Pingala Series

Donal Loitam - AI21BTECH11009

Abstract—**This manual provides a simple introduction to Transforms**

1 JEE 2019

Let

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \geq 1 \tag{1.1}$$

$$b_n = a_{n-1} + a_{n+1}, \quad n \geq 2, \quad b_1 = 1 \tag{1.2}$$

Verify the following using a python code.

1.1

$$\sum_{k=1}^{n} a_k = a_{n+2} - 1, \quad n \geq 1 \tag{1.3}$$

Solution: Download and run the following code.

```
wget https://github.com/Donal−08/EE3900/
    blob/main/Pingala/codes/1.1.py
```

run the above code using the command

```
python3 1.1.py
```

1.2

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{10}{89} \tag{1.4}$$

Solution: Download and run the following code.

```
wget https://github.com/Donal−08/EE3900/
    blob/main/Pingala/codes/1.2.py
```

run the above code using the command

```
python3 1.2.py
```

1.3

$$b_n = \alpha^n + \beta^n, \quad n \geq 1 \tag{1.5}$$

Solution: Download and run the following code.

```
wget https://github.com/Donal−08/EE3900/
    blob/main/Pingala/codes/1.3.py
```

run the above code using the command

```
python3 1.3.py
```

1.4

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{8}{89} \tag{1.6}$$

Solution: Download and run the following code.

```
wget https://github.com/Donal−08/EE3900/
    blob/main/Pingala/codes/1.4.py
```

run the above code using the command

```
python3 1.4.py
```

2 PINGALA SERIES

2.1 The *one sided* Z-transform of $x(n)$ is defined as

$$X^+(z) = \sum_{n=0}^{\infty} x(n)z^{-n}, \quad z \in \mathbb{C} \tag{2.1}$$

2.2 The *Pingala* series is generated using the difference equation

$$x(n + 2) = x(n + 1) + x(n), \quad x(0) = x(1) = 1, n \geq 0 \tag{2.2}$$

Generate a stem plot for $x(n)$.
Solution: Download and run the following code.Below code plots fig(2.2)

```
wget https://github.com/Donal−08/EE3900/
    blob/main/Pingala/Codes/2.2.py
```

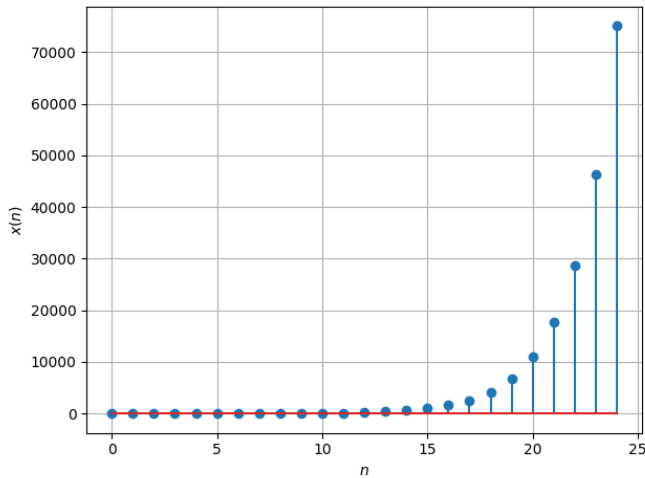run the above code using the command

```
python3 2.2.py
```


Fig. 2.2: Stem plot of $x(n)$

2.3 Find $X^+(z)$.
Solution: from eqn(2.1)

$$x(n) = x(n + 2) - x(n + 1) \qquad (2.3)$$

Taking one-sided z-transform on both sides.

$$X^+(z) = Z[x(n + 2)] - Z[x(n + 1)] \qquad (2.4)$$
$$(2.5)$$

Now, the R.H.S can be computed as :

$$Z[x(n + 2)] - Z[x(n + 1)] \qquad (2.6)$$

$$= \sum_{n=0}^{\infty} x(n + 2)z^{-n} - \sum_{n=0}^{\infty} x(n + 1)z^{-n} \qquad (2.7)$$

$$= \sum_{t=-2}^{\infty} x(t)z^{2-t} - \sum_{k=-1}^{\infty} x(k)z^{1-k} \qquad (2.8)$$

$$= z^2 \sum_{t=0}^{\infty} x(t)z^{-t} - z \sum_{k=0}^{\infty} x(k)z^{-k} - z^2 \qquad (2.9)$$

$$(2.10)$$

$$\implies X^+(z) = z^2 X^+(z) - zX^+(z) + z^2 \qquad (2.11)$$

$$X^+(Z)[z^2 - z - 1] = z^2 \qquad (2.12)$$

$$X^+(z) = \frac{z^2}{z^2 - z - 1} \qquad (2.13)$$

$$X^+(z) = \frac{1}{1 - z^{-1} - z^{-2}} \qquad (2.14)$$

Let α and β be the roots of the equation $1 - z^{-1} - z^{-2}$.

$$X^+(z) = \frac{1}{(1 - \alpha z^{-1})(1 - \beta z^{-1})}, \qquad |z| > \alpha \qquad (2.15)$$

2.4 Find $x(n)$.
Solution: Download and run the following code.

```
wget https://github.com/Donal−08/EE3900/
    blob/main/Pingala/codes/2.4.py
```

run the above code using the command

```
python3 2.4.py
```

Expanding $X^+(z)$ in (2.15) using partial fractions, we get

$$X^+(z) = \frac{1}{(\alpha - \beta)z^{-1}} \left[\frac{1}{1 - \alpha z^{-1}} - \frac{1}{1 - \beta z^{-1}} \right] \qquad (2.16)$$

$$= \frac{1}{(\alpha - \beta)} \sum_{n=0}^{\infty} (\alpha^n - \beta^n) z^{-n+1} \qquad (2.17)$$

$$= \sum_{n=1}^{\infty} \frac{\alpha^n - \beta^n}{\alpha - \beta} z^{-n+1} \qquad (2.18)$$

$$= \sum_{k=0}^{\infty} \frac{\alpha^{k+1} - \beta^{k+1}}{\alpha - \beta} z^{-k} \qquad (2.19)$$

where $k := n + 1$. Thus,

$$x(n) = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} u(n) = a_{n+1} u(n) \qquad (2.20)$$

2.5 Sketch

$$y(n) = x(n - 1) + x(n + 1), \qquad n \geq 0 \qquad (2.21)$$

Solution: Download and run the following code. Below code plots fig(2.5)

```
wget https://github.com/Donal−08/EE3900/
    blob/main/Pingala/Codes/2.5.py
```

run the above code using the command

```
python3 2.5.py
```

2.6 Find $Y^+(z)$.
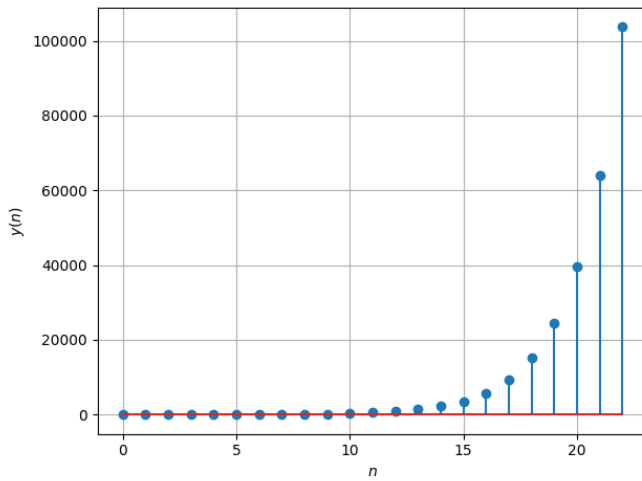**Solution:** Taking the one-sided Z-transform on

Fig. 2.5: Stem plot of $y(n)$

both sides of (2.21),

$$\mathcal{Z}^+[y(n)] = \mathcal{Z}^+[x(n+1)] + \mathcal{Z}^+[x(n-1)] \quad (2.22)$$

$$Y^+(z) = zX^+(z) - zx(0) + z^{-1}X^+(z) + zx(-1) \quad (2.23)$$

$$= \frac{z}{1 - z^{-1} - z^{-2}} - zx(0) + \frac{z^{-1}}{1 - z^{-1} - z^{-2}} + zx(-1) \quad (2.24)$$

$$= \frac{z + z^{-1}}{1 - z^{-1} - z^{-2}} - z \quad (2.25)$$

$$= \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}}, \quad |z| > \alpha \quad (2.26)$$

2.7 Find $y(n)$.

Solution: Download and run the following code.

```
wget https://github.com/Donal−08/EE3900/
    blob/main/Pingala/codes/2.7.py
```

run the above code using the command

```
python3 2.7.py
```

Using (2.15), and since $x(n) = 0 \; \forall \; n < 0$.

$$Y^+(z) = (1 + 2z^{-1}) \sum_{n=0}^{\infty} x(n)z^{-n} \quad (2.27)$$

$$= \sum_{n=0}^{\infty} x(n)z^{-n} + \sum_{n=1}^{\infty} 2x(n-1)z^{-n} \quad (2.28)$$

$$= x(0) + \sum_{n=1}^{\infty} (x(n) + 2x(n-1)) z^{-n} \quad (2.29)$$

Thus, $y(0) = x(0) = 1$ and for $n \geq 1$, using the fact that α and β are the roots of the equation $z^2 - z - 1 = 0$,

$$Y(z) = \sum_{n=1}^{\infty} (x(n) + 2x(n-1)) z^{-n} \quad (2.30)$$

$$\therefore y(n) = x(n) + 2x(n-1) \quad (2.31)$$

$$= \frac{\left(\alpha^{n+1} - \beta^{n+1}\right) + (2\alpha^n + 2\beta^n)}{\alpha - \beta} \quad (2.32)$$

$$= \frac{\left(\alpha^{n+2} - \beta^{n+2}\right) + (\alpha^n + \beta^n)}{\alpha - \beta} \quad (2.33)$$

$$= \frac{\left(\alpha^{n+2} - \beta^{n+2}\right) - \alpha\beta\left(\alpha^n + \beta^n\right)}{\alpha - \beta} \quad (2.34)$$

$$= \frac{(\alpha - \beta)\left(\alpha^{n+1} + \beta^{n+1}\right)}{\alpha - \beta} \quad (2.35)$$

$$= \alpha^{n+1} + \beta^{n+1} \quad (2.36)$$

Thus, $y(n) = \alpha^{n+1} + \beta^{n+1}$ for $n \geq 0$ as $\alpha + \beta = 1$. Comparing (2.33) with the definition of b_n, we see that $y(n) = b_{n+1}$. Hence, $b_n = \alpha^n + \beta^n$.

3 Power of the Z transform

3.1 Show that

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} x(k) = x(n) * u(n-1) \quad (3.1)$$

Solution: From (2.20), and noting that $x(n) = 0 \ \forall \ n < 0$,

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} x(k) \tag{3.2}$$

$$= \sum_{k=-\infty}^{n-1} x(k) \tag{3.3}$$

$$= \sum_{k=-\infty}^{\infty} x(k)u(n-1-k) \tag{3.4}$$

$$= x(n) * u(n-1) \tag{3.5}$$

3.2 Show that

$$a_{n+2} - 1, \quad n \geq 1 \tag{3.6}$$

can be expressed as

$$[x(n+1) - 1] u(n) \tag{3.7}$$

Solution: From (2.20),

$$a_{n+2} - 1 = [x(n+1) - 1], \quad n \geq 0 \tag{3.8}$$

and so, using the definition of $u(n)$,

$$a_{n+2} - 1 = [x(n+1) - 1] u(n) \tag{3.9}$$

3.3 Show that

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} = \frac{1}{10} X^+(10) \tag{3.10}$$

Solution:

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{a_{k+1}}{10^k} \tag{3.11}$$

$$= \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} \tag{3.12}$$

$$= \frac{1}{10} X^+(10) \tag{3.13}$$

$$= \frac{1}{10} \times \frac{100}{89} = \frac{10}{89} \tag{3.14}$$

3.4 Show that

$$\alpha^n + \beta^n, \quad n \geq 1 \tag{3.15}$$

can be expressed as

$$w(n) = \left(\alpha^{n+1} + \beta^{n+1}\right) u(n) \tag{3.16}$$

and find $W(z)$.
Solution: Putting $n = k+1$ in (3.15) and using

the definition of $u(n)$,

$$\alpha^n + \beta^n = \left(\alpha^{k+1} + \beta^{k+1}\right) u(k) \tag{3.17}$$

Hence, (3.15) can be expressed as

$$w(n) = \left(\alpha^{n+1} + \beta^{n+1}\right) u(n) = y(n) \tag{3.18}$$

Therefore,

$$W(z) = Y(z) = \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}} \tag{3.19}$$

3.5 Show that

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} = \frac{1}{10} Y^+(10) \tag{3.20}$$

Solution:

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{b_{k+1}}{10^k} \tag{3.21}$$

$$= \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} \tag{3.22}$$

$$= \frac{1}{10} Y^+(10) \tag{3.23}$$

$$= \frac{1}{10} \times \frac{120}{89} = \frac{12}{89} \tag{3.24}$$

3.6 Solve the JEE 2019 problem.
Which of the following options is/are correct?
a)

$$\sum_{k=1}^{n} a_k = a_{n+2} - 1, \quad n \geq 1 \tag{3.25}$$

b)

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{10}{89} \tag{3.26}$$

c)

$$b_n = \alpha^n + \beta^n, \quad n \geq 1 \tag{3.27}$$

d)

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{8}{89} \tag{3.28}$$

Solution: We know that

$$\sum_{k=1}^{n} a_k = x(n) * u(n-1) \tag{3.29}$$

But

$$x(n) * u(n-1) \overset{\mathcal{Z}}{\rightleftharpoons} X(z)z^{-1}U(z) \qquad (3.30)$$

$$= \frac{z^{-1}}{\left(1 - z^{-1} - z^{-2}\right)\left(1 - z^{-1}\right)} \qquad (3.31)$$

$$= z\left[\frac{1}{1 - z^{-1} - z^{-2}} - \frac{1}{1 - z^{-1}}\right] \qquad (3.32)$$

$$\overset{\mathcal{Z}}{\rightleftharpoons} z\sum_{n=0}^{\infty}(x(n) - 1)z^{-n} \qquad (3.33)$$

$$= \sum_{n=0}^{\infty}(x(n) - 1)z^{-n+1} \qquad (3.34)$$

$$= \sum_{n=0}^{\infty}(x(n+1) - 1)z^{-n} \qquad (3.35)$$

$$\qquad (3.36)$$

From (2.20), we get

$$\sum_{k=1}^{n} a_k = a_{n+2} - 1 \qquad (3.37)$$

We have already done proofs of remaining options in order in the problems (3.3), (2.7), (3.5). Therefore, options 1, 2, and 3 are correct and option 4 is incorrect.

The same thing can be verified from the code

```
wget https://github.com/Donal−08/EE3900/
    blob/main/Pingala/codes/3.6.py
```

run the code by using the command

```
python3 3.6.py
```

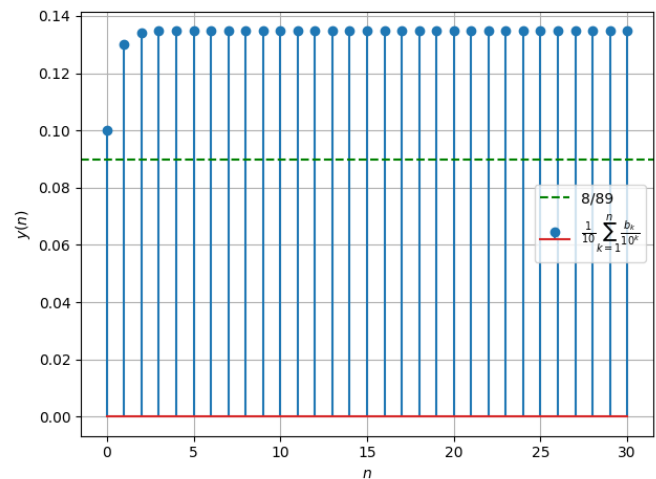Also, the stem plot given verifies that option 4 is incorrect as the series doesn't converge to 8/89

Fig. 3.6: Stem plot of $x(n)$