Math 104 - Spring 2025 - Homework 1

Due: April 9, 2025 at 08:00pm Pacific time on Gradescope.

Problem 1. Let $A \in \mathbb{R}^{n\times n}$ (i.e. A is an $n \times n$ matrix). The trace of A, denoted by $\mathrm{tr}(A)$, is the sum of its diagonal entries; that is,

$$\mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii}.$$

a) Show that the trace is a linear function; that is, prove that for any $A, B \in \mathbb{R}^{n\times n}$ and $c_1, c_2 \in \mathbb{R}$, $\mathrm{tr}(c_1A + c_2B) = c_1\mathrm{tr}(A) + c_2\mathrm{tr}(B)$.

We have that

$$\begin{aligned}
\mathrm{tr}(c_1A + c_2B) = \sum_{i=1}^{n}(c_1A + c_2B)_{ii} &= \sum_{i=1}^{n}(c_1a_{ii} + c_2b_{ii}) \\
&= c_1\sum_{i=1}^{n} a_{ii} + c_2\sum_{i=1}^{n} b_{ii} \\
&= c_1\mathrm{tr}(A) + c_2\mathrm{tr}(B).
\end{aligned}$$

b) Show that for any $A, B \in \mathbb{R}^{n\times n}$, $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

We have that

$$\mathrm{tr}(AB) = \sum_{i=1}^{n}(AB)_{ii} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}b_{ji} = \sum_{j=1}^{n}\sum_{i=1}^{n} b_{ji}a_{ij} = \sum_{j=1}^{n}(BA)_{jj} = \mathrm{tr}(BA).$$

c) Let $A \in \mathbb{R}^{n\times n}$ be a skew-symmetric matrix, that is, $A^\top = -A$. Show that $\mathrm{tr}(A) = 0$.

We have that $(A^\top)_{ii} = a_{ii}$ for all $i = 1, \ldots, n$. Since A is skew-symmetric, we also have that $(A^\top)_{ii} = (-A)_{ii} = -a_{ii}$. Thus, $a_{ii} = -a_{ii}$ which implies $a_{ii} = 0$ for all $i = 1, \ldots, n$; that is, all the entries on the diagonal of a skew-symmetric matrix are 0. Since the trace of A is the sum of the diagonal entries, we have that $\mathrm{tr}(A) = 0$.

Problem 2. Consider the following $n \times n$ tridiagonal matrix that shows up in the discretization of the heat equation

$$A_n = \begin{bmatrix} -2 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -2 \end{bmatrix}.$$

This is a tridiagonal matrix with -2's on the diagonal, and 1's on the first super-diagonal and the first sub-diagonal. Let $d_n = \det(A_n)$.

a) Write down A_1, A_2, and A_3, and then compute d_1, d_2, and d_3.

Note: The determinant of a 1×1 matrix is the value of the single entry itself.

The first three matrices are

$$A_1 = \begin{bmatrix} -2 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}.$$

So we have

$$d_1 = \det(A_1) = -2,$$

$$d_2 = \det(A_2) = (-2)(-2) - (1)(1) = 3,$$

$$d_3 = \det(A_3) = -2\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 0 & -2 \end{vmatrix} = -2(3) - (-2) = -4.$$

b) Express d_n in terms of d_{n-1} and d_{n-2}. Justify your answer.

Using Laplace expansion along the first row, we have

$$\det(A_n) = -2\det(A_{n-1}) - \begin{vmatrix} 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -2 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -2 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & -2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -2 \end{vmatrix}_{(n-1)\times(n-1)}$$

$$= -2\det(A_{n-1}) - \det(A_{n-2})$$

where for the second term (i.e. calculating the determinant of the $(n-1)\times(n-1)$ matrix), we used the Laplace expansion along the first column. So, we have

$$d_n = -2d_{n-1} - d_{n-2}.$$

c) Verify that the expression found in part (b) for d_n in terms of d_{n-1} and d_{n-2} applies to d_1, d_2, and d_3 calculated in part (a).

We want to verify that $d_3 = -2d_2 - d_1$. From part (a), we have that $d_1 = -2$, $d_2 = 3$, and $d_3 = -4$. Plugging these into the expression we have

$$\begin{aligned} d_3 &\stackrel{?}{=} -2d_2 - d_1 \\ -4 &\stackrel{?}{=} -2(3) - (-2) \\ -4 &\stackrel{?}{=} -6 + 2 \\ -4 &\stackrel{\checkmark}{=} -4 \end{aligned}$$

thus it satisfies the expression found in part (b).

Problem 3. A matrix $A \in \mathbb{R}^{n\times n}$ is said to be **idempotent** if $A^2 = A$.

a) Prove that the matrix $A = \frac{1}{2}\begin{bmatrix} 2\cos^2(\theta) & \sin(2\theta) \\ \sin(2\theta) & 2\sin^2(\theta) \end{bmatrix}$ is idempotent for all $\theta \in \mathbb{R}$.

Hint: You may need $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$.

We need to verify that $A^2 = A$. Computing A^2, we have

$$A^2 = \frac{1}{4}\begin{bmatrix} 2\cos^2(\theta) & \sin(2\theta) \\ \sin(2\theta) & 2\sin^2(\theta) \end{bmatrix}\begin{bmatrix} 2\cos^2(\theta) & \sin(2\theta) \\ \sin(2\theta) & 2\sin^2(\theta) \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4\cos^4(\theta) + \sin^2(2\theta) & 2\cos^2(\theta)\sin(2\theta) + 2\sin(2\theta)\sin^2(\theta) \\ 2\cos^2(\theta)\sin(2\theta) + 2\sin(2\theta)\sin^2(\theta) & \sin^2(2\theta) + 4\sin^4(\theta) \end{bmatrix}.$$

Simplifying the expressions in the entries of the resulting matrix using $\sin^2(\theta)+\cos^2(\theta) = 1$ and $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$, we have

$$4\cos^4(\theta) + \sin^2(2\theta) = 4\cos^4(\theta) + 4\sin^2(\theta)\cos^2(\theta) = 4\cos^2(\theta)(\cos^2(\theta) + \sin^2(\theta)) = 4\cos^2(\theta),$$

$$2\cos^2(\theta)\sin(2\theta) + 2\sin(2\theta)\sin^2(\theta) = 2\sin(2\theta)(\cos^2(\theta) + \sin^2(\theta)) = 2\sin(2\theta),$$

$$\sin^2(2\theta) + 4\sin^4(\theta) = 4\sin^2(\theta)\cos^2(\theta) + 4\sin^4(\theta) = 4\sin^2(\theta)(\cos^2(\theta) + \sin^2(\theta)) = 4\sin^2(\theta).$$

Thus,

$$A^2 = \frac{1}{4}\begin{bmatrix} 4\cos^2(\theta) & 2\sin(2\theta) \\ 2\sin(2\theta) & 4\sin^2(\theta) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2\cos^2(\theta) & \sin(2\theta) \\ \sin(2\theta) & 2\sin^2(\theta) \end{bmatrix} = A.$$

So we conclude that A is idempotent.

b) Prove that if A is an idempotent matrix and $A \neq I$, then A is not invertible.

Proof by contradiction: Let A be an idempotent matrix and assume that A is invertible (i.e. A^{-1} exists). Then, we have

$$\begin{aligned} A^2 &= A \\ A^{-1}A^2 &= A^{-1}A \\ A &= I \end{aligned}$$

which contradicts with the fact that $A \neq I$. Thus, we conclude that A is not invertible.

Problem 4. In many applications once the inverse of an $n \times n$ matrix A is computed, we seek to find the inverse of another $n \times n$ matrix B which differs from A only by a rank 1 update; that is, $B = A + \mathbf{u}\mathbf{v}^\top$ for some $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. The **Sherman-Morrison formula**

$$(A + \mathbf{u}\mathbf{v}^\top)^{-1} = A^{-1} - \frac{A^{-1}\mathbf{u}\mathbf{v}^\top A^{-1}}{1 + \mathbf{v}^\top A^{-1}\mathbf{u}}$$

provides a formula for finding the inverse of B using the inverse of A which has already been computed. This gives an algorithm that is computationally less expensive than computing the inverse of B from scratch.

a) Let $A \in \mathbb{R}^{n\times n}$ and let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. Show that

$$(A + \mathbf{u}\mathbf{v}^\top)^{-1} = A^{-1} - \frac{A^{-1}\mathbf{u}\mathbf{v}^\top A^{-1}}{1 + \mathbf{v}^\top A^{-1}\mathbf{u}}$$

by showing that $(A + \mathbf{u}\mathbf{v}^\top)^{-1}(A + \mathbf{u}\mathbf{v}^\top) = I_n$.

We want to show that

$$\left(A^{-1} - \frac{A^{-1}\mathbf{u}\mathbf{v}^\top A^{-1}}{1 + \mathbf{v}^\top A^{-1}\mathbf{u}}\right)(A + \mathbf{u}\mathbf{v}^\top) = I_n.$$

We have

$$\begin{aligned}
\left(A^{-1} - \frac{A^{-1}\mathbf{u}\mathbf{v}^\top A^{-1}}{1 + \mathbf{v}^\top A^{-1}\mathbf{u}}\right)(A + \mathbf{u}\mathbf{v}^\top) &= A^{-1}A + A^{-1}\mathbf{u}\mathbf{v}^\top - \frac{A^{-1}\mathbf{u}\mathbf{v}^\top A^{-1}A + A^{-1}\mathbf{u}\mathbf{v}^\top A^{-1}\mathbf{u}\mathbf{v}^\top}{1 + \mathbf{v}^\top A^{-1}\mathbf{u}} \\
&= I_n + A^{-1}\mathbf{u}\mathbf{v}^\top - \frac{A^{-1}\mathbf{u}\mathbf{v}^\top + A^{-1}\mathbf{u}\mathbf{v}^\top A^{-1}\mathbf{u}\mathbf{v}^\top}{1 + \mathbf{v}^\top A^{-1}\mathbf{u}} \\
&= I_n + A^{-1}\mathbf{u}\mathbf{v}^\top - \frac{A^{-1}\mathbf{u}(1 + \mathbf{v}^\top A^{-1}\mathbf{u})\mathbf{v}^\top}{1 + \mathbf{v}^\top A^{-1}\mathbf{u}} \\
&= I_n + A^{-1}\mathbf{u}\mathbf{v}^\top - A^{-1}\mathbf{u}\mathbf{v}^\top \\
&= I_n.
\end{aligned}$$

b) Given

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix} \qquad \text{and} \qquad A^{-1} = \begin{bmatrix} -3 & -1 & 1 \\ 14 & 2 & -3 \\ -10 & -1 & 2 \end{bmatrix},$$

find $(A + \mathbf{u}\mathbf{v}^\top)^{-1}$ where $\mathbf{u} = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix}^\top$ and $\mathbf{v} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^\top$.

We have that

$$\begin{aligned}
(A + \mathbf{u}\mathbf{v}^\top)^{-1} &= A^{-1} - \frac{A^{-1}\mathbf{u}\mathbf{v}^\top A^{-1}}{1 + \mathbf{v}^\top A^{-1}\mathbf{u}} \\
&= \begin{bmatrix} -3 & -1 & 1 \\ 14 & 2 & -3 \\ -10 & -1 & 2 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} -3 & -1 & 1 \\ 14 & 2 & -3 \\ -10 & -1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -3 & -1 & 1 \\ 14 & 2 & -3 \\ -10 & -1 & 2 \end{bmatrix} \\
&= \begin{bmatrix} -3 & -1 & 1 \\ 14 & 2 & -3 \\ -10 & -1 & 2 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} 3 \\ -14 \\ 10 \end{bmatrix} \begin{bmatrix} -3 & -1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -3 & -1 & 1 \\ 14 & 2 & -3 \\ -10 & -1 & 2 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} -9 & -3 & 3 \\ 42 & 14 & -14 \\ -30 & -10 & 10 \end{bmatrix} \\
&= \begin{bmatrix} -3/4 & -1/4 & 1/4 \\ 7/2 & -3/2 & 1/2 \\ -5/2 & 3/2 & -1/2 \end{bmatrix}.
\end{aligned}$$

Problem 5. Consider the following two matrices in $\mathbb{R}^{2\times 2}$:

(i) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$; a counterclockwise rotation by θ.

(ii) $\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$; a reflection across the x-axis followed by a counterclockwise rotation by θ (or equivalently, a reflection across the line that has an angle of $\theta/2$ with the x-axis).

Prove that all orthogonal matrices in $\mathbb{R}^{2\times 2}$ are of these two forms.

Hint: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an orthogonal matrix, so it has orthonormal columns and rows. Show that it has one of these two forms.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an orthogonal matrix, so it has orthonormal columns and rows. Since the first column should be a unit vector, we have that $a^2 + c^2 = 1$. Thus, there exists a $\theta \in [0, 2\pi)$ such that $a = \cos\theta$ and $c = \sin\theta$. So, we have $A = \begin{bmatrix} \cos\theta & b \\ \sin\theta & d \end{bmatrix}$. Since the first row should also be a unit vector, we have that $b^2 + \cos^2\theta = 1$ which implies $b = \pm\sin\theta$.

Let's consider the following three cases separately:

- **Case $b = 0$:** If $b = 0$, we have that $b = \pm\sin\theta = 0$ which implies $\sin\theta = 0$ so $\cos\theta = \pm 1$. In order for the second column of A to have unit norm, we have that $d = \pm 1$, so we have the following options

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

  which have the forms (i) with $\theta = 0$, (i) with $\theta = \pi$, (ii) with $\theta = 0$, and (ii) with $\theta = \pi$, respectively.

- **Case $b = \sin\theta \neq 0$:** Since the first column of A should be orthogonal to the second column, we have that $\cos\theta\sin\theta + d\sin\theta = 0$. Since $\sin\theta \neq 0$, we conclude that $d = -\cos\theta$, so we have a matrix of the form

$$\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$

- **Case $b = -\sin\theta \neq 0$:** Since the first column of A should be orthogonal to the second column, we have that $-\cos\theta\sin\theta + d\sin\theta = 0$. Since $\sin\theta \neq 0$, we conclude that $d = \cos\theta$, so we have a matrix of the form

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Problem 6. Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a set of orthonormal vectors in $\mathbb{R}^n$. Let $A \in \mathbb{R}^{n\times n}$ be a matrix. Show that $\{A\mathbf{v}_1, A\mathbf{v}_2, \ldots, A\mathbf{v}_n\}$ is a set of orthonormal vectors if and only if A is an orthogonal matrix.

Hint: Let

$$V = \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & & | \end{bmatrix}.$$

Since the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are orthonormal, the matrix V is orthogonal. Let

$$W = \begin{bmatrix} | & | & & | \\ A\mathbf{v}_1 & A\mathbf{v}_2 & \cdots & A\mathbf{v}_n \\ | & | & & | \end{bmatrix}.$$

Note that $W = AV$. The vectors $A\mathbf{v}_1, A\mathbf{v}_2, \ldots, A\mathbf{v}_n$ are orthonormal if and only if W is an orthogonal matrix. Show that $W^\top W = I$ if and only if $A^\top A = I$.

Following the hint, let

$$V = \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & & | \end{bmatrix}.$$

Since the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are orthonormal, the matrix V is orthogonal. Let

$$W = \begin{bmatrix} | & | & & | \\ A\mathbf{v}_1 & A\mathbf{v}_2 & \cdots & A\mathbf{v}_n \\ | & | & & | \end{bmatrix}.$$

Note that $W = AV$. The vectors $A\mathbf{v}_1, A\mathbf{v}_2, \ldots, A\mathbf{v}_n$ are orthonormal if and only if W is an orthogonal matrix. We want to show that $W^\top W = I$ if and only if $A^\top A = I$.

We have

$$\begin{aligned}
W^\top W = I &\iff (AV)^\top (AV) = I && \text{(since } W = AV\text{)} \\
&\iff V^\top A^\top A V = I && \text{(since } (AV)^\top = V^\top A^\top\text{)} \\
&\iff V(V^\top A^\top A V)V^\top = VV^\top && \text{(multiply by } V \text{ on the left and } V^\top \text{ on the right)} \\
&\iff (VV^\top)A^\top A(VV^\top) = I && (VV^\top = I \text{ since } V \text{ is orthogonal)} \\
&\iff A^\top A = I
\end{aligned}$$

Thus, W is an orthogonal matrix if and only if A is an orthogonal matrix.

Problem 7. Consider the matrix

$$A = \begin{bmatrix} 10 & 2 & 6 \\ 5 & -1 & 2 \\ 5 & 1 & 3 \\ -5 & -3 & -4 \end{bmatrix}.$$

a) Write the matrix A as a sum of rank 1 matrices, using the least number of factors. And, use the factors to write an LU decomposition of A.

A possible decomposition is the following:

$$A = \begin{bmatrix} 10 & 2 & 6 \\ 5 & -1 & 2 \\ 5 & 1 & 3 \\ -5 & -3 & -4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix} \begin{bmatrix} 10 & 2 & 6 \end{bmatrix} + B_1$$

$$= \begin{bmatrix} 10 & 2 & 6 \\ 5 & 1 & 3 \\ 5 & 1 & 3 \\ -5 & -1 & -3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \\ 0 & -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix} \begin{bmatrix} 10 & 2 & 6 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & -2 & -1 \end{bmatrix} + B_2$$

$$= \begin{bmatrix} 1 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix} \begin{bmatrix} 10 & 2 & 6 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \\ 0 & -2 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix} \begin{bmatrix} 10 & 2 & 6 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & -2 & -1 \end{bmatrix}$$

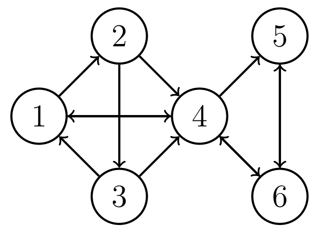
$$= \begin{bmatrix} 1 & 0 \\ 1/2 & 1 \\ 1/2 & 0 \\ -1/2 & 1 \end{bmatrix} \begin{bmatrix} 10 & 2 & 6 \\ 0 & -2 & -1 \end{bmatrix}$$

b) What is the rank of A?

The rank of A is 2.

Problem 8. The Google PageRank algorithm gives a method to "rank" webpages in the order of importance. This problem walks you through an example.

Assume that there are six webpages (the nodes in the graph below) and some of the pages link to each other (the edges in the graph). For instance, there is a link from webpage number 1 to 2 and 4, but not to 3, 5, or 6.



a) A user starts from webpage 1. At each time step, they follow one link uniformly at random and go to another webpage. Let p_i^k be the probability that the user is on webpage i after following k links. Let $\mathbf{p}_k = \begin{bmatrix} p_1^k & p_2^k & p_3^k & p_4^k & p_5^k & p_6^k \end{bmatrix}^\top$. Find the Markov matrix $M \in \mathbb{R}^{6\times 6}$ such that $\mathbf{p}_k = M\mathbf{p}_{k-1}$.

Using the graph, we have that

$$\begin{bmatrix} p_1^k \\ p_2^k \\ p_3^k \\ p_4^k \\ p_5^k \\ p_6^k \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1/2 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1/3 & 0 & 1/2 \\ 0 & 0 & 0 & 1/3 & 1 & 0 \end{bmatrix} \begin{bmatrix} p_1^{k-1} \\ p_2^{k-1} \\ p_3^{k-1} \\ p_4^{k-1} \\ p_5^{k-1} \\ p_6^{k-1} \end{bmatrix}, \text{ so } M = \begin{bmatrix} 0 & 0 & 1/2 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1/3 & 0 & 1/2 \\ 0 & 0 & 0 & 1/3 & 1 & 0 \end{bmatrix}.$$

b) On which webpage is the user most likely to be after 100 time steps?

Hint: Modify the code in this Google Colab link.

We should calculate $\mathbf{p}_{100} = M^{100}\mathbf{p}_0$, where $\mathbf{p}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^\top$ since the user is initially on webpage 1. Running the code in this Google Colab link, we have that the user is most likely to be on webpage 6 after 100 steps (with probability $\approx 33\%$).

c) Now, assume that the user follows a link as shown on the previous graph with probability 80%, and with probability 20%, they choose a new webpage uniformly at random. The Markov matrix for this new scenario is given by

$$M_{\text{new}} = \frac{80}{100}M + \frac{20}{100}A$$

for

$$A = \frac{1}{6}\mathbf{1}\mathbf{1}^\top, \quad \text{where } \mathbf{1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix};$$

that is, A is a 6×6 matrix with all entries $1/6$, which corresponds to an undirected complete graph with self-loops on each vertex (i.e. all vertices are connected to one another in both directions, and each vertex is also connected to itself).

Assume that the user starts in a webpage chosen uniformly at random. What are the probabilities of being on each webpage after 1000 time steps?

Note: The Perron-Frobenius Theorem ensures that there exists a vector $\mathbf{u}$ such that $M\mathbf{u} = \mathbf{u}$. The vector $\mathbf{u}$, which is called the stationary vector or the equilibrium vector, gives information on which pages are more important than others. Intuitively, if a page has incoming links from many other pages that are important, then it is important. If you are interested, you can watch this video or have a look here for an explanation of the PageRank idea. We will come back to this topic when we talk about eigenvalues!

We should calculate $\mathbf{p}_{1000} = M_{\text{new}}^{1000}\mathbf{p}_0$, where

$$\mathbf{p}_0 = \begin{bmatrix} 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \end{bmatrix}^\top = \frac{1}{6}\mathbf{1}$$

since the user starts in a webpage chosen uniformly at random. Running the code in this Google Colab link, we have that, approximately, the probability of being on the i-th webpage is 0.13, 0.08, 0.07, 0.25, 0.21, 0.27, respectively.