Digital Signal Processing

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Contents

***Abstract*—This manual provides a simple introduction to digital signal processing.**

1 Software Installation

Run the following commands

```
sudo apt −get update
sudo apt install libffi−dev libsndfile1 python3−
     scipy python3−numpy python3−matplotlib
sudo pip install cffi pysoundfile
```

2 Digital Filter

2.1 Download the sound file using

```
wget https://github.com/Donal−08/EE3900−
     assignments/blob/main/Assignment_1/
     codes/Sound_Noise.wav
```

2.2 You will find a spectrogram at https://academo.org/demos/spectrum-analyzer. Upload the sound file that you downloaded in Problem 2.1 in the spectrogram and play. Observe the spectrogram. What do you find?
Solution: There are a lot of yellow lines between 440 Hz to 5.1 KHz. These represent the synthesizer key tones. Also, the key strokes are audible along with background noise.

2.3 Write the python code for removal of out of band noise and execute the code.
Solution: Download the source code using

```
wget https://github.com/Donal−08/EE3900−
     assignments/blob/main/Assignment_1/
     codes/2_3.py_
```

and execute it using

```
$ python3 2_3.py
```

2.4 The output of the python script in Problem 2.3 is the audio file `Sound_With_ReducedNoise.wav`. Play the file in the spectrogram in Problem 2.2. What do you observe?
Solution: The key strokes as well as background noise is subdued in the audio. Also, the signal is blank for frequencies above 5.1 kHz.

3 Difference Equation

3.1 Let

$$x(n) = \left\{\underset{\uparrow}{1}, 2, 3, 4, 2, 1\right\} \tag{3.1}$$

Sketch $x(n)$.

3.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2), \quad y(n) = 0, n < 0 \tag{3.2}$$

Sketch $y(n)$.
Solution: The following code yields Fig. (3.2).

```
wget https://github.com/Donal−08/EE3900−
     assignments/blob/main/Assignment_1/
     codes/3_2.py
```

and execute it using

```
$ python3 3_2.py
```

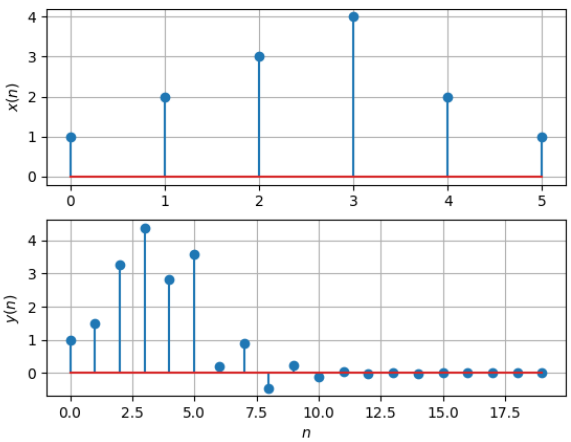

Fig. 3.2: Plot of $x(n)$ and $y(n)$

3.3 Repeat the above exercise using C code.
Solution: The following code yields Fig. (3.2).

```
wget https://github.com/Donal−08/EE3900−
    assignments/blob/main/Assignment_1/
    codes/3_2.py
wget https://github.com/Donal−08/EE3900−
    assignments/blob/main/Assignment_1/
    codes/3_2.c
```

4 Z-transform

4.1 The Z-transform of $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.1}$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \tag{4.2}$$

and find

$$\mathcal{Z}\{x(n-k)\} \tag{4.3}$$

Solution: From (4.1),

$$\mathcal{Z}\{x(n-k)\} = \sum_{n=-\infty}^{\infty} x(n-k)z^{-n} \tag{4.4}$$

$$= \sum_{n=-\infty}^{\infty} x(n)z^{-n-k} \tag{4.5}$$

$$= z^{-k} \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.6}$$

$$= z^{-k}X(n) \tag{4.7}$$

Putting $k = 1$ gives (4.2). or

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \tag{4.8}$$

4.2 Obtain $X(z)$ for $x(n)$ defined in problem 3.1
Solution: For the given $x(n)$, we have

$$\mathcal{Z}\{x(n)\} = \sum_{n=0}^{5} x(n)z^{-n} \tag{4.9}$$

$$X(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 2z^{-4} + z^{-5} \tag{4.10}$$

$$\implies \mathcal{Z}\{x(n-1)\} = z^{-1} + 2z^{-2} + 3z^{-3} + 4z^{-4} + 2z^{-5} + z^{-6} \tag{4.11}$$

$$= z^{-1}X(z) \tag{4.12}$$

which also proves 4.7

4.3 Find

$$H(z) = \frac{Y(z)}{X(z)} \tag{4.13}$$

from (3.2) assuming that the Z-transform is a linear operation.
Solution: Applying (4.7) in (3.2),

$$Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \tag{4.14}$$

$$\implies \frac{Y(z)}{X(z)} = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{4.15}$$

4.4 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.16}$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.17}$$

is

$$U(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.18}$$

Solution: We see using (4.16) that

$$\mathcal{Z}\{\delta(n)\} = \sum_{n=\infty}^{\infty} \delta(n)z^{-n} = \delta(0)z^{0} \tag{4.19}$$

$$= \delta(0) = 1 \tag{4.20}$$

and from (4.17),

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \tag{4.21}$$

$$= \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.22}$$

using the fomula for the sum of an infinite geometric progression.

4.5 Show that

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}} \quad |z| > |a| \tag{4.23}$$

Solution:

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \sum_{n=0}^{\infty} \left(az^{-1}\right)^n \tag{4.24}$$

$$= \frac{1}{1 - az^{-1}} \quad |z| > |a| \tag{4.25}$$

4.6 Let

$$H\left(e^{\jmath\omega}\right) = H\left(z = e^{\jmath\omega}\right). \tag{4.26}$$

Plot $|H\left(e^{\jmath\omega}\right)|$. Comment. $H(e^{\jmath\omega})$ is known as the *Discret Time Fourier Transform* (DTFT) of $x(n)$.

Solution: The following code plots Fig. (4.6).

```
wget https://github.com/Donal−08/
    EE3900−assignments/blob/main/
    Assignment_1/codes/4_5.py
```

The figure can be generated using

```
$ python3 4_5.py
```

Using (4.15), we observe that $|H\left(e^{\jmath\omega}\right)|$ is given by

$$|H\left(e^{\jmath\omega}\right)| = \left|\frac{1 + e^{-2\jmath\omega}}{1 + \frac{1}{2}e^{-\jmath\omega}}\right| \tag{4.27}$$

$$= \sqrt{\frac{(1 + \cos 2\omega)^2 + (\sin 2\omega)^2}{\left(1 + \frac{1}{2}\cos\omega\right)^2 + \left(\frac{1}{2}\sin\omega\right)^2}} \tag{4.28}$$

$$= \sqrt{\frac{2\left(1 + \cos 2\omega\right)}{\frac{5}{4} + \cos\omega}} \tag{4.29}$$

$$= \sqrt{\frac{2\left(2\cos^2\omega\right)}{\frac{5}{4} + \cos\omega}} \tag{4.30}$$

$$= \frac{4|\cos\omega|}{\sqrt{5 + 4\cos\omega}} \tag{4.31}$$

Now , we know that a function is periodic with period T if $f(t+T) = f(t)$, $\forall t \in D$ where D = Domain of $f(t)$

$$|cos(\omega + \pi)| = |-cos(\omega)| = |cos(\omega)| \tag{4.32}$$

$$cos\left(\omega + 2\pi\right) = cos(\omega) \tag{4.33}$$

$$L.C.M(2\pi, \pi) = 2\pi \tag{4.34}$$

Clearly the L.C.M of the fundamnetal period of numerator and denomintor is 2π and so its fundamental period is 2π.

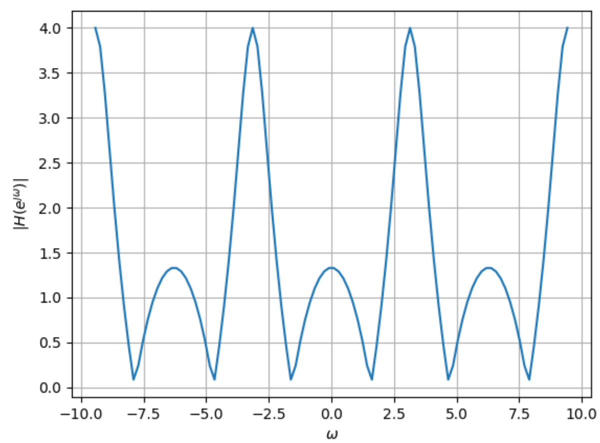

Fig. 4.6: Plot of $\left|H\left(e^{j\omega}\right)\right|$ against ω

4.7 Express $h(n)$ in terms of $H(e^{\jmath\omega})$.

Solution: Given

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n} \tag{4.35}$$

We can prove the IDFT equation as follows :

$$\int_{-\pi}^{\pi} H(e^{-j\omega})e^{j\omega k}d\omega \tag{4.36}$$

$$= \sum_{n=-\infty}^{\infty} h(n) \int_{-\pi}^{\pi} e^{-j\omega n}e^{j\omega k}d\omega \tag{4.37}$$

$$= \begin{cases} 0 & \text{if } n \neq k \\ h(n)\int_{-\pi}^{\pi} e^{-j\omega(n-n)}d\omega & , n = k \end{cases} \tag{4.38}$$

$$\text{i.e } \int_{-\pi}^{\pi} H(e^{-j\omega})e^{j\omega k}d\omega = h(n)(2\pi) \tag{4.39}$$

$$\therefore h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{\jmath\omega})e^{\jmath\omega n}d\omega \tag{4.40}$$

5 Impulse Response

5.1 Using long division, find

$$h(n), \quad n < 5 \tag{5.1}$$

Solution:

$$H(z) = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{5.2}$$

Substitute $z^{-1} = x$ for simplicity

$$\begin{array}{r} 2x - 4 \\ 1 + \frac{x}{2} \overline{\big) \, x^2 + 1} \\ - x^2 - 2x \\ \hline - 2x + 1 \\ 2x + 4 \\ \hline 5 \end{array}$$

$$\implies 1 + z^{-2} = \left(1 + \frac{1}{2}z^{-1}\right)(-4 + 2z^{-1}) + 5 \tag{5.3}$$

$$\implies H(z) = -4 + 2z^{-1} + \frac{5}{1 + \frac{1}{2}z^{-1}} \tag{5.4}$$

$$\frac{5}{1 + \frac{1}{2}z^{-1}} = 5\left(1 + \frac{1}{2}z^{-1}\right)^{-1} \tag{5.5}$$

$$= 5\sum_{n=0}^{\infty}\left(-\frac{z^{-1}}{2}\right)^n \tag{5.6}$$

$$H(z) = -4 + 2z^{-1} + 5 - \frac{5}{2}z^{-1} + \frac{5}{4}z^{-2} - \frac{5}{8}z^{-3} + \frac{5}{16}z^{-4} - \frac{5}{32}z^{-5} + \cdots \tag{5.7}$$

Therefore, by comparing coefficients

$$h(n) = \begin{cases} 1 & n = 0 \\ -\frac{1}{2} & n = 1 \\ \frac{5}{4} & n = 2 \\ -\frac{5}{8} & n = 3 \\ \frac{5}{16} & n = 4 \end{cases} \tag{5.8}$$

Alternatively, on applying the inverse Z-transform on both sides of the equation

$$H(z) \overset{\mathcal{Z}}{\rightleftharpoons} h(n) \tag{5.9}$$

$$-4 \overset{\mathcal{Z}}{\rightleftharpoons} -4\delta(n) \tag{5.10}$$

$$2z^{-1} \overset{\mathcal{Z}}{\rightleftharpoons} 2\delta(n-1) \tag{5.11}$$

$$\frac{5}{1 + \frac{1}{2}z^{-1}} \overset{\mathcal{Z}}{\rightleftharpoons} 5\left(-\frac{1}{2}\right)^n u(n) \tag{5.12}$$

$$\tag{5.13}$$

Therefore,

$$h(n) = -4\delta(n) + 2\delta(n-1) + 5\left(-\frac{1}{2}\right)^n u(n) \tag{5.14}$$

Download the following Python code that plots Fig. 5.1.

```
wget https://github.com/Donal−08/
    EE3900−assignments/tree/main/
    Assignment_1/codes/5_1.py
```

Run the code by executing

```
python 5.1.py
```

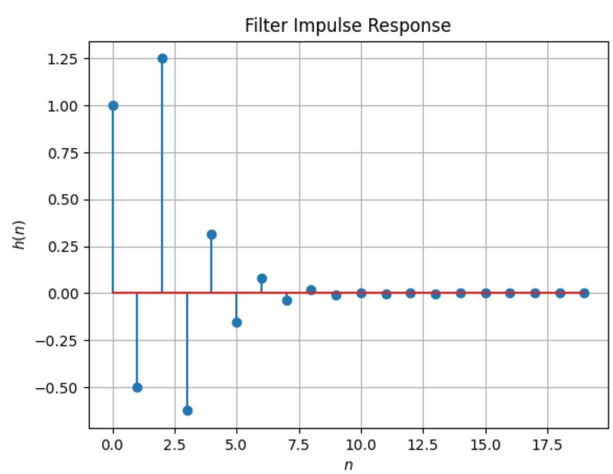

Fig. 5.1: Plot of $h(n)$

5.2 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \overset{\mathcal{Z}}{\rightleftharpoons} H(z) \tag{5.15}$$

and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the *impulse response* of the system defined by (3.2).

Solution: From (4.15),

$$H(z) = \frac{1}{1+\frac{1}{2}z^{-1}} + \frac{z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{5.16}$$

$$\implies h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.17}$$

using (4.23) and (4.7).

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1-az^{-1}} \quad |z| > |a| \tag{5.18}$$

Since Z-transform is a linear operator

5.3 Sketch $h(n)$. Is it bounded? Convergent?
Solution: The following code plots Fig. (5.3).

```
$ wget https://github.com/Donal−08/EE3900−
    assignments/blob/main/Assignment_1/
    codes/5_2.py
```

and execute it using

```
$ python3 5_2.py
```

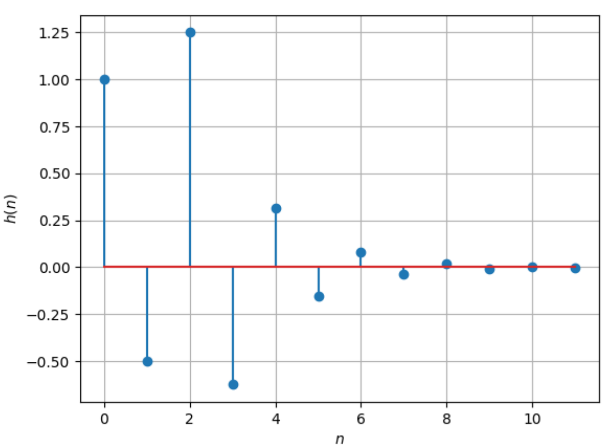

Fig. 5.3: $h(n)$ as the inverse of $H(z)$

Theoretically,

$$|u(n)| \le 1 \tag{5.19}$$

$$\left|\left(-\frac{1}{2}\right)^n\right| \le 1 \tag{5.20}$$

$$\implies \left|\left(-\frac{1}{2}\right)^n u(n)\right| \le 1 \tag{5.21}$$

Similarly,

$$\left|\left(-\frac{1}{2}\right)^{n-2} u(n-2)\right| \le 1 \tag{5.22}$$

$$\implies h(n) \le 2 \tag{5.23}$$

Therefore $h(n)$ is bounded.

5.4 Is it convergent? Justify using the ratio test.
Solution: $h(n)$ is also convergent. For large n, $u(n) = 1$ and so,

$$h(n) = \left(-\frac{1}{2}\right)^n + \left(-\frac{1}{2}\right)^{n-2} \tag{5.24}$$

$$= \left(-\frac{1}{2}\right)^n (4+1) = 5\left(-\frac{1}{2}\right)^n \tag{5.25}$$

$$\implies \left|\frac{h(n+1)}{h(n)}\right| = \frac{1}{2} \tag{5.26}$$

$$\text{i.e } \lim_{n\to\infty}\left|\frac{h(n+1)}{h(n)}\right| = \frac{1}{2} < 1 \tag{5.27}$$

and since, $\lim_{n\to\infty}\left|\frac{h(n+1)}{h(n)}\right| = \frac{1}{2} < 1$,we can conclude by *Ratio Test* that $h(n)$ converges.

5.5 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \tag{5.28}$$

Is the system defined by (3.2) stable for the impulse response in (5.15)?
Solution: Note that

$$\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=-\infty}^{\infty} \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.29}$$

$$= \sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n + \sum_{n=2}^{\infty}\left(-\frac{1}{2}\right)^{n-2} \tag{5.30}$$

$$= \frac{1}{1-\left(-\frac{1}{2}\right)} + \frac{1}{1-\left(-\frac{1}{2}\right)} \tag{5.31}$$

$$= 2\left(\frac{1}{1+\frac{1}{2}}\right) = \frac{4}{3} \tag{5.32}$$

Thus, the given system is stable.

5.6 Verify the above result using a Python code.
Solution: The stability has been verified in the following code

```
$ wget https://github.com/
    Donal−08/EE3900−
    assignments/blob/main/
    Assignment_1/codes/5
```

```
_3.py
```

Run the code by executing

```
python 5.3.py
```

5.7 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \quad (5.33)$$

This is the definition of $h(n)$.
Solution: The following code plots Fig. (5.7). Note that this is the same as Fig. (5.3).

```
wget https://github.com/Donal−08/
    EE3900−assignments/blob/main/
    Assignment_1/codes/5_4.py
```

and executed using

```
$ python3 5_4.py
```

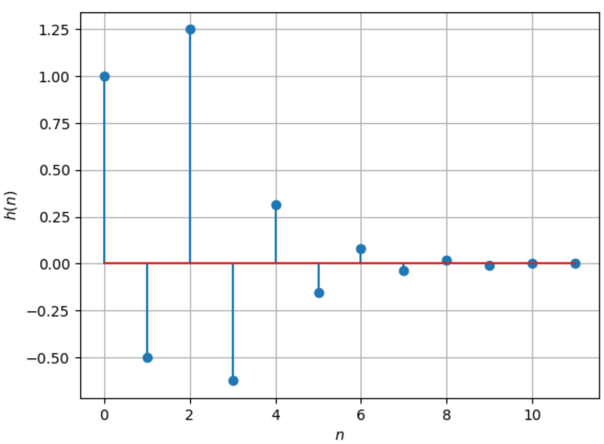

Fig. 5.7: $h(n)$ as the inverse of $H(z)$

5.8 Compute

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \quad (5.34)$$

Comment. The operation in (5.34) is known as *convolution*.
Solution: The following code plots Fig. (5.8). Note that this is the same as $y(n)$ in Fig. (3.2).

```
$ wget https://raw.githubusercontent.com/
    Donal−08/EE3900−assignments/main/
    Assignment_01/codes/5_5.py
```

and executed using

```
$ python3 5_5.py
```

We use Toeplitz matrices for convolution

$$\mathbf{y} = \mathbf{x} \circledast \mathbf{h} \quad (5.35)$$

$$\mathbf{y} = \begin{pmatrix} h_1 & 0 & . & . & . & 0 \\ h_2 & h_1 & . & . & . & 0 \\ h_3 & h_2 & h_1 & . & . & 0 \\ . & . & . & . & . & . \\ 0 & . & . & h_N & h_{N-1} & h_{N-2} \\ 0 & . & . & . & h_N & h_{N-1} \\ 0 & . & . & . & 0 & h_N \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad (5.36)$$

or, equivalently

$$y[k] = h[n] * x[n] = \sum_{i=-\infty}^{\infty} x[i]h[k-i] \quad (5.37)$$

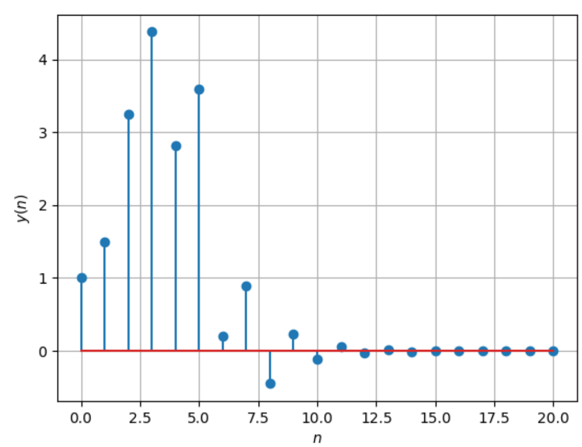

Fig. 5.8: $y(n)$ from the definition

5.9 Express the above convolution using a Toeplitz matrix.
Solution: Let

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \quad \mathbf{h} = \begin{pmatrix} 1 \\ -0.5 \\ 1.25 \\ -0.62 \\ 0.31 \\ -0.16 \end{pmatrix} \quad (5.38)$$

Their convolution is given by the product of the following Toeplitz matrix $\mathbf{T}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -0.5 & 1 & 0 & 0 & 0 & 0 \\ 1.25 & -0.5 & 1 & 0 & 0 & 0 \\ -0.62 & 1.25 & -0.5 & 1 & 0 & 0 \\ 0.31 & -0.62 & 1.25 & -0.5 & 1 & 0 \\ -0.16 & 0.31 & -0.62 & 1.25 & -0.5 & 1 \\ 0 & -0.16 & 0.31 & -0.62 & 1.25 & -0.5 \\ 0 & 0 & -0.16 & 0.31 & -0.62 & 1.25 \\ 0 & 0 & 0 & -0.16 & 0.31 & -0.62 \\ 0 & 0 & 0 & 0 & -0.16 & 0.31 \\ 0 & 0 & 0 & 0 & 0 & -0.16 \end{pmatrix} \tag{5.39}$$

and $\mathbf{x}$

$$\mathbf{y} = \mathbf{x} \circledast \mathbf{h} = \mathbf{T}\mathbf{x} = \begin{pmatrix} 1 \\ 1.5 \\ 3.25 \\ 4.38 \\ 2.81 \\ 3.59 \\ 0.12 \\ 0.78 \\ -0.62 \\ 0 \\ -0.16 \end{pmatrix} \tag{5.40}$$

Download the following Python code for computing the convolution by using a Toeplitz matrix and plotting Fig. 5.9

```
wget https://github.com/
    Donal−08/EE3900−
    assignments/tree/main/
    Assignment_1/codes/5
    _9.py
```

Run the Python code by executing

```
python 5.9.py
```

5.10 Show that

$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) \tag{5.41}$$

Solution: From (5.34), we substitute $k := n - k$

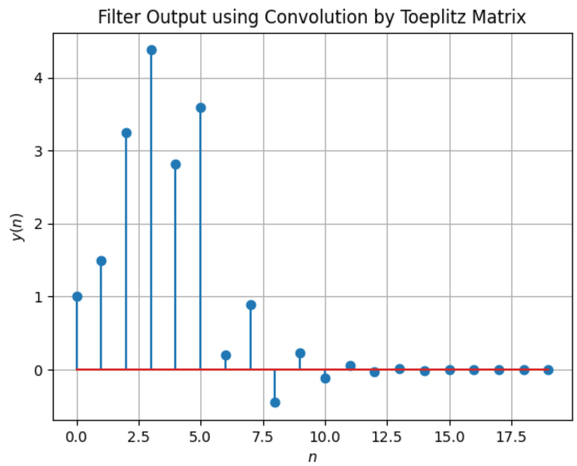

Fig. 5.9: Plot of the convolution of $x(n)$ and $h(n)$

to get

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \tag{5.42}$$

$$= \sum_{n-k=-\infty}^{\infty} x(n-k)\, h(k) \tag{5.43}$$

$$= \sum_{k=-\infty}^{\infty} x(n-k)\, h(k) \tag{5.44}$$

6 DFT and FFT

6.1 Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n) e^{-\jmath 2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \tag{6.1}$$

and $H(k)$ using $h(n)$. **Solution:** The following code plots Fig. (6.3). Note that this is the same as $y(n)$ in Fig. (3.2).

```
$ wget https://github.com/Donal−08/EE3900−
    assignments/blob/main/Assignment_1/
    codes/6_1.py
```

and executed using

```
$ python3 6_1.py
```

6.2 Compute

$$Y(k) = X(k)H(k) \tag{6.2}$$

Solution: The following code plots Fig. (6.3). Note that this is the same as $y(n)$ in Fig. (3.2).

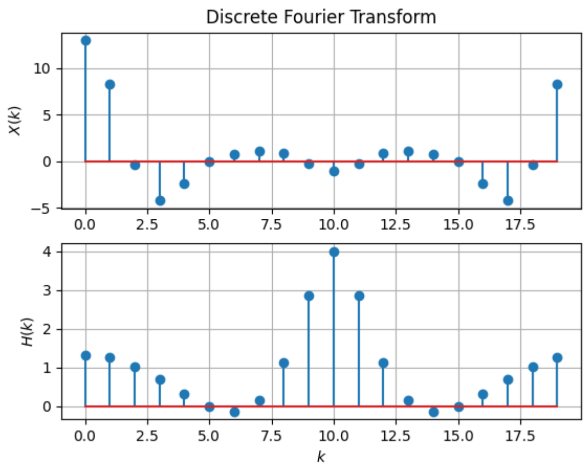

Fig. 6.1: DFT of $x(n)$ and $h(n)$

```
$ wget https://github.com/Donal-08/EE3900-
    assignments/blob/main/Assignment_1/
    codes/6_2.py
```

and executed using

```
$ python3 6_2.py
```

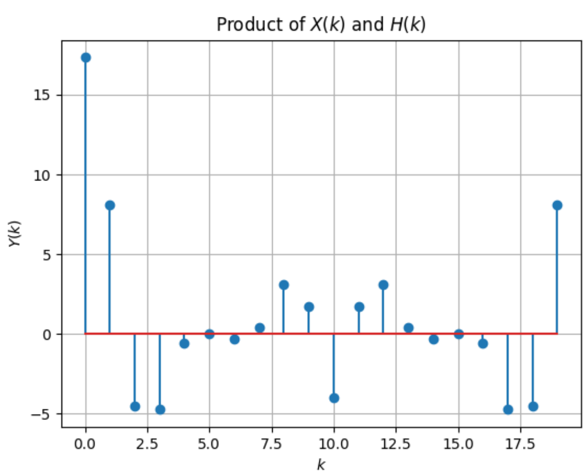

Fig. 6.2: $X(k) * H(k)$

6.3 Compute

$$y(n) = \frac{1}{N} \sum_{k=0}^{N-1} Y(k) \cdot e^{j2\pi kn/N}, \quad n = 0, 1, \ldots, N-1 \tag{6.3}$$

Solution: The following code plots Fig. (6.3). Note that this is the same as $y(n)$ in Fig. (3.2).

```
$ wget https://github.com/Donal-08/EE3900-
    assignments/blob/main/Assignment_1/
    codes/6_3.py
```

and executed using

```
$ python3 6_3.py
```

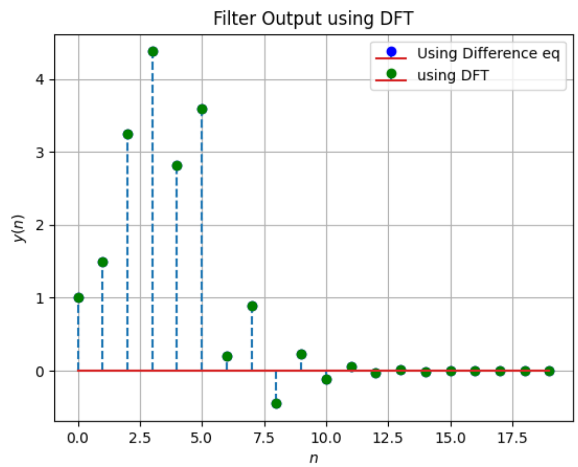

Fig. 6.3: $y(n)$ from the IDFT

6.4 Repeat the previous exercise by computing $X(k), H(k)$ and $y(n)$ through FFT and
Solution: Download the code from

```
$ wget https://github.com/Donal-08/EE3900-
    assignments/blob/main/Assignment_1/
    codes/6_4.py
```

and execute it using

```
$ python3 6_4.py
```

Observe that Fig. (6.4) is the same as $y(n)$ in Fig. (3.2).

7 FFT

7.1 The DFT of $x(n)$ is given by

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots, N-1 \tag{7.1}$$

7.2 Let

$$W_N = e^{-j2\pi/N} \tag{7.2}$$

Then the N-point *DFT matrix* is defined as

$$\mathbf{F}_N = [W_N^{mn}], \quad 0 \leq m, n \leq N-1 \tag{7.3}$$

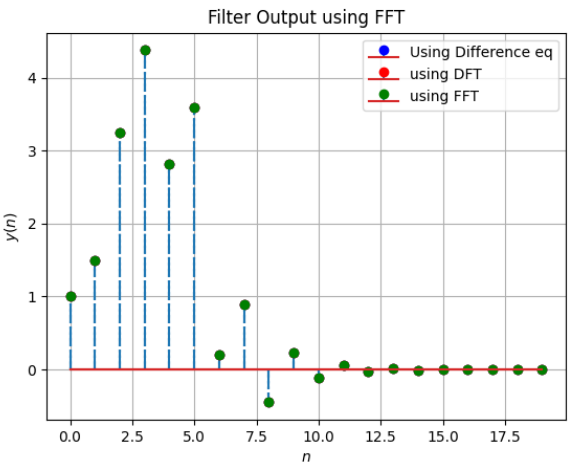

Fig. 6.4: $y(n)$ using FFT and IFFT

where W_N^{mn} are the elements of $\mathbf{F}_N$.

7.3 Let

$$\mathbf{I}_4 = \begin{pmatrix} \mathbf{e}_4^1 & \mathbf{e}_4^2 & \mathbf{e}_4^3 & \mathbf{e}_4^4 \end{pmatrix} \tag{7.4}$$

be the 4×4 identity matrix. Then the 4 point *DFT permutation matrix* is defined as

$$\mathbf{P}_4 = \begin{pmatrix} \mathbf{e}_4^1 & \mathbf{e}_4^3 & \mathbf{e}_4^2 & \mathbf{e}_4^4 \end{pmatrix} \tag{7.5}$$

7.4 The 4 point *DFT diagonal matrix* is defined as

$$\mathbf{D}_4 = \operatorname{diag}\begin{pmatrix} W_8^0 & W_8^1 & W_8^2 & W_8^3 \end{pmatrix} \tag{7.6}$$

7.5 Show that

$$W_N^2 = W_{N/2} \tag{7.7}$$

Solution:

$$W_N^2 = \left(\exp\left(-j\frac{2\pi}{N}\right)\right)^2 \tag{7.8}$$

$$= \exp\left(-j\frac{2\pi}{N} \cdot 2\right) \tag{7.9}$$

$$= \exp\left(-j\frac{2\pi}{N/2}\right) \tag{7.10}$$

$$= W_{N/2} \tag{7.11}$$

7.6 Show that

$$\mathbf{F}_4 = \begin{bmatrix} \mathbf{I}_2 & \mathbf{D}_2 \\ \mathbf{I}_2 & -\mathbf{D}_2 \end{bmatrix} \begin{bmatrix} \mathbf{F}_2 & 0 \\ 0 & \mathbf{F}_2 \end{bmatrix} \mathbf{P}_4 \tag{7.12}$$

Solution:

$$\begin{bmatrix} \mathbf{I}_2 & \mathbf{D}_2 \\ \mathbf{I}_2 & -\mathbf{D}_2 \end{bmatrix} \begin{bmatrix} \mathbf{F}_2 & 0 \\ 0 & \mathbf{F}_2 \end{bmatrix} \tag{7.13}$$

$$= \begin{bmatrix} \mathbf{F}_2 & \mathbf{D}_2\mathbf{F}_2 \\ \mathbf{F}_2 & -\mathbf{D}_2\mathbf{F}_2 \end{bmatrix} \tag{7.14}$$

$$= \begin{bmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & -\jmath \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} & -\begin{pmatrix} 1 & 0 \\ 0 & -\jmath \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \end{bmatrix} \tag{7.15}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -\jmath & \jmath \\ 1 & 1 & -1 & -1 \\ 1 & -1 & \jmath & -\jmath \end{bmatrix} \tag{7.16}$$

because $W_2^0 = 1$ and $W_2^1 = e^{-\jmath\pi} = -1$

Now

$$\begin{bmatrix} \mathbf{I}_2 & \mathbf{D}_2 \\ \mathbf{I}_2 & -\mathbf{D}_2 \end{bmatrix} \begin{bmatrix} \mathbf{F}_2 & 0 \\ 0 & \mathbf{F}_2 \end{bmatrix} \mathbf{P}_4 \tag{7.17}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -\jmath & \jmath \\ 1 & 1 & -1 & -1 \\ 1 & -1 & \jmath & -\jmath \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.18}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -\jmath & -1 & \jmath \\ 1 & -1 & 1 & -1 \\ 1 & \jmath & -1 & -\jmath \end{bmatrix} \tag{7.19}$$

$$= \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \tag{7.20}$$

$$= \mathbf{F}_4 \tag{7.21}$$

because

$$W_4^0 = 1 \tag{7.22}$$

$$W_4^1 = e^{-\jmath\frac{\pi}{2}} = -\jmath \tag{7.23}$$

$$W_4^2 = e^{-\jmath\pi} = -1 \tag{7.24}$$

$$W_4^3 = e^{-\jmath\frac{3\pi}{2}} = \jmath \tag{7.25}$$

$$W_4^n = W_4^{n-4} \qquad \forall n \geq 4 \tag{7.26}$$

7.7 Show that

$$\mathbf{F}_N = \begin{bmatrix} \mathbf{I}_{N/2} & \mathbf{D}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{N/2} & 0 \\ 0 & \mathbf{F}_{N/2} \end{bmatrix} \mathbf{P}_N \tag{7.27}$$

Solution:

$$\begin{bmatrix} \mathbf{I}_{N/2} & \mathbf{D}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{N/2} & 0 \\ 0 & \mathbf{F}_{N/2} \end{bmatrix} \tag{7.28}$$

$$= \begin{bmatrix} \mathbf{F}_{N/2} & \mathbf{D}_{N/2}\mathbf{F}_{N/2} \\ \mathbf{F}_{N/2} & -\mathbf{D}_{N/2}\mathbf{F}_{N/2} \end{bmatrix} \tag{7.29}$$

Now

$$\mathbf{D}_{N/2}\mathbf{F}_{N/2} \tag{7.30}$$

$$= \begin{bmatrix} W_N^0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & W_N^{N/2-1} \end{bmatrix} \begin{bmatrix} W_{N/2}^0 & \cdots & W_{N/2}^0 \\ \vdots & \ddots & \vdots \\ W_{N/2}^0 & \cdots & W_{N/2}^{(N/2-1)^2} \end{bmatrix} \tag{7.31}$$

$$= \begin{bmatrix} W_N^0 W_{N/2}^0 & \cdots & W_N^0 W_{N/2}^0 \\ \vdots & \ddots & \vdots \\ W_N^{N/2-1} W_{N/2}^0 & \cdots & W_N^{N/2-1} W_{N/2}^{(N/2-1)^2} \end{bmatrix} \tag{7.32}$$

Thus

$$(\mathbf{D}_{N/2}\mathbf{F}_{N/2})_{ij} = W_N^i W_{N/2}^{ij} \tag{7.33}$$

$$= W_N^i W_N^{2ij} \tag{7.34}$$

$$= W_N^{i(2j+1)} \tag{7.35}$$

where $i, j = 0, \dots, N/2 - 1$
Therefore, $\mathbf{D}_{N/2}\mathbf{F}_{N/2}$ forms the first $N/2$ rows of the odd-indexed columns of $\mathbf{F}_N$

$$W_N^{(i+N/2)(2j+1)} = \exp\left(-\jmath\frac{2\pi}{N}(2j+1)\left(i + \frac{N}{2}\right)\right) \tag{7.36}$$

$$= \exp\left(-\jmath\left(\frac{2\pi}{N}(2j+1)i + (2j+1)\pi\right)\right) \tag{7.37}$$

$$= -\exp\left(-\jmath\frac{2\pi}{N}(2j+1)i\right) \tag{7.38}$$

$$= -W_N^{i(2j+1)} \tag{7.39}$$

Thus, the remaining $N/2$ rows will be the negatives of the first $N/2$ rows

$$(\mathbf{F}_{N/2})_{ij} = W_{N/2}^{ij} \tag{7.40}$$

$$= W_N^{i(2j)} \tag{7.41}$$

where $i, j = 0, \dots, N/2 - 1$
Therefore, $\mathbf{F}_{N/2}$ forms the first $N/2$ rows of the even-indexed columns of $\mathbf{F}_N$

$$W_N^{(i+N/2)(2j)} = \exp\left(-\jmath\frac{2\pi}{N}(2j)\left(i + \frac{N}{2}\right)\right) \tag{7.42}$$

$$= \exp\left(-\jmath\left(\frac{2\pi}{N}(2j)i + (2j)\pi\right)\right) \tag{7.43}$$

$$= \exp\left(-\jmath\frac{2\pi}{N}(2j)i\right) \tag{7.44}$$

$$= W_N^{i(2j)} \tag{7.45}$$

Thus, the remaining $N/2$ rows will be the same as the first $N/2$ rows
Therefore

$$\begin{bmatrix} \mathbf{F}_{N/2} & \mathbf{D}_{N/2}\mathbf{F}_{N/2} \\ \mathbf{F}_{N/2} & -\mathbf{D}_{N/2}\mathbf{F}_{N/2} \end{bmatrix} = \mathbf{F}_N\mathbf{P}_N \tag{7.46}$$

where

$$\mathbf{P}_N = \begin{pmatrix} \mathbf{e}_N^1 & \mathbf{e}_N^3 & \cdots & \mathbf{e}_N^{N-1} & \mathbf{e}_N^2 & \mathbf{e}_N^4 & \cdots & \mathbf{e}_N^N \end{pmatrix} \tag{7.47}$$

Hence

$$\begin{bmatrix} \mathbf{F}_{N/2} & \mathbf{D}_{N/2}\mathbf{F}_{N/2} \\ \mathbf{F}_{N/2} & -\mathbf{D}_{N/2}\mathbf{F}_{N/2} \end{bmatrix} \mathbf{P}_N = \mathbf{F}_N\mathbf{P}_N^2 = \mathbf{F}_N \tag{7.48}$$

$$\therefore \mathbf{F}_N = \begin{bmatrix} \mathbf{I}_{N/2} & \mathbf{D}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{N/2} & 0 \\ 0 & \mathbf{F}_{N/2} \end{bmatrix} \mathbf{P}_N \tag{7.49}$$

for even N

7.8 Find

$$\mathbf{P}_4\mathbf{x} \tag{7.50}$$

Solution: Let $\mathbf{x} = \begin{pmatrix} x(0) & x(1) & x(2) & x(3) \end{pmatrix}^\top$

$$\mathbf{P}_4\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} \tag{7.51}$$

$$= \begin{bmatrix} x(0) \\ x(2) \\ x(1) \\ x(3) \end{bmatrix} \tag{7.52}$$

7.9 Show that

$$\mathbf{X} = \mathbf{F}_N\mathbf{x} \tag{7.53}$$

where $\mathbf{x}, \mathbf{X}$ are the vector representations of $x(n), X(k)$ respectively.

Solution:

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots, N-1 \tag{7.54}$$

$$\implies \mathbf{X} = \begin{bmatrix} \sum_{n=0}^{N-1} x(n)e^{-j2\pi n(0)/N} \\ \vdots \\ \sum_{n=0}^{N-1} x(n)e^{-j2\pi n(N-1)/N} \end{bmatrix} \tag{7.55}$$

$$= \begin{bmatrix} x(0) + \cdots + x(N-1) \\ \vdots \\ x(0) + \cdots + x(N-1)e^{-j2\pi(N-1)^2/N} \end{bmatrix} \tag{7.56}$$

$$\mathbf{X} = x(0)\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} + \cdots + x(N-1)\begin{bmatrix} 1 \\ \vdots \\ e^{-j2\pi(N-1)^2/N} \end{bmatrix} \tag{7.57}$$

$$= \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & e^{-j2\pi(N-1)^2/N} \end{bmatrix} \begin{bmatrix} x(0) \\ \vdots \\ x(N-1) \end{bmatrix} \tag{7.58}$$

$$= \mathbf{F}_N \mathbf{x} \tag{7.59}$$

7.10 Derive the following step-by-step visualisation of 8-point FFTs into 4-point FFTs and so on

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} + \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.60}$$

$$\begin{bmatrix} X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} - \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.61}$$

4-point FFTs into 2-point FFTs

$$\begin{bmatrix} X_1(0) \\ X_1(1) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.62}$$

$$\begin{bmatrix} X_1(2) \\ X_1(3) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.63}$$

$$\begin{bmatrix} X_2(0) \\ X_2(1) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.64}$$

$$\begin{bmatrix} X_2(2) \\ X_2(3) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.65}$$

$$\mathbf{P}_8 \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \\ x(6) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \\ x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} \tag{7.66}$$

$$\mathbf{P}_4 \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(4) \\ x(2) \\ x(6) \end{bmatrix} \tag{7.67}$$

$$\mathbf{P}_4 \begin{bmatrix} x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(1) \\ x(5) \\ x(3) \\ x(7) \end{bmatrix} \tag{7.68}$$

Therefore,

$$\begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} = \mathbf{F}_2 \begin{bmatrix} x(0) \\ x(4) \end{bmatrix} \tag{7.69}$$

$$\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} = \mathbf{F}_2 \begin{bmatrix} x(2) \\ x(6) \end{bmatrix} \tag{7.70}$$

$$\begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} = \mathbf{F}_2 \begin{bmatrix} x(1) \\ x(5) \end{bmatrix} \tag{7.71}$$

$$\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} = \mathbf{F}_2 \begin{bmatrix} x(3) \\ x(7) \end{bmatrix} \tag{7.72}$$

Solution:

$$X(k) = \sum_{n=0}^{7} x(n)e^{-j2\pi kn/8}, \quad k = 0, \ldots, 7 \tag{7.73}$$

$$= \sum_{n=0}^{7} x(n)W_8^{kn} \tag{7.74}$$

$$= \sum_{n \text{ is even}} x(n)W_8^{kn} + \sum_{n \text{ is odd}} x(n)W_8^{kn} \tag{7.75}$$

$$= \sum_{m=0}^{3} x(2m)W_8^{2km} + \sum_{m=0}^{3} x(2m+1)W_8^{2km+k} \tag{7.76}$$

Now substitute $W_8^2 = W_4$

$$X(k) = \sum_{m=0}^{3} x(2m)W_4^{km} + W_8^k \sum_{m=0}^{3} x(2m+1)W_4^{km} \tag{7.77}$$

Consider

$$x_1(n) = \{x(0), x(2), x(4), x(6)\} \tag{7.78}$$

$$x_2(n) = \{x(1), x(3), x(5), x(7)\} \tag{7.79}$$

Thus

$$X(k) = X_1(k) + W_8^k X_2(k) \qquad k = 0, \ldots, 7 \tag{7.80}$$

Now, $X_1(k)$ and $X_2(k)$ are 4-point DFTs which means they are periodic with period 4

$$X(k+4) = X_1(k+4) + W_8^{k+4} X_2(k+4) \tag{7.81}$$

$$= X_1(k) + e^{-\jmath 2\pi (k+4)/8} X_2(k) \tag{7.82}$$

$$= X_1(k) + e^{-\jmath (2\pi k/8 + \pi)} X_2(k) \tag{7.83}$$

$$= X_1(k) - e^{-\jmath 2\pi k/8} X_2(k) \tag{7.84}$$

$$= X_1(k) - W_8^k X_2(k) \tag{7.85}$$

Therefore, for $k = 0, 1, 2, 3$

$$X(k) = X_1(k) + W_8^k X_2(k) \tag{7.86}$$

$$X(k+4) = X_1(k) - W_8^k X_2(k) \tag{7.87}$$

which is the same as

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} + \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.88}$$

$$\begin{bmatrix} X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} - \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.89}$$

Similarly, we can divide $x_1(n)$ into

$$x_3(n) = \{x(0), x(4)\} \tag{7.90}$$

$$x_4(n) = \{x(2), x(6)\} \tag{7.91}$$

i.e.,

$$\begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} = \mathbf{F}_2 \begin{bmatrix} x(0) \\ x(4) \end{bmatrix} \tag{7.92}$$

$$\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} = \mathbf{F}_2 \begin{bmatrix} x(2) \\ x(6) \end{bmatrix} \tag{7.93}$$

to get

$$X_1(k) = X_3(k) + W_4^k X_4(k) \tag{7.94}$$

$$X_1(k+2) = X_3(k) - W_4^k X_4(k) \tag{7.95}$$

for $k = 0, 1$

$$\begin{bmatrix} X_1(0) \\ X_1(1) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.96}$$

$$\begin{bmatrix} X_1(2) \\ X_1(3) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.97}$$

And on dividing $x_2(n)$ into

$$x_5(n) = \{x(1), x(5)\} \tag{7.98}$$

$$x_6(n) = \{x(3), x(7)\} \tag{7.99}$$

i.e.,

$$\begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} = \mathbf{F}_2 \begin{bmatrix} x(1) \\ x(5) \end{bmatrix} \tag{7.100}$$

$$\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} = \mathbf{F}_2 \begin{bmatrix} x(3) \\ x(7) \end{bmatrix} \tag{7.101}$$

to get

$$X_2(k) = X_5(k) + W_4^k X_6(k) \tag{7.102}$$

$$X_2(k+2) = X_5(k) - W_4^k X_6(k) \tag{7.103}$$

for $k = 0, 1$

$$\begin{bmatrix} X_2(0) \\ X_2(1) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.104}$$

$$\begin{bmatrix} X_2(2) \\ X_2(3) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.105}$$

7.11 For

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{7.106}$$

compute the DFT using (7.53)
Solution: Download the following Python code that plots Fig. 7.11.

```
wget https://github.com/Donal−08/
    EE3900/raw/main/Assignment_1
    /codes/7.11.py
```

Run the code by executing

```
python 7.11.py
```

7.12 Repeat the above exercise using the FFT after zero padding **x**.
Solution: Download the following Python code that plots Fig. 7.12.

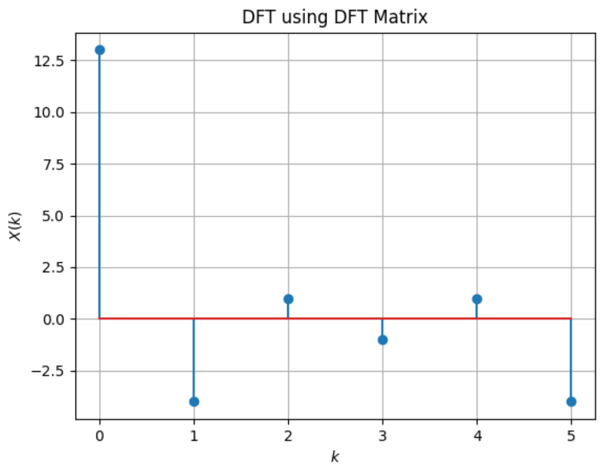

Fig. 7.11: Plot of DFT of **x**

```
wget https://github.
    com/Donal−08/
    EE3900/raw/
    main/
    Assignment_1/
    codes/7.12.py
```

Run the code by executing

```
python 7.12.py
```

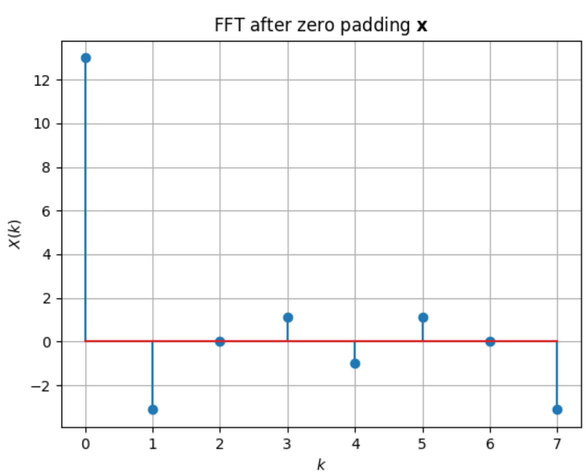

Fig. 7.12: Plot of the fast fourier transform of **x** after zero padding

7.13 Write a C program to compute the 8-point FFT.
Solution: Download the following C codes that generate the values of $X(k)$ using 8-point FFT

```
wget https://github.
    com/Donal−08/
    EE3900/raw/
    main/
    Assignment_1/
    codes/header.h
wget https://github.
    com/Donal−08/
    EE3900/raw/
    main/
    Assignment_1/
    codes/7.13.c
```

Compile and run the C program by executing the following

```
cc −lm 7.13.c
./a.out
```

Download the following Python code that plots Fig. 7.13 using the data generated by the above C code

```
wget https://github.
    com/Donal−08/
    EE3900/raw/
    main/
    Assignment_1/
    codes/7.13.py
```

Run the code by executing

```
python 7.13.py
```

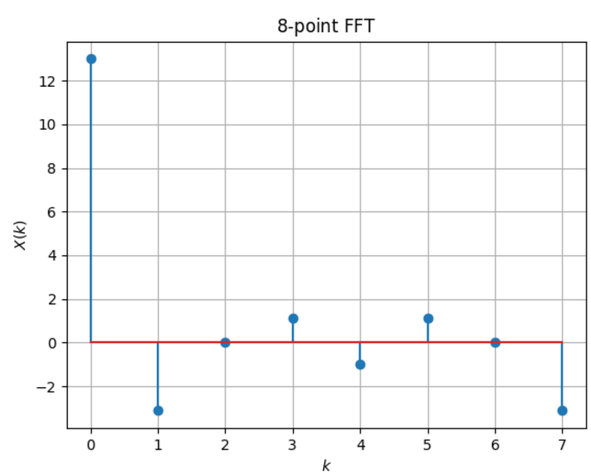

Fig. 7.13: Plot of **X** by 8-point FFT

8 Exercises

Answer the following questions by looking at the python code in Problem 2.3

8.1 The command

```
output_signal = signal.
    lfilter(b, a, input_signal
    )
```

in Problem 2.3 is executed through the following difference equation

$$\sum_{m=0}^{M} a(m)\, y(n-m) = \sum_{k=0}^{N} b(k)\, x(n-k) \tag{8.1}$$

where the input signal is $x(n)$ and the output signal is $y(n)$ with initial values all 0. Replace **signal.filtfilt** with your own routine and verify.
Solution: On taking the Z-transform on both sides of the difference equation

$$\sum_{m=0}^{M} a(m)\, z^{-m} Y(z) = \sum_{k=0}^{N} b(k)\, z^{-k} X(z) \tag{8.2}$$

$$\implies H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{N} b(k)\, z^{-k}}{\sum_{m=0}^{M} a(m)\, z^{-m}} \tag{8.3}$$

For obtaining the discrete Fourier transform, put $z = \mathrm{j}\frac{2\pi i}{I}$ where I is the length of the input signal and $i = 0, 1, \dots, I-1$
Download the following Python code that does the above

```
wget https://github.com/
    Donal−08/EE3900/raw/
    main/Assignment_1/
    codes/7.1.py
```

Run the code by executing

```
python 7.1.py
```

8.2 Repeat all the exercises in the previous sections for the above a and b
Solution: The polynomial coefficients obtained are

$$\mathbf{a} = \begin{pmatrix} 1.000 \\ -2.519 \\ 2.561 \\ -1.206 \\ 0.220 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 0.003 \\ 0.014 \\ 0.021 \\ 0.014 \\ 0.003 \end{pmatrix} \tag{8.4}$$

The difference equation is then given by

$$\mathbf{a}^\top \mathbf{y} = \mathbf{b}^\top \mathbf{x} \tag{8.5}$$

where

$$\mathbf{y} = \begin{pmatrix} y(n) \\ y(n-1) \\ y(n-2) \\ y(n-3) \\ y(n-4) \end{pmatrix} \qquad \mathbf{x} = \begin{pmatrix} x(n) \\ x(n-1) \\ x(n-2) \\ x(n-3) \\ x(n-4) \end{pmatrix} \tag{8.6}$$

We have

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{N} b(k)\, z^{-k}}{\sum_{m=0}^{M} a(m)\, z^{-m}} \tag{8.7}$$

By using partial fraction decomposition, we can write this as

$$H(z) = \sum_i \frac{r(i)}{1 - p(i) z^{-1}} + \sum_j k(j) z^{-j} \tag{8.8}$$

On taking the inverse Z-transform on both sides by using (4.23)

$$H(z) \overset{\mathcal{Z}}{\rightleftharpoons} h(n) \tag{8.9}$$

$$\frac{1}{1 - p(i) z^{-1}} \overset{\mathcal{Z}}{\rightleftharpoons} (p(i))^n\, u(n) \tag{8.10}$$

$$z^{-j} \overset{\mathcal{Z}}{\rightleftharpoons} \delta(n-j) \tag{8.11}$$

Thus

$$h(n) = \sum_i r(i)\, (p(i))^n\, u(n) + \sum_j k(j) \delta(n-j) \tag{8.12}$$

Download the following Python code

```
wget https://github.com/Donal−08/EE3900/
    raw/main/Assignment_1/codes/7.2.py
```

Run the code by executing

```
python 7.2.py
```

The above code outputs the values of $r(i), p(i), k(i)$

$$\begin{aligned} h(n) &= (0.24 - 0.71\mathrm{j})(0.56 + 0.14\mathrm{j})^n u(n) \\ &+ (0.24 + 0.71\mathrm{j})(0.56 - 0.14\mathrm{j})^n u(n) \\ &+ (-0.25 + 0.12\mathrm{j})(0.70 + 0.41\mathrm{j})^n u(n) \\ &+ (-0.25 - 0.12\mathrm{j})(0.70 - 0.41\mathrm{j})^n u(n) \\ &+ 0.016\delta(n) \end{aligned} \tag{8.13}$$

8.3 What is the sampling frequency of the input signal?
Solution: The sampling frequency of the input signal is 44 100 Hz = 44.1 kHz

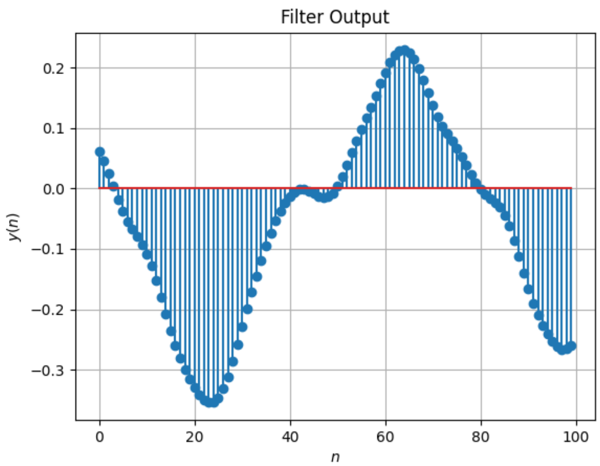

Fig. 8.2: Plot of $y(n)$

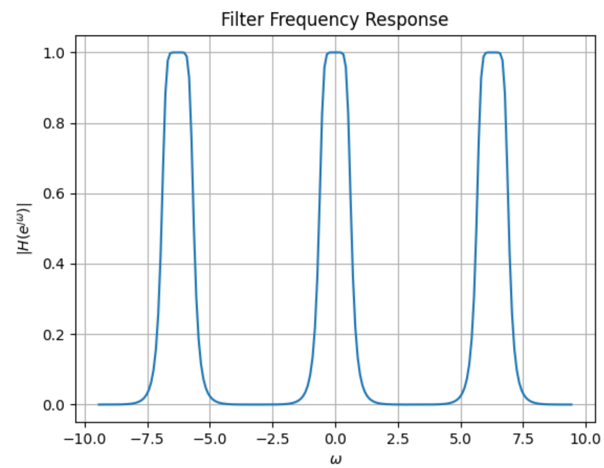

Fig. 8.2: Plot of $|H(e^{\jmath\omega})|$

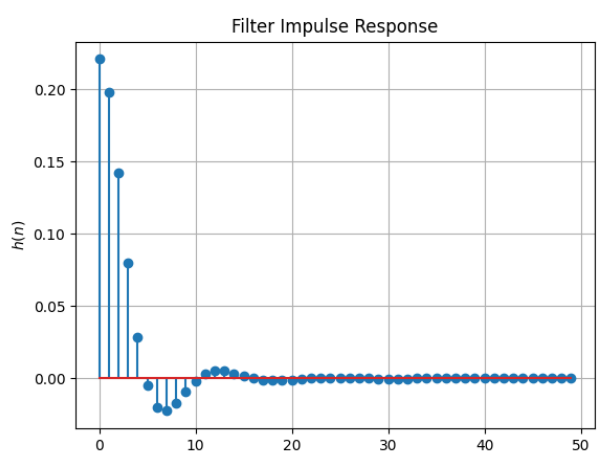

Fig. 8.2: Plot of $h(n)$

8.4 What is the type, order and cutoff frequency of the above Butterworth filter?
Solution:
Type: low-pass
Order: 4
Cutoff frequency: 4000 Hz = 4 kHz

8.5 Modify the code with different input parameters to get the best possible output.
Solution:
Order: 10
Cutoff frequency: 3000 Hz = 3 kHz