Uncertainty

CS4881 Artificial Intelligence

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Outline

- Uncertainty
- Probability
- Syntax and Semantics
- Inference
- Independence and Bayes' Rule

Uncertainty

Let action A_t = leave for airport t minutes before flight

Will A_t get me there on time?

Uncertainty

Let action A_t = leave for airport *t* minutes before flight
Will A_t get me there on time?

Problems:

1. partial observability (road state, other drivers' plans, etc.)
2. noisy sensors (traffic reports), Google Maps traffic layer
3. uncertainty in action outcomes (flat tire, etc.)
4. immense complexity of modeling and predicting traffic

Hence a purely logical approach either:

1. risks falsehood: "A_{25} will get me there on time", or
2. leads to conclusions that are too weak for decision making:

"A_{25} will get me there on time if there's *no accident on the bridge* and it *doesn't rain* and *my tires remain intact*, etc., etc."

(A_{1440} might reasonably be said to get me there on time, but I'd have to stay overnight in the airport …)

Methods for handling uncertainty

- Default or non-monotonic logic:
  - Assume my car does not have a flat tire
  - Assume A_{25} works unless contradicted by evidence
- Issues: What assumptions are reasonable? How to handle contradiction?

- Rules with fudge factors:
  - $A_{25} \mapsto_{0.3}$ get there on time
  - $Sprinkler \mapsto_{0.99} WetGrass$
  - $WetGrass \mapsto_{0.7} Rain$
- Issues: Problems with combination, e.g., *Sprinkler* causes *Rain*??

- Probability
  - Model agent's *degree of belief*
  - Given the available evidence, A_{25} will get me there on time with probability 0.04

Probability

Probabilistic assertions summarize effects of:

- laziness: failure to enumerate exceptions, qualifications, etc.
- ignorance: lack of relevant facts, initial conditions, etc.

Subjective probability:

- Probabilities relate propositions to agent's own state of knowledge (*belief*)

  e.g., $P(A_{25} \mid \text{no reported accidents}) = 0.06$

These are not assertions about the world

Probabilities of propositions change with new evidence:

e.g., $P(A_{25} \mid \text{no reported accidents, 5 a.m.}) = 0.15$

Making decisions under uncertainty

Suppose I believe the following:

$P(A_{25}$ gets me there on time | …) = 0.04
$P(A_{90}$ gets me there on time | …) = 0.70
$P(A_{120}$ gets me there on time | …) = 0.95
$P(A_{1440}$ gets me there on time | …) = 0.9999

- Which action to choose?

  Depends on my *preferences* for missing flight vs. time spent waiting, etc.
  - Utility theory is used to represent and infer preferences
  - Decision theory = probability theory + utility theory

Syntax

- Basic element: *random variable*

- Similar to propositional logic: possible worlds defined by assignment of values to random variables.

- Boolean random variables
  e.g., *Cavity* (do I have a cavity?)
- Discrete random variables
  e.g., *Weather* is one of <*sunny, rainy, cloudy, snow*>

- Domain values must be exhaustive, mutually exclusive, and sum to 1.0

- Elementary proposition constructed by assignment of a value to a random variable: e.g., *Weather = sunny*, *Cavity = false* (abbreviated as ¬*cavity*)

- Complex propositions formed from elementary propositions and standard logical connectives e.g., *Weather = sunny ^ Cavity = false*

Syntax

- Atomic event: A complete specification of the state of the world about which the agent is uncertain.

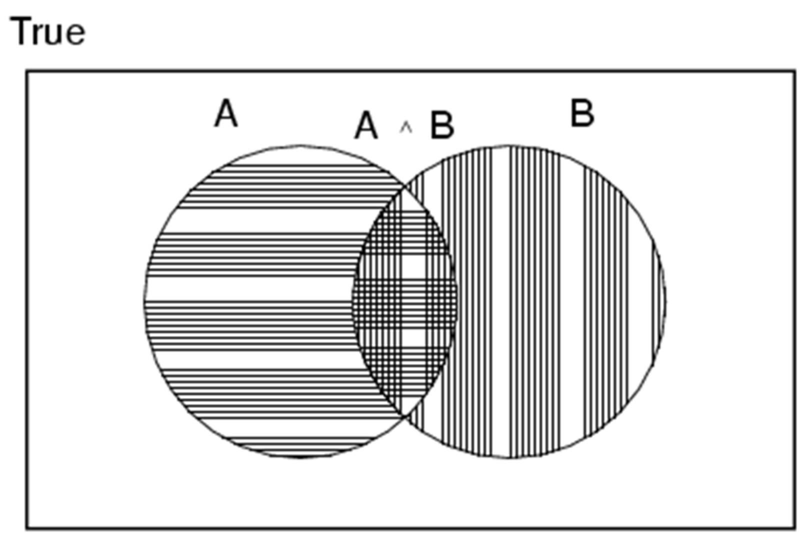
  E.g., if the world consists of only two Boolean variables *Cavity* and *Toothache*, then there are 4 distinct atomic events:

  $(\, Cavity = false \,) \wedge (\, Toothache = false \,)$
  $(\, Cavity = false \,) \wedge (\, Toothache = true \,)$
  $(\, Cavity = true \,) \wedge (\, Toothache = false \,)$
  $(\, Cavity = true \,) \wedge (\, Toothache = true \,)$

- Atomic events are *mutually exclusive* and *exhaustive*

Axioms of probability

- For any propositions *A*, *B*
  - $0 \leq P(A) \leq 1$
  - $P(true) = 1$ or $P(false) = 0$
  - $P(A \vee B) = P(A) + P(B) - P(A \wedge B)$



Prior probability

- Prior or unconditional probabilities of propositions
  e.g., **P(*Cavity* = true) = 0.1** and **P(*Weather* = sunny) = 0.72** correspond to my belief ***prior*** to arrival of any (new) evidence.

- Probability distribution gives values for all possible assignments:
  P(*Weather*) = <0.72,0.1,0.08,0.1> (normalized, i.e., sums to 1)

- Joint probability distribution for a set of random variables gives the probability of every atomic event on those random variables
  P(*Weather,Cavity*) = a 4 × 2 matrix of values:

Weather =	sunny	rainy	cloudy	snow	
Cavity = true	0.144	0.02	0.016	0.02	0.2
Cavity = false	0.576	0.08	0.064	0.08	0.8
	0.72	0.1	0.08	0.1	1

- Every question about a domain can be answered by the joint distribution
- *uncertainty.xlsx*

Conditional probability

- Conditional or posterior probabilities
  e.g., P(*cavity* | *toothache*) = 0.8
  i.e., given that *toothache* is all I know
- Notation for conditional distributions:
  P(*Cavity* | *Toothache*) = 2-element vector of 2-element vectors)
- If we know more, e.g., *cavity* is also given, then we have
  P(*cavity* | *toothache,cavity*) = 1
- New evidence may be irrelevant, allowing simplification, e.g.,
  P(*cavity* | *toothache, sunny*) = P(*cavity* | *toothache*) = 0.8
  In this case we say that cavity is independent of sunny given toothache.
- This kind of inference, sanctioned by domain knowledge, is crucial.

Conditional probability

- Definition of conditional probability:
  $P(a \mid b) = P(a \wedge b) / P(b)$ if $P(b) > 0$
- Product rule gives an alternative formulation:
  $P(a \wedge b) = P(a \mid b) P(b) = P(b \mid a) P(a)$
- A general version holds for whole distributions, e.g.,
  $\mathbf{P}(Weather, Cavity) = \mathbf{P}(Weather \mid Cavity)\, \mathbf{P}(Cavity)$
- E.g., Toothache, Cavity, Catch can be viewed as a set of 4 × 2 equations, *not* matrix multiplication.
- Chain rule is derived by successive application of product rule:

$$
\begin{aligned}
\mathbf{P}(X_1, \ldots, X_n) &= \mathbf{P}(X_1,\ldots,X_{n-1})\, \mathbf{P}(X_n \mid X_1,\ldots,X_{n-1}) \\
&= \mathbf{P}(X_1,\ldots,X_{n-2})\, \mathbf{P}(X_{n-1} \mid X_1,\ldots,X_{n-2})\, \mathbf{P}(X_n \mid X_1,\ldots,X_{n-1}) \\
&= \ldots \\
&= \prod_{i=1} \mathbf{P}(X_i \mid X_1, \ldots, X_{i-1})
\end{aligned}
$$

Inference by enumeration

- Start with the joint probability distribution:

	toothache		*¬ toothache*	
	catch	*¬ catch*	*catch*	*¬ catch*
cavity	.108	.012	.072	.008
¬ cavity	.016	.064	.144	.576

- For any proposition φ, sum the atomic events where it is true: $P(\varphi) = \Sigma_{\omega:\omega \models \varphi} P(\omega)$

Inference by enumeration

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- For any proposition φ, sum the atomic events where it is true: $P(\varphi) = \Sigma_{\omega:\omega \models \varphi} P(\omega)$
- P(*toothache*) = 0.108 + 0.012 + 0.016 + 0.064 = 0.2

Inference by enumeration

- Start with the joint probability distribution:

	toothache		$\neg$ *toothache*	
	catch	$\neg$ *catch*	*catch*	$\neg$ *catch*
cavity	.108	.012	.072	.008
$\neg$ *cavity*	.016	.064	.144	.576

- Can also compute conditional probabilities:

$$P(\neg cavity \mid toothache) = \frac{P(\neg cavity \wedge toothache)}{P(toothache)}$$

$$= \frac{0.016+0.064}{0.108 + 0.012 + 0.016 + 0.064}$$

$$= 0.4$$

Normalization

	toothache		¬ toothache	
	catch	¬ catch	catch	¬ catch
cavity	.108	.012	.072	.008
¬ cavity	.016	.064	.144	.576

- Denominator can be viewed as a normalization constant α
- α = 1/(0.108 + 0.012 + 0.016 + 0.064) = 5

P(*Cavity* | *toothache*) = α, **P**(*Cavity,toothache*)

= α, [**P**(*Cavity,toothache,catch*) + **P**(*Cavity,toothache,*¬ *catch*)]

= α, [<0.108,0.016> + <0.012,0.064>]

= α, <0.12,0.08> = <0.6,0.4>

General idea: compute distribution on query variable by fixing evidence variables and summing over hidden variables.

General Inference by enumeration

Typically, we are interested in:

- Query contains single variable **X** (e.g., cavity)
- List of evidence variables **E** (e.g., toothache)
- Let **e** be the list of observed values for the evidence variables

Let the hidden variables be **H** = (Set of all random variables) - **X** - **E**

Then the required summation of joint entries is done by summing out the hidden variables (*marginalization*):

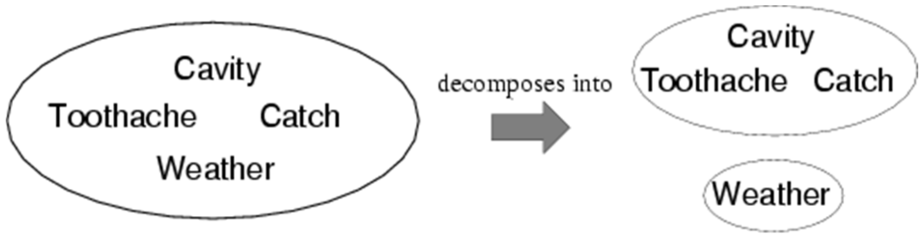
$\mathbf{P}(\mathbf{X} \mid \mathbf{E} = \mathbf{e}) = \alpha\mathbf{P}(\mathbf{X},\mathbf{E} = \mathbf{e}) = \alpha\Sigma_h\mathbf{P}(\mathbf{X},\mathbf{E}= \mathbf{e}, \mathbf{H} = \mathbf{h})$

- The terms in the summation are joint entries because **X**, **E** and **H** together exhaust the set of all random variables

- Obvious problems:
  1. Worst-case time complexity $O(d^n)$ where d is the largest *arity* domain variable
  2. Space complexity $O(d^n)$ to store the joint distribution
  3. How to find the numbers for $O(d^n)$ entries?

Independence

- A and B are independent *iff*
  $\mathbf{P}(A|B) = \mathbf{P}(A)$ or $\mathbf{P}(B|A) = \mathbf{P}(B)$ or $\mathbf{P}(A, B) = \mathbf{P}(A)\,\mathbf{P}(B)$



$\mathbf{P}(Toothache, Catch, Cavity, Weather)$
$= \mathbf{P}(Toothache, Catch, Cavity)\,\mathbf{P}(Weather)$

- 32 entries reduced to 12; for n independent biased coins, $O(2^n) \rightarrow O(n)$
  - *2^3*4 = 32, 32 element table (all possible combinations)*
  - *2^3 = 8 + one 4 element table (sunny, cloudy, rain, snow) = 12*

- Absolute independence powerful but rare

- *Dentistry is a large field with hundreds of variables, none of which are independent. What to do?*

Conditional independence

- $\mathbf{P}(\mathit{Toothache, Cavity, Catch})$ has $2^3 - 1 = 7$ independent entries
- If I have a cavity, the probability that the probe catches in it doesn't depend on whether I have a toothache:

  (1) $\mathbf{P}(\mathit{catch} \mid \mathit{toothache}, \mathit{cavity}) = \mathbf{P}(\mathit{catch} \mid \mathit{cavity})$
- The same independence holds if I haven't got a cavity:

  (2) $\mathbf{P}(\mathit{catch} \mid \mathit{toothache}, \neg \mathit{cavity}) = \mathbf{P}(\mathit{catch} \mid \neg \mathit{cavity})$
- *Catch* is conditionally independent of *Toothache* given *Cavity*:

  $\mathbf{P}(\mathit{Catch} \mid \mathit{Toothache}, \mathit{Cavity}) = \mathbf{P}(\mathit{Catch} \mid \mathit{Cavity})$
- Equivalent statements:

  $\mathbf{P}(\mathit{Toothache} \mid \mathit{Catch}, \mathit{Cavity}) = \mathbf{P}(\mathit{Toothache} \mid \mathit{Cavity})$

  $\mathbf{P}(\mathit{Toothache}, \mathit{Catch} \mid \mathit{Cavity}) = \mathbf{P}(\mathit{Toothache} \mid \mathit{Cavity})\, \mathbf{P}(\mathit{Catch} \mid \mathit{Cavity})$

Conditional independence contd.

- Write out full joint distribution using chain rule:
  P(*Toothache, Catch, Cavity*)
  = **P**(*Toothache* | *Catch, Cavity*) **P**(*Catch* | *Cavity*) **P**(*Cavity*)
  Using conditional independence:
  = **P**(*Toothache* | *Cavity*) **P**(*Catch* | *Cavity*) **P**(Cavity)
  I.e., 2 + 2 + 1 = 5 independent numbers
- In most cases, the use of conditional independence reduces the size of the representation of the joint distribution **from exponential in *n* to linear in *n***.
- Conditional independence is our most basic and robust form of knowledge about uncertain environments.

Bayes' Rule

- Product rule $P(a \wedge b) = P(a \mid b)\, P(b) = P(b \mid a)\, P(a)$
  $\Rightarrow$ Bayes' rule: $P(a \mid b) = P(b \mid a)\, P(a) / P(b)$

- or in distribution form

  $$\mathbf{P}(Y|X) = \mathbf{P}(X|Y)\, \mathbf{P}(Y) / \mathbf{P}(X) = \alpha \mathbf{P}(X|Y)\, \mathbf{P}(Y)$$

- Useful for assessing diagnostic probability from causal probability
  - $P(Cause|Effect) = P(Effect|Cause)\, P(Cause) / P(Effect)$
  - E.g., let *M* be meningitis, *S* be stiff neck:
    $P(m|s) = P(s|m)\, P(m) / P(s) = 0.8 \times 0.0001 / 0.1 = 0.0008$
  - Note: posterior probability of meningitis still very small!

Bayes' Rule and conditional independence

$\mathbf{P}(Cavity \mid toothache \wedge catch)$

$= \alpha\mathbf{P}(toothache \wedge catch \mid Cavity)\ \mathbf{P}(Cavity)$

$= \alpha\mathbf{P}(toothache \mid Cavity)\ \mathbf{P}(catch \mid Cavity)\ \mathbf{P}(Cavity)$

- This is an example of a naïve Bayes model:

  $\mathbf{P}(\text{Cause}, \text{Effect}_1, \ldots, \text{Effect}_n) = \mathbf{P}(\text{Cause})\ \pi_i \mathbf{P}(\text{Effect}_i \mid \text{Cause})$

- Total number of parameters is linear in *n*

Summary

- Probability is a rigorous formalism for uncertain knowledge.
- Joint probability distribution specifies probability of every atomic event.
- Queries can be answered by summing over atomic events.
- For nontrivial domains, we must find a way to reduce the joint size.
- Independence and conditional independence provide the tools.