Assignment: Random Variables

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1 SOLUTION:

Let U be a uniform random variable between 0 and 1.

(1.1) Solution: Download the following files and execute the C program

The C code - exrand.c

The Header - coeffs.h

(1.2) Solution: The following code plots Fig. 0

Python code - cdf_plot.py

(1.3) U is an uniform random variable.So,

$P_U(x_i) = P_U(x_j) = \frac{1}{b-a}, \ \forall i,j \quad , U \in [a,b]$

$F_U(x)$ = CDF of $P_U(x)$

$f_U(x) = \frac{1}{b-a} = 1 \qquad [\because \text{ here } a = 0, b = 1]$

$$\therefore F_U(x) = \int_0^x f_U(x)dx \tag{1.1}$$

$$= \int_0^x dx \tag{1.2}$$

$$= x \tag{1.3}$$

$$\therefore F_U(x) = x \qquad \forall\, 0 \le x \le 1 \tag{1.4}$$

Therefore,

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases} \tag{1.5}$$

(1.4) Solution: Download and run the following C code.

Mean and variance - mean_uni.c

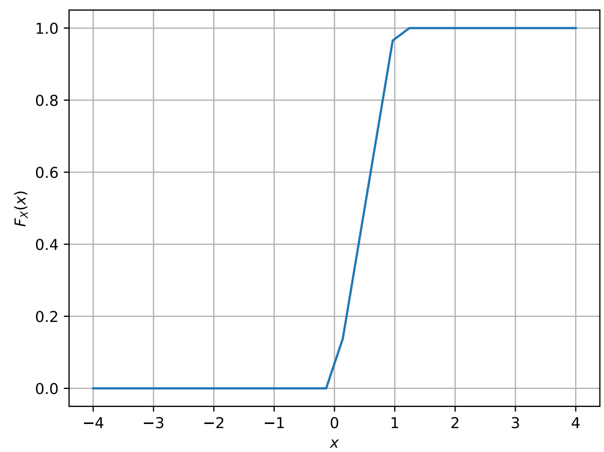

Fig. 0. CDF for (1)

(1.5) From 1.4, $\quad dF_U(x) = dx$

$$E[U^k] = \int_{-\infty}^{\infty} x^k dx \tag{1.6}$$

$$\therefore E[U] = \int_0^1 x dx = \frac{1}{2} = \boxed{0.5} \tag{1.7}$$

$$E[U^2] = \int_0^1 x^2 dx = \frac{1}{3} \tag{1.8}$$

$$\because P_X(x) = 0, \ \forall x \in (-\infty, 0) \cap (1, \infty) \tag{1.9}$$

$$Var(X) = E[U^2] - (E[U])^2 = \frac{1}{12} = \boxed{0.083} \tag{1.10}$$

We can observe that these theoretical values are quite close to what we obtained numerically

2.2 Load gau.dat in python and plot the empirical CDF of X using the sample in gau.dat. What properties does a CDF have?

Solution: The CDF of X is plotted in Fig. 0

properties of cdf

- $F_X(x)$ is a nondecreasing function of x for $-\infty < x < \infty$.

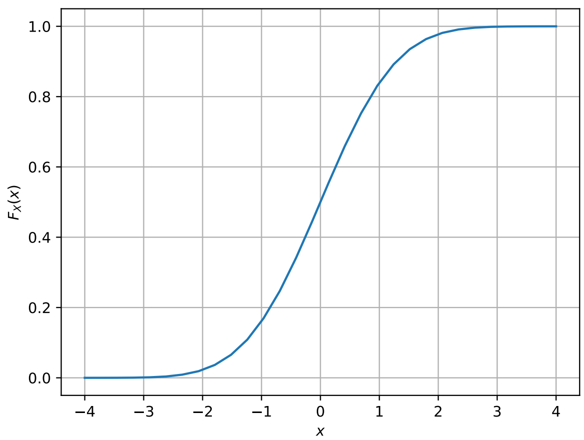

Fig. 0. The CDF of X

- The CDF, $F_X(x)$ ranges from 0 to 1. This makes sense since $F_X(x)$ is a probability.
- If the maximum value of X is b, then $F_X(b)$ = 1

(2.3)

1) PDF is symmetric about $x = 0$
2) Bell shaped Graph
3) mean of graph is situated at the apex point of the bell

(2.5) Given, $p_X(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$
$F_X(x) = \int_{-\infty}^{x} p_X(x)dx$

$$= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx \tag{1.11}$$

$$= \frac{1}{2} erf\left(\frac{x}{\sqrt{2}}\right) \tag{1.12}$$

$$E[x] = \int_{-\infty}^{\infty} x p_X(x)dx \tag{1.13}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x e^{-\frac{-x^2}{2}} \tag{1.14}$$

$$\because x e^{-\frac{-x^2}{2}} \text{is a odd function,} \tag{1.15}$$

$$\therefore E[X] = 0$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 p_X(x)dx \tag{1.16}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x(x e^{-\frac{-x^2}{2}})dx \tag{1.17}$$

Using integration by parts:

$$= \frac{1}{\sqrt{2\pi}} \left[-x e^{-\frac{-x^2}{2}} + \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx \right]_{-\infty}^{\infty} \tag{1.18}$$

$$= 1 \tag{1.19}$$

$$\text{Substituting limits we get, } E[X^2] = 1 \tag{1.20}$$

$$\therefore \text{Variance} = E[X^2] - E[X]^2 = 1 - 0 = 1 \tag{1.21}$$

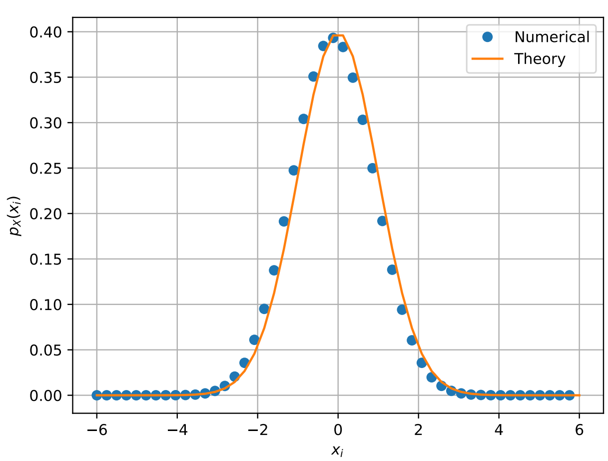

Fig. 3. PDF for (2)

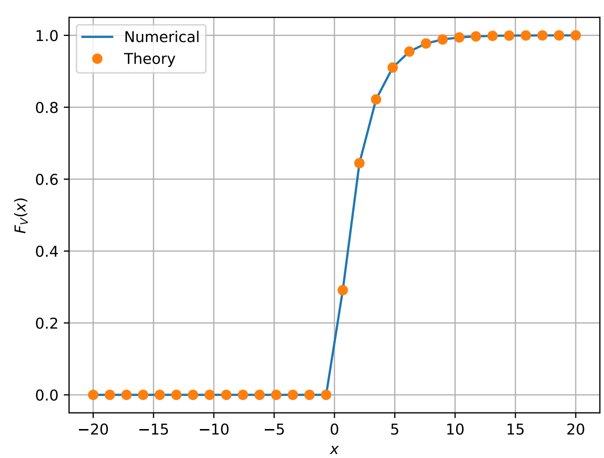

Fig. 3. CDF for (3)

(3.2)

$$F_V(x) = P(V \leq x) \tag{1.22}$$
$$= P(-2ln(1-U) \leq x) \tag{1.23}$$
$$= P(1 - e^{\frac{-x}{2}} \geq U) \tag{1.24}$$
$$\because P(U < x) = \int_0^x dx = x \tag{1.25}$$
$$\therefore \Pr\left(1 - e^{\frac{-x}{2}} \geq U\right) = 1 - e^{\frac{-x}{2}}, \forall x \geq 0 \tag{1.26}$$

So, $F_V(x) = 1 - e^{\frac{-x}{2}}$