Gravitational Compton scattering at zero and finite temperature

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The Lagrangian formulation of Gravitoelectromagnetism (GEM) theory is considered. GEM is a gravitational theory constructed based on the similarities between gravity and electromagnetism. In this framework, we investigate gravitational Compton scattering by calculating its cross section at both zero and finite temperatures. Thermal effects are introduced via the Thermo Field Dynamics formalism. Some comparisons between GEM theory and QED have been developed. The limits of high temperature have been analyzed.

I. INTRODUCTION

The Standard Model (SM) is currently the quantum theory that describes the interactions between known fundamental particles. It is renowned for unifying the three main forces of nature (electromagnetic, strong, and weak), enabling mathematical calculations that have been used to predict particles that were purely hypothetical until then. One example is Quantum Electrodynamics (QED), which precisely describes the interaction of charged particles, such as electrons and positrons, allowing a quantum description of Maxwell's electromagnetic theory. This theory is of great value to particle physics due to its accuracy concerning experimental results. However, although the Standard Model describes the primary interactions in quantum nature, it does not unify the gravitational force with the other three fundamental forces. This is because the gravitational force is a classical theory, while the other three have a quantum nature. As a result, various studies have been developed to seek a theory of quantum gravity. One of these theories is Gravitoelectromagnetism (GEM), which aims to describe gravitons, the hypothetical fundamental particles of gravity, using concepts from electromagnetic theory.

GEM is a theory that seeks to describe gravity through a framework similar to electromagnetism. Its development dates back to the period before Einstein's general relativity postulates. The first ideas were discussed by Faraday [1] and Maxwell [2]. In 1893, Heaviside proposed ways to explain the rapid advancement of Mercury's perihelion, developing a set of equations similar to Maxwell's

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equations of electromagnetic theory [3]. However, this attempt was unsuccessful, as Lorentz later demonstrated that this new force, called the gravitomagnetic force, was too weak to explain such a phenomenon [4]. Despite this, these concepts seemed to be related in some way. In 1918, Lense and Thirring showed that the existence of a gravitomagnetic field, generated by a massive rotating body, could affect the orientation of the orbit of another body in its vicinity, and this effect became known as the Lense-Thirring effect [5]. Starting in the 1950s, new ideas emerged based on the decomposition of the Weyl tensor into its gravitoelectric and gravitomagnetic components [6, 7]. This gravitational theory relies on two key assumptions: (i) a gravitomagnetic field is associated with moving masses, and (ii) the gravitational field propagates at the speed of light.

Although GEM theory can be derived through various methods, this discussion will focus on the approach based on the decomposition of the Weyl tensor. Using this approach, a Lagrangian formulation for GEM theory has been developed [8]. In this formalism, the gravitational field is described by the tensor potential $A^{\mu\nu}$. From the Lagrangian formulation, various studies involving the graviton can be explored. For example, there are studies discussing gravitational electron-positron scattering [9], gravitational Bhabha scattering [10], gravitational Möller scattering [11], gravitational Casimir effect [12], among others. In this work, we will use the Lagrangian formulation of GEM to investigate gravitational Compton scattering.

Compton scattering is a fundamental process in physics where a photon collides with an electron. The scattering was first experimentally observed by Arthur Compton in 1923 [13], confirming the particle-like nature of photons and providing direct evidence for the quantum theory of light. Compton scattering plays a crucial role in fields such as astrophysics, quantum electrodynamics, contributing significantly to our understanding of electromagnetic interactions at the microscopic level. Here, the gravitational version of this scattering is investigated. This involves replacing the photon with a graviton in the scattering process, i.e., $g + f \rightarrow g + f$. The main aim of this work is to calculate the cross section for this gravitational scattering at zero and finite temperature. Thermal effect will be introduced using the Thermo Field Dynamics (TFD) formalism.

TFD is a quantum field theory at finite temperature constructed from the idea that the statistical average of an arbitrary operator is equivalent to its vacuum expectation value [14–21]. In this formalism, a thermal vacuum state $|0(\beta)\rangle$ is constructed. However, two key ingredients are required: doubling the Hilbert space and implementing the Bogoliubov transformation. The main advantage of the TFD formalism is that it allows temporal information to be studied together with thermal effects. Furthermore, the real time propagators consist of a sum of two parts: one corresponding to the propagator at zero temperature and the other corresponding to a temperature dependent part. Using this formalism the thermal cross section for gravitational Compton scattering can be calculated.

This paper is organized as follows. In Section II, the GEM theory and its Lagrangian formalism are presented. In Section III, gravitational Compton scattering at zero temperature is investigated. The cross section is calculated, and some discussions are developed. In section IV, the TFD formalism is introduced. Then the gravitational Compton scattering at finite temperature is considered. The differential cross section at finite temperature is determined. Some limits of temperature are analyzed. In section V, some concluding remarks are made.

II. GRAVITOELECTROMAGNETISM (GEM)

In this section, a brief introduction to gravitoelectromagnetism (GEM) is presented. The GEM theory is a gravitational theory constructed by following the similarities between the electromagnetic force and the gravitational force. This gravitational model can be built based on three different situations: (i) considering the similarity between the linearized Einstein and Maxwell equations [22]; (ii) using tidal tensors [23]; and (iii) decomposing the Weyl tensor (C_{ijkl}) into $\mathbf{B}_{ij}$ and $\mathbf{E}_{ij}$, representing the gravitomagnetic and gravitoelectric components, respectively [24]. Here, the third approach is considered. Thus, the GEM theory is based on the decomposition of the Weyl tensor, which is defined as

$$\begin{aligned} C_{\alpha\sigma\mu\nu} &= R_{\alpha\sigma\mu\nu} - \frac{1}{2}\left(R_{\nu\alpha}g_{\mu\sigma} + R_{\mu\sigma}g_{\nu\alpha} - R_{\nu\sigma}g_{\mu\alpha} - R_{\mu\alpha}g_{\nu\sigma}\right) \\ &+ \frac{1}{6}R\left(g_{\nu\alpha}g_{\mu\sigma} - g_{\nu\sigma}g_{\mu\alpha}\right), \end{aligned} \tag{1}$$

where $R_{\alpha\sigma\mu\nu}$ is the Riemann tensor, $R_{\mu\nu}$ is the Ricci tensor and R is the Ricci scalar. Using this quantities, the gravitational fields are written as

$$\mathbf{B}_{ij} = \frac{1}{2}\epsilon_{ikl}C^{kl}_{0j}, \tag{2}$$

$$\mathbf{E}_{ij} = -C_{0i0j}. \tag{3}$$

The Weyl tensor shares the same symmetry as the Riemann tensor and has invariance under conformal changes in the metric [25]. In this context, the GEM, or Maxwell-like, equations are given as

$$
\begin{aligned}
\partial^i \mathbf{E}^{ij} &= 4\pi G \rho^j, && (4)\\
\partial^i \mathbf{B}^{ij} &= 0, && (5)\\
\varepsilon^{\langle ikl}\partial^k \mathbf{B}^{lj\rangle} - \frac{1}{c}\frac{\partial \mathbf{E}^{ij}}{\partial t} &= \frac{4\pi G}{c}\mathbf{J}^{ij}, && (6)\\
\varepsilon^{\langle ikl}\partial^k \mathbf{E}^{lj\rangle} + \frac{1}{c}\frac{\partial \mathbf{B}^{ij}}{\partial t} &= 0, && (7)
\end{aligned}
$$

where $\mathbf{E}^{ij}$ and $\mathbf{B}^{ij}$ are traceless second-order symmetric tensors, ρ^j is the mass density vector, $\mathbf{J}^{ij}$ is a traceless second-order tensor representing the mass current density, and G is the gravitational constant. Here, the symbol $\langle i \cdots j\rangle$ denotes the symmetrization of the first and last indices.

To investigate gravitational applications using the GEM theory, the Lagrangian formulation has been developed [8]. To achieve this objective, a gravitoelectromagnetic tensor potential $A^{\mu\nu}$ is defined. From this gravitational tensor potential and considering the GEM counterpart of the electromagnetic scalar potential φ, the GEM fields can be written as

$$
\begin{aligned}
\mathbf{E} &= -grad\,\varphi - \frac{1}{c}\frac{\partial \widetilde{A}}{\partial t}\\
\mathbf{B} &= curl\,\widetilde{A}, && (8)
\end{aligned}
$$

where $\tilde{A}$ has components A^{ij}, with $i,j = 1,2,3$. From this, we can define the gravitoelectromagnetic tensor $F^{\mu\nu\rho}$ as

$$
\mathbf{F}^{\mu\nu\rho} = \partial^\mu A^{\nu\rho} - \partial^\nu A^{\mu\rho}, \tag{9}
$$

where $\mu,\nu,\rho = 0,1,2,3$. Thus, we can rewrite the Maxwell-like equations as

$$
\begin{aligned}
\partial_\mu \mathbf{F}^{\mu\nu\rho} &= 4\pi G \mathcal{J}^{\nu\rho}, && (10)\\
\partial_\mu \mathbf{G}^{\mu\langle\nu\rho\rangle} &= 0, && (11)
\end{aligned}
$$

where $\mathcal{J}^{\nu\rho}$ is the mass density tensor depending on ρ^j and $\mathbf{J}^{ij}$. Additionally, $\mathbf{G}^{\mu\nu\rho}$ is the dual tensor of GEM, defined as follows

$$
\mathbf{G}^{\mu\nu\rho} = \frac{1}{2}\epsilon^{\mu\nu\gamma\sigma}\eta^{\rho\lambda}\mathbf{F}_{\gamma\sigma\lambda}. \tag{12}
$$

With these ingredients, the GEM Lagrangian is defined as

$$
\mathcal{L}_{\mathrm{GEM}} = -\frac{1}{16\pi}\mathbf{F}_{\mu\nu\rho}\mathbf{F}^{\mu\nu\rho} - G\mathcal{J}^{\nu\rho}A_{\nu\rho}. \tag{13}
$$

Using this Lagrangian in the Euler-Lagrange equations recovers the GEM equations (10) and (11). Furthermore, this Lagrangian formulation allows the study of interactions between gravitons and other fundamental particles. Since our main objective is to investigate the interaction between gravitons and fermions via gravitational Compton scattering, the complete Lagrangian that describes gravitons, fermions, and their interaction is given as

$$\mathcal{L}_{Total} = \mathcal{L}_{\mathrm{G}} + \mathcal{L}_{\mathrm{F}} + \mathcal{L}_{\mathrm{I}}, \tag{14}$$

where

$$\mathcal{L}_{\mathrm{G}} = -\frac{1}{16\pi}\mathbf{F}_{\mu\nu\rho}\mathbf{F}^{\mu\nu\rho}, \tag{15}$$

is the free part of the GEM Lagrangian,

$$\mathcal{L}_{\mathrm{F}} = -\frac{i}{2}(\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - \partial_{\mu}\bar{\psi}\gamma^{\mu}\psi) + m\bar{\psi}\psi \tag{16}$$

describes the fermions ψ with mass m and

$$\mathcal{L}_{\mathrm{I}} = -\frac{i\kappa}{4}A_{\mu\nu}(\bar{\psi}\gamma^{\mu}\partial^{\nu}\psi - \partial^{\mu}\bar{\psi}\gamma^{\nu}\psi), \tag{17}$$

is the interaction part, with $\kappa = \sqrt{8\pi G}$ being the coupling constant.

From the interaction Lagrangian, we can investigate gravitational Compton scattering, which describes the interactions between gravitons and fermions. The main objective is to calculate the cross-section for this scattering at zero and finite temperature. In the next section, the cross-section for gravitational Compton scattering at zero temperature is determined.

III. GRAVITATIONAL COMPTON SCATTERING AT ZERO TEMPERATURE

In this section, we will study the interaction between gravitons and fermions through Compton scattering. This investigation is conducted without considering temperature effects. This scattering process consists of $e^- + g \to e^- + g$, where e^- and g represent a electron and a graviton, respectively. Figure 1 shows the Feynman diagram that describes this process.

In order to calculate the cross section for this scattering process, let's first determine the transition amplitude, which is defined as

$$\mathcal{M} = \langle f|\,\hat{S}^{(2)}\,|i\rangle\,, \tag{18}$$

where $\hat{S}^{(2)}$ is the second order scattering matrix given as

$$\hat{S}^{(2)} = -\frac{1}{2}\int\int d^4x\, d^4y\; \tau[\mathcal{L}_I(x)\mathcal{L}_I(y)], \tag{19}$$

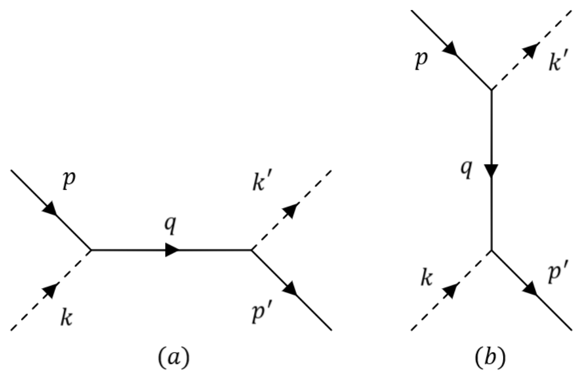

Figure 1. Feynman diagram for gravitational Compton scattering. The diagrams (a) and (b) represent the s-channel and u-channel, respectively. Here, $p_i = p$ and $k_i = k$ represent the initial momenta of the fermion and graviton, respectively, while $p_f = p'$ and $k_f = k'$ represent the final momenta. And q is the momentum of the fermion in the propagator.

with τ being the time-ordering operator. The final and initial asymptotic states of the particles, $\langle f|$ and $|i\rangle$, are written as

$$\begin{aligned} |i\rangle &= |e_i, g_i\rangle = b^\dagger_{s,p} a^\dagger_{\lambda,k} |0\rangle \,, \\ |f\rangle &= |e_f, g_f\rangle = b^\dagger_{s',p'} a^\dagger_{\lambda',k'} |0\rangle \,, \end{aligned} \tag{20}$$

where e_i, e_f, g_i and g_f represent the initial and final states of the electron and graviton, respectively. Here, $a^\dagger$ is the graviton creation operator and $b^\dagger$ is the electron creation operator.

Using the interaction Lagrangian (17), the transition amplitude is written as

$$\begin{aligned} \mathcal{M} &= \frac{\kappa^2}{16} \int d^4x \, d^4y \, \langle f| \, \tau \big[A_{\mu\nu} A_{\alpha\beta} (\bar{\psi}_x \gamma^\mu \partial^\nu \psi_x - \partial^\mu \bar{\psi}_x \gamma^\nu \psi_x) \\ &\quad \times (\bar{\psi}_y \gamma^\alpha \partial^\beta \psi_y - \partial^\alpha \bar{\psi}_y \gamma^\beta \psi_y) \big] \, |i\rangle \,. \end{aligned} \tag{21}$$

The only terms that will contribute to the process are those represented by the Feynman diagrams. Therefore, after analyzing the operators contained in $\mathcal{M}$, we can state that the probability amplitude becomes

$$\begin{aligned} \mathcal{M} &= \frac{\kappa^2}{16} \int d^4x \, d^4y \, \langle f| \, \tau \big[\big(A^-_{\mu\nu} A^+_{\alpha\beta} + A^+_{\mu\nu} A^-_{\alpha\beta} \big) \\ &\quad \times \big(\bar{\psi}^-_x \gamma^\mu [\partial^\nu \psi_x \bar{\psi}_y] \gamma^\alpha \partial^\beta \psi^+_y - \bar{\psi}^-_x \gamma^\mu [\partial^\nu \psi_x \partial^\alpha \bar{\psi}_y] \gamma^\beta \psi^+_y \\ &\quad - \partial^\mu \bar{\psi}^-_x \gamma^\nu [\psi_x \bar{\psi}_y] \gamma^\alpha \partial^\beta \psi^+_y + \partial^\mu \bar{\psi}^-_x \gamma^\nu [\psi_x \partial^\alpha \bar{\psi}_y] \gamma^\beta \psi^+_y \big) \big] \, |i\rangle \,, \end{aligned} \tag{22}$$

where negative and positive indices represent the parts of the fields associated with their respective frequencies. In other words, for the fields of GEM, we have that

$$
\begin{aligned}
A^{-}_{\mu\nu} &= \int d^3k\; N_k \sum_{\lambda} \epsilon^{(\lambda)}_{\mu\nu} a^{\dagger}_{\lambda,k}\; e^{ikx} \\
A^{+}_{\mu\nu} &= \int d^3k\; N_k \sum_{\lambda} \epsilon^{(\lambda)}_{\mu\nu} a_{\lambda,k}\; e^{-ikx}
\end{aligned}
\tag{23}
$$

with N_k being the normalization constant and $\epsilon_{\mu\nu}$ the graviton polarization tensor. For the fermionic field, on the other hand, we will have that

$$
\begin{aligned}
\bar{\psi}^{-}_{x} &= \int d^3p\; N_p \sum_{s} \bar{u}^{(s)}_{p} b^{\dagger}_{s,p}\; e^{ipx} \\
\psi^{+}_{x} &= \int d^3p\; N_p \sum_{s} u^{(s)}_{p} b_{s,p}\; e^{-ipx}.
\end{aligned}
\tag{24}
$$

Here, N_p is a normalization constant and $u^{(s)}_p$ is the Dirac spinor. With these ingredients, the transition amplitude can written as

$$
\mathcal{M} = \mathcal{M}_I + \mathcal{M}_{II}, \tag{25}
$$

where $\mathcal{M}_I$ and $\mathcal{M}_{II}$ represent Feynman diagrams (a) and (b) from Figure 1, respectively, and are given as

$$
\begin{aligned}
\mathcal{M}_I = &-\frac{\kappa^2}{16}\frac{1}{(k_f+p_f)^2-m^2}\sum_{\lambda s}\epsilon^{(\lambda_f)}_{\mu\nu}\epsilon^{(\lambda_i)}_{\alpha\beta}\big[\bar{u}^{(s_f)}_{p_f}\gamma^{\mu}(k^{\nu}_f+p^{\nu}_f)(\not{k}_f+\not{p}_f+m)\gamma^{\alpha}p^{\beta}_i u^{(s_i)}_{p_i} \\
&+\bar{u}^{(s_f)}_{p_f}\gamma^{\mu}(k^{\nu}_f+p^{\nu}_f)(\not{k}_f+\not{p}_f+m)(k^{\alpha}_f+p^{\alpha}_f)\gamma^{\beta}u^{(s_i)}_{p_i}+p^{\mu}_f\bar{u}^{(s_f)}_{p_f}\gamma^{\nu}(\not{k}_f+\not{p}_f+m)\gamma^{\alpha}p^{\beta}_i u^{(s_i)}_{p_i} \\
&+p^{\mu}_f\bar{u}^{(s_f)}_{p_f}\gamma^{\nu}(\not{k}_f+\not{p}_f+m)(k^{\alpha}_f+p^{\alpha}_f)\gamma^{\beta}u^{(s_i)}_{p_i}\big]
\end{aligned}
\tag{26}
$$

and

$$
\begin{aligned}
\mathcal{M}_{II} = &-\frac{\kappa^2}{16}\frac{1}{(p_i-k_f)^2-m^2}\sum_{\lambda s}\epsilon^{(\lambda_f)}_{\mu\nu}\epsilon^{(\lambda_i)}_{\alpha\beta}\big[\bar{u}^{(s_f)}_{p_f}\gamma^{\mu}(p^{\nu}_f-k^{\nu}_i)(\not{p}_i-\not{k}_f+m)\gamma^{\alpha}p^{\beta}_i u^{(s_i)}_{p_i} \\
&+\bar{u}^{(s_f)}_{p_f}\gamma^{\mu}(p^{\nu}_i-k^{\nu}_f)(\not{p}_i-\not{k}_f+m)(p^{\alpha}_f-k^{\alpha}_i)\gamma^{\beta}u^{(s_i)}_{p_i}+p^{\mu}_f\bar{u}^{(s_f)}_{p_f}\gamma^{\nu}(\not{p}_i-\not{k}_f+m)\gamma^{\alpha}p^{\beta}_i u^{(s_i)}_{p_i} \\
&+p^{\mu}_f\bar{u}^{(s_f)}_{p_f}\gamma^{\nu}(\not{p}_i-\not{k}_f+m)(p^{\alpha}_i-k^{\alpha}_f)\gamma^{\beta}u^{(s_i)}_{p_i}\big],
\end{aligned}
\tag{27}
$$

where we use the fact that the fermion propagator is defined by

$$
\langle 0|\,\tau[\psi_x\bar{\psi}_y]\,|0\rangle = \int \frac{d^4q}{(2\pi)^4}e^{-iq(x-y)}\frac{\not{q}+m}{q^2-m^2}, \tag{28}
$$

with q being the fermion momentum. After some calculations, the total transition amplitude (25) becomes

$$\mathcal{M} = - \frac{\kappa^2}{16}\frac{1}{(k_f+p_f)^2-m^2}\sum_{\lambda s}[\bar{u}^{(s_f)}_{p_f}(2p_f+k_f)\cdot\epsilon^*_f\not{\epsilon}^*_f](\not{k}_f+\not{p}_f+m)$$
$$\times\ [\not{\epsilon}_i\epsilon_i\cdot(2p_i+k_i)u^{(s_i)}_{p_i}]$$
$$- \frac{\kappa^2}{16}\frac{1}{(p_i-k_f)^2-m^2}\sum_{\lambda s}[\bar{u}^{(s_f)}_{p_f}(2p_f-k_i)\cdot\epsilon_i\not{\epsilon}_i](\not{p}_i-\not{k}_f+m)$$
$$\times\ [\not{\epsilon}^*_i\epsilon^*_i\cdot(2p_i-k_f)u^{(s_i)}_{p_i}], \tag{29}$$

where have been used that $\epsilon_{\mu\nu}=\epsilon_\mu\epsilon_\nu$ and $\not{\epsilon}=\gamma^\mu\epsilon_\mu$.

With the transition amplitude, we can calculate the cross section for gravitational Compton scattering at zero temperature. This approach will be carried out considering the center-of-mass (CM) reference frame, where the momenta are given by

$$\begin{aligned} p_i &= (E,0,0,\omega),\\ k_i &= (\omega,0,0,-\omega),\\ p_f &= (E,-\omega\sin\theta,0,-\omega\cos\theta),\\ k_f &= (\omega,\omega\sin\theta,0,\omega\cos\theta), \end{aligned} \tag{30}$$

where ω represents the frequency and θ the scattering angle. It is important to note that, due to the graviton's mass being zero, $m_g=0$, it follows from the dispersion relation that $|E_k|^2=|\vec{p}|^2=|\vec{p'}|^2$. Furthermore, by definition, the cross section is expressed as

$$\left(\frac{d\sigma}{d\Omega}\right)=\frac{1}{64\pi^2 s}\langle|\mathcal{M}|^2\rangle, \tag{31}$$

with $s=(2E)^2$. The parameter $\langle|\mathcal{M}|^2\rangle$ represents the probability density, expressed in terms of the modulus of the transition amplitude, that is,

$$\langle|\mathcal{M}|^2\rangle=\frac{1}{4}\sum_s|\mathcal{M}|^2=\langle|\mathcal{M}_I|^2\rangle+\langle|\mathcal{M}_{II}|^2\rangle+\langle 2\,\mathrm{Re}\,\mathcal{M}_I^\dagger\mathcal{M}_{II}\rangle. \tag{32}$$

Now let's calculate these terms separately. The first term is given as

$$\langle|\mathcal{M}_I|^2\rangle = \frac{1}{4}\sum_\lambda\sum_s\left(\frac{\kappa^2}{16}\frac{1}{(k_f+p_f)^2-m^2}\right)^2[(2p^\nu_f+k^\nu_f)\epsilon^*_\nu\epsilon^*_\mu][(2p^\beta_i+k^\beta_i)\epsilon_\beta\epsilon_\alpha]$$
$$\times\ [(2p^\sigma_i+k^\sigma_i)\epsilon^*_\sigma\epsilon^*_\rho][(2p^\xi_f+k^\xi_f)\epsilon_\xi\epsilon_\lambda][\bar{u}^{(s_f)}_{p_f}\gamma^\mu(\not{k}_f+\not{p}_f+m)\gamma^\alpha u^{(s_i)}_{p_i}$$
$$\times\ \bar{u}^{(s_i)}_{p_i}\gamma^\rho(\not{k}_f+\not{p}_f+m)\gamma^\lambda u^{(s_f)}_{p_f}]. \tag{33}$$

Using the property of the trace of gamma matrices

$$\sum_s[\bar{u}^{(s_f)}_{p_f}\gamma^\mu(\not{k}_f+\not{p}_f+m)\gamma^\alpha u^{(s_i)}_{p_i}\bar{u}^{(s_i)}_{p_i}\gamma^\rho(\not{k}_f+\not{p}_f+m)\gamma^\lambda u^{(s_f)}_{p_f}]$$
$$=\mathrm{Tr}[\sum_s u^{(s_f)}_{p_f}\bar{u}^{(s_f)}_{p_f}\gamma^\mu(\not{k}_f+\not{p}_f+m)\gamma^\alpha u^{(s_i)}_{p_i}\bar{u}^{(s_i)}_{p_i}\gamma^\rho(\not{k}_f+\not{p}_f+m)\gamma^\lambda], \tag{34}$$

the completeness relation

$$\sum_s u(p,s)\bar{u}(p,s) \; = \; \not{p}_i + m, \tag{35}$$

and the summation over the polarization tensor

$$\sum_\lambda \epsilon_{\mu\nu}(k,\lambda)\epsilon_{\alpha\rho}(k,\lambda) \; = \; \frac{1}{2}(g_{\mu\nu}g_{\nu\rho} + g_{\mu\rho}g_{\nu\alpha} - g_{\mu\nu}g_{\alpha\rho}), \tag{36}$$

Eq. (33) becomes

$$\begin{aligned} \langle|\mathcal{M}_I|^2\rangle \; &= \; \frac{\kappa^4}{4096\left((k_f+p_f)^2-m^2\right)^2}\left({k_f}^\nu + 2{p_f}^\nu\right)\left({k_f}^\xi + 2{p_f}^\xi\right)\left({k_i}^\beta + 2{p_i}^\beta\right)\left({k_i}^\sigma + 2{p_i}^\sigma\right) \\ &\times \; \left(g^{\alpha\sigma}g^{\beta\rho} + g^{\alpha\rho}g^{\beta\sigma} - g^{\alpha\beta}g^{\rho\sigma}\right)\left(-g^{\lambda\xi}g^{\mu\nu} + g^{\lambda\nu}g^{\mu\xi} + g^{\lambda\mu}g^{\nu\xi}\right) \\ &\times \; \mathrm{Tr}[(\gamma\cdot p_f + m)\,\gamma^\mu\,(\gamma\cdot(k_f+p_f)+m)\,\gamma^\alpha\,(\gamma\cdot p_i + m)\,\gamma^\rho\,(\gamma\cdot(k_f+p_f)+m)\,\gamma^\lambda]. \end{aligned} \tag{37}$$

Introducing the Mandelstam variables, defined by

$$s = 2p_i\cdot k_i = 2p_f\cdot k_f; \quad t = -2p_i\cdot p_f = -2k_i\cdot k_f; \quad u = -2p_i\cdot k_f = -2p_f\cdot k_i, \tag{38}$$

or, in terms of the scattering angle θ,

$$s = (2\omega)^2; \quad t = -2\omega^2(1-\cos\theta); \quad u = -2\omega^2(1+\cos\theta), \tag{39}$$

we obtain

$$\langle|\mathcal{M}_I|^2\rangle = -\frac{1}{16}\kappa^4\omega^4(\cos\theta - 1). \tag{40}$$

Similarly, we find that

$$\langle|\mathcal{M}_{II}|^2\rangle \; = \; -\frac{1}{16}\kappa^4\omega^4(\cos\theta - 1), \tag{41}$$

$$\langle 2\,\mathrm{Re}\,\mathcal{M}_I^\dagger\mathcal{M}_{II}\rangle \; = \; \frac{1}{128}\kappa^4\omega^4(28\cos\theta + 3\cos 2\theta - 7). \tag{42}$$

Therefore, the total transition amplitude is given as

$$\langle|\mathcal{M}|^2\rangle = \frac{3}{16}\kappa^4\omega^4\cos^4\left(\frac{\theta}{2}\right). \tag{43}$$

Thus, the differential cross section is obtained as

$$\left(\frac{d\sigma}{d\Omega}\right) \; = \; \frac{3\kappa^4\omega^2\cos^4\left(\frac{\theta}{2}\right)}{4096\pi^2}. \tag{44}$$

Performing the integration over the solid angle, the cross section of the gravitational Compton scattering is

$$\sigma = \frac{\kappa^4\omega^2}{1024\pi}. \tag{45}$$

Although we are using a gravitational formalism similar to that of electromagnetism, there are fundamental differences between the two. While in electromagnetism charges generate the field, in GEM, the field is generated by masses. Additionally, in electromagnetism, the field is described by vectors, while in GEM it is described by tensors. Furthermore, the coupling constants are different: the electromagnetic coupling constant is dimensionless, whereas the gravitational coupling constant κ has the dimension of the inverse of energy. Therefore, to compare the cross section of GEM with the QED result, the GEM result needs to be multiplied by a characteristic energy scale to obtain the same unit as the QED cross section. These distinctions implies that the cross section for gravitational Compton scattering may present significant differences between the two theories.

In the next section, we will analyze the thermal effects in this gravitational scattering process.

IV. GRAVITATIONAL COMPTON SCATTERING AT FINITE TEMPERATURE

The main objective here is to incorporate the effects of temperature into the cross section of gravitational Compton scattering. The thermal effects are introduced using the TFD formalism. This is a real-time formalism that introduces temperature effects into a system without losing temporal information. TFD is characterized by two ingredients: (1) duplication of the Hilbert space and (2) the Bogoliubov transformation. The doubled Hilbert space, or thermal Hilbert space, is defined as $\mathcal{S}_T = \mathcal{S} \otimes \tilde{\mathcal{S}}$, where $\mathcal{S}$ is the standard Hilbert space and $\tilde{\mathcal{S}}$ is the dual or tilde Hilbert space. The relation between these two spaces is governed by the tilde conjugation rules, which are defined as follows:

$$\begin{aligned}
&(A_iA_j)^{\tilde{}} = \tilde{A}_i\tilde{A}_j;\\
&(cA_i + A_j)^{\tilde{}} = c^*\tilde{A}_i + \tilde{A}_j;\\
&(A_i^\dagger)^{\tilde{}} = (\tilde{A}_i)^\dagger;\\
&(\tilde{A}_i)^{\tilde{}} = \pm A_i;\\
&\left[A_i, \tilde{A}_j\right] = 0.
\end{aligned} \tag{46}$$

The Bogoliubov transformation introduces temperature effects through a rotation between the tilde and non-tilde variables. As an example, let's consider the transformations for fermions operators

$$\begin{aligned}
b_{s,p} &= U(\beta)b_{s,p}(\beta) + V(\beta)\tilde{b}^{\dagger}_{s,p}(\beta),\\
\tilde{b}_{s,p} &= U(\beta)\tilde{b}_{s,p}(\beta) - V(\beta)b^{\dagger}_{s,p}(\beta),\\
b^{\dagger}_{s,p} &= U(\beta)b^{\dagger}_{s,p}(\beta) + V(\beta)\tilde{b}_{s,p}(\beta),\\
\tilde{b}^{\dagger}_{s,p} &= U(\beta)\tilde{b}^{\dagger}_{s,p}(\beta) - V(\beta)b_{s,p}(\beta),
\end{aligned} \tag{47}$$

where $U(\beta) = \cos\theta(\beta)$ and $V(\beta) = \sin\theta(\beta)$. In a similar way, for bosons we have

$$\begin{aligned}
a_{\lambda,k} &= U'(\beta)a_{\lambda,k}(\beta) + V'(\beta)\tilde{a}^{\dagger}_{\lambda,k}(\beta),\\
\tilde{a}_{\lambda,k} &= U'(\beta)\tilde{a}_{\lambda,k}(\beta) + V'(\beta)a^{\dagger}_{\lambda,k}(\beta),\\
a^{\dagger}_{\lambda,k} &= U'(\beta)a^{\dagger}_{\lambda,k}(\beta) + V'(\beta)\tilde{a}_{\lambda,k}(\beta),\\
\tilde{a}^{\dagger}_{\lambda,k} &= U'(\beta)\tilde{a}^{\dagger}_{\lambda,k}(\beta) + V'(\beta)a_{\lambda,k}(\beta),
\end{aligned} \tag{48}$$

with $U'(\beta) = \cosh\theta(\beta)$ and $V'(\beta) = \sinh\theta(\beta)$.

Another quantity that will be modified in the TFD formalism is the propagator, which will be decomposed into two distinct parts: one representing the spatial contribution and the other describing the thermal effects. Therefore, it will be written as

$$\langle 0(\beta)|\, \tau[\psi_x\bar{\psi}_y]\, |0(\beta)\rangle = i\int \frac{d^4q}{(2\pi)^4} e^{-iq(x-y)} \varDelta(q,\beta), \tag{49}$$

where $\varDelta(q,\beta) = S^{(0)}(q) + S^{(\beta)}(q)$ with

$$S^{(0)}(q) \;=\; \frac{\not{q} + m}{q^2 - m^2} \tag{50}$$

being the fermion propagator at zero temperature, and

$$S^{(\beta)}(q) \;=\; \frac{2\pi i}{e^{\beta q_0}+1}\left[\frac{(\gamma^0\xi - \gamma\cdot\vec{q} + m)}{2\xi}\varDelta_1\delta(q^0-\xi) + \frac{(\gamma^0\xi + \gamma\cdot\vec{q} + m)}{2\xi}\varDelta_2\delta(q^0+\xi)\right] \tag{51}$$

are the corrections for the propagator due to the temperature effects. Here, have been used that

$$\varDelta_1 = \begin{pmatrix} 1 & e^{\beta q_0/2} \\ e^{\beta q_0/2} & -1 \end{pmatrix}, \qquad \varDelta_2 = \begin{pmatrix} -1 & e^{\beta q_0/2} \\ e^{\beta q_0/2} & 1 \end{pmatrix}. \tag{52}$$

Now we can calculate the transition amplitude for the gravitational Compton scattering at finite temperature. In this approach, Eq. (18) becomes

$$\mathcal{M}(\beta) = \langle f,\beta|\, \hat{S}^{(2)}\, |i,\beta\rangle\,, \tag{53}$$

where the thermal asymptotic states are given by

$$\begin{aligned} |f,\beta\rangle &= b_f^\dagger(\beta) a_f^\dagger(\beta)\,|0(\beta)\rangle\,, \\ |i,\beta\rangle &= b_i^\dagger(\beta) a_i^\dagger(\beta)\,|0(\beta)\rangle\,. \end{aligned} \tag{54}$$

The second-order term of the scattering matrix will be

$$\hat{S}^{(2)} = -\frac{1}{2}\int d^4x\; d^4y\; \tau[\hat{\mathcal{L}}_I(x)\hat{\mathcal{L}}_I(y)], \tag{55}$$

with the interaction Lagrangian for the doubled space, representing the graviton-fermion interaction, being presented in the form

$$\begin{aligned} \hat{\mathcal{L}}_I(x) &= \mathcal{L}_I(x) - \tilde{\mathcal{L}}_I(x) \\ &= -\frac{i\kappa}{4} A_{\mu\nu}(\bar{\psi}_x\gamma^\mu\partial^\nu\psi_x - \partial^\mu\bar{\psi}_x\gamma^\nu\psi_x) + \frac{i\kappa}{4}\tilde{A}_{\mu\nu}(\tilde{\bar{\psi}}_x\gamma^\mu\partial^\nu\tilde{\psi}_x - \partial^\mu\tilde{\bar{\psi}}_x\gamma^\nu\tilde{\psi}_x). \end{aligned} \tag{56}$$

Then the transition amplitude becomes

$$\mathcal{M}(\beta) = \mathcal{M}_{\rm I}(\beta) + \mathcal{M}_{\rm II}(\beta). \tag{57}$$

The procedure to calculate the transition amplitude at finite temperature is the same as described for the case at zero temperature. Considering only the contributions relevant to the analyzed process, the transition amplitudes at finite temperature corresponding to the Feynman diagrams in Figure 1 are given as

$$\begin{aligned} \mathcal{M}_{\rm I}(\beta) = &- \frac{i\kappa^2}{16}U^2(\beta)U'^2(\beta)\sum_{\lambda s}[\bar{u}_{p_f}^{(s_f)}(2p_f+k_f)\cdot\epsilon_f^{*g}\not{\epsilon}_f^{*g}] \\ &\times\ \varDelta_{(p_f+k_f)}[\not{\epsilon}_i^{g}\epsilon_i^{g}\cdot(2p_i+k_i)u_{p_i}^{(s_i)}], \end{aligned} \tag{58}$$

$$\begin{aligned} \mathcal{M}_{\rm II}(\beta) = &- \frac{i\kappa^2}{16}U^2(\beta)U'^2(\beta)\sum_{\lambda s}[\bar{u}_{p_f}^{(s_f)}(2p_f-k_i)\cdot\epsilon_i^{g}\not{\epsilon}_i^{g}] \\ &\times\ \varDelta^{ab}_{(p_f-k_i)}[\not{\epsilon}_f^{*g}\epsilon_f^{*g}\cdot(2p_i-k_f)u_{p_i}^{(s_i)}]. \end{aligned} \tag{59}$$

To obtain the cross section at finite temperature, we need the quantity

$$\langle|\mathcal{M}(\beta)|^2\rangle = \frac{1}{4}\sum_s|\mathcal{M}(\beta)|^2 = \langle|\mathcal{M}_{\rm I}(\beta)|^2\rangle + \langle|\mathcal{M}_{\rm II}(\beta)|^2\rangle + \langle 2\,{\rm Re}\,\mathcal{M}_{\rm I}{}^\dagger(\beta)\mathcal{M}_{\rm II}(\beta)\rangle. \tag{60}$$

Taking the center of mass as the reference frame, we obtain the first term of Eq. (60) as

$$\langle|\mathcal{M}_{\rm I}(\beta)|^2\rangle \;=\; -\frac{1}{16}\kappa^4\omega^4(\cos\theta-1)\frac{1+\mathcal{A}(\beta)}{(1+e^{-2\beta\omega})^2}\left(\frac{1+\coth\beta\omega}{2}\right)^2, \tag{61}$$

where

$$\mathcal{A}(\beta) = e^{-2\beta\omega} + e^{-4\beta\omega}\left(4\pi^2\omega^2\varGamma_1^2+1\right), \tag{62}$$

with $\mathit{\Gamma}_1 = \big(\delta(2\omega)\mathit{\Delta}_1 - \delta(2\omega)\mathit{\Delta}_2\big)$. The second term of Eq. (60) reads

$$\langle|\mathcal{M}_{\rm II}(\beta)|^2\rangle \;=\; -\frac{1}{16}\kappa^4\omega^4(\cos\theta-1)\frac{1+\mathcal{B}(\beta)}{(1+e^{-2\beta\omega})^2}\left(\frac{1+\coth\beta\omega}{2}\right)^2, \tag{63}$$

where

$$\begin{aligned}\mathcal{B}(\beta) \;&=\; e^{-2\beta\omega} - e^{-4\beta\omega}\bigg\{\frac{\pi^2\omega^2}{2}\Big[1+\big(\delta(2\omega-\xi_2)\mathit{\Delta}_1' - 3\delta(2\omega+\xi_2)\mathit{\Delta}_2'\big)\mathit{\Gamma}_2\\ &-\; \xi_2^{-2}\Big(2\cos\theta\big(\omega^2\mathit{\Gamma}_3^2 - 2\xi_2^2\delta(2\omega+\xi_2)\mathit{\Delta}_2'\mathit{\Gamma}_2\big) + \cos2\theta\big(\xi_2^2\mathit{\Gamma}_2^2 - 4\omega^2\mathit{\Gamma}_3^2\\ &-\; 2\omega^2\cos3\theta\mathit{\Gamma}_3^2 + 2\xi_2\omega\mathit{\Gamma}_3\mathit{\Gamma}_4\big) + 4\omega^2\mathit{\Gamma}_3^2 - 2\omega\xi_2\mathit{\Gamma}_3\mathit{\Gamma}_4\Big)\Big]\bigg\},\end{aligned} \tag{64}$$

with $\mathit{\Gamma}_2 = \big(\delta(2\omega-\xi_2)\mathit{\Delta}_1' + \delta(2\omega+\xi_2)\mathit{\Delta}_2'\big)$, $\mathit{\Gamma}_3 = \big(\delta(2\omega-\xi_2)\mathit{\Delta}_1' - \delta(2\omega+\xi_2)\mathit{\Delta}_2'\big)$ and $\mathit{\Gamma}_4 = \big(\delta(2\omega-\xi_2)\mathit{\Delta}_1' + 2\delta(2\omega+\xi_2)\mathit{\Delta}_2'\big)$. Finally, the third term is written as

$$\langle 2\,\mathrm{Re}\,\mathcal{M}_{\rm I}{}^{\dagger}(\beta)\mathcal{M}_{\rm II}(\beta)\rangle \;=\; \frac{1}{128}\kappa^4\omega^4(28\cos\theta+3\cos2\theta-7)\frac{1+\eta\,\mathcal{C}(\beta)}{(1+e^{-2\beta\omega})^2}\left(\frac{1+\coth\beta\omega}{2}\right)^2, \tag{65}$$

with

$$\begin{aligned}\mathcal{C}(\beta) \;&=\; e^{-2\beta\omega}\big(56\cos\theta+6\cos2\theta-14\big) - e^{-4\beta\omega}\Big\{2\pi^2\omega^2\Big[2\cos\theta\mathit{\Gamma}_1\big(2\delta(2\omega-\xi_2)\mathit{\Delta}_1'\\ &+\; 17\delta(2\omega+\xi_2)\mathit{\Delta}_2'\big) - \xi_2^{-1}\Big(14\omega\mathit{\Gamma}_1\mathit{\Gamma}_3 - 45\omega\cos\theta\mathit{\Gamma}_5\mathit{\Gamma}_3 + \cos2\theta\mathit{\Gamma}_1\big(21\delta(2\omega-\xi_2)\mathit{\Delta}_1'\\ &+\; 16\delta(2\omega+\xi_2)\mathit{\Delta}_2'\big) + 34\omega\cos2\theta\mathit{\Gamma}_1\mathit{\Gamma}_3 + \cos3\theta\mathit{\Gamma}_1\big(3\omega\mathit{\Gamma}_3+2\xi_2\mathit{\Gamma}_2\big)\Big)\\ &+\; \mathit{\Gamma}_1\big(9\delta(2\omega-\xi_2)\mathit{\Delta}_1' + 16\delta(2\omega+\xi_2)\mathit{\Delta}_2'\big)\Big] + 3\xi_2^{-1} - 28\cos\theta + 7\Big\},\end{aligned} \tag{66}$$

where $\mathit{\Gamma}_5 = \big(\delta(2\omega)\mathit{\Delta}_1 + \delta(2\omega)\mathit{\Delta}_2\big)$ and $\eta = 1/(28\cos\theta+3\cos2\theta-7)$. Thus, using Eqs. (61), (63) and (65) the total transition amplitude becomes

$$\langle|\mathcal{M}(\beta)|^2\rangle \;=\; \frac{3}{16}\kappa^4\omega^4\cos^4\left(\frac{\theta}{2}\right)\frac{1-\mathcal{D}(\beta)}{(1+e^{-2\beta\omega})^2}\left(\frac{1+\coth\beta\omega}{2}\right)^2, \tag{67}$$

where

$$\mathcal{D}(\beta) = (\cos\theta-1)(\mathcal{A}(\beta)+\mathcal{B}(\beta)) - \frac{\eta\mathcal{C}(\beta)}{8}. \tag{68}$$

It is important to note that, in these results, we have $\xi_1 = m_g = 0$ e $\xi_2 = (2\omega^2(1+\sin\theta+\cos\theta) + \omega^2\sin2\theta)^{1/2}$.

From these results, our objective is to obtain the differential cross section at finite temperature, which is defined by

$$\left(\frac{d\sigma}{d\Omega}\right)_\beta \;=\; \frac{1}{64\pi^2 s}\langle|\mathcal{M}(\beta)|^2\rangle. \tag{69}$$

From the total transition amplitude, the differential cross section for the gravitational Compton scattering at finite temperature is given as

$$\left(\frac{d\sigma}{d\Omega}\right)_\beta = \alpha(\beta)\left(\frac{d\sigma}{d\Omega}\right)_0 \tag{70}$$

where $\left(\frac{d\sigma}{d\Omega}\right)_0$ is the differential cross section at zero temperature as found in Eq. (44) and

$$\alpha(\beta) \equiv \frac{1-\mathcal{D}(\beta)}{(1+e^{-2\beta\omega})^2}\left(\frac{1+\coth\beta\omega}{2}\right)^2 \tag{71}$$

is the modifications due to temperature effects.

It is important to note that our result for the cross section of gravitational Compton scattering is strongly influenced by temperature effects. Two limits must be investigated: (i) the limit of zero temperature and (ii) the limit of very high temperature. First, we note that when the temperature goes to zero, or $\beta \to \infty$, we have $\alpha(\beta) \to 1$ and the different cross section at zero temperature, Eq. (44), is recovered. On the other hand, in the limit of very high temperature, or $\beta \to 0$, the function $\coth\beta\omega$ becomes very large, and the effect of temperature becomes dominant.

Although the study presented here is theoretical, gravitational Compton scattering could conceivably occur in regions of space with high temperatures, such as near black holes or neutron stars. Furthermore, it's important to emphasize that the energies and conditions required to observe such interactions currently exceed our technological capabilities and observational techniques. Therefore, gravitational Compton scattering remains primarily a theoretical concept within the realm of quantum gravity and fundamental physics research.

V. CONCLUSIONS

The Compton scattering process is a fundamental and well-tested phenomenon in QED. To investigate a gravitational version of Compton scattering, the GEM theory is considered. GEM is a gravitational theory based on the similarities between electromagnetism and gravity. Assuming that the GEM equations are derived from the decomposition of the Weyl tensor, a Lagrangian formulation for this theory has been developed. From this Lagrangian, we can study the interaction between gravitons and other fundamental particles, such as electrons. Here, the gravitational Compton scattering, which describes the interaction between a graviton and an electron, is investigated. Two main results are obtained. First, the cross section at zero temperature is determined. Second, the thermal effects on the cross section are calculated. The temperature effects are introduced using the TFD formalism. Our results at zero temperature show that the gravitational cross section has a structure similar to that of the QED process. For the second result, it is evident that the temperature effects alter the cross section. Furthermore, at very high energy temperatures, this effect becomes dominant. Although temperature effects are crucial, it's important to emphasize that scattering processes analyzed in laboratories today are typically considered at zero temperature. Therefore, our results cannot be directly tested in current experiments. However, in astrophysical phenomena, temperature effects are significant and often dominant. Moreover, on a large scale, gravitational effects prevail, making gravitational processes like the one studied here both relevant and observable in such contexts.

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