Credit Derivative Theory & Practice - A Credit Primer & Review of the Impact of ISDA Standardization on Credit Default Swap Pricing & Credit Model Calibration

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Abstract

In this paper we review the pricing and model calibration of Credit Default Swaps referring to both the International Swaps and Derivatives Association (ISDA) CDS contract and credit model standardization guidelines. Furthermore we provide an Excel pricing workbook to supplement the materials discussed. The main goal is for this paper to act as a credit primer and to review the impact and purpose of ISDA contract and model standardization on credit pricing and modelling techniques.

We review the Credit Default Swap (CDS) product highlighting contract specifications, terminology and how the product has been standardized for increased liquidity and XVA capital cost reduction. We perform a fundamental review of probability and credit modelling, outlining standard market assumptions and techniques used by traders and other market practitioners. Furthermore we demonstrate how to price CDS contracts, calibrate credit models and discuss the ISDA Standard Model, ISDA Fair Value Model and Bloomberg Fair Value Models in particular. In conclusion we discuss CDS liquidity, the need for credit index proxies to hedge credit risk and outline liquidity alternatives to this such as the use of sector and index CDS contracts.

Contents

Notation

The notation in table (1) will be used for pricing formulae.

$\Delta(t_{i-1}, t_i)$	The year fraction of the CDS cashflow coupon from time t_{i-1} to t_i in years
s	The fixed premium rate in % of a credit default swap fixed premium cashflow
$\lambda(t)$	The hazard rate or instantaneous probability of default of a credit entity at time t
n	The total number of fixed coupons in a CDS contract
p	The par CDS spread i.e. the value of s that makes the premium and protection legs equal
$P(t, T)$	The discount factor evaluated at time t for a cashflow payable at time T, where $t < T$
$\bar{P}(t, T)$	The risky discount factor evaluated at time t for a cashflow payable at time T, where $t < T$
$Q(t)$	The cumulative probability of survival of a credit entity until time t
R	The recovery rate in % for the credit entity
$\bar{A}_N$	The risky annuity; the present value of the premium leg with a unitary fixed rate i.e. 100%
τ	The time to the credit event or default in years

Table 1: Notation

Introduction

Financial markets have undergone a significant overhaul following the global credit crisis of 2007-2008 and the subsequent collapse of Lehman Brothers. Governments and central banks responded to the crisis with unprecedented fiscal stimulus, monetary policy expansion and numerous large-scale bailouts of prominent financial institutions all over the world. Consequently financial institutions and market participants have come under increased credit, capital and regulatory scrutiny with the Dodd-Frank Wall Street Reform and Consumer Protection Act (2010) being enacted in the US in the aftermath of the crisis to promote the financial stability of the United States. Likewise the Basel III capital and liquidity standards were adopted in Europe and in many other countries around the world.

One of the slogans repeatedly in news headlines to justify the government bailouts of large conglomerates was the term 'to big to fail' and subsequently there was a push towards exchange or cleared transactions to decentralize risk[1]. The introduction of strict capital controls and credit requirements prompted a mass migration from bilateral trading to exchange or cleared trading. This reduced credit risk, capital charges and other associated XVA costs. To facilitate exchange and cleared trading contracts required standardization.

The market for credit default swap transactions went from being a bespoke bilateral market to a predominantly standardized and cleared market. This provided the credit market with increased liquidity and made it possible to reduce capital costs. Trades could be netted easily and portfolios compressed reducing the overall portfolio size and associated capital, margining, hedging and maintenance costs.

In this paper we review the credit default swap product with an emphasis on the ISDA contract standardization and its impact on pricing and credit model calibration. It was also desirable for this paper to act as a credit primer.

Firstly we discuss the credit default swaps and how they are used to both mitigate and speculate on credit risk. We also highlight the International Swaps and Derivatives Association (ISDA) standard CDS contract specifications for European and North American contracts and review CDS market specifications and terminology.

Credit default swaps provide insurance against credit default on a bond or loan position. We discuss the credit risks insured by CDS contracts and the definition of a credit default, which we more accurately describe as a credit event as this term makes it more transparent that credit risk is not limited to a default and also includes other events such as debt restructuring, late repayment or refusal to pay debt, to name but a few scenarios.

Secondly we discuss how to measure credit risk from first principles. We review how we calculate credit default and survival probabilities, making a clear distinction between conditional and unconditional (or marginal) probability. We review the fundamentals of credit risk modelling in a discrete binomial and continuous setting, which is highly relevant when working with piecewise constant default probabilities or hazard rates, which is standard market practice.

Thirdly we present how to price a CDS contract looking at the premium leg, accrued interest payable on default and the protection legs in isolation. In the context of pricing we apply the

[1]Interestingly this seems to further concentrate risk at the exchange or clearing house.

probability theory learned and demonstrate how to account for credit risk when pricing CDS cashflows. We also provide an Excel CDS pricer, available on request, to supplement and demonstrate calculations.

Fourthly we discuss credit modelling and consider structural and reduced form intensity models, highlighting clearly standard assumptions made. The main output of the credit model is the hazard rate or instantaneous default probability. It is the critical ingredient required to evaluate credit risk and price CDS and other credit derivative instruments. We look at how hazard rates are calibrated and summarize the approaches used in the ISDA Standard, ISDA Fair Value and also the Bloomberg Fair Value credit models.

In conclusion we discuss CDS liquidity and the need for credit index proxies to hedge credit risk. We also outline liquidity alternatives to this such as sector and index CDS contracts. It is desirable for this paper to act as a credit primer and to assess the impact and purpose of ISDA contract and model standardization on credit pricing and modelling techniques.

1 Credit Default Swaps

A Credit Default Swap (CDS) is a financial contract used to mitigate credit risk and protect against the risk of default of sovereign or corporate debt. The underlying bond or loan is called the credit reference entity or credit index. Interestingly a CDS can be loosely considered as an equity put option on the credit entity.

CDS contracts are insurance policies; one party pays a fixed insurance premium in exchange for reimbursement of any potential losses suffered in the event of a credit entity default. The premium is called a credit spread and considered as a spread over a risk-free bond.

In the event of a default a contingent payment is made from the protection seller to the protection buyer to reimburse default losses. A priori this loss given default (LGD) is estimated for CDS pricing purposes with the key component being an empirical estimation of the expected recovery rate in the event of a default.

The LGD is evaluated as $N(1 - R)$, where N is the notional and R the recovery rate, which is often estimated to be 40% for senior debt and 20% for subordinated debt. In the event of a default the prevailing LGD is paid and no further premium or protection payments are due; the CDS is effectively torn-up, however the CDS holder is expected to pay pro-rated insurance premia up to and including the day of default, which may result in a accrued interest or stub payment to the protection seller. We illustrate typical CDS transaction cashflows in figure (1) below.

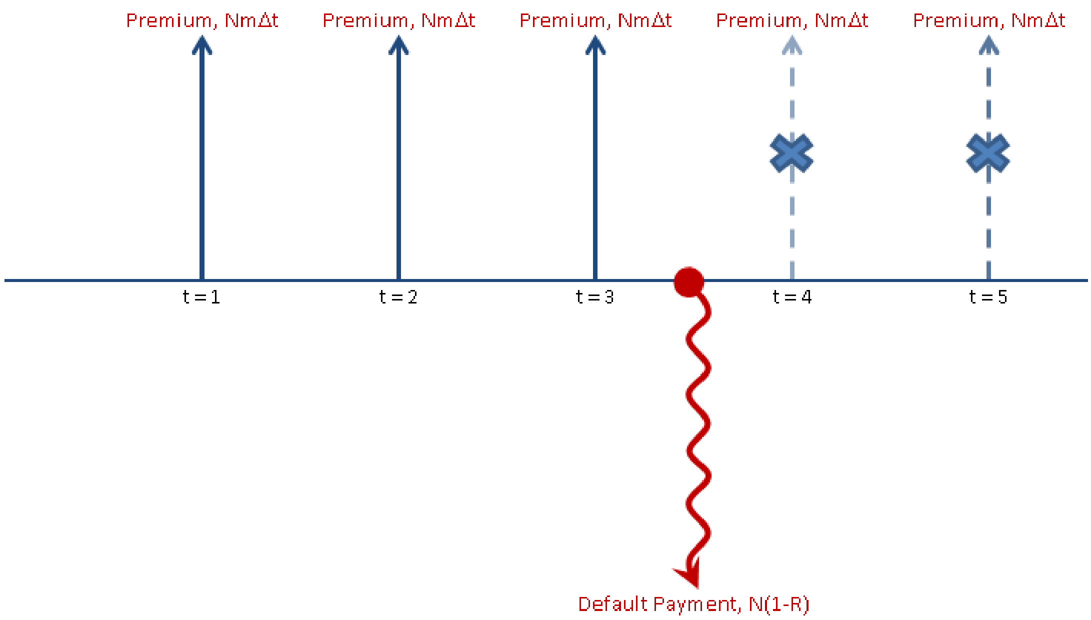

Figure 1: Typical CDS Cashflow Structure

Credit default swaps not only provide a mechanism to mitigate credit risk arising from the ownership of bonds or loans, but can also be used speculatively to express a positive or negative credit view on a credit entity. The larger the perceived or estimated credit risk posed by credit entity the larger the insurance premium required and respective transaction credit spread.

Typically the longer (shorter) the CDS contract tenor the higher (lower) the credit premium and respective credit spread, unless the issuer is in distress and / or the credit curve is inverted. Of course supply, demand and other liquidity pressures can also influence credit spread levels. Currently US Treasury Bonds quote with the CDS spreads as shown in figure (2) below.

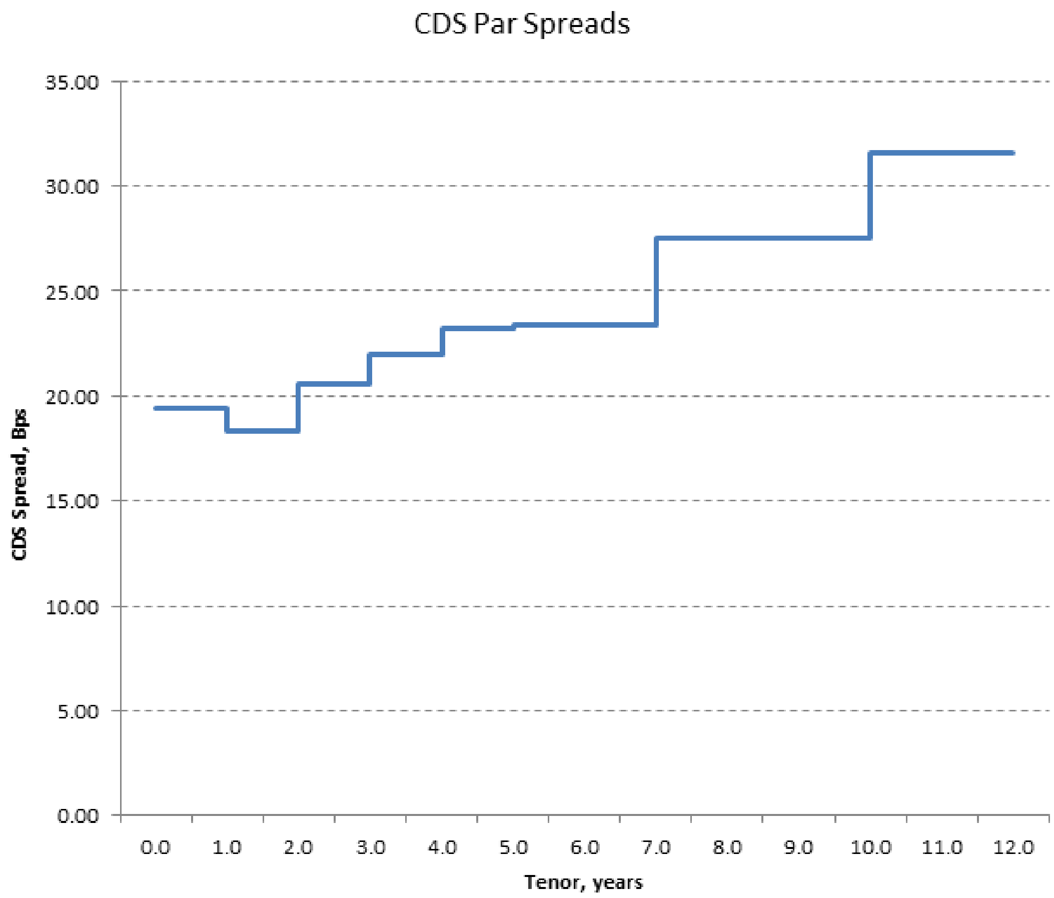

Figure 2: CDS Term-Structure for US Treasury Bonds

Speculative investors trade the CDS spread to bet on credit quality of the underlying credit entity improving or deteriorating. Should the credit quality of the credit entity improve we say that the credit entity experiences credit tightening whereby the credit spread reduces. Likewise if the credit quality deteriorates we say the credit entity experiences credit widening whereby the credit spread increases. The credit spread has a term-structure that varies with CDS contract maturity; the credit spread term-structure is referred to as the credit curve. Generally credit spreads increase for longer maturities, unless the underlying credit reference is in distress; in such cases we say the credit curve is inverted.

Remark 1: Long / Short Credit Risk
Long / Short refers to the credit risk position.

Remark2: Buy / Sell Protection
Buy / Sell refers to the protection leg insurance

Being long the CDS means a party is long the credit risk and is the protection seller. Similarly being short the CDS is tantamount to being short the credit risk making the party the protection buyer. The protection seller receives regular insurance premia in exchange for taking credit risk. Investors often sell CDS as part of a yield enhancement strategy, Credit Linked Note (CLN) issuance are a typical example of this.

CDS were traditionally a bespoke and bilateral product traded between two counterparties. Customizable and bespoke CDS features would sometimes make it difficult to find a buyer or seller of protection and often introduce liquidity issues. This would often make it expensive to place or unwind a CDS transaction as a result. Today CDS are highly standardized contracts and are primarily traded & cleared on exchange. Product standardization has increased CDS liquidity, reduced trading bid-offer spreads, enabled exchange and cleared trading and made it easier to find a counterparty when trading bilatarally.

The CDS contract does not involve the bond issuer or credit reference directly, since issuers cannot guarantee credit default losses on themselves. Credit default swaps can still be traded directly between two parties (bilateral trading), however most CDS transactions are cleared on exchange and are highly standardized to allow trade netting and to mitigate portfolio credit risk. This can significantly reduce XVA and capital charges on the trading book.

1.1 CDS Standardization

Credit Default Swaps are predominantly exchange traded and cleared with bilateral trading between counterparties becoming ever less popular. Exchange traded transactions are standardized and margined transactions. Standardization allows exchanges to compress portfolios and net transactions to reduce trading exposures and associated costs. In particular counterparties trade on exchange to mitigate and reduce credit, XVA and capital requirements, which can be significant. Furthermore standardization makes it cheaper and easier for counterparties to find a counterparty to unwind or novate transactions with.

Full standardization features for CDS contracts are outlined in the ISDA standard CDS contract documentation, see [4] and [5] for further details. ISDA also recommend a standardized pricing model see [3]. Standardized trade features include,

1. **Standard Maturity Dates**
   Credit Default Swap maturities have been standardized. Yearly CDS maturities are quoted with 5Y being the standard and most liquid contract maturity traded.

2. **Standard Coupon Dates**
   Quarterly IMM dates; 3rd Wednesday of March, June, September and December.

3. **Accrued Interest Payments**
   CDS contracts start from the last IMM date with full premium coupons payable even for contracts traded on non-IMM dates. New trades compensate and offset for this by quoting clean and dirty prices. Accrued interest payments are required to compensate for the full first coupon[2], in a similar manner to that of bond markets.

[2] A full first coupon must be paid, accruing from the previous IMM date even though we may have transacted on a non-IMM date.

4. **Standard Credit Spread Denominations**
   CDS trades are transacted with fixed credit spread denominations of typically 25, 100, 300, 500 and 1000 basis points[3]. This means CDS contracts rarely trade at par.

5. **Upfront Payments**
   Due to the introduction of standard CDS spreads being used for the premium leg fixed rate, it is seldom the case where CDS contracts transact at par[4]. As such CDS trades require an upfront payment is required to enter the transaction. This is identical to interest rate market convention, whereby interest rate swaps not traded at par require an upfront fee to enter the trade. The upfront fee is equal to the present value of the transaction.

6. **Identical Credit Event Triggers**
   Includes Default, Bankruptcy, Refusal to Pay, Restructuring, Repudiation and Moratorium, see [3] for more information.

7. **Default Protocols**
   Standard settlement and evaluation protocols for default protection payments, see [3] for more information.

CDS transactions are traded with quarterly premium coupon dates paying on market standard IMM dates. The International Monetary Market (IMM) dates are also used for most exchange traded future and option maturity dates and are scheduled as the third Wednesday of the March, June, September and December to avoid holidays and weekend settlement rolls. The most liquid CDS maturity traded is the 5Y contract and most CDS transactions have yearly maturities.

Further standardization includes the regularization of premium coupons, achieved by only allowing the fixed rate or credit spread to transact in regularized denominations of 25, 100, 300, 500 or 1000 bps. Also the definition of credit event triggers has been standardized so that the protection payment triggers are equivalent between contracts as much as possible.

CDS contract standardization is so extensive for Credit Default Swaps that portfolio compression and trade netting is highly efficient and effective in reducing credit risk, associated trade and risk management charges. However to accommodate standardization we have to adjust CDS transactions to make the standardization possible.

Firstly since the contracts start on the last IMM date we must adjust our trade for accrued interest very much in the way we would adjust a bond transaction. Spot CDS contracts start on the last quarterly IMM date (3rd Wednesday of March, June, September, December). However if we trade on a non-IMM date, which is almost almost the case, we are required to cash settle the accrued interest from the last IMM date to the transaction date to compensate the premium leg payer for the full first premium coupon required to be paid on the next IMM date.

Secondly investors are required to transact using regularized credit spread denominations. This makes it almost impossible to transact at par. We are required to transact a CDS contract with a fixed denomination of credit spread in basis points, namely at 25, 100, 300, 500, 1,000 bps.

[3] One Basis Point (bps) is 1/100th of a percent.
[4] A par CDS rate is the rate at which the premium and protection legs are equal in value.

If the current CDS spread is 40 bps and we transact at 25 bps say, we are required to cash settle the difference i.e. 15 bps as part of an upfront payment, since the trade is no-longer at par. This is no different from other products such as interest rate swaps, whereby non-par swaps transact with an upfront cost equal to the present value of the swap.

1.2 CDS Specifications & Terminology

Credit Default Swaps features are specified and described with the following key terms and features as specified in detail within [2], [7], and [8].

1. **Credit Event**
   A credit event triggers the contingent or protection payment. Possible credit events include bankruptcy, failure to pay, restructuring, repudiation and moratorium, see [4], [7] and [8] for more information.

2. **Credit Spread, s**
   The fixed premium rate to be paid in % on the premium leg.

3. **Credit Tightening**
   An improvement in the credit quality and reduction in credit spread

4. **Credit Widening**
   A deterioration in credit quality and increase in the credit spread %

5. **Hazard Rate, λ**
   The instantaneous probability of default in %. Hazard rates cannot be negative and must be in the range [0,1] to be considered a valid probability measure.

6. **Loss Given Default, LGD**
   The expected loss on the underlying credit reference in the event of a default, measured as $N(1 - R)$

7. **Notional, N**
   The notional amount of the underlying bond or credit reference to be insured.

8. **Par CDS Spread, p**
   The credit spread that makes the CDS premium and protection legs have equal value and the CDS price to zero or par.

9. **Premium Leg**
   The fixed leg of the credit default swap, which is also the insurance premium leg.

10. **Protection Leg**
    The protection or contingent leg of the CDS, which models and represents the loss given default to be paid on default.

11. **Protection Payment**
The loss given default payment to be made in the event of a default. Also known as the contingent payment. On default the actual loss given default is evaluated and paid.

12. **Recovery Rate, R**
An empirical measure of the % recovery of the credit reference notional in the event of a default. Typically estimated as 40% for senior debt and 20% for subordinated debt.

13. **Risky Annuity,** $\bar{A}_N$
The risky annuity or risky PV01 is the value of the premium leg with a unit credit spread i.e. 100%. This is a useful for calculating the par CDS spread, p.

$$\bar{A}_N = \sum_{i=1}^{n} N\Delta(t_{i-1}, t_i)Q(t_i)P(t_0, t_i) \tag{1}$$

giving

$$\begin{aligned} PV(\textit{Premium Leg}) &= \sum_{i=1}^{n} Ns\Delta(t_{i-1}, t_i)Q(t_i)P(t_0, t_i) \\ &= s\sum_{i=1}^{n} N\Delta(t_{i-1}, t_i)Q(t_i)P(t_0, t_i) \\ &= s\bar{A}_N \end{aligned} \tag{2}$$

14. **Risky Discount Factor,** $\bar{P}(t,T)$
The risky discount factor is the regular discount factor $P(t,T)$ scaled by the unconditional or cumulative probability of survival. It accounts for both the time value of money and counterparty default. Mathematically we represent this as,

$$\bar{P}(t,T) = e^{-\int_t^T f(u)+\lambda(u)\ du} \tag{3}$$

In the simplified case, where we have a constant zero rate r and constant hazard rate λ the spot discount factor would be evaluated as,

$$\bar{P}(t,T) = e^{-(r+\lambda)(T-t)} \tag{4}$$

1.3 Credit Events & Triggers

Credit default swaps insure the protection buyer from credit events. Credit events are not limited to the default of the underlying credit reference. A credit event triggers the protection payment. Prior to trading we explicitly specify the credit event being insured against in precise terms for the avoidance of doubt and of course law suits.

Possible credit event triggers include the following, for a full list of credit events and further details we refer the reader to [7] and [8].

- Default
- Restructuring
- Credit Spread Widening
- Ratings Downgrade
- Repudiation
- Moratorium

Repudiation and moratorium refers to an issuer's refusal to pay or postponement of payment respectively. These terms are usually applied to sovereign entities, which can print new money to satisfy debt obligations but instead refuse or postpone payment.

2 Credit Default & Survival Probabilities

To model and price CDS contracts we need to be able to assess the probability of a credit event. Mathematically we measure credit risk in terms of the probability of default or the probability of survival of the credit reference entity. Probabilities can be conditional or unconditional (marginal).

Unconditional or Marginal Probability
This is the probability of an event A independent of other events in the probability space. Also known as marginal probability it measures the chance of an occurrence of event A ignoring any knowledge of previous or external events. We can evaluate this as,

$$P(A) = \frac{\textit{Number of times A occurs}}{\textit{Total number of possible outcomes}} \tag{5}$$

Conditional Probability
Conditional probability evaluates the likelihood of an event A given an event B has occurred. This may be visualized by restricting the sigma-field of the probability space to the event B space, given $P(B) > 0$. Mathematically we write this as,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \tag{6}$$

Graphically we can visualize the probability of an event on a tree diagram with the branch probabilities being conditional on the parent node. Unconditional probabilities evaluate events from the view point of the parent node, whereas conditional probabilities restrict the sample space and start from the conditioning node.

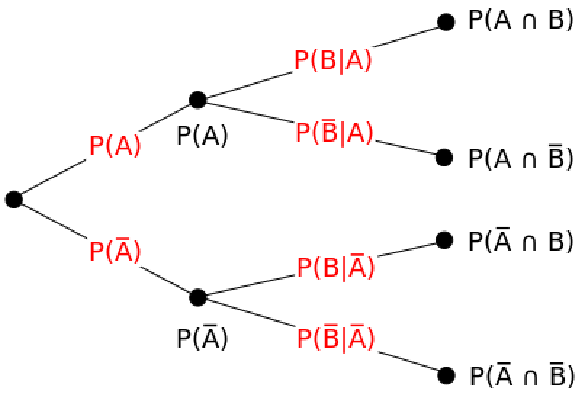

Figure 3: Probability Tree Diagram

2.1 Binomial Credit Model

When modelling credit default we typically work with unconditional (marginal) probabilities. As shown in [8] we can model the marginal probability of default and represent the default process using a binomial tree as illustrated below,

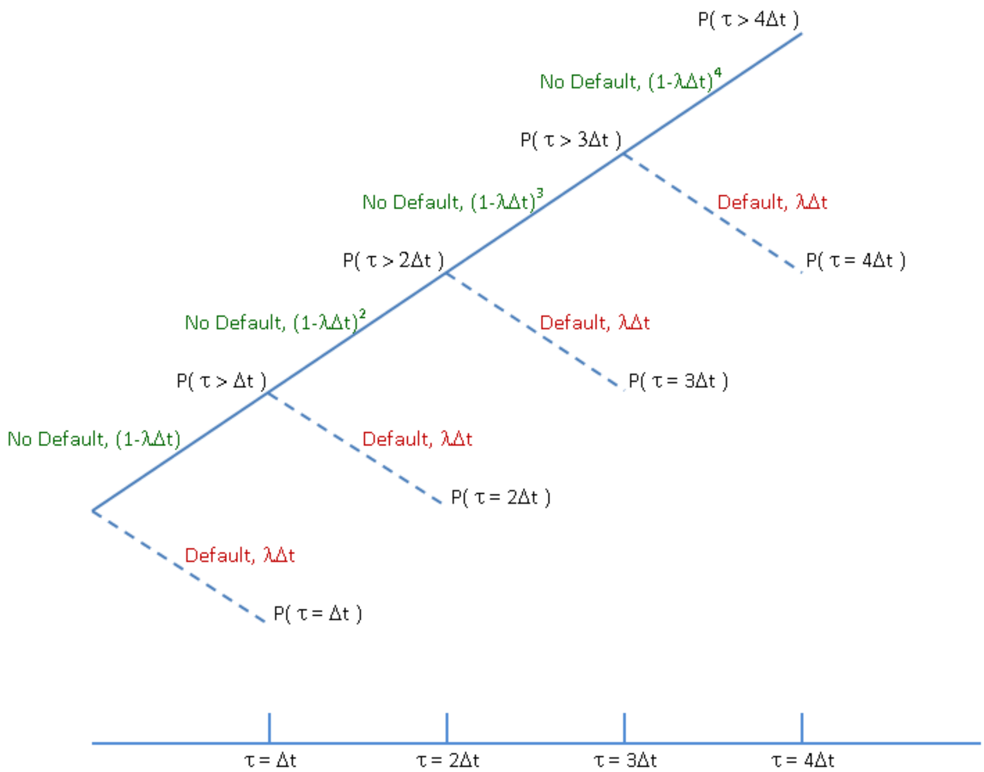

Figure 4: Default Probability Tree

2.2 Unconditional Survival Probability

In the discrete case we define the hazard rate λ as the probability of default within any given discrete time-step and in the continuous case it represents the instantaneous probability of default. In this paper we assume the hazard rate process is deterministic and independent of both interest & recovery rates. This assumption is a widely used and acceptable practice for most market participants; the pricing impact is typically within the bid-offer spread for credit default swaps as noted in [8].

The probability of survival using the binomial model approach and assuming a constant hazard rate λ is illustrated in figure (4). It is the probability of following the survival path denoted by the solid blue line. The unconditional probability of survival to time t is evaluated as the probability of time to default τ being larger than t namely,

$$P(\textit{Survive to time t}) = P(\tau > t) = (1 - \lambda \Delta t)^n \tag{7}$$

where n is the number of time-steps and $\Delta t = \frac{t}{n}$

Using the following identities we can evaluate the continuous unconditional survival probability equivalent of equation (7). Firstly knowing that,

$$\lim_{n \to \infty} \left(\frac{1}{\left(1 - \frac{x}{n}\right)^n} \right) = e^x \tag{8}$$

naturally leads to an equivalent identity,

$$\lim_{n \to \infty} \left(1 - \frac{x}{n}\right)^n = e^{-x} \tag{9}$$

Therefore as we increase the number of time-steps in the binomial model we know that in the limiting case as $n \to \infty$ we can apply (9) to (7) to give the continuous equivalent,

$$\begin{aligned} P(\textit{Survive to time t}) &= P(\tau > t) \\ &= \lim_{n \to \infty} (1 - \lambda \Delta t)^n \\ &= \lim_{n \to \infty} \left(1 - \frac{\lambda t}{n}\right)^n \\ &= e^{-\lambda t} \end{aligned} \tag{10}$$

2.3 Survival Probability Functional Form

Visually the continuous probability of survival from (10) is an exponential function looking as follows for a constant λ. Interestingly the probability of survival is bounded in the region [0,1] and a monotonically decreasing function, which is consistent with a probability measure and the monotonically decreasing probability of survival.

Furthermore $P(\tau = 0) = 1$ and $P(\tau = \infty) = 0$, which implies the immediate probability of default is zero, conversely the immediate probability of survival is one and that all credit references default eventually in the limiting case.

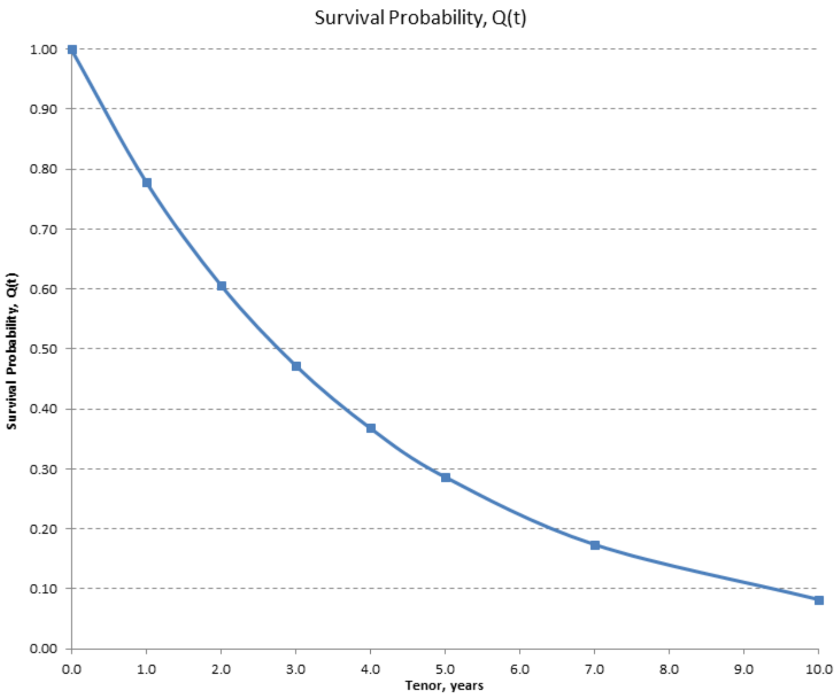

Figure 5: Survival Probability Function with a Constant and Large Hazard Rate

Hazard rates are usually very low and not typically constant making the real-world survival function look rather linear and less smooth, albeit monotonically decreasing as shown below,

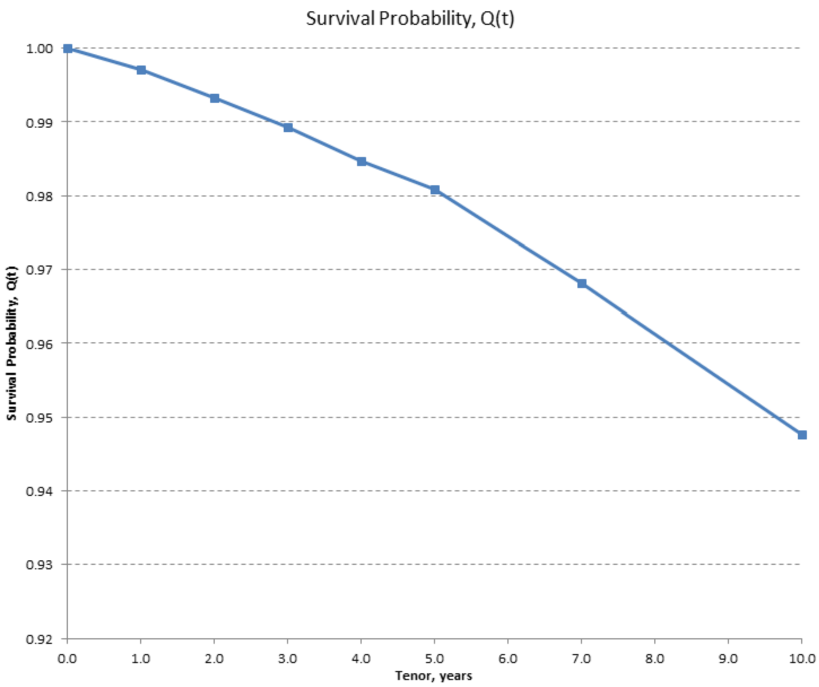

Figure 6: Typical Survival Probability Function Non-Constant and Low Hazard Rate

2.4 Marginal Conditional Default Probability

The marginal probability of default in a discrete binomial setting was illustrated in figure (4). We can represent the marginal probability of default, assuming a constant or piecewise-constant hazard rate λ, mathematically as,

$$\begin{aligned} P(\tau < t) &= \sum_{i=1}^{n} Q(t_i)\lambda(t_i)\Delta t \\ &= 1 - Q(t) \end{aligned} \tag{11}$$

To prove the equivalence of the representations in (11), subject to the assumption of constant or piecewise constant hazard rates, we proceed as follows,

$$\begin{aligned} P(\tau < t) &= \sum_{i=1}^{n} Q(t_i)\lambda(t_i)\Delta t \\ &= \sum_{i=1}^{n} (1 - \lambda\Delta t)^i \lambda\Delta t \\ &= \lambda\Delta t \sum_{i=1}^{n} (1 - \lambda\Delta t)^i \end{aligned} \tag{12}$$

Next we make use of the following identity, which can be verified by multiplying the LHS and RHS of the identity by $(1 - x)$ and noting most terms cancel,

$$1 + x + x^2 + ... + x^n = \left(\frac{1 - x^{n+1}}{1 - x}\right) \tag{13}$$

Applying identity (13) with $x = (1 - \lambda\Delta t)$ to equation (12) gives

$$\begin{aligned} P(\tau < t) &= \lambda\Delta t \left(\frac{1 - (1 - \lambda\Delta t)^{n+1}}{1 - (1 - \lambda\Delta t)}\right) \\ &= \lambda\Delta t \left(\frac{1 - (1 - \lambda\Delta t)^{n+1}}{1 - 1 + \lambda\Delta t)}\right) \\ &= \lambda\Delta t \left(\frac{1 - (1 - \lambda\Delta t)^{n+1}}{\lambda\Delta t}\right) \\ &= 1 - (1 - \lambda\Delta t)^{n+1} \end{aligned} \tag{14}$$

using (7) we have that,

$$\begin{aligned} P(\tau < t) &= 1 - (1 - \lambda\Delta t)^{n+1} \\ &= 1 - Q(t) \end{aligned} \tag{15}$$

and therefore equation (11) holds in the discrete case.

Furthermore we can show that the above also holds in the continuous case in the limit as we increase the number of time-steps n to ∞.

By definition we know that $\Delta t = \left(\frac{t}{n}\right)$ which gives,

$$P(\tau < t) = 1 - \left(1 - \frac{\lambda t}{n}\right)^{n+1} \tag{16}$$

using equation (9) we have that limit as $n \to \infty$,

$$\begin{aligned} P(\tau < t) &= \lim_{n\to\infty}\left[1 - \left(1 - \frac{\lambda t}{n}\right)^{n+1}\right] \\ &= 1 - e^{-\lambda t} \\ &= 1 - Q(t) \end{aligned} \tag{17}$$

therefore

$$P(\tau < t) = \sum_{i=1}^{n} Q(t_i)\lambda(t_i)\Delta t \tag{18}$$

and

$$P(\tau < t) = 1 - Q(t) \tag{19}$$

are equivalent expressions for the marginal default probability in the limit as $n \to \infty$.

Finally for completeness we can evaluate equation (11) in the continuous case directly in integral form. Using the result from (10) we can deduce in the continuous case that the marginal probability of default equals

$$\begin{aligned} P(\tau < t) &= 1 - P(\tau \geq t) \\ &= 1 - e^{-\lambda t} \end{aligned} \tag{20}$$

Next assuming constant λ, using equation (10) and knowing that λ is defined as the instantaneous probability of default gives,

$$\begin{aligned} P(\tau < t) &= \int_0^t Q(u)\lambda\, du = \lambda \int_0^t Q(u)\, du = \lambda \int_0^t e^{-\lambda t}\, du = -\lambda\left[\frac{e^{-\lambda t}}{\lambda}\right]_0^t \\ &= -e^{-\lambda t} + e^0 \\ &= 1 - e^{-\lambda t} \\ &= 1 - Q(t) \end{aligned} \tag{21}$$

giving

$$\begin{aligned} P(\tau = t) &= \int_0^t Q(u)\lambda\, du \\ &= 1 - Q(t) \end{aligned} \tag{22}$$

In the case where we have a non-constant hazard rate $\lambda(t)$ the marginal default probability is given by,

$$\begin{aligned} P(\tau = t) &= \int_0^t Q(u)\lambda(u)\, du \\ &= 1 - Q(t) \end{aligned} \tag{23}$$

3 Credit Default Swap Pricing

The price of a Credit Default Swap can be evaluated as shown below. The CDS contract is made up of two trade legs, namely the premium leg and protection leg. The premium leg represents insurance payments which are exchanged for a protection leg contingent payment if a credit event is triggered. As illustrated in figure (1) premium payments are made on credit index survival and cease if there is a credit event, conversely the contingent payment is made on credit index default. Additionally one should note that the protection buyer has to pay pro-rated insurance premia up to the day of default, which leads to a premium accrued interest payment.

3.1 Premium Leg

The premium leg of a credit default swap represents the stream of insurance payments payable for credit protection. These payments typically cease on default and are evaluated as follows,

$$PV(\textit{Premium Leg}) = \sum_{i=1}^{n} Ns\Delta(t_{i-1}, t_i)Q(t_i)P(t_0, t_i) + \textit{ Accrued Interest on Default} \tag{24}$$

3.2 Premium Accrued Interest on Default

We assume any premium accrued interest payable on default is paid on a regular premium coupon date and that on average the accrued premium to be due is half that of a regular premium. This is very similar, but not identical, to assuming defaults occur mid-coupon. This assumption simplifies CDS calculations greatly. We further assume that hazard rates are piecewise constant having a left-continuous functional form during each premium period.

$$PV(\textit{Premium Accrued Interest}) = \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} Ns \left(\frac{\Delta(t_{i-1}, t_i)}{2} \right) \underbrace{Q(u)\lambda(u)}_{P(Default)} P(t_0, t_i)\, du \tag{25}$$

which we can equivalently evaluate as,

$$PV(\textit{Premium Accrued Interest}) = \sum_{i=1}^{n} Ns \left(\frac{\Delta(t_{i-1}, t_i)}{2} \right) \underbrace{[Q(t_{i-1}) - Q(t_i)]}_{P(Default)} P(t_0, t_i) \tag{26}$$

3.3 Protection Leg

The protection or contingent leg models the loss given default (LGD). This is the payout to be made to reimburse the protection buyer for losses in the event of a credit trigger being breached. We evaluate the probability of default as a piecewise constant left-continuous function of the hazard rate over each premium coupon period. The protection leg has a price or present value determined as follows,

$$\begin{aligned} PV(\textit{Protection Leg}) &= \int_0^t N\,(1-R)\,\underbrace{Q(u)\lambda(u)}_{P(Default)}\,P(t_0,u)\,du \\ &= \sum_{i=1}^{n} N\,(1-R)\,\underbrace{Q(t_i)\lambda(t_i)\Delta(t_{i-1},t_i)}_{P(Default)}\,P(t_0,t_i) \end{aligned} \tag{27}$$

where $P(Default)$ denotes marginal unconditional default probability at time u, that is the cumulative probability of survival $Q(u)$ from time 0 to u followed by an instantaneous default $\lambda(u)$ at time u with $0 \leq u \leq t$.

Equivalently we can price the protection leg as,

$$PV(\textit{Protection Leg}) = \sum_{i=1}^{n} N\,(1-R)\,\underbrace{\left[Q(t_{i-1}) - Q(t_i)\right]}_{P(Default)}\,P(t_0,t_i) \tag{28}$$

3.4 Credit Default Swap Present Value

The present value of a Credit Default swap is therefore the sum total of its premium leg, accrued premium on default and protection leg. For a protection seller we would calculate the present value of the CDS as,

$$\begin{aligned} PV(\textit{CDS Protection Seller}) &= PV(\textit{Premium Leg}) \\ &+ PV(\textit{Premium Accrued Interest}) \\ &- PV(\textit{Protection Leg}) \end{aligned} \tag{29}$$

Therefore using equations (24), (26) and (28) we have that the present value of a CDS for the protection seller is,

$$\begin{aligned} PV(\textit{CDS Protection Seller}) &= \sum_{i=1}^{n} Ns\Delta(t_{i-1},t_i)Q(t_i)P(t_0,t_i) \\ &+ \sum_{i=1}^{n} Ns\left(\frac{\Delta(t_{i-1},t_i)}{2}\right)\left[Q(t_{i-1}) - Q(t_i)\right]P(t_0,t_i) \\ &- \sum_{i=1}^{n} N\,(1-R)\left[Q(t_{i-1}) - Q(t_i)\right]P(t_0,t_i) \end{aligned} \tag{30}$$

Similarly for a protection buyer we have that,

$$\begin{aligned} PV(\textit{CDS Protection Buyer}) &= PV(\textit{Protection Leg}) \\ &- PV(\textit{Premium Leg}) \\ &- PV(\textit{Premium Accrued Interest}) \end{aligned} \tag{31}$$

which gives a present value from the perspective of the protection buyer as,

$$
\begin{aligned}
PV(\textit{CDS Protection Seller}) = &\sum_{i=1}^{n} N\,(1-R)\,[Q(t_{i-1}) - Q(t_i)]\,P(t_0,t_i) \\
&- \sum_{i=1}^{n} Ns\Delta(t_{i-1},t_i)Q(t_i)P(t_0,t_i) \\
&- \sum_{i=1}^{n} Ns\left(\frac{\Delta(t_{i-1},t_i)}{2}\right)[Q(t_{i-1}) - Q(t_i)]\,P(t_0,t_i)
\end{aligned}
\tag{32}
$$

4 Credit Models

Credit models generally fall into two categories, namely structural models and reduced form intensity models. Both are well documented see [2] and [8] for further details.

4.1 Structural Models

The approach taken in structural models is to characterize bond defaults as being the consequence of some company credit event and are usually based upon or extensions of Merton's (1974) firm-value model [9]. Structural models lead towards the idea that a bond default can be modelled as an equity put option on the issuer. They require information about the balance sheet of the firm to formalize relationships between equity and debt markets. Such models are difficult to calibrate since internal company data is often only published at most quarterly.

4.2 Reduced Form Intensity Models

In this paper we only consider the reduced form approach. Reduced form models evaluate the credit-event process directly and extract default probability information from bond and CDS market prices. This allows us to generate a full term-structure of credit-risk, subject to calibration instrument availability. The reduced form intensity model approach is based on [6], whereby the credit event is modelled as a Poisson process with a hazard rate $\lambda(t)$, which models the instantaneous probability of default occurring at time τ as shown below.

Conditional Default Probability
The conditional probability of default calculates the probability of default occurring in the time interval $(t, dt]$ conditional on survival to time t and is given by,

$$P(\tau < t + dt | \tau \geq t) = \lambda(t)dt \tag{33}$$

Marginal Default Probability
Typically when pricing CDS and modelling credit risk we work with marginal or unconditional default probabilities,

$$P(\tau < t) = \int_0^t Q(u)\lambda(t)\,du = 1 - Q(t) \tag{34}$$

where $Q(t)$ is the cumulative probability of survival until time t.

4.3 Credit Model Assumptions

The calibration output of a credit model is the hazard rate function required to price credit default swaps and other credit derivatives. In this paper we make several hazard rate assumptions, which are relatively standard amongst practitioners see [2], [3], [7], [8], namely,

- **Deterministic Hazard Rates**
  Hazard rates are deterministic and independent of interest rates and recovery rates.
- **Piecewise-Constant Hazard Rates**
  The term-structure of hazard rates is piecewise-constant[5] with a left-continuous functional form i.e. continuous when approaching the limit or calibration point from the left.
- **Extrapolation**
  Hazard rate extrapolation is piecewise-constant i.e. flat extrapolation.

4.4 Hazard Rate Calibration

These assumptions, the piecewise-constant term-structure of hazard rates in particular, allow us to simplify the integral within equation (34) as follows,

$$P(\tau < t + dt) = \sum_{i=1}^{n} Q(t_i)\lambda(t_i)\Delta(t_{i-1}, t_i) = 1 - Q(t) \tag{35}$$

Equation (35) numerically evaluates the integral from (34) by partitioning the integral into n steps. Furthermore this allows us to apply adaptive numerical integration techniques, whereby we evaluate the integral as a series of irregular rectangular strips with n equal to the number of calibration instruments as illustrated in figure (7).

[5] Linear and piecewise constant are market standard, since credit curves often consist of a single calibration point or potentially no more than 5 calibration points for liquid government bonds.

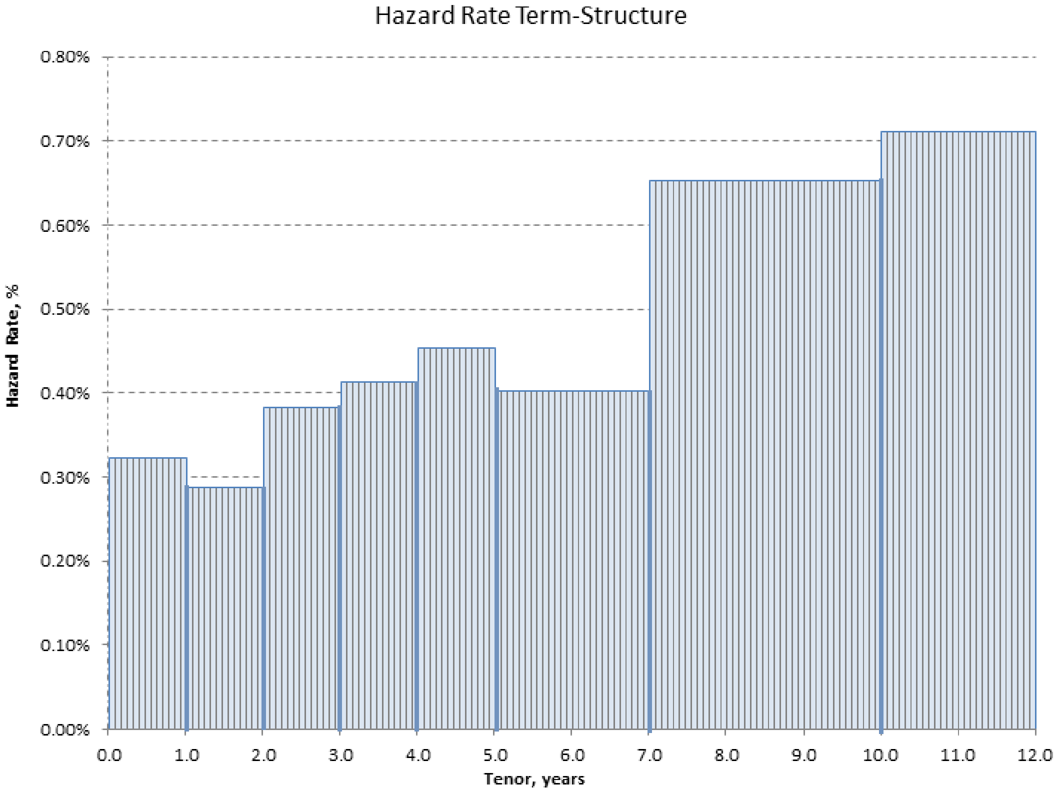

Figure 7: Hazard Rate Term-Structure for US Treasury Bonds

Forward Default Probability

Additionally forward default probabilities can be evaluated as follows,

$$\begin{aligned} P(s < \tau < t) &= \int_s^t Q(u)\lambda(t)\,du \\ &= \sum_{i=s'}^{t'} Q(t_i)\lambda\Delta(t_{i-1}, t_i) \\ &= Q(s) - Q(t) \end{aligned} \tag{36}$$

where $s \leq \tau \leq t$ and s' & t' correspond to the ith discrete bucket corresponding to time s and t respectively.

As shown in equation (36) the forward probability of default between s and t is calculated as the probability of survival to time s less the probability of survival to time t for $s < t$. We illustrate the forward probability of default for an arbitrary credit index between year 5 and 6 with $s = 5$ and $t = 6$ in figure (8).

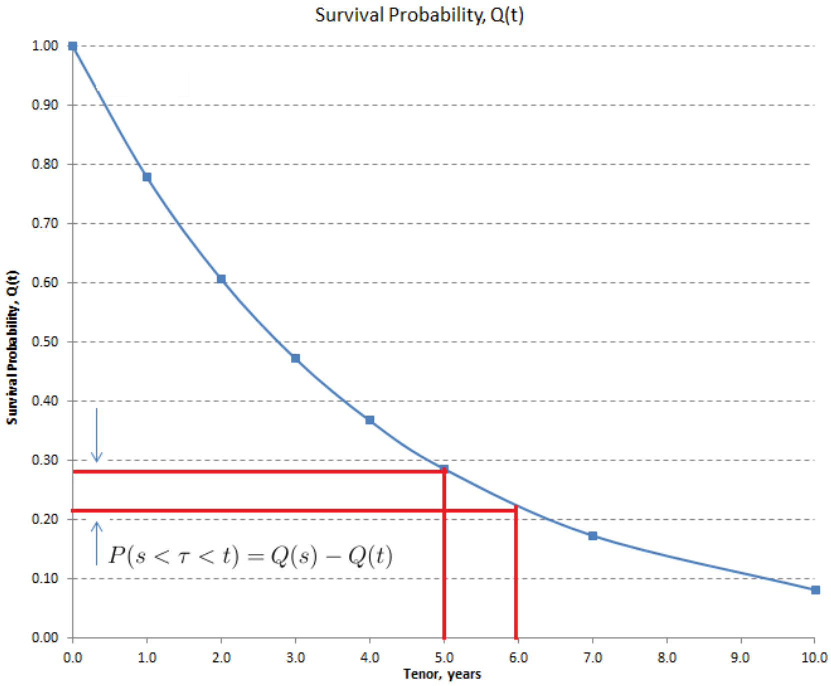

Figure 8: Forward Default Probability

Hazard Rate Approximation

We can approximate the hazard rate given the par CDS spread as follows. For a par CDS the present value of the premium and protection legs are equal. If we assume that there is no premium accrued interest to be paid on default we can approximate a constant or average hazard rate as,

$$\begin{aligned}
PV(\textit{Premium Leg}) &= PV(\textit{Protection Leg}) \\
\sum_{i=1}^{n} Ns\Delta(t_{i-1},t_i)Q(t_i)P(t_0,t_i) &= \sum_{i=1}^{n} N\left(1-R\right)\underbrace{Q(t_i)\lambda\Delta(t_{i-1},t_i)}_{P(Default)}P(t_0,t_i)\,du \\
s\sum_{i=1}^{n} N\Delta(t_{i-1},t_i)Q(t_i)P(t_0,t_i) &= \lambda\left(1-R\right)\sum_{i=1}^{n} N\Delta(t_{i-1},t_i)Q(t_i)P(t_0,t_i)\,du \\
s\bar{A}_N &= \lambda\left(1-R\right)\bar{A}_N \\
s &= \lambda\left(1-R\right)
\end{aligned} \tag{37}$$

this gives

$$\lambda = \frac{s}{(1-R)} \tag{38}$$

where s is the par CDS spread. This is a popular approximation used as an initial guess for the hazard rate when optimizing and calibrating credit models.

4.5 Standard Credit Market Models

In credit market the following standard models are used by many practitioners for calibrating credit risk and for pricing CDS see [1], [3], [10] for example.

- **ISDA Standard Model (I Model)**
  This is the market standard for valuing credit default swaps. It assumes a flat CDS curve. Therefore only one calibration instrument is needed to value a CDS. Typically the 5Y CDS point is used, since this is the most liquid instrument tenor. This model is owned and fully defined by the International Swaps and Derivatives Association (ISDA). For more information on this model see [3].

- **ISDA Fair Value Model (V Model)**
  The ISDA Fair Value Model uses the ISDA Standard Model to value credit default swaps, but allows a full term structure of par swap instruments to be used for calibration. Furthermore the model provides for customization of model input parameters such as coupon frequency, schedule and daycount parameters. For more information see [3].

- **Bloomberg Fair Value Model (B Model)**
  Many market participants use the Bloomberg terminal for pricing CDS and the associated CDSW pricing page. The Bloomberg Fair Value Model is very similar to the model described in this paper. It uses a full term structure of par CDS swaps for calibration, such as the ISDA Fair Value Model, but with an alternative yet similar numerical implementation as discussed in [1].

5 Credit Proxies, Sector and Index CDS

Despite market efforts to standardize CDS contracts to increase liquidity and facilitate cleared credit transactions; many credit indices have no actively traded CDS instruments and others may only have a single 5Y instrument quote. Therefore it is commonplace to substitute the credit index with a suitable proxy credit index that is as similar to and with a credit profile as highly correlated to the credit index of choice as possible. Selecting a suitable credit proxy is a fine art indeed, with practitioners often having no hard and fast rules for credit index proxy selection for credit risk hedging and CDS pricing. As alternative to credit proxy selection some traders turn to liquid sector and index CDS instruments and hedge credit risk by sector or by using Index CDS contracts. We refer the interested reader to [10] for more information.

6 Conclusion

In conclusion we examined Credit Default Swaps outlining their cashflow structure and contract specifications. We looked at how the product has evolved into a standardized exchange traded and cleared product. The ISDA standardized contract features, typical CDS specifications and market terminology were also discussed.

Furthermore reviewed probability fundamentals and explained the difference between conditional and unconditional probabilities. The later being used for credit default evaluation and often referred to as marginal probabilities. The binomial default model was reviewed from a fundamental perspective to help us understand the origins of survival probability modelling and better understand the survival probability functional form. Discrete case probabilities were discussed and we extended the concepts to the continuous case. This was a relevant exercise since in credit markets models are assumed to have piecewise constant default probabilities or hazard rates, which greatly simplify the continuous case models making them more in-line with discrete probability models. The resulting continuous case models are therefore a simple extension of the discrete case and are conveniently analytically tractable.

In the second half of this paper we reviewed standard CDS pricing formulae and bifurcated the Credit Default Swap into its principal components, namely the premium leg, accrued on default coupons and the protection leg. We provided an overview of credit modelling approaches namely structural and reduced form models, with a focus on the later. Reduced form or intensity models work directly with and evaluate marginal default probabilities or hazard rates. We outlined the standard model assumptions used and discussed hazard rate calibration. Several standard models used by traders and market participants were also discussed, namely the ISDA Standard Model, ISDA Fair Value Model and Bloomberg Fair Value Models.

Finally we concluded with a discussion on credit index proxies and liquidity issues. It is commonplace to have no active CDS contract for many credit reference entities. In such cases it is typical for traders and risk managers to select a credit index proxy or look at alternative approaches such as sector and index CDS hedging.

The main purpose for this paper was to provide the reader with a credit overview or primer with an emphasis on the more recent standardization features of CDS contracts and an assessment of impact of this on credit pricing and modelling.

Appendix

With this paper we provide an Excel example Credit Default Swap pricer, screenshots below, kindly email the author should you wish to receive a copy.

CDS Trade Parameters & Price

BuyOrSellProtection	BUY
BuySellIndicator, φ	1
Notional, N	10,000,000
HazardRate, λ	**0.4000%**
RecoveryRate, R	40.00%
CreditSpread, s	**0.2400%**
PremiumFrequency	QUARTERLY
PremiumYearFraction, Δt	0.2500
Maturity, years	5.0000
ZeroRate, z	1.86%

Present Value (PV)

CreditDefaultSwap	0
PremiumLeg	-113,148
AccruedInterest	-57
ProtectionLeg	113,205

Par CDS Rate

Par CDS Rate, p	0.2400%
ProtectionPV - AccruedInterest	-113,148
RiskyFixedAnnuity	-47,145,033

Figure 9: Excel CDS Price for US Treasury Protection

Time, T	Notional, N	CreditSpread, s	YearFraction, δt	Coupon	P(Survive), Q(T)	DiscFact, P(t,T)	PV
0.25	-10,000,000	0.2400%	0.2500	-6,000	0.999000	0.995361	-5,966
0.50	-10,000,000	0.2400%	0.2500	-6,000	0.998002	0.990743	-5,933
0.75	-10,000,000	0.2400%	0.2500	-6,000	0.997004	0.986147	-5,899
1.00	-10,000,000	0.2400%	0.2500	-6,000	0.996008	0.981572	-5,866
1.25	-10,000,000	0.2400%	0.2500	-6,000	0.995012	0.977018	-5,833
1.50	-10,000,000	0.2400%	0.2500	-6,000	0.994018	0.972486	-5,800
1.75	-10,000,000	0.2400%	0.2500	-6,000	0.993024	0.967974	-5,767
2.00	-10,000,000	0.2400%	0.2500	-6,000	0.992032	0.963483	-5,735
2.25	-10,000,000	0.2400%	0.2500	-6,000	0.991040	0.959014	-5,703
2.50	-10,000,000	0.2400%	0.2500	-6,000	0.990050	0.954565	-5,670
2.75	-10,000,000	0.2400%	0.2500	-6,000	0.989060	0.950136	-5,638
3.00	-10,000,000	0.2400%	0.2500	-6,000	0.988072	0.945728	-5,607
3.25	-10,000,000	0.2400%	0.2500	-6,000	0.987084	0.941341	-5,575
3.50	-10,000,000	0.2400%	0.2500	-6,000	0.986098	0.936974	-5,544
3.75	-10,000,000	0.2400%	0.2500	-6,000	0.985112	0.932627	-5,512
4.00	-10,000,000	0.2400%	0.2500	-6,000	0.984127	0.928300	-5,481
4.25	-10,000,000	0.2400%	0.2500	-6,000	0.983144	0.923994	-5,451
4.50	-10,000,000	0.2400%	0.2500	-6,000	0.982161	0.919707	-5,420
4.75	-10,000,000	0.2400%	0.2500	-6,000	0.981179	0.915440	-5,389
5.00	-10,000,000	0.2400%	0.2500	-6,000	0.980199	0.911194	-5,359

Figure 10: Premium Leg Cashflows

Time, T	Notional, N	CreditSpread, s	YearFraction, δt	Accrued Interest	P(Default), Q(t)λ(t)	DiscFact, P(t,T)	PV
0.2500	-10,000,000	0.2400%	0.1250	-3,000	0.001000	0.995361	-3
0.5000	-10,000,000	0.2400%	0.1250	-3,000	0.000999	0.990743	-3
0.7500	-10,000,000	0.2400%	0.1250	-3,000	0.000998	0.986147	-3
1.0000	-10,000,000	0.2400%	0.1250	-3,000	0.000997	0.981572	-3
1.2500	-10,000,000	0.2400%	0.1250	-3,000	0.000996	0.977018	-3
1.5000	-10,000,000	0.2400%	0.1250	-3,000	0.000995	0.972486	-3
1.7500	-10,000,000	0.2400%	0.1250	-3,000	0.000994	0.967974	-3
2.0000	-10,000,000	0.2400%	0.1250	-3,000	0.000993	0.963483	-3
2.2500	-10,000,000	0.2400%	0.1250	-3,000	0.000992	0.959014	-3
2.5000	-10,000,000	0.2400%	0.1250	-3,000	0.000991	0.954565	-3
2.7500	-10,000,000	0.2400%	0.1250	-3,000	0.000990	0.950136	-3
3.0000	-10,000,000	0.2400%	0.1250	-3,000	0.000989	0.945728	-3
3.2500	-10,000,000	0.2400%	0.1250	-3,000	0.000988	0.941341	-3
3.5000	-10,000,000	0.2400%	0.1250	-3,000	0.000987	0.936974	-3
3.7500	-10,000,000	0.2400%	0.1250	-3,000	0.000986	0.932627	-3
4.0000	-10,000,000	0.2400%	0.1250	-3,000	0.000985	0.928300	-3
4.2500	-10,000,000	0.2400%	0.1250	-3,000	0.000984	0.923994	-3
4.5000	-10,000,000	0.2400%	0.1250	-3,000	0.000983	0.919707	-3
4.7500	-10,000,000	0.2400%	0.1250	-3,000	0.000982	0.915440	-3
5.0000	-10,000,000	0.2400%	0.1250	-3,000	0.000981	0.911194	-3

Figure 11: Accrued Interest on Default

Time, T	Notional, N	LossGivenDefault, LGD	P(Default), Q(t)λ(t)	DiscFact, P(t,T)	PV
0.2500	10,000,000	6,000,000	0.001000	0.995361	5,969
0.5000	10,000,000	6,000,000	0.000999	0.990743	5,936
0.7500	10,000,000	6,000,000	0.000998	0.986147	5,902
1.0000	10,000,000	6,000,000	0.000997	0.981572	5,869
1.2500	10,000,000	6,000,000	0.000996	0.977018	5,836
1.5000	10,000,000	6,000,000	0.000995	0.972486	5,803
1.7500	10,000,000	6,000,000	0.000994	0.967974	5,770
2.0000	10,000,000	6,000,000	0.000993	0.963483	5,738
2.2500	10,000,000	6,000,000	0.000992	0.959014	5,705
2.5000	10,000,000	6,000,000	0.000991	0.954565	5,673
2.7500	10,000,000	6,000,000	0.000990	0.950136	5,641
3.0000	10,000,000	6,000,000	0.000989	0.945728	5,609
3.2500	10,000,000	6,000,000	0.000988	0.941341	5,578
3.5000	10,000,000	6,000,000	0.000987	0.936974	5,546
3.7500	10,000,000	6,000,000	0.000986	0.932627	5,515
4.0000	10,000,000	6,000,000	0.000985	0.928300	5,484
4.2500	10,000,000	6,000,000	0.000984	0.923994	5,453
4.5000	10,000,000	6,000,000	0.000983	0.919707	5,423
4.7500	10,000,000	6,000,000	0.000982	0.915440	5,392
5.0000	10,000,000	6,000,000	0.000981	0.911194	5,362

Figure 12: Protection Leg Modelled Cashflows

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