Credibility Theory

20.1 – Classic/Full/Limited Fluctuation Credibility

$P = Z\bar{X} + (1-Z)M$

Z = credibility factor
n = minimum sample size for full cred

Full Credibility (Z = 1)
Sample mean does not differ from stationary mean by a small amount r, with a high probability p. Commonly, $r = 0.05$, $p = 0.9$.
Set z as the smallest value such that:

$$P\left(\left|\frac{(1-r)\mu-\mu}{\sigma/\sqrt{n}}\right| \le z_{1-\alpha/2}\right) \ge p$$

$$\Rightarrow z_{1-\alpha/2} \le \frac{r\mu\sqrt{n}}{\sigma}$$

$$\Rightarrow n \le \left(\frac{z_{1-\alpha/2}}{r}\right)^2\left(\frac{\sigma}{\mu}\right)^2$$

This standard for full credibility applies for claim count, loss or aggregate severity, regardless of d.f.

Loss Measure	Standard for Full Cred	Partial-Cred factor Z
Claim Count (Poisson)	$n_{freq} \ge \left(\frac{z}{r}\right)^2$	$Z = \left(\frac{r}{z}\right)\sqrt{\lambda_N}$
Loss	$n_{loss} \ge \left(\frac{z}{r}\right)^2\left(\frac{\sigma_X}{\mu_X}\right)^2$	$Z = \sqrt{\frac{N}{\left(\frac{z}{r}\right)^2\left(\frac{\sigma_X}{\mu_X}\right)^2}}$
Aggregate Severity	$\frac{\mu_S}{\sigma_S} = \frac{\mu_X\sqrt{\lambda}}{\sqrt{\mu_X^2+\sigma_X^2}}$ $n_{sev} = \left(\frac{z}{r}\right)^2\left[1+\left(\frac{\sigma_X^2}{\mu_X^2}\right)\right] = n_{freq} + n_{loss}$	$Z = \sqrt{\frac{\lambda_N}{\left(\frac{z}{r}\right)^2\left(1+c_X^2\right)}} \equiv \sqrt{\frac{n}{n_X}}$

Note that n_{sev} does not need to take into account the *claim count df parameter* if claim counts are poisson or negative binomial.

20.3. Buhlmann-Straub Model (Greatest Accuracy/Least Squares)

Define loss r.v. **X** = $X_1 \ldots X_n$, of *i.n.i.d* loss variables, each with exposure weights $\mathbf{m_i} = m_1 \ldots m_n$.

$P_C = Z\bar{X} + (1-Z)\mu,$

$$\bar{X} = \sum_{i=1}^{n}\frac{X_i}{m_i}, Z = \frac{m}{m+k}$$

Vanilla Buhlmann is a special case where $m_i = 1$ for all i.

First Principles
The Buhlmann model is the least MSE model:

$$\hat{X}_{n+1} = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \ldots \beta_n X_n$$
$$= \hat{\beta}_0 + \hat{\beta}_S'\bar{X}$$
$$= \frac{1}{n+k}\bar{1}'\bar{X} + \frac{k}{n+k}\mu_X$$
$$= Z\bar{X} + (1-Z)\mu_X$$

NB: $Cov(X_i, X_j) = VHM$

	Analytical Results
Hypothetical means	$E[X_i \mid \Theta = \theta]$
Process variances	$\frac{Var[X_i \mid \Theta = \theta]}{m_i}$
EHM	$M = E[E(\Theta)] = E[\Theta](LIE)$
Expected Process Variance (v)	$EPV = E_\Theta[Var(X_i \mid \Theta = \theta)]$
Variance of hypothetical means (a)	$VHM = Var_\Theta[E[X_i \mid \Theta]]$ For analytical distributions, use the associated variance identity. For discrete data, use LTV: $VHM = E[\Theta^2] - E[\Theta]^2$

B.S. prediction for loss $X_{n+1} = E_{n+1}$ * B.S. Premium$_{n+1}$

2. Bayesian Premium

Bayesian premium = EHM, weighted by posterior probabilities = Conditional expected predictive, conditioned on **X**.

$$P = \sum_{\theta\in\Theta}\mu_n\pi(\theta \mid X)$$

Basic Bayesian Analysis

Categorical Class	$P(C) = c$
Prior Probabilities (actually conditional on class)	$\pi_C(\theta)$
Likelihood of observed data {**X**}	$f(\mathrm{X} = x \mid \Theta)$
Joint (Numerator of Bayes' Thm.)	$f(\mathrm{X} = x \mid \Theta)\pi_C(\theta)$
Posterior	$\pi_C(\Theta \mid \mathrm{X}) = \frac{f(\mathrm{X} = x \mid \Theta)\pi_C(\theta)}{f(\mathrm{X} = x)}$
Hypothetical Means	$E_{X\mid\Theta}[x]$
EHM	$P = \sum_{c\in C} E_{X\mid\Theta}\pi_c(\theta \mid X)$

$$E[X_{N+1} = a \mid X_N = x_n] = \int\left[\int x_{n+1} f_{X_{n+1}\mid\Theta}\, dx_{n+1}\right]\pi(\Theta \mid X_n)\, d\theta$$

The (unobservable) hypothetical means (actually conditional means) are:

$$\mu_n(\theta) = E[X_n \mid \theta] = \sum_{i=1}^{n} x_i p(x_i \mid \theta)$$

2.4 – Empirical Bayes Estimation

In 2.3, nice distributions for the model and prior distributions are known. Terminology:

- **Nonparametric** – if both the model and prior are unknown.
- **Semi-parametric** – a distribution for the loss variable X is known.
- **Parametric** – Both the model and the prior are known.

NB.:
Buhlmann estimates are least-squares approximations to Bayesian estimates, so will always form a straight line, cannot be entirely above or below the Bayesian Estimate.
Bayesian estimates cannot fall outside the range of the prior.

Setup

Group/Observation	$j = 1$	$j = 2$	…	$j = n_i$
Group i = 1	$m_{1,1}$ $X_{1,1}$	$m_{1,2}$ $X_{1,2}$	…	$m_{1,21}$ $X_{1,n1}$
…				
Group i = r	$m_{r,1}$ $X_{r,1}$	$m_{r,2}$ $X_{r,2}$	…	$m_{r,nr}$ $X_{r,nr}$

Notation

Obs Period $j=1\ldots n_i$	Indexes the j-th loss observation in the i-th risk group. The unwieldy notation n_i is used because some groups may have longer times to maturity than others. j can be a time index.
Groups $i=1\ldots r$	Indexes the classes/groups of policy holders
X_{ij}	Observed loss for the i-th group, out of r, during the j-th period, out of n_i.
m_{ij}	Amount of exposure for group i during observation period j. X_{ij} is then the loss per unit of exposure.

1. Full Non-Parametric Bayes

Unbiased estimators for EPV and VHM:

$$\hat{\mu}_{PV} = \frac{\sum_{i=1}^{r}\sum_{j=1}^{n_i} m_{ij}\left(X_{ij}-\bar{X}_i\right)^2}{\sum_{i=1}^{r}\left(n_i-1\right)}$$

$$\hat{\sigma}^2_{HM} = \frac{\left[\sum_{i=1}^{r} m_i\left(\bar{X}_i-\bar{X}\right)^2\right]-\left(r-1\right)\hat{\mu}_{PV}}{m-\frac{1}{m}\sum_{i=1}^{r} m_i^2}, m_i = \sum_{i=1}^{n_i} m_{ij}$$

$$\Rightarrow \hat{\sigma}_i^2 = \frac{1}{\left(n_i-1\right)}\left[\sum_{j=1}^{n_i} m_{ij}\left(X_{ij}-\bar{X}_i\right)^2\right], w_i = \frac{n_i-1}{\sum_{i=1}^{r}\left(n_i-1\right)}$$

$$\Rightarrow \hat{\mu}_{PV} = \sum_{i=1}^{r} w_i\hat{\sigma}_i^2$$

$$\Rightarrow k_i = \frac{\hat{\mu}_{PV}}{\hat{\sigma}^2_{HM}}, P_i = Z_i\bar{X}_i + \left(1-Z_i\right)\bar{X}$$

1a. All Risk Groups Have The Same Exposure

$$\hat{\mu}_{PV} = \frac{\sum_{i=1}^{r}\sum_{j=1}^{n_i}\left(X_{ij}-\bar{X}_i\right)^2}{\sum_{i=1}^{r}\left(n_i-1\right)}$$

$$\hat{\sigma}^2_{HM} = \frac{\left[\sum_{i=1}^{r} n_i\left(\bar{X}_i-\bar{X}\right)^2\right]-\left(r-1\right)\hat{\mu}_{PV}}{n-\frac{1}{n}\sum_{i=1}^{r} n_i^2}$$

1b. All Risk Groups have the same Exposure & n

$$\hat{\mu}_{PV} = \frac{1}{r}\sum_{i=1}^{r}\left\{\frac{\sum_{j=1}^{n}\left(X_{ij}-\bar{X}_i\right)^2}{\left(n-1\right)}\right\} = \frac{1}{r}\left[\sum_{i=1}^{r}\hat{\sigma}_i^2\right]$$

$$\hat{\sigma}^2_{HM} = \frac{1}{r-1}\left[\sum_{i=1}^{r}\left(\bar{X}_i-\bar{X}\right)^2\right]-\frac{\hat{\mu}_{PV}}{n}$$

Semi-Parametric

A conditional distribution about the model/likelihood is known/assumed.

e.g. For a Poisson claim count, and non-parametric loss df with total population size n:

$$EPV = \frac{\sum x_i}{n}, VHM = \frac{1}{n-1}\left[\sum_{i=1}^{n} p\left(x_i\right)\left(x_i-\bar{x}\right)\right]$$

For grouped claimed counts in r groups, where each group has size m_i,

$$EPV = \frac{\sum x_i}{n}, VHM = \frac{1}{m-1}\left[\sum_{i=1}^{r} m_i\left(X_i-\bar{X}\right)^2\right]-\bar{X}$$

Parametric