Calculus I, Notes

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Calculus, Class Notes

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Contents

Contents i

Preface vii

1 Limits and The Main Elementary Functions 3

1.1 Basic Elementary Functions and Elementary Functions 3

1.2 Sequences and their limits 6

1.2.1 Limits in the geometry of curves 14

1.2.2 Exponential Function 16

1.2.3 Identifying $f(x)$ with e^x 17

1.3 Fundamental Limits of real valued function 18

1.3.1 Problems 25

1.3.2 Solutions 28

1.4 Continuity and piecewise functions 33

1.4.1 Problems 35

1.4.2 Solutions to 1.4.1 Problems 36

1.4.3 **Sample Test 1 and Solutions** 36

2 Derivatives and the rules of differentiation 41

2.1 Derivatives of the basic elementary functions 41

2.2 Derivatives under algebraic operations 44

2.2.1 Problems 47

2.3 Implicit Differentiation 49

2.3.1 Problems 53

2.4 Derivatives of higher order 53
2.4.1 Problems 55
2.5 Related rates problems 55
2.5.1 Problems 56
2.6 Newton's Approximation Scheme 56

3 Applications 61
3.1 Fermat's Theorem, Rolle's Theorem, Mean Value Theorem, Cauchy Theorem 61
3.1.1 Problems 64
3.2 L'Hospital's Rule 65
3.2.1 Problems 67
3.3 Optimization Problems 69
3.4 Sketching Graphs of Elementary Functions 70

4 Definite Integral 73
4.1 Antiderivative and some previous formulae 73
4.1.1 More Homework Problems 76
4.2 Definite Integral 77
4.2.1 Homework Problems 80
4.3 Integration using a substitution 82
4.4 Integration by parts 83
4.4.1 Sample Final Exam and solutions 84
4.5 The logarithmic function 87
4.6 The trigonometric functions 88

5 Parametric Equations 89
5.1 Some classical parametrizations 90

6 Curves in space, Curvature and TNB-frame 95

7 Inequalities 99

Bibliography 103

List of Figures

1.1 The history of $\sqrt{2}$ 8

1.2 Plot of $f(x) = \left(1 + \frac{1}{x}\right)^x$, $x = 10000..300000$ 19

1.3 Graph of f in Problem 2. 35

2.1 Unit circle and a tangent line 50

2.2 Cubic curve 50

2.3 Another Cubic curve 52

2.4 Problem 1 53

2.5 Ellipse $\frac{x^2}{400} + \frac{y^2}{225} = 1$ 55

3.1 Example for Rolle's Theorem 62

3.2 Graph of $y = \frac{\sin x - x + \frac{x^3}{6}}{x^5}$, $x \neq 0, x \in [-10, 10]$ 67

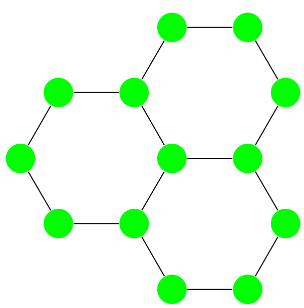
3.3 Cone inscribed in a sphere 70

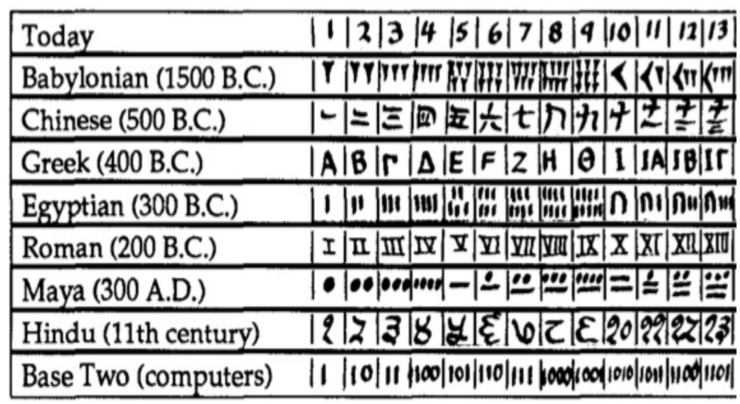
3.4 Graph of $y = \frac{x^3 - x}{x^2 + 1}$ 71

3.5 Graph of $y = g(x)$, $x \in [-5, 5]$ 72

List of Tables

Preface





"It is impossible to be a mathematician without being a poet in soul." Sofia Kovalevskaya

These lecture notes were written during a period of time beginning in 2007 and continuing, for my students enroled in the Calculus classes. There are very many good calculus books out there having lots and lots of information and beautiful problems. We will refer to some of them for various proofs and problems. Despite the fact that there is quite a number of topics in Calculus, yet, the main concepts are just a few: *limit*, *continuity*, *derivative* and the *definite integral*. In these notes we would like to take an approach that goes to the matter of things most of the time. For applications will take problems from various texts such as: [5] or [6]. The idea of using all transcendental functions from the start has nevertheless good pedagogical advantages. Some of the usual definitions one needs to have are:

$$\ln x := \int_1^x \frac{1}{t}dt, x > 0 \quad \textit{and} \quad \arcsin x := \int_0^x \frac{1}{\sqrt{1-t^2}}dt, x \in [-1, 1],$$

and the rest of the properties of all the elementary functions follow from these definitions once the concept of definite integral is introduced and the Fundamental Theorem of Calculus is established . We are going to fill out the details of this approach in that section (see Section 4.5 and Section 4.6). Also, the treatment of series and sequences is left after the integral calculus is developed but we will try to introduce some of those results as earlier as possible to help with the understanding of the concept of limit.

We begin with the concept of limits and introduce the so called fundamental limits. Exemplifying the concept of limit with nontrivial situations is not just a matter of taste but also a choice that we make to show the connection with the derivatives of the elementary functions. Continuity is briefly studied and some applications of the Intermediate Value Theorem are given. This is mostly a prelude for the work needed with the definition of the derivative and the study of all differentiation rules. We then continue with usual applications such as related rates problems, implicit differentiation, Newton's approximation technique and the Mean Value Theorem and its corollaries. Finally the concept of the Riemann integral and a few techniques of integration are given after the Fundamental Theorem of Calculus is discussed.

Chapter 1

Limits and The Main Elementary Functions

Quotation: *"To many, mathematics is a collection of theorems. For me, mathematics is a collection of examples; a theorem is a statement about a collection of examples and the purpose of proving theorems is to classify and explain the examples..." John B. Conway (Subnormal Operators, Pitman Advanced Publishing Program, 1981)*

1.1 Basic Elementary Functions and Elementary Functions

What kind of functions do we have as examples in calculus? Most of the textbooks are called *Calculus with early transcendentals*. Perhaps they should be called *Calculus with early non-algebraic functions*. In this section we are going to explain the rationale for such titles.

In general we can divide the class of functions into two sets: algebraic and non-algebraic. The basic algebraic functions are characterized by the following four properties

(a) the output for every real number in its domain can be obtained in a finite number algebraic operations (addition, subtraction, multiplication or division) from the input,

(b) the rule or the algorithm of obtaining the output is the same for every value of the input,

(c) the domain of such a function is the maximum posible within the real number system,

(d) if all the constants involved in the rule were rational numbers, then the output is rational for every rational input.

A big class of such functions are polynomial functions. These are functions f defined everywhere by a rule of the form

$$f(x) = a_0x^n + a_1x^{n-1} + ... + a_n$$

where a_0, a_1, ..., a_n are given and fixed real numbers and n is non-negative integer (which is called the degree of the polynomial f). In particular we have constant functions ($n = 0$), linear functions ($n = 1$), quadratic functions ($n = 2$), cubic functions ($n = 3$), quartic, quintic, etc.

Another big class satisfying these properties is the set of rational functions. A **rational** function g has a rule of the form

$$g(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials with no common factors and the domain is the set of all real numbers x such that $Q(x) \neq 0$. Of a particular interest are those rational functions for which Q has no real roots and as a result these are functions are defined for all real numbers. For instance, a function given by the rule $g(x) = \frac{x^3-x}{x^2+1}$, $x \in \mathbb{R}$.

One may consider piecewisely defined functions by using rules that are either polynomials or rational functions satisfying property (a), but such a function does not satisfy property (b) if there are at least two rules used. One such function that actually can be defined piecewisely is the absolute value:

$$|x| = \begin{cases} x \text{ for } x \geq 0 \\ \\ -x \text{ for } x < 0. \end{cases} \qquad (1.1)$$

It can be also defined with only one rule as $|x| = \sqrt{x^2}$, but the square-root is not an algebraic operation. So, although this is a pretty simple function we will classify it as non-algebraic.

What are other basic non-algebraic functions? Well, let us start with power functions, such as functions h defined by a rule of the form

$$h(x) = x^\alpha, \quad x \in D$$

where α is a real number that is not an integer, and D is the maximum possible domain, which in general contains the set of positive real numbers. One of the key problems in this rule is how do we calculate the output for let's say a simple input. Suppose that $\alpha = \frac{1}{2}$, so we are talking about the square root function $h(x) = x^{\frac{1}{2}}$, which has $D = [0, \infty)$. How do we calculate $h(2) = \sqrt{2}$? This number has a long history in mathematics going back to the Greek school and basically the discovery of irrational numbers. The Greeks had the false idea, for a long time, that all numbers were rational and they built every single theorem in geometry based on that. They were very puzeled when $\sqrt{2}$ came around as the diagonal of the unit square and they clearly proved to be not rational (see the next section for their proof by **infinite descent**). To compute the output we know that it requires an infinite sequence of algebraic operations. We have the following formula that can be used to obtain square roots:

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \frac{7}{256}x^5 - ... \quad |x| < 1, \tag{1.2}$$

but this requires an infinite number of terms and we will see the precise meaning of an identity like (1.2) later in the calculus sequence. So most of these non-algebraic functions are going to be defined in a formal precise way later on. That is the reason for calling the calculus approach using *early transcendental functions*, and we will use this approach too. It is true that most of the outputs are transcendental numbers (a number which is not a zero of a polynomial with integer coefficients as opposed to those numbers which are and they are called **algebraic numbers**) but for instance $\sqrt{2}$ is algebraic (since it is one of the zeros of the equation $x^2 - 2 = 0$). However, to compute exactly the decimals of $\sqrt{2}$ will require an infinite non-periodic number of digits (steps).

As a result, we will assume that the basic elementary functions we will be mentionig next are well defined and all of their properties are already established. These functions are the exponential and logarithmic functions, the trigonometric and inverse trigonometric functions.

Let us list a few properties of the power and exponential functions:

$$(ab)^\alpha = a^\alpha b^\alpha, \ (a^\alpha)^\beta = a^{\alpha\beta}, \ a^{\alpha+\beta} = a^\alpha a^\beta, \quad a, \ b > 0 \tag{1.3}$$

and α, β are real numbers. Let us remember that we can define the power function $(x \to x^\alpha)$ in terms of the exponential function as

$$x \to e^{\alpha \ln x}$$

and for that reason the maximum domain contains $(0, \infty)$. The correspoding properties for the logarithmic functions are

$$\log_a u = \frac{\log_{ab} u}{(1 - \log_{ab} b)}, \ \log a^\alpha = \alpha \log a, \ \log_a uv = \log_a u + \log_a v, \tag{1.4}$$

for all a, b, u and $v > 0$. The first property above is nothing else but the change of base formula, which is usually written (if we denote ab by c) as

$$\log_a u = \frac{\log_c u}{\log_c a}, \ \text{for all} \ \ a, \ c, \ u > 0.$$

The most important properties of the trigonometric functions are

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta,$$
$$(1.5)\qquad \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta,$$

for all $\alpha, \beta \in \mathbb{R}$. These formulae are usually called the *the addition trigonometric formulae*, from which all of the other trigonometric identities can be derived. For instance, one can easily obtain the addition formula for the tangent function:

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta},$$

true for all $\alpha, \beta \in \mathbb{R}$ such that $\alpha+\beta$ is not an odd multiple of $\frac{\pi}{2}$.

The elementary function that we are going to use are then the functions obtained from the basic ones using alll algebraic operations and in addition the composition operation. In top or that we will see examples where the piecewise elementary functions are glued together. Let us give just two examples:

$$j(x) = \left[\log_2(x^3+2x) + \sin(x)\right]^{\frac{2}{e^x}}, \quad \text{and} \quad k(x) = \frac{\arcsin(2^x + 3^{2x})}{\arctan(x) + \ln(2x+1)}.$$

Since the hyperbolic functions and their inverses are less known we are going to include them here with their domains:

- $\sinh(x) = \frac{e^x - e^{-x}}{2}, \quad x \in \mathbb{R},$
- $\cosh(x) = \frac{e^x + e^{-x}}{2}, \quad x \in \mathbb{R},$

To avoid circular reasoning we have to be careful and only use the results in the development of calculus that are not related to any of the non-algebraic functions when we are going to define them.

1.2 Sequences and their limits

The idea of a limit is closely related to the concept of infinity in mathematics, and that has a long history going back to the Greek school of mathematics. By the way, they didn't like to talk about infinity at all. There are mathematicians nowadays that only accept the discrete mathematics and stay away from the concepts that involve the continuum. The existence itself of numbers like $\sqrt{2}$, π or more general of irrationals is at the heart of this notion. Modern mathematics has various constructions which incorporate this idea and that is the usually referred to as the construction of the real number system ($\mathbb{R}$).

Lets look at the number $\sqrt{2}$ which is known to be irrational and if we write it in base 10, we can list a few decimals down

$$\sqrt{2} = 1.4142135623730950488016887242096980785696718753769480731766797379907324784621070388503875343276415 73\cdots$$

but we will never be able to get the exact number this way since the decimals follow some pattern that it is not periodic or easy to describe. The only exact definition is, that whatever number is, it is positive, and its square is equal to 2. In other words, it is the solution of

$$x > 0,\ x^2 = 2.$$

Then, the question is "how do we work with this number then ?" Even the existence of such a number was questioned from the day of its inception, so to speak when the Pythagorean school calculated the diagonal of a square of sides, say a, they realized that the diagonal (Figure1.1) BD can be expressed as

$$BD = \sqrt{BC^2 + DC^2} = \sqrt{2BC^2} = BC\sqrt{2}.$$

But the big surprise was when they discovered that $\sqrt{2}$ was not rational (the language they used was *commensurable segments*, referring to DB and DC). The reason of their bewilderment was the belief that every number is rational, or in their words every two segments are commensurable, i.e., they can be measured by some unit of measure in an exact number of steps. This assumption leads one to conclude that for some, possibly very small, unit of measurement $DB = a$ units and $DC = b$ units. Then the segment $BE = (a - b)$ and $BF = b - (a - b) = 2b - a$. The triangle BEF is also a right isosceles triangle (similar to ABD) and so the diagonal BF and the side BE are smaller and still commensurable with the same unit. One can repeat this construction over and over again until the two segments become so small that they are smaller than the unit. Therefore, they cannot be measured exactly with that unit. This contradiction shows that the side of the square and its diagonal cannot be commensurable. This was the first proof of the irrationality of $\sqrt{2}$ which nowadays it is called a proof by the method of (infinite) descent.

But, from a calculus point of view this method tells us more. Since the diagonal $BF = BE\sqrt{2}$, we conclude that

$$\sqrt{2} = \frac{DB}{DC} = \frac{a}{b} = \frac{BF}{BE} = \frac{2b - a}{a - b}.$$

Let us reverse this instead of descent, we want to do an ascent: let us set $2b - a = m$ and $a - b = n$ and solve for a and b. We obtain $b = m + n$ and $a = n + b = n + m + n = m + 2n$. Hence, we obtain

$$\sqrt{2} = \frac{m}{n} = \frac{m + 2n}{m + n}.$$

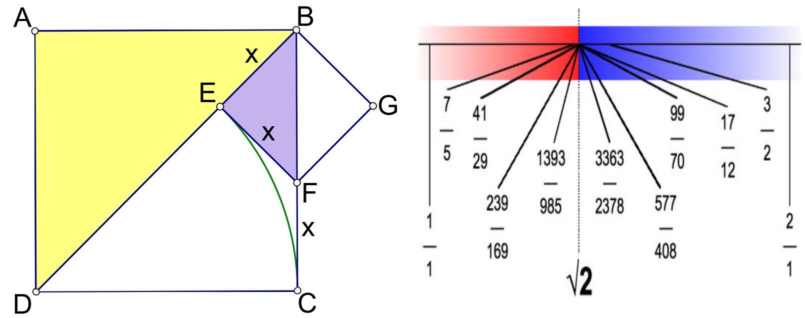


Figure 1.1: The history of $\sqrt{2}$

Let us see what happens if we apply this ascent and start with a fraction $\frac{m}{n}$ which is just an approximation of $\sqrt{2}$, say $\frac{3}{2}$. Then the next fraction is $x_2 = \frac{3+2\cdot 2}{3+2} = \frac{7}{5} \approx 1.4$. Then, we get $x_3 = \frac{7+2\cdot 5}{7+5} = \frac{17}{12} \approx 1.41\overline{6}$ which is a better approximation of $\sqrt{2}$. In fact, let us look at $x_3^2 = \frac{289}{144} = 2 + \frac{1}{144}$ and something similar can be said about $x_2^2 = \frac{49}{25} = 2 - \frac{1}{25}$. One can calculate the next iteration and obtain $x_4 = \frac{41}{29} \approx 1.4137931$ with $x_4^2 = 2 - \frac{1}{841}$ which is very close to 2. The sequence of iterations continues

$$\frac{99}{70}, \frac{239}{169}, \frac{577}{408}, \frac{1393}{985}, \ldots$$

a classical sequence in The On-Line Encyclopedia of Integer Sequences which is cataloged by A155046. We say that $\{x_n\}$ is a sequence which is **convergent** to $\sqrt{2}$ and $\{x_n^2\}$ is **convergent** to 2. We notice that $\{x_n\}$ is a sequence of fractions and they approximate $\sqrt{2}$ as good as we want but simply increasing the index n.

So, to answer our question, instead of working with $\sqrt{2}$, in practice, we simply work with an approximation of it and it is convenient to select certain approximations. The approximations above are optimal in the sense that the denominators are the smallest in order to achieve a certain desired error. We have a very specific definition of the convergence of a sequence and it may look complicated, and if it does no wonder because it took quite a long time in the development of mathematics to arrive at it (due to Augustin-Louis Cauchy)

Definition 1.2.1. *We say that the number L is the limit of the sequence $\{a_n\}$ if for every $\epsilon > 0$ there exists an index n (which depends on ϵ) such that $|a_m - L| < \epsilon$ for all $m \geq n$. A short way to express that $\{a_n\}$ has limit L (or a_n converges to L) is $\lim\limits_{n\to\infty} a_n = L$.*

We are not going to use this definition that much but it is arguably one of the most important concepts in calculus and it is usually the basis of checking all the properties that limits of sequences have. Let us list the main properties and then use the definition to prove one which is less standard (see Theorem 1.2.6):

$$(1.6)\qquad \begin{array}{c} \boxed{1.}\lim_{n\to\infty} c = c \qquad \boxed{2.}\lim_{n\to\infty} x_n \pm y_n = \lim_{n\to\infty} x_n \pm \lim_{n\to\infty} y_n \qquad \boxed{3.}\lim_{n\to\infty} cx_n = c\lim_{n\to\infty} x_n \\ \boxed{4.}\lim_{n\to\infty} x_n y_n = (\lim_{n\to\infty} x_n)(\lim_{n\to\infty} y_n), \qquad \boxed{5.}\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n}, \end{array}$$

provided that $\lim_{n\to\infty} x_n$ and $\lim_{n\to\infty} y_n$ exist and for the Property 5, $\{y_n\}$ is a a sequence of non-zero real values and $\lim_{n\to\infty} y_n \neq 0$. These properties are indeed properties as long the arithmetic operations make sense. So as long as $\{x_n\}$ and $\{y_n\}$ are rational numbers and their limits too, these are real properties. But if we take the limits to be real numbers which are not rational, these have to turn into definitions. So, for instance $\sqrt{2}+\sqrt{3}$ is the limit of the sum of two sequences of rational numbers convergent to $\sqrt{2}$, and $\sqrt{3}$ respectively. We are not going to go into these kind of details since this is part of the construction of real numbers which can be done in several ways. We will just refer the interested reader to the text of Walter Rudin (see [7]) for an account of the so called the Dedekind cuts construction.

On simple corollary of the definition of convergence is the so called Squeeze Theorem (we will see this again for functions).

Corollary 1.2.2. *Given three sequences $\{x_n\}$, $\{y_n\}$,$\{z_n\}$, such that*

$$x_n \le y_n \le z_n, \quad \textit{for all} \ \ n \in \mathbb{N}, \textit{and}$$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = L,$$

then $\{y_n\}$ is also convergent to L.

The same concept of convergence works for a sequence of complex numbers. The set of complex numbers

$$\mathbb{C} = \{a+bi | a, b \in \mathbb{R}\}$$

can be organized with addition $((a+bi)+(c+di) = (a+c)+(b+d)i)$ and multiplication $(a+bi)(c+di) = (ac-bd)+(ad+bc)i)$ that are typically taught in college algebra together with some of their properties which are very similar to the properties of addition and multiplication of real numbers. The only difference is that the absolute value here is defined to be $|a+bi| = \sqrt{a^2+b^2}$ which is nothing else but the Euclidean distance from the origin to the point (a, b).

For a fixed complex number $z = x+iy$ two very important sequences in calculus are defined by

$$(1.7)\qquad \boxed{z_n = \left(1+\frac{z}{n}\right)^n, \ \text{ and } \ w_n = 1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots\frac{z^n}{n!}.}$$

It turns out that these two sequences have the same limit which is denoted by e^z. We have this way a function which has the property

$$e^{z+w} = e^z \cdot e^w, \quad z, w \in \mathbb{C}.$$

The fact that the limits in (1.10) exist and the above property takes place is usually proved in an upper level course in analysis. This allows to introduce the transcendental functions in a different way: $e^x = |e^z|$ and $\cos y + \sin y i = e^{iy}$, $x, y \in \mathbb{R}$.

In order to be able to identify sequences which are convergent, especially when we do not know what their limit migth be, there are few ingredients that one can use (they are based on the axiomatics of the real numbers).

Theorem 1.2.3. *Every monotone and bounded sequence is convergent.*

A **monotone sequence** $\{x_n\}$ is a sequence which satisfies $x_{n+1} \le x_n$ for all $n \in \mathbb{N}$ (monotone non-increasing), or $x_n \le x_{n+1}$ for all $n \in \mathbb{N}$ (monotone non-decreasing). A **bounded sequence** $\{x_n\}$ is a sequence with the property that for some bound M, $|x_n| \le M$ for all $n \in \mathbb{N}$. When the inequalities are strict we say the sequence is **strictly increasing** or **strictly decreasing** (and sometime simply increasing or decreasing). For an unbounded sequence, a particular situation appears when the sequence is said to **converge to infinity**. This actually means that no matter how big M is, one can find an index n so that $x_m > M$ for all $m \ge n$.

Let us take an example here which is classic and goes back to L. Euler (Leonard Euler). Suppose that our sequence $\{x_n\}$ is defined by

$$x_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}.$$

A few terms of this sequence are listed next

$$\{1, \frac{5}{4}, \frac{49}{36}, \frac{205}{144}, \frac{5269}{3600}, \frac{5369}{3600}, \frac{266681}{176400}, \frac{1077749}{705600}, \frac{9778141}{6350400}, \frac{1968329}{1270080}, \cdots\}$$

This sequence is clearly strictly increasing since $x_{n+1} = x_n + \frac{1}{(n+1)^2} > x_n$ for all n. To see that it is bounded we will use the following trick (which is called **telescopic sums**):

$$x_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 1 + \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{(n-1)n} =$$

$$1 + (\frac{1}{1} - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \cdots + (\frac{1}{n-1} - \frac{1}{n}) = 2 - \frac{1}{n} < 2.$$

Hence, $x_1 < x_2 < \cdots < x_n < 2$, and so by Theorem 1.2.3 this sequence must be convergent to something. It was not known what the limit was for a long time and L. Euler arrived by some argument which was really clever, that the limit must be $\frac{\pi^2}{6}$. If we use a computer we can see that

$$x_{100} \approx 1.634983900..., \quad \text{and} \quad \frac{\pi^2}{6} \approx 1.644934...,$$

so the convergence is quite slow (i.e., it takes a lot of terms to get close to the limit, whithin a decimal point). The proof of this fact is included in the section about series since it requires a little more knowledge in calculus. At this point it is not known where does the following sequence converge

$$1+\frac{1}{2^3}+\frac{1}{3^3}+\cdots+\frac{1}{n^3}\to\zeta(3)=?,$$

but its limit is denoted by $\zeta(3)$.

Exercise 1: Show that the sequence defined recursively by $z_1 = 1$, and $z_{n+1} = \sqrt{2+z_n}$, for $n \geq 1$ is convergent to 2.

Another classical sequence is the one which is related to adding numbers in a geometric progression. We consider r a real number such that $r \in [0,1)$, and define the sequence $s_n = 1+r+r^2+\cdots+r^n$, $n \in \mathbb{R}$. We prove that this sequence converges to $\frac{1}{1-r}$. This is true since $s_n = \frac{1-r^{n+1}}{1-r}$ and using Bernoulli's Inequality (7.1) one can show that r^n is convergent to zero (left as an exercise to the reader). We usually write this as

$$1+r+r^2+\cdots+r^n+\cdots=\frac{1}{1-r}, \tag{1.8}$$

which is in fact true for all real $r \in (-1,1)$.

What if the sequence is not monotone? Cauchy had the following simple answer to this question.

Theorem 1.2.4. *Every Cauchy sequence is convergent. Aslo, every convergent sequence is Cauchy.*

Definition 1.2.5. *A sequence $\{x_n\}$ is Cauchy if for every fixed $\epsilon > 0$, we have $|x_m - x_n| \leq \epsilon$ for all $m, n \geq k$ with $k \in \mathbb{N}$ and index which depends on ϵ.*

We can see that a lot of these definitions and techniques that deal with convergence involve inequalities. Perhaps it is not far from true that calculus, in its basic proofs, boils down to inequalities. I would say that the "magic" of mathematics is like the magician's trick: it consists of three acts. The **pledge**, **the turn**, and **the prestige**. Using this analogy, the calculus results are the prestige of turning inequalities into equalities. A lot of inequalities that we are going to use are proved in Chapter 7.

A simple corollary of Theorem 1.2.4 (or simply just the definition) is that if a sequence $\{x_n\}$ has the property that $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are convergent to ℓ then the $\{x_n\}$ is convergent to ℓ.

Using the definition of convergence and Theorem 1.2.4, to exemplify, let us prove a property which can be generalized easily.

Theorem 1.2.6. *Suppose that $\{a_n\}$ is a sequence of non-negative rational numbers which has the following property that a_n^2 is convergent to L. Then, the sequence $\{a_n\}$ is convergent to some real number a and $a^2 = L$.*

Proof: We need to treat the case $L = 0$ separately. In this case $a = 0$, indeed $a_n < \epsilon$ is equivalent to $a_n^2 < \epsilon^2$ and so since we know that for some n_{ϵ^2}, the last inequality is true for $n > n_{\epsilon^2}$, it shows that $a_n \to 0$. Hence, we may assume that $L > 0$.

Since $L > 0$ there exists $q \in \mathbb{N}$ big enough so that $\frac{1}{q^2} < L$. Using the definition of limit we know that $a_n^2 > \frac{1}{q^2}$ for $n > n_1$. Since First let us show that $\{a_n\}$ is Cauchy. We have

$$|a_n - a_m| = \frac{|a_n^2 - a_m^2|}{a_n + a_m} < \frac{|a_n^2 - a_m^2|}{\frac{2}{q}}, \ m, n > n_1.$$

By triangle inequality $|a_n^2 - a_m^2| \le |a_n^2 - L| + |L - a_m^2|$, and so from the definition of the convergence $a_k^2 \to L$ we can find an n_2 such that $|a_n^2 - L| \le \epsilon/2$ for all $n > n_2$. Putting together these facts gives

$$|a_n - a_m| < \frac{q\epsilon}{4}$$

for all $m, n > max(n_1, n_2)$. This shows that $\{a_n\}$ is Cauchy and hence it is convergent to a limit say a. By Property (4) in 1.6 (or by definition) we get $a^2 = L$. ∎

It is not hard to see that for every non-negative real number L there exists a sequence a_n of rationals such that $a_n^2 \to L$ (using the idea of averaging or the mediant inequality 7.0.1). Also, the number a does not depend of the sequence of rationals $\{a_n\}$ we choose. Then we can apply this theorem and define the "square root" function by $f(L) = \lim_n a_n$ which has the property $f(L)^2 = L$. The notation for this function is $\sqrt{L}$, i.e., $f(L) = \sqrt{L}$. It is then true that $\lim_n \sqrt{x_n} = \sqrt{\lim_n x_n}$ with essentially the same proof as in the Theorem 1.2.6. In a similar way we can construct any root function $g(x) = x^{\frac{1}{m}}$ where $m \in \mathbb{N}$. We notice that in the case m odd we can define these function on the whole real line. So, we have a new property that we can add to those in (1.6):

$$\lim_{n\to\infty} (x_n)^{\frac{1}{m}} = (\lim_{n\to\infty} x_n)^{\frac{1}{m}}.$$

Exercise 2: Show that the sequence defined earlier by $x_1 = \frac{3}{2}$ and $x_{n+1} = \frac{x_n+2}{x_n+1}$ for $n \ge 1$, is convergent to $\sqrt{2}$.

This sequence is known in the literature as the sequence of **continued fractions** of $\sqrt{2}$. One may ask in connection with this continued fraction sequence "how fast" is it convergent to $\sqrt{2}$? It turns out this is a little difficult to answer but in some other situations we can use the following important tool called the Stolz–Cesàro theorem.

Theorem 1.2.7. *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers. Assume now that $a_n \to 0$ and $b_n \to 0$ while $\{b_n\}$ is strictly decreasing. If*

$$\lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \ell \quad \textit{then} \quad \lim_{n\to\infty} \frac{a_n}{b_n} = \ell.$$

There is another version of this:

Theorem 1.2.8. *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers. Assume now that $b_n \to \infty$ while $\{b_n\}$ is strictly increasing. If*

$$\lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \ell \quad \textit{then} \quad \lim_{n\to\infty} \frac{a_n}{b_n} = \ell.$$

Let us work an example here to see how this theorem can be used. Suppose that we want to compute the following limit

$$\lim_{n\to\infty} \frac{1^k + 2^k + \ldots + n^k}{n^{k+1}}$$

where $k \in \mathbb{N}$. We can apply Theorem 1.2.8, and obtain

$$\lim_{n\to\infty} \frac{1^k + 2^k + \ldots + n^k}{n^{k+1}} = \lim_{n\to\infty} \frac{(n+1)^k}{(n+1)^{k+1} - n^{k+1}} = \frac{1}{k+1},$$

using the binomial formula $(n+1)^{k+1} = n^{k+1} + (k+1)n^k + \cdots$ and the simple fact that if two polynomials P and Q have the same degree then $\frac{P(n)}{Q(n)} \to \frac{p_0}{q_0}$ where p_0 is the leading coefficient of P and q_0 is the leading coefficient of Q.

Exercise 3: Use Stolz–Cesàro to prove that

$$\lim_{n\to\infty} \frac{1^1 + 2^2 + \ldots + n^n}{n^n} = 1.$$

Exercise 4: Use Bernoulli's Inequality to prove that if $r \in [0,1)$ then $\lim_{n\to\infty} nr^n = 0$ and then prove that

$$1 + 2r + 3r^2 + \cdots + nr^{n-1} + \ldots = \frac{1}{(1-r)^2}. \tag{1.9}$$

Exercise 5: Use Bernoulli's Inequality and Squeeze Theorem to prove that if $a > 0$ then $\lim_{n\to\infty} a^{1/n} = 1$

Exercise 6: Consider the recurrent sequence $\{x_n\}$ defined by $x_1 \in (0,1)$ and $x_{n+1} = x_n(1-x_n)$ for $n \geq 2$. Show that $x_n \to 0$ and $nx_n \to 1$.

Exercise 7: Consider the recurrent sequence $\{x_n\}$ defined by $x_1 \in (0,1)$, $\lambda \in (0,1)$ and $x_{n+1} = x_n(1-\lambda x_n)$ for $n \geq 2$. Show that $x_n \to 0$ and find the limit of $\{nx_n\}$.

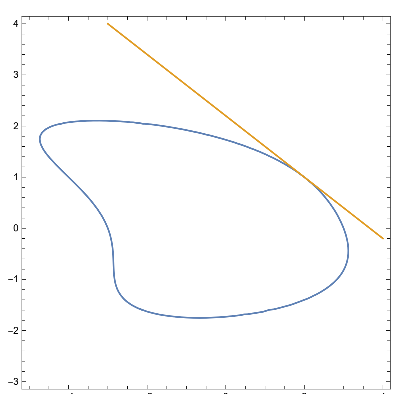
1.2.1 Limits in the geometry of curves

Historically speaking the idea of limit arived also in the geometry, first with Archimedes who wanted to compute various areas and volumes of regions or solids that where bounded by more complicated surfaces than planar regions. For instance, the volume of a pyramid is such an example. One can say that Archimedes was the discoverer of the integral calculus. Unfortunately most of his work was not known to the wide world until recently The method of Archimedes.

Another problem in mathematics which led to the concept of limit was the construction of a tangent line to a curve which is not a circle. For a circle we know that the tangent line at a point on a circle, to this circle, is the line perpendicular to the radius coresponding to that point. The question at this point is "how do we define the tangent line to a curve in general and how do we construct it?" Let us look into a simple curve which is not that much different of the circle but it was studied a lot by the ancient Greeks: the ellipse. It is well known that the equation of a general ellipse having the origin as its center and the semi-axes the axes of coordinates is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

This is an algebraic curve, i.e., a curve given implicitly by an equation of the form $P(x,y)=0$ where P is a polynomial in two variables. For example the graph of the algebraic curve $xy+x^2+xy^2+y^4=1$ is shown below



together with the tangent line at $(2,1)$. For an algebraic curve we can take the definition of the tangent line be the line $y=b+m(x-a)$ where (a,b) is a point on the curve $P(x,y)=0$ where m is determined with the property the equation $P(x,b+m(x-a))=0$ has $x=a$ a root of multiplicity at least 2. we know that $x-a$ has is a root because $P(a,b)=0$. Hence, we have the factorization $P(x,b+m(x-a))=(x-a)Q(x)$ and so m is determined by the condition $Q(a)=0$. One can work it out and figure out that the equation of the tangent line in the above figure is $3x+5y=11$. Let us prove the following theorem.

Theorem 1.2.9. *The equation of the tangent line to the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *at the point on the ellipse* (u,v) *is given by*

$$\frac{ux}{a^2}+\frac{vy}{b^2}=1.$$

Indeed, let us check that the equation $\frac{x^2}{a^2}+\frac{[v+m(x-u)]^2}{b^2}=1$ has a double root at $x=u$ if and only if $m=-\frac{ub^2}{a^2v}$ $(v\neq 0)$. The equation is equivalent to

$$\frac{x^2}{a^2}+\frac{v^2}{b^2}+\frac{2vm(x-u)}{b^2}+\frac{m^2(x-u)^2}{b^2}=1.$$

But since $\frac{u^2}{a^2}+\frac{v^2}{b^2}=1$ we see that the equation above turns into

$$\frac{(x-u)(x+u)}{a^2}+\frac{2vm(x-u)}{b^2}+\frac{m^2(x-u)^2}{b^2}=0,$$

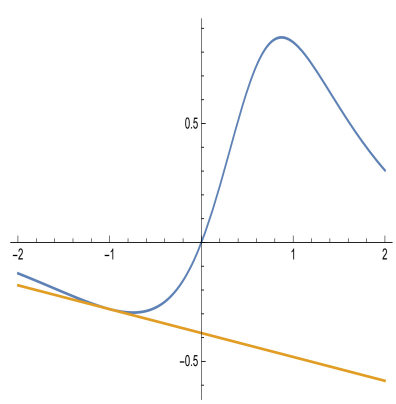
and after we factor out $x-u$, what is left is then

$$\frac{x+u}{a^2}+\frac{2vm}{b^2}+\frac{m^2(x-u)}{b^2}=0.$$

Setting $x=u$ gives the equation $\frac{2u}{a^2}+\frac{2vm}{b^2}=0$ which solved for m gives exactly $m=-\frac{ub^2}{a^2v}$ $(v\neq 0)$. If $v=0$, the statement is easily seen to be true. ■

Excercise: Show that for a parabola $y=mx^2$ the equation of the tangent line at (a,ma^2) is given by $y=ma(2x-a)$.

The definition of the tangent line in general cannot be using the concept of multiplicity, for instance in the case of transcendental functions, like in the figure below, where the function is given by $f(x)=\frac{\sin x}{x^2-x+1}$ and the tangent line is at $(-1,f(-1))$:



The working definition that we will use a lot and it is fundamental in differential calculus is the following:

Definition 1.2.10. *Given the graph of $y = f(x)$ (with f and elementary function) and $(a, f(a))$ a point in the interior of its domain, the tangent line at $(a, f(a))$ has equation $y = f(a)+m(x-a)$ where the slope m is the limit of the sequence $(f(x_n)-f(a))/(x_n-a)$ where x_n is a sequence in the domain (except a) convergent to a.*

In what follows we will see why the limit in the above definition exists and it is independent of the sequence $\{x_n\}$ for all elementary functions, and how do we calculate it using special rules which we call *differentiation rules*, hence the first part of calculus sometimes named *differential calculus.*

Of course, the concept of limit also appears naturally in the movement of an object through space as the idea of instantaneous velocity.

1.2.2 Exponential Function

Given a positive real number x, let us consider the sequence

$$f_n(x) := (1 + \frac{x}{n})^n, \ n \in \mathbb{N}. \tag{1.10}$$

Using the AM-GM inequality we have

$$\left[1 \cdot (1 + \frac{x}{n})^n)\right]^{\frac{1}{n+1}} < \frac{1 + n(1 + \frac{x}{n})}{n+1} = 1 + \frac{x}{n+1} \implies$$

$$(1 + \frac{x}{n})^n < (1 + \frac{x}{n+1})^{n+1} \implies f_n(x) < f_{n+1}(x).$$

Hence, the sequence $\{f_n(x)\}$ is strictly increasing. We will show next that this sequence is also bounded. Let m be the ceiling of x, or the smallest integer m such that $x \le m$.

If $x < 0$, what we have above is still valid but we need to makes sure $(1 + \frac{x}{n}) > 0$. Since $\frac{x}{n} \to 0$ we can accomplish this by taking n big enough. However, we need this information for $x = -1$, so for $n \ge 2$ we have $f_n(-1) < f_{n+1}(-1)$. This is equivalent to $(\frac{n-1}{n})^n < (\frac{n}{n+1})^{n+1}$. Taking reciprocals, we obtain $(1 + \frac{1}{n-1})^n > (1 + \frac{1}{n})^{n+1}$ for all $n \ge 2$. This is saying that the sequence $E_n = (1 + \frac{1}{n})^{n+1}$ is strictly decreasing and it is bounded below by zero. Therefore, it is convergent to a number which we will denote by e (in honor of L. Euler). Since, $f_n(1) = E_n/(1 + \frac{1}{n})$ this is also convergent to e. So, we get for $x \ge 0$

$$f_n(x) \le f_n(m) = (1+\frac{m}{n})^n < \lim_{n\to\infty} f_n(m) = \lim_{k\to\infty} f_{km}(m) = \lim_{k\to\infty}(1+\frac{m}{km})^{km} = \lim_{k\to\infty} f_k(1)^m = e^m.$$

For $x < 0$, $f_n(x) < f_n(0) = 1$, which means either way the sequence is bounded. As a result, the sequence $\{f_n(x)\}$ is convergent and we will set its limits as $f(x)$:

$$f(x) := \lim_{n\to\infty} (1 + \frac{x}{n})^n, \quad x \in \mathbb{R}. \tag{1.11}$$

Now, we will show some properties which will allow us to conclude that f is the natural exponential function. The most important property is

$$f(x+y) = f(x)f(y), \quad x, y \in \mathbb{R}. \tag{1.12}$$

By properties of limits

$$f(x)f(y) = \lim_{n\to\infty} (1 + \frac{x}{n})^n (1 + \frac{y}{n})^n = \lim_{n\to\infty} \left[(1 + \frac{x}{n})(1 + \frac{y}{n})\right]^n = \lim_{n\to\infty} \left(1 + \frac{x+y}{n} + \frac{xy}{n^2}\right)^n.$$

On the other hand, by definition $f(x+y) = \lim_{n\to\infty}(1 + \frac{x+y}{n})^n$ so (1.12) follows if we prove that

$$\lim_{n\to\infty} \frac{\left(1 + \frac{x+y}{n} + \frac{xy}{n^2}\right)^n}{(1 + \frac{x+y}{n})^n} = 1 \Leftrightarrow$$

$$\lim_{n\to\infty} \left(1 + \frac{xy}{(1 + \frac{x+y}{n})n^2}\right)^n = 1.$$

We have $\lim_{n\to\infty} \frac{xy}{(1+\frac{x+y}{n})} = xy$ so if we let $|xy| + 1 = C$ we have

$$(1 - C/n^2)^n \le \left(1 + \frac{xy}{(1 + \frac{x+y}{n})n^2}\right)^n \le (1 + C/n^2)^n \Leftrightarrow$$

$$f_{n^2}(-C)^{1/n} \le \left(1 + \frac{xy}{(1 + \frac{x+y}{n})n^2}\right)^n \le f_{n^2}(C)^{1/n} < f(C)^{1/n},$$

for n big enough, say $n \ge n_0$. But we know that $\lim_{n\to\infty} f(C)^{1/n} = 1$, and for the left inequality $f_{n^2}(-C)^{1/n} > f_{n_0^2}(-C)^{1/n}$ if $n \ge n_0$. Using squeeze theorem the desired limit follows.

1.2.3 Identifying $f(x)$ with e^x

We observe that (1.12) implies $f(nx) = f(x)^n$ for every x and every $n\mathbb{N}$. If we let $x = \frac{1}{n}$ we obtain $f(1) = e = f(1/n)^n$. From here $f(1/n) = e^{1/n}$. Now, letting $x = \frac{1}{m}$, we get

$$f(n/m) = f(1/m)^n = (e^{1/m})^n = e^{n/m}.$$

So, $f(r) = e^r$ for very rational r. This means $f(x)$ extends the natural exponential function from rationals to all real numbers.

There are other properties that determine this function uniquely. Let us show that f is strictly increasing. If $x < y$ then $f(y) = f(y-x+x) = f(y-x)f(x) > f(x)$ provided $f(y-x) > 1$. So, we reduced the property to the case $t > 0$ implies $f(t) > 1$. Since we can find a big enough $q \in \mathbb{N}$ such that $\frac{1}{q} < t$ we have $f_n(t) > f_n(1/q) = f_{nq}(1)^{1/q}$. Letting n go to infinity gives $f(t) \geq e^{1/q} > 1$.

Exercise 1: Use the same idea as above to prove that $\lim_{n\to\infty} f(x_n) = 1$ for every sequence $\{x_n\}$ convergent to 0.

We have $f : \mathbb{R} \to (0, \infty)$, one-to-one and the property above shows that f is also surjective. Therefore, f is a bijection and we can refer to its inverse. As usual, from now one we will denote these function by their standard names $f(x) = e^x$ and $f^{-1}(t) = \ln t$, $t > 0$.

Also, the following property is essential for the concept of derivative that follows:

$$\frac{e^{x_n} - 1}{x_n} \to 1$$

for every sequence $\{x_n\}$ convergent to 0. We will leave this for the section dealing with continuity of the exponential function and the derivative of it.

1.3 Fundamental Limits of real valued function

Quotation: *"The result of the mathematician's creative work is demonstrative reasoning, a proof, but the proof is discovered by plausible reasoning, by GUESSING" –George Polya, Mathematics and Plausible Reasoning, 1953.)*

The concept of limit is essential in the investigation of this mathematical subject called Calculus. The idea of limit can be intuitively given by some important examples.

Example 1: Let us consider the function

$$f(x) = \left(1 + \frac{1}{x}\right)^x$$

defined for all $x > 0$. Its graph is included in Figure 1.2.

From the graph of f we see that $f(x)$ gets closer and closer to a horizontal line, $y = 2.71...$, as x gets bigger and bigger; we formally say in a mathematical language that x goes or tends to infinity (symbol used for infinity is ∞). We assume this pattern continues as x grows indefinitely. This number that appears here magically is an important constant in mathematics and it is denoted by e. Leonhard Euler (1707-1783) was the first mathematician who used this notation.

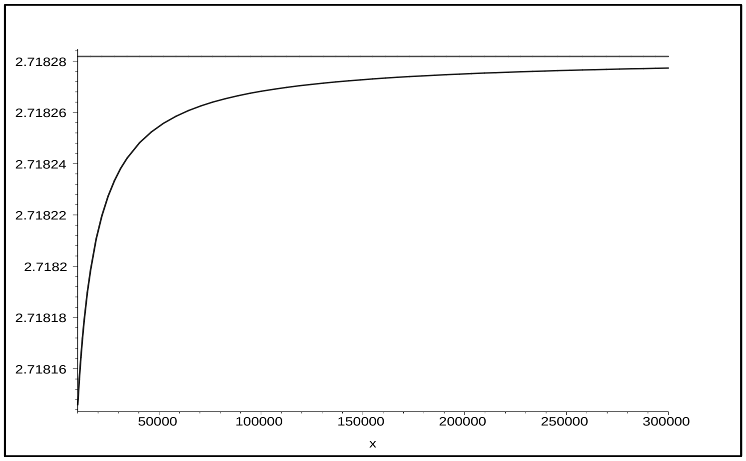


Figure 1.2: Plot of $f(x) = \left(1 + \frac{1}{x}\right)^x$, $x = 10000..300000$

This number is transcendental, i.e., there is no polynomial equation with integer coefficients that has e as one of its roots. The truncation to 20 decimals of e is

$$e \approx 2.71828182845904523536\ldots.$$

The fact about the behavior of the function f is recorded mathematically by writing

$$\boxed{\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = e.} \tag{1.13}$$

This is one of the fundamental limits that connects the behavior of polynomial functions with the exponential functions. In general the exponential functions are functions of the form $g(x) = a^x$ with $a \in (0,1) \cup (1,\infty)$. If a is the number e then the function is called the natural exponential function. In order to be able to show such a limiting behavior for $f(x) = \left(1 + \frac{1}{x}\right)^x$, we would need a rigorous definition for the exponential functions which is not a trivial matter at all. Think that in particular that will have to include what it means to calculate $\pi^{\sqrt{7}}$. (By the way, it seems incredible but we don't know what the unit digit of $10\pi^{\pi^{\pi^{\pi}}}$ is). We will come back to all these properties of limits and prove all the properties of the elementary functions as known, when we will have the concept of definite integral.

In the theory of limits for functions one can first introduce the limit of a particular type of functions which are called sequences. In general by *a sequence of real numbers* we just understand an infinite list $a_1, a_2, \ldots, a_n, \ldots$ where a_k are real numbers. As one of the simplest examples is $a_n = \frac{1}{n}$. As n goes to ∞ then a_n gets closer and closer to zero. We write this like $\lim_{n\to\infty} \frac{1}{n} = 0$.

The precise meaning of the limit of a sequence is given in the following definition:

An equivalent way of writing (1.13) is

$$\lim_{n\to\infty} \left(1 + \frac{1}{a_n}\right)^{a_n} = e \text{ for every sequence } a_n \to \infty.$$

Definition 1.3.1. *In general, we say that a function f has limit L at $x = a$ (which can be or not in the domain of the function) if the sequence $f(a_n)$ converges to L for every sequence a_n convergent to a, which is not eventually a constant sequence (so, it is implicit that the domain of the function allows for something like this to happen otherwise the concept is vacuous).*

The following equivalent statement for the definition of limit of a function at a finite point, it is usually known as the $\epsilon - \delta$ definition: `f has L as limit at a, if for every ϵ > 0 there exists δ > 0 such that for every x in the domain of the function such that 0 < |x − a| < δ, we have |f(x) − L| < ϵ.`

We write this information about the limit in the form

$$\lim_{x\to a} f(x) = L.$$

We are going to prove (1.13) later on in the course after the formal definition of exponential functions by use of definite integrals has been introduced. At this point we are just going to take (1.13) as fact. To avoid circular reasoning we have to avoid using (1.13) as an important fact in the process of defining the exponential function and of course all of its properties that lead to this fundamental limit.

There are other fundamental limits which will be introduced later. At this point we would like to derive some other elementary limits using properties of limits and these fundamental limits.

A list of the basic properties of limits of sequences or functions which can be derived (except the last one since it involves the power function) from the definitions of limit is given below:

1. $\lim_{x\to a} constant = constant$
2. $\lim_{x\to a} constant\ f(x) = constant \lim_{x\to a} f(x)$, $\lim_{n\to\infty} constant\ a_n = constant \lim_{n\to\infty} a_n$
3. $\lim_{x\to a} f(x) \pm g(x) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$

4. $\lim\limits_{x\to a} f(x)g(x) = (\lim\limits_{x\to a} f(x))(\lim\limits_{x\to a} g(x))$

5. $\lim\limits_{x\to a} f(x)/g(x) = (\lim\limits_{x\to a} f(x))/(\lim\limits_{x\to a} g(x))$ assuming that $\lim_{x\to a} g(x) \neq 0$

6. $\lim\limits_{x\to a} f(x)^r = (\lim\limits_{x\to a} f(x))^r$ if we have $\lim_{x\to a} f(x) > 0$

All these formulae are correct provided that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Let us work out an example in which these properties are used.

Example: Compute $\lim\limits_{x\to\infty} \left(\dfrac{x+3}{x}\right)^x$.

Since

$$\lim_{x\to\infty} \left(\frac{x+3}{x}\right)^x = \lim_{x\to\infty} \left(1+\frac{3}{x}\right)^x = \lim_{x\to\infty} \left[\left(1+\frac{3}{x}\right)^{\frac{x}{3}}\right]^3$$

using Property 6, and the substitution $\frac{x}{3} = t$ $(t \to \infty)$ we get

$$\lim_{x\to\infty} \left(\frac{x+3}{x}\right)^x = \left[\lim_{t\to\infty} \left(1+\frac{1}{t}\right)^t\right]^3 = e^3.$$

The Property 6 above can be extended to all the elementary functions. Let us include the list of the basic elementary functions and their domain here:

1. Polynomials: $p(x) = c_0x^n + c_1x^{n-1} + \cdots + c_{n-1}x + c_n$, $Domain = \mathbb{R}$;

2. Rational: $R(x) = \frac{P(x)}{Q(x)}$ where P and Q have no common linear factors, and the domain is $\{x \in \mathbb{R} | Q(x) \neq 0\}$;

3. Exponential Functions: $h(x) = a^x$, $x \in \mathbb{R}$, $a > 0$, $a \neq 1$; $Domain = \mathbb{R}$;

4. Power Functions: $g(x) = x^r = e^{r\ln x}$, $Domain = (0, \infty)$, $r \in \mathbb{R}$; In some cases the maximum domain is bigger. For example, if $r = \frac{1}{3}$, we define $g(x) = x^{\frac{1}{3}}$ simply as the inverse of $g^{-1}(x) = x^3$ whose domain and range is $\mathbb{R}$. Similarly, the domain of $x \to \sqrt{x}$ is $[0, \infty)$.

5. Trigonometric Functions: *sine*, *cosine*, *tangent*, *cotangent*, *secant*, *cosecant*;

6. Logarithmic Functions: $i(x) = \log_a(x)$, $Domain = (0, \infty)$;

7. Inverse Trigonometric Functions: *arcsin*, *arccos*, *arctan*

Such functions may have complicated domains but whatever these domains are they will play an important role in what follows. The Property 6 for limits can be extended (shown to hold true) to any elementary function as above, say F, in the following way:

$$\lim_{x\to a} F(f(x)) = F(\lim_{x\to a} f(x)), \tag{1.14}$$

whenever $\lim_{x\to a} f(x)$ is in the domain of F and the composition $F(f(x))$ makes sense. The reason for which (1.14) happens is in fact a more general (at least formally) property:

$$\lim_{\substack{x\to b\\ x\in D}} F(x) = F(b),\ \ b\in D = Domain(F), \tag{1.15}$$

which is called continuity of F at the point b. In other words we have the following theorem:

Theorem 1.3.2. *Every elementary function is continuous at each point in its domain of definition.*

As an application of this theorem let us derive another fundamental limit which is equivalent to (1.13) and it is an intimate connection between polynomials and the natural logarithmic function:

$$\boxed{\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1.} \tag{1.16}$$

Since ln is continuous at the point e we obtain

$$\lim_{x\to\infty} \ln(1+\frac{1}{x})^x = \ln e = 1,\ \ or\ \lim_{x\to\infty} x\ln(1+\frac{1}{x}) = 1,$$

and if we substitute $y=\frac{1}{x}\to 0$ we get $\lim_{y\to 0}\frac{\ln(1+y)}{y}=1$ which is nothing else but (1.16). Of course, if one assumes that (1.16) is true, the first fundamental limit (1.13), follows.

The third fundamental limit can be derived from (1.16) and it intimately connects the polynomials with the exponentials:

$$\boxed{\lim_{x\to 0} \frac{a^x-1}{x} = \ln a,\ \ a>0,\ \ a\neq 1.} \tag{1.17}$$

Indeed, if we set $y=\log_a(1+x)=\frac{\ln(1+x)}{\ln a}\to 0$ as $x\to 0$ (continuity of ln at the point 1) we obtain $x=a^y-1$ and so (1.16) becomes

$$\lim_{y\to 0}\frac{y}{a^y-1} = \lim_{x\to 0}\frac{\ln(1+x)}{x\ln a} = \frac{1}{\ln a},$$

which proves that we must have (1.17).

Next, let us derive the fourth fundamental limit which intimately connects the polynomials with the power functions:

$$\boxed{\lim_{x\to 0}\frac{(1+x)^\alpha - 1}{x} = \alpha, \quad \alpha\in\mathbb{R}.} \tag{1.18}$$

Using the fact that the logarithmic function is the inverse of the exponential function, i.e. $a = e^{\ln a}$, we have

$$\lim_{x\to 0}\frac{(1+x)^\alpha - 1}{x} = \lim_{x\to 0}\frac{e^{\ln(1+x)^\alpha} - 1}{x} = \lim_{x\to 0}\frac{e^{\alpha\ln(1+x)} - 1}{\alpha\ln(1+x)}\frac{\alpha\ln(1+x)}{x}.$$

Because $t = \alpha\ln(1+x) \to 0$ as $x \to 0$ and using (1.16) and (1.17) we obtain

$$\lim_{x\to 0}\frac{(1+x)^\alpha - 1}{x} = \lim_{y\to 0}\frac{e^y - 1}{y}\lim_{x\to 0}\frac{\alpha\ln(1+x)}{x} = \alpha.$$

Let us work an exercise in which (1.18) plays an important role.

Exercise: Calculate the limit $\lim\limits_{x\to 1}\dfrac{\sqrt[3]{x}-1}{x-1}$.

Solution: Changing the variable of the limit to $y = x-1$ we see that while $x \to 1$ then $y \to 0$. Hence the limit becomes

$$\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{x-1} = \lim_{y\to 0}\frac{(1+y)^{1/3}-1}{y} = \frac{1}{3}.$$

The fifth fundamental limit which cannot be derived from the previous ones is one that intimately connects the polynomials with trigonometric functions:

$$\boxed{\lim_{x\to 0}\frac{\sin x}{x} = 1.} \tag{1.19}$$

We are going to show this property when the trigonometric functions will be defined rigorously with the concept of definite integral.

A few simple corollaries of (1.19) are worth mentioning. First, for every $a \neq 0$, a simple substitution gives

$$\lim_{x\to 0}\frac{\sin ax}{x} = \lim_{t\to 0}\frac{\sin t}{t/a} = a. \tag{1.20}$$

Also, using the double angle formula $1-\cos(\alpha) = 2\sin(\alpha/2)^2$ leads us into another important trigonometric limit:

$$\lim_{x\to 0}\frac{1-\cos ax}{x^2}=\lim_{x\to 0}\frac{2\sin^2 ax/2}{x^2}=2(a/2)^2=\frac{a^2}{2}. \tag{1.21}$$

Finally, another important tool used in computing limits is the so called Squeeze Theorem.

Theorem 1.3.3. *Given three functions f, g and h defined on a domain D which has a as limiting point (there exist a non-constant sequence in D, which is convergent to a), and*

$$f(x)\le g(x)\le h(x), for\ all\ x\in D\setminus\{a\}.$$

If $\lim_{x\to a} f(x)=\lim_{x\to a} h(x)=L$ *then* $\lim_{x\to a} g(x)=L$.

This theorem can be easily shown directly from the definition of the limit (1.3.1). One needs to use the following inequality which is left as an exercise: for all a, b and c such that $a\le b\le c$ and for every $x\in\mathbb{R}$ we have

$$|x-b|\le|x-a|+|x-c|. \tag{1.22}$$

Proof Sketch: We let $\epsilon>0$ be arbitrary and choose $\delta_1>0$ such that $|f(x)-L|<\epsilon/2$ for all $x\in D$ such that $0<|x-a|<\delta_1$. Also, because $\lim_{x\to a}h(x)=L$, we can find $\delta_2>0$ such that $|h(x)-L|<\epsilon/2$ for all $x\in D$ such that $0<|x-a|<\delta_2$. Therefore, for $x\in D$ such that $0<|x-a|<\delta:=min\{\delta_1,\delta_2\}$, using 1.22, we have

$$|g(x)-L|\le|f(x)-L|+|h(x)-L|<\epsilon/2+\epsilon/2=\epsilon. \quad\blacksquare$$

Here is an example of how this theorem can be used to show that $\lim_{x\to\infty}\frac{\sin x}{x}=0$. We observe that $-\frac{1}{x}\le\frac{\sin x}{x}\le\frac{1}{x}$ for $x>0$. Also, since $\lim_{x\to\infty}\frac{1}{x}=0$, the claim follows by applying the theorem to $f(x)=-\frac{1}{x}$, $g(x)=\frac{\sin x}{x}$ and $h(x)=\frac{1}{x}$.

Finally some important limits, which deal with the end behavior of elementary are listed next:

$$\lim_{x\to\infty}\frac{x^\alpha}{a^x}=0,\ a>1,\ and\ \alpha\in\mathbb{R}, \tag{1.23}$$

$$\lim_{x\to\infty}\frac{\ln x}{x^\alpha}=0,\quad \alpha\in(0,\infty), \tag{1.24}$$

(1.25) $$\lim_{x\to\infty}\arctan x = \frac{\pi}{2},\ \textit{and}\ \ \lim_{x\to-\infty}\arctan x = -\frac{\pi}{2}.$$

The last two limits follow from the properties of the function $f(x) = \tan x$ and the limits $\lim_{x\to\frac{\pi}{2}^-} f(x) = \infty$ and $\lim_{x\to\frac{\pi}{2}^+} f(x) = -\infty$.

Let us prove next (1.23) using the Squeeze Theorem. We need the so called Bernoulli's Inequality:

(1.26) $$(1+\epsilon)^n \ge 1 + n\epsilon,$$

for every $\epsilon > -1$ and $n \in \mathbb{N}$.

One can prove this by induction on n. It is clear for $n = 1$ and for the induction step,

$$(1+\epsilon)^{n+1} \ge (1+n\epsilon)(1+\epsilon) = 1 + (n+1)\epsilon + n\epsilon^2 \ge 1 + (n+1)\epsilon.$$

Then for every $a > 1$ we can write $a = (1+\epsilon)^2$ for some $\epsilon > 0$. Then

$$0 \le \frac{n}{a^n} \le \frac{1}{(1+\epsilon)^n}\frac{n}{1+n\epsilon} < \frac{1}{(1+\epsilon)^n}\frac{1}{\epsilon} \to 0.$$

This shows that $\frac{n}{a^n} \to 0$ as $n \to \infty$. Hence,

$$0 \le \frac{x}{a^x} \le \frac{\lfloor x\rfloor + 1}{a^{\lfloor x\rfloor}} \to 0,\ \text{as } x \to \infty,$$

where $\lfloor x\rfloor$ is the greatest integer part of x, $x > 0$. This shows (1.23) for $\alpha = 1$. Then for α arbitrary we observe that it is true for $\alpha \le 0$ and for $\alpha > 0$ we have

$$\lim_{x\to\infty}\frac{x^\alpha}{a^x} = \lim_{x\to\infty}\left(\frac{x}{(a^{1/\alpha})^x}\right)^\alpha = 0,$$

because we still have $a^{1/\alpha} > 1$. Using a substitution, $x = e^t$, one reduces (1.24) to (1.23).

1.3.1 Problems

In these exercises assume that all the fundamental limits discussed earlier are true, and all the properties of limits take place including the theorem about elementary functions. Although all of these limits can be computed easily later, by L'Hospital's rule, look at these limits as a simple opportunity to brush up on your algebra skills.

$\boxed{1.}$ Calculate the following limit

$$\lim_{x\to\infty}\left(\frac{2x+3}{2x-1}\right)^{3x-1}$$

$\boxed{2.}$ Show that

$$\lim_{x\to 0}\frac{1}{x}\ln\left(\frac{1+2x}{1-3x}\right)=5.$$

$\boxed{3.}$ Use the fundamental limits to obtain the equality

$$\lim_{x\to 0}\frac{4^x-2^x}{x}=\ln 2.$$

$\boxed{4.}$ Find the limit

$$\lim_{x\to 1}\frac{x^5-1}{x^7-1}.$$

$\boxed{5.}$ Use the last fundamental limit to prove that

$$\lim_{x\to 0}\frac{\tan 6x}{\tan 3x}=2.$$

$\boxed{6.}$ Use any of the fundamental limits and properties of limits to show that

$$\lim_{x\to 0}\frac{4^x-2^{x+1}+1}{x^2}=(\ln 2)^2.$$

$\boxed{7.}$ Prove that

$$\lim_{x\to 0}\frac{\sqrt[3]{1+3x}-\sqrt[3]{1-3x}}{x}=2.$$

$\boxed{8.}$ Use a simple substitution to calculate

$$\lim_{x\to 1}\frac{x^2-3x+2}{x-1}.$$

$\boxed{9.}$ Use any methods to find

$$\lim_{x\to 4}\frac{\sqrt{x+5}-3}{\sqrt{x}-2}.$$

$\boxed{10.}$ (More challenging one) Assuming that the limit

$$L = \lim_{x\to 0} \frac{e^x - 1 - x}{x^2},$$

exists, prove that $L = \frac{1}{2}$.

$\boxed{11.}$ Determine the following limits numerically and analytically:

$$(a) \lim_{x\to\infty} \left(\frac{2x+3}{2x-3}\right)^{x/2} \qquad (b) \lim_{x\to 0} \frac{\cos 3x - \cos 5x}{x^2} \qquad (c) \lim_{x\to 1} \frac{\sqrt[5]{x}-1}{\sqrt[7]{x}-1}$$

$\boxed{12.}$ Determine if the following function is continuous or not. If it is not continuous find the points of discontinuity.

$$f(x) = \begin{cases} (x-2)(x-3) & if \quad x \geq 0 \\ \dfrac{(\sin 2x)(\sin 3x)}{x^2} & if \quad x < 0 \end{cases}.$$

$\boxed{13.}$ Find all values of a such that the following function is continuous:

$$h(x) = \begin{cases} \dfrac{ax}{3+a^2x} & if \quad x \geq 1 \\ \dfrac{\sqrt[4]{|x|}-1}{x-1} & if \quad x < 1 \end{cases}.$$

$\boxed{14.}$ Use the Intermediate Value Theorem to show that the following equation has a solution in the specified interval:

$$e^x = 2 - 2x \qquad in \;\; (0,1).$$

$\boxed{15.}$ Use Squeeze Theorem to show that following limit is equal to 1:

$$\lim_{x\to\infty} \left(\frac{x}{x+\cos x}\right)^{\frac{1}{x}}$$

$\boxed{16.}$ Show that for all a, b and c such that $a \leq b \leq c$ and for every $x \in \mathbb{R}$ we have

$$|x-b| \le |x-a| + |x-c|.$$

1.3.2 Solutions

$\boxed{1.}$ We substitute $\frac{2x+3}{2x-1} = 1+t$. We observe that since $x \to \infty$, then $\frac{1}{2x-1} \to 0$. Hence $t = \frac{2x+3}{2x-1} - 1 = \frac{2x+3-2x+1}{2x-1} = \frac{4}{2x-1} \to 0$. Solving for x, we obtain $x = \frac{1}{2} + \frac{2}{t}$. Then the limit given can be written in terms of t

$$\lim_{x\to\infty}\left(\frac{2x+3}{2x-1}\right)^{3x-1} = \lim_{t\to 0}(1+t)^{1/2+6/t} = \lim_{t\to 0}(1+t)^{1/2}\lim_{t\to 0}\left((1+t)^{1/t}\right)^6 = 1(e^6) = e^6.$$

We used several of the property of the limits listed on page 6, including a variation of the first fundamental limit (1.13): $\lim\limits_{t\to 0}(1+t)^{1/t} = e$.

$\boxed{2.}$ We use the property of the logarithmic functions, $\ln(a/b) = \ln a - \ln b$, $(a, b > 0)$, and separate the given limit into two limits which have basically the same nature:

$$L = \lim_{x\to 0}\frac{1}{x}\ln\left(\frac{1+2x}{1-3x}\right) = \lim_{x\to 0}\frac{\ln(1+2x)}{x} - \lim_{x\to 0}\frac{\ln 1-3x}{x}.$$

So, because for every real number $k \neq 0$, we have by 1.16

$$\lim_{x\to 0}\frac{\ln(1+kx)}{x} = \lim_{t\to 0}\frac{\ln(1+t)}{t/k} = \lim_{t\to 0}k\frac{\ln(1+t)}{t} = k,$$

we see that the required limit is $L = 2 - (-3) = 5$.

$\boxed{3.}$ Since we know that $\lim\limits_{x\to 0}\frac{a^x-1}{x} = \ln a$, we have

$$\lim_{x\to 0}\frac{4^x-2^x}{x} = \lim_{x\to 0}\frac{2^x(2^x-1)}{x} = \left(\lim_{x\to 0}2^x\right)\lim_{x\to 0}\frac{2^x-1}{x} = 2^0\ln 2 = \ln 2.$$

$\boxed{4.}$ We use the fundamental limit $\lim\limits_{t\to 0}\frac{(1+t)^a-1}{t} = a$ and with the substitution $t = x-1 \to 0$, we get

$$\lim_{x\to 1}\frac{x^5-1}{x^7-1} = \lim_{t\to 0}\frac{(1+t)^5-1}{(1+t)^7-1} = \frac{\lim\limits_{t\to 0}\frac{(1+t)^5-1}{t}}{\lim\limits_{t\to 0}\frac{(1+t)^7-1}{t}} = \frac{5}{7}.$$

$\boxed{5.}$ Let us first observe that

$$\lim_{x\to 0}\frac{\tan ax}{x} = \lim_{x\to 0}\frac{\sin ax}{x}\lim_{x\to 0}\frac{1}{\cos ax} = 1.$$

Hence, we have

$$\lim_{x\to 0}\frac{\tan 6x}{\tan 3x} = \frac{\lim_{x\to 0}\frac{\tan 6x}{x}}{\lim_{x\to 0}\frac{\tan 3x}{x}} = \frac{6}{3} = 2.$$

$\boxed{6.}$ Let us observe that $4^x - 2^{x+1} + 1 = 2^{2x} - 2(2^x) + 1 = (2^x - 1)^2$, and so

$$\lim_{x\to 0}\left(\frac{2^x - 1}{x}\right)^2 = (\ln 2)^2.$$

$\boxed{7.}$ We observe that for $a \neq 0$, we have

$$\lim_{t\to 0}\frac{(1+at)^\alpha - 1}{t} = \lim_{x\to 0}\frac{(1+x)^\alpha - 1}{x/a} = a\alpha.$$

So, the required limit L can be calculated as shown below

$$L = \lim_{x\to 0}\frac{\sqrt[3]{1+3x} - 1}{x} - \lim_{x\to 0}\frac{\sqrt[3]{1-3x} - 1}{x} = 3(1/3) - (-3)(1/3) = 2.$$

$\boxed{8.}$ We substitute $x - 1 = t$, and observe that

$$\lim_{x\to 1}\frac{x^2 - 3x + 2}{x - 1} = \lim_{t\to 0}\frac{(1+t)^2 - 3(1+t) + 2}{t} = \lim_{t\to 0} t - 1 = -1.$$

$\boxed{9.}$ Let's multiply by the conjugate top and bottom and get rid of the differences of square roots by using the formula $(\sqrt{a} - b)(\sqrt{a} + b) = a - b^2$:

$$\lim_{x\to 4}\frac{\sqrt{x+5} - 3}{\sqrt{x} - 2} = \lim_{x\to 4}\frac{(x-4)(\sqrt{x} + 2)}{(x-4)(\sqrt{x+5} + 3)} = \lim_{x\to 4}\frac{\sqrt{x} + 2}{\sqrt{x+5} + 3} = 4/6 = 2/3.$$

$\boxed{10.}$ Assuming that the limit, let us first observe that if we change the variable $x \to 2x$, we have

$$L = \lim_{x\to 0}\frac{e^{2x} - 1 - (2x)}{4x^2},$$

and so

$$4L = \lim_{x\to 0} \frac{e^{2x} - 1 - (2x)}{x^2}.$$

On the other hand, using a fundamental limit, we have

$$1 = \lim_{x\to 0} \frac{(e^x - 1)^2}{x^2} = \lim_{x\to 0} \frac{e^{2x} - 2e^x + 1}{x^2}.$$

Subtracting the two equalities we get

$$4L - 1 = \lim_{x\to 0} \frac{2e^x - 2x - 2}{x^2} = 2L.$$

From here we solve for L: $2L = 1$ or $L = 1/2$.

$\boxed{11.}$ We will demonstrate similar problems:

$$(a) \lim_{x\to 0} \left(\frac{3x+1}{1+2x}\right)^{1/x} \qquad (b) \lim_{x\to 0} \frac{\cos 5x - \cos 7x}{x^2} \qquad (c) \lim_{x\to 1} \frac{\sqrt[3]{x} - 1}{\sqrt[7]{x} - 1}$$

(a) Let $f(x) = \left(\frac{3x+1}{1+2x}\right)^{\frac{1}{x}}$ for $x \in (-1/2, 1/2)$. Some of the values of f at inputs that are getting closer to zero are tabulated next:

f(0.1)	2.226491601
f(-0.1)	3.801189052
f(0.01)	2.652804911
f(-0.01)	2.788907699
f(0.001)	2.711511762
f(-0.001)	2.725103316

It seems to be the case that the limit is ≈ 2.71. To do this algebraically we use the first fundamental limit:

$$\lim_{x\to 0} \left(\frac{3x+1}{1+2x}\right)^{1/x} = \lim_{x\to 0} \left(1 + \frac{x}{1+2x}\right)^{1/x} = \lim_{x\to 0} \left[\left(1 + \frac{1}{(1+2x)/x}\right)^{\frac{1+2x}{x}}\right]^{\frac{x}{1+2x}\frac{1}{x}} = e^{\lim_{x\to 0} \frac{1}{1+2x}} = e$$

so

$$\boxed{\lim_{x\to\infty} \left(\frac{3x+1}{1+2x}\right)^{2x} = e}.$$

■

(b) If $g(x) = \dfrac{\cos 5x - \cos 7x}{x^2}$ if $x \neq 0$. Some of the values of g for inputs getting closer and closer to zero are included in the next table:

g(0.1)	11.27403746
g(0.01)	11.99260100
g(0.001)	11.99990000

We can guess that this limit must be equal to 36. This is indeed the case since

$$\lim_{x\to 0} \frac{\cos 3x - \cos 9x}{x^2} = \lim_{x\to 0} \frac{2\sin x \sin 6x}{x^2} = \lim_{x\to 0} 2\frac{\sin x}{x} \lim_{x\to 0} 6\frac{\sin 6x}{6x} = 12.$$

Hence $\boxed{\lim_{x\to 0} \dfrac{\cos 5x - \cos 7x}{x^2} = 12}$. ■

(c) Finally if $h(x) = \dfrac{\sqrt[3]{x} - 1}{\sqrt[7]{x} - 1}$ for every real number $x \neq 1$. Some of the values of g around zero are shown below:

h(0.9)	2.310133871
h(1.1)	2.354690647
h(0.99)	2.331101813
h(1.01)	2.335546124

So it is reasonable to conclude that the limit of this function at $x = 1$ is $2.\overline{3} = \frac{7}{3}$. This is true since if we make the change of variable $x = (1+t)^7$ we see that $t \to 0$ and

$$\lim_{x\to 1} \frac{\sqrt[3]{x} - 1}{\sqrt[7]{x} - 1} = \lim_{t\to 0} \frac{(1+t)^{\frac{7}{3}} - 1}{t} = \frac{7}{3}.$$

Therefore, $\boxed{\lim_{x\to 1} \dfrac{\sqrt[3]{x} - 1}{\sqrt[7]{x} - 1} = \dfrac{7}{3}.}$ ■

$\boxed{12.}$ We observe that there are not problems with the continuity at points a other than $a = 0$. Clearly $\lim_{x\to 0^+} f(x) = 6 = f(0)$ and

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{(\sin 2x)(\sin 3x)}{x^2} = \lim_{x\to 0^-} \frac{\sin 2x}{x} \lim_{x\to 0^-} \frac{\sin 3x}{x} = 2(3) = 6.$$

$\boxed{13.}$ We observe that h is continuous everywhere except possibly at $x = 1$. Next, we see that $\lim_{x\to 1^+} h(x) = h(1) = a/(3 + a^2)$. Also, we get

$$\lim_{x\to 1^-} h(x) = \lim_{t\to 0} \frac{(1+t)^{1/4} - 1}{t} = 1/4.$$

In order for h to be continuous, we need $a/(3+a^2) = 1/4$ or $a^2 - 4a + 3 = 0$. This quadratic has two solutions: $a = 1$ and $a = 3$. Therefore, h is continuous if and only if $a \in \{1, 3\}$.

$\boxed{14.}$ We consider $f(x) = e^x - 2 + 2x$ defined on $[0,1]$. This is a continuous function, being elementary. We notice that $f(0) = e^0 - 2 = 1 - 2 = -1 < 0$ and $f(1) = e - 2 + 2 = e > 0$. Thus, we can apply IVT to f on $[0,1]$ and $y = 0$. We conclude that there exists $x_0 \in (0,1)$ such that $f(x_0) = 0$. This is equivalent to $e^{x_0} = 2 - 2x_0$.

$\boxed{15.}$ Since $-1 \leq cosx \leq 1$ we see that, for $x > 1$, we have

$$\left(\frac{x}{x-1}\right)^{\frac{1}{x}} \leq \left(\frac{x}{x+\cos x}\right)^{\frac{1}{x}} \leq \left(\frac{x}{x+1}\right)^{\frac{1}{x}}$$

But $\lim_{x\to\infty}\left(\frac{x}{x-1}\right) = \lim_{x\to\infty}\left(\frac{1}{1-1/x}\right) = 1$ and similarly $\lim_{x\to\infty}\left(\frac{x}{x+1}\right) = \lim_{x\to\infty}\left(\frac{1}{1+1/x}\right) = 1$. Hence

$$\lim_{x\to\infty}\left(\frac{x}{x-1}\right)^{\frac{1}{x}} = 1^0 = 1 \text{ and } \lim_{x\to\infty}\left(\frac{x}{x+1}\right)^{\frac{1}{x}} = 1^0 = 1.$$

This forces

$$\lim_{x\to\infty}\left(\frac{x}{x+\cos x}\right)^{\frac{1}{x}} = 1.$$

$\boxed{16.}$ There are four possibilities: $x \in (-\infty, a]$, $x \in (a, b]$, $x \in (b, c]$ and $x \in (c, \infty)$.

Case I So, for $x \in (-\infty, a]$ the inequality becomes equivalent to $b - x \leq a - x + c - x$ or $x \leq a + c - b$. This is true since $x \leq a$ and $c - b \geq 0$.

Case II, $x \in (a, b]$ The inequality is the same as $b - x + x - a + c - x$. This is the same as $b \leq (x - a) + c$ which is true because $b \leq c$ and $x - a \geq 0$. Similarly, one can analyze the other two cases.

Fundamental Limits

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e \tag{1.27}$$

$$\lim_{x\to 0}\frac{\ln(1+x)}{x} = 1 \tag{1.28}$$

$$\lim_{x\to 0}\frac{a^x - 1}{x} = \ln a, a \neq 0 \tag{1.29}$$

$$\lim_{x\to 0}\frac{(1+x)^{\alpha}-1}{x}=\alpha, x\in\mathbb{R} \tag{1.30}$$

$$\lim_{x\to 0}\frac{\sin(x)}{x}=1 \tag{1.31}$$

$$\lim_{x\to 0}\frac{1-\cos(\alpha x)}{x^2}=\frac{\alpha^2}{2} \tag{1.32}$$

1.4 Continuity and piecewise functions

Calculating limits from the fundamental limits may turn out to be a real challenge We have seen in Theorem 1.3.2 that every elementary functions is continuous on its domain of definition.

A new class of functions which appears often in applications we will refer to it here as *piecewise functions*. This set of functions is important also within mathematics as a theoretical tool since it provides a good pool for examples and counterexamples.

Let us consider such an example:

$$f(x)=\begin{cases}\frac{\sin 2x}{x} & if\ x>0,\\ 2 & if\ x=0,\\ \frac{1-\ln(1-2x)}{x} & if\ x<0\end{cases}.$$

This function is continuous at every point different of zero since the rules for each branch are elementary functions well defined on those intervals. At $x=0$ we have

$$\lim_{x\to 0^+}\frac{1-\ln(1-2x)}{x}=2\lim_{x\to 0}\frac{\ln(1-2x)-1}{-2x}=2$$

and

$$\lim_{x\to 0^-}\frac{\sin 2x}{x}=2\lim_{x\to 0}\frac{\sin 2x}{2x}=2.$$

Hence we conclude that $\lim_{x\to 0} f(x)=2=f(0)$ and so this function is continuous.

On the other hand if we simply change the definition of f to

$$\hat{f}(x) = \begin{cases} \frac{\sin 2x}{x} \;\; if\; x > 0, \\ \\ 3 \;\; if\; x = 0, \\ \\ \frac{1-\ln(1-2x)}{x} \;\; if\; x < 0 \end{cases} .$$

In this case clearly $\hat{f}$ is not continuous, we say it is *discontinuous*, and since the limit exists at this point we call such a point a *removable discontinuity*.

A more interesting example is the following function which does not have a limit at zero:

$$g(x) = \begin{cases} \sin(\frac{1}{x}) \;\; if\; x > 0, \\ \\ 2 \;\; if\; x = 0, \\ \\ x\sin(\frac{1}{x}) \;\; if\; x < 0 \end{cases} ,$$

although the left hand side limit exists since $|x\sin(\frac{1}{x})| \le |x| \to 0$. This forces $\lim_{x\to 0^-} g(x) = 0$. This principle is known as the *squeeze theorem*. So, g is discontinuous at $x = 0$ and such a discontinuity is called an *essential discontinuity*.

An important theorem that is used often in mathematics is the Intermediate Value Theorem:

Theorem 1.4.1. *(***IVT***) Consider a continuous function on a closed interval $[a, b]$ and a number c between $f(a)$ and $f(b)$. Then there exists a value $x \in (a, b)$ such that $f(x) = c$.*

The proof of this theorem is beyond the scope of the course so we invite the interested students read a proof of it from a real analysis textbook.

As an application let us work the following problem:

If a and b are positive numbers, prove that the equation

$$\frac{a}{x^3 + 2x^2 - 1} + \frac{b}{x^3 + x - 2} = 0 \tag{1.33}$$

has at least one solution in the interval $(-1, 1)$.

The equation is equivalent to $a(x^3 + x - 2) + b(x^3 + 2x^2 - 1) = 0$. So, if we denote by $p(x) = a(x^3 + x - 2) + b(x^3 + 2x^2 - 1)$ we notice that, p is continuous on $[-1, 1]$ and

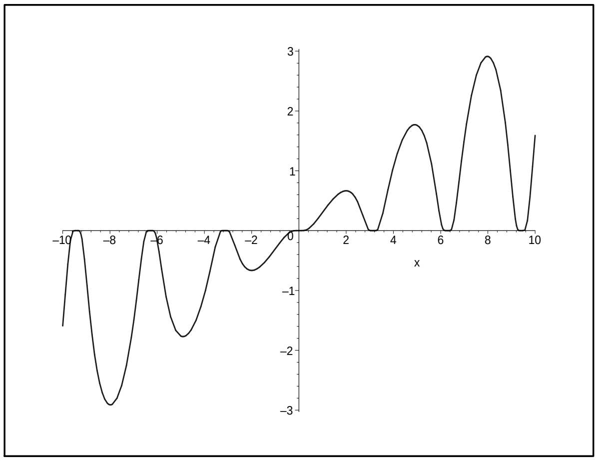


Figure 1.3: Graph of f in Problem 2.

$p(-1) = -4a < 0$ and $p(1) = 2b > 0$. Hence 0 is in between the two values of p at the endpoints of the interval $[-1, 1]$ and so, by the Intermediate Value Theorem, there must be a $c \in (-1, 1)$ such that $p(c) = 0$. This means c is a solution of the original equation.

A related problem and a more precise statement about the possible zeroes of (1.33) will be two show that the equation (1.33) has at least one solution in the interval $(\alpha, 1)$ where $\alpha = \frac{\sqrt{5}-1}{2} \approx 0.6180$ (reciprocal of the so called golden ratio number).

Indeed, the polynomial above can be written in the form $p(x) = a(x-1)(x^2+x+2)+b(x+1)(x^2+x-1)$ and α is a root of the polynomial x^2+x-1. Hence $p(\alpha) = a(\alpha-1)3 < 0$ and $p(1) = 2b > 0$. Therefore the same argument applies for the interval $(\alpha, 1)$.

1.4.1 Problems

$\boxed{1.}$ *Use the IVT to prove that every continuous function $f : [a, b] \to [a, b]$ has a fixed point, i.e. a point $c \in [a, b]$ such that $f(c) = c$.*

$\boxed{2.}$ *Consider the function f defined in the following way:*

$$f(x) = \begin{cases} xe^{-\frac{1}{|\sin x|}}, \ if \quad x \neq k\pi, \ k \in \mathbb{Z}, \\ \\ 0 \ if \ x = k\pi, \quad k \in \mathbb{Z}. \end{cases}$$

Show that f is continuous on $\mathbb{R}$.

$\boxed{3.}$ *Prove that every continuous function on $[a, b]$ which is one-to-one, must be strictly monotone.*

(A *one-to-one function* is a function with the property that $f(u) = f(v)$ can happen only if $u = v$ and a strictly monotone function is either strictly increasing or strictly decreasing. A *strictly increasing* function is a function with the property that for every u and v in its domain such that $u < v$, then $f(u) < f(v)$.)

1.4.2 Solutions to 1.4.1 Problems

$\boxed{1.}$ We consider the function $g(x) = f(x) - x$ and observe that g is continuous on $[a, b]$, $g(a) = f(a) - a \geq 0$ and $g(b) = f(b) - b \leq 0$. If either $g(a) = 0$ or $g(b) = 0$, then we found a fixed point: a or b. If $g(a) > 0$ and $g(b) < 0$ then we can use IVT for $c = 0$ and obtain a point $x_0 \in (a, b)$ such that $g(x_0) = 0$. Hence, x_0 is a fixed point for f.

$\boxed{2.}$ First let us observe show that we don't have a problem with the continuity except for points of the form $a = k\pi$. In order to prove the continuity at a we need to show that $\lim_{x\to a} f(x) = 0$. Since $|\sin x| \to 0$ when $x \to a$, we conclude that $\lim_{x\to a} f(x) = a \lim_{t\to\infty} 1/e^t = 0$.

1.4.3 Sample Test 1 and Solutions

$\boxed{1.}$ *Determine the following limits numerically and analytically:*

$$(a) \lim_{x\to 0} \frac{1 + \cos x - 2\cos 2x}{x^2} \qquad (b) \lim_{x\to 1} \frac{\sqrt[3]{x} - 1}{x^7 - 1}$$

Solutions: (a) If $g(x) = \dfrac{1 + \cos x - 2\cos 2x}{x^2}$ if $x \neq 0$. Some of the values of g for inputs getting closer and closer to zero are included in the next table:

g(0.1)	3.4871
g(0.01)	3.4998
g(0.001)	3.4999

We can guess that this limit must be equal to 7/2. We split the limit in two:

$$\lim_{x\to 0} \frac{1 + \cos x - 2\cos 2x}{x^2} = \lim_{x\to 0} \frac{2 - 2\cos 2x}{x^2} - \lim_{x\to 0} \frac{1 - \cos(x)}{x^2} =$$

$$2\lim_{x\to 0} \frac{1 - \cos 2x}{x^2} - \lim_{x\to 0} \frac{1 - \cos(x)}{x^2} = 2(4/2) - 1/2 = 7/2 = \boxed{7/2},$$

(b) Finally if $g(x) = \dfrac{\sqrt[3]{x}-1}{x^7-1}$ for every real number $x \neq 1$. Some of the values of g around 1 are shown below:

g(0.9)	0.066
h(1.1)	0.034
h(0.99)	0.049
h(1.01)	0.046

If we make the change of variables $x = (1+t)^(1/7)$, we see that $t \to 0$ and

$$\lim_{x\to 1} \frac{\sqrt[3]{x}-1}{x^7-1} = \lim_{t\to 0} \frac{(1+t)^{\frac{1}{21}}-1}{t} = \frac{1}{21} \approx 0.\overline{047619}.$$

Therefore, $\boxed{\lim_{x\to 1} \frac{\sqrt[3]{x}-1}{x^7-1} = \frac{1}{21}.}$ ■

$\boxed{2.}$ *Determine if the following function is continuous or not. If it is not continuous find the points of discontinuity.*

$$f(x) = \begin{cases} \frac{x}{\pi} & if \quad x \geq \pi/2 \\ \\ \dfrac{\cos x}{\pi/2 - x} & if \quad x < \pi/2 \end{cases}.$$

Solution: The function is defined everywhere and it is continuous at every point other than $\pi/2$ since it is elementary defined there. For $x = \pi/2$ we observe that

$$f(\pi/2) = 1/2, \lim_{x\searrow\pi/2} f(x) = 1/2, \lim_{x\nearrow\pi/2} f(x) = \lim_{x\nearrow\pi/2} \frac{\cos x}{\pi/2 - x} = \lim_{t\to 0} \frac{\cos(\pi/2 - t)}{t}.$$

Since $\cos(\pi/2 - t) = \cos(\pi/2)\cos t + \sin(\pi/2)\sin t = \sin t$ we see that

$$\lim_{x\nearrow\pi/2} f(x) = \lim_{t\to 0} \frac{\sin t}{t} = 1.$$

So, the function is not continuous at $x = \pi/2$. ■

The graph of f is shown next:

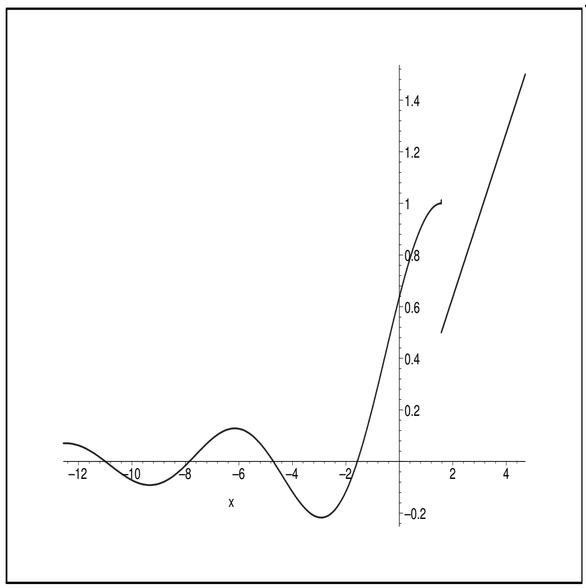


Figure 1

We can see that there is only one point of discontinuity, namely, at $x = \pi/2$.

$\boxed{3.}$ *Use the Intermediate Value Theorem to show that the following equation has a solution in the specified interval:*

$$2\cos x = 1 + \sin x \qquad \textit{in} \ \ (0, \frac{\pi}{2}).$$

Solution: We consider the function $g(x) = 2\cos x - 1 - \sin x$ which is defined for all $x \in [0, \frac{\pi}{2}]$. This function is continuous since it is in terms of elementary trigonometric functions whose domain of definition is the whole real line. Because $g(0) = 1 > 0$ and $g(\frac{\pi}{2}) = -2 > 0$, we can apply IVT to g on $[0, \frac{\pi}{2}]$ and for $y = 0$, to conclude that there exists a $c \in (0, \frac{\pi}{2})$ such that $g(c) = 0$. This implies that the equation

$$2\cos x = 1 + \sin x$$

has a solution in $(0, \frac{\pi}{2})$. It turns out that this solution is $\arctan(3/4) \approx 0.644$. ■

The graph of the two functions, $x \to 2\cos x$ and $x \to 1 + \sin x$, on the interval $[0, \frac{\pi}{2}]$ is shown below:

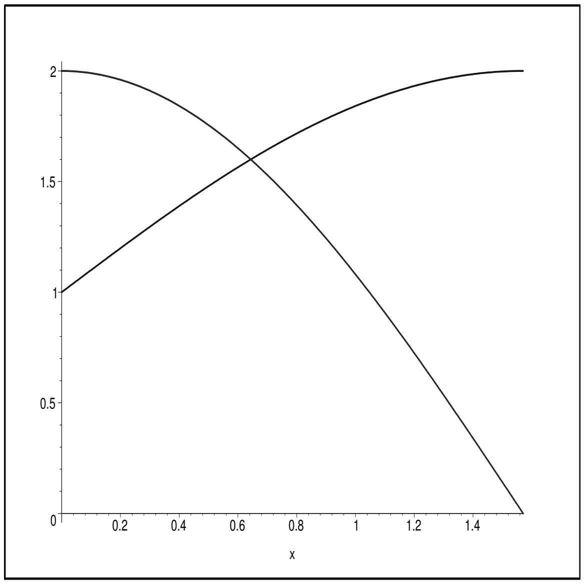


Figure 2

Clearly the point of their intersection, which is unique, is the value given by the IVT. $\boxed{4.}$ *Calculate the derivatives of the following functions:*

(a) $g(x) = x^2 - 3x^3 + x\sqrt{x} + 1, \quad x \in [0, \infty)$

(b) $h(x) = x^3 \ln x - x^2 e^x, \quad x \in \mathbb{R}$

(c) $l(x) = 2^3 - 3^{2x}\ , \quad x \in \mathbb{R}$

(d) $m(x) = (\ln x)^3, \quad x > 0$

(e) $n(x) = (2x + 1)^3, \quad x \in \mathbb{R}.$

Solution: (a) First we write $g(x) = x^2 - 3x^3 + x^{3/2} + 1$ and so using the power rule, we obtain $\boxed{g'(x) = 2x - 9x^2 + \frac{3}{2}\sqrt{x}}$ for all $x \in [0, \infty)$.

(b) The product rule gives $\boxed{h'(x) = 3x^2 \ln(x) + x^2 - 2xe^x - x^2 e^x}$ for all $x > 0$.

(c) Using the derivative of b^x, we have $l'(x) = 0 - (3^{2x})' = -(9^x)' = -9^x \ln 9 = \boxed{-2(3^{2x}) \ln 3}$, $x \in \mathbb{R}$.

(d) We know that $(fg)' = f'g + fg'$, and then $(f^2)' = f'f + ff' = 2ff'$. Therefore, $(f^3)' = (f^2 f)' = 2ff'f + f^2 f' = 2f^2 f' + f^2 f' = 3f^2 f'$ which gives us the formula

$(f^3)' = 3f^2 f'$. In particular, $m'(x) = 3(\ln x)^2(1/x) = \boxed{\dfrac{3(\ln x)^2}{x}}$ for all $x > 0$.

(e) Similarly, $n'(x) = 3(2x+1)^2(2x+1)' = \boxed{6(2x+1)^2}$, $x \in \mathbb{R}$.

One could just rewrite $n(x) = 8x^3 + 12x^2 + 6x + 1$ and differentiate term by term using the power rule: $\boxed{n'(x) = 24x^2 + 24x + 6}$, which is the same answer as before but in the foiled form:

$$6(2x+1)^2 = 6(4x^2 + 4x + 1) = 24x^2 + 24x + 6.$$

Chapter 2

Derivatives and the rules of differentiation

2.1 Derivatives of the basic elementary functions

> **Quotation:** *A great discovery solves a great problem but there is a grain of discovery in the solution of any problem. Your problem may be modest; but if it challenges your curiosity and brings into play inventive faculties, and if you solve it by your own means, you may experience the tension and enjoy the triumph of discovery. –George Polya*

The concept of differentiation is nevertheless the most important in calculus. We are going to start with the geometric question that leads to this notion. Consider one of the important curves that one plays with in geometry: the circle. Taking a point on this circle one can draw several lines passing through this point but only one will intersect the circle at only that particular point. We usually call this line the tangent line to the circle at the given point. We know that such a line can be obtained by just taking the perpendicular to the corresponding radius of the point where the tangent is to be drawn.

What if we have some other types of curves? First, how do we even define the concept of tangent line and how do we compute it's equation?

Let us start with the curve of equation $y = f(x)$ and suppose we take $P = (a, f(a))$ a point on this curve. For another point close to P, say $Q = (x, f(x))$ we can calculate the slope of the secant line $\overline{PQ}$:

$$\frac{f(x) - f(a)}{x - a}.$$

Intuitively, when $x \to a$, this slope tends to have the limiting value of the slope of the "tangent" line to the curve at this point. This is actually what we will take by definition to be the tangent line at $(a, f(a))$ to $y = f(x)$:

$$\boxed{y - f(a) = f'(a)(x - a)}$$

where $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ if this last limit exists. We call this limit the derivative of f at a. Other notations used for this limit are: $\frac{df}{dx}(a)$ or $\frac{df}{dx}|_{x=a}$. We may also look at this calculation as a function if we define f' (the derivative of f) as being

$$\boxed{f'(x) = \lim_{t\to x} \frac{f(t) - f(x)}{t - x}} \tag{2.1}$$

for all $x \in Domain(f') := \{x|$ *all real* x *where the limit* (2.1) *exists*$\}$.

We can say that calculus is the study of the operation $f \to f'$ as applied mainly to elementary functions. *There are quite a few surprises and interesting stories about this "simple" transformation.*

One of the beginning stories is that each of the fundamental limits, that we have identified in Chapter I, represents the derivative of one of the basic elementary functions at a certain point. Not only that but each such limit is basically reflected into the derivative at other point in one way or another. Let us be more specific.

We start with the derivative of a power function:

$$\alpha = \lim_{t\to 0} \frac{(1+t)^\alpha - 1}{t} = \lim_{x\to 1} \frac{x^\alpha - 1}{x - 1} = f'(1)$$

where $f(x) = x^\alpha$, $x > 0$.

Let us calculate the derivative at any other point $a > 0$:

$$f'(a) = \lim_{x\to a} \frac{x^\alpha - a^\alpha}{x - a} = \lim_{x\to a} \frac{a^\alpha((\frac{x}{a})^\alpha - 1)}{a(\frac{x}{a} - 1)} = a^{\alpha-1} \lim_{t\to 1} \frac{t^\alpha - 1}{t - 1} = \alpha a^{\alpha-1}.$$

Hence, we have the derivative of a power function, also known as the power rule:

$$\boxed{\frac{d}{dx}(x^\alpha) = \alpha x^{\alpha-1},\ x > 0.}$$

Next, let us find out the derivative of the exponential function.

Consider $g(x) = e^x$ and $a \in \mathbb{R}$ arbitrary. Then

$$g'(a) = \lim_{t\to a} \frac{g(t) - g(a)}{t - a} = \lim_{t\to a} \frac{e^t - e^a}{t - a} = \lim_{t\to a} \frac{e^a(e^{t-a} - 1)}{t - a}$$

and after the substitution $t-a=x$, since $x \to 0$ we obtain, using the second fundamental limit (1.17):

$$g'(a) = e^a \lim_{x\to 0} \frac{e^x - 1}{x} = e^a.$$

Therefore we have $\boxed{\dfrac{d}{dx}e^x = e^x, x \in \mathbb{R}.}$

But what if we have a simple change in the base of the exponential function? Say, $g(x) = b^x$ with $b > 0$ and $b \neq 1$.

Then, using again (1.17), we get

$$g'(a) = \lim_{t\to a} \frac{g(t) - g(a)}{t-a} = \lim_{t\to a} \frac{b^t - b^a}{t-a} = \lim_{t\to a} \frac{b^a(b^{t-a} - 1)}{t-a} =$$

$$b^a \lim_{x\to 0} \frac{b^x - 1}{x} = b^a \lim_{x\to 0} \frac{e^{x\ln b} - 1}{x \ln b} \ln b = b^a \ln b.$$

Hence, $\boxed{\dfrac{d}{dx}b^x = b^x \ln b, x \in \mathbb{R}.}$

We will find next the derivative of the most common trigonometric function: $h(x) = \sin x$ defined for all radian angles $x \in \mathbb{R}$.

For fixed $a \in \mathbb{R}$ we have

$$h'(a) = \lim_{t\to a} \frac{\sin t - \sin a}{t-a} = \lim_{t\to a} \frac{2\sin\frac{t-a}{2}\cos\frac{t+a}{2}}{t-a}$$

using the formula from trigonometry $\sin\alpha - \sin\beta = 2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}$. Then we change the variable $\frac{t-a}{2} = x$ and notice that $x \to 0$ as $t \to a$. That gives

$$h'(a) = \lim_{x\to 0} \frac{\sin x}{x}\cos(x+a) = \cos a,$$

and so $\boxed{\dfrac{d}{dx}\sin x = \cos x, x \in \mathbb{R}.}$

For the cosine we can do a similar calculation. Let $i(x) = \cos x$ with $x \in \mathbb{R}$. The formula from trigonometry we need is $\cos\alpha - \cos\beta = -2\sin\frac{\alpha-\beta}{2}\sin\frac{\alpha+\beta}{2}$. We have, for fixed $a \in \mathbb{R}$,

$$i'(a) = \lim_{t\to a} \frac{\cos t - \cos a}{t-a} = \lim_{t\to a} \frac{-2\sin\frac{t-a}{2}\sin\frac{t+a}{2}}{t-a}.$$

After changing the variable as before we see that

$$i'(a) = -\lim_{x\to 0} \frac{\sin x}{x} \sin(x+a) = -\sin a.$$

2.2 Derivatives under algebraic operations

The basic algebraic operations that we do with numbers as addition, multiplication, subtraction and division can be done with functions. The derivative behaves nicely under these operations. One can observe that straight from the properties of the limit we get

$$(\alpha f + \beta g)' = \alpha f' + \beta g'$$

at every point where f' and g' exist. One rule that is a little unexpected is the so called, the *product rule*:

$$(fg)' = f'g + fg'$$

again, as long as f' and g' exist. Let us see where this is coming from. Suppose we have a point a at which $f'(a)$ and $g'(a)$ exist. Then

$$(fg)'(a) = \lim_{t\to a} \frac{f(t)g(t) - f(a)g(a)}{t-a} = \lim_{t\to a} \frac{f(t)g(t) - f(t)g(a) + f(t)g(a) - f(a)g(a)}{t-a} =$$

$$\lim_{t\to a} f(t)\frac{g(t)-g(a)}{t-a} + \lim_{t\to a} g(a)\frac{f(t)-f(a)}{t-a}.$$

One can observe that since $f'(a)$ exists then $\lim_{t\to a} f(t) = f(a)$. So, the limit

$$(fg)'(a) = \lim_{t\to a} \frac{f(t)g(t) - f(a)g(a)}{t-a} = f(a)g'(a) + f'(a)g(a)$$

which proves the product rule.

We apply the product rule now to find the derivative of functions that are products in different basic elementary functions. As an example let us compute $\frac{d}{dx}[(x^2-x)e^x]$:

$$\frac{d}{dx}[(x^2-x)e^x] = (2x-1)e^x + (x^2-x)e^x = (x^2+x-1)e^x.$$

The *quotient rule* can be stated like this:

$$(\frac{f}{g})' = \frac{f'g - fg'}{g^2},$$

of course, whenever the derivatives involved exist. The proof of this is similar to the one we did for the product rule so we let that to the reader as an exercise. This rule allows us to compute now the derivative of the rest of the trigonometric functions:

$$\frac{d}{dx}(\tan)(x) = \frac{\sin' x \cos x - \sin x \cos' x}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

Similarly, we get $\frac{d}{dx}(\cot)(x) = -\csc^2 x$, whenever the $\sin x \neq 0$. Finally we can show that $\sec' x = \sec x \tan x$ and $\csc' x = -\csc x \cot x$.

Can any function be a derivative? Derivatives have the special property that we talked about at the end of the previous section on continuity.

Theorem 2.2.1. *The derivative f' of a differentiable function f on $[a, b]$ has the, so called, Darboux property, or the intermediate value property, i.e. for y in between $f'(x_1)$ and $f'(x_2)$ $(a \leq x_1 < x_2 \leq b)$, there exists $c \in [x_1, x_2]$ such that $f'(c) = y$.*

We will include a proof of this in the next section. Let us make the observation that a function which has jump discontinuities such as

$$signum(x) = \begin{cases} 1 \; if \; x > 0 \\ \\ 0 \; if \; x = 0 \\ \\ -1 \; if \;\; x < 0 \end{cases}$$

cannot be the derivative of any function.

Finally, we have one more but the most trickier rule which deals with the composition of two functions: *the chain rue.* Suppose that $f : D(f) \to A \subset D(g) \xrightarrow{g} \mathbb{R}$, are two differentiable functions on their domain. Then $(g \circ f)' = (g' \circ f)f'$ or written a certain x in the domain of f:

$$(g \circ f)'(x) = (g' \circ f)(x) f'(x).$$

One example, let us say, $g(x) = x^{10}$ and $f(x) = x^2 + 2x + 1$. We observe that $(g \circ f)(x) = (x^2 + 2x + 1)^{10}$ so $\frac{d}{dx}(x^2 + 2x + 1)^{10} = 10((x^2 + 2x + 1)^9(2x + 2) = 20(x^2 + 2x + 1)^9(x + 1)$. Let us observe that $(g \circ f)(x) = (x + 1)^{20}$ so we can apply the chain rule into different functions and get $(g \circ f)'(x) = 20(x + 1)^{19}(x + 1)' = 20(x + 1)^{19}$, which is the same answer as we have gotten before.

One important application of the chain rule is the formula for computing the derivative of the inverse of a function. Let us assume that $f : I \to J$ and $g : J \to I$ is the inverse of f, which is assumed to be differentiable with the derivative not zero at every point in the interval I. It is possible to show that g is differentiable and so $g(f(x)) = x$ implies $g'(f(x))f'(x) = 1$. Hence,

$$\boxed{g'(y) = \frac{1}{f'(g(y))}, y \in J}.$$

As an example, if we take $f(x) = \sin x$, $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$ with the inverse $g(x) = \arcsin x$, $x \in (-1, 1)$. The formula above gives:

$$g'(x) = \frac{d}{dx}(\arcsin)(x) = \frac{1}{\cos(\arcsin x)} = \frac{1}{\sqrt{1-\sin^2(\arcsin x)}} = \frac{1}{\sqrt{1-x^2}},$$

so we have the formula

$$\boxed{\frac{d}{dx}(\arcsin)(x) = \frac{1}{\sqrt{1-x^2}}, \ x \in (-1, 1).}$$

One can similarly find the following two similar formulae:

$$\boxed{\frac{d}{dx}(\arccos)(x) = -\frac{1}{\sqrt{1-x^2}}, \ x \in (-1, 1), \ \textit{and}}$$

$$\boxed{\frac{d}{dx}(\arctan)(x) = \frac{1}{1+x^2}, \ x \in \mathbb{R}.}$$

Finally, let us show another important formula that can be derived from the chain rule, which is helpful when we differentiate functions of the form $u(x)^{v(x)}$. So, let us assume that u and v are two differentiable functions defined on I and $u(x) > 0$ for all $x \in I$ (some interval). Then

$$\frac{d}{dx}(u^v) = vu^{v-1}u' + u^v \ln(u)v' \ . \tag{2.2}$$

We can derive this by the so called logarithmic differentiation method: we set $w = u^v$ and apply ln both sides to get $\ln w = v \ln u$. Differentiating we obtain $\frac{w'}{w} = v' \ln u + v\frac{u'}{u}$. Solving for w' we obtain formula (2.3).

Let us see how this formula works for $f(x) = x^x$ defined for $x \in (0, \infty)$. We see that $f'(x) = x(x^{x-1}) + x^x \ln x$ or $f'(x) = x^x(1 + \ln x)$. We will see later that this implies to following interesting inequality

$$x^x \geq e^{-\frac{1}{e}} \approx 0.6922006276, \quad x > 0. \tag{2.3}$$

2.2.1 Problems

$\boxed{1.}$ *Find the derivative of the following functions at every (interior) point in their natural domain using the definition of the derivative:*

$$(a) f(x) = \frac{1}{x} \qquad (b) \quad g(x) = \sqrt{x}$$

$\boxed{2.}$ *Calculate the derivatives of the following functions with the appropriate rules:*

(a) $f(x) = \frac{2+x^2}{x^5}, \quad x \neq 0,$

(b) $g(x) = e^x \sin x, \quad x \in \mathbb{R}$

(c) $h(x) = x^2 \tan x, \quad x \in (0, \frac{\pi}{2})$

(d) $k(x) = \frac{2x-1}{x^2+1}, \quad x \in \mathbb{R}$

(e) $l(x) = (3x^2 - 2x)\ln(x), \quad x > 0$

(f) $m(x) = 3\sec x - 2\cot x, \quad x \in (0, \frac{\pi}{2})$

(g) $n(x) = (\sinh x)(\cosh x), \quad x \in \mathbb{R}.$

(h) $o(x) = e^{x^2+2x}, \quad x \in \mathbb{R}.$

(i) $p(x) = \ln(x^2 + 3), \quad x \in \mathbb{R}.$

(j) $q(x) = \sin(x + \cos x), \quad x \in \mathbb{R}.$

(k) $r(x) = \arcsin(2x - 1), \quad x \in (0, 1).$

(l) $s(x) = \arccos(\frac{2x}{1+x^2}), \quad x \in (-1, 1).$

(m) $t(x) = \arctan(\tanh x), \quad x \in \mathbb{R}.$

$\boxed{3.}$ *Determine if the following function is differentiable or not. If it is, calculate its derivative.*

$$f(x) = \begin{cases} x^2 \quad if \quad x \geq 0 \\ \\ -x^2 \quad if \quad x < 0 \end{cases} .$$

Is this function twice differentiable?

$\boxed{4.}$ *Find all values of a such that the following function is differentiable:*

$$h(x) = \begin{cases} (x+a)^2 \quad if \quad x \geq 1 \\ \\ 2a + a^2 + x \quad if \quad x < 1 \end{cases} .$$

$\boxed{5.}$ *If $f(x) = \dfrac{x^2 - 3x + 2}{x^2 + 1}$, $x \in \mathbb{R}$ then find the equation of a line which is tangent to the graph of $y = f(x)$ at the point $(0, 2)$. Draw the graphs of both the function and its tangent line.*

$\boxed{6.}$ *Let $g(x) = u(x)v(x)$, with x in some interval domain which is a common domain for the two "highly" differentiable functions u and v. Calculate $g''(x)$ in terms of the derivatives of u and v. What about $g'''(x)$, can you guess what is that going to be with calculating it?*

$\boxed{7.}$ *Let n be a non-negative integer. Prove that if P is a polynomial of degree n, and $a \neq 0$, then*

$$\frac{d}{dx}\left[\left(\frac{P(x)}{a} - \frac{P'(x)}{a^2} + ... + (-1)^n \frac{P^{(n)}(x)}{a^{n+1}}\right) e^{ax} \right] = P(x)e^{ax}, \quad x \in \mathbb{R}.$$

$\boxed{8.}$ *Prove the quotient rule.*

$\boxed{9.}$ *Prove the rule for the triple product $(fgh)' = f'gh + fg'h + fgh'$ and a similar one for the quotient:*

$$\left(\frac{1}{fgh}\right)' = -\frac{1}{fgh}\left(\frac{f'}{f} + \frac{g'}{g} + \frac{h'}{h}\right).$$

$\boxed{10.}$ *Prove the formula of differentiating the product of two functions: for $n \in \mathbb{N}$*

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^k g^{n-k}.$$

$\boxed{11.}$ *Find the derivative of the function* $g(x) = (\sin^2 x)^x$.

2.3 Implicit Differentiation

We are going to do four examples here. Let us start with a curve that looks implicit but it cab be treated as explicit, as we will see later: $x^2 + y^2 = 1$, the unit circle. Clearly the point $P := (3/5, 4/5)$ is a point on this circle. We are going to find the equation of the tangent line of this circle at the point P. For this purpose we employ a procedure which is going to be used in the examples of this type. The equation which we have for the circle, we think of it as a functional equation, i.e. $x^2 + y(x)^2 = 1$ and differentiate, we say implicitly, but it is really the chain rule that is used: $2x + 2yy' = 0$. At this time we substitute the coordinates of the point P: $2(\frac{3}{5}) + 2(\frac{4}{5})y' = 0$. The equation we get must be a solvable linear equation in y'. So, solving for y' gives $y' = -\frac{3}{4}$. Hence, the equation of the tangent line is $y - \frac{4}{5} = -\frac{3}{4}(x - \frac{3}{5})$ or

$$y = \frac{4}{5} + \frac{9}{20} - \frac{3x}{4} \Leftrightarrow \boxed{y = \frac{5}{4} - \frac{3x}{4}}.$$

The graph of the unit circle and the tangent line at P is included in Figure 2.1. The reason we said this is not really an implicit situation is because we can solve for y ($y > 0$) in terms of x and obtain an explicit expression $y = \sqrt{1 - x^2}$. Then $y'(x) = -\frac{2x}{2\sqrt{1-x^2}}$. So, $y' = -\frac{3}{4}$ as before.

If we want to take an example that would be really difficult to do it in explicit form (but possible, since in general algebraic equations cannot be solved in explicit form, i.e in terms of the elementary functions we have, if their degree is more or equal to 5), we may take the following curve: $x^3 + y^3 = 9y + x - 2$ and the point of tangency is $P := (2, 1)$. Differentiating implicitly gives $3x^2 + 3y^2y' = 9y' + 1$. Next, we substitute with the coordinates of P: $12 + 3y' = 9y' + 1$ which gives $y' = \frac{11}{6}$. Hence the equation of the tangent line if

$$y - 1 = \frac{11}{6}(x - 2) \Leftrightarrow \boxed{y = \frac{11x - 16}{6}}.$$

The graph of this cubic and the tangent line at P is included in Figure 2.2.

Let us take a look at a situation in which both x and y are related by an implicit equation and the third variable, the time t, is the independent variable. It is known

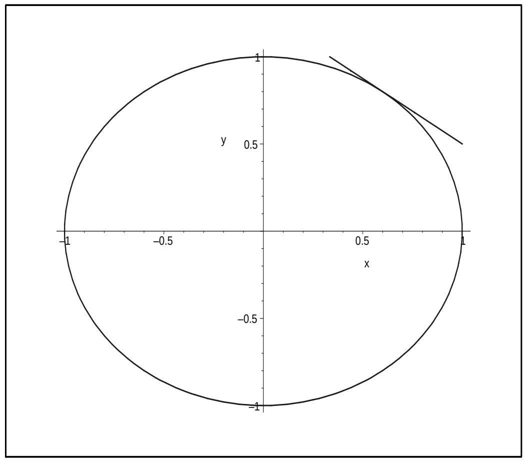


Figure 2.1: Unit circle and a tangent line

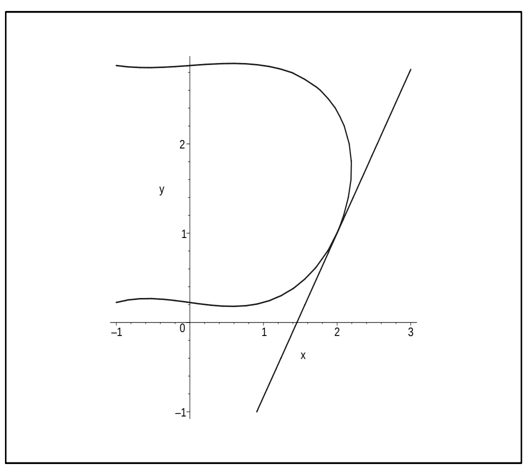


Figure 2.2: Cubic curve

that the planets revolve around the Sun in elliptical orbits and they move according to Kepler's law: the radius connecting the planet to the Sun wipes out an area that varies proportionally with time. Let us suppose that the equation of the trajectory of a planet is given in polar form by the equation

$$r = \frac{a(1-e^2)}{1+e\cos\theta}, 0 \le e < 1,$$

where e is usually called eccentricity (and is pretty small for the planets closer to the Sun), a is the semi-major axis.

It is easy to see that the formula of the area of a triangle ABC is given by $A = \frac{bc\sin A}{2}$ and so if we assume that the triangle has the vertex A at the origin (the Sun) and vertices B and C on the trajectory at time t and $t+\epsilon$, with $\epsilon > 0$ very small, we see that

$$\frac{d}{dt}A(t) = \frac{r^2}{2}\frac{d\theta}{dt}.$$

Let us suppose that it takes T days (Earth days) to complete a full revolution. Then $\theta(T) = 2\pi$ and $A(t) = area(Ellipse)\frac{t}{T}$ so

$$\frac{d\theta}{dt} = \frac{area(Ellipse)}{T}\frac{2}{r^2}$$

.

The area of the ellipse, is in this case, equal to $\pi a^2\sqrt{1-e^2}$. We will learn in Calculus II and Calculus III that the equation of the arc-length is given by $L = \int_{\theta_1}^{\theta_2}\sqrt{r^2 + \frac{dr}{d\theta}^2}d\theta$. So, the linear speed of the a planet is given by

$$v = \frac{dL}{dt} = \sqrt{r^2 + \frac{dr}{d\theta}^2}\frac{d\theta}{dt} = \sqrt{r^2 + \frac{dr}{d\theta}^2}\frac{2\pi a^2\sqrt{1-e^2}}{Tr^2}.$$

Let us compute the speed at $t = 0$, in other words, when the planet is at the closest distance to the Sun. Differentiating with respect to θ, we get

$$\frac{dr}{d\theta} = \frac{a(1-e^2)e\sin\theta}{(1+e\cos\theta)^2} \Rightarrow \frac{dr}{d\theta}|_{\theta=0} = 0.$$

Therefore,

$$v(0) = \frac{2\pi a^2\sqrt{1-e^2}}{Ta(1-e)} = \frac{2\pi a}{T}\sqrt{\frac{1+e}{1-e}}.$$

We can think of the quantity $\frac{2\pi a}{T}$ as the average speed and call it v_{av}. We get the following formulae for the speed of a planet at the Aphelion and Perihelion

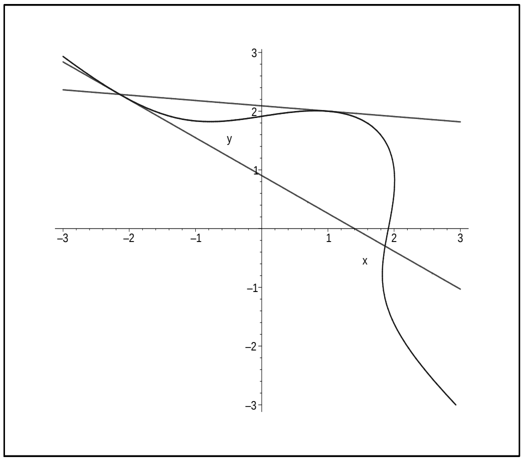


Figure 2.3: Another Cubic curve

$$v_{ap} = v_{av}\sqrt{\frac{1-e}{1+e}}, \quad v_{peri} = v_{av}\sqrt{\frac{1+e}{1-e}}.$$

A nice applet that let you check the movement for an arbitrary planet around the Sun can be found at

$$http://galileo.phys.virginia.edu/classes/109N/more\_stuff/flashlets/kepler6.htm$$

Finally let us take a look at a problem which provides a great deal of ideas in mathematics. We consider the curve $x^3+y^3=xy+7$. A point on this curve of integer coordinates is $P(1,2)$. The usual technique to determine the equation the tangent line to this curve at P gives: $3x^2+3y^2y'=y+xy'$ or $3+12y'=2+y'$. Solving for y' we get $y'=-\frac{1}{11}$. Hence the tangent line has equation $y=2-(x-1)/11=\frac{23-x}{11}$. We include a picture of this curve and the tangent line at P in Figure 2.3. Let us observe that the tangent line intersects the curve at another point. What is interesting is that this point has rational coordinates too. In other words the equation $x^3+(23-x)^3/11^3=x(23-x)/11+7$ has a double zero at $x=1$ and the third zero at $x=-\frac{15}{7}$. This gives the point of intersection of the tangent line with the curve at $Q(-\frac{15}{7},\frac{16}{7})$. Now we can repeat the procedure with the tangent line at Q. We see that this shows that the equation $x^3+y^3=xy+7$ has possibly infinitely many points on it of rational coordinates (unless we get back to P or other such point already constructed). It turns out that one can define some algebraic structure (similar to the addition of numbers) on such points and the part of mathematics which studies these structures is usually referred to as *Elliptic Curves*. These days there are applications of this theory in Cryptography (see [1]).

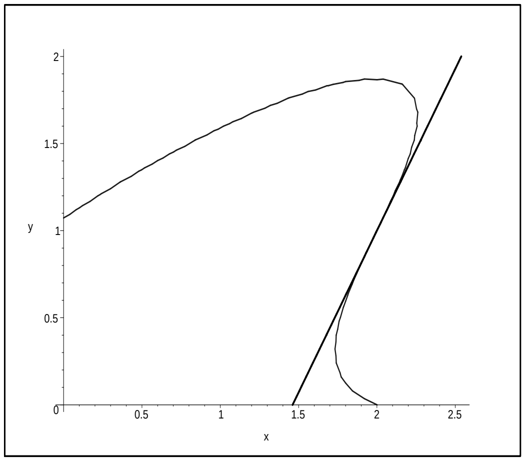


Figure 2.4: Problem 1

2.3.1 Problems

$\boxed{1.}$ *Find the equation of the tangent line at the point $P := (2,1)$ to the curve $(x-2y)^3 - (2x-y)^2 - x + y + 10 = 0$ (see Figure 2.4). Answer: $7y + 19 - 13x = 0$.*

$\boxed{1.}$ *Find the point of intersection of the tangent line to*

$$\mathcal{C}: \quad x^3 + y^3 - xy = 7$$

at $Q(-\frac{15}{7}, \frac{16}{7})$ with $\mathcal{C}$. Answer: $R = (\frac{97455}{52297}, -\frac{15584}{52297})$ (Maple problem)

2.4 Derivatives of higher order

In this section we will take a look at some of the functions whose derivatives can be computed for all orders. The simplest case is $f(x) = e^x$, $x \in \mathbb{R}$. It is clear that $f^{(n)}(x) = e^x$ for all $n \in \mathbb{N}$.

The next situation when we can find basically all the derivatives is a polynomial function p. If the degree of this polynomial has degree d, $d \in mathbbN$, then $p^{(n)}(x) = 0$ for all $n \geq d+1$. The first d derivatives can be calculated with the Power Rule. This has a certain consequence later on then we are going to talk about the Taylor polynomial and Taylor series for real analytical functions.

One other case which is really simple is $g(x) = \frac{1}{x}$ for, say, $x > 0$. One can check that $g'(x) = -\frac{1}{x^2}$, $g''(x) = \frac{2}{x^3}$, $g'''(x) = -\frac{6}{x^4}$, and so the pattern we have here is

$$g^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}}, \quad x > 0, \ n \in \mathbb{N}.$$

One example which is a little more difficult: $f(x) = xe^x$, $x \in \mathbb{R}$. Using the Product Rule, one can find the first few derivatives and obtain $f'(x) = (x+1)e^x$, $f''(x) = (x+2)e^x$,.... Hence, we guess that the general formula is $f^{(n)}(x) = (x+n)e^x$.

In general, to establish a formula like these, we use in formal mathematics, a proof, most of the time in these kind of examples, called (mathematical) induction proof or proof by induction. The name comes from the fact that the proof is based on the Mathematical Induction Principle (PMI).

Let us do an example like that. Suppose we take the function $f(x) = (1+x)^{1/2}$, $x \geq 0$. If we calculate the first derivative we get $f'(x) = \frac{1}{2}(1+x)^{-1/2}$, $x \geq 0$. Then, the second derivative is $f''(x) = \frac{-1}{4}(1+x)^{-3/2}$, $x \geq 0$. Another step will give us the idea of how the derivative is going to look in general: $f'''(x) = \frac{3}{8}(1+x)^{-5/2}$, $x \geq 0$. We want to show by induction on $n \geq 2$ that

$$f^{(n)}(x) = (-1)^{n+1}\frac{(2n-3)!!}{2^n}(1+x)^{-\frac{2n-1}{2}}, \ x \geq 0. \tag{2.4}$$

(We used the notation $(2k-1)!! = 1(3)(5)\cdots(2k-1)$ for $k \in \mathbb{N}$.)

We see that (2.4) is true for $n = 2$. Assume (2.4) is true for some $n \geq 2$. Then

$$f^{(n+1)}(x) = (-1)^{n+2}\frac{(2n-3)!!(2n-1)}{2^{n+1}}(1+x)^{-\frac{2n+1}{2}}, \ x \geq 0$$

which is (2.4) for $n+1$ instead of n. The PMI applies and we conclude the (2.4) is true for all $n \geq 2$.

The possibility of computing all the derivatives of a function is related to the Taylor expansion which we will see later in Calculus II. We include here a few such expansions:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots, \ x \in \mathbb{R},$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ..., \ x \in \mathbb{R},$$

$$\arctan(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - ..., \quad |x| < 1.$$

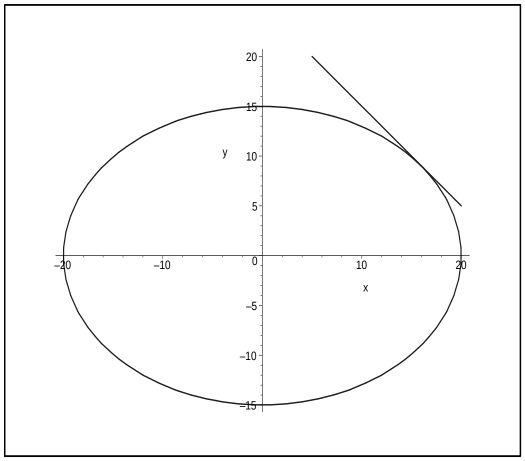


Figure 2.5: Ellipse $\frac{x^2}{400} + \frac{y^2}{225} = 1$

2.4.1 Problems

$\boxed{1.}$ *Find the nth-derivative of* $g(x) = \frac{1}{1+\frac{x}{2}}$, $x > 0$. *Use the pattern you discovered to give a reasonable calculational formula for* $g^{(2011)}(736)$.

$\boxed{2.}$ *Find the nth-derivative of* $h(x) = xe^{-x}$, x *real number.*

$\boxed{3.}$ *Let* f *be the function defined for all* x: $f(x) = x \sin x$. *What is the 100th derivative of* f*?*

2.5 Related rates problems

In this section we are going to show how the derivative concepts can be used to arrive at some answers for reasonable questions involving movement. First, let us start with a geometry question similar to the movement of the planets around the Sun. Suppose a point P of coordinates (x, y) rotates on the ellipse (Figure 2.5)

$$\frac{x^2}{20^2} + \frac{y^2}{15^2} = 1$$

in counterclockwise direction in such a way the distance to the origin changes in a constant way ($|\frac{d}{dt}OP|=1$). The question is, what are the values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ at the point $(16,9)$? We know that $PO=\sqrt{x^2+y^2}$ and so $-1=\frac{d}{dt}OP=\frac{2x\frac{dx}{dt}+2y\frac{dy}{dt}}{2\sqrt{x^2+y^2}}$. Also, if we differentiate the the equation of the ellipse, implicitly with respect to t, we get $\frac{2x}{20^2}\frac{dx}{dt}+\frac{2y}{15^2}\frac{dy}{dt}=0$, or $\frac{dx}{dt}=-\frac{20^2(9)}{15^2(16)}\frac{dy}{dt}=-\frac{dy}{dt}$. Hence, $\frac{dx}{dt}=-\frac{dy}{dt}=-\frac{\sqrt{337}}{7}\approx -2.622508537$.

2.5.1 Problems

$\boxed{1.}$ *Let a and b be positive real numbers such that $a>b$. The point $P(x,y)$ moves on the ellipse of equation*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

in such a way the distance to the origin has equation $PO=\frac{a+b}{2}+\frac{a-b}{2}\cos 2t$ where t is the time measured from initial position $(a,0)$ at $t=0$. How fast is the point P moving at time $t=\frac{\pi}{4}$? In other words, what is $v=\sqrt{\frac{dx}{dt}^2+\frac{dy}{dt}^2}$ when $t=\frac{\pi}{4}$?

$\boxed{2.}$ This problem appears in [6] (Problem 39, page 170). *A conical watering pail has a grid of holes uniformly distributed over all of its surface. The water flows out through the holes at a rate of kA m^3/min, where k is a constant and A is the surface area in contact with the water. Calculate the rate at which the water level changes ($\frac{dh}{dt}$) at a level of the water of h meters.*

2.6 Newton's Approximation Scheme

In general equations of the form $f(x)=0$, with f an elementary function, are not solvable in terms of our elementary functions (in other words, f^{-1} may exist locally but it is not elementary), and so we usually have to approximate the solutions when we know they exist. One of the methods of approximating such solutions is given by the Newton's Method which consists of taking a first guess, say x_0, and then constructing the tangent line at $(x_0,f(x_0))$ to the the graph of $y=f(x)$, $y=f(x_0)+f'(x_0)(x-x_0)$, and then taking the intersection of this line with $y=0$, i.e. solving the equation $0=f(x_0)+f'(x_0)(x-x_0)$ for x and considering this intersection the next iteration:

$$x_1:=x_0-\frac{f(x_0)}{f'(x_0)}.$$

Of course, we need to assume that $f'(x_0)\neq 0$ and that is usually happening if we are in an interval I (containing the solution of $f(x)=0$) where the sequence of iterations defined recursively by

$$x_{n+1} := x_n - \frac{f(x_n)}{f'(x_n)}, n \geq 0, \tag{2.5}$$

is well defined and the derivative of f is bounded away from zero ($|f'(x)| \geq \delta > 0$ for all $x \in I$), and one can study the convergence of the sequence $\{x_n\}$ to the solution of $f(x) = 0$, say α.

Usually the convergence is quadratic, in the sense that the error sequence $\epsilon_n = |x_n - \alpha|$, satisfies some inequality of the form $\epsilon_{n+1} \leq C\epsilon_n^2$ for some constant C.

One classical result here is the following theorem

Theorem 2.6.1. *(Newton-Raphson Theorem). Assume that $f : [a, b] \to \mathbb{R}$ is a twice differentiable function, and $f(\alpha) = 0$ form some $\alpha \in [a, b]$. If $f'(\alpha) \neq 0$, then there exists an $\epsilon > 0$ such that the sequence defined by the iteration (2.5) converges to α for any initial approximation $x_0 \in (\alpha - \epsilon, \alpha + \epsilon)$.*

Let us look at an example which goes back to Babylonians: approximating the square root of a number. Suppose that $a > 0$ and $f(x) = x^2 - a$. Then the iteration (2.5) can be written in the form

$$x_{n+1} = x_n - \frac{x_n^2 - a}{2x_n} = \frac{1}{2}(x_n + \frac{a}{x_n}).$$

Sample Test II and Solutions

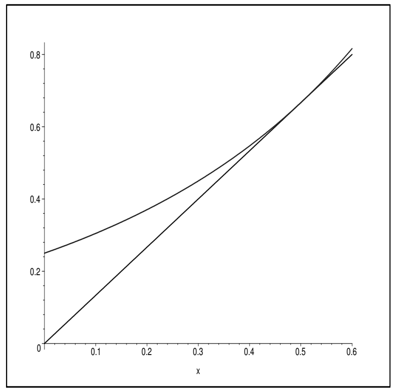
$\boxed{\mathbf{1.}}$ *Consider the function $f(x) = \dfrac{1 + x}{(2 - x)^2}$. Find the equation of the tangent line to the graph of $y = f(x)$ at the point $(\frac{1}{2}, \frac{2}{3})$. (**Bonus:** Use a graphing calculator to draw the graph of $y = f(x)$ and the above tangent line.)*

Solution: Using the quotient rule and the product rule, we get

$$f'(x) = \frac{(2 - x)^2 - (1 + x)2(2 - x)(-1)}{(2 - x)^4} = \frac{2 - x + 2 + 2x}{(2 - x)^3} \text{ or}$$

$$f'(x) = \frac{4 + x}{(2 - x)^3}.$$

which gives $f'(1/2) = \frac{9}{2}\frac{8}{27} = \frac{4}{3}$. Hence, the equation is $y = \frac{2}{3} + \frac{4}{3}(x - \frac{1}{2})$ or $\boxed{y = \dfrac{4x}{3}}$. The graph f and of the tangent line at $(1/2, 2/3)$ on the interval $[0, 0.6]$ is:



$\boxed{\mathbf{2.}}$ *Compute the derivative for each of the following functions:*

(a) $f(x) = (\cos 2x)^3(\sin 3x)^2$ (b) $g(x) = \sqrt{x - x^2}$

(c) $h(x) = \ln(x + \sqrt{x^2+1})$ (d) $i(x) = e^{-2x}\sec(3x) - \arctan(\sin x)$

Solution: (a) Using the product formula and the derivatives of *sine* and *cosine*, we get

$$\boxed{f'(x) = -6(\cos 2x)^2 \sin 2x \sin^2 3x + 6(\cos 2x)^3 \sin 3x \cos 3x}.$$

(b) The derivative of g is

$$g'(x) = \frac{1-2x}{2\sqrt{x-x^2}}$$

(c) We have seen that $h'(x) = \frac{1}{\sqrt{x^2+1}}$.

(d) The derivative is

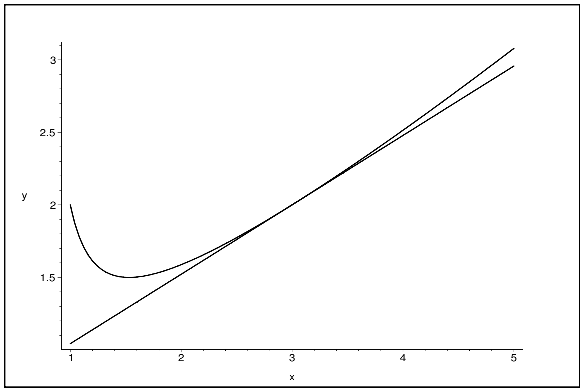
$$\boxed{i'(x) = -2e^{-2x}\sec(3x) + 3e^{-2x}\sec(3x)\tan(3x) - \frac{\cos x}{1+\sin^2 x}.}$$

$\boxed{\mathbf{3.}}$ *Determine the equation of the tangent line to the graph of equation*

$$x\ln(y+2) - y\ln(3x-1) = 0$$

at the point $(3, 2)$.

Solution: We use implicit differentiation to get $\ln(y+2) + x\frac{y'}{y+2} - y'\ln(3x-1) - y\frac{3}{3x-1} = 0$. Substituting $x = 3$ and $y = 2$ gives $\ln 4 + 3y'/4 - y'\ln 8 - 6/8 = 0$. Solving for y', we obtain $y' = \frac{3/4-\ln 4}{3/4-\ln 8} \approx 0.479$. Hence the equation of the tangent line is $\boxed{y = 2 + (x-3)\frac{3-4\ln 4}{3-4\ln 8}}$.
The graph of the implicit equation and of the tangent line is included next:



$\boxed{4.}$ *Find the n^{th} derivative of $f(x) = \frac{x+1}{3x+2}$.*

Solution: Since

$$f(x) = \frac{3x+3}{3(3x+2)} = \frac{3x+2}{3(3x+2)} + \frac{1}{3(3x+2)} = \frac{1}{3} + \frac{1}{3(3x+2)}$$

we get $f'(x) = -\frac{3}{3(3x+2)^2} = -\frac{1}{(3x+2)^2} = -(3x+2)^{-2}$. Then the second derivative

$f''(x) = 2(3)(3x+2)^{-3}$, $f'''(x) = -(3!)(3^2)(3x+2)^{-4}$, $f^{(4)}(x) = (4!)3^3(3x+2)^{-5}$, ... and in general

$$\boxed{f^{(n)}(x) = (-1)^n n! 3^{n-1}(3x+2)^{-(n+1)},\ n \geq 1.}$$

$\boxed{5.}$ *Differentiate $y = (1+2x)^{x^2}$.*

Solution: Using the formula $(u^v)' = vu^{v-1}u' + u^v(\ln u)v'$, we obtain

$$\boxed{y' = 2x^2(1+2x)^{x^2-1} + 2x(1+2x)^{x^2}\ln(1+2x)}.$$

Chapter 3

Applications

Quotation: *Euclid taught me that without assumptions there is no proof. Therefore, in any argument, examine the assumptions. —Eric Temple Bell (1883-1960)*

"The word theorem in English derives from the Greek word theoreo which is a verb that has to do with "the quality of attention that has the intention of mind which contemplates an object studiously and attentively." From Bullinger, E. W "A Critical Lexicon and Concordance to the ENglish and Greek New Testament", Kregel Publications Grand Rapids, Michigan 1908.

"Like fire in a piece of flint, knowledge exists in the mind. Suggestion is the friction which brings it out." Vivekananda

3.1 Fermat's Theorem, Rolle's Theorem, Mean Value Theorem, Cauchy Theorem

Let us start with a theorem that is essential in showing all the important theorems in this section. In what follows we assume that a, b are two real numbers such that $a < b$.

Theorem 3.1.1. *(***Extreme Value Theorem***)Every continuous function f on a closed interval $[a, b]$ is bounded. Moreover, the bounds of f are attained, e.i. there exist two points α and β in $[a, b]$ such that $f(\alpha) \le f(x) \le f(\beta)$ for all $c \in [a, b]$.*

Sketch of proof. If the function is not bounded then there exists a sequence x_n such that $|f(x_n)| \to \infty$. There must be a point in $[a, b]$ to which the sequence x_n accumulates, or in other words, there must be a subsequences x_{n_k} convergent to a point $c \in [a, b]$. Since f is assumed to be continuous $|f(c)| = \lim_{k\to\infty} |f(x_{n_k})| = \infty$. This is not possible. Hence the range of f must be a bounded interval because of the Intermediate Value Theorem

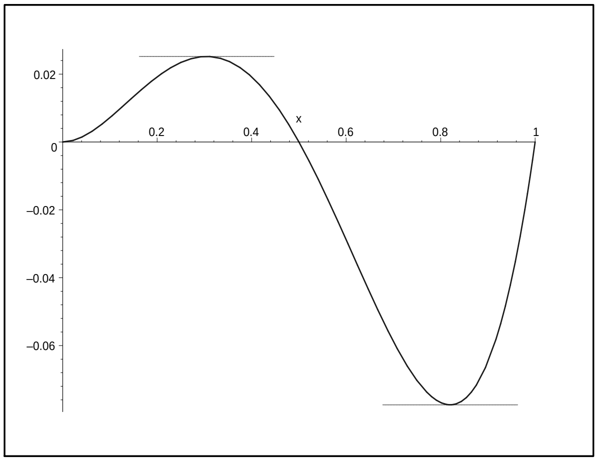


Figure 3.1: Example for Rolle's Theorem

(which we have seen before). This interval cannot be of the form $[c, d)$ because of the continuity argument used above. ■

The assumption that we have a closed interval is critical. If we only take an open interval, like $f(x) = \frac{1}{x(1-x)}$ defined on $(0, 1)$, we see that this function is continuous and unbounded.

Theorem 3.1.2. *(***Fermat's Theorem***) Let f be a differentiable function on (a, b) and continuous on $[a, b]$. If $c \in (a, b)$ is a point of local maximum or local minimum, then $f'(c) = 0$.*

Proof. Without loss of generality we may assume that $f(c) \leq f(x)$ for all $x \in (c-\epsilon, c+\epsilon)$ for some small $\epsilon > 0$. By definition of the derivative we must have

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

If we let $x < c$, we have $x - c < 0$ and so $\frac{f(x)-f(c)}{x-c} \leq 0$ which implies $f'(c) \leq 0$. If we let $x > c$, then $x - c > 0$ and $\frac{f(x)-f(c)}{x-c} \geq 0$ which implies so $f'(c) \geq 0$. This is possible only if $f'(c) = 0$. ■

Theorem 3.1.3. *(***Rolle's Theorem***) Let f be a differentiable function on (a, b) and continuous on $[a, b]$. If $f(a) = f(b)$, then, there exists a $c \in (a, b)$ such that $f'(c) = 0$.*

Proof. The function f is either a constant function, in which case the conclusion is clearly true, or a non constant function. Hence, we have a point x_0 where either $f(x_0) < f(a) =$

$f(b)$ or $f(x_0) > f(a) = f(b)$. Without loss of generality we may assume that $f(x_0) < f(a) = f(b)$. Then let c be the point given by Theorem 3.1.1 such that $f(c) \leq f(x)$ for all $x \in [a, b]$. Since $f(c) \leq f(x_0) < f(a) = f(b)$ we must have $c \in (a, b)$. By Fermat's Theorem, we must have $f'(c) = 0$. ■

Theorem 3.1.4. *(***Mean Value Theorem***) Let f be a differentiable function on (a, b) and continuous on $[a, b]$. Then, there exists a $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Sketch of Proof. Let us consider the function $g(x) = f(x) - mx$ where $m = \frac{f(b)-f(a)}{b-a}$. One can see that this function satisfies the hypothesis of Role's theorem. Hence, there must be a $c \in (a, b)$ such that $g'(c) = 0$. This implies the desired conclusion. ■

The next corollary is very close to the First Law of Classical Mechanics: "The velocity of a body remains constant unless the body is acted upon by an external force."

Corollary 3.1.5. *(***"First Principle of Classical Mechanics"***) Let f be a differentiable function on (a, b) and continuous on $[a, b]$. If $f'(x) = 0$ for all $x \in (a, b)$, then there exists a constant C such that $f(x) = C$ for all $x \in [a, b]$.*

Proof. Suppose by way of contradiction that f is not a constant. Then we can find $x_1 < x_2$, $a \leq x_1 < x_2 \leq b$, such that $f(x_1) \neq f(x_2)$. Then by Mean Value Theorem applied to f on $[x_1, x_2]$ we find a $c \in (x_1, x_2)$ such that $f'(c) = \frac{f(x_2)-f(x_1)}{x_2-x_1} \neq 0$. This contradiction gives the result. ■

Radioactive Decay: Here is an application of this result. Let us assume $a \in \mathbb{R}$, $a > 0$ (decay constant). Suppose that we have a function f which satisfy the differential equation:

$$f'(x) = -af(x) \text{ for all } x \in \mathbb{R},$$

which is saying that the amount of radioactive substance rate of change (decreasing) is proportional to the amount of radioactive substance left. Let us show that the only functions which satisfy this equation are $f(x) = Ce^{-ax}$, for all $x \in \mathbb{R}$. Indeed, we look at the newly defined function $g(x) = f(x)e^{ax}$ and compute its derivative: $g'(x) = f'(x)e^{ax} + af(x)e^{ax} = 0$ for all $x \in \mathbb{R}$. By Corollary 3.1.5, we must have $g(x) = C$ for all $x \in \mathbb{R}$. Hence $f(x) = Ce^{-ax}$ for all $x \in \mathbb{R}$.

"Propagation of light": Let us show that the differential equation $f'' + f = 0$ has only the solution $f(x) = C_1 \sin x + C_2 \cos x$, $x \in \mathbb{R}$. We consider the new function $g(x) = f'(0) \sin x + f(0) \cos x - f(x)$. Let us observe that $g(0) = g'(0) = 0$ and $g'' + g = 0$. Let us look at another function $h(x) = g(x)^2 + g'(x)^2$, $x \in \mathbb{R}$. We observe that $h'(x) = 2g(x)g'(x) + 2g'(x)g''(x) = 0$, $x \in \mathbb{R}$. Hence by Corollary 3.1.5, $h(x) = C$ for all $x \in \mathbb{R}$. Since $h(0) = 0$ we see that $h(x) = 0$ for all $x \in \mathbb{R}$. Therefore $g(x) = 0$ for all $x \in \mathbb{R}$. So, $f(x) = C_1 \sin x + C_2 \cos x$, $x \in \mathbb{R}$.

Theorem 3.1.6. (**Cauchy's Theorem**) *Let f, g be two functions continuous on $[a,b]$ ($a<b$), differentiable on (a,b) such that $g'(x)\neq 0$ for all $x\in(a,b)$. Then there exists a $\xi\in(a,b)$ such that*

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(\xi)}{g'(\xi)}.$$

Proof. Let us consider the function $h(x)=f(x)-kg(x)$ where $k=\frac{f(b)-f(a)}{g(b)-g(a)}$. This number is well defined since $g(b)=g(a)$ would imply by Rolle's Theorem that $g'(c)=0$ for some $c\in(a,b)$, which is not possible by our assumption. We apply Rolle's Theorem to h on $[a,b]$. Clearly h is continuous and differentiable on $[a,b]$ (resp (a,b)). Also, $h(b)=h(a)$ is equivalent to $f(b)-kg(b)=f(a)-kg(a)$ or $k=\frac{f(b)-f(a)}{g(b)-g(a)}$ (true by definition of k). Hence we must have a $\xi\in(a,b)$ in such a way, that $h'(\xi)=0$. This is equivalent to $f'(\xi)-kg(\xi)=0$ or $\frac{f'(\xi)}{g'(\xi)}=k$. ■

Here is another application of the sort of differential equation we have seen before.

Problem: *Let us assume that f is a differential function on some interval $I=(a,b)$ such that $f'(x)=f(x)^2$ and $f(x)\neq 0$ for all $x\in I$. Show that there exists a constant $C\notin I$ such that $f(x)=\frac{1}{C-x}$ for all $x\in I$.*

Proof. We consider $g(x)=\frac{1}{f(x)}$ which is well defined for $x\in I$. Then $g'(x)=\frac{-f'(x)}{f(x)^2}=-1$ and so $(g(x)+x)'(x)=0$. Hence $g(x)+x=C$ for some constant C. This implies $f(x)=\frac{1}{C-x}$ for $x\in I$. It is clear that $C\notin I$. ■

3.1.1 Problems

$\boxed{1.}$ *Let $a>0$ and f a function twice differentiable on $\mathbb{R}$ such that $f''(x)+a^2f(x)=0$ for all $x\in\mathbb{R}$. Show that there exists two constants C_1 and C_2 such that $f(x)=C_1\sin ax+C_2\sin ax$ for all $x\in\mathbb{R}$.*

$\boxed{2.}$ *Consider a differentiable function f is a differential function on some interval $I=(a,b)$ such that $f'(x)=f(x)^3$ and $f(x)\neq 0$ for all $x\in I$. Show that there exists a constant C, such that $f(x)=\frac{\pm 1}{\sqrt{C-2x}}$, $x\in I$.*

$\boxed{3.}$ [**Darboux Property for derivatives**] *Consider $f:[a,b]\to\mathbb{R}$ a differentiable function and some real number s such that $f'(a)<s<f'(b)$. Follow the following steps to prove the Darboux Property for derivatives (see Lars Olsen [4]):*

(i) show that $u(x)=\begin{cases}\frac{f(x)-f(a)}{x-a} & \text{if } x>a\\ f'(a) & \text{if } x=a\end{cases}$ *and* $v(x)=\begin{cases}\frac{f(b)-f(x)}{b-x} & \text{if } x<b\\ f'(b) & \text{if } x=b\end{cases}$ *are continuous functions.*

(ii) check that $t=u(b)=v(a)$ and if $s=t$ then we can apply MVT to f and

conclude that $m = f'(c)$ for some $c \in (a, b)$.

(iii) if $s < t$ we can apply IVT to u and then MVT to f and conclude that $m = f'(c)$ for some $c \in (a, b)$.

(ii) if $t < s$ we can apply IVT to v and then MVT to f and conclude that $m = f'(c)$ for some $c \in (a, b)$.

$\boxed{4.}$ *Consider a differentiable function f on $[-1, 1]$ such that $f(-1) = -3$, $f(0) = -5$ and $f(1) = 2$. Prove that there is a point $c \in (-1, 1)$ such that $f'(c) = 4$.*

$\boxed{5.}$ **[Putnam Exam]***Let f be a three times differentiable function on $\mathbb{R}$ having at least five distinct real zeroes. Show that*

$$f + 6f' + 12f'' + 8f'''$$

has at least two distinct real zeroes.

3.2 L'Hospital's Rule

L'Hospital Rule is a technique used in the computation of limits in order to reduce them to elementary ones. There are two main cases in which one uses L'Hospital's Rule. Let us start with the case when the limit of the second function is ∞.

Theorem 3.2.1. *Let us assume that f and g are two functions defined on some domain D containing a as a limit point. In addition we know that $g(x) \nearrow \infty$ (it goes increasingly to infinity, i.e. $g'(x) > 0$, as $x \in D$ goes to a) and $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$. Then $\lim_{x\to a} \frac{f(x)}{g(x)} = L$.*

Proof Sketch: We fix an $\epsilon \in (0, 1)$. Let us assume that if $0 < |x - a| < \delta_1$ we have $|\frac{f'(x)}{g'(x)} - L| < \frac{\epsilon}{4}$. For u fixed but satisfying the same inequality, i.e. $0 < |u - a| < \delta_1$, we look at

$$|\frac{f(x)}{g(x)} - \frac{f(x) - f(u)}{g(x) - g(u)}| = \left|\frac{\frac{f(u)}{g(x)} - \frac{f(x)}{g(x)}\frac{g(u)}{g(x)}}{1 - \frac{g(u)}{g(x)}}\right| = \left|\frac{\frac{f(u)-Lg(u)}{g(x)} - (\frac{f(x)}{g(x)} - L)\frac{g(u)}{g(x)}}{1 - \frac{g(u)}{g(x)}}\right|,$$

we observe that if we let $0 < |x - a| < \delta_2 = \delta_2(u, \epsilon) < \delta_1$, $g(x)$ is big enough to insures that

$$|\frac{f(x)}{g(x)} - \frac{f(x) - f(u)}{g(x) - g(u)}| \leq \frac{\epsilon}{4} + |\frac{f(x)}{g(x)} - L|\frac{\epsilon}{4}.$$

By Cauchy's Theorem we have $\frac{f(x)-f(u)}{g(x)-g(u)} = \frac{f'(c_{c,u})}{g'(c_{x,c})}$ with $c_{x,u}$ between x and u which makes it satisfy $0 < |c_{x,u} - a| < \delta_1$. Hence

$$|\frac{f(x)-f(u)}{g(x)-g(u)} - L| = |\frac{f'(c_{c,u})}{g'(c_{x,c})} - L| < \frac{\epsilon}{4}.$$

Therefore, one can use the triangle inequality, and the above inequalities to get

$$|\frac{f(x)}{g(x)} - L| \le |\frac{f(x)}{g(x)} - \frac{f(x)-f(u)}{g(x)-g(u)}| + |\frac{f(x)-f(u)}{g(x)-g(u)} - L| < \frac{\epsilon}{2} + |\frac{f(x)}{g(x)} - L|\frac{\epsilon}{4} \implies$$

$$(1-\frac{\epsilon}{4})|\frac{f(x)}{g(x)} - L| \le \frac{\epsilon}{2} \implies |\frac{f(x)}{g(x)} - L| < \frac{\epsilon}{2}(\frac{4}{3}) < \epsilon. \quad \blacksquare$$

Let's see some applications of this very powerful rule. We have some limits in Chapter I which we now prove with this rule. For instance,

$$\lim_{x\to\infty} \frac{x^2}{e^x} \stackrel{L'H}{=} \lim_{x\to\infty} \frac{2x}{e^x} = \stackrel{L'H}{=} \lim_{x\to\infty} \frac{2}{e^x} = 0.$$

In a similar way one can show any of the cases in (1.23). Clearly, (1.24) follows from (1.23), but we can use L'Hospital, for example,

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} \stackrel{L'H}{=} \lim_{x\to\infty} \frac{1/x}{1/(2\sqrt{x})} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0.$$

The second version of L'Hospital Rule is when both functions approach 0.

Theorem 3.2.2. *Let us assume that f and g are two functions defined on some domain D containing a as a limit point. In addition we know that $f(x), g(x) \to 0$ and $g'(x) \neq 0$, as $x \in D$ goes to a. Finally, if $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$. Then $\lim_{x\to a} \frac{f(x)}{g(x)} = L$.*

The proof goes the same way as before and we let it as an exercise for the reader.

Let's look at some of the fundamental limits in Chapter I. First, we have for the second fundamental limit (1.16)

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} \stackrel{L'H}{=} \lim_{x\to 0} \frac{1/(1+x)}{1} = \lim_{x\to 0} \frac{1}{1+x} = 1.$$

Notice that we have a vicious circle here since we arrived at the derivatives of the elementary functions by using the fundamental limits. So, when we define these transcendental functions more precisely, we will have to prove those limits independent of the L'Hospital's Rule or any differentiation technique. Let us show one other example of how can we obtain pretty good information about a function with L'Hospital's Rule. Let us prove that $\sin x = x - \frac{x^3}{6} + O(x^5)$, here we used a classical notation, $f(x) =$

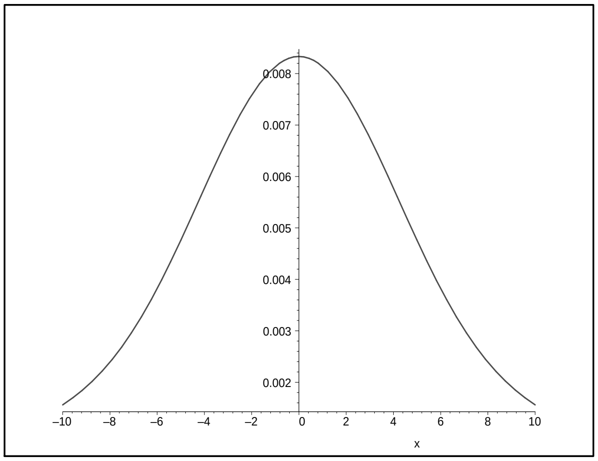


Figure 3.2: Graph of $y = \frac{\sin x - x + \frac{x^3}{6}}{x^5}$, $x \neq 0, x \in [-10, 10]$

$g(x) + O(h(x))$, which means that $\frac{f(x)-g(x)}{h(x)}$ is bounded as a function of x in a certain domain. Indeed, first

$$\lim_{x\to 0} \frac{\sin x - x}{x^3} \overset{L'H}{=} \lim_{x\to 0} \frac{\cos x - 1}{3x^2} \overset{L'H}{=} \lim_{x\to 0} \frac{-\sin x}{6x} = -\frac{1}{6}, \text{ and}$$

$$\lim_{x\to 0} \frac{\sin x - x + \frac{x^3}{6}}{x^5} \overset{L'H}{=} \lim_{x\to 0} \frac{\cos x - 1 + \frac{x^2}{2}}{5x^4} \overset{L'H}{=} \lim_{x\to 0} \frac{-\sin x + x}{20x^3} = \frac{1}{120}.$$

This implies that $\sin x = x - \frac{x^3}{6} + O(x^5)$ for all $x \in \mathbb{R}$. What is interesting is that a more precise statement is true, as the Figure 3.5 suggests, and its proof is left as an exercise:

$$|\sin x - x + \frac{x^3}{6}| \leq \frac{|x|^5}{120}, \; x \in \mathbb{R}.$$

3.2.1 Problems

$\boxed{1.}$ *Prove the second version of L'Hospital's Rule.*

$\boxed{2.}$ *Prove the inequality*

$$|\sin x - x + \frac{x^3}{6}| \le \frac{|x|^5}{120},\ \ x \in \mathbb{R}.$$

$\boxed{3.}$ *Prove the inequality*

$$|\cos x - 1 + \frac{x^2}{2}| \le \frac{x^4}{24},\ \ x \in \mathbb{R}.$$

$\boxed{4.}$ *Use L'Hospital's Rule to show that*

$$\lim_{x\to 0} \frac{e^x - 1 - x - \frac{x^2}{2}}{x^3} = \frac{1}{6}.$$

$\boxed{5.}$ *Use L'Hospital's Rule to show that*

$$\lim_{x\to 0} \frac{\tan x - x}{x^3} = \frac{1}{3}.$$

$\boxed{6.}$ *Use L'Hospital's Rule to show that*

$$\lim_{x\to 0} \frac{\arctan x - x + \frac{x^3}{3}}{x^5} = \frac{1}{5}.$$

$\boxed{7.}$ *Prove the inequality*

$$|\arctan x - x + \frac{x^3}{3}| \le \frac{|x|^5}{5},\ \ x \in \mathbb{R}.$$

$\boxed{8.}$ *Use L'Hospital's Rule to show that*

$$\lim_{x\to 0} \frac{\sqrt{1+x} - 1 - \frac{x}{2} + \frac{x^2}{8}}{x^3} = \frac{1}{16}.$$

$\boxed{9.}$ *Prove the inequality*

$$|\sqrt{1+x} - 1 - \frac{x}{2} + \frac{x^2}{8}| \le \frac{3|x|^3}{8},\quad x \in [-1, \infty).$$

$\boxed{10.}$ *Let $a > 0$ and f be differentiable on $(0, \infty)$ such that $f'(x) + af(x) \to L$. Show that $f(x) \to \frac{L}{a}$.*

3.3 Optimization Problems

Let us take a look at three optimization problems which are classic. First, we want to prove the Arithmetic-Geometric Mean inequality:

$$(3.1) \qquad n \in \mathbb{N}, n \geq 2, \quad a_1, a_2, ..., a_n \geq 0 \implies \frac{a_1 + a_2 + ... + a_n}{n} \geq \sqrt[n]{a_1 a_2 ... a_n}.$$

Consider the function $f(x) = \frac{a_1+a_2+...+x}{n} - \sqrt[n]{a_1 a_2 ... a_{n-1} x}$ defined for all $x \geq 0$. We see that $f(0) \geq 0$, and $f'(x) = \frac{1}{n} - \frac{1}{n}\sqrt[n]{a_1 a_2 ... a_{n-1}} x^{\frac{1-n}{n}}$. We may assume that $a_1, a_2, ..., a_{n-1} > 0$ and, in this case, we see that the only critical point of f is $x_0 = \sqrt[n-1]{a_1 a_2 ... a_{n-1}}$. This is clearly a point of minimum for f and if we calculate $f(x_0)$ we see that

$$f(x_0) = \frac{n-1}{n}\left(\frac{a_1 + a_2 + ... + a_{n-1}}{n-1} \geq \sqrt[n-1]{a_1 a_2 ... a_{n-1}}\right).$$

We observe that this reduces the problem to $n-1$ non-negative numbers. This argument can then be repeated until we arrive at only two numbers a_1 and a_2. It is clear that $(a_1 + a_2)/2 \geq \sqrt{a_1 a_2}$ is true because it is algebraically equivalent to $(\sqrt{a_1} - \sqrt{a_2})^2 \geq 0$.

Let us consider now the problem of finding the maximum volume cone inscribed in a sphere (see Figure 3.3). The radius of the sphere is $R > 0$ and the radius of the cone is $r > 0$. Hence the hight of the cone is $h = R + \sqrt{R^2 - r^2}$, and so the volume is $V = \frac{\pi r^2 h}{3} = \frac{\pi}{3}(r^2 R + r^2\sqrt{R^2 - r^2})$. We look at the derivative of V with respect to r

$$V'(r) = \frac{\pi}{3}(2rR + 2r\sqrt{R^2 - r^2} - \frac{r^3}{\sqrt{R^2 - r^2}}), \textit{ or}$$

$$V'(r) = \frac{r\pi}{3\sqrt{R^2 - r^2}}(2R\sqrt{R^2 - r^2} + 2(R^2 - r^2) - r^2).$$

The equation $V'(r) = 0$ is equivalent to $2R\sqrt{R^2 - r^2} = 3r^2 - 2R^2$ or

$$4R^4 - 4R^2 r^2 = 9r^4 - 12r^2 R^2 + 4R^4 \iff r = r_0 := \frac{2\sqrt{2}R}{3}.$$

We notice that $V(r_0) = \frac{\pi r_0^2 h}{3} = \frac{\pi (8) R^3}{27}(1 + \frac{1}{3}) = \frac{32\pi R^3}{81}$. Since we have $V(0) = 0$ and $V(R) = \frac{\pi R^3}{3} < \frac{32\pi R^3}{81}$ we see that we could assume that the center of the sphere is inside

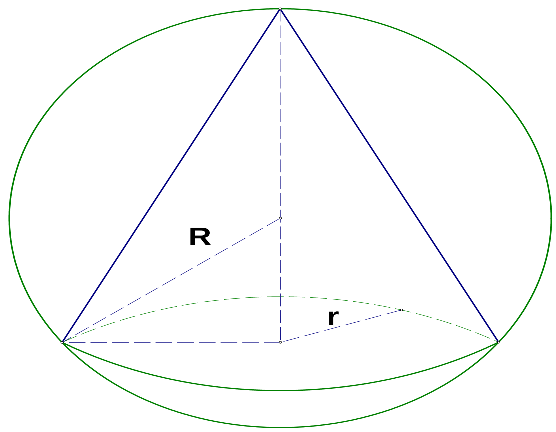


Figure 3.3: Cone inscribed in a sphere

the cone. We have only one critical point so this must be the maximum. If we denote this maximum by V_c and the volume of the sphere by V_s we observe that $\frac{V_c}{V_s} = (\frac{2}{3})^3$.

Finally, let us prove the Cauchy-Schwartz inequality:

$$a_1, a_2, ..., a_n, b_1, b_2, ...b_n \Longrightarrow (a_1^2+a_2^2+...+a_n^2)(b_1^2+b_2^2+...+b_n^2) \geq (a_1b_1+a_2b_2+...+a_nb_n)^2.$$

Let us consider the function $g(x) = (a_1 - b_1x)^2 + (a_2 - b_2x)^2 + ... + (a_n - b_nx)^2$ which satisfies clearly $g(x) \geq 0$ for all real numbers x. The function g is a quadratic since $g(x) = (a_1^2+a_2^2+...+a_n^2)-2(a_1b_1+a_2b_2+...+a_nb_n)x+(b_1^2+b_2^2+...+b_n^2)x^2 = A-2Bx+Cx^2$. Its minimum is attained at x_0 which is the solution of $g'(x) = 0$. We can assume that $C > 0$, otherwise the inequality is trivially satisfied. Then $x_0 = \frac{B}{C}$ and so, in particular, $g(x_0) = \frac{AC-B^2}{C} \geq 0$, which is equivalent to our inequality of interest.

3.4 Sketching Graphs of Elementary Functions

For some simple functions, if we use the information about the function, such as the x-intercepts, y-intercept, asymptotes, symmetry, the information about the derivative and its second derivative, we can draw the graph of the function with pretty good accuracy. We are going to exemplifying this first with $f(x) = \frac{x^3-x}{x^2+1}$, $x \in \mathbb{R}$. We see that the x-intercepts are $x = 0$, $x = 1$ and $x = -1$. The function is odd because $f(-x) = -f(x)$, so

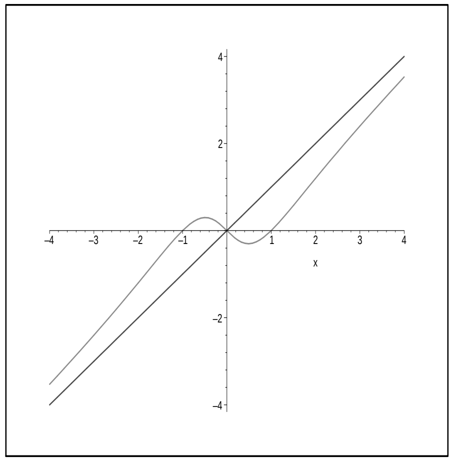


Figure 3.4: Graph of $y = \frac{x^3-x}{x^2+1}$

its graph is symmetric with respect to the origin. We have $f'(x) = \frac{x^4+4x^2-1}{(x^2+1)^2}$, which gives the critical points $x_{1,2} = \pm\sqrt{\sqrt{5}-2} \approx \pm 0.4858682712$. The second derivative is given by $f''(x) = -\frac{4x(x^2-3)}{(x^2+1)^3}$ which gives inflection points $x_3 = 0$, $x_{4,5} = \pm\sqrt{3} \approx \pm 1.732050808$. We have a slant asymptote since $f(x) = x - \frac{2x}{x^2+1}$. This identity shows that $y = x$ is the slant asymptote. All the information leads to the graph of f shown in Figure 3.4.

Let us mention that $y = mx + n$ is a slant asymptote of f at ∞, if $m = \lim_{x\to\infty} \frac{f(x)}{x}$ and $n = \lim_{x\to\infty} f(x) - mx$. The same definition goes for $-\infty$.

Next, we are going to look at an example of an elementary function which has a horizontal asymptote at ∞ and a slant asymptote at $-\infty$. Let $g(x) = \frac{\sqrt{x^6+1}-x^3}{x^2+1}$, $x \in \mathbb{R}$. We are going to use Maple to do some computations here, for getting

$$g'(x) = \frac{(x^3 - \sqrt{x^6+1})x(3x^3 + 3x + 2\sqrt{x^6+1})}{(x^2+1)^2\sqrt{x^6+1}}.$$

There are only two critical which can be computed exactly $x_1 = 0$ and $x_2 = -\sqrt{10\sqrt{249} - 130}/10 \approx -0.5272318124$. One can check that $y = -2x$ is a slant asymptote at $-\infty$ and $y = 0$ is a horizontal asymptote at ∞. We are not going to look at the second derivative. The graph of $y = g(x)$ is shown in Figure 3.5.

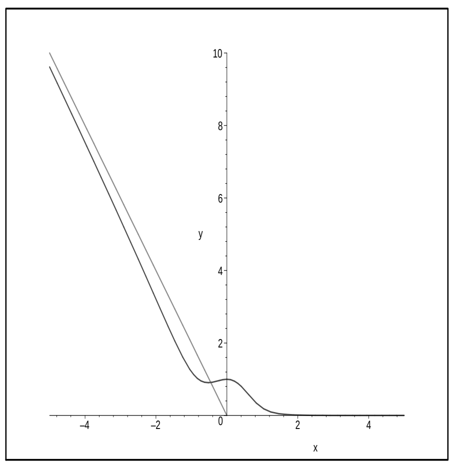


Figure 3.5: Graph of $y = g(x)$, $x \in [-5, 5]$

Chapter 4

Definite Integral

Quotation: *"Every minute dies a man, Every minute one is born;" I need hardly point out to you that this calculation would tend to keep the sum total of the world's population in a state of perpetual equipoise, whereas it is a well-known fact that the said sum total is constantly on the increase. I would therefore take the liberty of suggesting that in the next edition of your excellent poem the erroneous calculation to which I refer should be corrected as follows: "Every moment dies a man, And one and a sixteenth is born." I may add that the exact figures are 1.067, but something must, of course, be conceded to the laws of metre. Charles Babbage, letter to Alfred, Lord Tennyson, about a couplet in his "The Vision of Sin"*

4.1 Antiderivative and some previous formulae

Let us start with the definition of the *anti-derivative* of a function. We say that F differentiable on D (in general a union of intervals) is the *antiderivative* of f defined on D, if $F'(x) = f(x)$ for all $x \in D$. It is clear that if F is an antiderivative of f then $F + c$ is too, for every constant c. The notation used to go from a function f to its antiderivative F, if it exist, is $\int f(x)dx = F(x) + C$. So we can write all the differentiation formulae we have seen so far with this new notation:

$$\int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1} + C, where\ \alpha \neq -1, and\ \int \frac{1}{x} dx = \ln|x| + C,$$

$$\int a^{\alpha x} dx = \frac{a^{\alpha x}}{\alpha \ln a} + C,\ a \neq 1, a > 0,\ \alpha \neq 0,$$

$$\int \ln x dx = x \ln x - x + C, x > 0,$$

$$\int \sin \alpha x dx = -\frac{\cos \alpha x}{\alpha}, \int \cos \alpha x dx = \frac{\sin \alpha x}{\alpha}, \ \alpha \neq 0,$$

$$\int \tan \alpha x dx = -\frac{\ln|\cos \alpha x|}{\alpha} + C, \ \int \cot \alpha x dx = \frac{\ln|\sin \alpha x|}{\alpha} + C, \ \alpha \neq 0,$$

$$\int \sec \alpha x dx = \frac{\ln|\sec \alpha x + \tan \alpha x|}{\alpha} + C, \ \int \csc \alpha x dx = -\frac{\ln|\csc \alpha x + \cot \alpha x|}{\alpha} + C, \ \alpha \neq 0,$$

$$\int \sec^2 \alpha x dx = \frac{\tan \alpha x}{\alpha} + C, \ \int \csc^2 \alpha x dx = -\frac{\cot \alpha x}{\alpha} + C, \ \alpha \neq 0,$$

$$\int \frac{1}{x^2 + \alpha^2} dx = \frac{1}{\alpha} \arctan \frac{x}{\alpha} + C, \ \alpha \neq 0,$$

$$\int \frac{1}{\sqrt{\alpha^2 + x^2}} dx = \ln(x + \sqrt{\alpha^2 + x^2}) + C,$$

$$\int \frac{1}{\sqrt{\alpha^2 - x^2}} dx = \arcsin(\frac{x}{\alpha}) + C. \tag{4.1}$$

$$\int \frac{1}{x^2 - \alpha^2} dx = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C.$$

Let us point out that these rules are basically just our pervious differentiation rules "in reverse". The whole process of integration becomes all of a sudden a lot trickier when we throw in the chain rule. For instance, let us look at the problem of finding the anti-derivative of $f(x) = \frac{e^x}{1+e^{2x}}$. We observe that $f(x) = \frac{g'(x)}{1+g(x)^2}$, where $g(x) = e^x$, so $\int f(x)dx = \arctan g(x) + C = \arctan e^x + C$. In Calculus II, we will study a variety of techniques that will make the process of integration more straightforward. We will see in this chapter just one of them, called, integration by substitution, but we will do it in the context of definite integrals.

Since the derivative is a linear operation we can easily observe that

$$\int [\alpha f(x) + \beta g(x)] dx = \alpha \int f(x) dx + \beta \int g(x) dx,$$

the equality is "up to a constant", i.e. one needs to add appropriate constants to get the equality.

Examples: *Compute an antiderivative of each of the following functions:*

(a) $f(x) = \dfrac{x^2 - 2x + 3}{x^4}$,

(b) $g(x) = x\sin(x^2) - \dfrac{1}{1 + 9x^2}$,

(c) $h(x) = \dfrac{x + 2}{x^2 - 1}$

Solutions: (a) We have $f(x) = x^{-2} - 2x^{-3} + 3x^{-4}$ and therefore $\int f(x)dx = -\frac{1}{x} - 2\frac{x^{-2}}{-2} + 3\frac{x^{-3}}{-3} + C$ or $\int f(x)dx = -\frac{1}{x} + \frac{1}{x^2} - \frac{1}{x^3} + C$. If we want to put the answer in the same form as the given function then

$$\boxed{\int f(x)dx = \frac{x - x^2 - 1}{x^3} + C}.$$

(b) Here we need to think of the chain rule in reverse. So we have

$$\int g(x)dx = \frac{1}{2}\int 2x\sin(x^2)dx - \frac{1}{3}\int \frac{3}{1 + (3x)^2}dx =$$

$$= \boxed{-\frac{\cos(x^2)}{2} - \frac{\arctan(3x)}{3} + C}.$$

(c) We split it as follows: $h(x) = \frac{x}{x^2-1} + \frac{1}{x-1} - \frac{1}{x+1}$ and then

$$\int h(x)dx = \frac{1}{2}\int \frac{2x}{x^2 - 1}dx + \int \frac{1}{x - 1}dx - \int \frac{1}{x + 1}dx =$$

$$\frac{\ln|x^2 - 1|}{2} + \ln|x - 1| - \ln|x + 1| + C = \frac{1}{2}\ln|x^2 - 1|\frac{|x - 1|^2}{|x + 1|^2} + C,$$

or

$$\boxed{\int h(x)dx = \frac{1}{2}\ln\frac{|x - 1|^3}{|x + 1|} + C.}$$

More examples (the techniques used here are going to be studied in more detail in the next several sections):

Compute an antiderivative of each of the following functions:

(a) $f(x) = \dfrac{x^3 - x^2 + 3x + 1}{x^4}$,

(b) $g(x) = x\cos(x^2) - \dfrac{1}{4+x^2}$,

(c) $h(x) = \dfrac{3x+5}{(x+1)(x-4)}$

Solutions: (a) Usual formulae for computing the antiderivative give

$$\int f(x)dx = \int \frac{1}{x}dx - \int x^{-2}dx + 3\int x^{-3}dx + \int x^{-4}dx = \ln|x| + \frac{1}{x} - \frac{3}{2x^2} - \frac{1}{3x^3} + C,$$

so,

$$\boxed{\int f(x)dx = \ln|x| + \frac{1}{x} - \frac{3}{2x^2} - \frac{1}{3x^3} + C.}$$

(b) For the antiderivative of g use the chain rule in "reverse" and remember the rule of differentiation of the $\tan^{-1}$:

$$\boxed{\int g(x)dx = \frac{1}{2}\sin x^2 - \frac{1}{2}\arctan(x/2) + C.}$$

(c) We first decompose $\frac{3x+5}{(x+1)(x-4)} = \frac{A}{x-4} + \frac{B}{x+1}$. Solving for A and B one gets $A = \frac{17}{5}$ and $B = -\frac{2}{5}$. Then

$$\int h(x)dx = \frac{17}{5}\int \frac{1}{x-4}dx - \frac{2}{5}\int \frac{1}{x+1}dx = \frac{17}{5}\ln|x-4| - \frac{2}{5}\ln|x+1| + C,$$

and so

$$\boxed{\int h(x)dx = \frac{1}{5}\ln\frac{|x-4|^{17}}{|x+1|^2} + C.}$$

4.1.1 More Homework Problems

$\boxed{1.}$ *Find an antiderivative of the following functions:*

(A) $f(x) = x^2 - 2x - \frac{1}{x} + \frac{3}{x^2},\ x \neq 0,$

(B) $g(x) = \frac{x+1}{x^3},\ x \neq 0,$

(C) $h(x) = 2\sin x - 3\cos 2x,\ x \in \mathbb{R},$

(D) $i(x) = \tan^2 x,\ x \in (-\pi/2, \pi/2),$

(E) $j(x) = e^{2x} + 2^{3x},\ x \in \mathbb{R},$

(F) $k(x) = \log_2 x$, $x > 0$,

(G) $l(x) = \frac{1}{x^2-4}$, $x > 2$,

(H) $m(x) = \frac{x}{x^2+1}$, $x \in \mathbb{R}$

$\boxed{2.}$ *Calculate* $\int x \sin x dx$ *and* $\int x \cos x dx$.

$\boxed{3.}$ *Calculate* $\int xe^x dx$ *and* $\int xe^{2x} dx$.

$\boxed{4.}$ *Find a twice differentiable function* f *such that* $f(1) = f'(1) = 0$ *and* $f''(x) = \frac{1}{x}$ *for all* $x > 0$.

$\boxed{5.}$ *Find a twice differentiable function* f *such that* $f(-1) = f'(-1) = 0$ *and* $f''(x) = \frac{1}{x}$ *for all* $x < 0$.

$\boxed{6.}$ *(Chain rule combinations) Find an antiderivative of the following functions:*

(a) $f(x) = 2\sin(2x+1) - 3\cos(3x-1)$, $x \in \mathbb{R}$,

(b) $g(x) = (2x+1)e^{x^2+x}$, $x \in \mathbb{R}$,

(c) $h(x) = \frac{2x+1}{x^2+x+1}$, $x \in \mathbb{R}$

4.2 Definite Integral

This is the third most important concept in Calculus besides the notions of limit and derivative. We are going to introduce it for the so called Riemann Integral, but it can be generalized to cover a bigger class of functions. At this point, let us assume that f is a real valued function defined on the closed interval $[a, b]$ with $a < b$. For $n \in \mathbb{N}$, we let $a = x_0 < x_1 < x_2 ... < x_n = b$ be a partition of $[a, b]$ into n-intervals, which are not necessarily equal in length, and some arbitrary points $c_k \in [x_{k-1}, x_k]$, $k = 1, ..., n$. The number $\delta = max\{x_i - x_{i-1} : i = 1, ..., n\}$ is called the <u>norm</u> of the partition $\Delta := (x_0, x_1, x_2, ..., x_n)$, $x_0 = a < x_1 < x_2 < ... < x_n = b$.

Definition 4.2.1. *We say that* f *is* **Riemann integrable** *on the interval* $[a, b]$ *if the limit*

$$\ell := \lim_{\delta \to 0} \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) \tag{4.2}$$

exists. The limit is understood in the sense that c_i *and the partition are arbitrary. The limit* ℓ *is usually denoted by*

$$\int_a^b f(x)dx.$$

Sums of the form $\sum_{i=1}^n f(c_i)(x_i - x_{i-1})$, as in (4.2), are called Riemann sums. It turns out that every continuous function on a closed interval is Riemann integrable. This is a result that is taught in a more advanced courses, like Real Analysis I or II (for mathematics majors). There are discontinuous functions which are still Riemann integrable. One interesting example is the function

$$g(x) = \begin{cases} \sin\frac{1}{x} \quad if \quad x \neq 0, \\ \\ 0 \quad if \quad x = 0 \end{cases},$$

which has an essential discontinuity at zero. However, the Riemann integral of g over $[0, \pi]$ exists and it is about 1.575936300. A finite number of discontinuities in $[a, b]$ (especially of the ones where sided limits exist) do not pose any problems for the Riemann integral. So, a function like

$$s(x) = \begin{cases} \frac{\sin x}{|\sin x|} \quad if \quad x \neq k\pi, k \in \mathbb{Z} \\ \\ 0 \quad if \quad x = k\pi, k \in \mathbb{Z} \end{cases},$$

is Riemann integrable for every interval $[a, b]$. (A nice exercise here is to compute $\int_{2000}^{2018} s(x)dx$).

One classical example of a function which is not Riemann integrable is given by

$$h(x) = \begin{cases} 1 \quad if \quad x \in \mathbb{Q}, \\ \\ 0 \quad if \quad x \notin \mathbb{Q} \end{cases}.$$

One can see that the limit in (4.2) doesn't exist since we can pick in every interval a rational c_k or an irrational c_k. That changes the sum from 1 to 0, for every partition.

One of the geometrical interpretations of the number $\int_a^b f(x)dx$ is the area under the graph of $y = f(x)$, x-axis $x = a$ and $x = b$. The next theorem gives a very interesting way of computing the above limits in terms of an antiderivative of f and at the same time gives the existence of an anti-derivative of a continuous function.

Theorem 4.2.2. (Fundamental Theorem of Calculus-FTC.)

(a) Let f be a real-valued function defined on $[a, b]$ which is continuous. If $F(x) = \int_a^x f(t)dt$ for all $x \in [a, b]$. Then $F'(x) = f(x)$ for all $x \in [a, b]$.

(b) If f is Riemann integrable and F is an anti-derivative of f, then $\int_a^b f(t)dt = F(b) - F(a)$.

Let us look at some applications of the FTC.

$\boxed{\text{Problem 1.}}$ *Let* $F(x) = \displaystyle\int_{x^2}^{3x^3-2x} \frac{1}{t+\ln t}dt$ *for* $x \in [1,\infty)$*. Find the derivative of* $F(x)$ *and then compute* $F(1)$*.*

Solution: The function $g(t) = \frac{1}{t+\ln t}$ is well defined and an elementary function on the interval $t \in [1,\infty)$. By FTC part (a) if we introduce $G(x) = \int_1^x \frac{1}{t+\ln t}dt$ for $x \geq 1$, we have $G'(t) = g(t)$, for all t. This means that G is an anti-derivative of g. By part (b) of FTC, we see that $F(x) = G3(x^3-2x) - G(x^2)$. As a result, chain rule gives

$$F'(x) = G'(3x^3-2x)(9x^2-2) - G'(x^2)(2x).$$

But $G'(3x^3-2x) = g(3x^3-2x) = \frac{1}{3x^3-2x+\ln(x^3-2x)}$ and $G(x^2) = g(x^2) = \frac{1}{x^2+\ln(x^2)}$. Substituting we obtain

$$F'(x) = \frac{9x^2-2}{3x^3-2x+\ln(x^3-2x)} - \frac{2x}{x^2+\ln(x^2)}, \quad x \geq 1.$$

From here we just substitute $x=1$ and obtain $F'(1) = 9-2-2 = \boxed{5}$. ■

This next problem is a little more trickier.

$\boxed{\text{Problem 2.}}$ *Let* $f(x) = \displaystyle\int_{\cos x}^{\sin x} \frac{1}{\sqrt{1-t^2}}dt$ *for* $x \in [0, \frac{\pi}{2}]$*. Find the derivative of* $f(x)$ *and then find* $f(x)$*.*

Solution: Using the FTC and the chain rule, we get

$$f'(x) = \tfrac{d}{dx}\int_0^{\sin x} \tfrac{1}{\sqrt{1-t^2}}dt - \tfrac{d}{dx}\int_0^{\cos x} \tfrac{1}{\sqrt{1-t^2}}dt =$$
$$\tfrac{1}{\sqrt{1-\sin(x)^2}}\cos x - \tfrac{1}{\sqrt{1-\cos(x)^2}}(-\sin x) = \tfrac{\cos x}{\cos x} + \tfrac{\sin x}{\sin x} = \boxed{2}, \quad x \in [0, \tfrac{\pi}{2}].$$

Hence $f(x) = 2x + C$. Since $f(\pi/4) = 0$, we must have $f(x) = 2x - \frac{\pi}{2}$.

Let us observe that one ca use FTC part (b) and formula (4.1) and arrive at the same result:

$$f(x) = \arcsin(\sin x)) - \arcsin(\cos x)) = x - \arcsin(\sin(\pi/2 - x)) =$$

$$x - (\pi/2 - x) = 2x - \pi/2, \quad x \in [0, \tfrac{\pi}{2}].$$

$\boxed{\text{Problem 3.}}$ *Differentiate the function* $F(x) = \displaystyle\int_{-\tan x}^{\tan x} \frac{1}{1+t^2}dt.$

Solution: We define $G(x) = \int_0^x \frac{1}{1+t^2}dt$ and observe that $G'(x) = \frac{1}{1+x^2}$ and $F(x) = G(\tan x) - G(-\tan x)$. Then

$$F'(x) = G'(\tan x)\sec^2 x - G'(-\tan x)(-\sec^2 x) = \frac{\sec^2 x}{1+\tan^2 x} + \frac{\sec^2 x}{1+\tan^2 x} = 2$$

So,

$$\boxed{F'(x) = 2}. \qquad \blacksquare$$

Another application of the FTC and the fact that continuous functions are Riemann integrable, is the next exercise of calculating a special type of limit.

$\boxed{Problem\ 4.}$ *Find the value of the limit* $\lim_{n\to\infty} n \sum_{k=1}^{2n} \frac{1}{4n^2+k^2}$.

Solution: We write this limit as the limit of a Riemann sums:

$$\lim_{n\to\infty} n \sum_{k=1}^{2n} \frac{1}{4n^2+k^2} = \lim_{n\to\infty} \sum_{k=1}^{2n} \frac{n}{4n^2} \frac{1}{1+(\frac{k}{2n})^2} =$$

$$= \tfrac{1}{2} \lim_{n\to\infty} \sum_{k=1}^{2n} \frac{1}{2n} \frac{1}{1+(\frac{k}{2n})^2} = \frac{1}{2} \int_0^1 \frac{1}{1+x^2} dx = \frac{1}{2} \arctan x|_0^1 = \frac{1}{2}\frac{\pi}{4} = \frac{\pi}{8}.$$

Therefore $\boxed{\lim_{n\to\infty} n \sum_{k=1}^{2n} \frac{1}{4n^2+k^2} = \frac{\pi}{8}}$.

$\boxed{Problem\ 5.}$ *Find the value of the limit*

$$\lim_{n\to\infty} \frac{1}{3n+1} + \frac{1}{3n+2} + \ldots + \frac{1}{4n}.$$

Solution: We use the sigma notation to rewrite this limit as

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{3n+k} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n} \frac{1}{3+k/n} = \int_0^1 \frac{1}{3+x},$$

so after computing the integral we get

$$\boxed{\lim_{n\to\infty} \frac{1}{3n+1} + \frac{1}{3n+2} + \ldots + \frac{1}{4n} = \ln(4/3).}$$

4.2.1 Homework Problems

$\boxed{Problem\ 1.}$ *Let* $f(x) = \int_{2x-1}^{3x-2} \frac{1}{t^2+2} dt$ *for* $x \in \mathbb{R}$. *Find the derivative of* $f(x)$.

$\boxed{\textit{Problem 2.}}$ *Let* $f(x) = \displaystyle\int_{-\tan x}^{\tan x} \sqrt{1+t^2}dt$ *for* $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$*. Find the derivative of* $f(x)$ *and then find* $f(\pi/6)$*.*

$\boxed{\textit{Problem 3.}}$ *Find the value of the following limit:*

$$\lim_{n\to\infty} \frac{\sin\frac{\pi}{4n} + \sin\frac{2\pi}{4n} + ... + \sin\frac{n\pi}{4n}}{n} \qquad \textit{Answer}: \boxed{\frac{4-2\sqrt{2}}{\pi}}$$

$\boxed{\textit{Problem 4.}}$ *Find the value of the following limit:*

$$\lim_{n\to\infty} \sum_{k=1}^{5n} \frac{1}{n+k} \qquad \textit{Answer}: \boxed{\ln(6)}$$

$\boxed{\textit{Problem 5.}}$ *What is the exact value of the limit:*

$$\lim_{n\to\infty} n \sum_{k=1}^{n} \frac{1}{n^2+k^2} \quad ? \qquad \textit{Answer}: \boxed{\frac{\pi}{4}}$$

$\boxed{\textit{Problem 6.}}$ *What is the exact value of the limit:*

$$\lim_{n\to\infty} n \sum_{k=1}^{3n} \frac{1}{n^2+k^2} \quad ? \qquad \textit{Answer}: \boxed{\arctan(3)}$$

$\boxed{\textit{Problem 7.}}$ *Find the value of the following limit:*

$$\lim_{n\to\infty} \sum_{k=1}^{4n} \frac{1}{\sqrt{9n^2+k^2}} \qquad \textit{Answer}: \boxed{\ln(3)}$$

$\boxed{\textit{Problem 8.}}$ *Find the value of the following limit:*

$$\lim_{n\to\infty} \left(\frac{1}{3n+1} + \frac{1}{3n+3} + \frac{1}{3n+5} + ... + \frac{1}{5n-1} \right) \qquad \textit{Answer}: \boxed{\frac{1}{2}\ln(5/3)}$$

$\boxed{\textit{Problem 9.}}$ *Find the value of the following limit:*

$$\lim_{n\to\infty} \sum_{k=1}^{4n} \frac{1}{\sqrt{25n^2-k^2}} \qquad \textit{Answer}: \boxed{\arcsin(4/5)}$$

$\boxed{\textit{Problem 10.}}$ *What is the exact value of the limit:*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{k}{\sqrt{n^2-k^2}} \quad ? \qquad \textit{Answer}: \boxed{1}$$

$\boxed{Problem\ 11.}$ *What is the exact value of the limit:*

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\sqrt{n^2+kn}} \quad ? \qquad\qquad Answer: \boxed{2\sqrt{2}-2}$$

4.3 Integration using a substitution

We are mainly concerned with the change of variables in the definite integral. The chain rule $\frac{d}{dt}F(u(t)) = f(u(t))u'(t)$ where F is an anti-derivative of f, and the FTC gives the following formula

$$\int_a^b f(x)dx = F(b) - F(a) = \int_{t_a}^{t_b} f(u(t))u'(t)dt$$

where $u(t_a) = a$ and $u(t_b) = b$, and $u : [t_a, t_b] \to [a, b]$ is a differentiable function called the substitution ($x = u(t)$).

$\boxed{Problem\ 1.}$ *Calculate the definite integral* $\displaystyle\int_0^5 \frac{x}{\sqrt{3x+1}}dx.$

Solution: Changing the variable $3x + 1 = u^2$ gives $3dx = 2udu$

$$\int_0^5 \frac{x}{\sqrt{3x+1}}dx = \int_1^4 \frac{u^2-1}{3u}\frac{2udu}{3} = \frac{2}{9}(\frac{u^3}{3}|_1^4 - u|_1^4) = 4,$$

so

$$\boxed{\int_0^5 \frac{x}{\sqrt{3x+1}}dx = 4}.$$

$\boxed{Problem\ 2.}$ *Calculate the definite integral* $\displaystyle\int_0^4 \frac{9-5x}{\sqrt{2x+1}}dx.$

Solution: We change the variable $t^2 = 2x + 1$ ($tdt = dx$)and obtain

$$\int_0^4 \frac{9-5x}{\sqrt{2x+1}}dx = \int_1^3 \frac{9-5\frac{t^2-1}{2}}{t}tdt = \frac{1}{2}\int_1^3 (23-5t^2)dt =$$

$$\tfrac{1}{2}[23t|_1^3 - \tfrac{5}{3}t^3|_1^3] = \tfrac{1}{2}[23(2) - \tfrac{5(26)}{3}] = 23 - \tfrac{65}{3} = \tfrac{4}{3}.$$

Hence,

$$\boxed{\int_0^4 \frac{9-5x}{\sqrt{2x+1}}dx = \frac{4}{3}.}$$

$\boxed{\textit{Problem 3.}}$ *Find the value of the definite integral* $\displaystyle\int_0^1 \frac{(x+2)dx}{\sqrt{4+5x}}$.

Solution: We make a substitution $4+5x=u^2$ which means $5dx=2udu$ and so

$$\int_0^1 \frac{(x+2)dx}{\sqrt{4+5x}} = \frac{1}{5}\int_2^3 \frac{\frac{u^2-4}{5}+2}{u}2udu = \frac{2}{25}\int_2^3 (u^2+6)du =$$

$$= \tfrac{2}{25}(\tfrac{u^3}{3}|_2^3 + 6u|_2^3) = \tfrac{2}{25}(\tfrac{19}{3}+6) = \tfrac{74}{75}.$$

Hence $\boxed{\displaystyle\int_0^1 \frac{(x+2)dx}{\sqrt{4+5x}} = \frac{74}{75}}$

$\boxed{\textit{Problem 4.}}$ *Find the value of the definite integral* $\displaystyle\int_0^8 \frac{3x-2}{\sqrt{9+2x}}dx$.

4.4 Integration by parts

The idea of this technique is based on the product rule of differentiation from Calculus I: $(fg)'=f'g+fg'$ where f and g are differentiable functions. We are mostly concerned with definite integrals, so by FTC, we have

$$\int_a^b f(x)g'(x)dx = f(b)g(b)-f(a)g(a)-\int_a^b f'(x)g(x)dx. \tag{4.3}$$

Let us see the most standard applications of this formula.

$\boxed{\textit{Problem 1.}}$ *Find the value of the definite integral* $\displaystyle\int_0^\pi x\cos x dx$.

Solution: We can write the integral as $\displaystyle\int_0^\pi x\frac{d}{dx}(\sin x)dx$ and so, we can use the formula (4.3), for $f(x)=x$ and $g(x)=\sin x$. We can continue,

$$\int_0^\pi x\frac{d}{dx}(\sin x)dx = f(x)g(x)|_0^\pi - \int_0^\pi \frac{d}{dx}(x)(\sin x)dx \Rightarrow$$

$$\int_0^\pi x\cos x dx = \cos x|_0^\pi = \boxed{-2}.$$

$\boxed{\textit{Problem 2.}}$ *Find the value of the definite integral* $\displaystyle\int_1^e \frac{\ln x}{x^2}dx$.

Solution: We can write the integral as $\displaystyle\int_1^e (\ln x)\frac{d}{dx}(-\frac{1}{x})dx$ and so, we can use the formula (4.3), for $f(x) = \ln x$ and $g(x) = -\frac{1}{x}$. We can continue,

$$\int_1^e \frac{\ln x}{x^2}dx = f(x)g(x)|_1^e - \int_1^e \frac{d}{dx}(\ln x)(-\frac{1}{x})dx \Rightarrow$$

$$\int_1^e \frac{\ln x}{x^2}dx = -\frac{1}{e} + \int_1^e \frac{1}{x^2} = (-\frac{1}{x})|_1^e - \frac{1}{e} = \boxed{\frac{e-2}{e} \approx 0.264241118}.$$

4.4.1 Sample Final Exam and solutions

$\boxed{1.}$ *Find the first derivative of the functions:*

(a) $f(x) = (\cos 2x)^3(\sin 3x)^2$ $\qquad$ (b) $g(x) = \sqrt{x - x^2}$

(c) $h(x) = \ln(x + \sqrt{x^2+1})$ $\qquad$ (d) $i(x) = e^{-2x}\sec(3x) - \arctan(\sin x)$

$(e) j(x) = \dfrac{1+x}{(2-x)^2}$ $\qquad$ $(f) k(x) = \arcsin(x^2)$

Solution: (a) Using the product formula and the derivatives of *sine* and *cosine*, we get

$$\boxed{f'(x) = -6(\cos 2x)^2 \sin 2x \sin^2 3x + 6(\cos 2x)^3 \sin 3x \cos 3x}.$$

(b) The derivative of g is

$$\boxed{g'(x) = \frac{1-2x}{2\sqrt{x-x^2}}}.$$

(c) We have seen, several times, that $\boxed{h'(x) = \dfrac{1}{\sqrt{x^2+1}}}$.

(d) The derivative is

$$\boxed{i'(x) = -2e^{-2x}\sec(3x) + 3e^{-2x}\sec(3x)\tan(3x) - \frac{\cos x}{1+\sin^2 x}}.$$

(e) Using the quotient rule and the product rule, we get

$$j'(x) = \frac{(2-x)^2 - (1+x)2(2-x)(-1)}{(2-x)^4} = \frac{2-x+2+2x}{(2-x)^3} \text{ or}$$

$$\boxed{j'(x) = \frac{4+x}{(2-x)^3}}.$$

(f) Simply, he chain rule gives

$$\boxed{k'(x) = \frac{2x}{1-x^4}}.$$

$\boxed{2.}$ *Compute an antiderivative of each of the following functions:*

$$(a)\ f(x) = \frac{x^2+2x+3}{(x+1)^2} \qquad (b)\ g(x) = x\sec^2(x^2+1)$$

$$(c)\ h(x) = \tfrac{2x+1}{x^2-4} \qquad (d)\ i(x) = \frac{1}{1+4x^2}$$

Solution: (a) We observe that $f(x) = \frac{x^2+2x+1+2}{(x+1)^2} = 1 + \frac{2}{(x+1)^2}$. This implies that

$$\boxed{\int f(x)dx = x - \frac{2}{x+1} + C}.$$

(b) Using the chain rule in reverse we see that $\boxed{\displaystyle\int g(x)dx = \frac{1}{2}\tan(x^2+1) + C}$.

(c) We write $h(x) = \frac{2x+1}{(x-2)(x+2)} = \frac{1}{4}(\frac{5}{x-2} + \frac{3}{x+2})$ which gives

$$\boxed{\int h(x)dx = \frac{5}{4}\ln|x-2| + \frac{3}{4}\ln|x+2| + C}.$$

(d) Using the chain rule in reverse we see that $\boxed{\displaystyle\int i(x)dx = \frac{1}{2}\arctan(2x) + C}$.

$\boxed{3.}$ *Water is filling up a pool in the shape shown below, at a rate of* $5\ ft^3/min$. *How fast is the water level rising when it is* $4\ ft$ *deep (at the deep end)?*

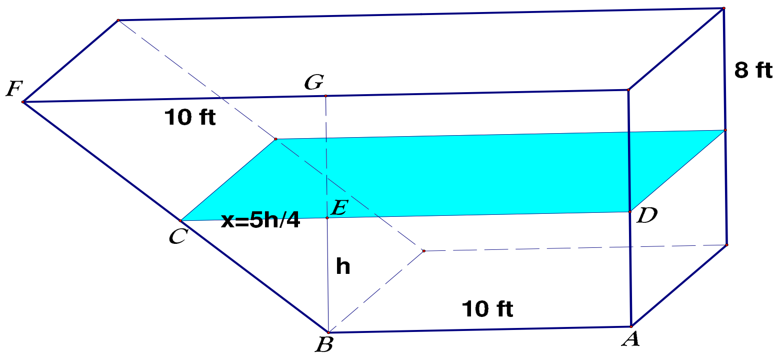


Figure 1

Solution: We refer to Figure 1 above. We need to find the volume of the water in terms of h (the depth at the deep end). The formula is simply the volume of a prism, i.e., the area of the trapezoid $ABCD$ times the other dimension, which is 10 ft: $V = Area(ABCD)10$. From the similarity of the triangles CEB and FGB, we get $\frac{CE}{FG} = \frac{h}{BG}$ or $CE = \frac{10h}{8} = \frac{5h}{4}$. Then the area of the trapezoid $ABCD$ is $A = \frac{(CD+AB)h}{2}$. Hence, the volume is

$$V = 10\frac{[\frac{5h}{4} + 10 + 10]h}{2} = \frac{25}{4}h(h+16) = \frac{25}{4}(h^2 + 16h).$$

Differentiating, we obtain $\frac{dV}{dt} = \frac{25}{4}(2h+16)\frac{dh}{dt} = \frac{25}{2}(h+8)\frac{dh}{dt}$. Substituting $h = 4$ and $\frac{dV}{dt} = 5$ gives $\boxed{\frac{dh}{dt} = \frac{1}{30} \approx 0.03\ ft/min}$.

$\boxed{4.}$ *Calculate the definite integral* $I := \int_1^e (x^2 + \frac{1}{x^2}) \ln x dx$.

Solution: Changing the variable $x = e^s$

$$I = \int_0^1 (e^{2s} + e^{-2s})se^s ds = \int_0^1 (e^{3s} + e^{-s})sds.$$

We use the formula we did in class and get

$$I = e^{3s}(s/3 - 1/9) + e^{-s}(s/(-1) - 1/(-1)^2)|_0^1 \Rightarrow$$

$$\boxed{I = \frac{2e^3}{9} - \frac{2}{e} + \frac{10}{9}}.$$

$\boxed{5.}$ *Fundamental Theorem of Calculus.*

Solution: (i) Given a Riemman integrable function f on $[a,b]$, with an antiderivative F, then $\int_a^b f(x)dx = F(b) - F(a)$.

(ii) If f is continuous on $[a,b]$ then $F(x) = \int_a^x f(t)dt$ has the property that $F'(x) = f(x)$ for all $x \in [a,b]$.

4.5 The logarithmic function

Let us define the logarithmic function by $f(x) := \int_1^x \frac{1}{t}dt$, for all $x > 0$. First, we want to prove that $f(ab) = f(a) + f(b)$ for all $a, b > 0$. This can be accomplished by observing that

$$f(ab) - f(a) = \int_a^{ab} \frac{1}{t}dt \overset{t=as}{=} \int_1^b \frac{1}{as}ads = \int_1^b \frac{1}{s}ds = f(b),$$

which proves the essential identy which characterizes logarithmic functions. We observe that $f(1) = 0$ and $f'(x) = \frac{1}{x}$ by the Fundamental Theorem of Calculus. This implies that $f'(x) > 0$ for all x. We conclude that f is a stritly increasing function on $(0,\infty)$. By the property we have established $f(2^n) = nf(2) \to \infty$ and so we have $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to 0} f(x) = -\infty$ and so f is a bijection from $(0,\infty)$ into $(-\infty,\infty)$. Let us denote by e the solution of the equation $f(x) = 1$. In other words, we have $f(e) = 1$. Consider now the inverse function of f and let us denote that by g. Then $g : (-\infty,\infty) \to (0,\infty)$ is a function with the following properties:

$$g(1) = e, g(0) = 1, \quad g(x+y) = g(x)g(y), \quad x, y \in \mathbb{R}.$$

If we set $a = g(x)$ and $b = g(y)$, we observe that $f(ab) = f(a) + f(b)$ or $g(x)g(y) = ab = g(f(a) + f(b)) = g(x+y)$ proving the above identity. This implies $g(nx) = g(x)^n$ for every real number x and every natural number n. In particular is $x = \frac{1}{n}$ we obtain $e = g(1) = g(1/n)^n$ or $g(1/n) = e^{1/n}$. Also, if $x = \frac{1}{m}$ then $g(n/m) = g(1/m)^n = (e^{1/m})^n = e^{n/m}$ so $g(r) = e^r$ for every positve rational number r. Hence, $g(x) = e^x$ for all x. In order to conclude that g is what we consider to be the natural exponential function we have to prove that $e = \lim_{n\to\infty}(1 + \frac{1}{n})^n$. This is equivalent to $f(e) = \lim_{n\to\infty} nf(1 + \frac{1}{n})$ or $f(e) = \lim_{x\to 0} \frac{f(1+x) - f(1)}{x} = f'(1) = 1$. This is correct by the definition of e. So, we conclude that $f(x) = \ln x$ and $g(x) = e^x$ where $e = \lim_{n\to\infty}(1 + \frac{1}{n})^n \approx 2.7182818284590452354$.

4.6 The trigonometric functions

In this section we want to do something similar and build the trigonometric functions using integration. Let us define first $f(x) = \int_0^x \frac{1}{\sqrt{1-t^2}}dt$ for all $x \in (-1,1)$. We observe that $f'(x) = \frac{1}{\sqrt{1-x^2}} > 0$ if $x \in (-1,1)$ and so f is strictly increasing. The integral above for $x = 1$ is an improper integral of second type which is convergent. Hence, $f(1)$ is well-defined by $\int_0^1 \frac{1}{\sqrt{1-t^2}}dt$ value which we will denote by a. Then f is a one-to-one map from $[-1,1]$ into $[-a,a]$, and we can call its inverse g. We observe that $g'(x) = \frac{1}{f'(g(x))} = \sqrt{1-g(x)^2}$ for all $x \in (-a,a)$. We can introduce $h(x) = \sqrt{1-g(x)^2}$, for all $x \in [-a,a]$. This gives the usual Pythagorean identity $g(x)^2 + h(x)^2 = 1$ if $x \in [-a,a]$. One may check easily that

$$\int \sqrt{1-t^2}dt = \frac{1}{2}(x\sqrt{1-x^2} + f(x)) + C,$$

Which implies that

$$\frac{1}{2}Area(Unit\ Disk) = \int_{-1}^{1} \sqrt{1-t^2}dt = \frac{\pi}{2} = a \implies \boxed{x = \frac{\pi}{2}}.$$

Then $h'(x) = -\frac{g(x)g'(x)}{\sqrt{1-g(x)^2}} = -g(x)$ for all $x \in (-a,a)$. Then $g''(x) = h'(x) = -g(x)$, which means g is a twice differentiable function satisfying $g'' + g = 0$, $g(0) = 0$ and $g'(x) = 1$. There is only one such function so $g(x) = \sin(x)$, $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$. We then have $h(x) = \cos x$, and we can extend these two function by symmetry first, $g(a-x) = g(a+x)$, and then by periodicity $g(x+4a) = g(x)$. One can prove the usual formulae

$$\cos(x+y) = \cos x \cos y - \sin x \sin y, \quad x, y \in \mathbb{R},$$

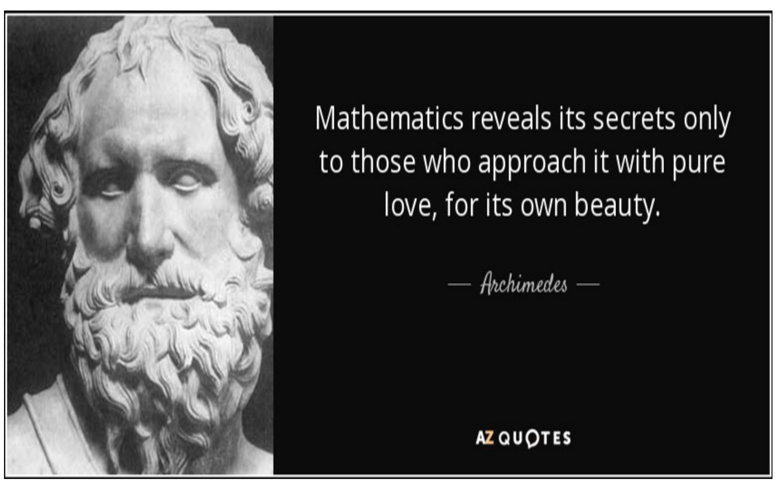
$$\sin(x+y) = \sin x \cos y + \cos x \sin y, \quad x, y \in \mathbb{R},$$

using the uniquencess of the solution of the differential equation $u'' + u = 0$ with initial conditions. Once we have identified g and h we can construct easily the other trigonometric functions: $\tan x = \frac{g(x)}{h(x)}$ defined for all x where h is not zero, $\cot x = \frac{g(x)}{h(x)}$, $x \neq k\pi$, etc.

Chapter 5

Parametric Equations

Quotation:



"One of the greatest minds of all times!" Norman John Wildberger
www.youtube.com/user/njwildberger

5.1 Some classical parametrizations

The most elementary curves that are at the heart of Euclidean geometry are clearly lines and circles. For a line in the plane the most classic equation is given by

$$ax + by + c = 0 \tag{5.1}$$

where either a or b is not equal to zero. Suppose that we have a point on this line, say $P(x_0, y_0)$. Then a classical parametrization of the line (5.1) is given by

$$\begin{cases} x = x_0 + bt \\ y = y_0 - at, t \in \mathbb{R}. \end{cases} \tag{5.2}$$

One can easily see that substituting x and y, given by (5.2), into (5.1), results in the equality $ax_0 + by_0 + c = 0$ which is assumed to be true.

In practice, we often need to write the equation of a line determined by two distinct points, say $A(x_A, y_A)$ and $B(x_B, y_B)$. A nice and convenient way to write such an equation is to employ the conventional linear algebra approach (using determinants):

$$\begin{vmatrix} x & y & 1 \\ x_A & y_A & 1 \\ x_B & y_B & 1 \end{vmatrix} = 0 \quad \Leftrightarrow \quad (y_A - y_B)x - (x_A - x_B)y + (x_A y_B - x_B y_A) = 0. \tag{5.3}$$

We can see that A and B are points on this line, either by an algebra exercise or by recalling some properties of determinants (a determinant with two identical rows is zero). We also have that a determinant whose row is a linear combination of the others is equal to zero too. Hence, the next parametrization comes naturally

$$\begin{cases} x = (1-t)x_A + tx_B \\ y = (1-t)y_A + ty_B, t \in \mathbb{R}. \end{cases} \tag{5.4}$$

This parametrization has the advantage that for $t = 0$ we obtain the point A and for $t = 1$ we end up at B. Moreover, every point on the segment $\overline{AB}$ is given by a value of the parameter $t \in [0, 1]$, and vice versa.

Suppose we have two other points, $C(x_C, y_C)$ and $D(x_D, y_D)$. How do we determine the intersection of $\overline{AB}$ and $\overline{CD}$? Of course, assuming that the two segments are not parallel, we can simply plug into the equation of $\overline{CD}$ the parametrization (5.4)

$$\begin{vmatrix} (1-t)x_A + tx_B & (1-t)y_A + ty_B & 1-t+t \\ x_C & y_C & 1 \\ x_D & y_D & 1 \end{vmatrix} = 0 \tag{5.5}$$

and solve for t:

$$t = t_i = \frac{\begin{vmatrix} x_A & y_A & 1 \\ x_C & y_C & 1 \\ x_D & y_D & 1 \end{vmatrix}}{\begin{vmatrix} x_A - x_B & y_A - y_B & 0 \\ x_C & y_C & 1 \\ x_D & y_D & 1 \end{vmatrix}}, \tag{5.6}$$

which exists precisely when $\begin{vmatrix} x_A - x_B & y_A - y_B & 0 \\ x_C & y_C & 1 \\ x_D & y_D & 1 \end{vmatrix} \neq 0.$

Problem 1

`Use this information to prove Menlaus' Theorem and Ceva's Theorem.`

The standard parametrization of the unit circle:

$$\begin{cases} x = \cos t \\ y = \sin t, t \in [0, 2\pi). \end{cases} \tag{5.7}$$

Problem 2

A cow is tied to a silo with radius by a rope just long enough to reach the opposite side of the silo (cylinder of radius r). Find the area available for grazing by the cow.

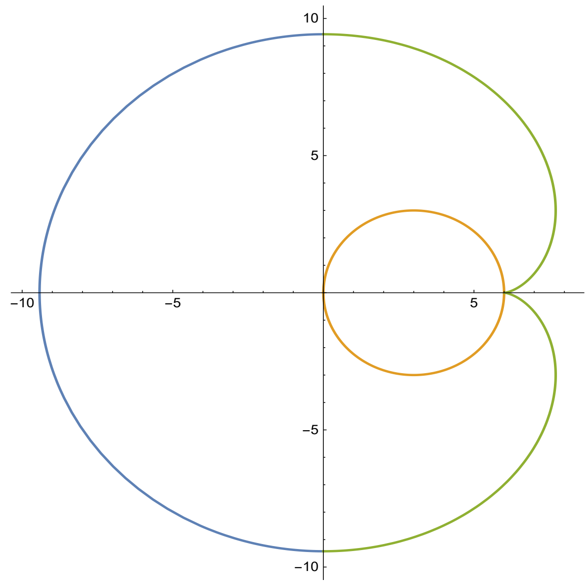


Figure 1

Solution: The cow is going to graze an area which is the inside of the curve depicted in Figure 1, which is formed by an involute and a semicircle, minus the area of the circle of radius r (the inside of the silo). We are going to choose the axes as in Figure 1 above. So, the parametrization of the curve that is to the right of the y-axis (the involute) is given by

$$\begin{cases} x(t) = r + r(\cos t + t\sin t) \\ \\ y(t) = r(\sin t - t\cos t), \end{cases} \qquad t \in [-\pi, \pi],$$

and the part to the left is a semicircle or radius πr centered at the origin. We observe that the involute is a curve explicit with respect to y and so we can use the formula for area given by

$$\int_{-\pi r}^{\pi r} x dy.$$

Then, the area grazed is given by

$$A = \frac{(\pi r)^2\pi}{2} + \int_{-\pi}^{\pi} x(t)y'(t)dt - \pi r^2.$$

Since $y'(t) = r(\cos t - \cos t + t\sin t) = rt\sin t$ we see that the integral we have to compute equals

$$I = \int_{-\pi}^{\pi} x(t)y'(t)dt = r^2\int_{-\pi}^{\pi}(1+\cos t+t\sin t)t\sin t dt = r^2\int_{-\pi}^{\pi}(t\sin t+t\sin t\cos t+t^2\sin^2 t)dt.$$

So, let us compute first (by parts)

$$I_1 = \int_{-\pi}^{\pi} (t \sin t) dt = t(-\cos t)|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} (\cos t) dt = \pi - (-\pi) = 2\pi.$$

Then the second (also by parts) is

$$I_2 = \int_{-\pi}^{\pi} (t \sin t \cos t) dt = \frac{1}{2} \int_{-\pi}^{\pi} (t \sin 2t) dt = \frac{1}{2} [t(-\frac{\cos 2t}{2})|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{\cos 2t}{2} dt] = -\frac{\pi}{2}.$$

Finally the last integral is given by

$$I_3 = \int_{-\pi}^{\pi} (t^2 \sin^2 t dt = \frac{1}{2} \int_{-\pi}^{\pi} t^2 (1 - \cos 2t) dt = \frac{1}{2} [\frac{t^3}{3}|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} t^2 \cos 2t dt] \Rightarrow$$

$$I_3 = \frac{1}{2} [2\frac{\pi^3}{3} - t^2 \frac{\sin 2t}{2}|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} 2t \frac{\sin 2t}{2} dt] = \frac{\pi^3}{3} - \frac{\pi}{2}.$$

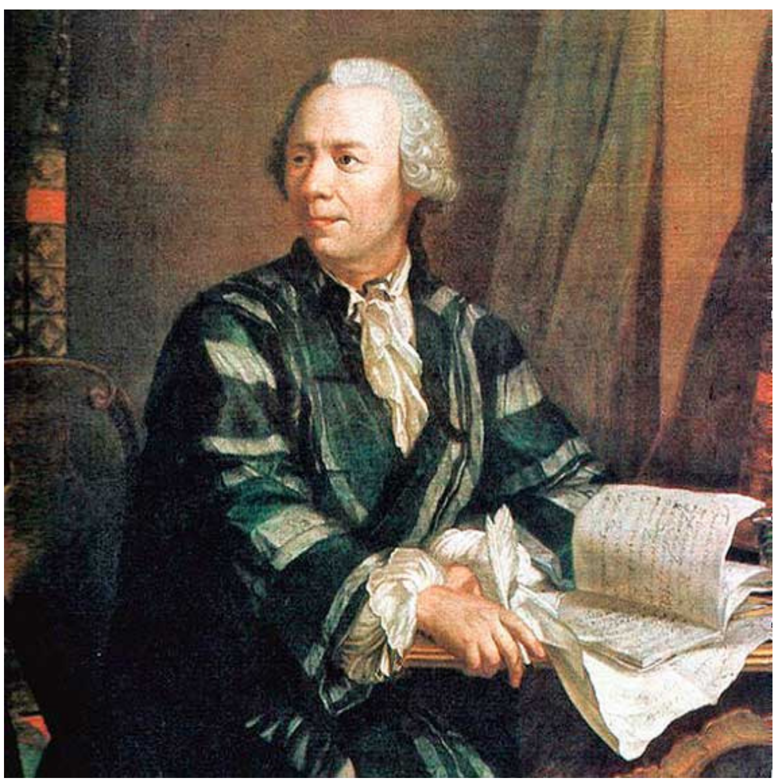
Therefore, the area A is

$$A = r^2 (\frac{\pi^3}{2} + \frac{\pi^3}{3} + 2\pi - \frac{\pi}{2} - \frac{\pi}{2}) - \pi r^2 = \boxed{\frac{5\pi^3 r^2}{6}}. \quad \square$$

Chapter 6

Curves in space, Curvature and TNB-frame

Quotation:



"Read Euler, read Euler, he is the master of us all!" Pierre-Simon Laplce

Suppose the curve is given parametrically by $r(t) = [f(t), g(t), h(t)]$. The derivative of r is $r'(t) = [f'(t), g'(t), h'(t)]$ and the length of the curve between point $A = r(t_0)$ and point $B = r(t)$ is given by

$$s(t) = \int_{t_0}^{t} |r'(u)|du = \int_{t_0}^{t} \sqrt{f'(u)^2 + g'(u)^2 + h'(t)^2}du.$$

As an example, let's compute the length of a helix with parametric equations $r = [r\cos t, r\sin t, ht]$, for $t \in [0, 2\pi]$:

$$L = \int_{0}^{2\pi} \sqrt{r^2 + h^2}du = 2\pi\sqrt{r^2 + h^2}.$$

The unit tangent vector is defined by

$$T(t) = \frac{r'(t)}{|r'(t)|}$$

provided $r'(t) \neq 0$ (a smooth curve). The **curvature** is defined intrinsically in terms of the arc-length parametrization

$$\kappa = |\frac{T(s)}{ds}| \quad \Rightarrow \quad \boxed{\kappa = \frac{|r' \times r''|}{|r'|^3}}. \tag{6.1}$$

Indeed, $\frac{ds}{dt} = |r'|$ and since $r' = |r'(t)|T(t) = \frac{ds}{dt}T(t)$ if we differentiate again we get $r'' = \frac{d^2s}{dt^2}T(t) + \frac{ds}{dt}T'(t)$. Then taking the cross product with r' implies

$$r' \times r'' = (\frac{ds}{dt})^2 T(t) \times T'(t) = |r'|^2 T(t) \times T'(t).$$

Hence, we have $|r' \times r'| = |r'|^2|T(t) \times T'(t)| = |r'|^2|T'(t)|$ becasue T is a unit vector and T and T' are perpendicular. Therefore, we have

$$k = |\frac{T(s)}{ds}| = \frac{|T'|}{|r'|} = \frac{|r' \times r''|}{|r'|^3}.$$

By definition $N = \frac{T'}{|T'|}$ the derivative being taken with respect to t (the parameter). Using chain rule

$$T' = \frac{dT}{dt} = \frac{dT}{ds}\frac{ds}{dt} = |r'|\frac{dT}{ds}.$$

Hence, we obtain

$$\frac{dT}{ds} = \frac{T'}{|r'|} = \frac{|T'}{|r'|}|N = \kappa N \Rightarrow$$

$$\frac{dT}{ds} = \kappa N. \tag{6.2}$$

Let us see a nice corollary of (6.2). Suppose that a unit vector u is fixed in space and let φ be the angle that T makes with u, and ψ be the angle that N makes with u. Then we have

$$\kappa = |\frac{d\varphi}{ds}\frac{\sin\varphi}{\cos\psi}|, \tag{6.3}$$

which reduces to $\kappa = |\frac{d\varphi}{ds}|$ for a plane curve and u a unit vector in that plane. This can be derived in the following way: first differentiate $\cos\varphi = T \cdot u$ with respect to s. Since u is fixed, we get

$$(-\sin\varphi)\frac{d\varphi}{ds} = \frac{dT}{ds} \cdot u = \kappa N \cdot u = \kappa \cos\psi,$$

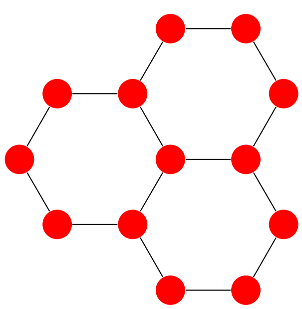
which gives (6.3).

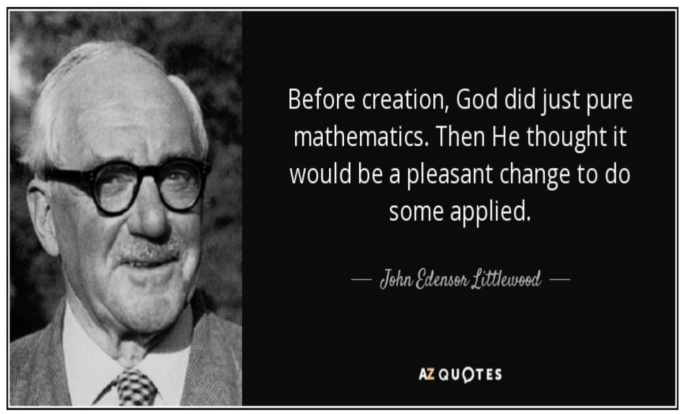
$\boxed{\textit{Joke Time}!}$

Break for a short shot!

Chapter 7

Inequalities





"All analysts spend half their time hunting through the literature for inequalities which they want to use and cannot prove." - G.H. Hardy

Let us start with the classical inequality of Bernoulli:

$$(1+x)^n \geq 1+nx, \quad n \in \mathbb{N}, x > -1. \tag{7.1}$$

We are going to use induction for this. For $n = 1$, the inequality (7.1) is trivial. Assuming that $(1+x)^n \geq 1+nx$ we multiply by $(1+x) > 0$ and get $(1+x)^{n+1} \geq (1+nx)(1+x) = 1+(n+1)x+nx^2 \geq 1+(n+1)x$ since $nx^2 \geq 0$. Then, we conclude that $(1+x)^{n+1} \geq 1+(n+1)x$ which ends the inductive step. Therefore by PMI, (7.1) is true for all $n \in \mathbb{N}$. ■

Theorem 7.0.1. **[Mediant Inequality -Farey fractions]** Given four positive real numbers a, b, c and d such that $\frac{a}{b} < \frac{c}{d}$ then $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.

The proof of it is simply algebra. We observe that $D = \frac{c}{d} - \frac{a}{b} = \frac{bc-ad}{bd}$ and by hypothesis $D > 0$. Then, we have

$$\frac{a+c}{b+d} - \frac{a}{b} = \frac{ab+bc-(ab+ad)}{b(b+d)} = \frac{bc-ad}{b(b+d)} = D\frac{d}{b+d} > 0 \text{ and}$$

$$\frac{c}{d} - \frac{a+c}{b+d} = \frac{bc+dc-(ad+dc)}{d(b+d)} = \frac{bc-ad}{b(b+d)} = D\frac{b}{b+d} > 0.$$

This shows the inequality but also something more, which is an estimate of the distance of $\frac{a+c}{b+d}$ relative to the endpoints of the interval $(\frac{a}{b}, \frac{c}{d})$. ∎

Remark: If we apply this construction several times and start with rational numbers, this gives a simple way of constructing more rational points inside of that interval. Of course, one can accomplish that same thing with regular averaging: $u < v$ implies $u < \frac{u+v}{2} < v$.

Some less standard application of the induction principle is the following proof of AM-GM-inequality (arithmetic mean-geometric mean inequality). We need to show that given a_1, a_2, ...,a_n non-negative numbers we have

$$\frac{1}{n}\sum_{i=1}^{n} a_i \geq (\prod_{i=1}^{n} a_i)^{1/n}. \tag{7.2}$$

First let us observe that we can assume that the numbers are strictly positive (if one of the numbers is zero, the right hand side of (7.2) is zero). Without loss of generality, we may assume that $\prod_{i=1}^{n} a_i = 1$. Indeed, if the product is not equal to one but say P, we can reduce to this situation by emplying the substitution $b_i = a_i/P^{1/n}$, $i = 1, 2, ..., n$.

For the Basis Step, we need to prove that $(1/2)(a+b) \geq 1$ if $ab = 1$. This is true since we can write $(1/2)(a+b) \geq 1$ as $(\sqrt{a}-\sqrt{b})^2 \geq 0$. For the Inductive Step, we assume that for n positive numbers $\{a_i\}$ whose product is 1, we have $a_1+a_2+...+a_n \geq n$. We need to show that given $n+1$ positive numbers b_j, whose product is 1, then $b_1+b_2+...+b_n+b_{n+1} \geq n+1$.

We know that $b_1b_2...b_nb_{n+1} = 1$. We notice that not all these numbers can be greater than 1. Otherwise the product is strictly greater than one. Hence, there exists $b_i \leq 1$. Similarly, not all the $b's$ can be less than 1. Thus, there exists b_j $(i \neq j)$ such that $b_j \geq 1$. Without loss of generality, we may assume that i and j are 1 and 2. By the induction hypothesis, $b_1b_2 + b_3 + ... + b_{n+1} \geq n$. Now, let us observe that $b_1 + b_2 \geq b_1b_2 + 1$ is equivalent to $0 \geq (b_1 - 1)(b_2 - 1)$ (true by our assumption on b_1 and b_2). Therefore,

$$b_1 + b_2 + b_3 + ... + b_{n+1} \geq b_1b_2 + 1 + b_3 + ... + b_{n+1} \geq n + 1,$$

which finishes the Induction Step. Hence by PMI, we must have (7.2) true for every n non-negative numbers. ∎

Remark: It is important to observe that equality in (7.2) happens only if all numbers are equal. This is indeed the case if $n = 2$ since $(\sqrt{a} - \sqrt{b})^2 = 0$ implies $a = b$. If $n > 2$, let us check this claim by induction. In the reduction we did above having the product of the numbers involved equal to 1, we may disregard the numbers which are already equal to one. So, if all the numbers involved are different from 1, the inequality used, is a strict inequality: $b_1 + b_2 > b_1 b_2 + 1$. As a result, $b_1 + b_2 + b_3 + ... + b_{n+1} > b_1 b_2 + 1 + b_3 + ... + b_{n+1} \geq n + 1$ which contradicts the induction hypothesis. It remains that all numbers must be equal if we have an identity in (7.2).

Excercise: Use induction to rearrange the above proof reducing (7.2) to the case $a_1 + a_2 + \cdots + a_n = n$.

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