C3	C5

C3

3.1 – Moments

Raw Moments

$$E[X_k]=u_k'=\begin{cases}\int_{-\infty}^{+\infty} x^k f(x)\,dx & \textit{Continuous}\\ \sum_i x_i^k p(x_i) & \textit{Discrete}\end{cases}$$

Central Moments

$$\kappa_k=E\left[(X-\mu)^k\right]=\begin{cases}\int_{-\infty}^{+\infty}(x-\mu)^k f(x)\,dx & \textit{Continuous}\\ \sum_i (x_i-\mu)^k p(x_i) & \textit{Discrete}\end{cases}$$

Skewness:

$$\gamma_1=\frac{\kappa_3}{(\kappa_2)^{3/2}},\kappa_3=E(X^3)-3E(X^2)E(X)+2E(X)^3$$

- Symmetric distributions have 0 skew
- Positive skewness → heavier on the right

Kurtosis:

$$\gamma_2=\frac{\kappa_4}{\kappa_2^2}$$

- Measures flatness of distribution relative to normal (which has kurtosis of 3)
- Kurtosis > 3 → flatter distribution

PGF & MGF

$$\left.\begin{array}{ll}MGF: & M_X(t)=E(e^{tX})\\ PGF: & P_X(t)=E(z^X)\end{array}\right\}\begin{array}{l}M_X(t)=P_X(e^t)\\ P_X(z)=M_X(\ln z)\end{array}$$

3.5 – Risk Measures & VaR

Coherent Risk Measure

$$\begin{array}{ll}\textit{Subadditivity}: & \rho(X+Y)\le\rho(X)+\rho(Y)\\ \textit{Monotonicity}: & X\le Y\to\rho(X)\le\rho(Y)\\ \textit{Homogeneity}: & \rho(cX)=c\rho(X)\\ \textit{Translation}: & \rho(X+c)=\rho(X)+c\end{array}$$

Tail-VAR

Utilized because VaR is not sub-additive. Basically just excess loss.

$$TVar(X)=E[X|X>q]=\frac{\int_q^\infty xf(x)\,dx}{S(q)}$$

For an arbitrary loss probability *p*, and loss quantile *q*.

4.2a – K-point Mixture

A r.v. Y is a k-point mixture of *k* r.v.s $X_1 \dots X_k$ if its CDF is given by:

$$F_Y(y)=p_1F_{X_1}(y)+p_2F_{X_2}(y)+\dots p_kF_{X_k}(y)$$

$$p_1+p_2+\dots p_k=1$$

F_Y is a weighted average of k different distributions. The raw moments of F_Y are a weighted average of the raw moments of the constituent CDFs.

4.2b – Variable-Component Mixture

$$F(x)=\sum_{i=1}^K a_iF_i(x),\sum_{i=1}^K a_i=1$$

$$i=1\dots K,K=1,2\dots$$

That is, a k-point mixture, but *K* is not known *a priori*.

C5

Creating New Distributions

Scaling

$$Y=nX,$$

$$F_Y(y)=F_X\left(\frac{y}{n}\right),f_Y(y)=\frac{1}{n}f_X\left(\frac{y}{n}\right)$$

Raising to a power

$$Y=X^{\frac{1}{\tau}},\tau>0,$$

$$F_Y(y)=F_X(y^\tau),$$

$$f_Y(y)=\tau y^{\tau-1}f_X(y^\tau),y>0$$

Transformed: τ > 0
Inverse: τ = -1
Inverse-transformed: τ < 0, but not -1

Exponent

$$Y=e^X$$

$$F_Y(y)=F_X(\ln y),$$

$$f_Y(y)=\frac{1}{y}f_X(\ln y)$$

Of which the most famous is the *log-normal*, though a more apt name would be *exp-normal*.

Mixing (A Marginal from a Model & Prior)

$$X\sim f_{X|\Theta}(x|\Theta=\theta),\theta=\pi(\Theta)$$

Where θ is a parameter of X, and is itself a r.v. with pdf $f_\Theta(\theta)$
Then the unconditional pdf of X is:

$$f_X(x)=\int f_{X|\Theta}(x|\Theta=\theta)\pi(\theta)\,d\theta$$

Where the integral is taken over all possible values of θ.
The 1st moments of the resulting *mixture distribution* are simply LIE and LTV respectively:

$$E(X^k)=E\left[E(X^k|\Theta)\right]$$

$$Var(X)=E\left[Var(X|\Theta)\right]+Var\left[E(X|\Theta)\right]$$

Frailty Models

A frailty model is first specified by an analytic hazard rate function, then the survival rate function.

$$h_{X|\theta}(x|\Theta)=\theta a(x),\theta=\pi(\Theta)$$

$$Then: S(x|\Theta)=e^{-\theta A(x)}$$

Splicing (Probit)

$$f_X(x)=\begin{cases}a_1f_1(x) & c_0<x<c_1\\ a_2f_2(x) & c_1<x<c_2\\ \vdots & \vdots\\ a_kf_k(x) & c_{k-1}<x<c_k\end{cases}$$

$$a_1+a_2+\dots a_k=1$$

Linear Exponential Family

Can be expressed as:

$$f(x;\theta)=\frac{a(x)e^{r(\theta)x}}{b(\theta)}$$

Moments

$$E[X]=\frac{b'(\theta)}{r'(\theta)b(\theta)}=\mu(x)$$

$$Var(X)=\frac{\mu'(x)}{r'(x)}$$