geometric background

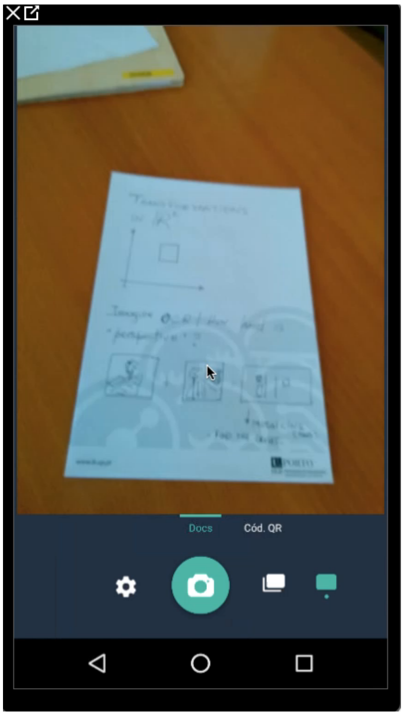
Summary

- geometric primitives
- rigid body transformations
- other geometric transformations
- rotation matrices in 3D
- homogeneous coordinates

adapted from www.zonaenduro.ro

We need to understand how an object moves in 3D world and how this motion is projected in the image plane.

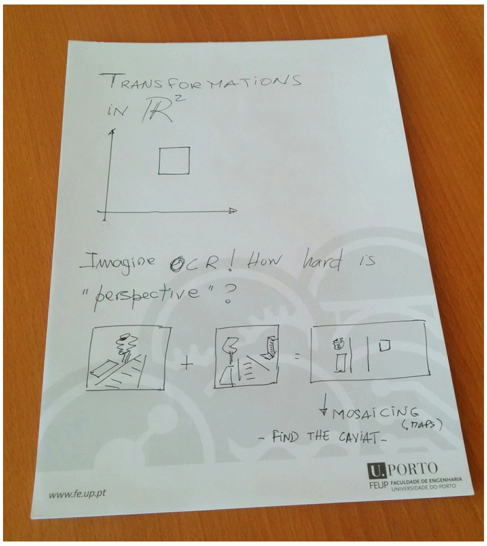
Something trivial: a scanner on my cellphone

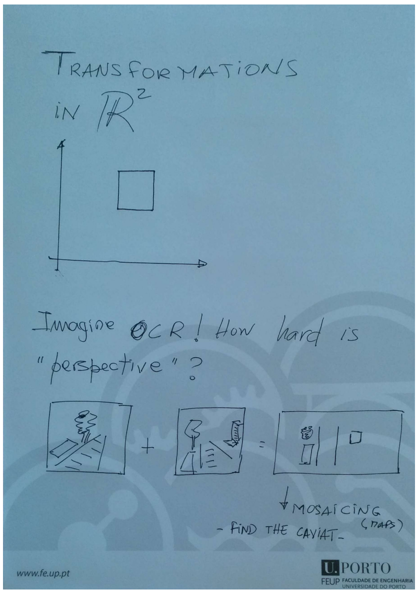


How would you do it ?

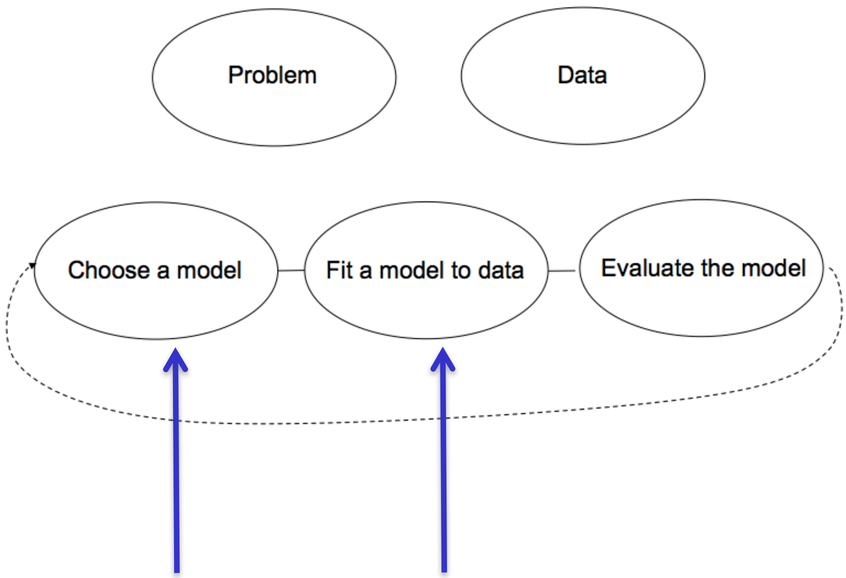
The geeks that did this app took a computer vision course…

Can be done with a few matlab lines of code … Where is the secret ?





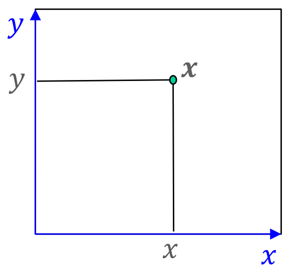
Remember from last class…



That is why we need “math”

Back to “material”

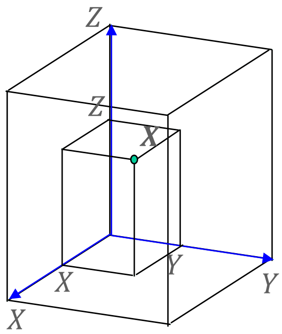
2D points

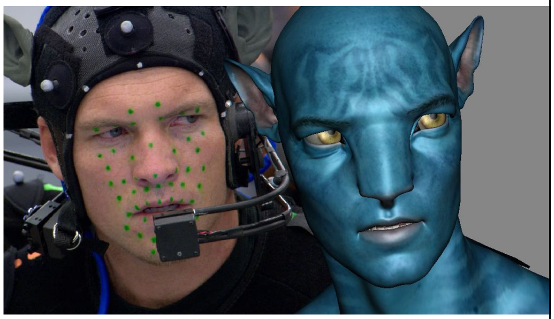


$$\boldsymbol{x} = (x, y) = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$$

- the x, y axis are orthogonal and the rotation from x to y axes is counter-clockwise
- x, y are known as Cartesian coordinates

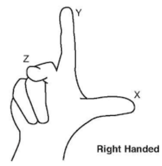
3D points



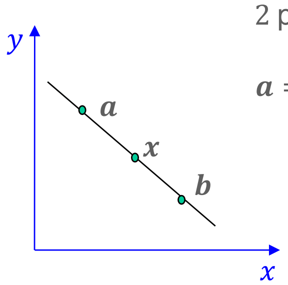


$$\boldsymbol{X} = (X, Y, Z) = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \in \mathbb{R}^3$$

- X, Y, Z are orthogonal axes and obey the right handed rule
- X, Y, Z are known as Cartesian coordinates



2D lines



2 points

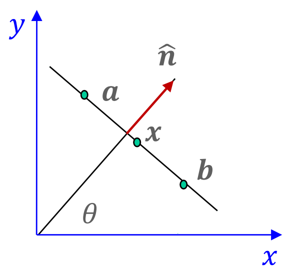
$$\boldsymbol{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

a line is defined by two (different) points: $\boldsymbol{a}, \boldsymbol{b}$.

$$\boldsymbol{x} = (1-\alpha)\boldsymbol{a} + \alpha\boldsymbol{b}$$

$$\boldsymbol{x} = \boldsymbol{a} - \alpha\boldsymbol{a} + \alpha\boldsymbol{b} = \boldsymbol{a} + \alpha(\boldsymbol{b} - \boldsymbol{a})$$

$$\boldsymbol{x} = \boldsymbol{a} + \alpha\boldsymbol{t}\,, \qquad \mathbf{t} = \mathbf{b} - \mathbf{a}$$



unit normal

$$\hat{\boldsymbol{n}} = \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

inner product

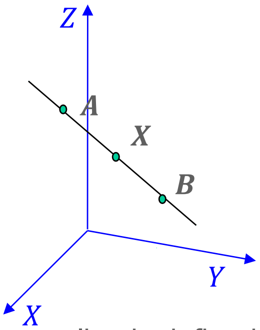
$$\hat{\boldsymbol{n}} \cdot \boldsymbol{x} = \hat{\boldsymbol{n}} \cdot (\boldsymbol{a} + \alpha\boldsymbol{t})$$

$$= \underbrace{\hat{\boldsymbol{n}} \cdot \boldsymbol{a}}_{= d} + \alpha\underbrace{\hat{\boldsymbol{n}} \cdot \boldsymbol{t}}_{= 0}$$

$$\hat{\boldsymbol{n}} \cdot \boldsymbol{x} = d$$

d is the (signed) distance of the straight line to the origin

3D lines



2 points

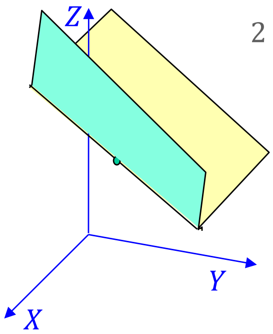
$$\boldsymbol{A} = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix},$$

a line is defined by two (different) points: $\boldsymbol{A}, \boldsymbol{B}$.

$$\boldsymbol{X} = (1-\alpha)\boldsymbol{A} + \alpha\boldsymbol{B}$$

$$= \boldsymbol{A} - \alpha\boldsymbol{A} + \alpha\boldsymbol{B} = \boldsymbol{A} + \alpha(\boldsymbol{B} - \boldsymbol{A})$$

$$\boldsymbol{X} = \boldsymbol{A} + \alpha\boldsymbol{T}\,, \qquad \mathbf{T} = \mathbf{B} - \mathbf{A}$$

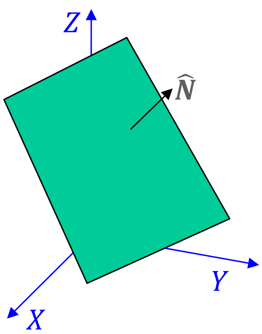


2 planes

a line is defined by the intersection of two non-parallel planes

see next slide.

3D planes



unit normal

$$\widehat{\boldsymbol{N}} = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} \qquad \boldsymbol{P} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

plane equation

$$N_1X + N_2Y + N_3Z = d$$

$$\widehat{\boldsymbol{N}} \cdot \boldsymbol{P} = d$$

d – signed distance from the plane to the origin

motion of rigid objects

When a rigid object moves, each point *X* moves according to

$$\boldsymbol{X} \xrightarrow{\psi} \boldsymbol{X}'$$

$$\boldsymbol{X}' = \psi(\boldsymbol{X})$$

We distinguish between shape and pose (position + orientation).

When a rigid object moves, the shape does not change only the pose.

rigid body transformation

Definition

A map $\psi: \mathbb{R}^p \to \mathbb{R}^p$ $(p = 2 \; or \; 3)$ is called a rigid body transformation if it preserves

i) distances between points

$$\|\psi(\boldsymbol{a}) - \psi(\boldsymbol{b})\| = \|\boldsymbol{a} - \boldsymbol{b}\| \text{ , for all } \boldsymbol{a}, \boldsymbol{b}.$$

ii) orientation (no reflections)

Rigid body transformations can be written as a rotation followed by a translation (or vice-versa)

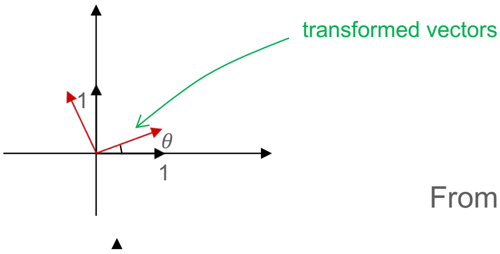
$$\psi(\boldsymbol{x}) = R\boldsymbol{x} + \boldsymbol{T}$$

where $R \in \mathbb{R}^{p \times p}$ is a rotation matrix and $\boldsymbol{T} \in \mathbb{R}^p$ is a translation vector.

rotation matrices are a special class of matrices that will be discussed later.

Example: rotation matrices in the plane

Rotation matrices in the plane are very simple.



rotation

$$\psi(x) = Rx = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

From the figure we conclude,

$$\psi\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \Longrightarrow \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

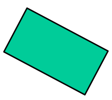
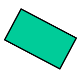
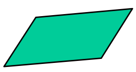
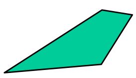
$$\psi\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \Longrightarrow \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

General expression

$$\mathrm{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

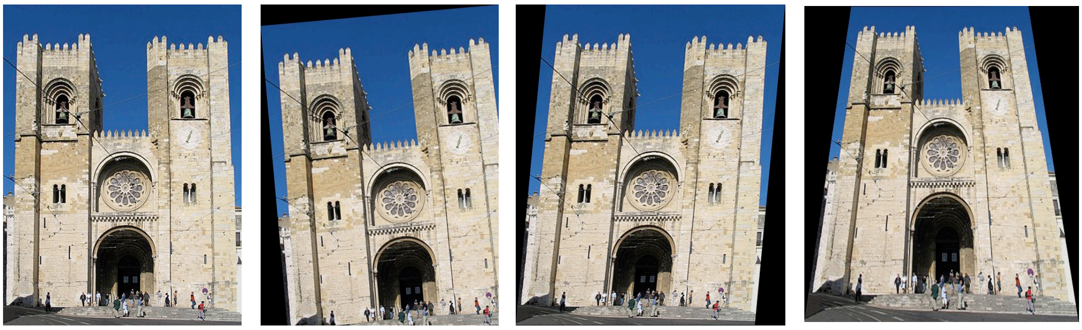
A rotation matrix depends on a single parameter.

Geometric transformations in the plane

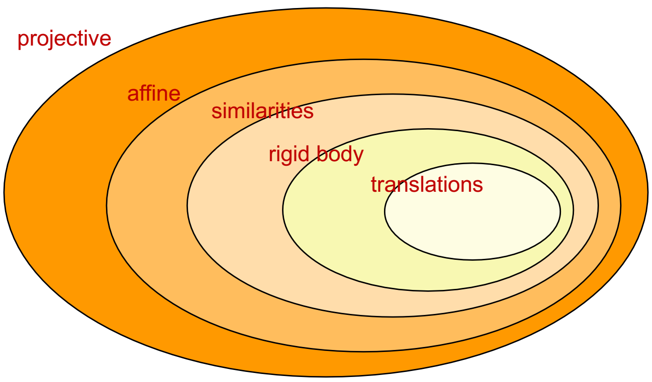
	original object 	expression		degrees of freedom
translation		$\psi(x) = x + t$	$t \epsilon \mathbb{R}^2$	2
rigid body		$\psi(x) = Rx + t$	$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$	3
Euclidean similarity		$\psi(x) = Ax + t$	$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$	4
affine		$\psi(x) = Ax + t$	$A \epsilon \mathbb{R}^{2\times 2}$	6
projective (homography)		$\psi(x) = \begin{bmatrix} \dfrac{a_{11}x + a_{12}y + a_{13}}{a_{31}x + a_{32}y + a_{33}} \\ \dfrac{a_{21}x + a_{22}y + a_{23}}{a_{31}x + a_{32}y + a_{33}} \end{bmatrix}$		8

Examples

rigid body affine projective



hierarchy of transforms



geometric invariants

transforms	lines	paralelism	angles	length
T	Y	Y	Y	Y
RB	Y	Y	Y	Y
ES	Y	Y	Y	N
A	Y	Y	N	N
P	Y	N	N	N

exercises

1. A plane is characterized by a normal vector $N = [1 \quad 3 \quad 2]^T$ (unnormalized) and a signed distance $d = 1/2$.
   a) Determine the equation of the plane,
   b) Check if the point $X = [\sqrt{3} \quad -2 \quad 3]^T$ belongs to the plane.

2. A straight line contains the points $X_1 = [1 \quad 2 \quad 1]^T, X_2 = [-1 \quad 1 \quad 0]^T$. Write two equations for the straight line.

3. A 2D point X is translated by an amount $[-1 \quad 1]^T$, followed by a rotation of amplitude $\theta = \frac{\pi}{6}$.
   a) Derive an expression for the transformed point.
   b) Is there a point that remains invariant after the transformation?

4. The following transformation (similarity) $x' = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} x$ transforms the canonical base $e_1 = [1 \quad 0]^T, e_2 = [0 \quad 1]^T$ into $e_1' = [2 \quad 0]^T, e_2' = [0 \quad 2]^T$.
   a) determine a, b.
   b) Could the problem be solved if the two points e_1, e_2 were randomly chosen.

5. Extend the following 2D transformations to transformations in 3D space: translation, rigid body, affine, homography (projective).

representation 3D rotation matrices

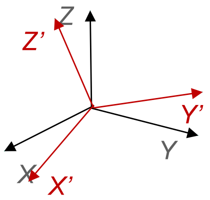
the representation of 3D rotations is a more difficult problem.

several approaches have been adopted:

- orthogonal matrix
- Euler angles
- quaternions
- axis and angle (Rodriguez formula)
- exponential of a matrix (exponential twist)

3D rotation matrices

Given a 3 orthogonal axis, we wish to move them to another orientations, keeping the angles between each pair of them.



$$X' = RX = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{12} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

Property

A rotation matrix R is a 3×3 matrix that has the following properties:

i) R is an orthogonal matrix

ii) $\det R = 1$

orthogonal matrices contain two subclasses: rotation and reflection matrices. The second condition discriminates between both classes.

orthogonal matrices

an orthogonal transformation is a linear map $\psi(\boldsymbol{x}) = A\boldsymbol{x}$ that preserves inner products between pairs of vectors.

$$\psi(\boldsymbol{a}) \cdot \psi(\boldsymbol{b}) = \boldsymbol{a} \cdot \boldsymbol{b} \quad \forall \boldsymbol{a}, \boldsymbol{b}$$

Inner product

$$\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{a}^T \boldsymbol{b}$$

The matrix A is called an orthogonal matrix.

Properties:

i) $\|\psi(\boldsymbol{a}) - \psi(\boldsymbol{b})\| = \|\boldsymbol{a} - \boldsymbol{b}\|\,, \quad \forall a, b$

ii) $A^T A = I, \ \ A^{-1} = A^T$

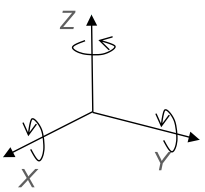
iii) the columns of A define an orthonormal basis of $\mathbb{R}^3$

iv) $\det(A) = 1$ or $\det(A) = -1$

vv) If λ is an eigenvalue of A, $|\lambda| = 1$

proof

Euler angles



We can obtain na arbitrary rotation R by performing 3 simple rotations with respect to each coordinate axis.

The three rotation angles are called Euler angles.

rotation axis X:

$$R_\alpha = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$

rotation axis Y:

$$R_\beta = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

rotation axis Z:

$$R_\gamma = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Euler angles

Combining the three rotations (order is important!)

$$R = R_\gamma R_\beta R_\alpha = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$

$$R = \begin{bmatrix} \cos\gamma\cos\beta & \cos\gamma\sin\beta\sin\alpha - \sin\gamma\cos\alpha & \cos\gamma\sin\beta\cos\alpha + \sin\gamma\sin\alpha \\ \sin\gamma\cos\beta & \sin\gamma\sin\beta\sin\alpha + \cos\gamma\cos\alpha & \sin\gamma\sin\beta\cos\gamma - \cos\gamma\sin\alpha \\ -\sin\beta & \cos\beta\sin\alpha & \cos\beta\cos\alpha \end{bmatrix}$$

This expression has several drawbacks:

- complicated,
- the rotation matrix depends on the order of the elementary rotations,
- smooth changes of the rotation axis may lead to drastic parameter changes

small rotations

Euler angles lead to a complicated expression.

However, if the three angles α, β, γ are small, we can linearize the sine and cosine as follows: $\cos\theta \simeq 1, \sin\theta \simeq \theta$.

Furthermore, the product of two sines can be considered as zero.

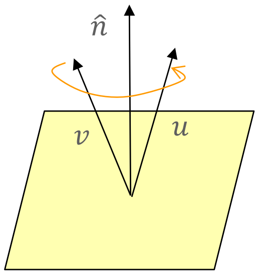
$$R = \begin{bmatrix} 1 & -\gamma & \beta \\ \gamma & 1 & -\alpha \\ -\beta & \alpha & 1 \end{bmatrix}$$

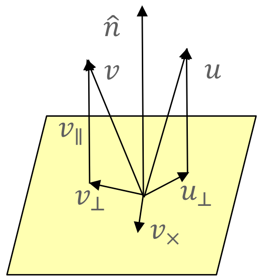
This is the identity plus an anti-symmetric matrix $\quad \boldsymbol{a} = [\alpha \quad \beta \quad \gamma]^t$

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -\gamma & \beta \\ \gamma & 0 & -\alpha \\ -\beta & \alpha & 0 \end{bmatrix} = I + [\boldsymbol{a}]_\times \qquad [\boldsymbol{a}]_\times = \begin{bmatrix} 0 & -\gamma & \beta \\ \gamma & 0 & -\alpha \\ -\beta & \alpha & 0 \end{bmatrix}$$

an anti-symmetric matrix $[\boldsymbol{a}]_\times$ applied to a vector v computes the external product $\boldsymbol{a} \times \boldsymbol{v}$

rotation axis and angle





$\hat{n}$ - rotation axis (unit norm)
θ - angle

$v = v_{\parallel} + v_{\perp} \qquad u = v_{\parallel} + u_{\perp}$

how can we compute $u_{\perp}, v_{\parallel}$?

$$v_{\times} = \hat{n} \times v = [\hat{n}]_{\times} v \qquad v_{\perp} = -\hat{n} \times v_{\times} = -[\hat{n}]_{\times}^2 v$$

$$u_{\perp} = \cos\theta \, v_{\perp} + \sin\theta \, v_{\times} = \left(\sin\theta \, [\hat{n}]_{\times} - \cos\theta \, [\hat{n}]_{\times}^2\right) v \qquad v_{\parallel} = v - v_{\perp} = v + [\hat{n}]_{\times}^2 v$$

$$u = \left(I + [\hat{n}]_{\times}^2 + \sin\theta \, [\hat{n}]_{\times} - \cos\theta \, [\hat{n}]_{\times}^2\right) v$$

Rodriguez's formula

$$R = I + \sin\theta \, [\hat{n}]_{\times} + (1 - \cos\theta)[\hat{n}]_{\times}^2$$

small angles

In this case we can assume that $\cos\theta \simeq 1, \sin\theta \simeq \theta$

Rodriguez's formula becomes

$$R \simeq I + [\theta\hat{n}]_{\times} = I + [\omega]_{\times}$$

$$R \simeq \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

$$u = Rv = v + \omega \times v$$

exponential of a matrix

Let us consider the exponential of the matrix $[\omega]_\times = \theta[\hat{n}]_\times$

$$exp[\omega]_\times = I + \theta[\hat{n}]_\times + \frac{\theta^2}{2!}[\hat{n}]_\times^2 + \frac{\theta^3}{3!}[\hat{n}]_\times^3 + \cdots$$

Since $[\hat{n}]_\times^{k+2} = -[\hat{n}]_\times^k$ we can separate the odd and even powers of the series

$$exp[\omega]_\times = I + \left(\theta - \frac{\theta^3}{3!} + \cdots\right)[\hat{n}]_\times + \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \cdots\right)[\hat{n}]_\times^2$$

$$exp[\omega]_\times = I + \sin\theta\,[\hat{n}]_\times + (1-\cos\theta)\,[\hat{n}]_\times^2$$

We obtained the expression of Rodriguez's rule. Therefore

$$R = exp[\theta\hat{n}]_\times$$

Homogeneous coordinates

The homogeneous coordinates provide a simple way to characterize transformations.

	2D	3D
Cartesian coordinates	$\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix} \epsilon \mathbb{R}^2$	$\boldsymbol{X} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \epsilon \mathbb{R}^3$
homogeneous coordinates	$\tilde{\boldsymbol{x}} = \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \lambda \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \\ \lambda \end{bmatrix} \epsilon \mathbb{R}^3 \setminus \{0\}$	$\tilde{\boldsymbol{X}} = \begin{bmatrix} \tilde{X} \\ \tilde{Y} \\ \tilde{Z} \\ \lambda \end{bmatrix} = \begin{bmatrix} \lambda X \\ \lambda Y \\ \lambda Z \\ \lambda \end{bmatrix} \epsilon \mathbb{R}^4 \setminus \{0\}$

This is a redundant representation: the same physical point is represented by an infinite number of homogeneous vectors.

Two homogeneous vectors $\tilde{x}, \tilde{y}$ represent the same point iif $\tilde{\boldsymbol{x}} = \alpha\tilde{\boldsymbol{y}}$. This will be denoted as

$$\tilde{\boldsymbol{x}} \sim \tilde{\boldsymbol{y}}$$

Points of the form $\tilde{\mathbf{x}} = [\tilde{x} \quad \tilde{y} \quad 0]^T, \tilde{\boldsymbol{X}} = [\tilde{X} \quad \tilde{Y} \quad \tilde{Z} \quad 0]^T$ are called points at infinity since they do not represent (finite) Cartesian points.

Geometric transformations in space

	Cartesian coordinates	homogeneous coordinates	degrees of freedom
translation	$\psi(\boldsymbol{X}) = \boldsymbol{X} + \boldsymbol{T}$	$\lambda \begin{bmatrix} \boldsymbol{X}' \\ 1 \end{bmatrix} = \begin{bmatrix} I & \boldsymbol{T} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{X} \\ 1 \end{bmatrix}$	3
rigid body	$\psi(\boldsymbol{X}) = R\boldsymbol{X} + \boldsymbol{T}$	$\lambda \begin{bmatrix} \boldsymbol{X}' \\ 1 \end{bmatrix} = \begin{bmatrix} R & \boldsymbol{T} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{X} \\ 1 \end{bmatrix}$	6
affine	$\psi(\boldsymbol{X}) = A\boldsymbol{X} + \boldsymbol{T}$	$\lambda \begin{bmatrix} \boldsymbol{X}' \\ 1 \end{bmatrix} = \begin{bmatrix} A & \boldsymbol{T} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{X} \\ 1 \end{bmatrix}$	12
projective (homography)	$\psi(\boldsymbol{X}) = \begin{bmatrix} \frac{h_{11}X + h_{12}Y + h_{13}Z + h_{14}}{h_{41}X + h_{42}Y + h_{43}Z + h_{44}} \\ \frac{h_{21}X + h_{22}Y + h_{23}Z + h_{24}}{h_{41}X + h_{42}Y + h_{43}Z + h_{44}} \\ \frac{h_{31}X + h_{32}Y + h_{33}Z + h_{34}}{h_{41}X + h_{42}Y + h_{43}Z + h_{44}} \end{bmatrix}$	$\lambda \begin{bmatrix} \boldsymbol{X}' \\ 1 \end{bmatrix} = H \begin{bmatrix} \boldsymbol{X} \\ 1 \end{bmatrix}$	15

all transforms:
$$\lambda \widetilde{\boldsymbol{X}}' = M \widetilde{\boldsymbol{X}}$$

Exercises

1. Check if the following matrix is a rotation matrix $M = \frac{1}{2}\begin{bmatrix} \sqrt{3} & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & \sqrt{3} \end{bmatrix}$

2. Determine the rotation matrix knowing that the rotation axis is $\hat{n} = \frac{1}{3}[1 \quad 2 \quad 2]^T$ and the rotation angle is $\theta = \frac{\pi}{3}$.

3. Prove the following properties of orthogonal transformations $\psi(x) = Ax$:

   a) $||\psi(b) - \psi(a)|| = ||b - a||$

   b) $A^T A = I, A^{-1} = A^T$

   c) The columns of A form na orthonormal basis of $\mathbb{R}^3$.

   d) $\det(A) = +1$ or $\det(A) = -1$

   e) All eigenvalues of A verify the constraint $|\lambda|$=1.