

Exotic Option Pricing using Heston Simulation

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Executive Summary

- ▶ Heston Model Calibration is well presented in academic literature such as ([Mrazek & Pospisil 2017](#))
- ▶ In these slides we review, derive and present the discretized Heston model simulation process
- ▶ This is to allow us to perform Heston simulations for bespoke payoffs and exotic option pricing
- ▶ With this presentation we provide the corresponding python implementation

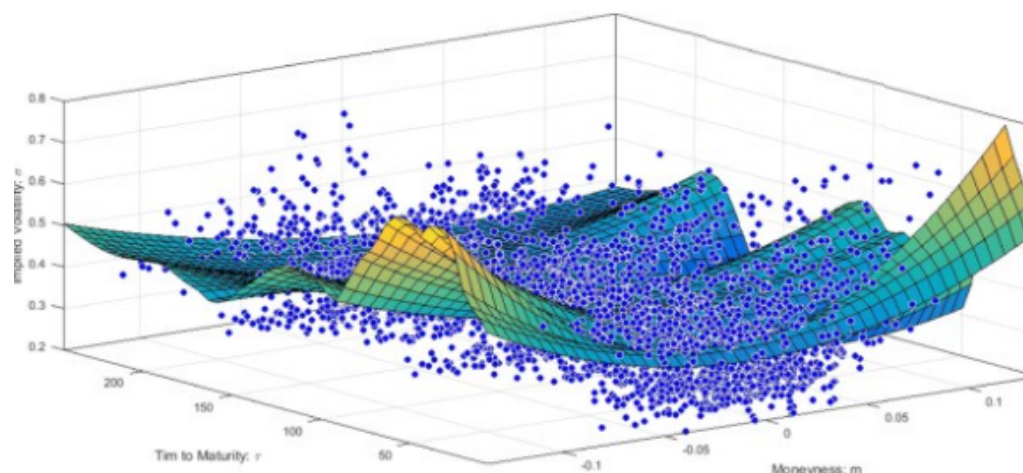
Calibration & Pricing Process

Heston Model Parameters

- ▶ Calibrate to Vanilla Options
- ▶ Use Closed-Form Methods
- ▶ Imply Model Parameters

Exotic Option Pricing

- ▶ To Price Complex Payoffs
- ▶ Use Heston Simulation
- ▶ Given Model Parameters



$$dS(t) = rS(t)dt + s(t)\sqrt{v(t)}dW_S^Q(t)$$
$$dv(t) = \kappa(\bar{v} - v(t))dt + \gamma\sqrt{v(t)}dW_V^Q(t)$$

Heston Model

Heston Stochastic Volatility (SV) Process

For a stock process (S) and a volatility process (v) we have,

$$\begin{aligned}dS(t) &= rS(t)dt + s(t)\sqrt{v(t)}dW_S^Q(t) \\ dv(t) &= \kappa(\bar{v} - v(t))dt + \gamma\sqrt{v(t)}dW_V^Q(t)\end{aligned}\tag{1}$$

with correlated Brownian motions $dW_S^Q(t)dW_V^Q(t) = \rho dt$

Model Parameters

- ▶ κ - speed of mean reversion
- ▶ \bar{v} - long-term volatility level
- ▶ γ - volatility of the volatility

Model Properties

- ▶ Recovers implied volatility smile/skew observed in the market
- ▶ Especially good for pricing medium and long-dated options¹
- ▶ Stochastic volatility process is mean-reverting and non-negative (CIR Model)
- ▶ Market volatilities usually move in opposite direction to underlying asset; the model supports this with negative correlation $\rho_{S,V}$

¹Heston extensions such as adding jumps helps to better models options with short-dated maturities e.g. Bates model

Log-Normal Process for Better Simulation Convergence

- ▶ The Heston model has the following dynamics,

$$\begin{aligned}dS(t) &= rS(t)dt + s(t)\sqrt{v(t)}dW_S^Q(t) \\ dv(t) &= \kappa(\bar{v} - v(t))dt + \gamma\sqrt{v(t)}dW_V^Q(t)\end{aligned}\tag{2}$$

- ▶ Defining the log-normal process $X(t) := \log(S(t))$ and using Itô's Lemma gives,

$$\begin{aligned}dX(t) &= \left(r - \frac{1}{2}v(t)\right)dt + \sqrt{v(t)}dW_X^Q(t) \\ dv(t) &= \kappa(\bar{v} - v(t))dt + \gamma\sqrt{v(t)}dW_V^Q(t)\end{aligned}\tag{3}$$

with correlation $dW_X^Q(t)dW_V^Q(t) = \rho dt$

Correlated Heston Process

- ▶ We use '**Cholesky Decomposition**' to correlate our Brownian motions, but apply this to the log-normal process $X(t)$.
- ▶ This leaves the CIR variance process unmodified and allows exact simulation to be used for the variance process $v(t)$.

$$\begin{aligned}
 dX(t) &= \left(r - \frac{1}{2}v(t) \right) dt + \sqrt{v(t)} \left[\rho_{X,v} d\tilde{W}_V^Q(t) + \sqrt{1 - \rho_{X,v}^2} d\tilde{W}_X^Q(t) \right] \\
 dv(t) &= \kappa(\bar{v} - v(t))dt + \gamma\sqrt{v(t)} dW_V^Q(t)
 \end{aligned}
 \tag{4}$$

where $d\tilde{W}^Q$ denotes an independent Brownian motion under Q , the risk-neutral measure

CIR Variance Process

The CIR variance process follows a non-central chi-squared distribution with $v(t)$ conditional on $v(s)$ for $s < t$ as follows,

$$v(t)|v(s) = \bar{c}(t, s) \cdot \chi^2(\delta, \bar{\kappa}(t, s)) \quad (5)$$

where δ represents the degrees of freedom and $\bar{\kappa}$ the critical value of the chi-squared distribution.

The model parameters are: $\bar{c}(t, s) = \frac{\gamma^2}{4\kappa} (1 - e^{-\kappa(t-s)})$, $\delta = \frac{4\kappa\bar{v}}{\gamma^2}$

and $\kappa(t, s) = \left(\frac{4\kappa e^{-\kappa(t-s)}}{\gamma^2(1 - e^{-\kappa(t-s)})} \right) v(s)$

Exact Simulation of the CIR Process

- ▶ Exact simulation allows us perform Monte Carlo simulation with large time steps with no loss of accuracy
- ▶ Bypassing '**Euler Discretation**', requiring small incremental time-steps, gives a significant performance speed-up
- ▶ For exact simulation we order the CIR parameters as follows to manage model dependencies,

$$\begin{aligned}\bar{c}(t_{i+1} - t_i) &= \frac{\gamma^2}{4\kappa} \left(1 - e^{-\kappa(t_{i+1} - t_i)}\right) \\ \kappa(t_{i+1} - t_i) &= \left(\frac{4\kappa e^{-\kappa(t_{i+1} - t_i)}}{\gamma^2(1 - e^{-\kappa(t_{i+1} - t_i)})} \right) \boxed{v_i}\end{aligned}\tag{6}$$

$$\boxed{v_{i+1}} = \bar{c}(t_{i+1}, t_i) \cdot \chi^2(\delta, \bar{\kappa}(t_{i+1}, t_i))$$

- ▶ For some given initial value $v(t_0) = v_0$ and constant parameter $\delta = 4\kappa\bar{v}/\gamma^2$

Almost Exact Simulation of the Heston Process I

Integrating the Heston processes from (4) leads to,

$$\begin{aligned}
 X_{i+1} = X_i &+ \int_{t+i}^{t_{i+1}} \left(r - \frac{1}{2} v(t) \right) dt \\
 &+ \rho_{X,V} \boxed{\int_{t+i}^{t_{i+1}} \sqrt{v(t)} d\tilde{W}_V^Q(t)} + \sqrt{1 - \rho_{X,V}^2} \int_{t+i}^{t_{i+1}} \sqrt{v(t)} d\tilde{W}_X^Q(t)
 \end{aligned} \tag{7}$$

and

$$v_{i+1} = v_i + \kappa \int_{t+i}^{t_{i+1}} (\bar{v} - v(t)) dt + \gamma \boxed{\int_{t+i}^{t_{i+1}} \sqrt{v(t)} dW_V^Q(t)} \tag{8}$$

Heston Variance Integral

The variance in the Heston asset process $X(t)$ from (7) is the same as that in the CIR variance process (8), which we rearrange to give,

$$\boxed{\int_{t+i}^{t_{i+1}} \sqrt{v(t)} dW_V^Q(t)} = \frac{1}{\gamma} \left(v_{i+1} - v_i - \kappa \int_{t+i}^{t_{i+1}} (\bar{v} - v(t)) dt \right) \quad (9)$$

We can simulate the variance v_{i+1} for a given v_i using the CIR process dynamics or via the CIR sequence (6), which employs the non-central chi-squared distribution.

Almost Exact Simulation of the Heston Process II

Simulating the Heston asset process $X(t)$ from (7) and applying the variance integral definition (9) gives,

$$\begin{aligned}
 X_{i+1} = & X_i + \int_{t+i}^{t_{i+1}} \left(r - \frac{1}{2} v(t) \right) dt \\
 & + \frac{\rho_{X,V}}{\gamma} \left(v_{i+1} - v_i - \kappa \int_{t+i}^{t_{i+1}} (\bar{v} - v(t)) dt \right) \\
 & + \sqrt{1 - \rho_{X,V}^2} \int_{t+i}^{t_{i+1}} \sqrt{v(t)} d\tilde{W}_X^Q(t)
 \end{aligned} \tag{10}$$

Integral Approximation

- ▶ Evaluating the integrals in (10) numerically is computationally expensive requiring function evaluation at many time-steps
- ▶ Therefore we evaluate the integrals using a 'freezing' approximation, which is consistent with Euler discretisation schemes, where left-integration boundaries are used to perform piecewise-constant integration
- ▶ Consequently we cannot perform exact simulation with large time-steps without losing accuracy. However with a moderate number of time-steps this approximation gives good results
- ▶ Performance is greatly improved with negligible approximation error

Heston Model Simulation Process I

Applying the freezing approximation to the Heston model simulation of $X(t)$ from (10) leads to the following, where $v(t)$ terms become v_i ,

$$\begin{aligned}
 X_{i+1} \approx & X_i + \int_{t+i}^{t_{i+1}} \left(r - \frac{1}{2} \boxed{v_i} \right) dt \\
 & + \frac{\rho_{X,V}}{\gamma} \left(v_{i+1} - v_i - \kappa \int_{t+i}^{t_{i+1}} (\bar{v} - \boxed{v_i}) dt \right) \\
 & + \sqrt{1 - \rho_{X,V}^2} \int_{t+i}^{t_{i+1}} \sqrt{\boxed{v_i}} d\tilde{W}_X^Q(t)
 \end{aligned} \tag{11}$$

Heston Model Simulation Process II

This leads to a trivial discretization for X_{i+1} as follows,

$$\begin{aligned} X_{i+1} \approx & X_i + \left(r - \frac{1}{2} v_i \right) \Delta t \\ & + \frac{\rho_{X,V}}{\gamma} (v_{i+1} - v_i - \kappa(\bar{v} - v_i) \Delta t) \\ & + \sqrt{\left(1 - \rho_{X,V}^2 \right) v_i} \left(\tilde{W}_X^Q(t_{i+1}) - \tilde{W}_X^Q(t_i) \right) \end{aligned} \quad (12)$$

where $\tilde{W}_X^Q(t_{i+1}) - \tilde{W}_X^Q(t_i) \stackrel{d}{=} \sqrt{\Delta t} Z_X$ with $Z_X \sim \mathcal{N}(0, 1)$

Discretization of Almost Exact Heston Model

To implement the Almost Exact Simulation (AES) of the Heston model, we discretize the Heston model as follows,

$$\begin{aligned} X_{i+1} &\approx X_i + k_0 + k_1 v_i + k_2 v_{i+1} + \sqrt{k_3 v_i} Z_X \\ v_{i+1} &= \bar{c}(t_{i+1}, t_i) \cdot \chi^2(\delta, \bar{\kappa}(t_{i+1}, t_i)) \end{aligned} \quad (13)$$

with constants $k_0 = \left(r - \frac{1}{2} v_i\right) \Delta t$, $k_1 = \left(\frac{\rho_{X,V}}{\gamma} - \frac{1}{2}\right) \Delta t - \frac{\rho_{X,V}}{\gamma}$,
 $k_2 = \frac{\rho_{X,V}}{\gamma}$ and $k_3 = \left(1 - \rho_{X,V}^2\right) \Delta t$,

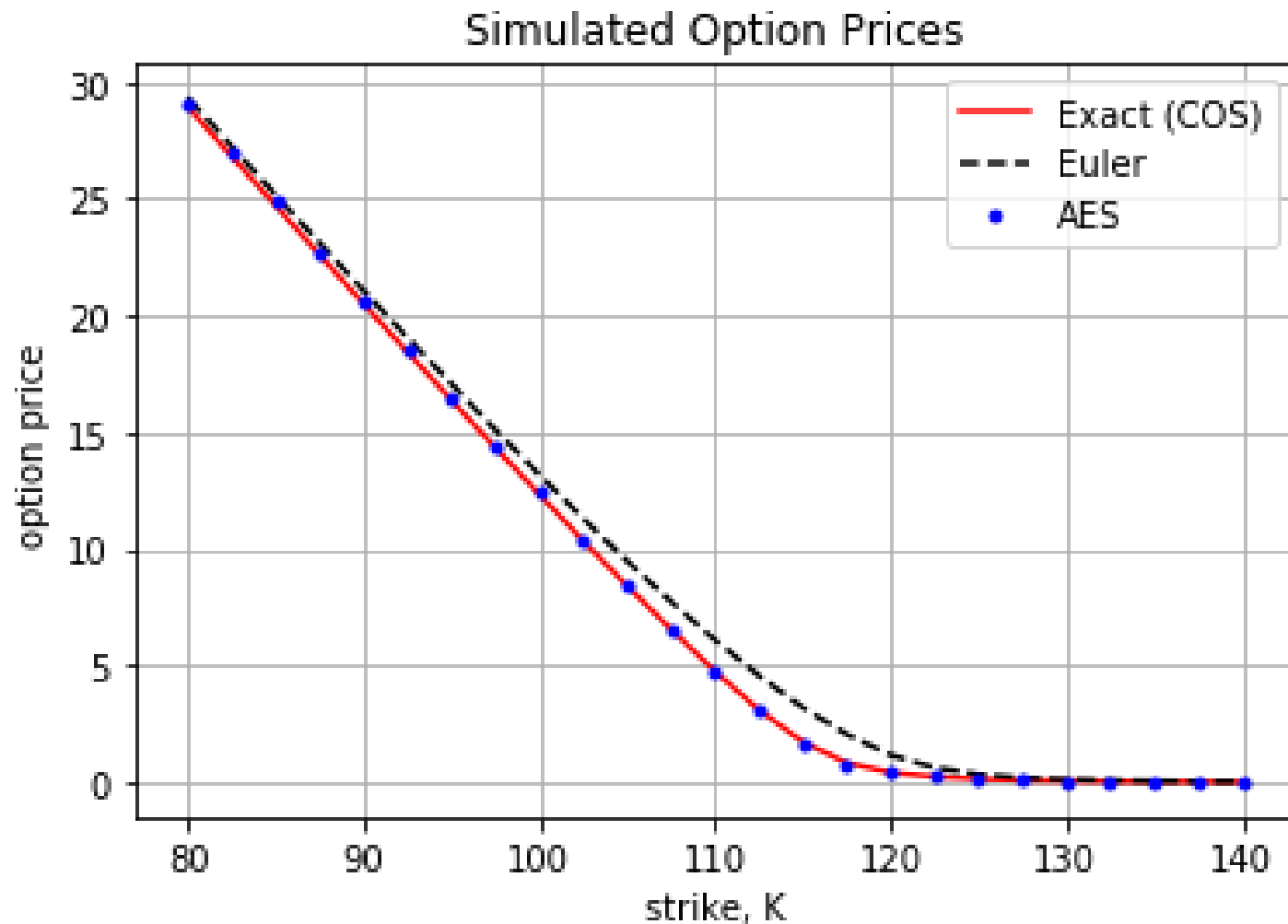
and the variance process simulated as: $\bar{c} = \frac{\gamma^2}{4\kappa} \left(1 - e^{-\kappa(t_{i+1}-t_i)}\right)$,
 $\delta = \frac{4\kappa\bar{v}}{\gamma^2}$, $\bar{\kappa} = \left(\frac{4\kappa e^{-\kappa\Delta t}}{\gamma^2(1-e^{-\kappa\Delta t})}\right) v_i$ and $\chi^2(\delta, \bar{\kappa})$ the non-central
 chi-squared distribution with δ degrees of freedom and
 non-centrality parameter $\bar{\kappa}$.

Simulation Remarks

- ▶ When simulating the CIR process variance cannot be negative
- ▶ Negative variance is avoided when the Feller condition is satisfied i.e. when $2\kappa\bar{v} > \gamma^2$
- ▶ This is often not the case, so we manage this problem using the 'absorption' technique where $v_{i+1} = v_{i+1}^+$ or equivalently $v_{i+1} = \max(v_{i+1}, 0)$
- ▶ We could also use the 'reflection' technique with $v_{i+1} = |v_{i+1}|$
- ▶ Empirically the absorption technique exhibits lower bias

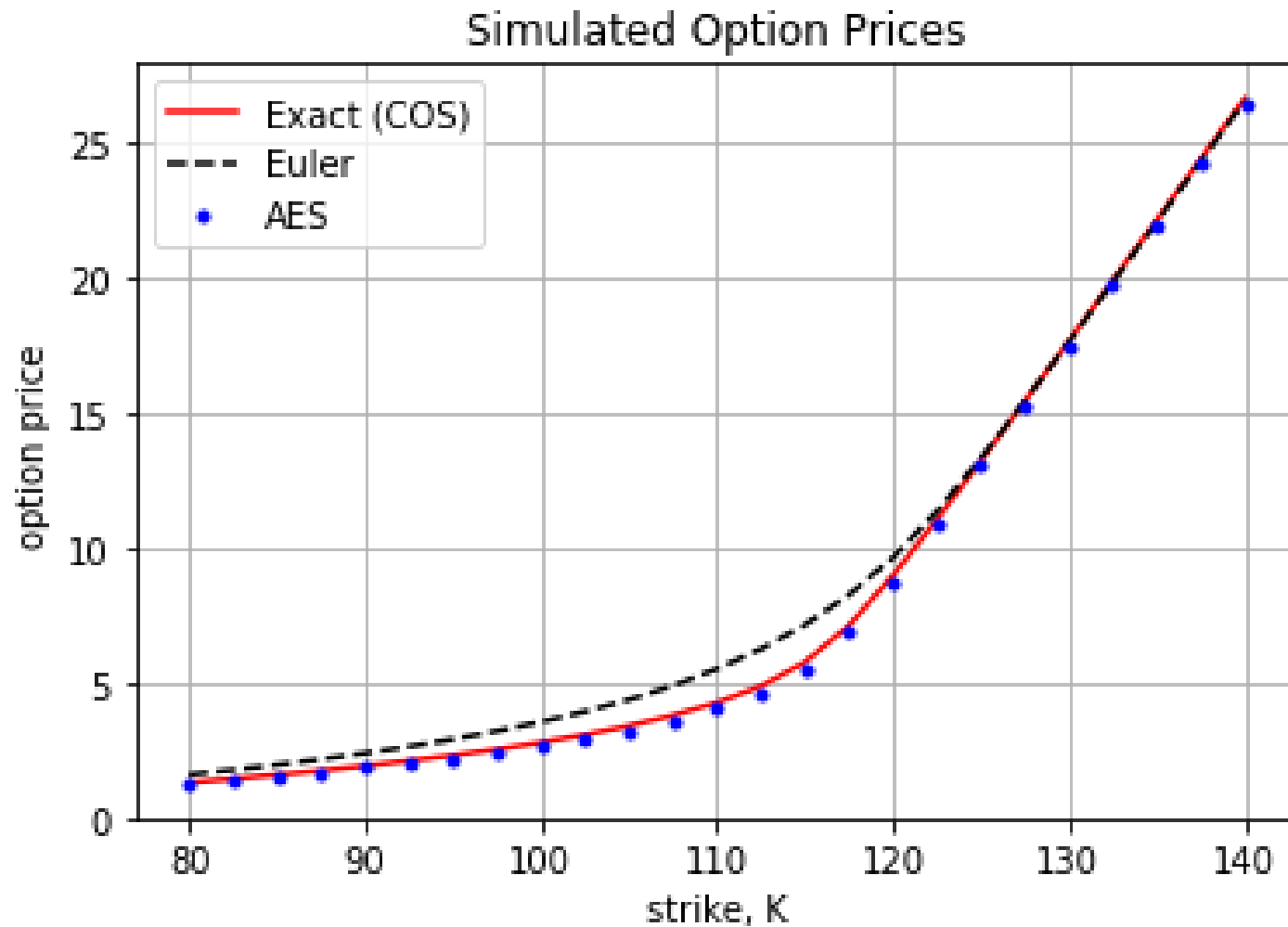
Benchmark Pricing Results I

European Call Option² Convergence: Exact, Euler and AES Comparison



Benchmark Pricing Results II

European Put Option³ Convergence: Exact, Euler and AES Comparison



Simulation Convergence Results

Standard Error by Timestep Size

Euler Scheme

Strike (K), Timestep (dt), Standard Error (eps)

Euler Scheme, K = [140], dt = 1.0, eps = [1.30476871]

Euler Scheme, K = [140], dt = 0.25, eps = [0.54404403]

Euler Scheme, K = [140], dt = 0.125, eps = [0.1720427]

Euler Scheme, K = [140], dt = 0.0625, eps = [0.08078707]

Euler Scheme, K = [140], dt = 0.03125, eps = [0.01260589]

Euler Scheme, K = [140], dt = 0.015625, eps = [0.00605332]

Almost Exact Simulation (AES)

Strike (K), Timestep (dt), Standard Error (eps)

AES Scheme, K = [140], dt = 1.0, eps = [0.00800533]

AES Scheme, K = [140], dt = 0.25, eps = [0.00985109]

AES Scheme, K = [140], dt = 0.125, eps = [0.00135139]

AES Scheme, K = [140], dt = 0.0625, eps = [0.00661993]

AES Scheme, K = [140], dt = 0.03125, eps = [0.0157029]

AES Scheme, K = [140], dt = 0.015625, eps = [0.00699352]

Final Remarks

Almost Exact Simulation (AES)

- ▶ Easily replicates exact option prices
- ▶ Integral freezing approx has negligible impact on price
- ▶ AES simulation is quick and requires few simulation time-steps

Euler Discretization Scheme

- ▶ Euler performs well, but slower & requires more time steps

Exotic Options & Bespoke Option Payoffs

- ▶ Knowing the Heston simulation & discretization process from (13) all that remains is to script the bespoke or exotic payoff, which is a trivial exercise.

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Calibration and Simulation of the Heston Model

Available at: <https://doi.org/10.1515/math-2017-0058>