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RESEARCH ARTICLE

On a Lagrangian formulation of gravitoelectromagnetism

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Abstract We examine a Lagrangian formulation of gravity based on an approach analogous to electromagnetism, called *Gravitoelectromagnetism* (GEM). The gravitational analogue of the electromagnetic field tensor is a three-index tensor, $\mathcal{F}_{\mu\nu\lambda}$, defined in terms of a two-index gravitoelectromagnetic potential, $\mathcal{A}_{\mu\nu}$. The energy-momentum tensor is derived and is symmetric. We construct a Lagrangian which allows us to describe interactions between fermions, photons and gravitons. We calculate transition amplitudes of various processes involving gravitons: gravitational Møller scattering, gravitational Compton scattering, and the graviton photoproduction.

Keywords Gravitoelectromagnetism · Lagrangian formalism · General relativity · Gauge fields

1 Introduction

In this paper, we consider a description of gravity based on an analogy with electromagnetism, called *Gravitoelectromagnetism* (GEM). Specifically, we exploit the analogous gauge structure of GEM (for which the potential and field tensor carry

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one more index than their electromagnetic equivalents) in order to construct a Lagrangian theory of GEM. The similarity with traditional gauge theories helps us to discuss various processes involving fermions, photons and gravitons. One of these processes is the graviton photoproduction, described by the reaction: $e + \gamma \rightarrow e + g$, which is of interest in modern experiments such as the International Linear Collider [1] and the Canadian Light Source [2]. A discussion of this process can be found in Ref. [3]. Electron-photon scattering using a high energy beam and a laser has been used to obtain a beam of polarised γ-rays that was used for various experiments in nuclear physics [4–7].

The approach to gravity provided by GEM has a long history. As early as 1832, Faraday considered a possible relation between electromagnetism and gravity when he conducted some unsuccessful experiments to detect the electric current induced in a massive circuit falling through the gravitational field at the Earth's surface [8]. Various aspects of the gravitation-electromagnetism analogy were considered, for instance, by Maxwell [9], Heaviside [10–12], Einstein [13], and Lense and Thirring [14–16]. Several formalisms have been investigated in order to unravel the rich analogy and remarkable correspondence between electromagnetism and gravitation [17–30]. An interesting series of experiments in order to detect gravito-magnetic fields were proposed by Forward [31]. Modern approaches to this analogy and some applications are described in [32–38]. This description of gravity in analogy with electromagnetism has also been studied with group theoretical methods [39,40]. An extensive list of references is provided in [41]. Recently, a comparative study among different formulations of the analogy, based on tidal tensor, has been considered [42].

The GEM approaches fall under two main categories. The first one relies on the similarity between the linearized Einstein's equations, in the harmonic gauge, and Maxwell's equations in the Lorentz gauge. This is valid within the framework of the weak-field approximation. The second type of analogy relies on the irreducible splitting of the Weyl tensor into two parts: electric and magnetic components. The field equations for the components of the Weyl tensor have a structure similar to Maxwell's equations of electromagnetism. In this paper, we use the second approach to investigate flat space–time treatment of the theory of gravitation in order to describe the interaction between gravitational field and other fields using a Lagrangian formalism.

As shown by Campbell and others [26–29], the Einstein field equations, $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}$, together with the Bianchi identities, $R_{\mu\nu\alpha\beta;\gamma}\,\epsilon^{\alpha\beta\gamma\delta} = 0$, can be combined to give the Weyl curvature tensor [43,44]:

$$C_{\alpha\beta\mu\nu;}{}^{\nu} = 4\pi G\left(-T_{\mu\beta;\alpha} + T_{\mu\alpha;\beta} + \frac{1}{3}T_{,\alpha}g_{\mu\beta} - \frac{1}{3}T_{,\beta}g_{\mu\alpha}\right),$$

where $T_{\mu\nu}$ is the energy-momentum tensor and $T = T^{\mu}_{\mu}$. The Weyl tensor can be decomposed into two symmetric traceless tensors: the electric-like tensor (or *gravito-electric* field), $\mathcal{E}_{ij} = -C_{0i0j}$, and the magnetic-like tensor (that is, *gravito-magnetic field*), $\mathcal{B}_{ij} = \frac{1}{2}\epsilon_{ikl}C^{kl}{}_{0j}$ where $i, j, k, \ldots = 1, 2, 3$ [26–29]. In a nearly flat space–time, this leads to the following Maxwell-like equations:

$$\begin{aligned}
\partial_j \mathcal{B}_{jk} &= -4\pi G \epsilon_{kjn} \partial_j T_{0n}, \\
\partial_j \mathcal{E}_{jk} &= 4\pi G \left[\partial_j T_{jk} - \frac{1}{3} \partial_k (T_{ii} + 2T_{00}) \right], \\
\epsilon_{klm} \partial_l \mathcal{B}_{jm} - \partial_t \mathcal{E}_{jk} &= 4\pi G \left(\partial_j T_{0k} - \partial_t T_{jk} + \frac{1}{3} \delta_{jk} \partial_t T \right), \\
\epsilon_{klm} \partial_l \mathcal{E}_{jm} + \partial_t \mathcal{B}_{jk} &= -4\pi G \left(\epsilon_{jlm} \partial_l T_{mk} + \frac{1}{3} \epsilon_{ljk} \partial_l T \right).
\end{aligned}$$

where ϵ^{klm} is the Levi-Civita symbol. In this paper, we are interested in similar equations where the right-hand side of the first and fourth equations is equal to zero. We will set up a Lagrangian that leads to these equations.

Somewhat surprisingly, it appears that there is no systematic investigation of a Lagrangian formulation of GEM in the literature, and this, in spite of numerous investigations of the underlying gauge structure and symmetries of GEM. We construct GEM Lagrangians by considering the symmetric gravitoelectromagnetic tensor potential $\mathcal{A}^{\mu\nu}$, as the fundamental field which describes the gravitational interaction. A symmetric energy-momentum tensor for the gravitational field is obtained. Hereafter, caligraphic symbols denote GEM quantities. Then we study the interaction of gravitons with fermions and photons, in the same way as in Ref. [45]; a crucial difference is that we use the GEM potential, $\mathcal{A}^{\mu\nu}$, as the fundamental gravitational field in flat space–time, instead of the small perturbation, $h_{\mu\nu}$, in the weak-field approximation of the Einstein field.

In Sect. 2, we construct the Lagrangian function for GEM and its energy-momentum tensor. In Sect. 3, we define a full Lagrangian which describes interactions between fermions, photons and gravitons and discuss its gauge invariance in the context of gravitation and give equations of motion. The transition amplitudes of various processes are calculated in Sect. 4. Finally, in Sect. 5, we present some conclusions.

2 Gravitoelectromagnetism and its free Lagrangian

The Maxwell-like equations of GEM without sources in flat space time [39] are given by

$$\begin{aligned}
\partial^i \mathcal{E}^{ij} &= 0, \\
\partial^i \mathcal{B}^{ij} &= 0, \\
\epsilon^{\langle ikl} \partial^k \mathcal{B}^{lj\rangle} - \frac{1}{c} \frac{\partial \mathcal{E}^{ij}}{\partial t} &= 0, \\
\epsilon^{\langle ikl} \partial^k \mathcal{E}^{lj\rangle} + \frac{1}{c} \frac{\partial \mathcal{B}^{ij}}{\partial t} &= 0,
\end{aligned}$$

where the gravito-electric field $\mathcal{E}^{ij}$ and the gravito-magnetic field $\mathcal{B}^{ij}$ are both symmetric and traceless tensors (STT) of rank two. The symbol $\langle \cdots \rangle$ denotes the symmetrization of the first and last indices i.e. i and j, in the equations above, and the traceless part of the tensor [34,35]. Henceforth, we work with Gaussian units, c is the speed of light in vacuum, and the metric of flat Minkowski space is given by $\eta_{\mu\nu}$, is $(-1, +1, +1, +1)$. In the presence of sources [40], the Maxwell-like equations become

$$\partial^i \mathcal{E}^{ij} = 4\pi G\rho^j, \tag{1}$$

$$\partial^i \mathcal{B}^{ij} = 0, \tag{2}$$

$$\epsilon^{\langle ikl}\partial^k \mathcal{B}^{lj\rangle} - \frac{1}{c}\frac{\partial \mathcal{E}^{ij}}{\partial t} = \frac{4\pi G}{c} J^{ij}, \tag{3}$$

$$\epsilon^{\langle ikl}\partial^k \mathcal{E}^{lj\rangle} + \frac{1}{c}\frac{\partial \mathcal{B}^{ij}}{\partial t} = 0, \tag{4}$$

where ρ^j is the vector mass density and J^{ij}, a rank-2 STT, is the mass current density. Here G is the gravitational constant.

We express the fields $\mathcal{E}^{ij}$ and $\mathcal{B}^{ij}$ in terms of a symmetric rank-2 tensor field, $\tilde{\mathcal{A}}$, with components $\mathcal{A}^{ij}$, where $i, j = 1, 2, 3$, such that

$$\mathcal{B} = \text{curl}\ \tilde{\mathcal{A}}\ , \tag{5}$$

in terms of components, Eq. (5) is expressed as $\mathcal{B}^{ij} = \epsilon^{\langle ikl}\partial^k \mathcal{A}^{lj\rangle}$ and the divergence of $\mathcal{B}$ is $\text{div}\,\mathcal{B} = \partial^i \mathcal{B}^{ij}$. (Later, we will define the non-tilde vector, $\mathcal{A}$, for all four space–time coordinates.) Note that $\text{div}\,\mathcal{B} = \text{div}\,\text{curl}\,\tilde{\mathcal{A}} \neq 0$ [34,35,40,46,47]. Indeed, we find

$$\text{div}\ \text{curl}\,\tilde{\mathcal{A}} = \frac{1}{2}\text{curl}\ \text{div}\,\tilde{\mathcal{A}}.$$

Therefore, we will take $\tilde{\mathcal{A}}$ such that $\text{div}\,\tilde{\mathcal{A}} = 0$. This implies that $\text{div}\,\mathcal{B} = \text{div}\ \text{curl}\,\tilde{\mathcal{A}} = 0$. With $\text{curl}\,\mathcal{E} = \epsilon^{\langle ikl}\partial^k \mathcal{E}^{lj\rangle}$ and by substituting Eq. (5) into Eq. (4), we find

$$\text{curl}\left(\mathcal{E} + \frac{1}{c}\frac{\partial \tilde{\mathcal{A}}}{\partial t}\right) = 0.$$

We define the gravito-electric field $\mathcal{E}$, with components $\mathcal{E}^{ij}$ $(i, j = 1, 2, 3)$, as

$$\mathcal{E} + \frac{1}{c}\frac{\partial \tilde{\mathcal{A}}}{\partial t} = -\text{grad}\,\boldsymbol{\varphi},$$

where $\text{grad}\,\boldsymbol{\varphi} = \partial^i \varphi^j$. Note that $\boldsymbol{\varphi}$ is the GEM vector counterpart of the EM scalar potential ϕ. As for $\tilde{\mathcal{A}}$, we observe here that $\text{curl}\ \text{grad}\,\boldsymbol{\varphi} \neq 0$ [40,46,47], since

$$\text{curl}\ \text{grad}\,\boldsymbol{\varphi} = \frac{1}{2}\text{grad}\ \text{curl}\,\boldsymbol{\varphi}.$$

As for the case of $\mathcal{B}$, if the vector $\boldsymbol{\varphi}$ satisfies

$$\operatorname{curl} \boldsymbol{\varphi} = 0,$$

then we find curl grad $\boldsymbol{\varphi} = 0$. From the definition of the tensor field $\mathcal{E}$, we observe that since $\mathcal{E}$ is a rank-2 STT, the tensors $\tilde{\mathcal{A}}$ and grad $\boldsymbol{\varphi}$ are also rank-2 STTs.

To summarize, the GEM fields $\mathcal{E}$ and $\mathcal{B}$ are defined as

$$\begin{aligned} \mathcal{E} &= -\operatorname{grad} \boldsymbol{\varphi} - \frac{1}{c}\frac{\partial \tilde{\mathcal{A}}}{\partial t}, \\ \mathcal{B} &= \operatorname{curl} \tilde{\mathcal{A}}, \end{aligned} \tag{6}$$

with the following conditions on the vector $\boldsymbol{\varphi}$ and the rank-2 tensor $\tilde{\mathcal{A}}$:

$$\operatorname{curl} \boldsymbol{\varphi} = 0 \quad \operatorname{div} \tilde{\mathcal{A}} = 0$$

Since $\tilde{\mathcal{A}}$ is a STT, its diagonal elements satisfy $\mathcal{A}^{11} + \mathcal{A}^{22} + \mathcal{A}^{33} = 0$. The same is true for rank-2 tensor grad φ where $\partial^1\varphi^1 + \partial^2\varphi^2 + \partial^3\varphi^3 = 0$. In terms of components, Eq. (6) becomes

$$\mathcal{E}^{ij} = -\begin{pmatrix} \partial^1\varphi^1 & \partial^1\varphi^2 & \partial^1\varphi^3 \\ \partial^1\varphi^2 & \partial^2\varphi^2 & \partial^2\varphi^3 \\ \partial^1\varphi^3 & \partial^2\varphi^3 & \partial^3\varphi^3 \end{pmatrix} - \frac{1}{c}\partial_t \begin{pmatrix} \mathcal{A}^{11} & \mathcal{A}^{12} & \mathcal{A}^{13} \\ \mathcal{A}^{12} & \mathcal{A}^{22} & \mathcal{A}^{23} \\ \mathcal{A}^{13} & \mathcal{A}^{23} & \mathcal{A}^{33} \end{pmatrix},$$

and

$$\mathcal{B}^{ij} = \epsilon^{\langle ikl}\partial^k A^{lj\rangle}.$$

The $\mathcal{E}$ and $\mathcal{B}$ tensor fields are elements of a rank-3 tensor, the *gravitoelectromagnetic tensor* $\mathcal{F}$, defined by

$$\mathcal{F}^{\mu\nu\alpha} = \partial^\mu \mathcal{A}^{\nu\alpha} - \partial^\nu \mathcal{A}^{\mu\alpha}, \tag{7}$$

where $\mu, \nu, \alpha = 0, 1, 2, 3$. We write the elements of the rank-2 symmetric tensor $\mathcal{A}^{\mu\nu}$ as

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}^{00} & \mathcal{A}^{01} & \mathcal{A}^{02} & \mathcal{A}^{03} \\ \mathcal{A}^{10} & \mathcal{A}^{11} & \mathcal{A}^{12} & \mathcal{A}^{13} \\ \mathcal{A}^{20} & \mathcal{A}^{21} & \mathcal{A}^{22} & \mathcal{A}^{23} \\ \mathcal{A}^{30} & \mathcal{A}^{31} & \mathcal{A}^{32} & \mathcal{A}^{33} \end{pmatrix} = \begin{pmatrix} \varphi^0 & \varphi^1 & \varphi^2 & \varphi^3 \\ \varphi^1 & \mathcal{A}^{11} & \mathcal{A}^{12} & \mathcal{A}^{13} \\ \varphi^2 & \mathcal{A}^{12} & \mathcal{A}^{22} & \mathcal{A}^{23} \\ \varphi^3 & \mathcal{A}^{13} & \mathcal{A}^{23} & \mathcal{A}^{33} \end{pmatrix},$$

Although the symmetry properties of $\mathcal{A}_{\mu\nu}$ are similar to those of $h_{\mu\nu}$, and the interactions terms in the Lagrangian look like those of the linear case, our approach is different, since the nature of $\mathcal{A}_{\mu\nu}$ is different from $h_{\mu\nu}$. Indeed, the tensor potential has nothing to do with the perturbation of the metric of space–time; instead, it is connected directly with the description of the gravitational field in flat space–time.

From Eq. (7), we observe that $\mathcal{F}$ is antisymmetric in its first two indices:

$$\mathcal{F}^{\mu\nu\alpha} = -\mathcal{F}^{\nu\mu\alpha},$$

and obeys the cyclic identity

$$\mathcal{F}^{\mu\nu\alpha} + \mathcal{F}^{\nu\alpha\mu} + \mathcal{F}^{\alpha\mu\nu} = 0.$$

The tensor $\mathcal{F}^{\mu\nu\alpha}$ has properties similar to a three-index tensor called the *Fierz tensor*, which is discussed earlier [48,49] in order to represent a spin-2 field interacting with an external graviton, whereas in our approach the symmetric tensor $\mathcal{A}^{\mu\nu}$ is the fundamental field which describes spin-2 gravitons.

The non-zero components of $\mathcal{F}$ are defined by

$$\mathcal{F}^{0ij} = \mathcal{E}^{ij}, \quad \mathcal{F}^{ijk} = \epsilon^{ijl}\mathcal{B}^{lk}.$$

Note that $\mathcal{F}^{100}$, $\mathcal{F}^{200}$, $\mathcal{F}^{300}$, $\mathcal{F}^{010}$, $\mathcal{F}^{020}$ and $\mathcal{F}^{030}$ all vanish and lead to the following condition: $\nabla\varphi^0 + \frac{1}{c}\frac{\partial\varphi^i}{\partial t} = 0$.

We define the *dual gravitoelectromagnetic tensor* $\mathcal{G}$ by

$$\mathcal{G}^{\mu\nu\alpha} = \frac{1}{2}\epsilon^{\mu\nu\gamma\sigma}\eta^{\alpha\beta}\mathcal{F}_{\gamma\sigma\beta},$$

where

$$\mathcal{F}_{\gamma\sigma\beta} = \eta_{\gamma\mu}\eta_{\sigma\nu}\eta_{\beta\alpha}\mathcal{F}^{\mu\nu\alpha}.$$

The components of $\mathcal{G}$ are related to those of $\mathcal{F}$ by changing $\mathcal{E} \to \mathcal{B}$ and $\mathcal{B} \to -\mathcal{E}$.

Therefore the Maxwell-like equations, given in Eqs. (1)–(4), are written in a covariant form:

$$\begin{aligned} \partial_\mu\mathcal{F}^{\mu\nu\alpha} &= -\frac{4\pi G}{c}\mathcal{J}^{\nu\alpha}, \\ \partial_\mu\mathcal{G}^{\mu\langle\nu\alpha\rangle} &= 0, \end{aligned} \tag{8}$$

where $\mathcal{J}^{\nu\alpha}$ is a rank-2 tensor which depends on the mass density $c\rho^i$ and mass current density J^{ij}.

The GEM Lagrangian density is then written as

$$\mathcal{L}_{\text{GEM}} = -\frac{1}{16\pi G}\mathcal{F}_{\mu\nu\alpha}\mathcal{F}^{\mu\nu\alpha} + \frac{1}{c}\mathcal{J}^{\nu\alpha}\mathcal{A}_{\nu\alpha},$$

where G is the gravitational constant. The Euler–Lagrange equations lead to Eq. (8). In addition, Eq. (8) leads to the continuity equation for GEM,

$$\partial_\nu\mathcal{J}^{\nu\alpha} = 0.$$

Quantisation of GEM with the symmetric tensor field $\mathcal{A}_{\mu\nu}$ leads to spin-2 gravitons, in analogy to the electromagnetism field, where the vector potential A^{μ} yields spin-1 photons. Then this allows us to consider a full Lagrangian that describes gravitons, fermions and photons along with their interactions.

Before studying the full interaction Lagrangian, we complete the Lagrangian formulation of GEM obtaining an expression for the energy-momentum tensor $T_G^{\mu\nu}$ of GEM.

Following the standard definition, the tensor $T_G^{\mu\nu}$ is expressed as

$$T_G^{\mu\nu} = \eta^{\mu\nu}\mathcal{L}_G - \frac{\partial \mathcal{L}_G}{\partial(\partial_\mu \mathcal{A}^{\alpha\beta})}(\partial^\nu \mathcal{A}^{\alpha\beta}),$$

where $\partial_\mu T_G^{\mu\nu} = 0$ and $\mathcal{L}_G$ is the free GEM Lagrangian

$$\mathcal{L}_G = -\frac{1}{16\pi G}\mathcal{F}_{\rho\sigma\theta}\mathcal{F}^{\rho\sigma\theta}.$$

Then $T_G^{\mu\nu}$ is explicitly given by

$$T_G^{\mu\nu} = \eta^{\mu\nu}\mathcal{L}_G + \frac{1}{4\pi G}\eta^{\gamma\mu}\mathcal{F}_{\gamma\alpha\beta}(\partial^\nu \mathcal{A}^{\alpha\beta}).$$

However, we notice that this tensor is not symmetric. We follow the standard prescription to symmetrize it as it is done for the energy-momentum tensor of electromagnetism. From the definition of $\mathcal{F}^{\alpha\nu\beta}$, we substitute $\partial^\nu \mathcal{A}^{\alpha\beta} = -\mathcal{F}^{\alpha\nu\beta} + \partial^\alpha \mathcal{A}^{\nu\beta}$ and obtain

$$T_G^{\mu\nu} = -\frac{1}{4\pi G}\left[\eta^{\gamma\mu}\mathcal{F}_{\gamma\alpha\beta}\mathcal{F}^{\alpha\nu\beta} + \frac{1}{4}\eta^{\mu\nu}\mathcal{F}_{\rho\sigma\theta}\mathcal{F}^{\rho\sigma\theta}\right] + \frac{1}{4\pi G}\eta^{\gamma\mu}\mathcal{F}_{\gamma\alpha\beta}\left(\partial^\alpha \mathcal{A}^{\nu\beta}\right). \tag{9}$$

The terms in square bracket are symmetric in μ and ν and gauge invariant. With the help of the source-free equations, $\partial^\alpha \mathcal{F}_{\alpha\gamma\beta} = 0$, the last term can be written as

$$t_G^{\mu\nu} = \frac{1}{4\pi G}\eta^{\gamma\mu}\mathcal{F}_{\gamma\alpha\beta}\partial^\alpha \mathcal{A}^{\nu\beta} = \frac{1}{4\pi G}\eta^{\gamma\mu}\partial^\alpha\left(\mathcal{F}_{\gamma\alpha\beta}\mathcal{A}^{\nu\beta}\right).$$

The tensor $t_G^{\mu\nu}$ satisfies the following properties:

$$\text{(i)}\quad \partial_\mu t_G^{\mu\nu} = 0, \quad \text{(ii)}\quad \int t_G^{0\nu} d^3x = 0.$$

Thus the conservation law will hold for $T_G^{\mu\nu} - t_G^{\mu\nu}$ if it holds for $T_G^{\mu\nu}$. Therefore, we define the symmetric energy-momentum tensor $\Theta_G^{\mu\nu}$ of GEM by $\Theta_G^{\mu\nu} = T_G^{\mu\nu} - t_G^{\mu\nu}$

or

$$\Theta_G^{\mu\nu} = -\frac{1}{4\pi G}\left[\eta^{\gamma\mu}\mathcal{F}_{\gamma\alpha\beta}\mathcal{F}^{\alpha\nu\beta} + \frac{1}{4}\eta^{\mu\nu}\mathcal{F}_{\rho\sigma\theta}\mathcal{F}^{\rho\sigma\theta}\right], \tag{10}$$

with $\partial_\mu \Theta_G^{\mu\nu} = 0$.

3 Full interaction Lagrangian

The full Lagrangian formulation of GEM including interactions of gravitons with fermions and photons leads us to calculate their scattering amplitudes and cross sections. The rank-2 symmetric tensor $\mathcal{A}^{\mu\nu}$ is the fundamental field which represents the gravitational field, and it plays the same role as the potential A^μ in electromagnetism.

3.1 Lagrangian

The full Lagrangian density is written as

$$\mathcal{L} = \mathcal{L}_G + \mathcal{L}_F + \mathcal{L}_A + \mathcal{L}_{FA} + \mathcal{L}_{GF} + \mathcal{L}_{GA} + \mathcal{L}_{GFA}, \tag{11}$$

where

$$\mathcal{L}_G = -\frac{1}{16\pi G}\mathcal{F}_{\mu\nu\alpha}\mathcal{F}^{\mu\nu\alpha}$$

is the kinetic-energy term for the gravito-electromagnetic field, $\mathcal{F}_{\mu\nu\alpha}$;

$$\mathcal{L}_F = -\frac{i\hbar c}{2}\left(\overline{\psi}\gamma^\mu\partial_\mu\psi - \partial_\mu\overline{\psi}\gamma^\mu\psi\right) + mc^2\overline{\psi}\psi$$

describes the Fermi field, ψ, with mass m. Here $\overline{\psi} = \psi^\dagger\gamma_0$ and γ_μ are the Dirac matrices that anticommute, $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$. The Lagrangian for the electromagnetic field is

$$\mathcal{L}_A = -\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and

$$\mathcal{L}_{FA} = e\overline{\psi}\gamma^\mu\psi A_\mu$$

is the interaction between the Fermi field, ψ, and the electromagnetic potential A_μ;

$$\mathcal{L}_{GF} = -\frac{i\hbar c\kappa}{4}\,\mathcal{A}_{\mu\nu}\left(\overline{\psi}\gamma^\mu\partial^\nu\psi - \partial^\mu\overline{\psi}\gamma^\nu\psi\right)$$

describes the interaction between the gravito-electromagnetic potential, $\mathcal{A}_{\mu\nu}$, and the Fermi field. The coupling constant κ is defined by

$$\kappa = \frac{\sqrt{8\pi G}}{c^2}.$$

The interaction term

$$\mathcal{L}_{GA} = \frac{\kappa}{4\pi}\mathcal{A}_{\mu\nu}\left(F^{\mu}_{\ \alpha}F^{\nu\alpha} - \frac{1}{4}\eta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right)$$

provides the photon–graviton interaction, and

$$\mathcal{L}_{GFA} = \frac{1}{2}e\kappa\ \overline{\psi}\gamma^{\mu}\psi\ A^{\nu}\mathcal{A}_{\mu\nu}$$

describes a photon–graviton–fermion interaction. Here and in the rest of the paper $e(e < 0)$ denotes the charge of the electron. The Euler–Lagrange equations are given in Sect. 3.3.

3.2 Gauge invariance

Here we discuss the gauge invariance of the full Lagrangian $\mathcal{L}$, thereby justifying the presence of the interaction terms. The full Lagrangian, Eq. (11), is invariant under the following gauge transformations:

$$\psi \rightarrow e^{-\frac{ie}{\hbar c}\theta(x)}\psi,$$

and

$$A^{\mu} \rightarrow A^{\mu} + \partial^{\mu}\theta(x).$$

The electron is described by the Dirac equation that is obtained from the Dirac Lagrangian density. If we impose local gauge invariance, we are forced to introduce a massless vector field A^{μ}, which at the same time undergoes a gauge transformation. Thus, the charge of the electron is directly related to the gauge transformation and the field A^{μ} is the mediator between charged particles.

On the other hand, the graviton is, of course, a massless and uncharged particle and is the mediator of the gravitational interaction between bodies with finite mass. Therefore it does not undergo a gauge transformation as a consequence of a phase change $\theta(x)$ in the Fermi field ψ. Thus, the gravito-electromagnetic potential $\mathcal{A}^{\mu\nu}$ is invariant,

$$\mathcal{A}_{\mu\nu} \rightarrow \mathcal{A}_{\mu\nu}.$$

Under these transformations, we observe that

$$\mathcal{L}'_G = \mathcal{L}_G, \quad \mathcal{L}'_A = \mathcal{L}_A, \quad \mathcal{L}'_{GA} = \mathcal{L}_{\mathrm{GA}}.$$

Note, however, that

$$\mathcal{L}'_F = \mathcal{L}_F - e\,\overline{\psi}\gamma^\mu\psi\ \partial_\mu\theta,$$

as it is the case for the usual Dirac Lagrangian. The situation is similar for the fermion–photon interaction term,

$$\mathcal{L}'_{FA} = \mathcal{L}_{FA} + e\,\overline{\psi}\gamma^\mu\psi\ \partial_\mu\theta,$$

the fermion–gravity interaction,

$$\mathcal{L}'_{GF} = \mathcal{L}_{GF} - \frac{e\kappa}{2}\,\mathcal{A}_{\mu\nu}\,\overline{\psi}\gamma^\mu\psi\ \partial^\nu\theta,$$

as well as the fermion–graviton–photon interaction,

$$\mathcal{L}'_{GFA} = \mathcal{L}_{GFA} + \frac{e\kappa}{2}\,\mathcal{A}_{\mu\nu}\,\overline{\psi}\gamma^\mu\psi\ \partial^\nu\theta.$$

The extra terms in the last two expressions cancel and the extra term in $\mathcal{L}'_F$ cancels the one in $\mathcal{L}'_{FA}$, so that the full Lagrangian, Eq. (11), is gauge-invariant.

3.3 Equations of motion

In this subsection, we give the Euler–Lagrange equation of motions for different fields, ψ, $\overline{\psi}$, A_μ, and $\mathcal{A}_{\mu\nu}$.

The Euler–Lagrange equation for ψ is

$$i\gamma^\mu\partial_\mu\psi - \left(\frac{mc}{\hbar}\right)\psi = \frac{e}{\hbar c}\gamma^\mu\psi A_\mu + \frac{1}{2}\frac{e\kappa}{\hbar c}\gamma^\mu\psi A^\nu\mathcal{A}_{\mu\nu} - \frac{i\kappa}{2}\mathcal{A}_{\mu\nu}\gamma^\mu\partial^\nu\psi \\ -\frac{i\kappa}{4}\gamma^\mu\psi\partial^\nu\mathcal{A}_{\mu\nu}$$

and for $\overline{\psi}$, we have

$$i\partial_\mu\overline{\psi}\gamma^\mu + \left(\frac{mc}{\hbar}\right)\overline{\psi} = -\frac{e}{\hbar c}\overline{\psi}\gamma^\mu A_\mu - \frac{1}{2}\frac{e\kappa}{\hbar c}\overline{\psi}\gamma^\mu A^\nu\mathcal{A}_{\mu\nu} - \frac{i\kappa}{2}\mathcal{A}_{\mu\nu}(\partial^\mu\overline{\psi})\gamma^\nu \\ -\frac{i\kappa}{4}\overline{\psi}\gamma^\mu\partial^\nu\mathcal{A}_{\mu\nu}.$$

The equation of motion for the electromagnetic gauge field, A_μ, is

$$\partial_\rho F^{\rho\sigma} = -4\pi \left(e\overline{\psi}\gamma^\sigma\psi + \frac{1}{2}e\kappa\overline{\psi}\gamma_\mu\psi\mathcal{A}^{\mu\sigma} \right) + 2\kappa\partial_\rho \left(\mathcal{A}^{\mu\rho} F_\mu^{\ \sigma} - \mathcal{A}^{\mu\sigma} F_\mu^\rho \right) + \frac{\kappa}{16\pi}\eta_{\mu\nu}\partial_\rho \left(\mathcal{A}^{\mu\nu} F^{\rho\sigma} \right).$$

The analogous equation for the gravito-electromagnetic field, $\mathcal{A}_{\mu\nu}$, is

$$\partial_\rho \mathcal{F}^{\rho\sigma\mu} = -4\pi G \left[\frac{1}{2}e\kappa\overline{\psi}\gamma^\sigma\psi A^\mu - \frac{i\hbar c\kappa}{4} \left(\overline{\psi}\gamma^\sigma\partial^\mu\psi - \partial^\sigma\overline{\psi}\gamma^\mu\psi \right) \right] - \kappa G \left(F^\sigma_{\ \alpha} F^{\mu\alpha} - \frac{1}{4}\eta^{\sigma\mu} F^{\alpha\beta} F_{\alpha\beta} \right).$$

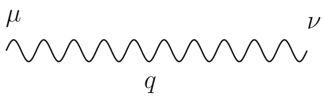
4 Gravitons interacting with photons and charged fermions

In this section, we calculate the transition amplitudes, $\mathcal{M}$, for some scattering processes: the gravitational Møller scattering and the gravitational Compton scattering and amplitudes for the graviton photoproduction. Before presenting the transition amplitudes, we present our notation for the propagators and the vertex factors.

4.1 Propagators

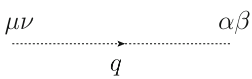
In this subsection, we give the expressions of the propagators of the photon (see Fig. 1), graviton (Fig. 2) and fermion (Fig. 3), that we will use in subsequent subsections.

Fig. 1 Photon propagator



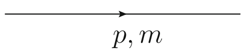
$$-\frac{i\eta^{\mu\nu}}{q^2}$$

Fig. 2 Graviton propagator



$$i\,\frac{\eta^{\mu\nu}\eta^{\alpha\beta} - \eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha}}{2q^2}$$

Fig. 3 Fermion propagator

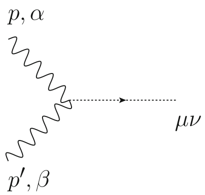


$$\frac{i(\not{p} + mc)}{p^2 - m^2c^2}$$

4.2 Vertex factors

The vertex factors are similar to those given earlier [50], where they have extra terms in the full interaction Lagrangian since they work to the third order approximation in h.

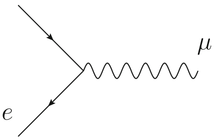
Here we only use the first approximation in the tensor potential $\mathcal{A}$ in the full Lagrangian. We enumerate the vertex factors for interactions between fermions, photons and gravitons. The graviton-photon (Fig. 4), fermion-photon (Fig. 5), graviton-fermion (Fig. 6) and graviton-photon-fermion (Fig. 7) vertices are displayed below.



$$-\frac{i\kappa}{4\pi}(\eta^{\alpha\beta}p^{\mu}p'^{\nu} - \eta^{\nu\beta}p^{\mu}p'^{\alpha} - \eta^{\mu\alpha}p^{\beta}p'^{\nu} + \eta^{\mu\alpha}\eta^{\nu\beta}p\cdot p' \\ -\tfrac{1}{2}\eta^{\mu\nu}\eta^{\beta\alpha}p\cdot p' + \tfrac{1}{2}\eta^{\mu\nu}p^{\beta}p'^{\alpha})$$

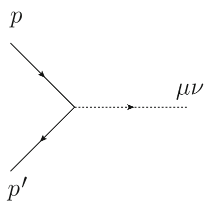
Fig. 4 Graviton–photon vertex factor

Fig. 5 Fermion–photon vertex factor



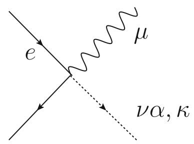
$$ie\gamma^{\mu}$$

Fig. 6 Graviton–fermion vertex factor



$$-\frac{i\hbar c\kappa}{4}(\gamma^{\mu}p^{\nu} + p'^{\mu}\gamma^{\nu})$$

Fig. 7 Graviton–fermion–photon vertex factor



$$\frac{ie\kappa}{2}\gamma^{\nu}\eta^{\alpha\mu}$$

4.3 Transition amplitudes

Formally, the S-matrix is defined as the unitary matrix connecting asymptotic physical particle states in the Hilbert space. In order to make the Lorentz invariance more transparent, it is convenient to move to the interaction picture where the S-matrix is calculated as a time-ordered exponential of the interacting Lagrangian. By taking into account the interaction terms of the full Lagrangian, Eq. (11), the S-matrix becomes

$$S = \hat{T}\left[\exp\left[i\int(\mathcal{L}_{FA} + \mathcal{L}_{GF} + \mathcal{L}_{GA} + \mathcal{L}_{GFA})d^4x\right]\right],$$

where $\hat{T}[\cdots]$ represents the time-ordered product. The Feynman diagrams which describe the scattering processes are obtain by a perturbative expansion of this expression. For non-interacting fields, the S-matrix reduces to the identity operator. When interactions are present, we write the S-matrix in the first approximation as

$$S = 1 - iT,$$

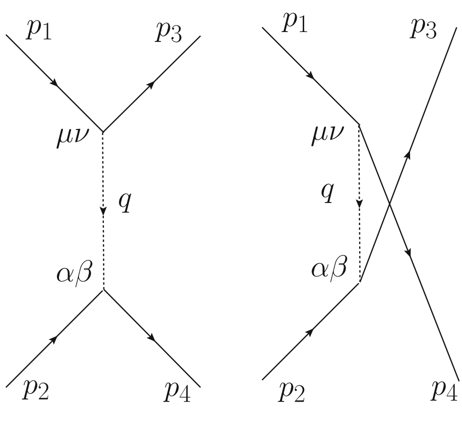
where the term iT is defined proportional to $-i\mathcal{M}$, with $\mathcal{M}$, the *invariant transition amplitude*.

In the following subsections, we calculate the amplitudes, $\mathcal{M}$, of various scattering processes involving gravitons. The Feynman diagrams below are drawn with the convention that the arrow of time is horizontal and points toward the right.

4.3.1 Gravitational Møller scattering

As mentioned at the beginning of Sect. 4, the diagrams are similar to the traditional Møller scattering, where the photon is replaced by a graviton. Then we must consider the two diagrams given in Fig. 8.

Fig. 8 Feynman diagrams for Møller scattering



The total transition amplitude for the gravitational version of the Møller scattering is

$$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2,$$

where the two terms are given by

$$\begin{aligned}\mathcal{M}_1 = \frac{\hbar^2 c^2 \kappa^2}{32(p_1 - p_3)^2} &\left[\bar{u}_3(\gamma^\mu p_1^\nu + p_3^\mu \gamma^\nu) u_1\right] \\ &\times \left[\bar{u}_4\{g_{\mu\nu}(\not{p}_2 + \not{p}_4) + \gamma_\nu (p_2 - p_4)_\mu - \gamma_\mu (p_2 - p_4)_\nu\} u_2\right]\end{aligned}$$

and

$$\mathcal{M}_2 = -\frac{\hbar^2 c^2 \kappa^2}{32(p_1 - p_4)^2} \left[\bar{u}_4(\gamma^\mu p_1^\nu + p_4^\mu \gamma^\nu) u_1\right] \\ \times \left[\bar{u}_3\{g_{\mu\nu}(\not{p}_2 + \not{p}_3) + \gamma_\nu (p_2 - p_3)_\mu - \gamma_\mu (p_2 - p_3)_\nu\} u_2\right].$$

4.3.2 Gravitational Compton scattering

Here we consider the gravitational version of the Compton scattering ($e+g \to e+g$). In this section, we consider only the lowest-order approximation, Fig. 9, while Holstein [52] has included higher order processes.

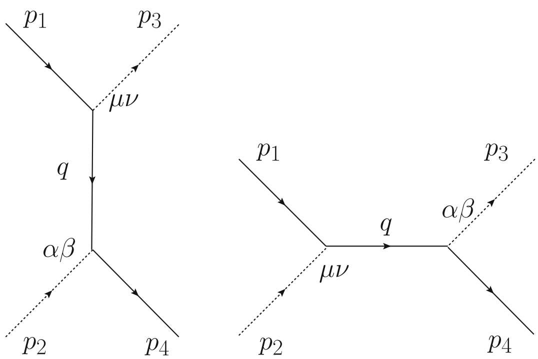


Fig. 9 Feynman diagrams for Compton scattering

We find the following total transition amplitude of the gravitational Compton scattering:

$$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2,$$

where

$$\mathcal{M}_1 = \frac{\hbar^2 c^2 \kappa^2}{16[(p_1 - p_3)^2 - m^2 c^2]} \left[\bar{u}_4(2p_4 - p_2) \cdot \epsilon_2^g \not{\epsilon}_2^{\,g}\right] (\not{p}_1 - \not{p}_3 + mc) \\ \times \left[\not{\epsilon}_3^{*g} \epsilon_3^{*g} \cdot (2p_1 - p_3) u_1\right]$$

and

$$\mathcal{M}_2 = \frac{\hbar^2 c^2 \kappa^2}{16[(p_1 + p_2)^2 - m^2 c^2]} \left[\bar{u}_4(2p_4 + p_3) \cdot \epsilon_3^{*g} \not{\epsilon}_3^{\,*g}\right] (\not{p}_1 + \not{p}_2 + mc) \\ \times \left[\not{\epsilon}_2^{g} \epsilon_2^{g} \cdot (2p_1 + p_2) u_1\right].$$

In these calculations we have considered that the graviton polarization tensor $\epsilon^g_{\mu\nu}$ can be taken to be the product [51] of two spin-1 polarization vectors ϵ^g_μ and ϵ^g_ν,

$$\epsilon^g_{\mu\nu} = \epsilon^g_\mu \epsilon^g_\nu.$$

4.3.3 Graviton photoproduction amplitudes

Now we consider the photoproduction of gravitons $e + \gamma \rightarrow e + g$. When viewed from the four possible directions, the Feynman diagrams in Figs. 10, 11, 12 and 13 lead to four different processes: $e + \gamma \rightarrow e + g$ (with time arrow to the right (Fig. 10)), $e + g \rightarrow e + \gamma$ (time arrow to the left (Fig. 11)), $g + \gamma \rightarrow e + \overline{e}$ (time arrow upward (Fig. 12)), and $e + \overline{e} \rightarrow g + \gamma$ (time arrow downward (Fig. 13)). Here, we consider the first process, which would be relevant in various experiments [1–3].

Fig. 10 Graviton photoproduction: Born diagram

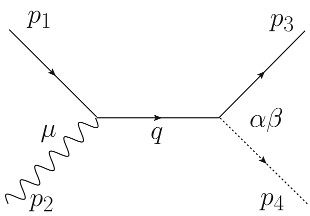


Fig. 11 Graviton photoproduction: Seagull diagram

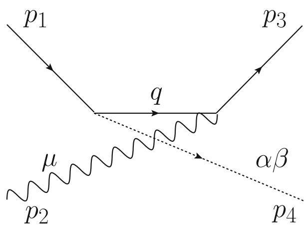
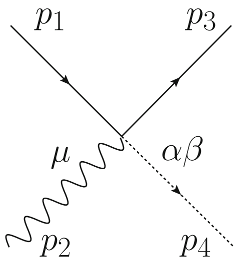


Fig. 12 Graviton photoproduction: four-point interaction



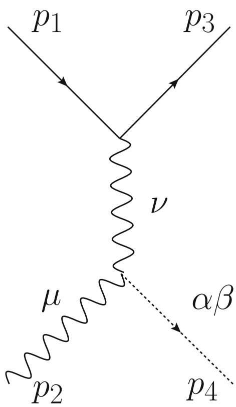


Fig. 13 Graviton photoproduction: $\gamma + e \to e + g$

First, we consider the Born diagram given in Fig. 10.

The transition amplitude for this process is

$$\mathcal{M} = -\frac{e\hbar c\kappa}{4[(p_1+p_2)^2 - m^2c^2]}\left[\bar{u}_3(p_1+p_2+p_3)\cdot\epsilon_4^{*g}\not{\epsilon}_4^{*g}\right](\not{p}_1+\not{p}_2+mc)[\not{\epsilon}_2 u_1].$$

This result is analogous to Eq. (2.9) of Ref. [45]. However, we remind once again that here, the graviton field is represented in a different fashion; that is, by a 2-index potential field, instead of the perturbation field, $h_{\mu\nu}$, of the weak-field approximation.

Next, we consider the seagull diagram of Fig. 11.

The transition amplitude is given by

$$\mathcal{M} = -\frac{e\hbar c\kappa}{4[(p_1-p_4)^2 - m^2c^2]}[\bar{u}_3\not{\epsilon}_2](\not{p}_1-\not{p}_4+mc)\left[\not{\epsilon}_4^{*g}\epsilon_4^{*g}\cdot(p_1+p_3-p_2)u_1\right].$$

This is similar to Eq. (2.10) of Ref. [45].

The third process that we consider is the four-point interaction, illustrated in Fig. 12.

The transition amplitude for this scattering is

$$\mathcal{M} = -\frac{e\kappa}{2}\left[\bar{u}_3\not{\epsilon}_4^{*g}\epsilon_4^{*g}\cdot\epsilon_2 u_1\right].$$

This is equivalent to Eq. (2.11) in Ref. [45].

Finally, we consider the following Born diagram, shown in Fig. 13, which is the only process involving the graviton–photon vertex factor of Fig. 4.

The transition amplitude for this graviton photoproduction process is

$$\mathcal{M} = \frac{e\kappa}{8\pi(p_4-p_2)^2}\Big[\bar{u}_3\Big\{-\epsilon_4^{*g}\cdot\epsilon_2\not{\epsilon}_4^{*g}(p_4-p_2)^2 + 2\epsilon_4^{*g} \cdot p_2(-\not{\epsilon}_4^{*g}p_4\cdot\epsilon_2 - \not{\epsilon}_2\epsilon_4^{*g}\cdot p_2 + \epsilon_4^{*g}\cdot\epsilon_2\not{p}_2)\Big\}u_1\Big].$$

which is similar to Eq. (2.12) in Ref. [45].

5 Conclusions

The Lagrangian formulation of the GEM, for which the graviton field is described in a way analogous to electromagnetism, provides us with an alternative way to study the interaction of the gravitational field with fermions and photons in flat space–time. This formulation then leads to a definition of an S-matrix, that helps us to formulate a perturbation series in order to define transition amplitudes for various scattering processes. We calculate the transition amplitudes for the gravitational Compton scattering, the gravitational Møller scattering, and graviton photoproduction. It is important to mention that the Lagrangian formulation leads to an energy-momentum tensor that is symmetric.

The main feature of our approach is the symmetric tensor potential, $\mathcal{A}^{\mu\nu}$, which describes a spin-2 particle, the graviton. This symmetric tensor is inspired by the formulation of gravity analogous to electromagnetism and is different from the traditional perturbation scheme based on the weak-field approximation of the metric with the small perturbation, $h_{\mu\nu}$, of the flat space–time. Both are symmetric and the terms in the Lagrangian involving the interactions among gravitons, fermions and photons, include $\mathcal{A}^{\mu\nu}$ and $h^{\mu\nu}$ in the same form. However, the nature and origin of these two potentials are quite different. In spite of this, there are some similarities between the two descriptions. We notice that the transition amplitudes of some scattering processes have the same structure except for numerical factors. This is the case with the graviton photoproduction. This fact reinforces the idea that the graviton behaves like a photon in the graviton interaction with charged fermions and a photons.

This Lagrangian formulation will be useful to study the Galilean limit and explore the Casimir effect in the framework of GEM. We have not included a true gravitational gauge invariance in our treatment. By 'true gravitational gauge', we mean that the tensor potential $\mathcal{A}^{\mu\nu}$ possesses a gauge transformation as a result of a phase transformation of the field as in the case of potential A^{μ}. However, by following both the guidelines of the local gauge invariance and the analogy to the electrodynamics, it would be possible to obtain a true gauge transformation for $\mathcal{A}^{\mu\nu}$ in similar way to A^{μ}.

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