Assignment 10

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TABLE OF CONTENTS

Problem

Ex 6.2: X and Y are independent and uniform in the interval $(0, a)$. Find the p.d.f of :-
(a) $\frac{X}{Y}$
(b) $\frac{Y}{(X+Y)}$
(c) $|X - Y|$

Solution - Part(a)

$$f_{XY}(x,y) = f_X(x)f_Y(y) = \frac{1}{a^2}, \qquad 0 < x \leq a, \quad 0 < y \leq a$$

(a) $F_Z(z) = \Pr\left(\frac{X}{Y} \leq z\right) = \Pr\left(X \leq zY\right)$

(i) $z < 1$

$$F_Z(z) = \Pr\left(X \leq zY\right) \tag{1}$$

$$= \int_0^a \int_0^{zy} \frac{1}{a}.\frac{1}{a}\,dx\,dy = \frac{z}{2}, \quad z \leq 1 \tag{2}$$

(i) $z < 1$

$$F_Z(z) = \Pr\left(X \leq zY\right) \tag{3}$$

$$= 1 - \int_0^a \int_0^{x/z} \frac{1}{a}.\frac{1}{a}\,dy\,dx \tag{4}$$

$$= 1 - \int_0^1 \frac{x}{2}\,dx = 1 - \frac{1}{2z}, \quad z > 1 \tag{5}$$

Solution - Part(a)(Contd.)

(a)

Hence, $f_Z(z)$ can be written as :-

$$f_Z(z) = \begin{cases} \frac{1}{2}, & z \leq 1 \\ \frac{1}{2z^2}, & z > 1 \end{cases}$$

Solution - Part(b)

(b)

$$F_Z(z) = \Pr(Z \leq z) = \Pr\left(\frac{Y}{X+Y} \leq z\right) \tag{6}$$
$$= \Pr\left(\frac{X}{Y} \geq \frac{1}{z} - 1\right) = 1 - \Pr\left(\frac{X}{Y} \leq \frac{1-z}{z}\right) \tag{7}$$
$$= \begin{cases} \frac{1}{2}\left(\frac{z}{1-z}\right), & 0 < z \leq 1/2 \\ 1 - \frac{1}{2}\left(\frac{1-z}{z}\right), & 1/2 < z < 1 \end{cases} \tag{8}$$

$$f_Z(z) = \begin{cases} \dfrac{1}{2(1-z)^2}, & 0 < z \leq 1/2 \\ \dfrac{1}{2z^2}, & 1/2 < z < 1 \end{cases}$$

Solution - Part(c)

(c)

$$F_Z(z) = \Pr(Z \leq z) = \Pr(|X - Y| \leq z) \tag{9}$$

$$= \Pr(\{|X - Y| \leq z\}\{X \geq Y\}) + \Pr(\{|X - Y| \leq z\}\{X < Y\}) \tag{10}$$

$$= \Pr(X - Y \leq z, X \geq Y) + \Pr(Y - X \leq z, X < Y) \tag{11}$$

$$= \int_0^{\infty}\int_y^{y+z} f_{XY}(x, y)\, dx\, dy + \int_0^{\infty}\int_x^{x+z} f_{XY}(x, y)\, dy\, dx \tag{12}$$

$$= \int_0^{\infty}\int_y^{y+z} f_{XY}(x, y)\, dx\, dy + \int_0^{\infty}\int_y^{y+z} f_{XY}(y, x)\, dx\, dy \tag{13}$$

$$= \int_0^{\infty}\int_y^{y+z} \{f_{XY}(x, y) + f_{XY}(y, x)\}\, dx\, dy \tag{14}$$

Solution - Part(c)(Contd.)

In general,

$$f_Z(z) = \int_0^{\infty} \frac{d}{dz} \int_y^{y+z} f_{XY}(x,y) + f_{XY}(y,x)\, dx\, dy \tag{15}$$

$$= \int_0^{\infty} \left\{ f_{XY}(y+z,y) + f_{XY}(y,y+z) \right\}\, dy \tag{16}$$

Here, $X \sim U(0,a), \quad Y \sim U(0,a)$

$$F_Z(z) = 1 - \frac{1}{a_2}.2.\frac{(a-z)^2}{2} = 1 - \left(1 - \frac{z}{a}\right)^2 \tag{17}$$

$$f_Z(z) = \frac{2}{a}\left(1 - \frac{z}{a}\right) \qquad 0 \le z \le a \tag{18}$$

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