9.3 – Compound Distributions

S is the sum of a discrete random number, N, of random loss variables, X:

$S = X_1 + X_2 + \ldots X_N$

Back Matter: Convolution

$$(f * g)(t) \overset{def}{=} \int_0^x f(t) g(x-t)\,dt$$

The Compound Model

$$\left.\begin{aligned} F_S &= \sum_n^\infty p_n F_X^{*n}(x) \\ f_S &= \sum_n^\infty p_n f_X^{*n}(x) \end{aligned}\right\} \begin{aligned} F_X^{*n} &= \int_0^x F_X^{*(n-1)}(x-t) f_X(t)\,dt, n = 2,3\ldots \\ f_X^{*n} &= \int_0^x f_X^{*(n-1)}(x-t) f_X(t)\,dt, n = 2,3\ldots \end{aligned}$$

Note that this the specific case where the X's are i.i.d.

This is computed recursively, since n is a random variable, so the aggregate severity distribution must be computed by brute force summing of all possible values of n.

P(S=0) has a discrete, possibly non-zero, probability at $x = 0$. N.B.: $P(S=0) \neq f_S(0)$.

Severity Distributions Closed Under Convolution

A severity distribution is closed under convolution if adding i.i.d members of a family produces another family of that family. Formally:

$$X_i \sim f_X(x;a), S = \sum_{i=1}^N X_i$$

$$f_S = \sum_{i=1}^\infty p_n f_X^{*n}(x;a) = \sum_{i=1}^\infty p_n f_X(x;na)$$

Properties of S

MGF & PGF

$$M_S(t) = M_N\left[\lg M_X(t)\right]$$

$$G_S(z) = G_N\left[G_X(t)\right]$$

Using PGF$_S$ can be a more convenient way to evaluate $f_S(s)$:

$$f_S(S=0) = G_S(0) = G_N\left[G_X(0)\right]$$

$$f_S(S=1) = G_S'(0) = \left\{G_N'\left[G_X(0)\right]\right\} G_X'(0)$$

$$\vdots$$

$$f_S(S=k) = \frac{G_S^{(k)}(0)}{k!}$$

Moments

$$E(S) = E(N)E(X)$$

$$Var(S) = E(N)Var(X) + E(X)^2 Var(N)$$

9.3b – Net stop-loss Premium

The deductible is applied to aggregate losses, instead of individual losses:

$$E\left[(S-d)_+\right] = \int_d^\infty 1 - F_S(x)\,dx$$

$$= \int_d^\infty (x-d) f_S(x)\,dx$$

Linear Interpolation

In case there is an interval with no aggregate probability, interpolation may be used.

Suppose P(a < S < b) = 0. Then, for a ≤ d ≤ b,

$$E\left[(S-d)_+\right] = \frac{b-d}{b-a} E\left[(S-a)_+\right] + \frac{d-a}{b-a} E\left[(S-b)_+\right]$$

9.4 – Analytic Shortcuts for Aggregate-Severity

1. Negative Binomial-Exponential

Frequency ~ NB, Severity ~ exponential

$$F_S(x) = 1 - \left[\sum_{n=1}^r \binom{r}{n} \left(\frac{\beta}{1+\beta}\right)^n \left(\frac{1}{1+\beta}\right)^{r-n}\right] \left[\sum_{j=0}^{n-1} \left[x\theta^{-1}(1-\beta)^{-1}\right]^j e^{-x\theta^{-1}(1+\beta)^{-1}} \Big/ j!\right]$$

If r = 1, this is a *geometric compound distribution*:

$$f_S = \frac{\beta}{\theta(1+\beta)^2} \exp\left[-\frac{x}{\theta(1+\beta)}\right], F_S = 1 - \frac{\beta}{1+\beta} \exp\left[-\frac{x}{\theta(1+\beta)}\right], x \geq 0$$

2. Exponential Severity

F_S is a gamma-mixture.

The MGF of the sum of N exponentially distributed r.v.s with same mean θ is gamma:

$$M_{X_1+X_2+\ldots X_N}(t) = (1-\theta t)^{-n}$$

$$\Rightarrow F_X^{*n}(x) = \Gamma(n; x/\theta) = 1 - \sum_{j=0}^{n-1} \frac{(x/\theta)^j}{j!} e^{-x/\theta}, n = 1,2,3\ldots$$

(Discrete form of lower incomplete Gamma function)

Distribution of aggregate claims:

$$f_S = \sum_{n=1}^\infty p_n \left\{\frac{1}{\theta} e^{-x/\theta} \frac{(x/\theta)^{n-1}}{(n-1)!}\right\}, F_S = p_0 + \sum_{n=1}^\infty p_n \Gamma(n; x/\theta)$$

Which is a weighted gamma-mixture.

3. Compound Poisson Aggregate Severity

$$X_i \sim Po(\lambda_i) \Rightarrow S \sim Po\left(\lambda = \sum_{i=1}^n \lambda_i\right)$$

$$\Rightarrow F_S = \sum_{i=1}^n \frac{\lambda_i}{\lambda} F_{X_i}(x)$$

Note that $P(N = n) = \dfrac{\lambda_i}{\lambda}$

9.6 – Computing Aggregate Distribution

1. Recursion for (a, b, 0), (a, b, 1) Class

(a, b, 1) *N* Distribution

Severity distribution is discrete and defined on 1, 2…,m (m can be infinite)

$$f_S = \frac{\left[p_1 - (a-b)p_0\right] f_X(x) + \sum_{y=1}^{\min(x,m)} \left[a + \frac{by}{x}\right] f_X(y) f_S(x-y)}{1 - a f_X(0)}$$

(a, b, 0)]*N* Distribution

$$f_S = \frac{\sum_{x=1}^{s} \left[a + \frac{bx}{s}\right] f_X(y) f_S(x-y)}{1 - a f_X(0)}, x \in \mathrm{Z}_+$$

Both cases start with calculating the value of $P(S = 0) = P_N(f_X(0))$.

2. Continuous Severity

(a, b, 1) & (a, b, 0) Frequency Distribution

$$f_S = p_1 f_X(x) + \int_0^x \left[a + \frac{by}{x}\right] f_X(y) f_S(x-y)\,dy$$

3. Discretization of Continuous Distributions

3.1 Mass Dispersal

For an arbitrary interval of span *h*, index $j = 0, 1, 2, \dots$

$$f_0 = P\left(X < \frac{h}{2}\right) = F_X\left(\frac{h}{2} - 0\right)$$

$$f_j = P\left(jh - \frac{h}{2} \le X < jh + \frac{h}{2}\right) = F_X\left(jh + \frac{h}{2} - 0\right) - F_X\left(jh - \frac{h}{2} - 0\right)$$

3.2 Method of Moment (matching)

The idea is to construct an arithmetic distribution function that matches the first *p* moments.

1. Construct an arbitrary interval of length *ph*, denoted $[x_k, x_k + ph)$.
2. Compute point masses $m_0, m_1 \dots m_p$ at points $x_k, x_k + h \dots x_k + ph$
3. Lagrange interpolation:

$$\sum_{j=0}^{p} (x_k + jh)^r m_j^k = \int_{x_k - 0}^{x_k + ph - 0} x^r f_X(x)\,dx, r = 0,1,2 \dots p$$

$$m_j^k = \int_{x_k - 0}^{x_k + ph - 0} \prod_{i \neq j} \frac{x - x_k - ih}{(j-i)h} f_X(x)\,dx, j = i \dots p$$

9.7 – Effect of Individual Modifiers on Aggregate Severity

Per-Payment and Per-Loss Relationships

$$F_{Y^L}(y) = (1-v) + vF_{Y^P}(y), y \ge 0$$

$$M_{Y^L}(t) = (1-v) + vM_{Y^P}(t)$$

$$Where: v = P(X > d)$$

Changes to Frequency

$$G_{N^P}(z) = G_{N^L}(1 - v + vz)$$

Moments

It is more convenient to use:

$$E[S] = E(N^L)E(Y^L)$$

$$Var(S) = E[N]Var(Y^L) + E[Y^L]^2 Var(N)$$

9.11 – Individual Risk Models

This is the compound model with one additional complication: severities are compound Bernoulli:

Parametric Approximation

That is, just assuming a normal, gamma, or lognormal distribution.