Centre for Robotics

EGB339 Part 2: Robotic Arms

Lecture 3: Inverse Kinematics

Chris Lehnert (Lecturer)

Outline

- Topics covered in this series of lectures
  - Rigid Body Motions (week 8)
  - Forward Kinematics (week 9)
  - Inverse Kinematics (week 10)
  - Velocity Kinematics (week 11)
  - Path and Trajectory Planning (week 12)
  - Revision (week 13)

- Topics not covered in this series of lectures
  - Dynamics
  - Control
  - Hardware
  - (Artificial) Intelligence
  - …

Watch these online videos

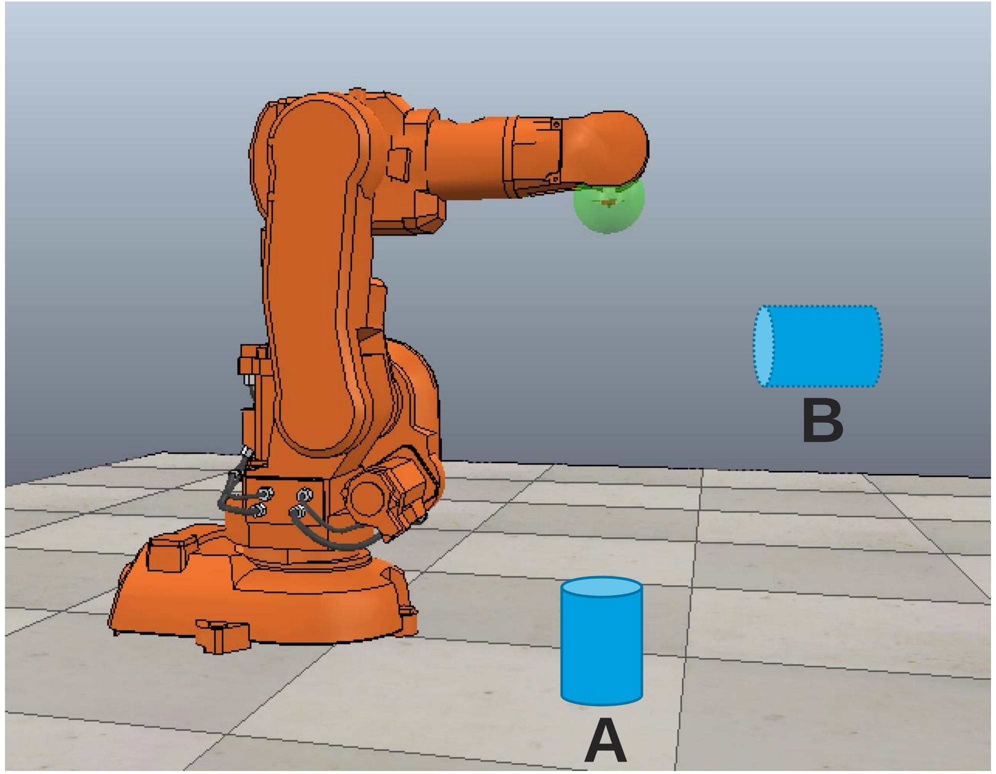
- QUT Robot Academy (by Prof Peter Corke)
  - Inverse Kinematics and Robot Motion
    - https://robotacademy.net.au/masterclass/inverse-kinematics-and-robot-motion/

Review of Week 9

- A robotic arm can be modelled by a kinematic chain.
- The kinematics can be parameterised by the Denavit-Hartenberg (D-H) convention.
- Four D-H parameters are used to parameterise a homogeneous transformation between two consecutive frames.
- Forward kinematics is able to calculate the pose of the tool with respect to the base given a set of joint variables.

Motivating Problem

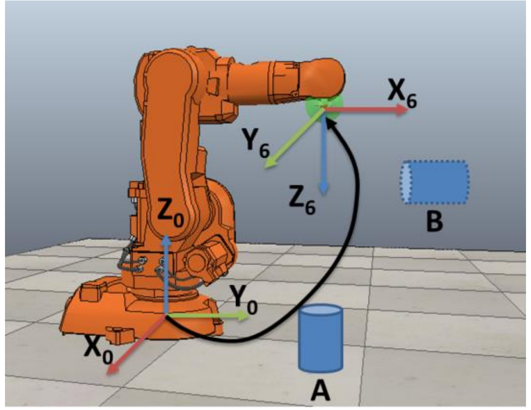
- Imagine one of your arms is replaced by a robotic arm. You are supposed to move an object from A to B.
- Now you know where the object is in front of you (homogeneous transformation).
- You also know where your “hand” is with respect to your “body” (forward kinematics).
- How can you move your “hand” to reach the object?



Forward Kinematics and Inverse Kinematics

Forward kinematics

$$f_{\theta,d,\alpha,a} : q \rightarrow p$$



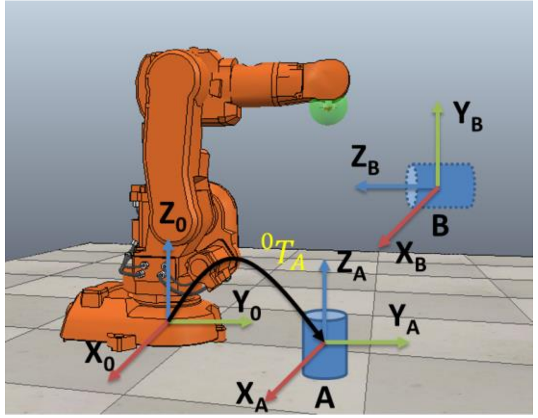
${}^0T_6 = fkine(q_1, q_2, q_3, q_4, q_5, q_6)$

Joint Space

$$q = [q_1, q_2, \ldots q_n]$$

Task Space

$$p = [R, t]$$



$q_A = ikine({}^0T_A)$

Inverse kinematics

$$f^{-1}_{\theta,d,\alpha,a} : p \rightarrow q$$

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Methods for Inverse Kinematics

- Geometric methods
  - For simple cases (such as robots with no more than 3DoFs)
  - Closed-form solutions
  - Most intuitive

- Algebraic methods
  - For simple cases and also some complicated cases
    - 6 DOF robot with 3 consecutive joints intersecting at one point
    - 6 DOF robot with 3 consecutive parallel joints
  - Closed-form solutions

- Numeric methods
  - For general cases
  - No closed-form solutions
  - Requires a good initial guess
  - No guarantee to get a correct solution (convergence problem)

Task Space

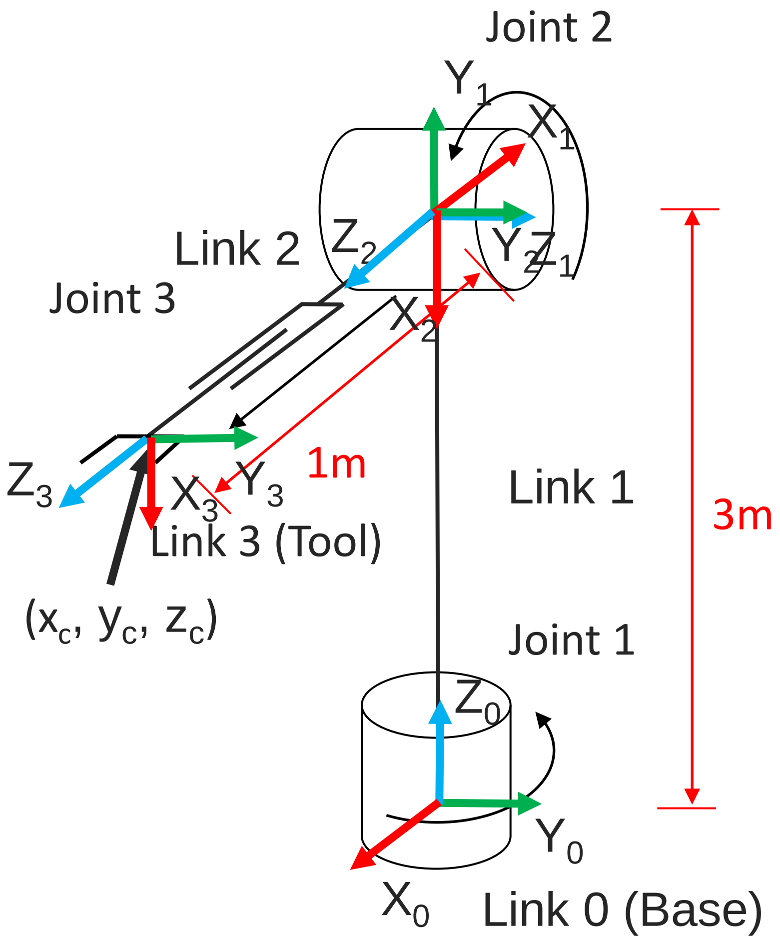
- Recall that the homogeneous transformation has the form

$$T = \begin{bmatrix} & R & & t \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- The task space consists of both a position subspace and an orientation subspace.
- For starters, we will only investigate the position inverse kinematics problem.

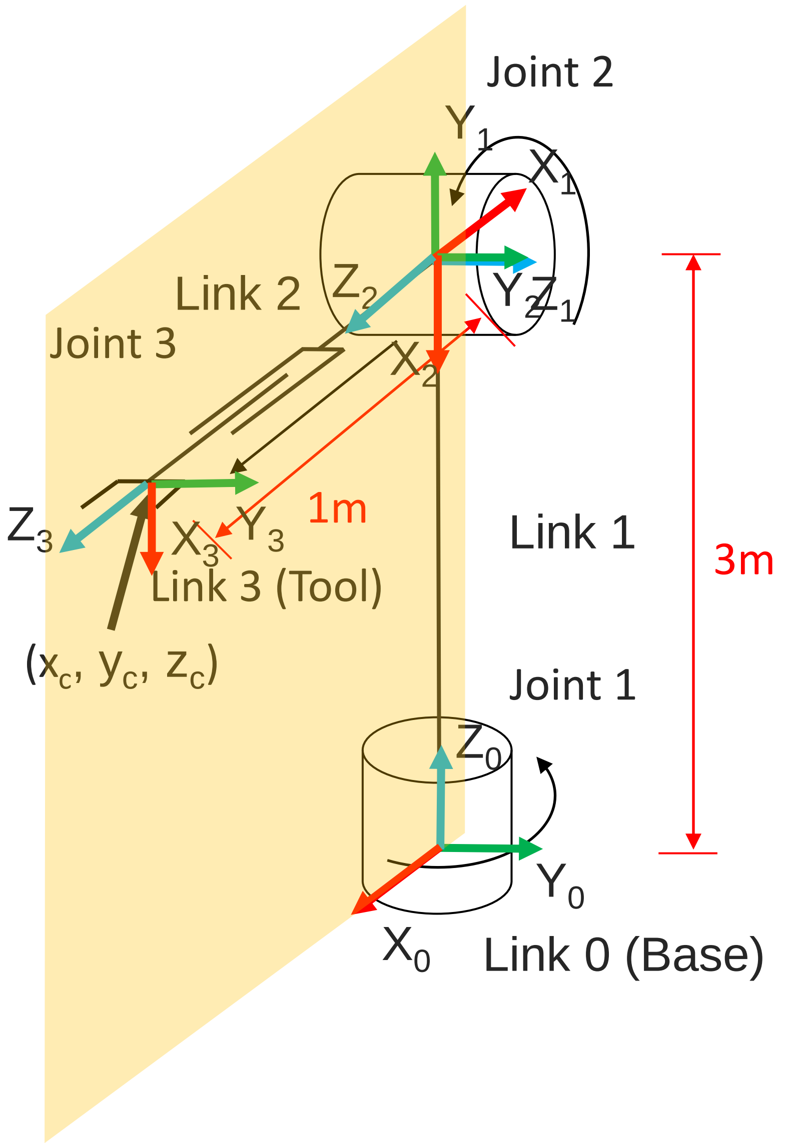
Geometric Methods - Example

- Given the 3D position of the tool (x_c, y_c, z_c), find the joint variables (q_1, q_2, q_3).



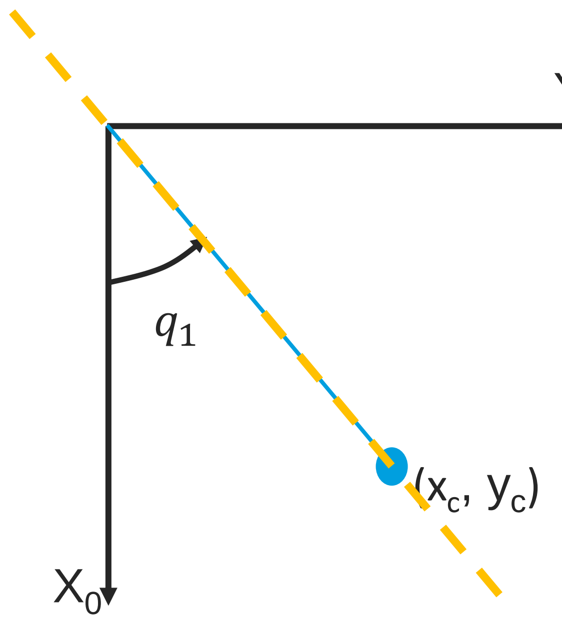
Geometric Methods - Example

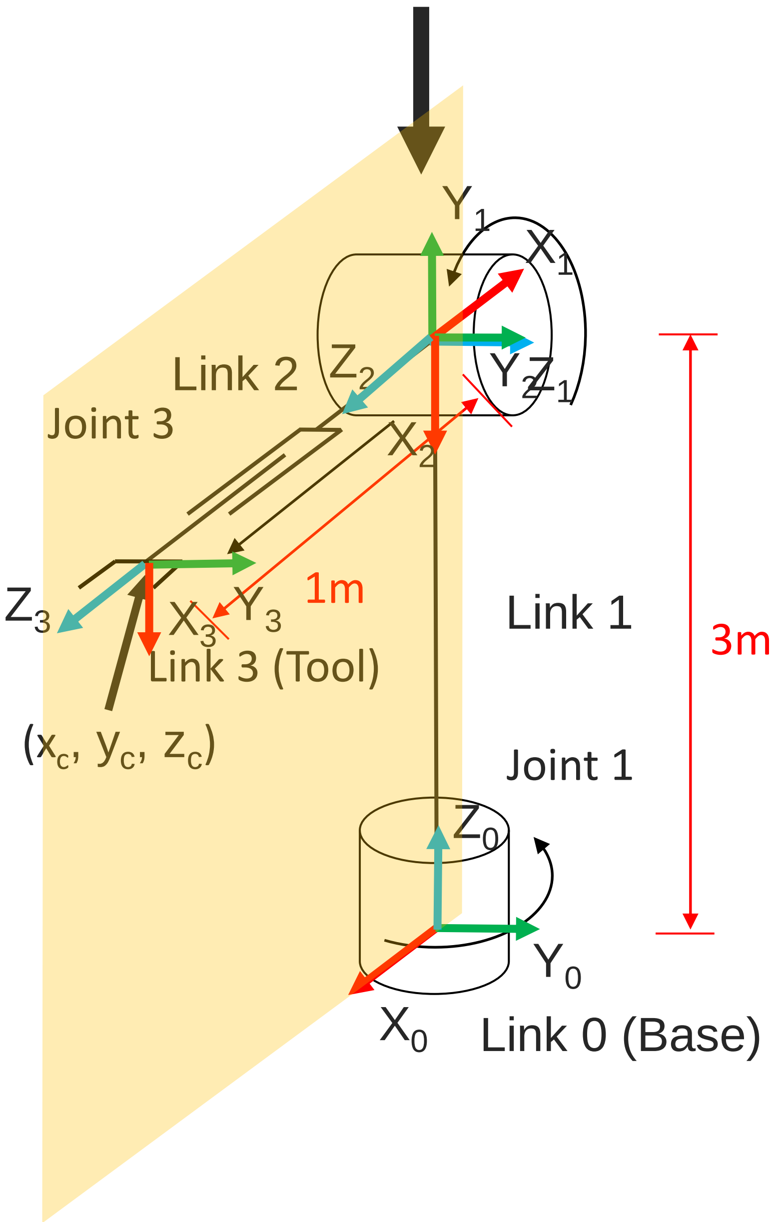
- By inspection, the position of the tool can be decoupled into the position within the yellow plane and the angle of the yellow plane with respect to the X_0Z_0 plane.
- q_1 is determined solely by the angle of the yellow plane.
- q_2 and q_3 are determined only by the position within the yellow plane.



1. Solve q_1

- Draw projection diagram looking from the top





Inverse Tangent Function

https://robotacademy.net.au/masterclass/inverse-kinematics-and-robot-motion/?lesson=293

The atan2(x,y) function

- *atan(x)* returns a value between -π/2 and π/2.
- *atan2(x, y)* returns a value between -π and π by considering the signs of x and y
- MATLAB uses the format **atan2(y, x)**
- **We will use the format atan2(x, y) or atan2(adjacent, opposite)**
- Also express as atan2(adj, opp)
- **Be <u>VERY CAREFUL WITH THE ORDER OF THE ARGUMENTS TO THE ATAN2 FUNCTION</u>,** they are not always consistent between different programming languages and calculators e.g. MATLAB

Coding atan2 using just the atan function

```
float atan2(float x, float y) {
        if (x > 0.0)
            return atan(y/x);
        if (x < 0.0) {
            if (y >= 0.0)
              return (PI + atan(y/x));
            else
              return (-PI + atan(y/x));
            }
        if (y > 0.0) // x == 0
            return PI_ON_TWO;
        if (y < 0.0)
            return -PI_ON_TWO;
        return 0.0; // Should be undefined
}
```

The atan2(x,y) function - example

- atan2(1,1)
- atan2(1,-1)
- atan2(0,1.5)
- atan2(-1.8,-1.8)
- atan2(-2,0)

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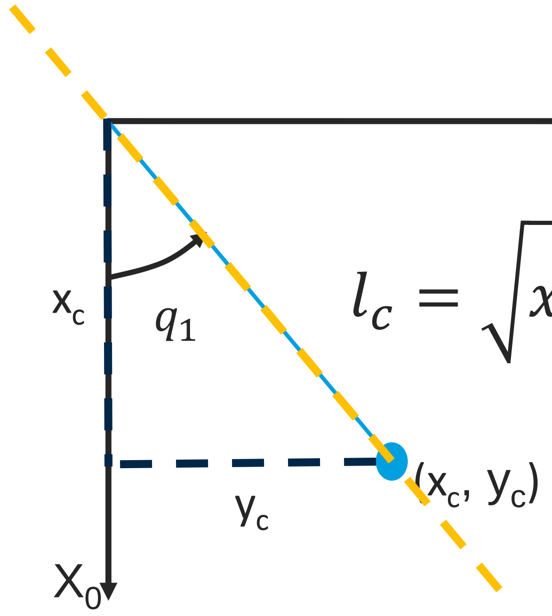
1. Solve q_1

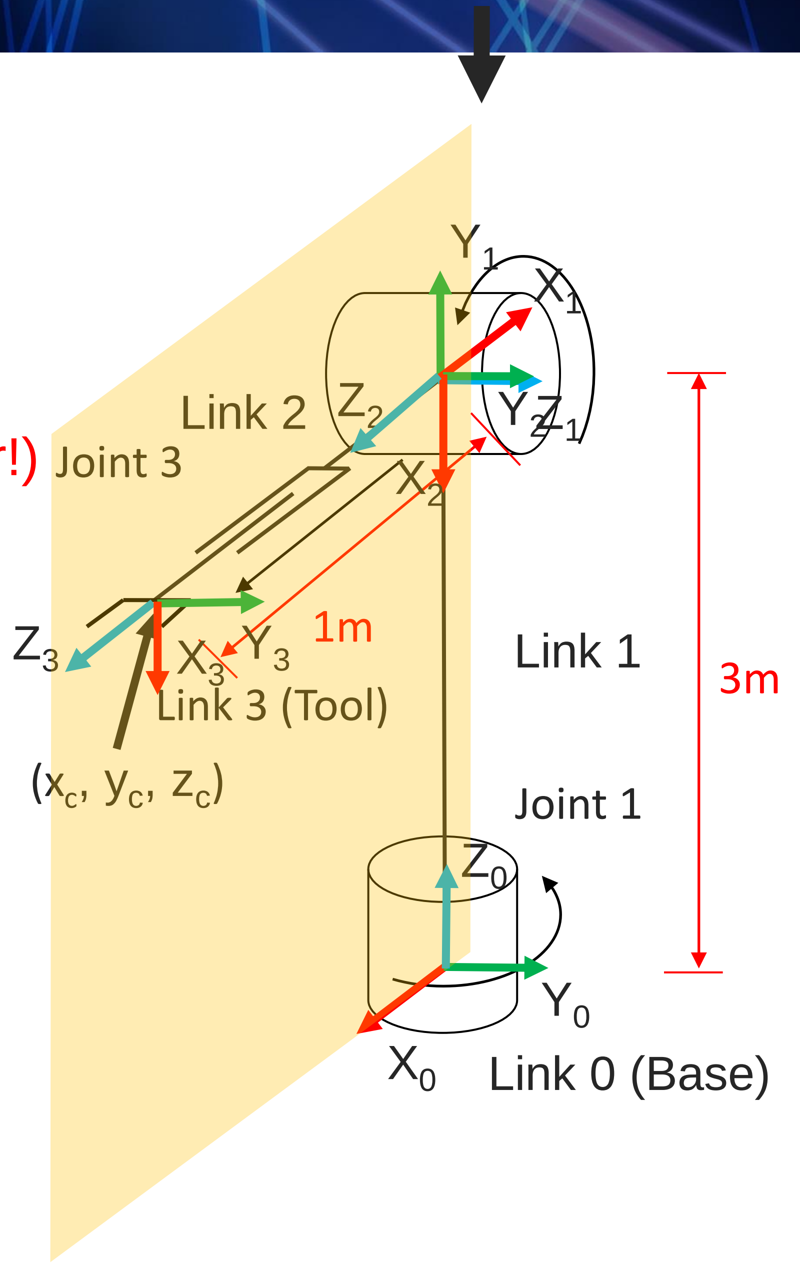
$$q_1 = atan2(x_c, y_c)$$

or $q_1 = atan2(x_c, y_c) + 180°$

(Because joint 2 could be flipped over!)

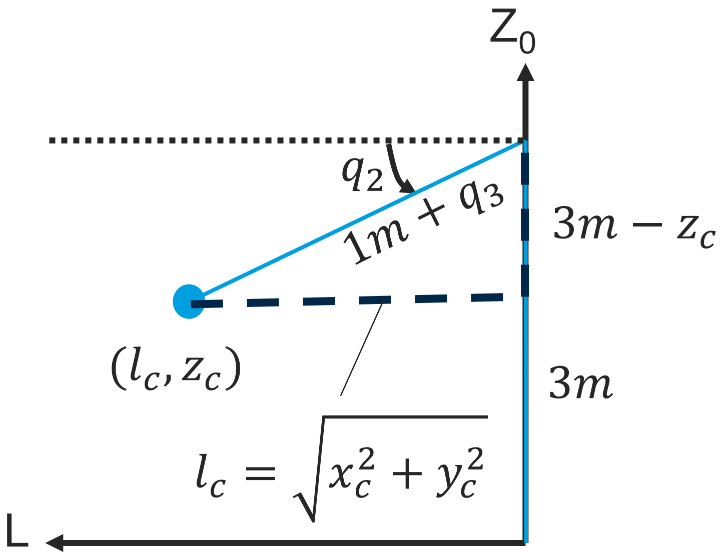


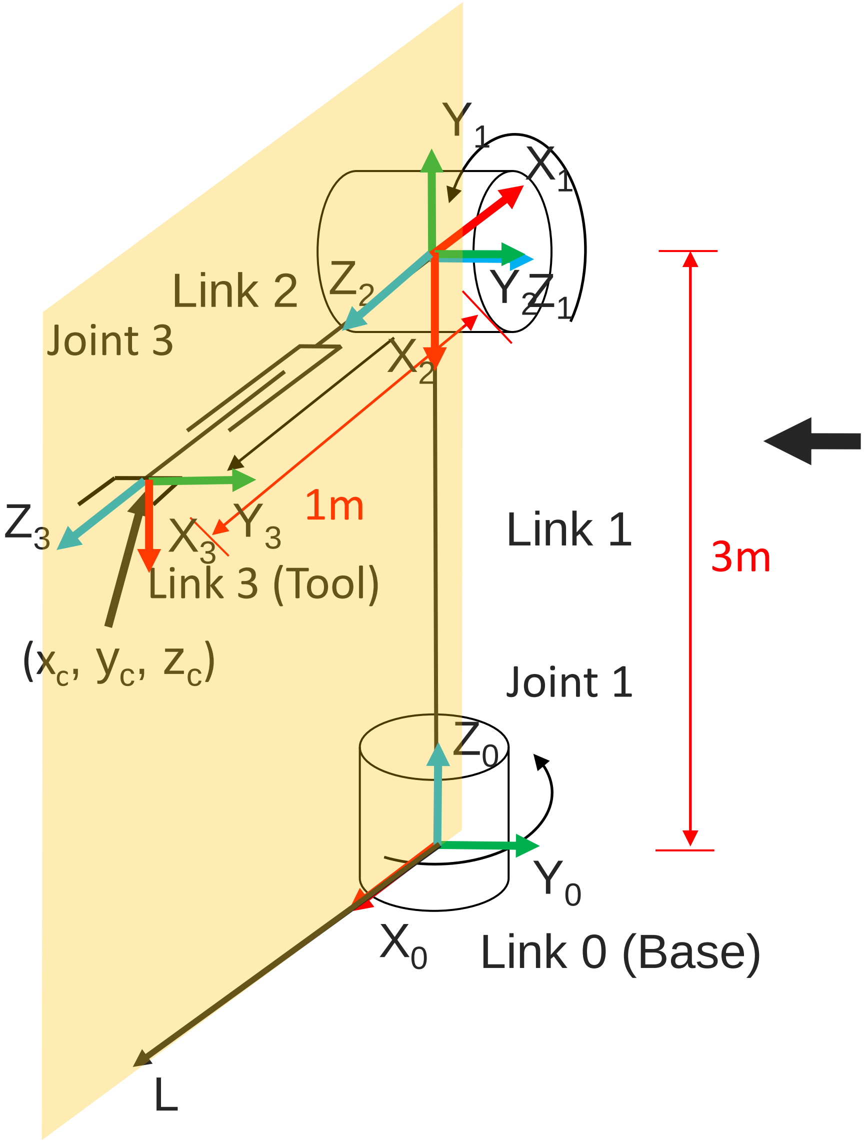
$$l_c = \sqrt{x_c^2 + y_c^2}$$



2. Solve q_2

- Draw projection diagram within the yellow plane looking from right normal



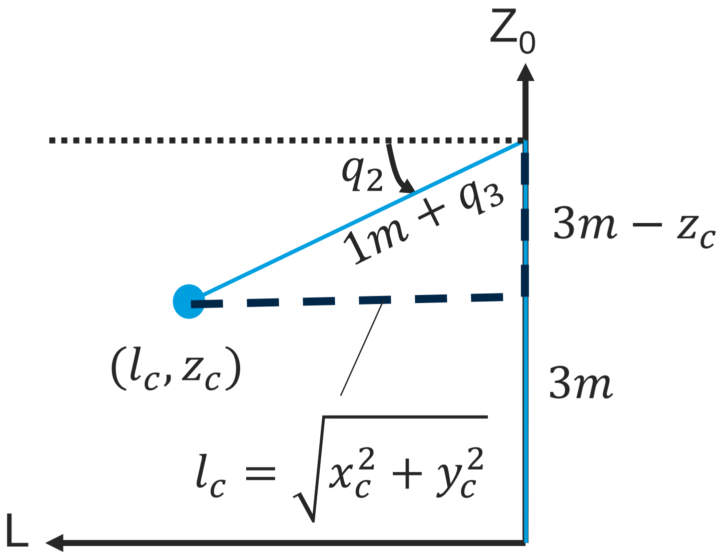


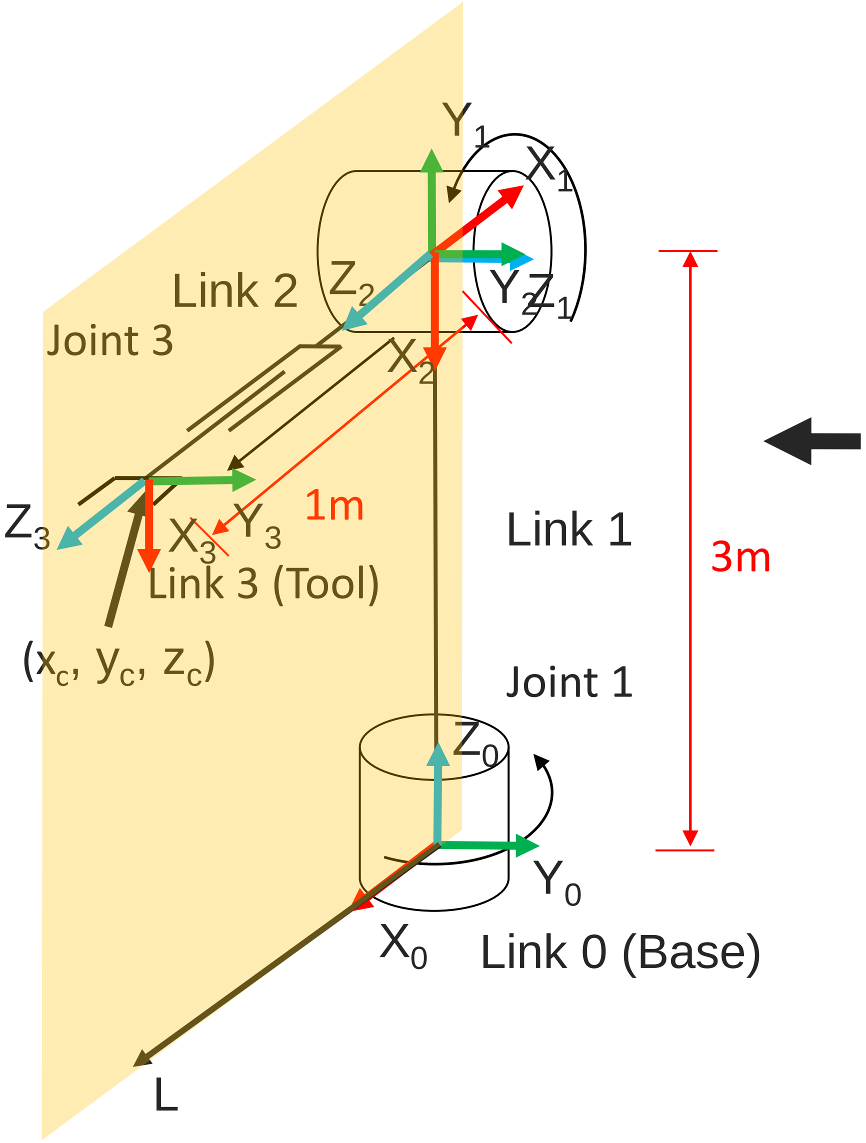
2. Solve q_2

$$q_2 = atan2(l_c, 3m - z_c)$$

or $$q_2 = -180° - atan2(l_c, 3m - z_c)$$

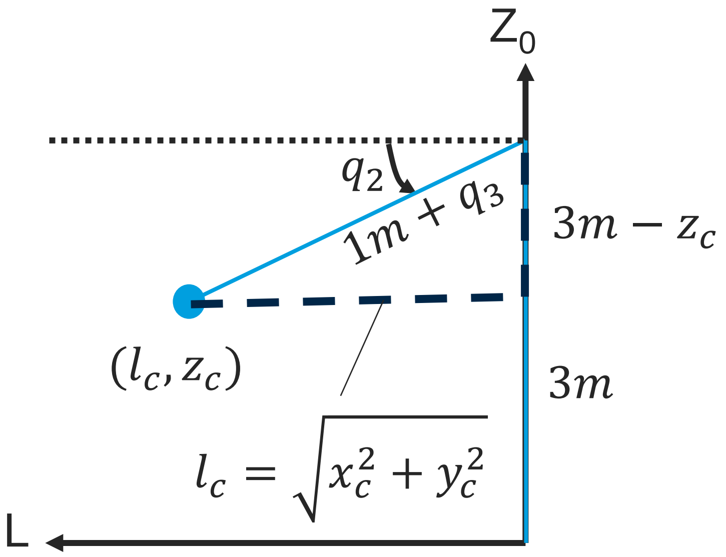
(Because joint 2 could be flipped over!)

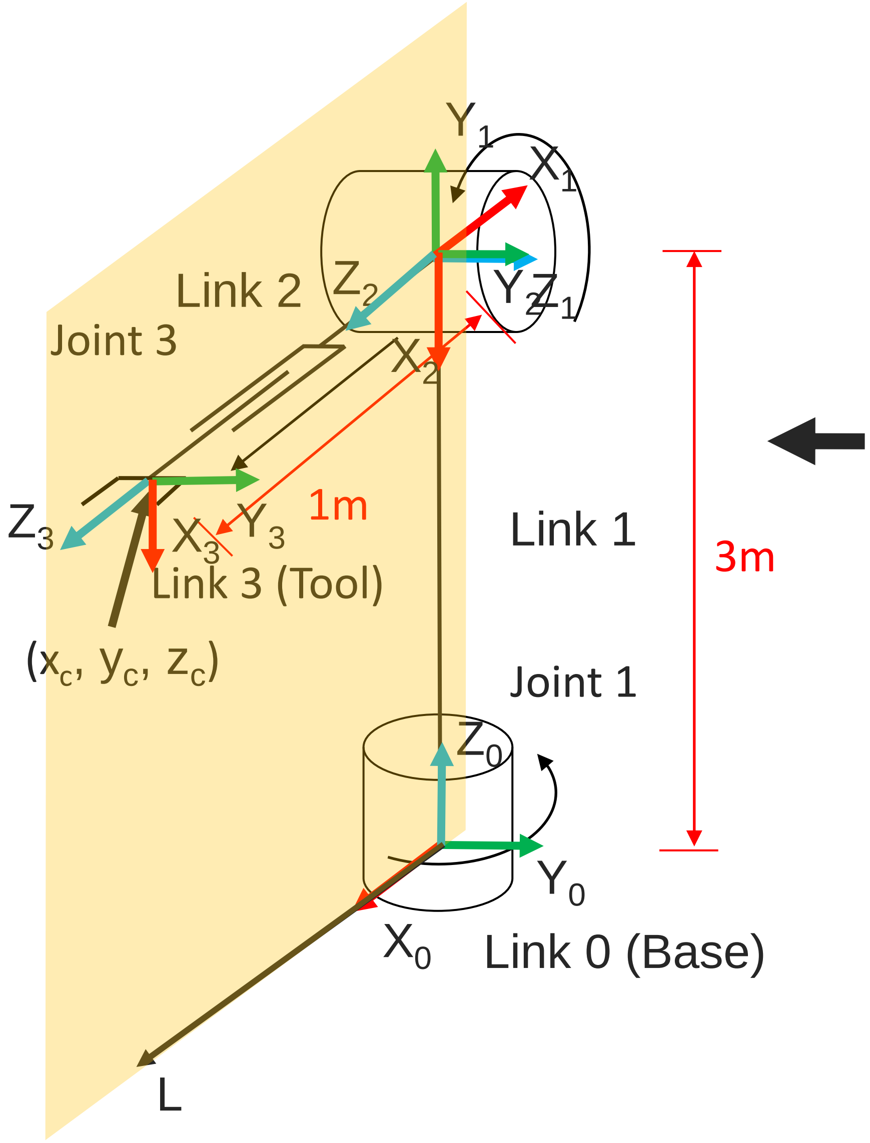




3. Solve q_3

$$q_3 = \sqrt{l_c^2 + (3m - z_c)^2} - 1m$$
$$= \sqrt{x_c^2 + y_c^2 + (3m - z_c)^2} - 1m$$





4. Summary

Solution 1:

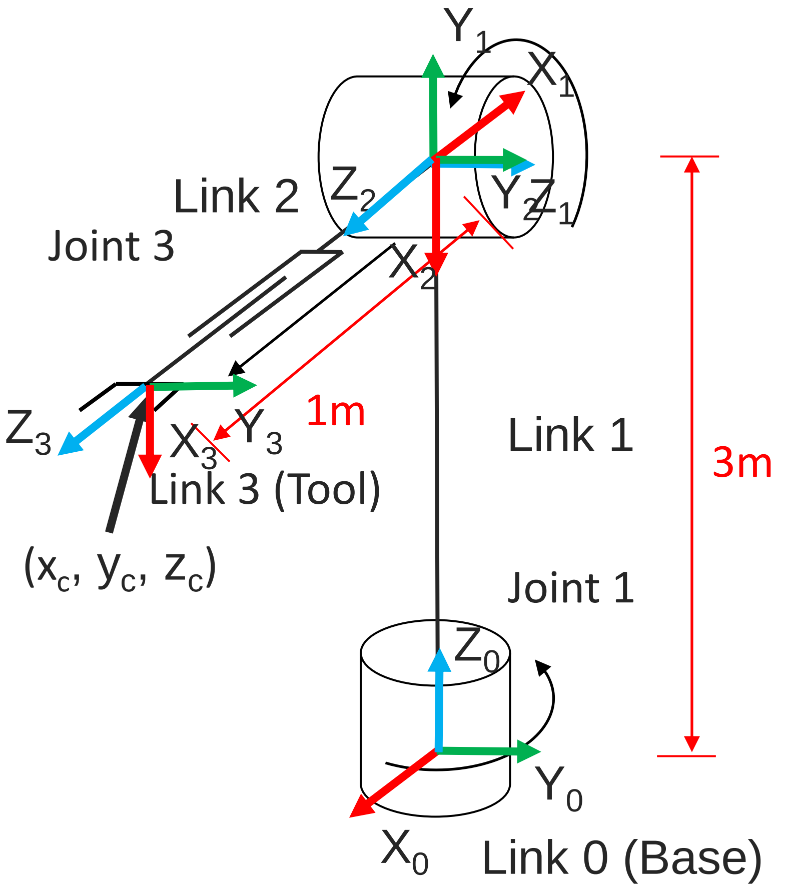
$$q_1 = atan2(x_c, y_c)$$
$$q_2 = atan2(\sqrt{x_c^2 + y_c^2}, 3m - z_c)$$
$$q_3 = \sqrt{x_c^2 + y_c^2 + (3m - z_c)^2} - 1m$$

Solution 2:

$$q_1 = atan2(x_c, y_c) + 180°$$
$$q_2 = -180° - atan2\left(\sqrt{x_c^2 + y_c^2}, 3m - z_c\right)$$
$$q_3 = \sqrt{x_c^2 + y_c^2 + (3m - z_c)^2} - 1m$$

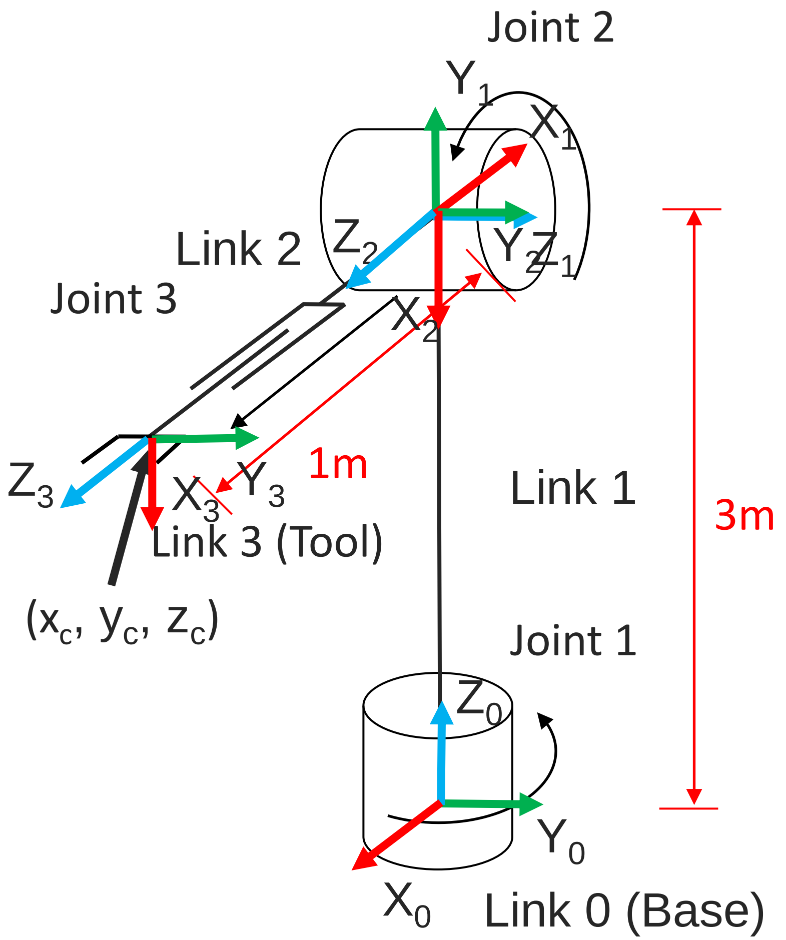
(Because joint 2 could be flipped over!)

$$\begin{cases} x_c = 1m \\ y_c = 1m \\ z_c = 3m \end{cases} \Rightarrow \begin{cases} q_1 = 45° \\ q_2 = 0° \\ q_3 = 0.414m \end{cases} \text{ or } \begin{cases} q_1 = 225° \\ q_2 = -180° \\ q_3 = 0.414m \end{cases}$$



Algebraic Methods - Example

- Given the 3D position of the tool (x_c, y_c, z_c), find the joint variables (q_1, q_2, q_3).

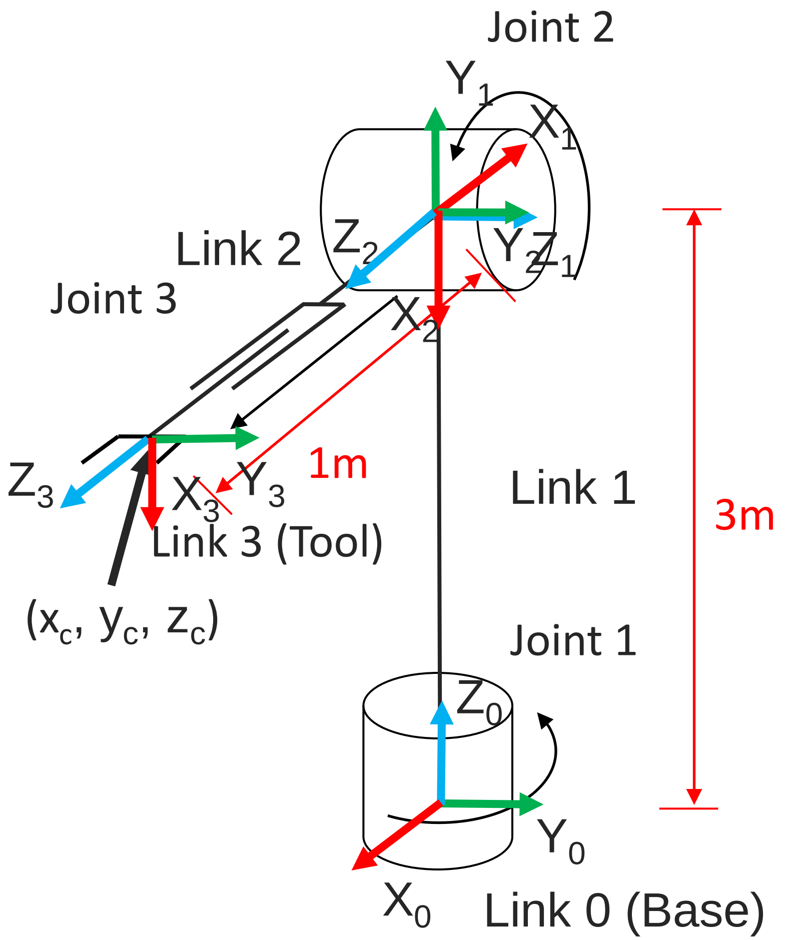


Algebraic Methods - Example

i	θ_i	d_i	α_i	a_i
1	$180^\circ + q_1$	3m	90°	0
2	$-90^\circ + q_2$	0	90°	0
3	0°	$1\text{m} + q_3$	0°	0

$$
{}^{0}T_3 = \begin{bmatrix} c_1c_2 & s_1 & c_1s_2 & d_3c_1s_2 \\ s_1c_2 & -c_1 & s_1s_2 & d_3s_1s_2 \\ s_2 & 0 & -c_2 & 3m - d_3c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\begin{cases} d_3c_1s_2 = x_c \\ d_3s_1s_2 = y_c \\ 3m - d_3c_2 = z_c \end{cases} \Rightarrow \begin{cases} d_3c_1s_2 = x_c & (1) \\ d_3s_1s_2 = y_c & (2) \\ d_3c_2 = 3m - z_c & (3) \end{cases}
$$



Note shorthand: $C_i = \cos(\theta_i)$, $S_i = \sin(\theta_i)$, $\theta_i \neq q_i$

1. Solve q_1

$$\begin{cases} d_3 c_1 s_2 = x_c & \text{①} \\ d_3 s_1 s_2 = y_c & \text{②} \\ d_3 c_2 = 3m - z_c & \text{③} \end{cases}$$

Assume $d_3 s_2 \neq 0$ (otherwise singular, we will discuss this later)

Change ① and ② to: $\begin{cases} c_1 = \dfrac{x_c}{d_3 s_2} \\ s_1 = \dfrac{y_c}{d_3 s_2} \end{cases}$

$$\theta_1 = atan2\left(\frac{x_c}{d_3 s_2}, \frac{y_c}{d_3 s_2}\right)$$

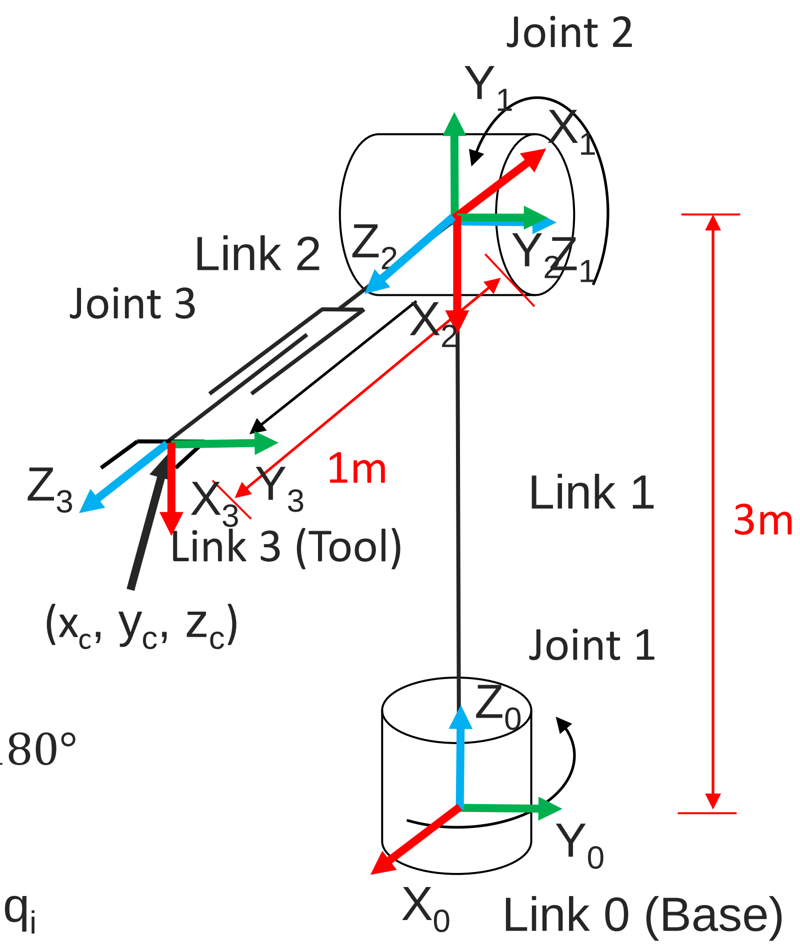
If $d_3 s_2 < 0$, i.e., $-180° < \theta_2 < 0°$, i.e., $-90° < q_2 < 90°$

$$\theta_1 = atan2(-x_c, -y_c) = atan2(x_c, y_c) + 180°$$
$$q_1 = \theta_1 - 180° = atan2(x_c, y_c)$$

If $d_3 s_2 > 0$, i.e., $0° < \theta_2 < 180°$, i.e., $90° < q_2 < 270°$

$$\theta_1 = atan2(x_c, y_c) \qquad q_1 = \theta_1 - 180° = atan2(x_c, y_c) - 180°$$

i	θ_i	d_i	α_i	a_i
1	180°+q_1	3m	90°	0
2	-90°+q_2	0	90°	0
3	0°	1m+q_3	0°	0



Note shorthand: $C_i = \cos(\theta_i)$, $S_i = \sin(\theta_i)$, $\theta_i \neq q_i$

2. Solve q_2

i	θ_i	d_i	α_i	a_i
1	$180^\circ + q_1$	3m	90°	0
2	$-90^\circ + q_2$	0	90°	0
3	0°	$1\text{m} + q_3$	0°	0

$$\begin{cases} d_3 c_1 s_2 = x_c & (1) \\ d_3 s_1 s_2 = y_c & (2) \\ d_3 c_2 = 3m - z_c & (3) \end{cases}$$

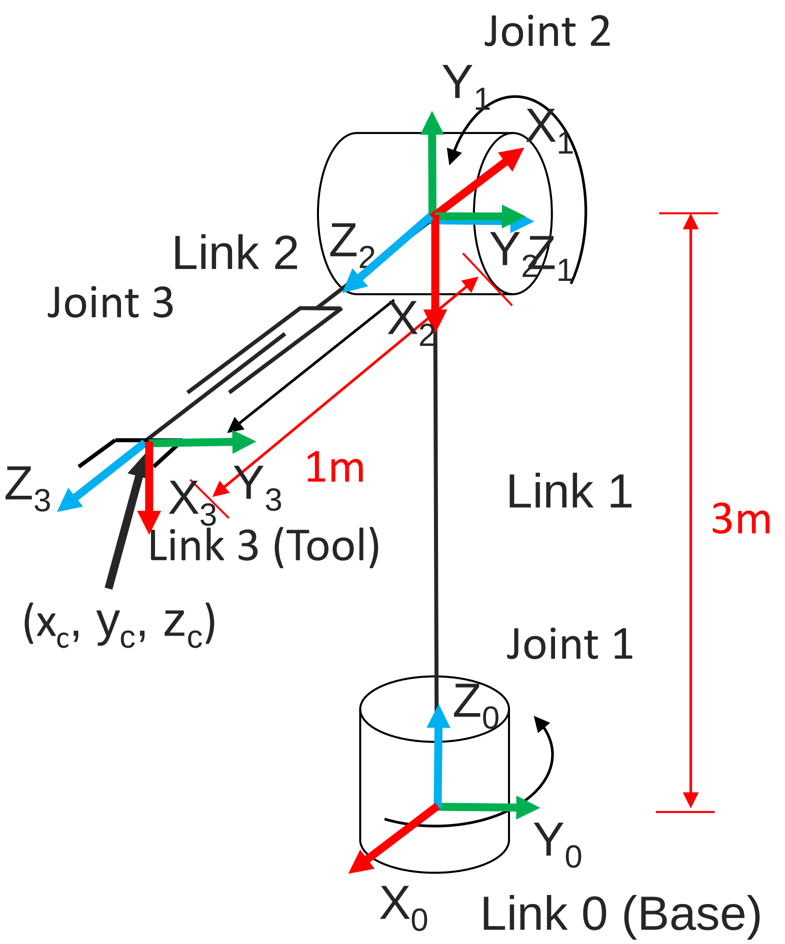
$$\sqrt{(1)^2 + (2)^2}: \quad \begin{cases} d_3 s_2 = \pm\sqrt{x_c^2 + y_c^2} \\ d_3 c_2 = 3m - z_c \end{cases}$$

If $d_3 s_2 < 0$, i.e., $-180^\circ < \theta_2 < 0^\circ$,
i.e., $-90^\circ < q_2 < 90^\circ$

$$\theta_2 = atan2(3m - z_c, -\sqrt{x_c^2 + y_c^2})$$
$$q_2 = \theta_2 + 90^\circ$$
$$= atan2\left(3m - z_c, -\sqrt{x_c^2 + y_c^2}\right) + 90^\circ$$

If $d_3 s_2 > 0$, i.e., $0^\circ < \theta_2 < 180^\circ$,
i.e., $90^\circ < q_2 < 270^\circ$

$$\theta_2 = atan2(3m - z_c, \sqrt{x_c^2 + y_c^2}),$$
$$q_2 = \theta_2 + 90^\circ = atan2\left(3m - z_c, \sqrt{x_c^2 + y_c^2}\right) + 90^\circ$$



Note shorthand: $C_i = \cos(\theta_i)$, $S_i = \sin(\theta_i)$, $\theta_i \neq q_i$

3. Solve q_3

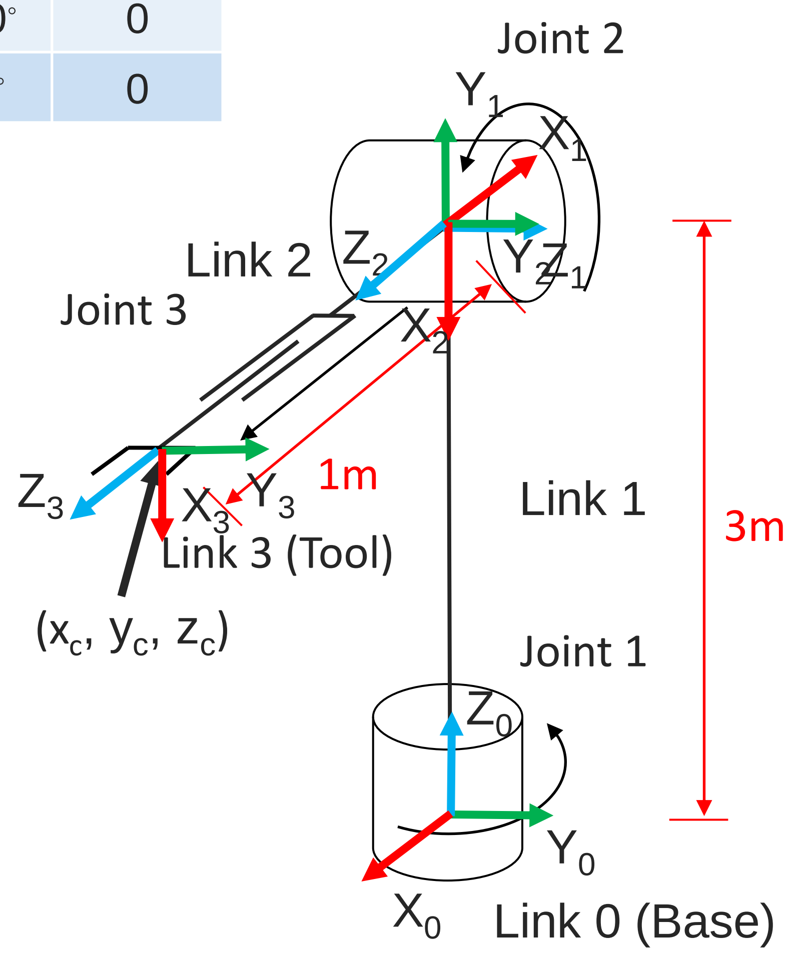
i	θ_i	d_i	α_i	a_i
1	$180^\circ + q_1$	3m	90°	0
2	$-90^\circ + q_2$	0	90°	0
3	0°	$1\text{m} + q_3$	0°	0

$$\begin{cases} d_3 c_1 s_2 = x_c & (1) \\ d_3 s_1 s_2 = y_c & (2) \\ d_3 c_2 = 3m - z_c & (3) \end{cases}$$

$$\sqrt{(1)^2 + (2)^2 + (3)^2}:$$

$$d_3 = \sqrt{x_c^2 + y_c^2 + (3m - z_c)^2}$$

$$q_3 = d_3 - 1m$$
$$= \sqrt{x_c^2 + y_c^2 + (3m - z_c)^2} - 1m$$



4. Summary

Solution 1: If $-90° < q_2 < 90°$

$$q_1 = atan2(x_c, y_c)$$
$$q_2 = atan2\left(3m - z_c, -\sqrt{x_c^2 + y_c^2}\right) + 90°$$
$$q_3 = \sqrt{x_c^2 + y_c^2 + (3m - z_c)^2} - 1m$$

Solution 2: If $90° < q_2 < 270°$

$$q_1 = atan2(x_c, y_c) - 180°$$
$$q_2 = atan2\left(3m - z_c, \sqrt{x_c^2 + y_c^2}\right) + 90°$$
$$q_3 = \sqrt{x_c^2 + y_c^2 + (3m - z_c)^2} - 1m$$

i	θ_i	d_i	α_i	a_i
1	180°+q_1	3m	90°	0
2	-90°+q_2	0	90°	0
3	0°	1m+q_3	0°	0

$$\begin{cases} x_c = 1m \\ y_c = 1m \\ z_c = 3m \end{cases} \Rightarrow \begin{cases} q_1 = 45° \\ q_2 = 0° \\ q_3 = 0.414m \end{cases}$$ or

i	θ_i	d_i	α_i	a_i
1	180°+q_1	3m	90°	0
2	-90°+q_2	0	90°	0
3	0°	1m+q_3	0°	0

Singularity

$$\begin{cases} d_3 c_1 s_2 = x_c & ① \\ d_3 s_1 s_2 = y_c & ② \\ d_3 c_2 = 3m - z_c & ③ \end{cases}$$

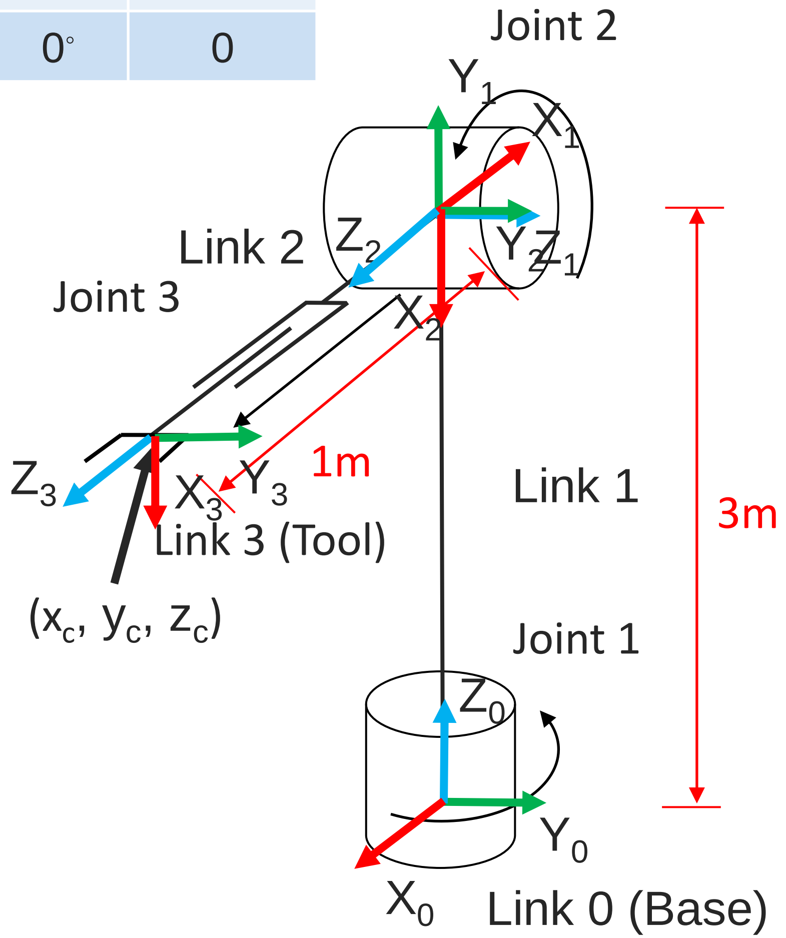
if $d_3 s_2 = 0$

if $d_3 = 0$ $\quad \begin{cases} 0 = x_c \\ 0 = y_c \\ 0 = 3m - z_c \end{cases}$

q_1 and q_2 have infinite solutions.

if $s_2 = 0$ $\quad \begin{cases} 0 = x_c \\ 0 = y_c \\ d_3 = \pm(3m - z_c) \end{cases}$

q_1 has infinite solutions.



Note shorthand: $C_i = \cos(\theta_i)$, $S_i = \sin(\theta_i)$, $\theta_i \neq q_i$

Singularity

- Many equivalent definitions

- From the viewpoint of inverse kinematics
  - For a non-redundant robot, singularity means a configuration of the robot in which the robot has an infinite number of inverse kinematics solutions.

- In most cases, singularity should be avoided.

The robot has an infinite number of inverse kinematics solutions.

IK Challenges - Singularities

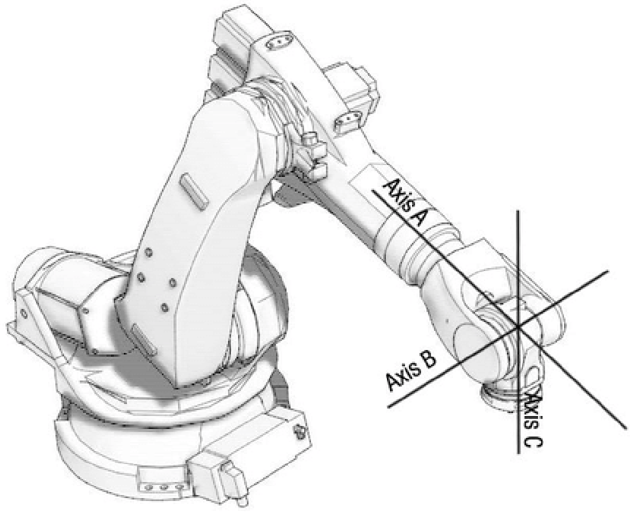
https://www.youtube.com/watch?v=zlGCurgsqg8

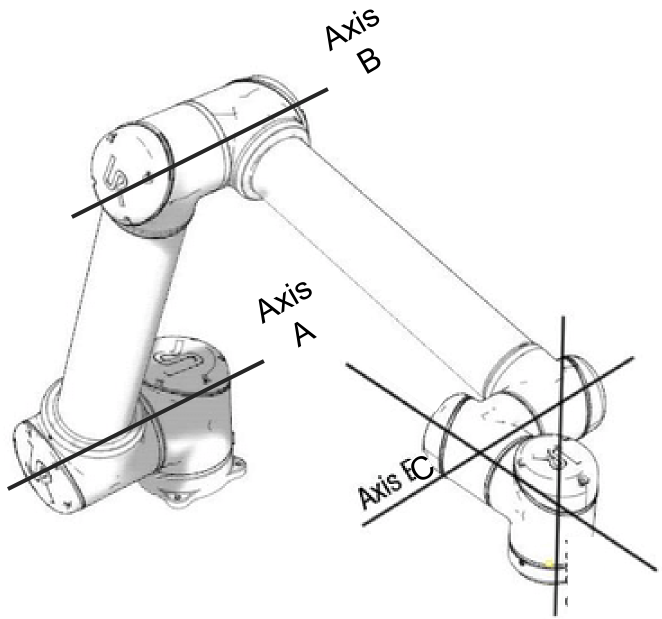
Special 6R Robotic Arms

- Can Have closed-form solutions

- With 3 consecutive joints intersecting at one point
- E.g., spherical wrist (very common)

- With 3 consecutive parallel joints
- E.g., UR robot





Elashry, Khaled & Glynn, Ruairi. (2014). An Approach to Automated Construction Using Adaptive Programing. 51-66. 10.1007/978-3-319-04663-1_4.

General 6R Robotic Arms

- Up to 16 inverse kinematics solutions to a given pose of the target
- Closed-form solutions generally do not exist
- Could be solved by numeric methods

Summary

- Given the pose of a target, we can calculate the joint variables that enable the robot to reach the target by inverse kinematics.

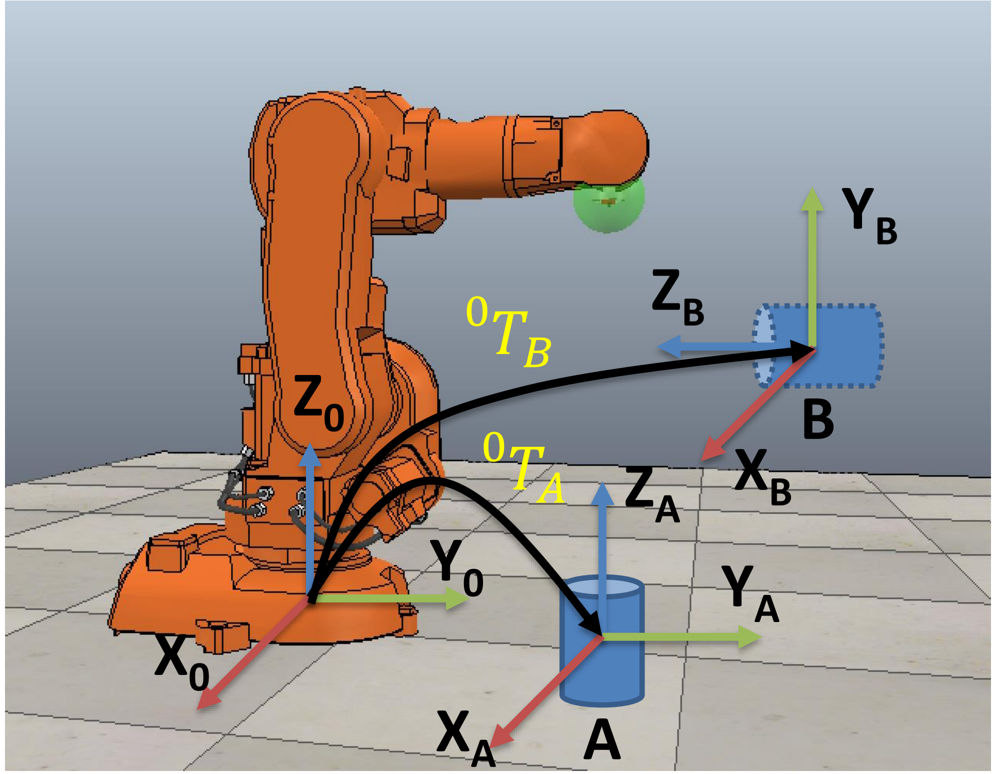
- There are three methods to solve the inverse kinematics.
  - Geometric method
  - Algebraic method
  - Numeric method

- A single pose of the target can correspond to multiple configurations of the robots.

- Singularity could occur when a non-redundant robot has an infinite number of inverse kinematics solutions.

Motivating Problem - Revisited

- Imagine one of your arms is replaced by a robotic arm. You are supposed to move an object from A to B.
- Now you know where the object is in front of you (homogeneous transformation).
- You also know where your “hand” is with respect to your “body” (forward kinematics).
- How can you move your “hand” to reach the object?
  - Inverse kinematics



$q_A = ikine(\,{}^0T_A)$ $\quad$ $q_B = ikine(\,{}^0T_B)$

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Final Remarks

- Acknowledgements
  - Some material of the slides was developed by the previous lecturers of EGB339 - Introduction to Robotics (Michael Milford, Peter Corke, and Leo Wu)