第七章 强化学习II—动态规划方法

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[based on David Silver and Sergey Levine's course]

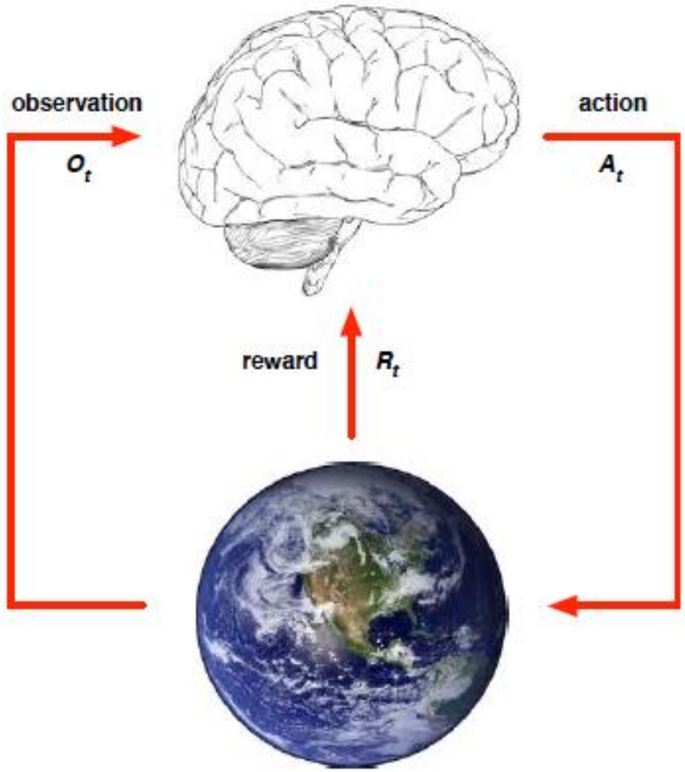
Recap

- Last lecture:
  - Course introduction
  - What's RL?
  - Broad applications of RL
  - Why RL?
  - Basic components of RL: Reward, State, Policy, Model, Value function

- This lecture:
  - The formal formulation of an RL problem as a Markov decision process
  - Making good decisions given a Markov decision process

The RL Problem

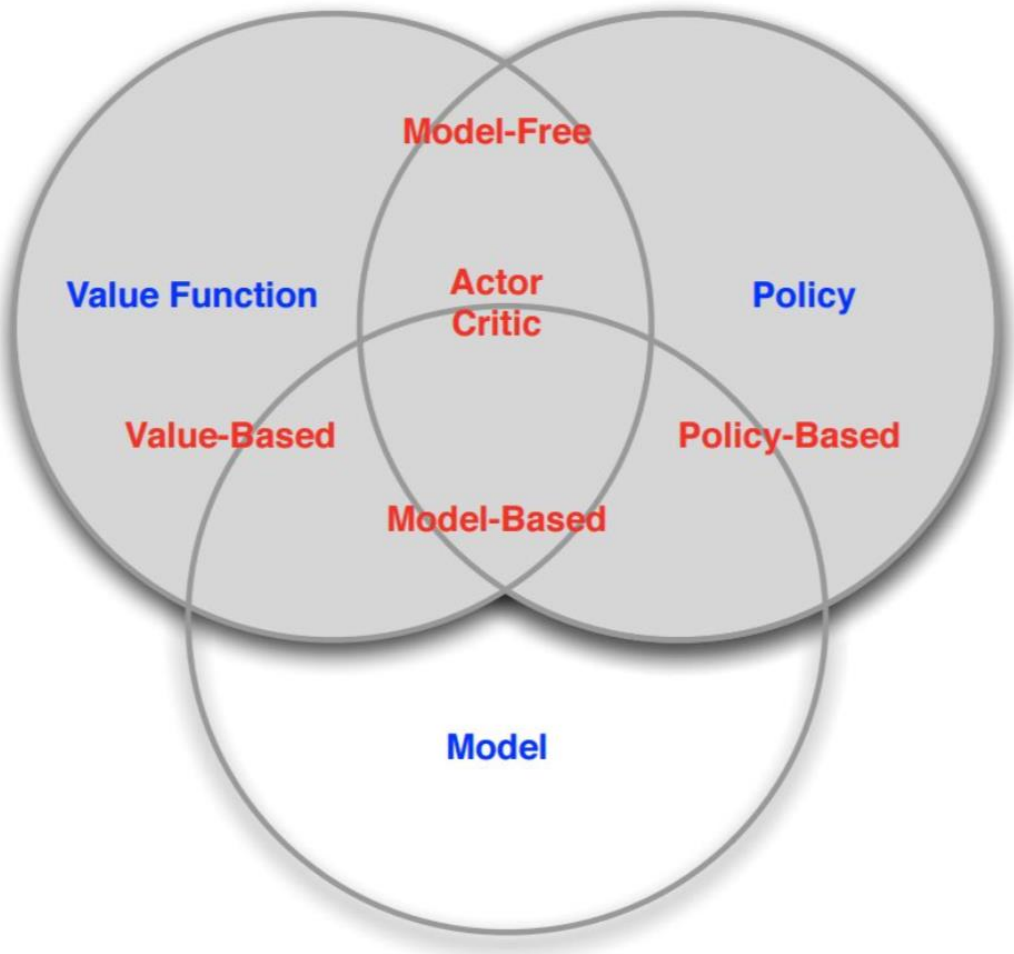
"learn to *make good sequences of decisions through trail-and-errors*"



- At each step t the agent:
  - Executes action A_t
  - Receives observation O_t
  - Receives scalar reward R_t
- The environment:
  - Receives action A_t
  - Emits observation O_{t+1}
  - Emits scalar reward R_{t+1}

Goal: learn a policy (*i.e.*, a mapping from observations to actions) to maximise total future reward

Categorizing RL Algorithms



[from David Silver's course]

Formulation of an RL Problem

- Markov Process
- Markov Reward Process (MRP)
- Markov Decision Process (MDP)
- Evaluation/Prediction and Improvement/Control in MDP

Recall: Markov Property

- A Markov state contains all useful information from the history, i.e., future is independent of past given present

A state S_t is Markov if and only if

$$\mathbb{P}[S_{t+1} \mid S_t] = \mathbb{P}[S_{t+1} \mid S_1, ..., S_t]$$

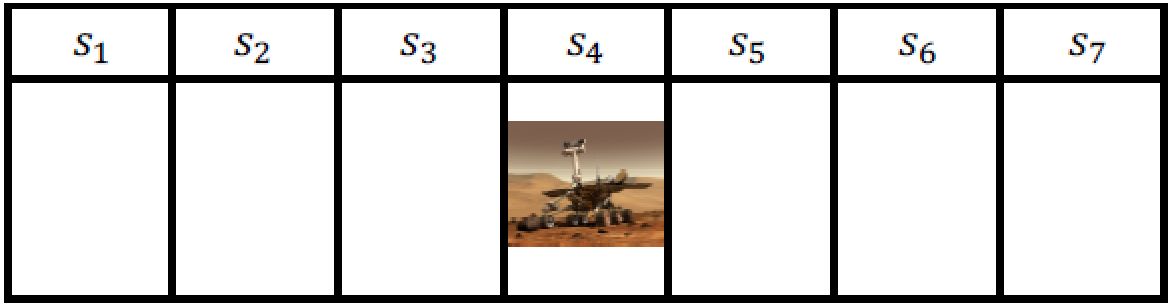
- Markov Process or Markov Chain
  - Sequence of random states with Markov property
  - P is dynamics/transition model that specifies

$$p(s_{t+1} = s' | s_t = s)$$

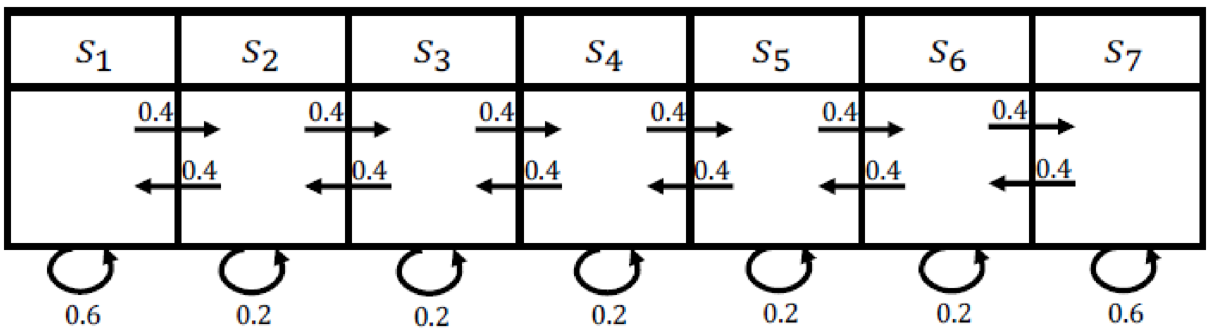
  - no rewards, no actions
  - P can be expressed as a matrix

$$P = \begin{pmatrix} P(s_1|s_1) & P(s_2|s_1) & \cdots & P(s_N|s_1) \\ P(s_1|s_2) & P(s_2|s_2) & \cdots & P(s_N|s_2) \\ \vdots & \vdots & \ddots & \vdots \\ P(s_1|s_N) & P(s_2|s_N) & \cdots & P(s_N|s_N) \end{pmatrix}$$

Markov Process



The Mars rover problem



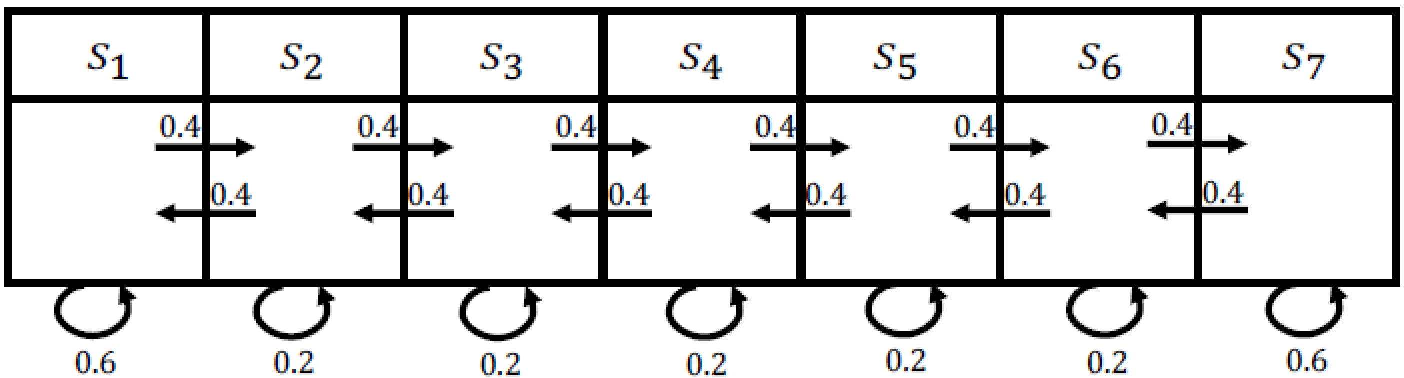
$$P = \begin{pmatrix} 0.6 & 0.4 & 0 & 0 & 0 & 0 & 0 \\ 0.4 & 0.2 & 0.4 & 0 & 0 & 0 & 0 \\ 0 & 0.4 & 0.2 & 0.4 & 0 & 0 & 0 \\ 0 & 0 & 0.4 & 0.2 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0.2 & 0.4 & 0 \\ 0 & 0 & 0 & 0 & 0.4 & 0.2 & 0.4 \\ 0 & 0 & 0 & 0 & 0 & 0.4 & 0.6 \end{pmatrix}$$

e.g., Sample episodes starting from *S4*

- $s_4, s_5, s_6, s_7, s_7, s_7, \ldots$
- $s_4, s_4, s_5, s_4, s_5, s_6, \ldots$
- $s_4, s_3, s_2, s_1, \ldots$

Markov Reward Process (MRP)

- Markov Reward Process is a Markov Process with rewards
  - P is dynamics/transition model that specifies
    $$p(s_{t+1} = s' | s_t = s)$$
  - R is a reward function $R(s_t = s) = \mathbb{E}[r_t | s_t = s]$
  - Discount factor $\gamma \in [0, 1]$
  - No actions



Reward: +1 in s_1, +10 in s_7, 0 in all other states

Return & Value Function

- Definition of Horizon (H)
  - Number of time steps in each episode
  - Can be infinite or finite
- Definition of Return
  - Discounted sum of rewards from time step t to horizon H

$$G_t = r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \cdots + \gamma^{H-1} r_{t+H-1}$$

- Definition of State Value Function V(s)
  - Expected return from starting in state s

$$V(s) = \mathbb{E}[G_t | s_t = s] = \mathbb{E}[r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \cdots + \gamma^{H-1} r_{t+H-1} | s_t = s]$$

Discount Factor

- Mathematically convenient
  - avoid infinite returns and values
- Model humans' behaviors
  - $\gamma = 0$: only care about immediate reward
  - $\gamma = 1$: future reward is with the same importance

$$G_t = r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \cdots + \gamma^{H-1} r_{t+H-1}$$

Computing the Value of MRP

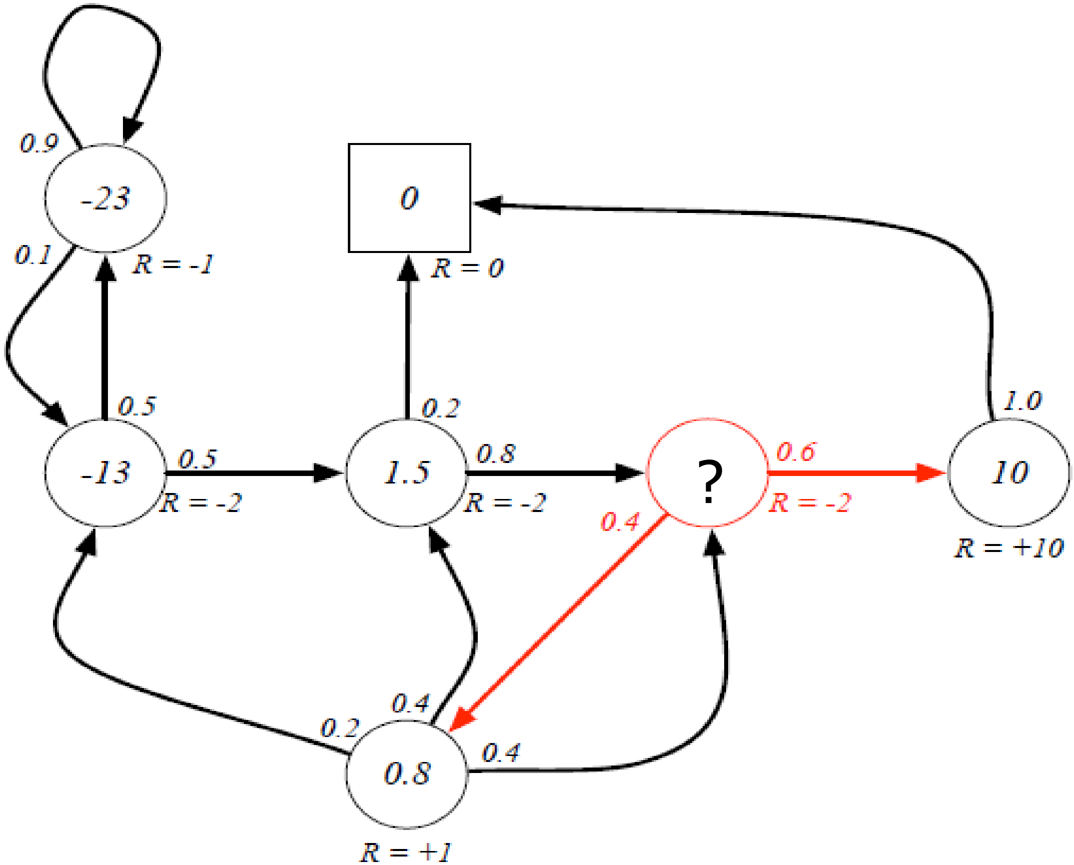
- MRP value function satisfies the Bellman Equation

$$V(s) = \mathbb{E}[G_t | s_t = s] = \mathbb{E}[r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \cdots + \gamma^{H-1} r_{t+H-1} | s_t = s]$$

$$V(s) = \underbrace{R(s)}_{\text{Immediate reward}} + \underbrace{\gamma \sum_{s' \in S} P(s'|s) V(s')}_{\text{Discounted sum of future rewards}}$$

Computing the Value of MRP

$$V(s) = \underbrace{R(s)}_{\text{Immediate reward}} + \underbrace{\gamma \sum_{s' \in S} P(s'|s) V(s')}_{\text{Discounted sum of future rewards}}$$



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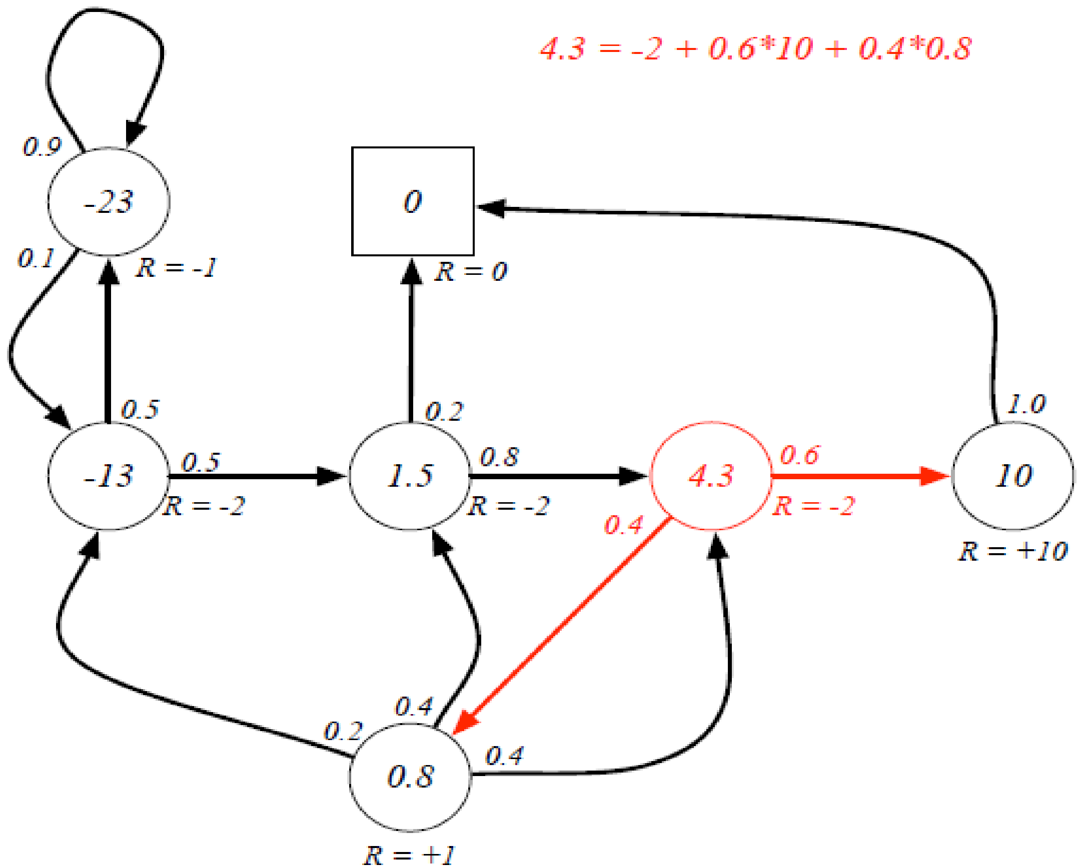
Computing the Value of MRP

$$V(s) = \underbrace{R(s)}_{\text{Immediate reward}} + \underbrace{\gamma \sum_{s' \in S} P(s'|s)V(s')}_{\text{Discounted sum of future rewards}}$$



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Computing the Value of MRP

- For finite state MRP, we can express *V*(s) in a matrix form

$$\begin{pmatrix} V(s_1) \\ \vdots \\ V(s_N) \end{pmatrix} = \begin{pmatrix} R(s_1) \\ \vdots \\ R(s_N) \end{pmatrix} + \gamma \begin{pmatrix} P(s_1|s_1) & \cdots & P(s_N|s_1) \\ P(s_1|s_2) & \cdots & P(s_N|s_2) \\ \vdots & \ddots & \vdots \\ P(s_1|s_N) & \cdots & P(s_N|s_N) \end{pmatrix} \begin{pmatrix} V(s_1) \\ \vdots \\ V(s_N) \end{pmatrix}$$

$$\begin{aligned} V &= R + \gamma P V \\ V - \gamma P V &= R \\ (I - \gamma P) V &= R \\ V &= (I - \gamma P)^{-1} R \end{aligned}$$

- There are many iterative methods for large MRPs, e.g.
  - Dynamic programming
  - Monte-Carlo evaluation
  - Temporal-Difference learning

Markov Decision Process (MDP)

- Markov Decision Process is Markov Reward Process with actions
  - P is dynamics/transition model for each action that specifies
    $P(s_{t+1} = s' | s_t = s, a_t = a)$
  - R is a reward function $R(s_t = s, a_t = a) = \mathbb{E}[r_t | s_t = s, a_t = a]$
  - Discount factor $\gamma \in [0, 1]$
  - MDP is a tuple: (S, A, P, R, γ)

$$P(s'|s, a_1) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad P(s'|s, a_2) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The transition matrix for the Mars rover problem (a_1 means moving left, and a_2 means moving right)

MDP Policies

- Policy specifies what action to take in each state
  - Can be deterministic or stochastic
  - Usually is a distribution over actions given states

    $\pi(a|s) = P(a_t = a|s_t = s)$
  - Given an MDP and a policy, then

$$R^{\pi}(s) = \sum_{a \in A} \pi(a|s) R(s, a)$$

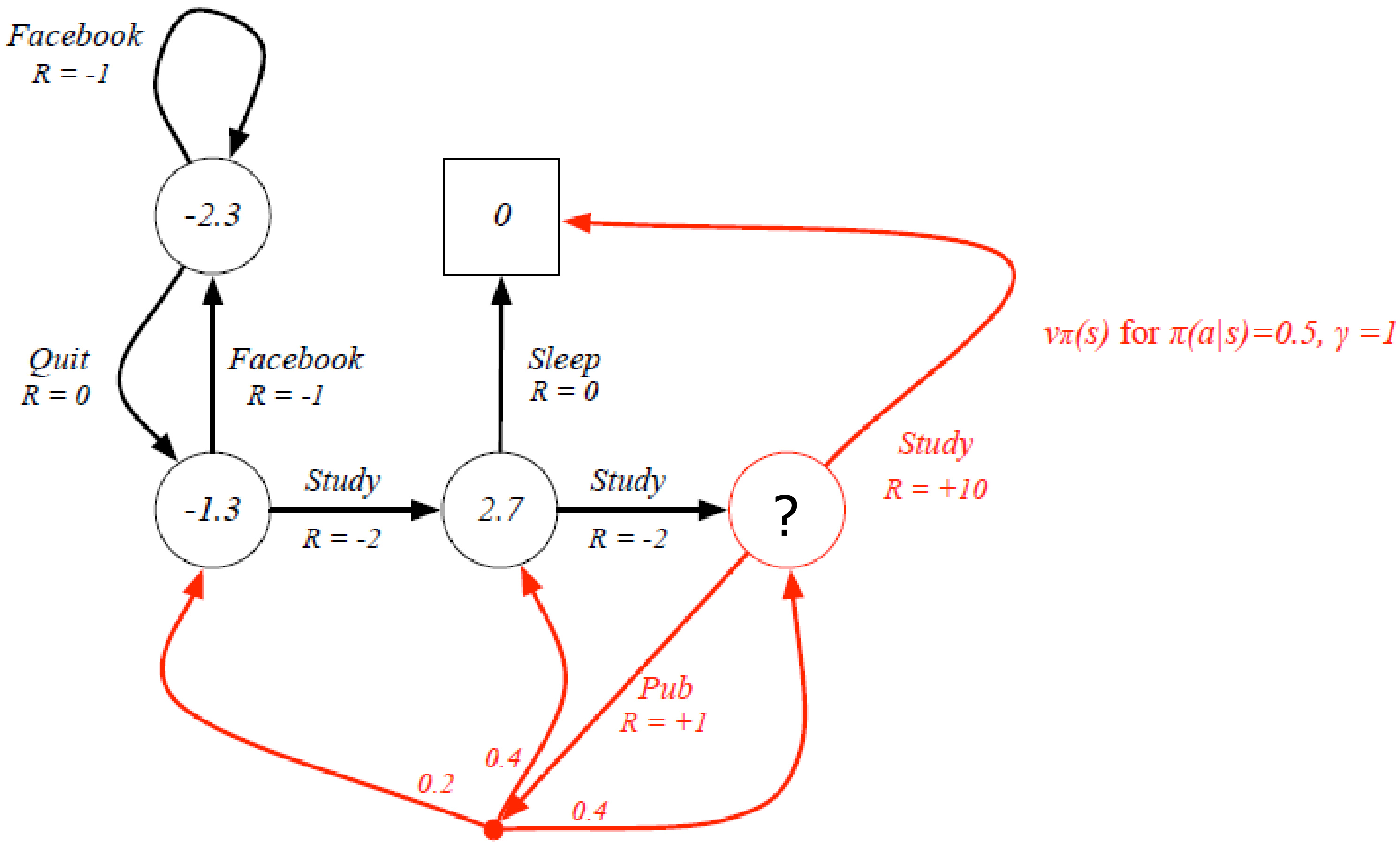
$$P^{\pi}(s'|s) = \sum_{a \in A} \pi(a|s) P(s'|s, a)$$

  - State-Action Value Q for a policy

$$Q^{\pi}(s, a) = R(s, a) + \gamma \sum_{s' \in S} P(s'|s, a) V^{\pi}(s')$$

Computing the Value of MDP

- The value of a state is the value of expected next state plus the reward expected along the way



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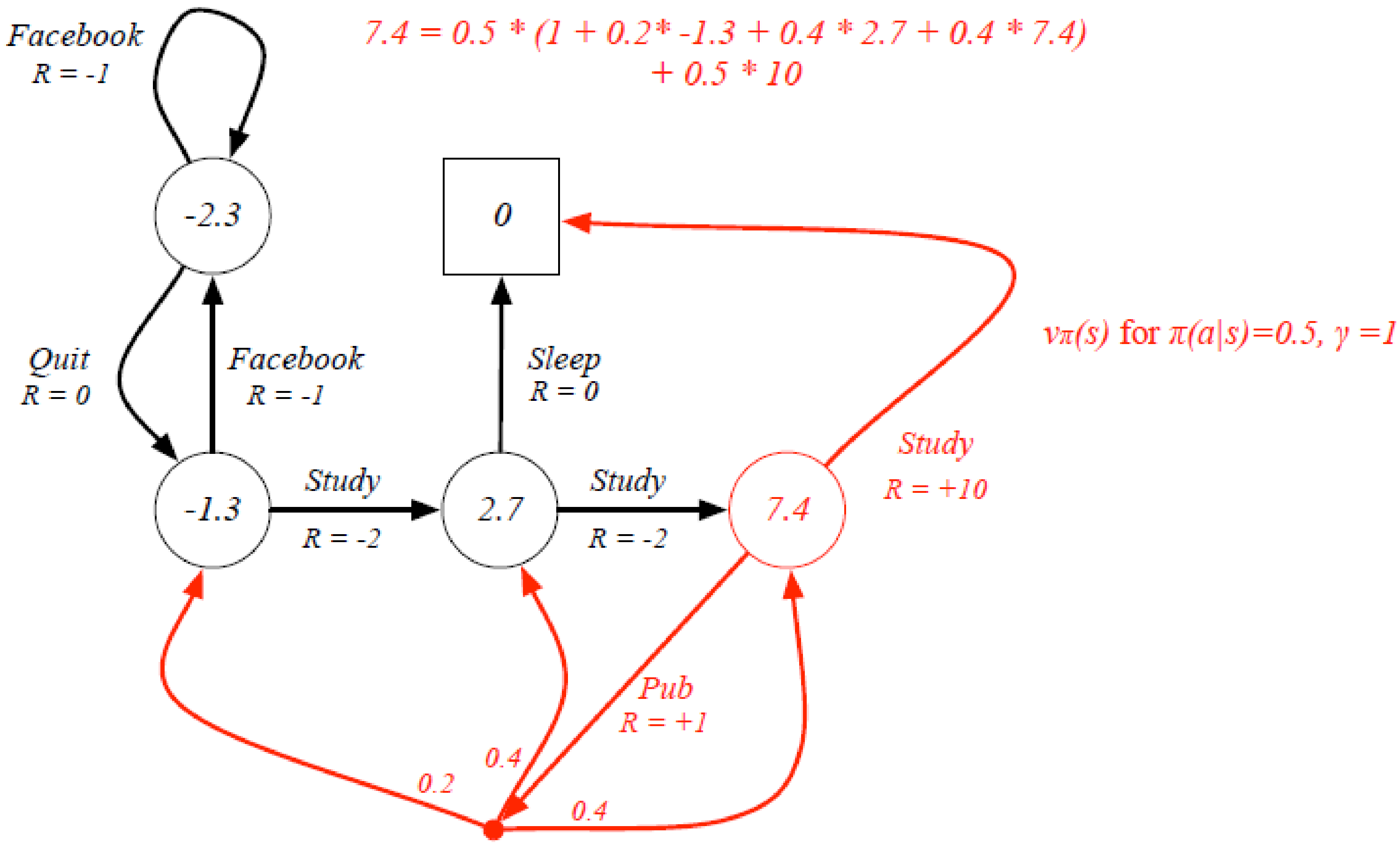
Computing the Value of MDP

- The value of a state is the value of expected next state plus the reward expected along the way



[from David Silver's course]

Optimal Policy

- Compute the optimal policy

$$\pi^*(s) = \arg\max_{\pi} V^{\pi}(s)$$

- There exists a unique optimal value function, which specifies the best possible performance in the MDP
- Optimal policy for an MDP in an infinite horizon problem is deterministic, but not necessarily unique
- One option is searching to compute best policy
- Number of deterministic policies is $|A|^{|S|}$
- Policy iteration is generally more efficient than enumeration

What’s Dynamic Programming (DP)?

- *Dynamic*: sequential or temporal component to the problem
- *Programming:* optimizing a “program", i.e. a policy
- A method for solving complex problems by
  - breaking them down into subproblems
  - combining solutions of subproblems
- DP is a general solution method for problems with two properties:
  - Optimal solution can be decomposed into subproblems
  - Subproblems recur many times and solutions can be cached and reused
- MDP satisfy both properties
  - Bellman equation gives recursive decomposition
  - Value function stores and reuses solutions

$$V(s) = \underbrace{R(s)}_{\text{Immediate reward}} + \underbrace{\gamma \sum_{s' \in S} P(s'|s)V(s')}_{\text{Discounted sum of future rewards}}$$

Solving MDP using DP

- DP assumes full knowledge of the MDP for planning
- DP is an iterative solution method to MDP
  - Policy Iteration (PI)
  - Value Iteration (VI)
- PI iterates between the following processes
  - Policy evaluation (prediction): Estimate/predict the expected rewards from following a given policy
  - Policy improvement (control): find a better policy
- VI iterates between the estimation of value functions and policy optimization, without explicit policy

Value Function for MDP

- The state-value function *v*(s) of an MDP is the expected return starting from state *s*, and following policy *π*

$$v^{\pi}(s) = \mathbb{E}_{\pi}[G_t | s_t = s]$$

where $G_t = R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \ldots$

- The action-value function *q*(s,a) is the expected return starting from state *s*, taking action *a*, and then following policy *π*

$$q^{\pi}(s, a) = \mathbb{E}_{\pi}[G_t | s_t = s, A_t = a]$$

- We have the relation

$$v^{\pi}(s) = \sum_{a \in A} \pi(a|s) q^{\pi}(s, a)$$

Bellman Expectation Equation

- The state-value function can be decomposed into immediate reward plus discounted value of the successor state,

$$v^{\pi}(s) = E_{\pi}[R_{t+1} + \gamma v^{\pi}(s_{t+1}) | s_t = s]$$

- The action-value function can similarly be decomposed

$$q^{\pi}(s, a) = E_{\pi}[R_{t+1} + \gamma q^{\pi}(s_{t+1}, A_{t+1}) | s_t = s, A_t = a]$$

Bellman Expectation Equation for V and Q

$$v^\pi(s) = \sum_{a \in A} \pi(a|s) q^\pi(s,a)$$

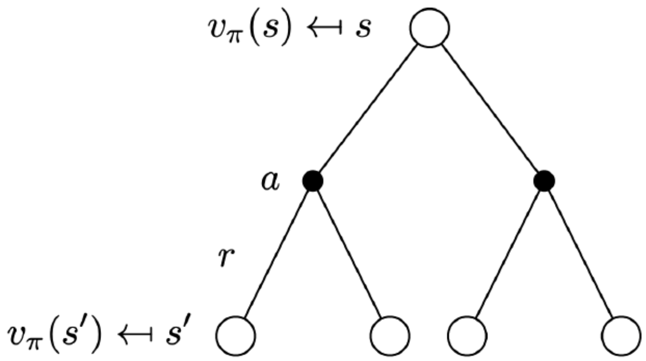
$$q^\pi(s,a) = R_s^a + \gamma \sum_{s' \in S} P(s'|s,a) v^\pi(s')$$

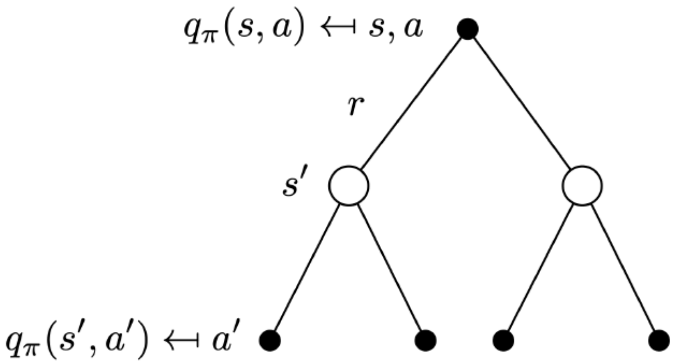
Thus

$$v^\pi(s) = \sum_{a \in A} \pi(a|s)(R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) v^\pi(s'))$$

$$q^\pi(s,a) = R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) \sum_{a' \in A} \pi(a'|s') q^\pi(s',a')$$

Backup Diagram for V and Q



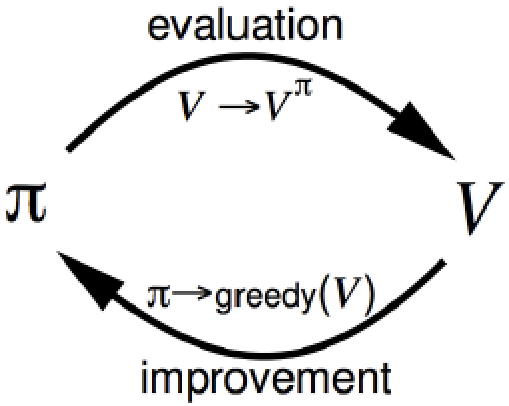


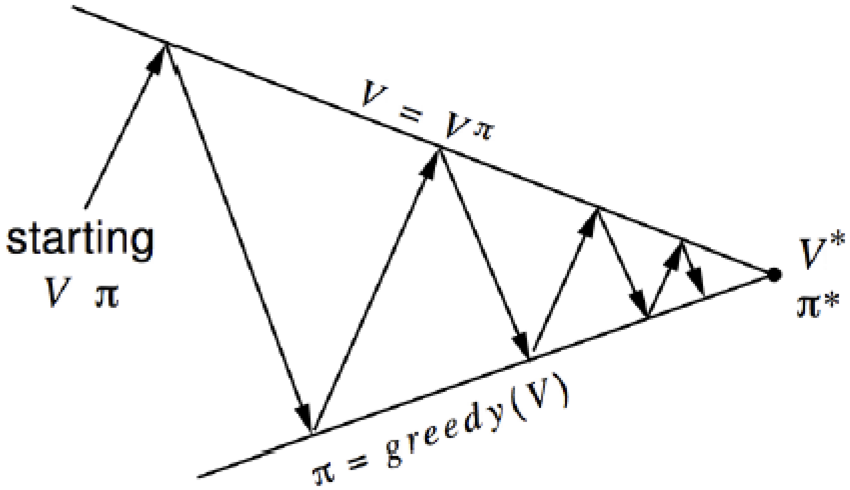
$$v^\pi(s) = \sum_{a \in A} \pi(a|s)(R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) v^\pi(s'))$$

$$q^\pi(s,a) = R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) \sum_{a' \in A} \pi(a'|s') q^\pi(s',a')$$

Policy Iteration (PI)

- Set $i = 0$
- Initialize $\pi_0(s)$ randomly for all states s
- While $i == 0$ or $\|\pi_i - \pi_{i-1}\|_1 > 0$ (L1-norm, measures if the policy changed for any state):
  - $V^{\pi_i} \leftarrow$ MDP V function policy **evaluation** of π_i
  - $\pi_{i+1} \leftarrow$ Policy **improvement**
  - $i = i + 1$





Policy Evaluation

- Objective: evaluate a given policy π for a MDP
- Output: the value function under policy π
- Solution: iteration on Bellman expectation backup
- Algorithm: Synchronous backup

1. At each iteration t+1
   update $v_{t+1}(s)$ from $v_t(s')$ for all states $s \in \mathcal{S}$ where s' is a successor state of s

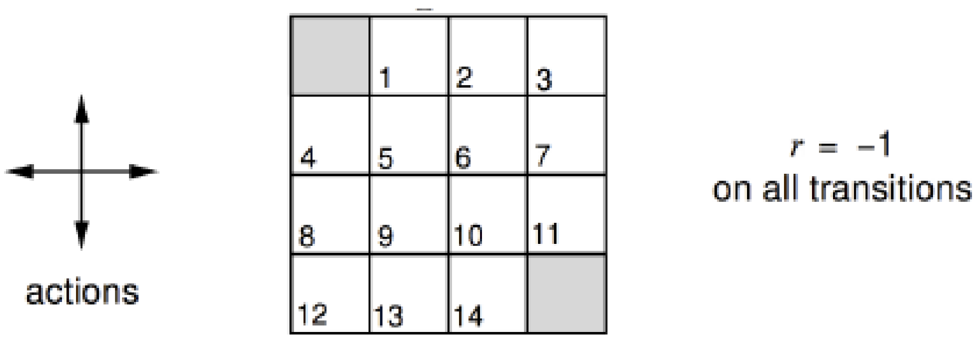
$$v_{t+1}(s) = \sum_{a \in \mathcal{A}} \pi(a|s)(R(s,a) + \gamma \sum_{s' \in \mathcal{S}} P(s'|s,a) v_t(s'))$$

- Convergence: $v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v^{\pi}$

Policy Evaluation

☐ Example 4.1 in the Sutton RL textbook.

- Actions leading out of the grid leave state unchanged
- Reward is -1 until the terminal state is reached
- Agent follows uniform random policy



$$v_{t+1}(s) = \sum_{a \in \mathcal{A}} \pi(a|s)(R(s,a) + \gamma \sum_{s' \in \mathcal{S}} P(s'|s,a) v_t(s'))$$

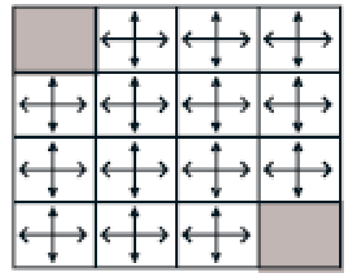
Policy Evaluation

v_k for the Random Policy

Greedy Policy w.r.t. v_k

$k = 0$

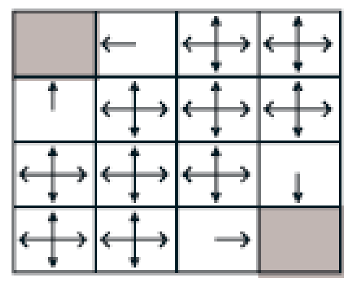
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0



random policy

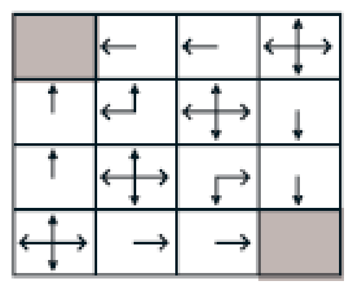
$k = 1$

0.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	0.0



$k = 2$

0.0	-1.7	-2.0	-2.0
-1.7	-2.0	-2.0	-2.0
-2.0	-2.0	-2.0	-1.7
-2.0	-2.0	-1.7	0.0



Policy Evaluation

$k = 3$

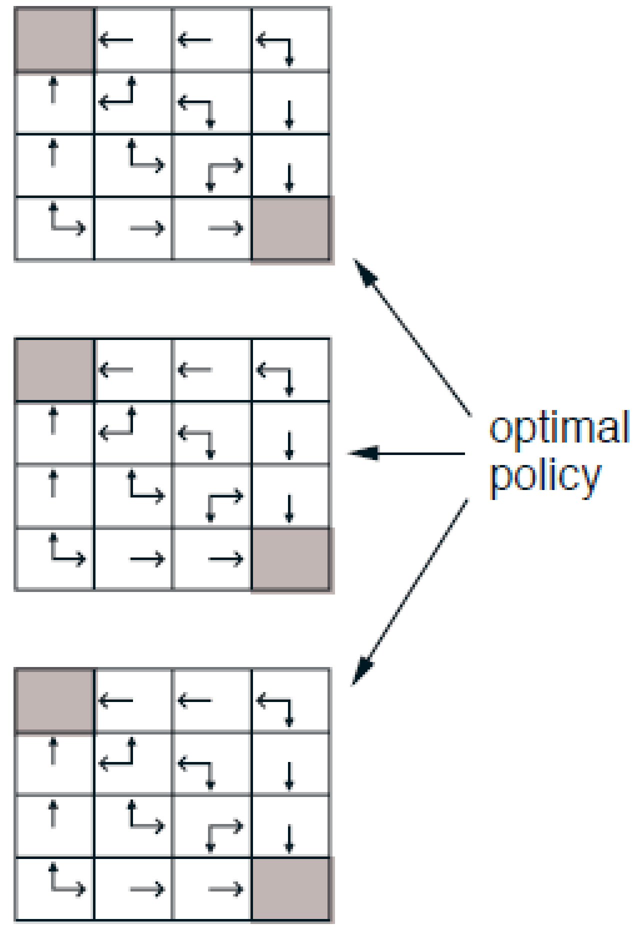
0.0	-2.4	-2.9	-3.0
-2.4	-2.9	-3.0	-2.9
-2.9	-3.0	-2.9	-2.4
-3.0	-2.9	-2.4	0.0

$k = 10$

0.0	-6.1	-8.4	-9.0
-6.1	-7.7	-8.4	-8.4
-8.4	-8.4	-7.7	-6.1
-9.0	-8.4	-6.1	0.0

$k = \infty$

0.0	-14.	-20.	-22.
-14.	-18.	-20.	-20.
-20.	-20.	-18.	-14.
-22.	-20.	-14.	0.0



Policy Improvement

- Consider a determinisitc policy $a = \pi(s)$
- We improve the policy through

$$\pi'(s) = \arg\max_a q^\pi(s, a)$$

- This improves the value from any state *s* over one step

$$q^\pi(s, \pi'(s)) = \max_{a\in\mathcal{A}} q^\pi(s, a) \geq q^\pi(s, \pi(s)) = v^\pi(s)$$

- It therefore improves the value function $v_{\pi'}(s) \geq v_\pi(s)$

$$\begin{aligned}
v_\pi(s) &\leq q_\pi(s, \pi'(s)) = \mathbb{E}_{\pi'}\left[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s\right] \\
&\leq \mathbb{E}_{\pi'}\left[R_{t+1} + \gamma q_\pi(S_{t+1}, \pi'(S_{t+1})) \mid S_t = s\right] \\
&\leq \mathbb{E}_{\pi'}\left[R_{t+1} + \gamma R_{t+2} + \gamma^2 q_\pi(S_{t+2}, \pi'(S_{t+2})) \mid S_t = s\right] \\
&\leq \mathbb{E}_{\pi'}\left[R_{t+1} + \gamma R_{t+2} + \ldots \mid S_t = s\right] = v_{\pi'}(s)
\end{aligned}$$

Bellman Optimality Equation

- The optimal value functic B^{π} are reached by the Bellman optimality equations

$$v^*(s) = \max_a q^*(s, a)$$

$$q^*(s, a) = R(s, a) + \gamma \sum_{s' \in S} P(s'|s, a) v^*(s')$$

thus

$$v^*(s) = \max_a R(s, a) + \gamma \sum_{s' \in S} P(s'|s, a) v^*(s')$$

$$q^*(s, a) = R(s, a) + \gamma \sum_{s' \in S} P(s'|s, a) \max_{a'} q^*(s', a')$$

Value Iteration (VI)

- Policy iteration computes optimal value and policy
- Value iteration is another technique
  - Maintain optimal value of starting in a state *s* if having a finite number of steps *k* left in the episode
  - Iterate to consider longer and longer episodes
- In other word, we assume we know the solution to subproblems and then find the optimal solution by one-step lookahead

$$v(s) \leftarrow \max_{a \in \mathcal{A}} \left(R(s,a) + \gamma \sum_{s' \in \mathcal{S}} P(s'|s,a) v(s') \right)$$

Value Iteration (VI)

Example: Shortest Path

g			

Problem

0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0

V_1

0	-1	-1	-1
-1	-1	-1	-1
-1	-1	-1	-1
-1	-1	-1	-1

V_2

0	-1	-2	-2
-1	-2	-2	-2
-2	-2	-2	-2
-2	-2	-2	-2

V_3

0	-1	-2	-3
-1	-2	-3	-3
-2	-3	-3	-3
-3	-3	-3	-3

V_4

0	-1	-2	-3
-1	-2	-3	-4
-2	-3	-4	-4
-3	-4	-4	-4

V_5

0	-1	-2	-3
-1	-2	-3	-4
-2	-3	-4	-5
-3	-4	-5	-5

V_6

0	-1	-2	-3
-1	-2	-3	-4
-2	-3	-4	-5
-3	-4	-5	-6

V_7

Value Iteration (VI)

1. Objective: find the optimal policy π
2. Solution: iteration on the Bellman optimality backup
3. Value Iteration algorithm:
    1. initialize $k = 1$ and $v_0(s) = 0$ for all states s
    2. For $k = 1 : H$
        1. for each state s

$$
\begin{aligned}
q_{k+1}(s,a) &= R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) v_k(s') \\
v_{k+1}(s) &= \max_a q_{k+1}(s,a)
\end{aligned}
$$

        2. $k \leftarrow k + 1$
    3. To retrieve the optimal policy after the value iteration:

$$\pi(s) = \arg\max_a R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) v_{k+1}(s')$$

Value vs Policy Iteration

- Policy iteration includes: policy evaluation + policy improvement, and the two are repeated iteratively until policy converges
- Value iteration includes: finding optimal value function + one policy extraction. There is no repeat of the two because once the value function is optimal, then the policy out of it should also be optimal (i.e. converged).
- Finding optimal value function can also be seen as a combination of policy improvement (due to max) and truncated policy valuation (the reassignment of v(s) after just one sweep of all states regardless of convergence).