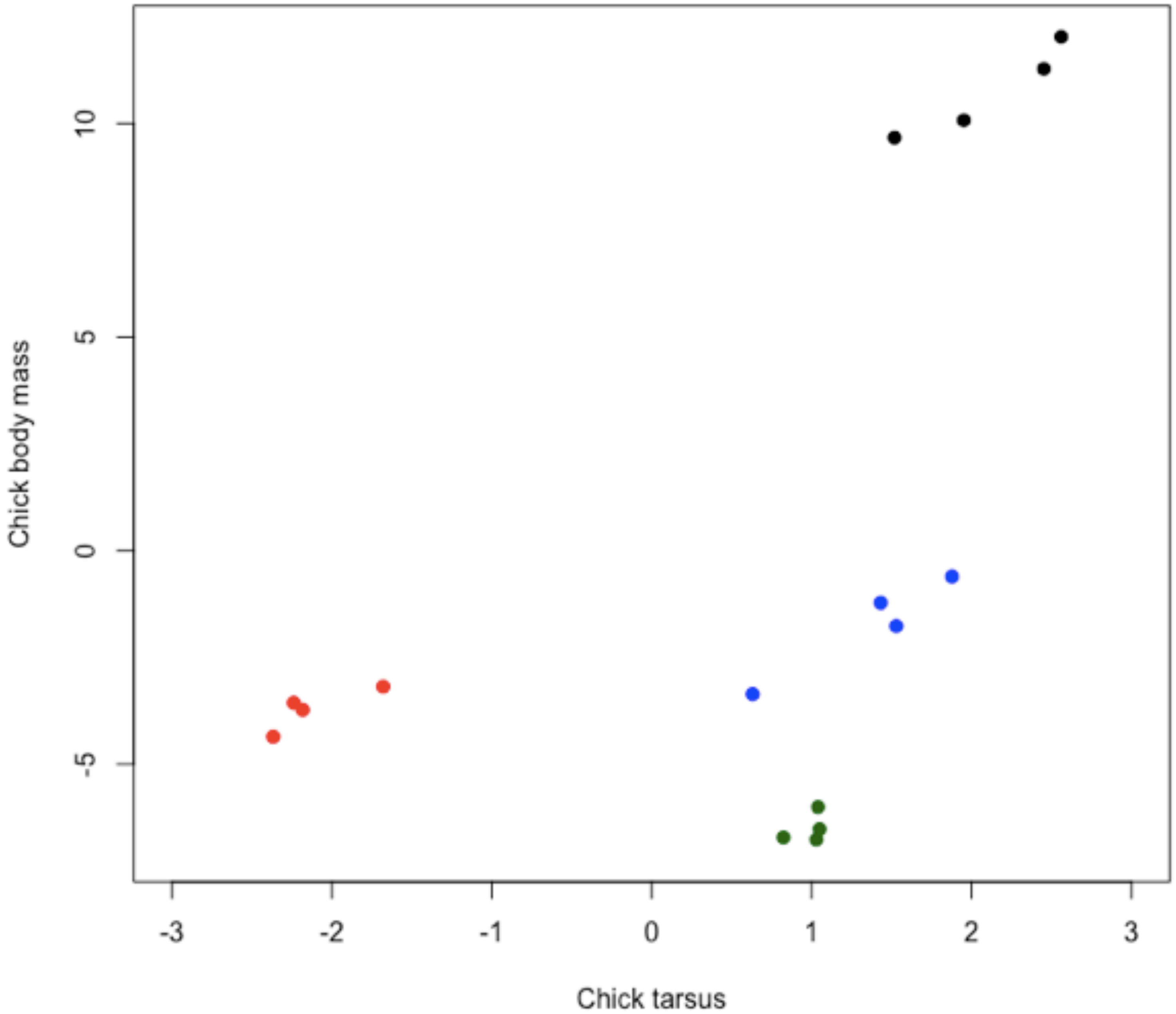
> summary(mm)
Linear mixed model fit by REML ['lmerMod']
Formula: bm ~ tarsus + (1 | nest)

REML criterion at convergence: 34.4

Scaled residuals:
     Min       1Q   Median       3Q      Max
-1.08739 -0.60062 -0.05266  0.58491  1.18030

Random effects:
 Groups   Name        Variance Std.Dev.
 nest     (Intercept) 42.1465  6.492
 Residual              0.1115  0.334
Number of obs: 16, groups:  nest, 4

Fixed effects:
            Estimate Std. Error t value
(Intercept)  -1.4939     3.2503   -0.46
tarsus        2.0325     0.2446    8.31
```



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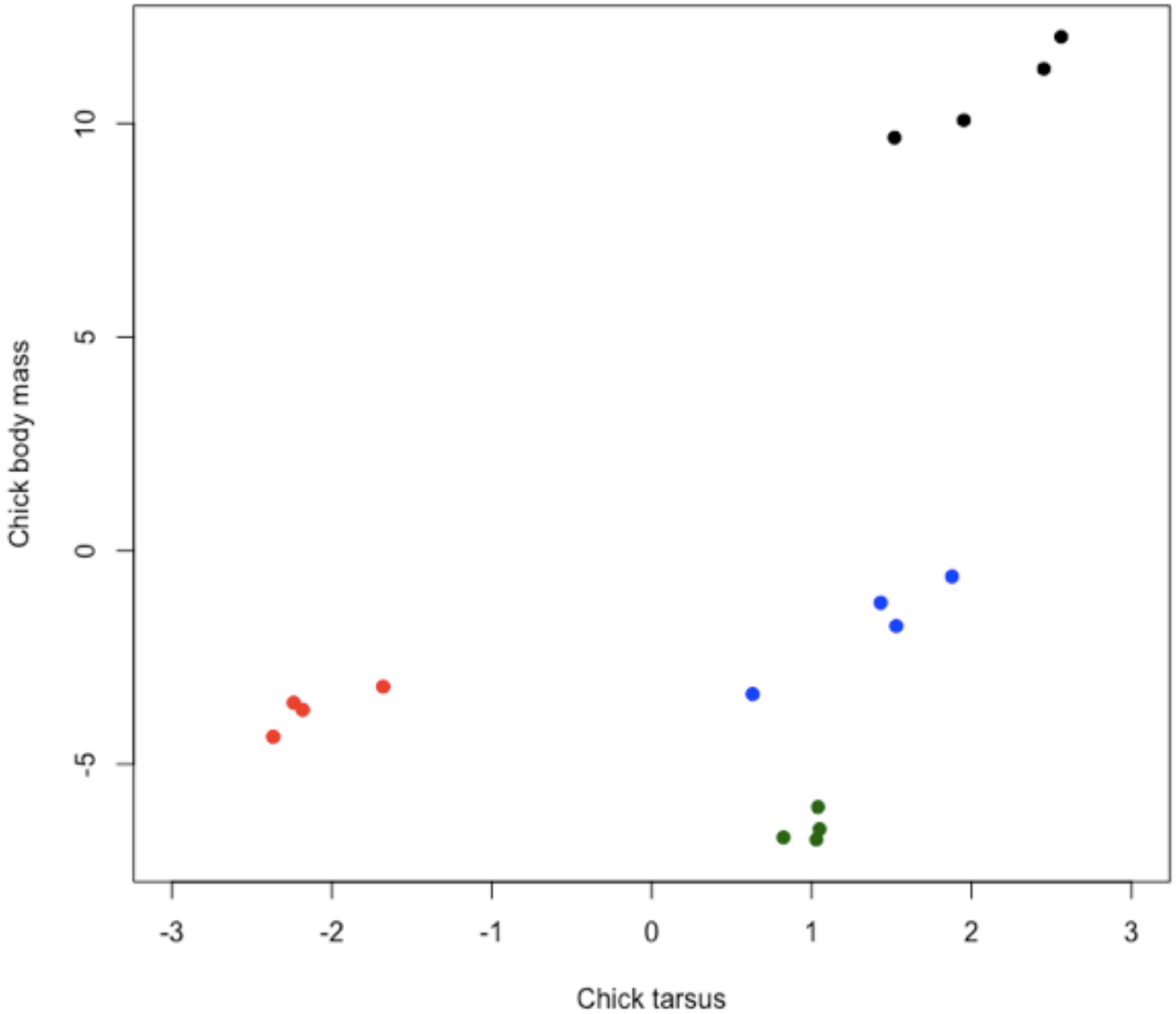

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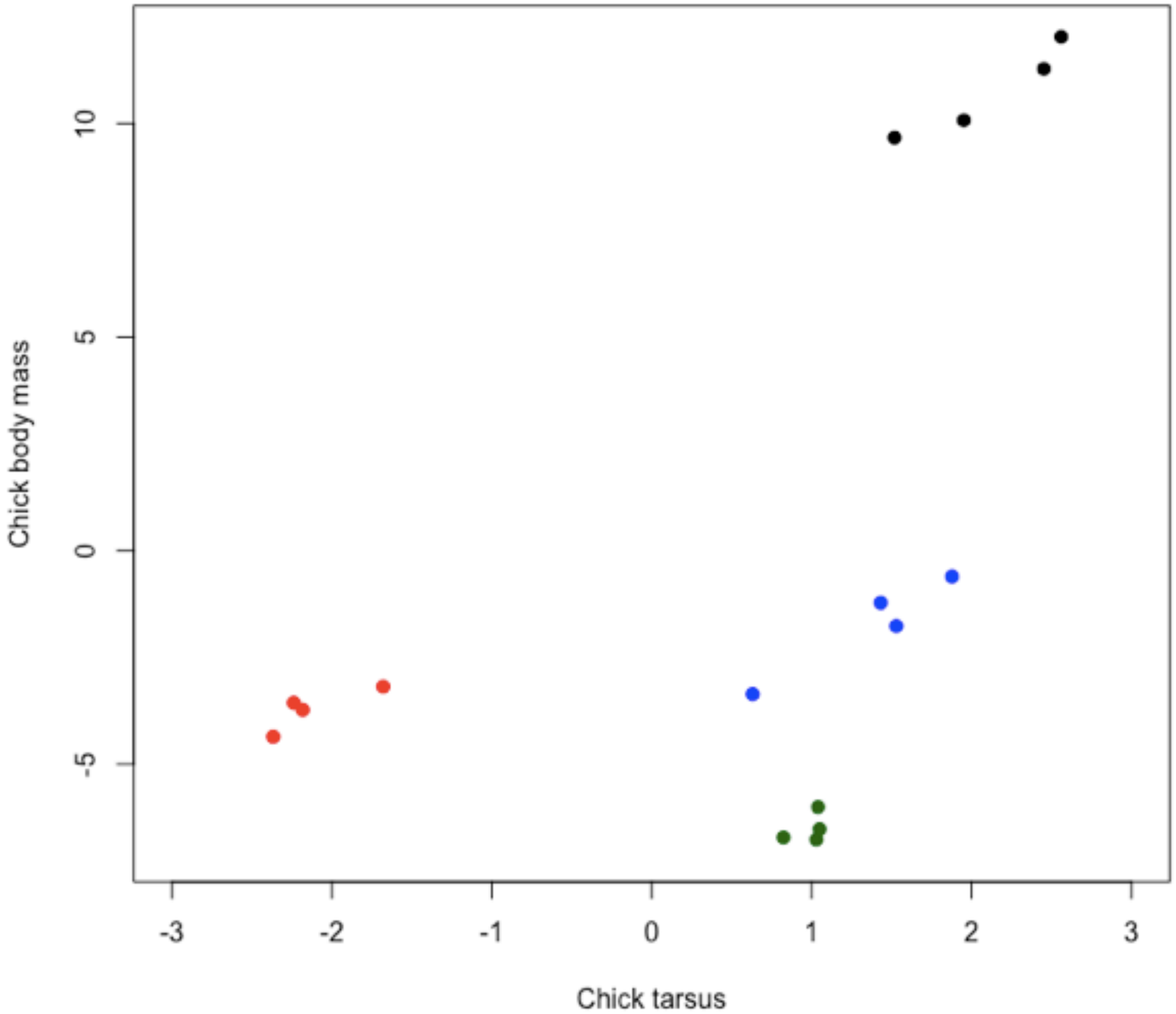

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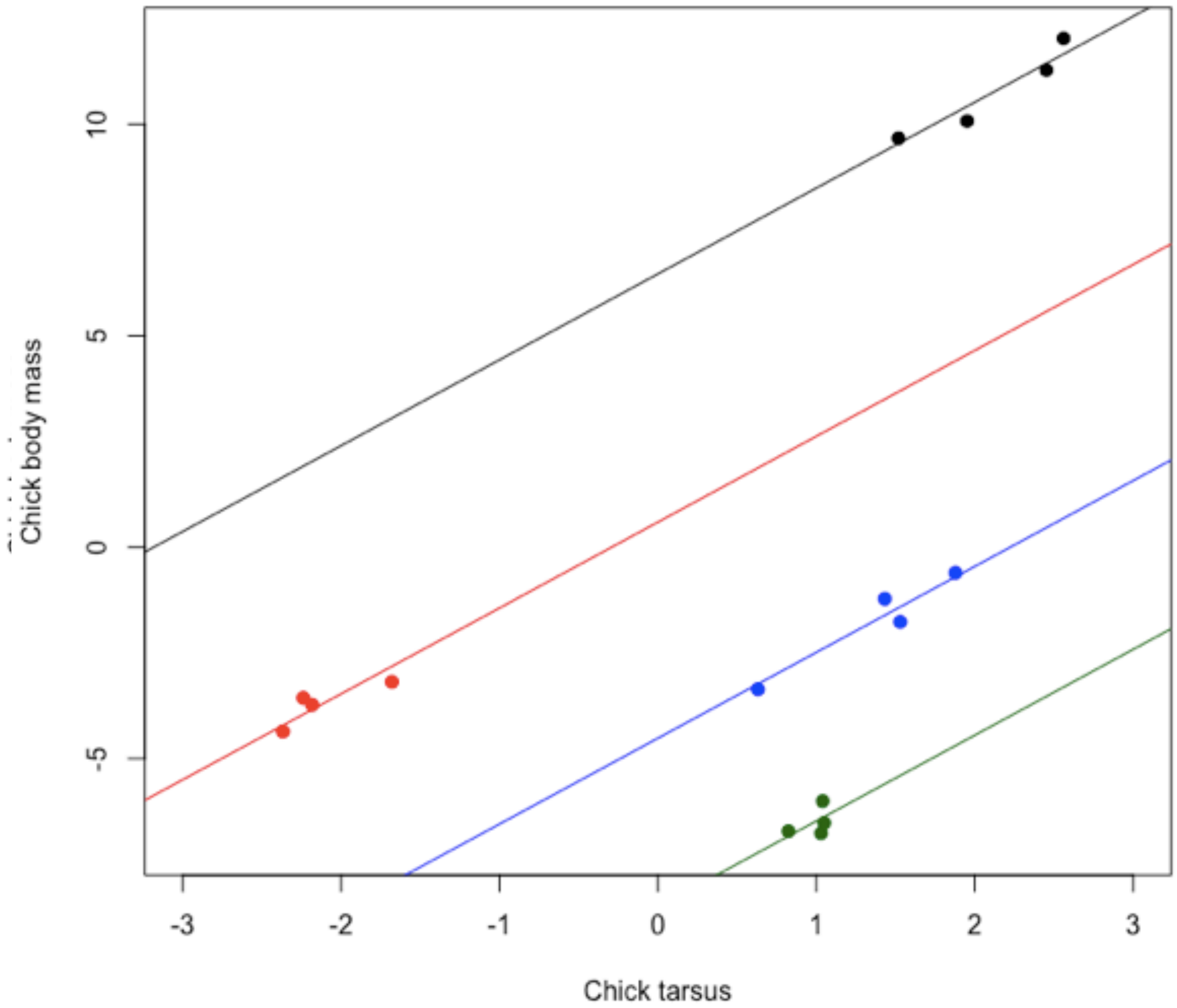

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Call:
lm(formula = bm ~ tarsus)

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    Min      1Q  Median      3Q     Max
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Coefficients:
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(Intercept)   -1.570      1.662  -0.944   0.3610
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---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

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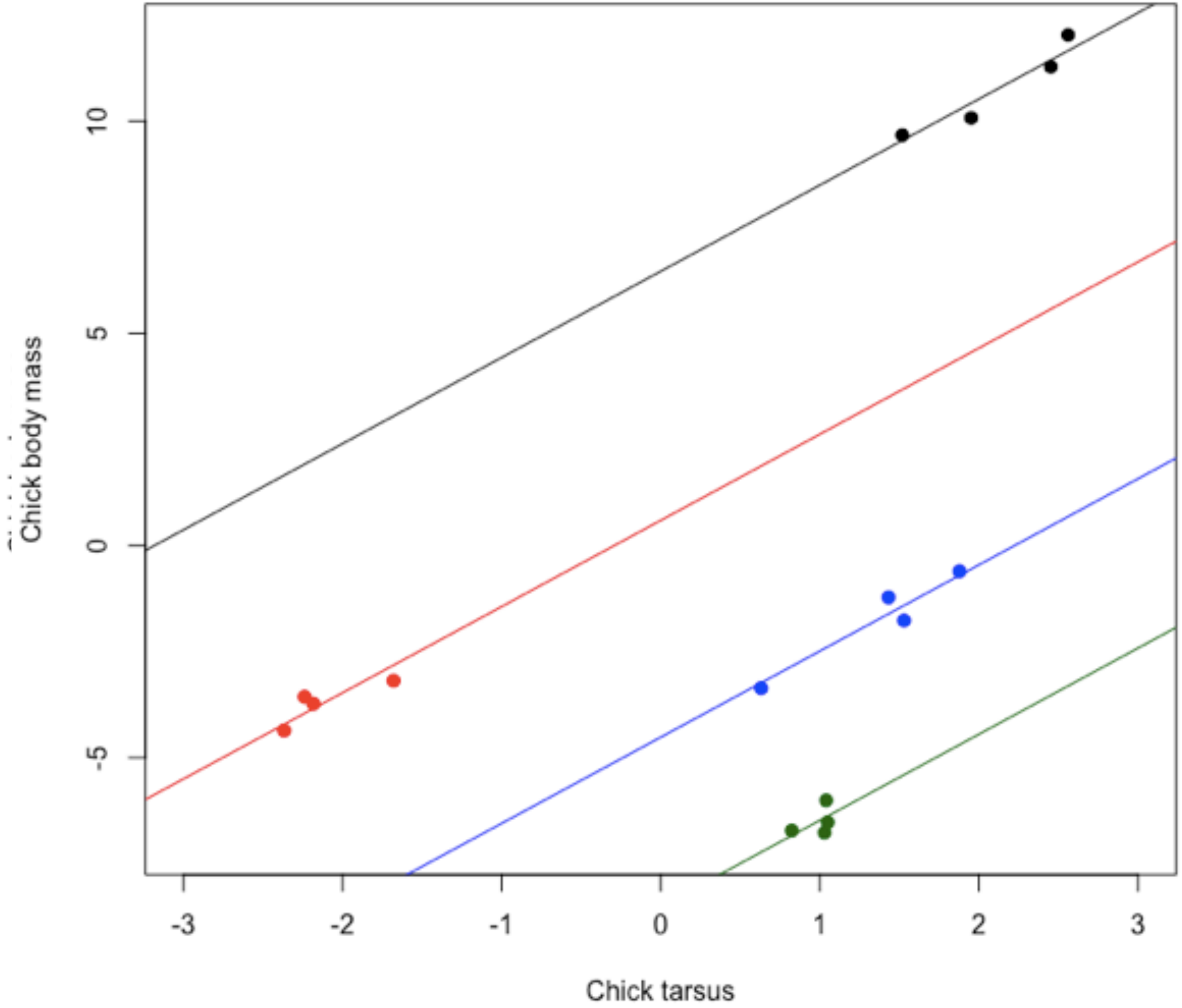
Multiple R-squared:  0.2123,    Adjusted R-squared:  0.1561
F-statistic: 3.774 on 1 and 14 DF,  p-value: 0.07246

> |
```

Mixed models

- Ok, so why don't we add nest as fixed factor instead?



Nest:

Random

```
> require(lme4)
> mm<-(lmer(bm~tarsus+(1|nest)))
> summary(mm)
Linear mixed model fit by REML ['lmerMod']
Formula: bm ~ tarsus + (1 | nest)

REML criterion at convergence: 34.4

Scaled residuals:
     Min       1Q   Median       3Q      Max 
-1.08739 -0.60062 -0.05266  0.58491  1.18030 

Random effects:
 Groups   Name        Variance Std.Dev.
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 Residual              0.1115  0.334   
Number of obs: 16, groups:  nest, 4

Fixed effects:
            Estimate Std. Error t value
(Intercept)  -1.4939     3.2503   -0.46
tarsus        2.0325     0.2446    8.31
```

Fixed

```
> summary(lm(bm~tarsus+nest))

Call:
lm(formula = bm ~ tarsus + nest)

Residuals:
     Min       1Q   Median       3Q      Max 
-0.36367 -0.20454 -0.01581  0.18941  0.39257 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)   6.4606     0.5490  11.767 1.42e-07 ***
tarsus        2.0305     0.2466   8.234 4.96e-06 ***
nestB        -5.8707     1.0715  -5.479 0.000192 ***
nestC       -10.9770     0.3005 -36.525 7.81e-13 ***
nestD       -14.9654     0.3663 -40.858 2.29e-13 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.3342 on 11 degrees of freedom
Multiple R-squared:  0.9983,	Adjusted R-squared:  0.9976 
F-statistic:  1581 on 4 and 11 DF,  p-value: 4.275e-15
```

Nest:

Random

```
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 Residual              0.1115  0.334
Number of obs: 16, groups:  nest, 4

Fixed effects:
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Fixed

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Call:
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     Min       1Q   Median       3Q      Max
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(Intercept)   6.4606     0.5490  11.767 1.42e-07 ***
tarsus        2.0305     0.2466   8.234 4.9[illegible]e-06 ***
nestB         5.8707     1.0715   5.479 0.000192 ***
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nestD       -14.9654     0.3663 -40.858 2.29e-13 ***
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(Intercept)  -1.4939     3.2503   -0.46
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> summary(lm(bm~tarsus+nest))

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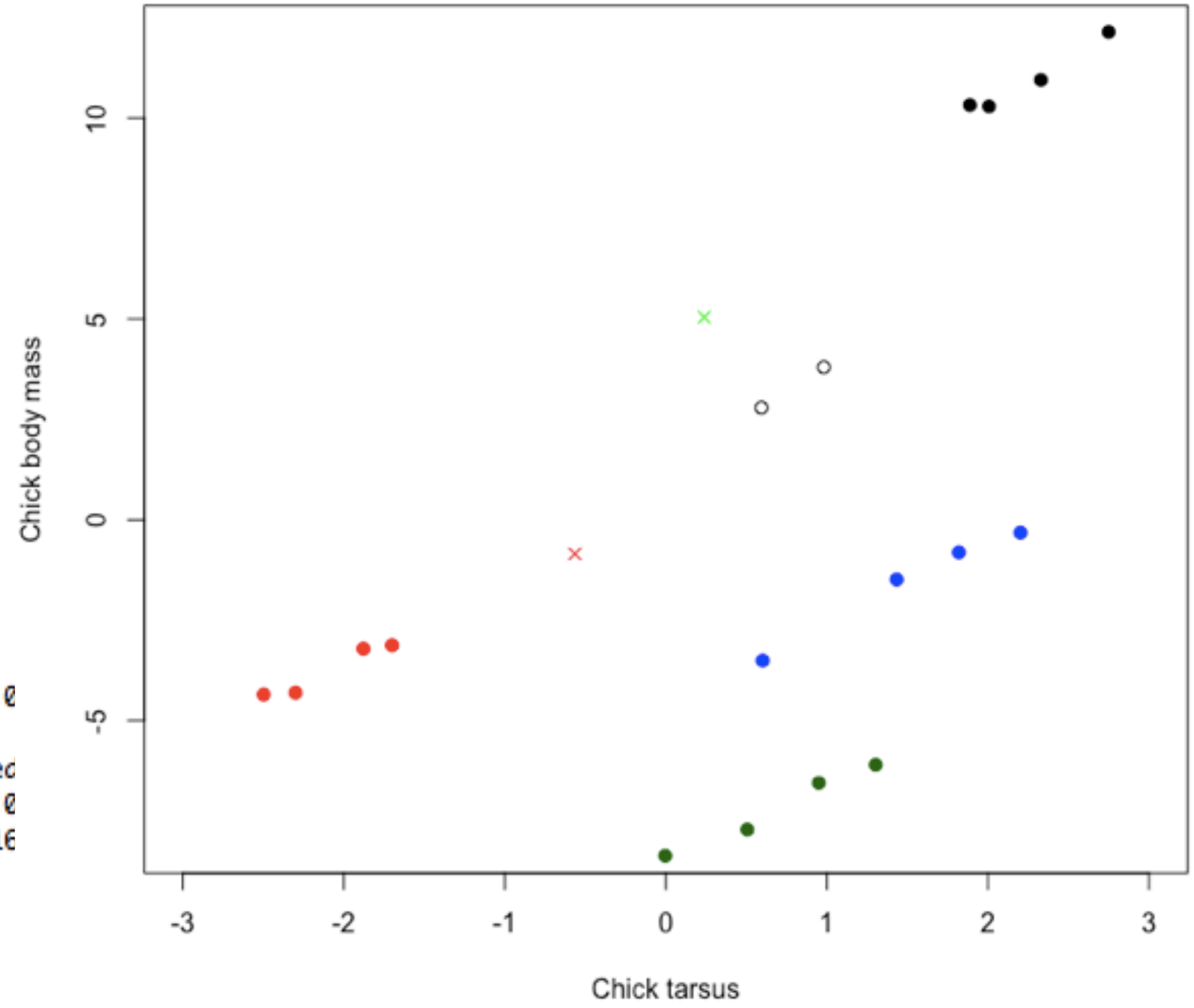
Multiple R-squared:  0.9983,	Adjusted R-squared:  0.9976 
F-statistic:  1581 on 4 and 11 DF,  p-value: 4.275e-15
```

```
Call:
lm(formula = bm ~ tarsus + nest)

Residuals:
    Min      1Q  Median      3Q     Max
-0.1731 -0.1472  0.0000  0.1178  0.2166

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   6.5022     0.2523  25.773 7.10e-12 ***
tarsus        1.9725     0.1043  18.919 2.67e-10 ***
nestB        -6.1191     0.4716 -12.974 2.02e-08 ***
nestC       -11.0200     0.1539 -71.617  < 2e-16 ***
nestD       -15.0427     0.2102 -71.571  < 2e-16 ***
nestE        -4.7637     0.2233 -21.331 6.57e-11 ***
nestF        -6.2356     0.3612 -17.262 7.72e-10 ***
nestG        -1.9313     0.2974  -6.494 2.96e-05 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0

Residual standard error: 0.1892 on 12 degrees of freed
Multiple R-squared:  0.9995,    Adjusted R-squared:  0
F-statistic:  3230 on 7 and 12 DF,  p-value: < 2.2e-16
```



Mixed models

- Estimates variance components simultaneously to fixed terms
- Allow to account for nested structure in data with lots of levels

Mixed models

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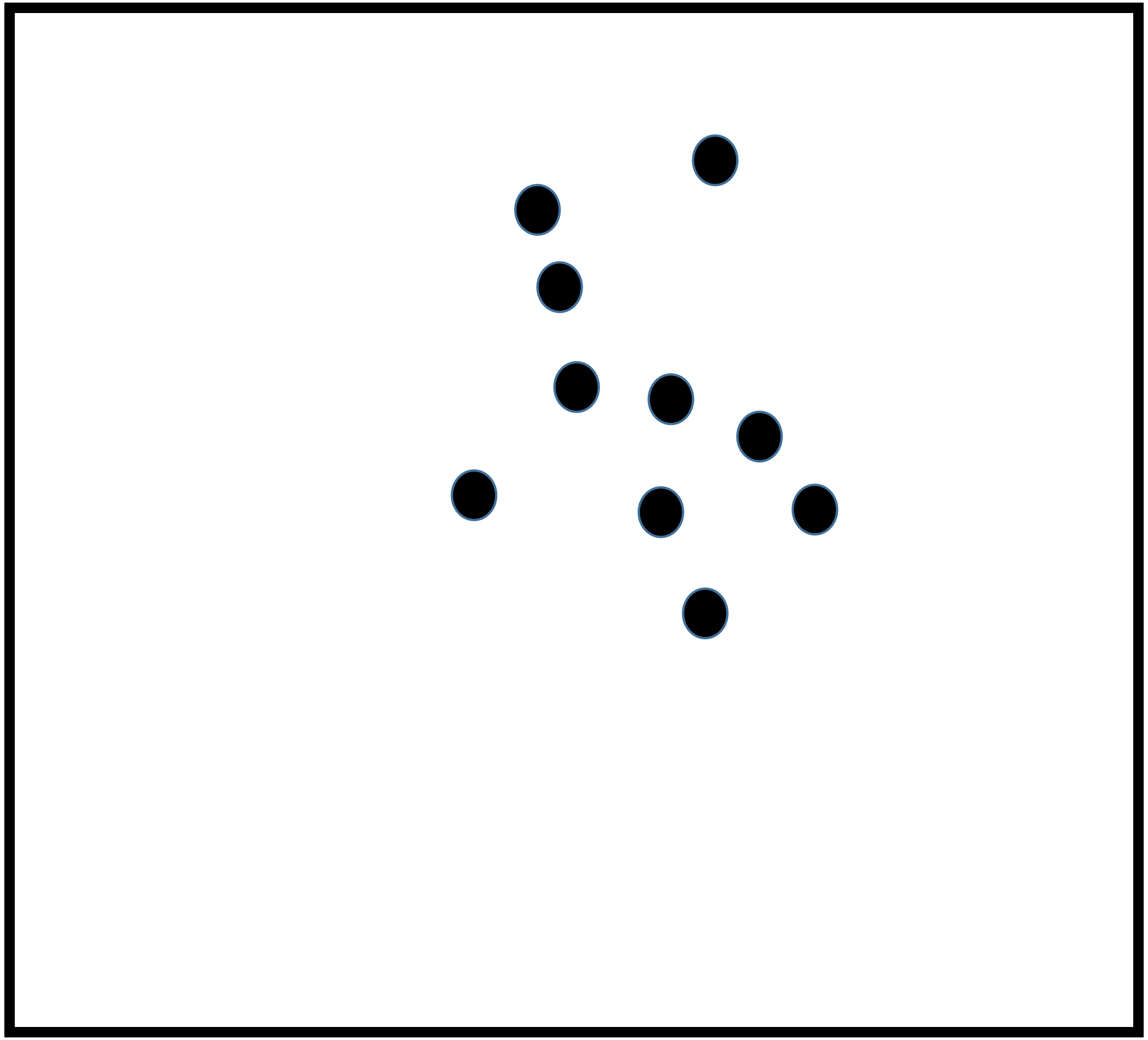
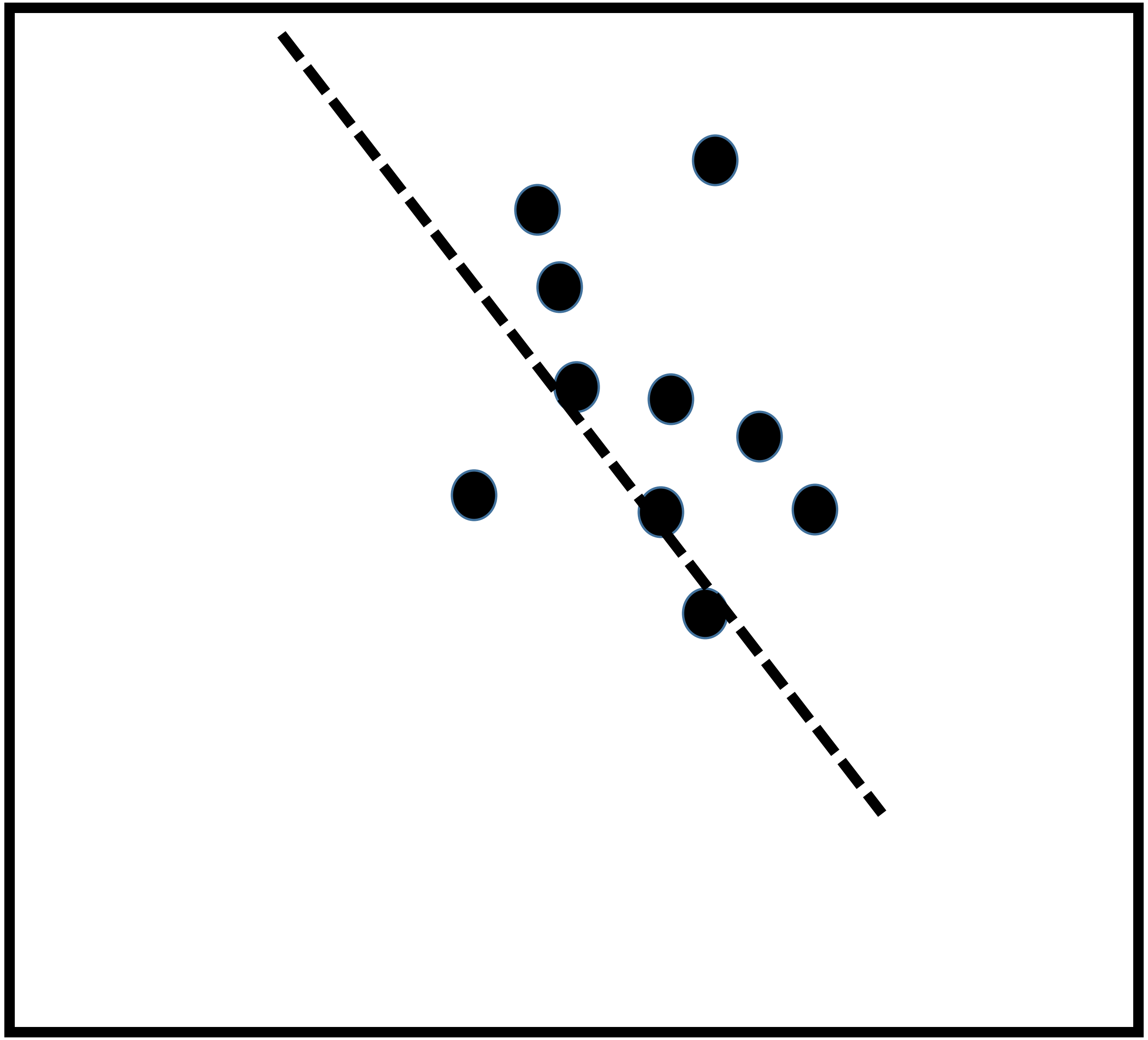
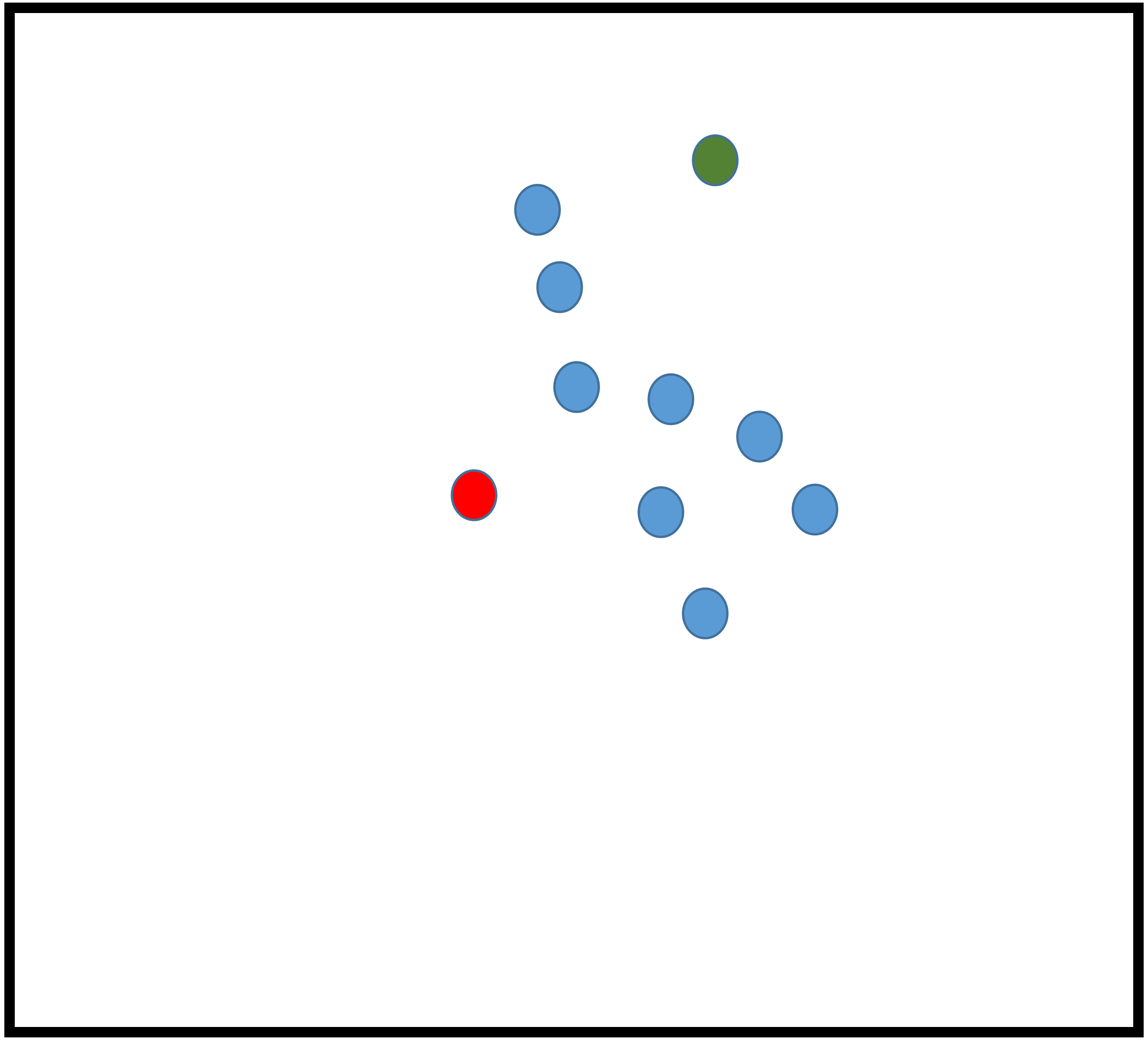
- Is robust against heterogeneous data

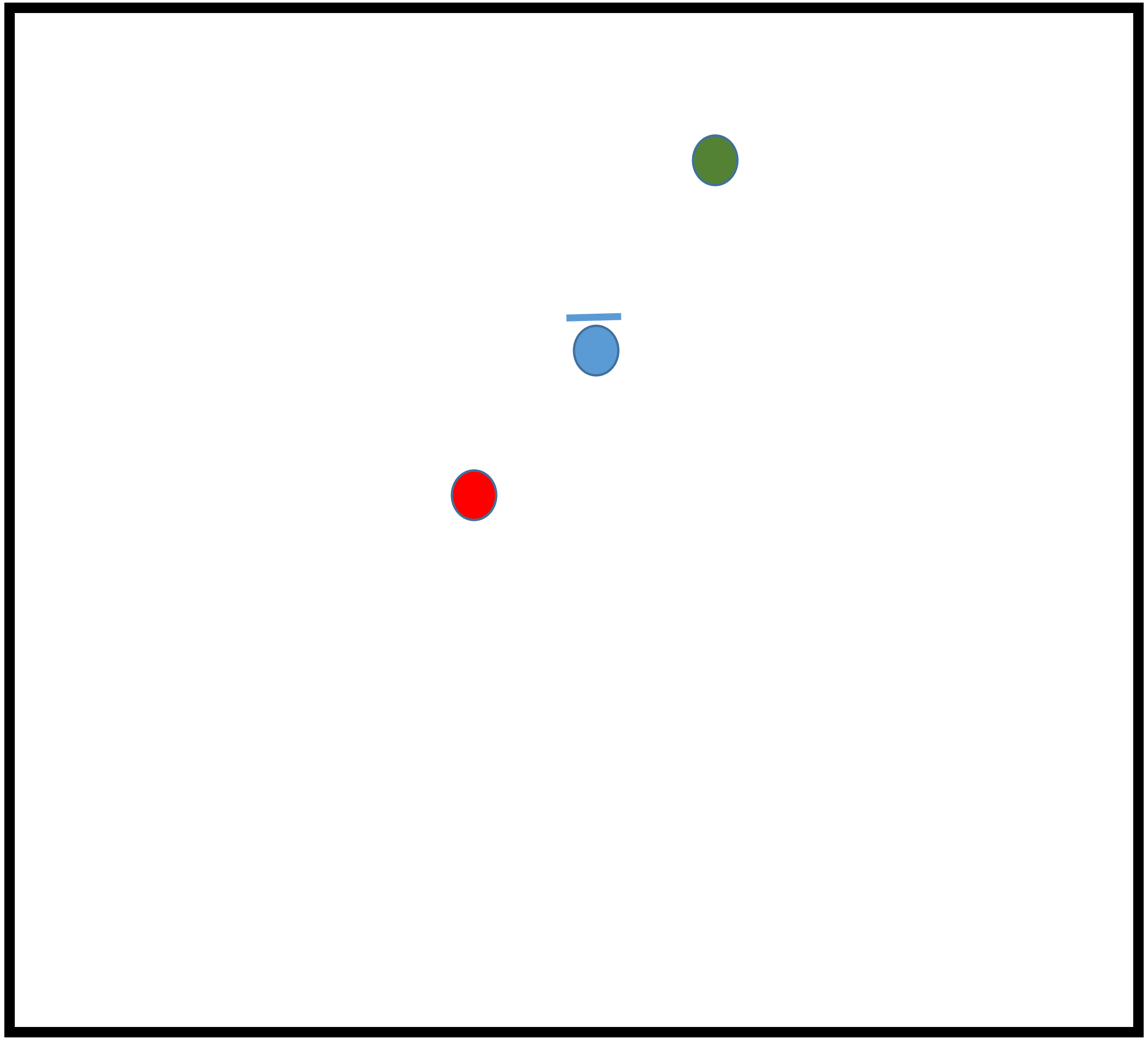
Mixed models

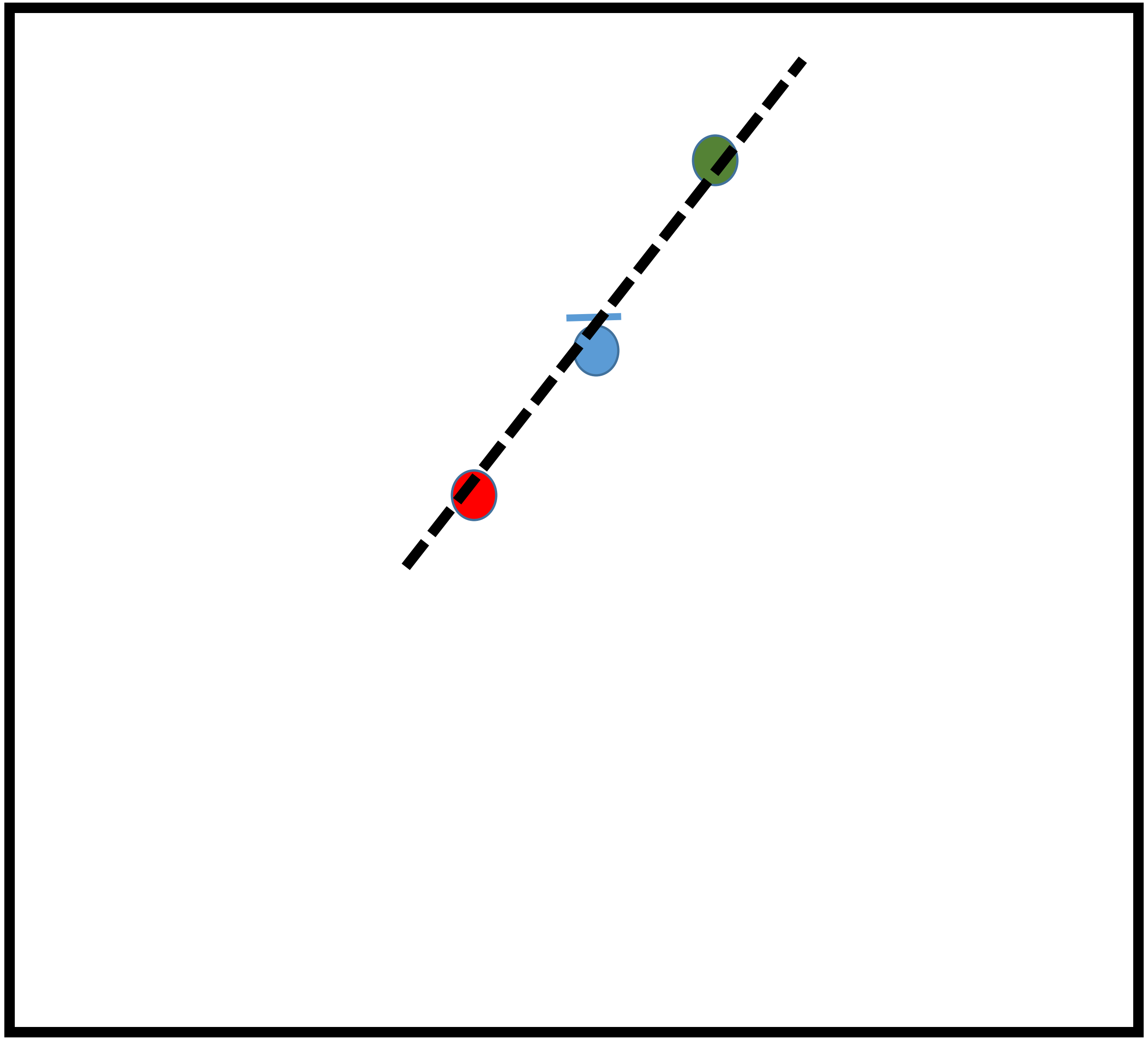
- Estimates variance components simultaneously to fixed terms
- Allow to account for nested structure in data with lots of levels
- Is robust against heterogeneous data
- Allow to account for pseudoreplication

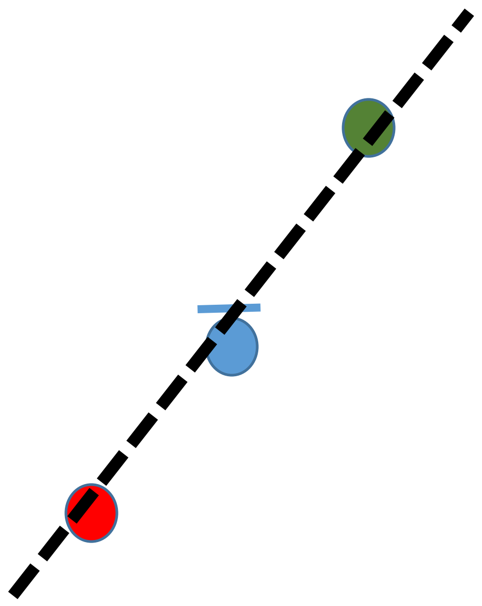




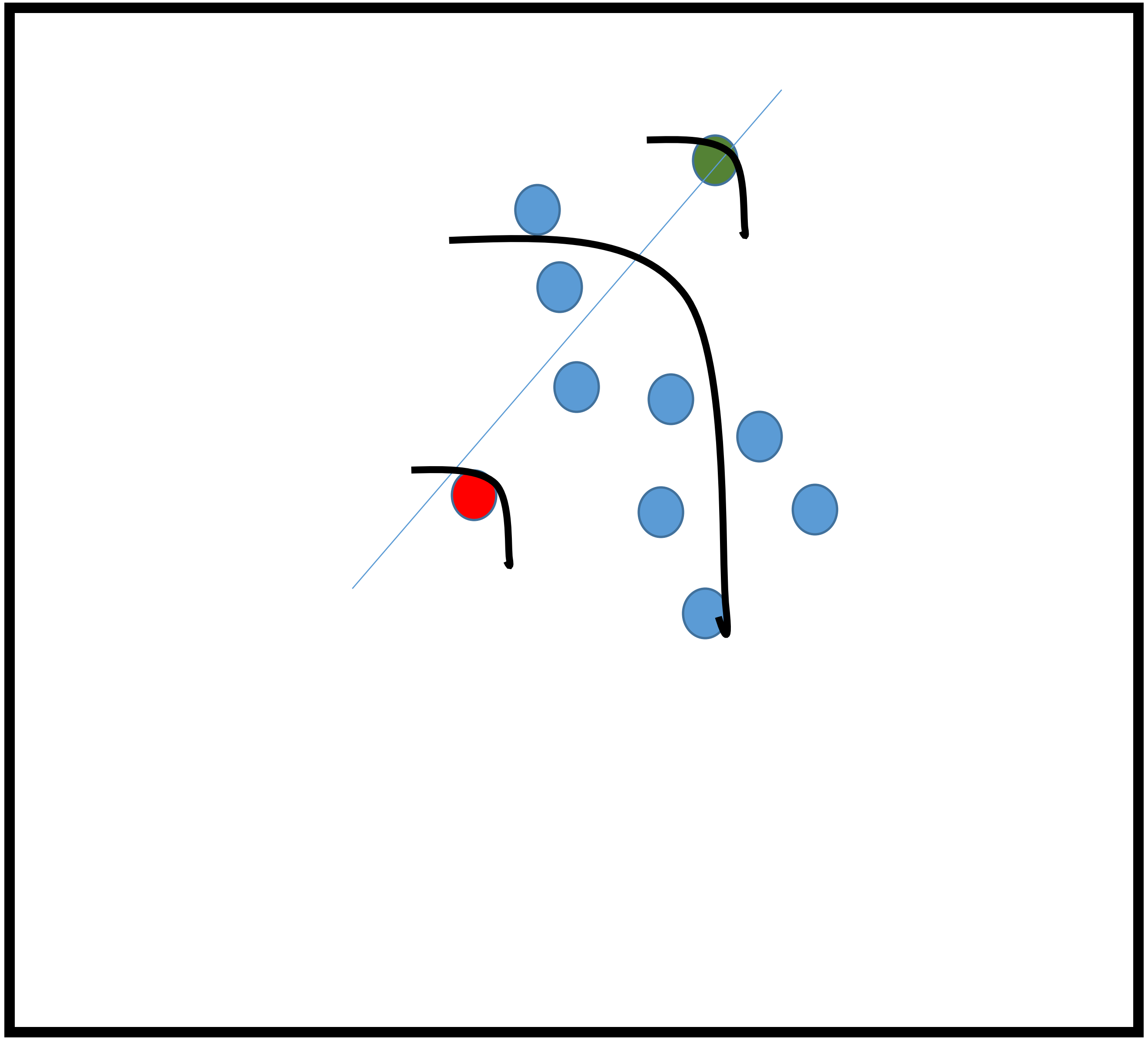








N = 3 !



Mixed models

- Estimates variance components simultaneously to fixed terms
- Allow to account for nested structure in data with lots of levels
- Is robust against heterogeneous data
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- Get information about variance components (e.g. to inform about repeatability)

When is something random factor and when fixed?

- Rules of thumb:
- Random effects are factors!
- Are you interested in means (fixed) or variance (random)?
- Do you want to correct for a factorial effect but it’s not in your questions specifically? -> random
- More than 5 levels: random
- LMM use only with large N (>50)

Mixed models – step by step

- Consider your hypothesis!
- Assume you want to know about fixed effects

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- 2) decide which fixed effects you want to have

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- 1) decide which random effects you want to have in your model
- 2) decide which fixed effects you want to have
- Run model with all fixed + random effects

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- 1) decide which random effects you want to have in your model
- 2) decide which fixed effects you want to have
- Run model with all fixed + random effects
- Use common sense to reduce model if needed
- Use likelihood ratio test or DIC

Likelihood ratio test

- Needs two models, with 1 parameter difference

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$$2(logLmodel\ - logLreduced)$$

Df for chi square = 1

LogL in R

```
> m0<-lm(y~1)
```

LogL in R

```
> m0<-lm(y~1)
> m1<-lm(y~x)
```

LogL in R

```
> m0<-lm(y~1)
> m1<-lm(y~x)
> lrtest(m0,m1)
```

LogL in R

```
> m0<-lm(y~1)
> m1<-lm(y~x)
> lrtest(m0,m1)
Likelihood ratio test

Model 1: y ~ 1
Model 2: y ~ x
  #Df  LogLik Df  Chisq Pr(>Chisq)
1   2 -17.488
2   3  -5.684  1 23.609  1.181e-06 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
>
```

Mixed models – step by step

- Consider your hypothesis!
- Assume you want to know whether or not to include random effects

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Mixed models – step by step

- Consider your hypothesis!
- Assume you want to know whether or not to include random effects
- Decide on fixed effects structure (best keep it to what’s known)
- Add all the random effects
- Compare with reduced model (always only drop one!), use LRT
- OR
- When using Bayesian (MCMCglmm) use BIC (equiv. to AIC)

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- Are hard

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- Not straightforward

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Mixed models - resources

- http://glmm.wikidot.com
- Gelman and Hill 2006 Data Analysis Using Regression and Multilevel/Hierarchical Models
- Zuur et al.: Mixed Effects Models and Extensions in Ecology 2009
- Bolker et al. Trends Ecol Evol 2009 Generalized linear mixed models: a practical guide for ecology and evolution

Mixed models