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Coordinación de Cálculo III y Cálculo Avanzado
para el Módulo Básico de Ingeniería

PEP 1 Cálculo Avanzado, Forma B
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Problema 1. Considere la función f definida por $f(x,y,z) = x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z}$, cuando $xyz \neq 0$. Obtenga y clasifique los puntos críticos de f contenidos en el conjunto $A = \{(x,y,z) : x > 0,\ y > 0\}$

Solución. $\frac{\partial f}{\partial x}(x,y,z) = 1 - \frac{1}{x^2} = 0$, $\frac{\partial f}{\partial y}(x,y,z) = 1 - \frac{1}{y^2} = 0$

$\frac{\partial f}{\partial z}(x,y,z) = 1 - \frac{1}{z^2} = 0$,

Luego, $x^2 = 1$, $y^2 = 1$, $z^2 = 1$ y por lo tanto $(1,1,\pm 1)$ son los puntos críticos de f en A

Para clasificarlos calculamos las derivadas de segundo orden

$f_{\mathrm{xx}} = \frac{2}{x^3}$, $f_{yx} = 0$, $f_{zx} = 0$

$f_{\mathrm{xy}} = 0$, $f_{yy} = \frac{2}{y^3}$, $f_{zy} = 0$

$f_{\mathrm{xz}} = 0$, $f_{yz} = 0$, $f_{zz} = \frac{2}{z^3}$

Luego, la matriz Hessiana esta dada por :

$$H(x,y,z) = \begin{pmatrix} \frac{2}{x^3} & 0 & 0 \\ 0 & \frac{2}{y^3} & 0 \\ 0 & 0 & \frac{2}{z^3} \end{pmatrix}$$

Evaluando en los puntos críticos , tenemos :

$H(1,1,1) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, se verifica que todos los determinantes de las submatrices tienen signo positivo. Por lo tanto $(1,1,1)$ es un mínimo local.

$H(1,1,-1) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$, se verifíca que $2 > 0$, Det$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 4 > 0$ y Det$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix} = -8 < 0$

Por lo tanto califica como punto silla.

□

Problema 2. Sean $u, v \in \mathbb{R}^3 \to \mathbb{R}$ dadas por $u(x,y,z) = \operatorname{sen}(xyz)$, $v(x,y,z) = \cos(xyz)$. Sea $w : \mathbb{R}^2 \to \mathbb{R}$ dado por $w(u,v) = (u+v)^2$. Defina $f(x,y,z) = w(u(x,y,z), v(x,y,z))$.

Determine los valores $\lambda_1,\ \lambda_2 \in \mathbb{R}$ tales que

$$\nabla f\left(\left(\frac{\pi}{2}\right)^{\frac{1}{3}},\left(\frac{\pi}{2}\right)^{\frac{1}{3}},\left(\frac{\pi}{2}\right)^{\frac{1}{3}}\right)=$$
$$=\lambda_1\nabla u\left(\left(\frac{\pi}{2}\right)^{\frac{1}{3}},\left(\frac{\pi}{2}\right)^{\frac{1}{3}},\left(\frac{\pi}{2}\right)^{\frac{1}{3}}\right)+\lambda_2\nabla v\left(\left(\frac{\pi}{2}\right)^{\frac{1}{3}},\left(\frac{\pi}{2}\right)^{\frac{1}{3}},\left(\frac{\pi}{2}\right)^{\frac{1}{3}}\right)$$

Solución. Tenemos $f(x,y,z)=w(u(x,y,z),v(x,y,z))$. Entonces

$$\begin{aligned}
&\frac{\partial f}{\partial x}(x,y,z)=\frac{\partial w}{\partial u}(u,v)\cdot\frac{\partial u}{\partial x}+\frac{\partial w}{\partial v}(u,v)\cdot\frac{\partial v}{\partial x},\\
&\frac{\partial f}{\partial y}(x,y,z)=\frac{\partial w}{\partial u}(u,v)\cdot\frac{\partial u}{\partial y}+\frac{\partial w}{\partial v}(u,v)\cdot\frac{\partial v}{\partial y},\\
&\frac{\partial f}{\partial z}(x,y,z)=\frac{\partial w}{\partial u}(u,v)\cdot\frac{\partial u}{\partial z}+\frac{\partial w}{\partial v}(u,v)\cdot\frac{\partial v}{\partial z},\\
\Rightarrow\qquad &\nabla f(x,y,z)=\frac{\partial w}{\partial u}(u,v)\cdot\nabla u(x,y,z)+\frac{\partial w}{\partial v}(u,v)\cdot\nabla v(x,y,z).
\end{aligned}$$

Así que, en general, los valores $\lambda_1=\frac{\partial w}{\partial u}(u,v),\lambda_2=\frac{\partial w}{\partial v}(u,v)$ satisfacen

$$\nabla f(x,y,z)=\lambda_1\nabla u(x,y,z)+\lambda_2\nabla v(x,y,z), \tag{1}$$

y son los únicos valores que satisfacen esta ecuación si y sólo si ∇u y ∇v son linealmente independientes.

En el caso de $(x,y,z)=\left(\left(\frac{\pi}{2}\right)^{\frac{1}{3}},\left(\frac{\pi}{2}\right)^{\frac{1}{3}},\left(\frac{\pi}{2}\right)^{\frac{1}{3}}\right)$ tenemos

$$\nabla u(x,y,z)=(yz\cos(xyz),xz\cos(xyz),xy\cos(xyz)),$$
$$\nabla u\left(\left(\tfrac{\pi}{2}\right)^{\frac{1}{3}},\left(\tfrac{\pi}{2}\right)^{\frac{1}{3}},\left(\tfrac{\pi}{2}\right)^{\frac{1}{3}}\right)=(0,0,0),$$

de donde λ_1 no es único, puesto que cualquier $\lambda_1\in\mathbb{R}$ satisface (1); y

$$\nabla v(x,y,z)=(-yz\operatorname{sen}(xyz),-xz\operatorname{sen}(xyz),-xy\operatorname{sen}(xyz)),$$
$$\nabla v\left(\left(\tfrac{\pi}{2}\right)^{\frac{1}{3}},\left(\tfrac{\pi}{2}\right)^{\frac{1}{3}},\left(\tfrac{\pi}{2}\right)^{\frac{1}{3}}\right)=\left(-\left(\tfrac{\pi}{2}\right)^{\frac{2}{3}},-\left(\tfrac{\pi}{2}\right)^{\frac{2}{3}},-\left(\tfrac{\pi}{2}\right)^{\frac{2}{3}}\right),$$

de donde

$$\begin{aligned}
\lambda_2&=\frac{\partial w}{\partial v}(u,v)=2(u+v)=2\left(\operatorname{sen}\left(\left(\tfrac{\pi}{2}\right)^{\frac{1}{3}}\left(\tfrac{\pi}{2}\right)^{\frac{1}{3}}\left(\tfrac{\pi}{2}\right)^{\frac{1}{3}}\right)+\cos\left(\left(\tfrac{\pi}{2}\right)^{\frac{1}{3}}\left(\tfrac{\pi}{2}\right)^{\frac{1}{3}}\left(\tfrac{\pi}{2}\right)^{\frac{1}{3}}\right)\right)\\
&=2\left(\operatorname{sen}\left(\tfrac{\pi}{2}\right)+\cos\left(\tfrac{\pi}{2}\right)\right)=2.
\end{aligned}$$

Por lo tanto, los valores λ_1,λ_2 pedidos, son $\lambda_1\in\mathbb{R}$, $\lambda_2=2$.

Otra solución: Tenemos

$$\begin{aligned} f(x,y,z) &= (\operatorname{sen}(xyz)+\cos(xyz))^2 = \\ &= \operatorname{sen}^2(x,y,z) + 2\operatorname{sen}(xyz)\cos(xyz) + \cos^2(xyz) = \\ &= 1 + 2\operatorname{sen}(xyz)\cos(xyz) \end{aligned}$$

Entonces

$$\begin{aligned} \frac{\partial f}{\partial x}(x,y,z) &= 2\cos^2(xyz)yz - 2\operatorname{sen}^2(xyz)yz \\ &= 2yz(\cos^2(xyz) - \operatorname{sen}^2(xyz)) = \\ &= 2yz\cos(2xyz). \\ \frac{\partial f}{\partial y}(x,y,z) &= 2\cos^2(xyz)xz - 2\operatorname{sen}^2(xyz)xz \\ &= 2xz(\cos^2(xyz) - \operatorname{sen}^2(xyz)) \\ &= 2xz\cos(2xyz) \\ \frac{\partial f}{\partial z}(x,y,z) &= 2\cos^2(xyz)xy - 2\operatorname{sen}^2(xyz)xy \\ &= 2xy(\cos^2(xyz) - \operatorname{sen}^2(xyz)) \\ &= 2xy\cos(2xyz) \end{aligned}$$

$$\begin{aligned} \Rightarrow \qquad \nabla f(x,y,z) &= (2yz\cos(2xyz), 2xz\cos(2xyz), 2xy\cos(2xyz)) \\ &= 2\cos(2xyz)(yz,xz,xy). \end{aligned}$$

Por otro lado,

$$\frac{\partial u}{\partial x}(x,y,z) = \cos(xyz)yz; \quad \frac{\partial u}{\partial y}(x,y,z) = \cos(xyz)xz, \quad \frac{\partial u}{\partial z}(x,y,z) = \cos(xyz)xy.$$
$$\frac{\partial v}{\partial x}(x,y,z) = -\operatorname{sen}(xyz)yz; \quad \frac{\partial u}{\partial y}(x,y,z) = -\operatorname{sen}(xyz)xz, \quad \frac{\partial u}{\partial z}(x,y,z) = -\operatorname{sen}(xyz)xy.$$

Tenemos:

$$\begin{aligned} \nabla f\left(\left(\tfrac{\pi}{2}\right)^{\frac13},\left(\tfrac{\pi}{2}\right)^{\frac13},\left(\tfrac{\pi}{2}\right)^{\frac13}\right) &= 2\cos\left(2\left(\tfrac{\pi}{2}\right)^{\frac13}\left(\tfrac{\pi}{2}\right)^{\frac13}\left(\tfrac{\pi}{2}\right)^{\frac13}\right)\cdot\left(\left(\tfrac{\pi}{2}\right)^{\frac23},\left(\tfrac{\pi}{2}\right)^{\frac23},\left(\tfrac{\pi}{2}\right)^{\frac23}\right) \\ &= 2\cos(\pi)\left(\left(\tfrac{\pi}{2}\right)^{\frac23},\left(\tfrac{\pi}{2}\right)^{\frac23},\left(\tfrac{\pi}{2}\right)^{\frac23}\right) \\ &= -2\left(\tfrac{\pi}{2}\right)^{\frac23}(1,1,1), \\ \text{y} \qquad & \lambda_1\nabla u\left(\left(\tfrac{\pi}{2}\right)^{\frac13},\left(\tfrac{\pi}{2}\right)^{\frac13},\left(\tfrac{\pi}{2}\right)^{\frac13}\right) + \lambda_2\nabla v\left(\left(\tfrac{\pi}{2}\right)^{\frac13},\left(\tfrac{\pi}{2}\right)^{\frac13},\left(\tfrac{\pi}{2}\right)^{\frac13}\right) \\ &= \lambda_1\cos\left(\tfrac{\pi}{2}\right)\left(\tfrac{\pi}{2}\right)^{\frac23}(1,1,1) + \lambda_2\operatorname{sen}\left(\tfrac{\pi}{2}\right)\left(\tfrac{\pi}{2}\right)^{\frac23}(-1,-1,-1) \\ &= -\lambda_2\left(\tfrac{\pi}{2}\right)^{\frac23}(1,1,1) \end{aligned}$$

Para tener $\nabla f\left(\left(\frac{\pi}{2}\right)^{\frac{1}{3}},\left(\frac{\pi}{2}\right)^{\frac{1}{3}},\left(\frac{\pi}{2}\right)^{\frac{1}{3}}\right)=\lambda_1\nabla u\left(\left(\frac{\pi}{2}\right)^{\frac{1}{3}},\left(\frac{\pi}{2}\right)^{\frac{1}{3}},\left(\frac{\pi}{2}\right)^{\frac{1}{3}}\right)+\lambda_2\nabla v\left(\left(\frac{\pi}{2}\right)^{\frac{1}{3}},\left(\frac{\pi}{2}\right)^{\frac{1}{3}},\left(\frac{\pi}{2}\right)^{\frac{1}{3}}\right)$, debemos tener:

$$-2\left(\frac{\pi}{2}\right)^{\frac{2}{3}}(1,1,1)=-\lambda_2\left(\frac{\pi}{2}\right)^{\frac{2}{3}}(1,1,1) \quad \Rightarrow \quad \lambda_2=2.$$

Así, para obtener la igualdad requerida en el punto $\left(\left(\frac{\pi}{2}\right)^{\frac{1}{3}},\left(\frac{\pi}{2}\right)^{\frac{1}{3}},\left(\frac{\pi}{2}\right)^{\frac{1}{3}}\right)$ sirve cualquier $\lambda_1\in\mathbb{R}$ y $\lambda_2=2$.

$\square$

Problema 3.

a) Desarrollar la serie de Fourier por senos para $f:[0,1/3]\to\mathbb{R}$ definida por $f(x)=\cos(3\pi x)$.
Solución. $a_0=a_n=0$ pues se usó la expansión impar. Para obtener el puntaje esto debió ser explicado .

$$b_n=\frac{2}{1/3}\int_0^{1/3}\cos(3\pi x)\operatorname{sen}(3n\pi x)dx \underset{u=3\pi x}{=} \frac{6}{3\pi}\int_0^{\pi}\cos(u)\operatorname{sen}(nu)du$$

$$=\frac{\cancel{2}}{\pi}\int_0^{\pi}\frac{1}{\cancel{2}}(\operatorname{sen}(nx+x)+\operatorname{sen}(nx-x))dx=\frac{1}{\pi}\int_0^{\pi}(\operatorname{sen}((n+1)x)+\operatorname{sen}((n-1)x))dx$$

$$=\begin{cases}-\frac{1}{\pi}\left(\frac{\cos((n+1)x)}{n+1}+\frac{\cos((n-1)x)}{n-1}\right)\Big|_0^{\pi}, & n\neq 1\\ -\frac{1}{\pi}\frac{\cos(2x)}{2}\Big|_0^{\pi}, & n=1\end{cases}$$

$$=\begin{cases}-\frac{1}{\pi}\left(\frac{(-1)^{n+1}-1}{n+1}+\frac{(-1)^{n-1}-1}{n-1}\right), & n\neq 1\\ -\frac{1}{\pi}\frac{(1-1)}{2}, & n=1\end{cases}$$

$$=\begin{cases}\frac{1-(-1)^{n+1}}{\pi}\left(\frac{1}{n+1}+\frac{1}{n-1}\right), & n\neq 1\\ 0, & n=1\end{cases} \quad (\text{notar que } (-1)^{n+1}=(-1)^{n-1})$$

$$=\begin{cases}\frac{1-(-1)^{n+1}}{\pi}\left(\frac{2n}{n^2-1}\right), & n\neq 1\\ 0, & n=1\end{cases}=\begin{cases}\frac{2}{\pi}\left(\frac{2n}{n^2-1}\right), & n \text{ par}\\ 0, & n \text{ impar diferente de } 1\\ 0, & n=1\end{cases}$$

$$=\begin{cases}\frac{1}{\pi}\left(\frac{4n}{n^2-1}\right), & n \text{ par}\\ 0, & n \text{ impar}\end{cases}$$

Así

$$\cos(3\pi x)\sim\sum_{n=1}^{\infty}b_n\operatorname{sen}(3\pi nx)=\sum_{\substack{n=1\\ n \text{ par}}}^{\infty}b_n\operatorname{sen}(3\pi nx)=\sum_{k=1}^{\infty}b_{2k}\operatorname{sen}(3\pi(2k)x)$$

$$= \sum_{k=1}^{\infty} \frac{1}{\pi} \left(\frac{8k}{4k^2 - 1} \right) \operatorname{sen}(6\pi k x)$$

$\square$

b) Determinar la convergencia de la serie

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{4n^2 - 1}$$

Solución. Por el teorema de convergencia, 1/12 es un punto de continuidad. Entonces:

$$\begin{aligned} 0 = f(1/12) &= \sum_{k=1}^{\infty} \frac{1}{\pi} \left(\frac{8n}{4n^2 - 1} \right) \operatorname{sen}(6\pi k(1/12)) \\ &= \sum_{n=1}^{\infty} \frac{1}{\pi} \left(\frac{8n}{4n^2 - 1} \right) \operatorname{sen}(n\pi/2) \\ &= \sum_{\substack{n=1 \\ n \text{ impar}}}^{\infty} \frac{1}{\pi} \left(\frac{8n}{4n^2 - 1} \right) (-1)^{(n-1)/2} \end{aligned}$$

Entonces:

$$\frac{\pi^2}{8} = \sum_{k=1}^{\infty} \frac{1}{(2k - 1)^2}$$

$\square$