Circuits and Transforms

EE3900: Linear Systems and Signal Processing

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1. Definitions

1.1 The unit step function is defined as

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases} \tag{1.1}$$

1.2 The Laplace transform of $g(t)$ is defined as

$$G(s) = \int_{-\infty}^{\infty} g(t) e^{-st} \, \mathrm{d}t \tag{1.2}$$

2. Laplace Transform

2.1. In the circuit, the switch S is connected to position P for a long time so that the charge on the capacitor becomes q_1 µC. Then S is switched to position Q. After a long time, the charge on the capacitor is q_2 µC

2.2. Draw the circuit using latex-tikz
Solution:

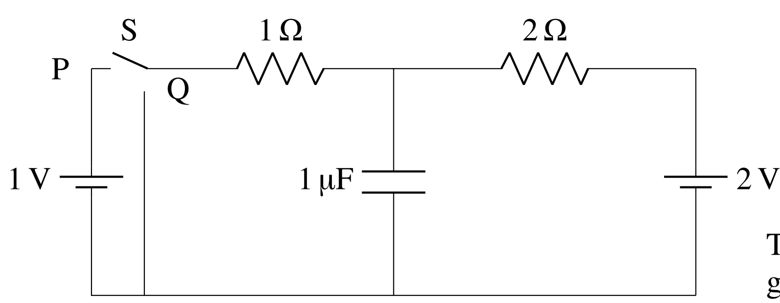
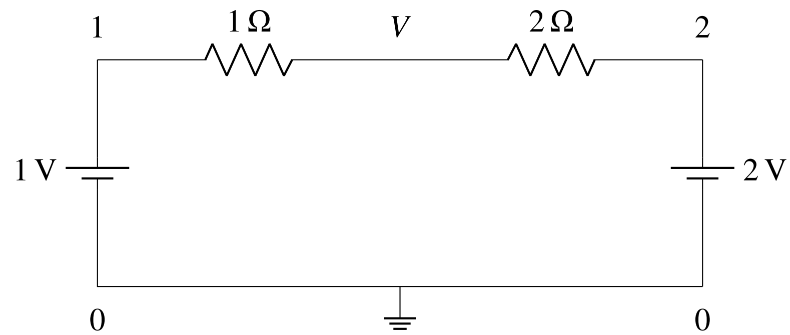


Fig. 2.2. Circuit diagram of the circuit in question

2.3. Find q_1
Solution: After a long time, when steady state is achieved, a capacitor behaves like an open circuit, i.e., current passing through it is zero



By Kirchoff's junction law, we get

$$\frac{V-1}{1} + \frac{V-2}{2} = 0 \tag{2.1}$$

$$\implies V = \frac{4}{3} \text{V} \tag{2.2}$$

$$\implies q_1 = CV = \frac{4}{3} \,\mu\text{C} \tag{2.3}$$

2.4. Show that the Laplace transform of $u(t)$ is $\frac{1}{s}$ and find the ROC
Solution: The Laplace transform of $u(t)$ is given by

$$\mathcal{L}\{u(t)\} = \int_{-\infty}^{\infty} u(t) e^{-st} \, \mathrm{d}t \tag{2.4}$$

$$= \int_{0}^{\infty} e^{-st} \, \mathrm{d}t \tag{2.5}$$

$$= \lim_{R \to \infty} \frac{1 - e^{-sR}}{s} \tag{2.6}$$

This limit is finite only if $\Re(s) > 0$, which is going to be its ROC
Therefore

$$u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s} \qquad \Re(s) > 0 \tag{2.7}$$

2.5. Show that

$$e^{-at} u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a} \qquad a > 0 \tag{2.8}$$

and find the ROC
Solution: The Laplace transform of $e^{-at}u(t)$ for $a > 0$ is given by

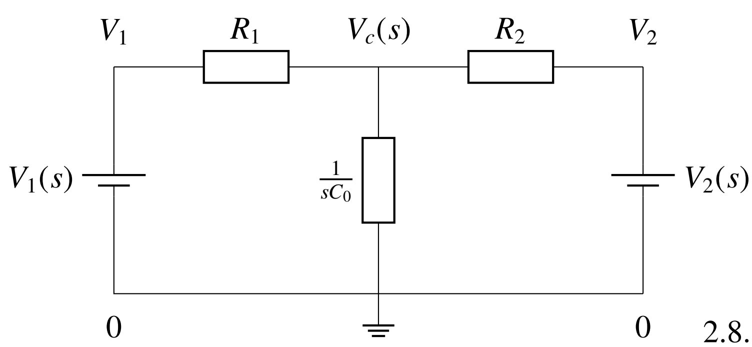
$$\mathcal{L}\{u(t)\} = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}\,\mathrm{d}t \tag{2.9}$$
$$= \int_{0}^{\infty} e^{-(s+a)t}\,\mathrm{d}t \tag{2.10}$$
$$= \lim_{R\to\infty} \frac{1 - e^{-(s+a)R}}{s+a} \tag{2.11}$$

This limit is finite only if $\Re(s+a) > 0$, which is going to be its ROC
Therefore

$$e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a} \qquad \Re(s) > -a \tag{2.12}$$

since a is real

2.6. Now consider the following resistive circuit transformed from Fig. 2.2



where

$$u(t) \overset{\mathcal{L}}{\longleftrightarrow} V_1(s) \tag{2.13}$$
$$2u(t) \overset{\mathcal{L}}{\longleftrightarrow} V_2(s) \tag{2.14}$$

Find the voltage across the capacitor $V_{C_0}(s)$
Solution:

$$V_1(s) = \frac{1}{s} \qquad \Re(s) > 0 \tag{2.15}$$
$$V_2(s) = \frac{2}{s} \qquad \Re(s) > 0 \tag{2.16}$$

By Kirchoff's junction law, we get

$$\frac{V_c - V_1}{R_1} + \frac{V_c - V_2}{R_2} + \frac{V_c - 0}{\frac{1}{sC_0}} = 0 \tag{2.17}$$
$$\implies V_c\left(\frac{1}{R_1} + \frac{1}{R_2} + sC_0\right) = \frac{V_1}{R_1} + \frac{V_2}{R_2} \tag{2.18}$$
$$\implies V_c(s) = \frac{\frac{1}{sR_1} + \frac{2}{sR_2}}{\frac{1}{R_1} + \frac{1}{R_2} + sC_0} \tag{2.19}$$
$$= \frac{\frac{1}{R_1C_0} + \frac{2}{R_2C_0}}{s\left(s + \frac{1}{R_1C_0} + \frac{1}{R_2C_0}\right)} \tag{2.20}$$

2.7. Find $v_c(t)$
Solution:

$$V_c(s) = \frac{\frac{1}{R_1C_0} + \frac{2}{R_2C_0}}{\frac{1}{R_1C_0} + \frac{1}{R_2C_0}}\left(\frac{1}{s} - \frac{1}{s + \frac{1}{R_1C_0} + \frac{1}{R_2C_0}}\right) \tag{2.21}$$
$$= \frac{\frac{1}{R_1} + \frac{2}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2}}\left(\frac{1}{s} - \frac{1}{\frac{1}{C_0}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + s}\right) \tag{2.22}$$

Taking the inverse Laplace transform in (2.22),

$$V(s) \overset{\mathcal{L}}{\longleftrightarrow} \frac{2R_1 + R_2}{R_1 + R_2}u(t)\left(1 - e^{-\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{t}{C_0}}\right) \tag{2.23}$$
$$= \frac{4}{3}\left(1 - e^{-\left(1.5\times10^6\right)t}\right)u(t) \tag{2.24}$$

The python code `codes/2.6.py` plots the graph below.

2.8. Verify your result using ngspice

3. Initial Conditions

3.1. Find q_2 in Fig. 2.2
Solution: After a long time, when steady state is achieved, a capacitor behaves like an open circuit, i.e., current passing through it is zero
By Kirchoff's junction law, we get

$$\frac{V - 0}{1} + \frac{V - 2}{2} = 0 \tag{3.1}$$
$$\implies V = \frac{2}{3}\,\text{V} \tag{3.2}$$
$$\implies q_2 = CV = \frac{2}{3}\,\mu\text{C} \tag{3.3}$$

3.2. Draw the equivalent s-domain resistive circuit when S is switched to position Q. Use variables

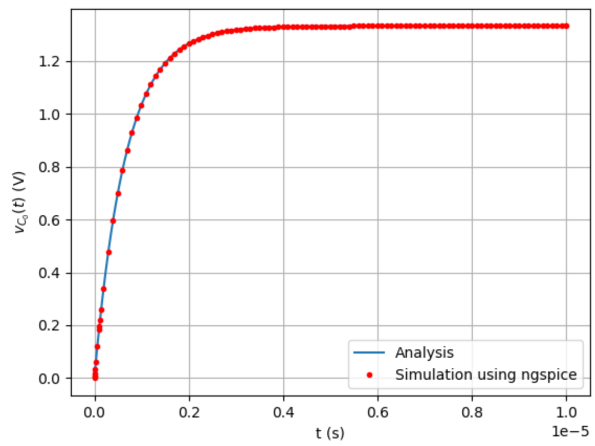
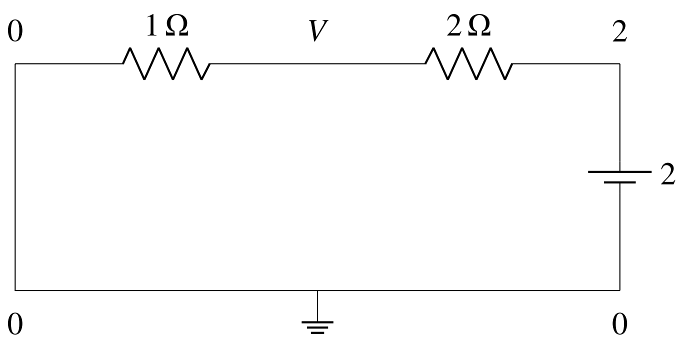


Fig. 2.7. $v_{C_0}(t)$ before the switch is flipped



R_1, R_2, C_0 for the passive elements. Use latex-tikz

Solution:

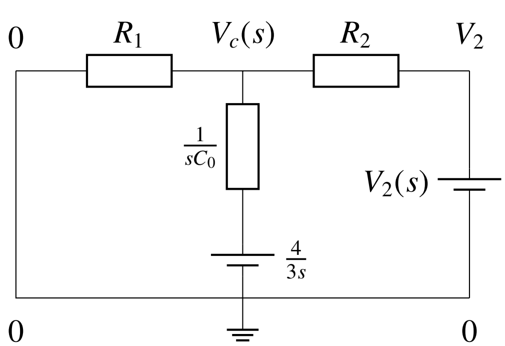


Fig. 3.2. Circuit diagram in s-domain

The battery $\frac{4}{3s}$ corresponds to the intial potential difference of $\frac{4}{3}$ V across the capacitor just before switching it to Q

3.3. Find $V_c(s)$

Solution: By Kirchoff's junction law, we get

$$\frac{V_c - 0}{R_1} + \frac{V_c - V_2}{R_2} + \frac{V_c - \frac{4}{3s}}{\frac{1}{sC_0}} = 0 \tag{3.4}$$

$$\implies V_c\left(\frac{1}{R_1} + \frac{1}{R_2} + sC_0\right) = \frac{V_2}{R_2} + \frac{4}{3}C_0 \tag{3.5}$$

$$\implies V_c(s) = \frac{\frac{2}{sR_2} + \frac{4}{3}C_0}{\frac{1}{R_1} + \frac{1}{R_2} + sC_0} \tag{3.6}$$

$$= \frac{\frac{2}{R_2C_0} + \frac{4}{3}s}{s\left(s + \frac{1}{R_1C_0} + \frac{1}{R_2C_0}\right)} \tag{3.7}$$

3.4. Find $v_c(t)$. Plot using Python

Solution: On performing partial fraction decomposition

$$V_c(s) = \frac{4}{3}\left(\frac{1}{s + \frac{1}{R_1C_0} + \frac{1}{R_2C_0}}\right) + \frac{\frac{2}{R_2C_0}}{\frac{1}{R_1C_0} + \frac{1}{R_2C_0}}\left(\frac{1}{s} - \frac{1}{s + \frac{1}{R_1C_0} + \frac{1}{R_2C_0}}\right) \tag{3.8}$$

for $\Re(s) > 0$

On taking the inverse Laplace transform, we get

$$v_c(t) = \frac{4}{3}e^{-\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{t}{C_0}}u(t) + \frac{2R_1}{R_1 + R_2}\left(u(t) - e^{-\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{t}{C_0}}u(t)\right) \tag{3.9}$$

Substitute the values $R_1 = 1\,\Omega, R_2 = 2\,\Omega, C_0 = 1\,\mu\text{F}$

$$v_c(t) = \frac{4}{3}e^{-\frac{3}{2}\times 10^6 t}u(t) + \frac{2}{3}\left(1 - e^{-\frac{3}{2}\times 10^6 t}\right)u(t) \tag{3.10}$$

$$= \frac{2}{3}\left(1 + e^{-\frac{3}{2}\times 10^6 t}\right)u(t)\text{ V} \tag{3.11}$$

The Python code `codes/3.4.py` plots the graph below.

3.5. Verify your result using ngspice **Solution:** The ngspice script `codes/3.5.cir` simulates the given circuit and the generated output is depicted in Fig. (3.4).

3.6. Find $v_c(0^-)$, $v_c(0^+)$ and $v_c(\infty)$

Solution: At $t = 0^-$, the switch still hasn't been switched to Q and the circuit is in steady state

$$v_c(0^-) = \frac{4}{3}\text{ V} \tag{3.12}$$

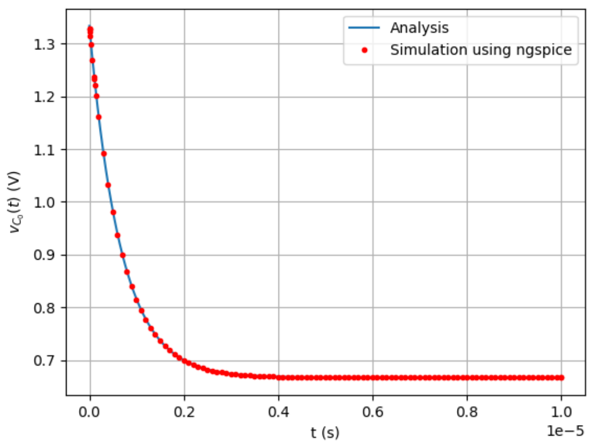


Fig. 3.4. $v_{C_0}(t)$ after the switch is flipped

For $t \geq 0$, we can use the above formula

$$v_c(0^+) = \lim_{t\to 0^+} v_c(t) = \frac{4}{3}\,\mathrm{V} \tag{3.13}$$

$$v_c(\infty) = \lim_{t\to \infty} v_c(t) = \frac{2}{3}\,\mathrm{V} \tag{3.14}$$

3.7. Obtain Fig. 3.2 using the equivalent differential equation
Solution: Using Kirchoff's junction law

$$\frac{v_c(t) - 0}{R_1} + \frac{v_c(t) - v_2(t)}{R_2} + \frac{\mathrm{d}q}{\mathrm{d}t} = 0 \tag{3.15}$$

where $q(t)$ is the charge on the capacitor
On taking the Laplace transform on both sides of this equation

$$\frac{V_c(s) - 0}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + \left(sQ(s) - q(0^-)\right) = 0 \tag{3.16}$$

But $q(0^-) = \frac{4}{3}C_0$ and

$$q(t) = C_0 v_c(t) \tag{3.17}$$

$$\implies Q(s) = C_0 V_c(s) \tag{3.18}$$

Thus

$$\frac{V_c(s) - 0}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + \tag{3.19}$$

$$\left(sC_0 V_c(s) - \frac{4}{3}C_0\right) = 0 \tag{3.20}$$

$$\implies \frac{V_c(s) - 0}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + \frac{V_c(s) - \frac{4}{3s}}{\frac{1}{sC_0}} = 0 \tag{3.21}$$

which is the same equation as the one we obtained from Fig. 3.2
The equivalent circuit in the t-domain is shown below.

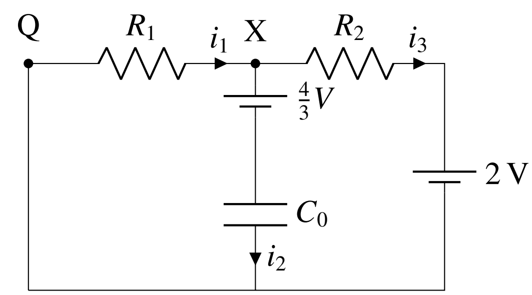


Fig. 3.7.

4. Bilinear Transform

4.1. In Fig. 2.2, consider the case when S is switched to Q right in the beginning. Formulate the differential equation
Solution: The differential equation is the same as before

$$\frac{v_c(t) - 0}{R_1} + \frac{v_c(t) - v_2(t)}{R_2} + \frac{\mathrm{d}q}{\mathrm{d}t} = 0 \tag{4.1}$$

$$\text{i.e., } \frac{v_c(t)}{R_1} + \frac{v_c(t) - v_2(t)}{R_2} + C_0\frac{\mathrm{d}v_c}{\mathrm{d}t} = 0 \tag{4.2}$$

but with a different initial condition

$$q(0^-) = q(0) = 0 \tag{4.3}$$

4.2. Find $H(s)$ considering the ouput voltage at the capacitor
Solution: The transfer function of a control system is defined as the ratio of the Laplace transform of the output variable to Laplace transform of the input variable assuming all initial conditions to be zero.
On taking the Laplace transform on both sides of this equation

$$\frac{V_c(s)}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + sQ(s) - 0 = 0 \tag{4.4}$$

$$\implies V_c(s)\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + sC_0 V_c(s) = \frac{V_2(s)}{R_2} \tag{4.5}$$

$$\implies \frac{V_c(s)}{V_2(s)} = \frac{\frac{1}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2} + sC_0} \tag{4.6}$$

The transfer function is thus

$$H(s) = \frac{\frac{1}{R_2C_0}}{s + \frac{1}{R_1C_0} + \frac{1}{R_2C_0}} \tag{4.7}$$

On substituting the values, we get

$$H(s) = \frac{5 \times 10^5}{s + 1.5 \times 10^6} \tag{4.8}$$

4.3. Plot $H(s)$. What kind of filter is it?
Solution: Download the following Python code that plots Fig. 4.3

```
wget https://github.com/Donal-08/EE3900/
    raw/main/Circuit/codes/4.3.py
```

Run the codes by executing

```
python 4.3.py
```

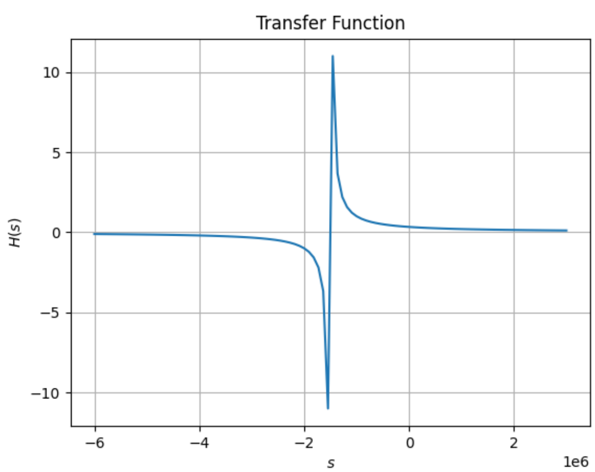


Fig. 4.3. Plot of $H(s)$

Consider the frequency-domain transfer function by putting $s = \jmath\omega$

$$H(\jmath\omega) = \frac{5 \times 10^5}{\jmath\omega + 1.5 \times 10^6} \tag{4.9}$$

$$\implies \left|H(\jmath\omega)\right| = \frac{5 \times 10^5}{\sqrt{\omega^2 + 2.25 \times 10^{12}}} \tag{4.10}$$

As ω increases, $\left|H(\jmath\omega)\right|$ decreases
In other words, the amplitude of high-frequency signals gets diminished and they get filtered out
Therefore, this is a low-pass filter

4.4. Using trapezoidal rule for integration, formulate the difference equation by considering

$$y(n) = y(t)|_{t=n} \tag{4.11}$$

Solution:

$$\frac{v_c(t)}{R_1} + \frac{v_c(t) - v_2(t)}{R_2} + C_0\frac{\mathrm{d}v_c}{\mathrm{d}t} = 0 \tag{4.12}$$

$$\implies C_0\frac{\mathrm{d}v_c}{\mathrm{d}t} = \frac{2u(t) - v_c(t)}{R_2} - \frac{v_c(t)}{R_1} \tag{4.13}$$

$$\implies v_c(t)|_{t=n}^{n+1} = \int_n^{n+1} \left(\frac{2u(t) - v_c(t)}{R_2C_0} - \frac{v_c(t)}{R_1C_0}\right)\mathrm{d}t \tag{4.14}$$

By the trapezoidal rule of integration

$$\int_a^b f(t)\mathrm{d}t \approx \frac{b-a}{2}(f(a) + f(b)) \tag{4.15}$$

Consider $y(t) = v_c(t)$

$$y(n+1) - y(n) = \frac{1}{R_2C_0}(u(n) + u(n+1)) - \frac{1}{2}(y(n+1) + y(n))\left(\frac{1}{R_1C_0} + \frac{1}{R_2C_0}\right) \tag{4.16}$$

Thus, the difference equation is

$$y(n+1)\left(1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0}\right) = y(n)\left(1 - \frac{1}{2R_1C_0} - \frac{1}{2R_2C_0}\right) + \frac{1}{R_2C_0}(u(n) + u(n+1)) \tag{4.17}$$

4.5. Find $H(z)$
Solution: Let $\mathcal{Z}\{y(n)\} = Y(z)$
On taking the Z-transform on both sides of the difference equation

$$zY(z)\left(1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0}\right) = Y(z)\left(1 - \frac{1}{2R_1C_0} - \frac{1}{2R_2C_0}\right) + \frac{1}{R_2C_0}\left(\frac{1}{1-z^{-1}} + \frac{z}{1-z^{-1}}\right) \tag{4.18}$$

$$Y(z)\left(z + \frac{z}{2R_1C_0} + \frac{z}{2R_2C_0} - 1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0}\right) = \frac{1}{R_2C_0}\frac{1+z}{1-z^{-1}} \tag{4.19}$$

Also

$$v_2(t) = 2 \qquad \forall t \geq 0 \tag{4.20}$$

$$\implies x(n) = 2u(n) \tag{4.21}$$

$$\implies X(z) = \frac{2}{1-z^{-1}} \qquad |z| > 1 \tag{4.22}$$

Thus, the transfer function in z-domain is

$$H(z) = \frac{Y(z)}{X(z)} \tag{4.23}$$

$$= \frac{\frac{1+z}{2R_2C_0}}{z + \frac{z}{2R_1C_0} + \frac{z}{2R_2C_0} - 1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0}} \tag{4.24}$$

$$= \frac{\frac{1+z^{-1}}{2R_2C_0}}{1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0} - z^{-1} + \frac{z^{-1}}{2R_1C_0} + \frac{z^{-1}}{2R_2C_0}} \tag{4.25}$$

On substituting the values

$$H(z) = \frac{2.5 \times 10^5(1 + z^{-1})}{7.5 \times 10^5 + 1 + (7.5 \times 10^5 - 1)z^{-1}} \tag{4.26}$$

with the ROC being

$$|z| > \max\left(1, \left|\frac{7.5 \times 10^5 - 1}{7.5 \times 10^5 + 1}\right|\right) \tag{4.27}$$

$$\implies |z| > 1 \tag{4.28}$$

4.6. How can you obtain $H(z)$ from $H(s)$?
Solution: The Z-transform can be obtained from the Laplace transform by the substitution

$$s = \frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}} \tag{4.29}$$

where T is the step size of the trapezoidal rule (1 in our case)
This is known as the bilinear transform
Thus

$$H(z) = \frac{\frac{1}{R_2C_0}}{2\frac{1-z^{-1}}{1+z^{-1}} + \frac{1}{R_1C_0} + \frac{1}{R_2C_0}} \tag{4.30}$$

$$= \frac{\frac{1+z^{-1}}{2R_2C_0}}{1 - z^{-1} + \left(\frac{1}{2R_1C_0} + \frac{1}{2R_2C_0}\right)(1 + z^{-1})} \tag{4.31}$$

$$= \frac{\frac{1+z^{-1}}{2R_2C_0}}{1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0} - z^{-1} + \frac{z^{-1}}{2R_1C_0} + \frac{z^{-1}}{2R_2C_0}} \tag{4.32}$$

$$= \frac{2.5 \times 10^5(1 + z^{-1})}{7.5 \times 10^5 + 1 + (7.5 \times 10^5 - 1)z^{-1}} \tag{4.33}$$

which is the same as what we obtained earlier