THE ASTRONOMICAL JOURNAL VOLUME 81, NUMBER 10 OCTOBER 1976

Tables of functions for a spherical galaxy obeying the $r^{1/4}$ law in projection

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(Received 12 May 1976)

A spherical galaxy with reduced surface brightness, $J=B(\alpha)/B_e$, obeying the $r^{1/4}$ law, $\log J=-3.3307$ $(\alpha^{1/4}-1)$, where α is the reduced radius, $\alpha=r/r_e$ (r_e is the effective radius), is deprojected to find the corresponding space density, mass, mean density, force, potential, escape velocity, and potential energy at each point in the galaxy. Numerical tabulations to five significant figures are given for 124 points in the range $10^{-6}\leq s=R/r_e\leq 260$. In addition the projected surface brightness $B(\alpha)$ and integrated luminosity within α are tabulated for the range $10^{-6}\leq\alpha=r/r_e\leq 260$. Conversion factors to cgs units and to $\mathcal{M}_\odot$, pc, km sec^{-1}, $\mathcal{L}_\odot$ units are given. Asymptotic expansions for the space density $\rho(s)$ in the ranges $s\lesssim 10^{-4}$ and $s\gtrsim 10^{-1}$ are derived, and it is demonstrated that the projection of the expansion for $s\lesssim 10^{-1}$ is almost indistinguishable from the $r^{1/4}$ law itself, apart from a small excess of luminosity in the central regions. Formulae and numerical tables of the luminosity distribution are given for use in galaxy photometry. Relations between the total galactic mass $\mathcal{M}_T$, the effective radius r_e, the velocity dispersion σ_v, the central density ρ_c, and the mass $\mathcal{M}_N$ and radius R_N of the nucleus are derived. Here the "nucleus" is defined as the region within which stars having a velocity equal to the mean velocity dispersion in space, σ_v, will be constrained to remain. Four numerical examples are given to illustrate the orders of magnitude of the above quantities for typical elliptical galaxies from supergiant E(e.g.,M87) to compact dwarf (e.g.,M32).

INTRODUCTION

IT IS now well established that the luminosity distribution in elliptical galaxies is closely approximated by the empirical law (de Vaucouleurs 1959, 1962)

$$\log J = \log B(\alpha)/B_e = -3.33071\ (\alpha^{1/4}-1), \quad (1)$$

where $\alpha = a/a_e$ is the reduced radius (or semimajor axis) in units of the effective radius of the isophote enclosing half the total luminosity and J is the surface brightness $B(\alpha)$ normalized to its value at the effective radius, $B_e = B(1)$.

This surface luminosity distribution also describes well the bulge component of early-type spirals, the spheroidal component of D galaxies and compact dwarf elliptical systems (de Vaucouleurs 1974a). It does not apply to the disks of late-type spirals, to Magellanic irregulars or to low-density dwarf elliptical systems.

The $r^{1/4}$ law has been established for the spheroidal bulge of M31 in the range $2\times10^{-3}\leqslant\alpha\leqslant 1.1$ (de Vaucouleurs 1958), and, within the errors of roughly deconvolved photographic photometry, perhaps even to $\alpha \simeq 0$ (de Vaucouleurs 1974b). Photometry of other early-type galaxies, including lenticulars, shows similarly that the $r^{1/4}$ law describes the luminosity distribution to small values of α (de Vaucouleurs 1975), when allowance is made for the convolution effects of seeing on the central peak of luminosity. It is surprising that the empirical law should hold in the nuclear regions at $\alpha\leqslant 10^{-3}$, but there is evidence of a central spike in the luminosity distribution even slightly above that of the $r^{1/4}$ law (de Vaucouleurs and Capaccioli 1977).

Since the $r^{1/4}$ law seems to be valid over an extremely large range of α, including $\alpha\to 0$, it was considered useful to extend the work of Poveda *et al.* (1960) who tabulated galaxy models obeying in projection the $r^{1/4}$ law, for the range $10^{-2}\leqslant\alpha\leqslant 28$, since it was thought at the time that the law might not apply to $\alpha<10^{-2}$ (de Vaucouleurs 1953, 1959). The range $10^{-6}\leqslant\alpha\leqslant 260$ is considered here as certainly bracketing the region where the $r^{1/4}$ law is valid, since a continuous model must break down when discreteness of the stellar population on a scale $\alpha\simeq 10^{-6}$ is no longer negligible, and for $\alpha>260$ when the overlapping with neighboring galaxies becomes significant.

I. DEPROJECTION OF THE $r^{1/4}$ LAW

A. Notation

The quantities below are expressed in units such that

(1) total galaxy mass $\mathcal{M}_T^* = 1$;
(2) effective radius $r_e^* = 1$;
(3) central potential $\psi_0^* = 1$;
(4) total luminosity $\mathcal{L}_T^* = 1$.

Note that in these units the mass-to-luminosity ratio $K^* = \mathcal{M}_T^*/\mathcal{L}_T^* = 1$. Furthermore, the normalization of the potential ψ_0^* introduces a dynamical scaling factor β^* which is defined and explained in Sec. I-B.

Table I gives a list of the notations and conversion factors to be used when it is desired to convert to (i) the cgs system, and (ii) a system in which masses are in $\mathcal{M}_\odot$, distances in parsecs, velocities in kilometers/sec, and luminosities in $\mathcal{L}_\odot$.

B. Numerical Computations

(1) The surface brightness law, Eq. (1), is equivalent to

TABLE I. Conversion factors.

Reduced quantity	$\mathcal{M}_\odot$, pc, km sec^{-1}, $\mathcal{L}_\odot$ system		cgs system	
$\mathcal{M}_T^* = 1$	$\mathcal{M}_T$ = total galaxy mass in $\mathcal{M}_\odot$		$\mathcal{M}_T'$ = total galaxy mass in grams	
$s = 1$	r_e = projected effective radius in parsecs		r_e' = projected effective radius in centimeters	
$\mathcal{L}_T^* = 1$	$\mathcal{L}_T$ = total luminosity in $\mathcal{L}_\odot$ (B system)		$\mathcal{L}_T'$ = total luminosity in ergs/sec (bolometric) $\mathcal{L}_T''$ = total luminosity in ergs/sec (B band)	
$K^* = 1$	$K = \mathcal{M}_T/\mathcal{L}_T$ (B system)		$K' = \mathcal{M}_T'/\mathcal{L}_T'$ (bolometric) $K'' = \mathcal{M}_T'/\mathcal{L}_T''$ (B band)	
$\beta^* = 2.62175$	$\beta = 1.1277 \times 10^{-2}$		$\beta' = 2.62175$ $G = 1.749 \times 10^{-7}$	
$B^*(\alpha)$	$B(r) = B^*(\alpha)\mathcal{M}_T K^{-1} r_e^{-2}$	$\mathcal{L}_\odot$ pc^{-2}	$B'(r') = B^*(\alpha)\mathcal{M}_T' K'^{-1} r_e'^{-2}$	erg sec^{-1} cm^{-2}
$f^*(\alpha)$	$f(r) = f^*(\alpha)\mathcal{M}_T K^{-1}$	$\mathcal{L}_\odot$	$f'(r') = f^*(\alpha)\mathcal{M}_T' K'^{-1}$	erg sec^{-1}
$\rho^*(s)$	$\rho(R) = \rho^*(s)\mathcal{M}_T r_e^{-3}$	$\mathcal{M}_\odot$pc^{-3}	$\rho'(R') = \rho^*(s)\mathcal{M}_T' r_e'^{-3}$	g cm^{-3}
$\mathcal{M}^*(s)$	$\mathcal{M}(R) = \mathcal{M}^*(s)\,\mathcal{M}_T$	$\mathcal{M}_\odot$	$\mathcal{M}'(R') = \mathcal{M}^*(s)\mathcal{M}_T'$	g
$\bar{\rho}^*(s)$	$\bar{\rho}(R) = \bar{\rho}^*(s)\mathcal{M}_T r_e^{-3}$	$\mathcal{M}_\odot$ pc^{-3}	$\bar{\rho}'(R') = \bar{\rho}^*(s)\mathcal{M}_T' r_e'^{-3}$	g cm^{-3}
$F^*(s)$	$F(R) = F^*(s)\beta\mathcal{M}_T r_e^{-2}$	(km sec^{-1})^2 pc^{-1}	$F'(R') = F^*(s)\beta'\mathcal{M}_T' r_e'^{-2}$	dyn
$\psi^*(s)$	$\psi(R) = \psi^*(s)\beta\mathcal{M}_T r_e^{-1}$	(km sec^{-1})^2	$\psi'(R') = \psi^*(s)\beta'\mathcal{M}_T' r_e'^{-1}$	(cm sec^{-1})^2
$w^*(s)$	$w(R) = w^*(s)\beta^{1/2}\mathcal{M}_T^{1/2} r_e^{-1/2}$	km sec^{-1}	$w'(R') = w^*(s)\beta'^{1/2}\mathcal{M}_T'^{1/2} r_e'^{-1/2}$	cm sec^{-1}
$\Omega^*(s)$	$\Omega(R) = \Omega^*(s)\beta\mathcal{M}_T^2 r_e^{-1}$	(km sec^{-1})^2	$\Omega'(R') = \Omega^*(s)\beta'\mathcal{M}_T'^2 r_e'^{-1}$	g (cm sec^{-1})^2

$$B^*(\alpha) = b^8 \exp(-b\alpha^{1/4})/(\pi 8!), \tag{2}$$

where

$$\mathcal{L}_T^* = \int_0^\infty 2\pi\alpha B^*(\alpha)\, d\alpha = 1. \tag{3}$$

(2) The relative integrated luminosity $\mathcal{L}(\alpha)/\mathcal{L}_T$, emitted within a circle of radius α, is

$$f^*(\alpha) = \int_0^\alpha 2\pi\alpha' B^*(\alpha')\, d\alpha' = 1 - \exp(-b\alpha^{1/4}) \sum_{n=0}^{7} \frac{b^n\alpha^{n/4}}{n!}. \tag{4}$$

By definition of the effective radius, $f^*(1) = ½$, it follows that

$$b = 7.66924944. \tag{5}$$

(3) The magnitude difference $m(\alpha) - m_T$ between the total (asymptotic) magnitude and that contained within radius α is

$$\Delta m(\alpha) = -2.5 \log f^*(\alpha). \tag{6}$$

(4) Poveda (1960) gives the mathematical derivation of the deprojection of the $r^{1/4}$ law to calculate the space density $\tilde{\rho}(s)$ (assuming $\mathcal{M}/\mathcal{L}$ = constant throughout the galaxy); he shows that

$$\tilde{\rho}(s) = (2j^3)^{-1} \int_1^\infty \frac{\exp(-bjt)\, dt}{(t^8 - 1)^{1/2}} \tag{7}$$

or

$$\tilde{\rho}(s) = j^{-3} \int_1^\infty \frac{\exp(-bjt)(t-1)^{1/2}\, dt}{[(t+1)(t^2+1)(t^4+1)]^{1/2}} \times \left[bj + \frac{1}{2(t+1)} + \frac{t}{t^2+1} + \frac{2t^3}{t^4+1} \right], \tag{8}$$

where $j^4 = s$.

We have dropped a constant included by Poveda, and also extracted the factor $1/j^3$. The integrals (7) and (8) should be multiplied by a factor of $(2b^9K/\pi^2 8!)$, where $K = \mathcal{M}/\mathcal{L}$, in order to represent the deprojection of the surface brightness $B^*(\alpha)$ of Eq. (2) normalized as in Eq. (3). This factor is dropped since a renormalization of $\tilde{\rho}(s)$ is imminent in Eq. (10). The tilde indicates that the density is not yet in reduced form. Equation (8) is in the form most useful for computations since the integrand is free from singularities. The exponential cutoff takes effect for $t \gtrsim t_0 = 1/bj$, and the integrations were made using Simpson's rule with 2500 steps out to $15t_0$.

(5) The (as yet unreduced) mass may be calculated from

$$\tilde{\mathcal{M}}(s) = 4\pi \int_0^s s^2\tilde{\rho}(s)\, ds, \tag{9}$$

which was evaluated by Simpson's rule with 32 steps between each tabulated value.

The total mass $\tilde{\mathcal{M}}_T = \tilde{\mathcal{M}}(\infty)$, 2.1676×10^{-3} was calculated and the renormalizations made

$$\rho^*(s) = \tilde{\rho}(s)/\tilde{\mathcal{M}}_T;\ \mathcal{M}^*(s) = \tilde{\mathcal{M}}(s)/\tilde{\mathcal{M}}_T. \tag{10}$$

(6) The reduced mean density of the sphere of radius s is

$$\bar{\rho}^*(s) = \mathcal{M}^*(s)/(4\pi s^3/3). \tag{11}$$

(7) The (unreduced) force at radius s is

$$\tilde{F}(s) = \tilde{\mathcal{M}}(s)/s^2. \tag{12}$$

(8) The (unreduced) potential is

$$\tilde{\psi}(s) = \int_0^s \tilde{\mathcal{M}}(s)/s^2\, ds = \int_0^s 4\pi s\tilde{\rho}(s) - \tilde{\mathcal{M}}(s)/s, \tag{13}$$

which, again, was computed with 32 steps per tabulated value.

The force and potential may now be reduced by setting $\tilde{\psi}_0 = \tilde{\psi}(\infty)$ and then,

$$\psi^*(s) = 1 - \tilde{\psi}(s)/\tilde{\psi}_0;\ F^*(s) = F(s)/\tilde{\psi}_0;$$
$$\psi^*(0) = 1;\ \psi^*(\infty) = 0;\ F^*(s) = -d\psi^*(s)/ds.$$

(9) The reduced escape velocity is

$$w^*(s) = [2\psi^*(s)]^{1/2}. \tag{14}$$

(10) The potential energy is

$$\Omega^*(s) = \frac{1}{2}\int_0^s \psi^*(s)\, d\mathcal{M}^*(s)$$
$$= \frac{1}{2}\mathcal{M}^*(s)\psi^*(s) - \frac{1}{2}\int_0^s [\mathcal{M}^*(s)/s]^2\, ds. \tag{15}$$

This was evaluated with a 32-point Simpson's rule.

The dynamical scaling factor

$$\beta^* = \tilde{\psi}_0/\tilde{\mathcal{M}}_{\mathrm{T}} = \int_0^\infty \frac{\mathcal{M}^*(s)}{s^2}\, ds = 2.62175 \tag{16}$$

connects the units of mass and length, via Newton's gravitational law, to the dynamical velocity units (which introduces the unit of time).

In Table II the above quantities are listed to five significant figures for a suitable grid of values of s in the range $10^{-6} \leqslant s \leqslant 260$.

With only a difference of normalization, the density values $\rho^*(s)$ agree with those of Poveda. However, since Poveda assumed $\rho^*(s \leqslant 10^{-2}) = \rho^*(10^{-2})$, there is a small discrepancy between the other values tabulated. This is smoothed out for large values of s since the mass neglected by Poveda's approximation is only 0.1% of the total mass.

However, in problems concerned with the presence and growth of a central massive object the region $s \leqslant 10^{-2}$ may be important.

The functions B^*, Δm, ρ^*, $\mathcal{M}^*$, ψ^* are depicted in Figs. 1–5.

II. NUMERICAL PROPERTIES OF THE $r^{1/4}$ LAW

(1) The effective radius in space s_e of the sphere containing half the total mass, i.e., $\mathcal{M}^*(s) = \frac{1}{2}$, is

$$s_e = 1.350. \tag{17}$$

(2) The sphere containing 99.9% of the mass has the radius

$$s = 51.2. \tag{18}$$

The circle within which 99.9% of the luminosity is emitted has the radius

$$\alpha = 43.1. \tag{19}$$

(3) At the center ($\alpha = 0$), the surface brightness in reduced units (corresponding to Table II) is

$$B^*(0) = 94.483. \tag{20}$$

At the effective radius in the image, the surface brightness is fainter by

$$2.5 \log B^*(0)/B_e^* = 8.3268 \text{ mag}, \tag{21}$$

and since, by definition, the luminosity enclosed by the effective radius ($\alpha = 1$) is half the total luminosity,

$$\Delta m(1) = 0.753 \text{ mag}. \tag{22}$$

(4) If M is the absolute magnitude of the galaxy and r_e the effective radius in parsecs, the central specific intensity expressed in magnitude arcsec^2 is

$$\mu(0) = 15.724 + M + 5 \log r_e. \tag{23}$$

At radius α, from Eq. (21), the specific intensity is

$$\mu(\alpha) = \mu(0) + 8.3268\alpha^{1/4}. \tag{24}$$

(5) In the B system, converting M_B to the luminosity of the galaxy $\mathcal{L}_{\mathrm{T}}$ (in $\mathcal{L}_\odot$),

$$\mu(0) = 21.13 - 2.5 \log \mathcal{L}_{\mathrm{T}} + 5 \log r_e, \tag{25}$$

if $m_B(\odot) = -26.80$, $M_B(\odot) = 5.41$.

(6) The total potential energy of the galaxy, in cgs units, is

$$\Omega = \Omega^*(\infty)\beta'\mathcal{M}_{\mathrm{T}}'^2/r_e' = 0.33611 G\mathcal{M}_{\mathrm{T}}'^2/r_e', \tag{26}$$

where $\Omega^*(s)$ is from Table II.

III. MATHEMATICAL EXPANSIONS AND APPROXIMATIONS

Although the space density corresponding to the $r^{1/4}$ law in projection has no simple analytic form, only that given by the integrals (7) or (8), it is possible to form expansions about the origin and the point at infinity. The latter expansion, in particular, may prove to be particularly useful as an approximation to the space density over a large range of the reduced radius.

A. Asymptotic Expansion about $s = 0$

From Eq. (7),

$$\tilde{\rho}(s) = \frac{1}{2j^3}\int_1^\infty \frac{\exp(-bjt)}{(t^8-1)^{1/2}}\, dt$$
$$= \frac{\exp(-bj)}{2j^3}\int_0^\infty \frac{\exp(-bju)}{[(u+1)^8-1]^{1/2}}\, du. \tag{27}$$

Thus,

$$\tilde{\rho}(s) = \frac{\exp(-bj)}{j^3}\sum_{n=0}^\infty \frac{(-bj)^n}{n!} \times \frac{1}{2}\int_0^\infty \frac{u^n du}{[(u+1)^8-1]^{1/2}}. \tag{28}$$

Note: $\rho^*(s) = 4.6132 \times 10^{-2}\, \tilde{\rho}(s)$.

The leading term gives

TABLE II QUANTITIES IN REDUCED UNITS FOR SPHERICAL GALAXY OBEYING $R^{\frac{1}{4}}$-LAW IN PROJECTION

Space/Plane Radius s/α	Surface Luminosity $B^*(\alpha)$	Enclosed Luminosity $f^*(\alpha)$	Magnitude Difference $\Delta m(\alpha)$	Density $\rho^*(s)$	Mass $\mathcal{M}^*(s)$	Mean Density $\bar{\rho}^*(s)$	Force $F^*(s)$	Potential $\psi^*(s)$	Escape Velocity $w^*(s)$	Potential Energy $\Omega^*(s)$
1.0E-06	7.4135E+01	2.3935E-10	2.4052E+01	2.5535E+06	1.4717E-11	3.5135E+06	5.6136E+00	1.0000E+00	1.4142E+00	7.3587E-12
2.0E-06	7.0810E+01	9.1921E-10	2.2591E+01	1.4321E+06	6.6420E-11	1.9821E+06	6.3335E+00	9.9999E-01	1.4142E+00	3.3210E-11
3.0E-06	6.8665E+01	2.0126E-09	2.1741E+01	1.0162E+06	1.5968E-10	1.4119E+06	6.7675E+00	9.9998E-01	1.4142E+00	7.9841E-11
4.0E-06	6.7050E+01	3.5033E-09	2.1139E+01	7.9473E+05	2.9691E-10	1.1075E+06	7.0781E+00	9.9998E-01	1.4142E+00	1.4845E-10
5.0E-06	6.5744E+01	5.3794E-09	2.0673E+01	6.5582E+05	4.7972E-10	9.1620E+05	7.3191E+00	9.9997E-01	1.4142E+00	2.3986E-10
6.0E-06	6.4641E+01	7.6312E-09	2.0294E+01	5.5998E+05	7.0932E-10	7.8398E+05	7.5154E+00	9.9996E-01	1.4142E+00	3.5465E-10
7.0E-06	6.3684E+01	1.0251E-08	1.9973E+01	4.8961E+05	9.8666E-10	6.8673E+05	7.6804E+00	9.9995E-01	1.4142E+00	4.9332E-10
8.0E-06	6.2837E+01	1.3231E-08	1.9696E+01	4.3559E+05	1.3125E-09	6.1199E+05	7.8222E+00	9.9995E-01	1.4142E+00	6.5623E-10
9.0E-06	6.2076E+01	1.6566E-08	1.9452E+01	3.9274E+05	1.6875E-09	5.5262E+05	7.9463E+00	9.9994E-01	1.4142E+00	8.4372E-10
1.0E-05	6.1384E+01	2.0250E-08	1.9234E+01	3.5786E+05	2.1122E-09	5.0425E+05	8.0564E+00	9.9993E-01	1.4142E+00	1.0560E-09
2.0E-05	5.6574E+01	7.5362E-08	1.7807E+01	1.9229E+05	9.1725E-09	2.7372E+05	8.7466E+00	9.9985E-01	1.4141E+00	4.5858E-09
3.0E-05	5.3561E+01	1.6156E-07	1.6979E+01	1.3263E+05	2.1497E-08	1.9007E+05	9.1105E+00	9.9976E-01	1.4140E+00	1.0747E-08
4.0E-05	5.1342E+01	2.7669E-07	1.6395E+01	1.0149E+05	3.9195E-08	1.4621E+05	9.3438E+00	9.9966E-01	1.4140E+00	1.9593E-08
5.0E-05	4.9576E+01	4.1918E-07	1.5944E+01	8.2256E+04	6.2317E-08	1.1902E+05	9.5077E+00	9.9957E-01	1.4139E+00	3.1150E-08
6.0E-05	4.8107E+01	5.8782E-07	1.5577E+01	6.9164E+04	9.0882E-08	1.0045E+05	9.6290E+00	9.9947E-01	1.4138E+00	4.5425E-08
7.0E-05	4.6848E+01	7.8160E-07	1.5268E+01	5.9660E+04	1.2489E-07	8.6928E+04	9.7220E+00	9.9938E-01	1.4138E+00	6.2422E-08
8.0E-05	4.5746E+01	9.9966E-07	1.5000E+01	5.2440E+04	1.6435E-07	7.6632E+04	9.7949E+00	9.9928E-01	1.4137E+00	8.2137E-08
9.0E-05	4.4765E+01	1.2413E-06	1.4765E+01	4.6764E+04	2.0924E-07	6.8521E+04	9.8529E+00	9.9918E-01	1.4136E+00	1.0456E-07
1.0E-04	4.3881E+01	1.5057E-06	1.4556E+01	4.2183E+04	2.5954E-07	6.1962E+04	9.8997E+00	9.9908E-01	1.4136E+00	1.2969E-07
2.0E-04	3.7954E+01	5.3008E-06	1.3189E+01	2.1075E+04	1.0561E-06	3.1516E+04	1.0071E+01	9.9808E-01	1.4129E+00	5.2740E-07
3.0E-04	3.4435E+01	1.0950E-05	1.2402E+01	1.3850E+04	2.3708E-06	2.0953E+04	1.0048E+01	9.9707E-01	1.4121E+00	1.1831E-06
4.0E-04	3.1939E+01	1.8221E-05	1.1849E+01	1.0212E+04	4.1822E-06	1.5600E+04	9.9699E+00	9.9607E-01	1.4114E+00	2.0857E-06
5.0E-04	3.0012E+01	2.6959E-05	1.1423E+01	8.0276E+03	6.4712E-06	1.2359E+04	9.8730E+00	9.9508E-01	1.4107E+00	3.2251E-06
6.0E-04	2.8449E+01	3.7044E-05	1.1078E+01	6.5757E+03	9.2208E-06	1.0191E+04	9.7695E+00	9.9410E-01	1.4100E+00	4.5924E-06
7.0E-04	2.7139E+01	4.8381E-05	1.0788E+01	5.5431E+03	1.2416E-05	8.6416E+03	9.6647E+00	9.9313E-01	1.4093E+00	6.1798E-06
8.0E-04	2.6013E+01	6.0892E-05	1.0539E+01	4.7730E+03	1.6043E-05	7.4804E+03	9.5611E+00	9.9217E-01	1.4087E+00	7.9799E-06
9.0E-04	2.5030E+01	7.4512E-05	1.0319E+01	4.1775E+03	2.0089E-05	6.5798E+03	9.4599E+00	9.9121E-01	1.4080E+00	9.9862E-06
1.0E-03	2.4158E+01	8.9182E-05	1.0124E+01	3.7042E+03	2.4544E-05	5.8593E+03	9.3615E+00	9.9027E-01	1.4073E+00	1.2193E-05
2.0E-03	1.8663E+01	2.8482E-04	8.8636E+00	1.6348E+03	8.9524E-05	2.6715E+03	8.5366E+00	9.8135E-01	1.4010E+00	4.4206E-05
3.0E-03	1.5698E+01	5.5130E-04	8.1465E+00	9.8936E+02	1.8695E-04	1.6530E+03	7.9229E+00	9.7313E-01	1.3951E+00	9.1796E-05
4.0E-03	1.3732E+01	8.7262E-04	7.6479E+00	6.8466E+02	3.1200E-04	1.1638E+03	7.4378E+00	9.6546E-01	1.3896E+00	1.5239E-04
5.0E-03	1.2293E+01	1.2388E-03	7.2675E+00	5.1087E+02	4.6129E-04	8.8100E+02	7.0379E+00	9.5823E-01	1.3844E+00	2.2418E-04
6.0E-03	1.1178E+01	1.6431E-03	6.9609E+00	4.0018E+02	6.3223E-04	6.9877E+02	6.6986E+00	9.5136E-01	1.3794E+00	3.0578E-04
7.0E-03	1.0279E+01	2.0801E-03	6.7048E+00	3.2434E+02	8.2275E-04	5.7264E+02	6.4044E+00	9.4481E-01	1.3746E+00	3.9608E-04
8.0E-03	9.5332E+00	2.5460E-03	6.4853E+00	2.6961E+02	1.0311E-03	4.8080E+02	6.1454E+00	9.3854E-01	1.3701E+00	4.9420E-04
9.0E-03	8.9016E+00	3.0376E-03	6.2937E+00	2.2855E+02	1.2560E-03	4.1131E+02	5.9143E+00	9.3251E-01	1.3657E+00	5.9936E-04
1.0E-02	8.3577E+00	3.5520E-03	6.1238E+00	1.9679E+02	1.4960E-03	3.5715E+02	5.7062E+00	9.2671E-01	1.3614E+00	7.1093E-04
2.0E-02	5.2821E+00	9.5884E-03	5.0456E+00	7.0217E+01	4.5473E-03	1.3570E+02	4.3362E+00	8.7730E-01	1.3246E+00	2.0832E-03
3.0E-02	3.8831E+00	1.6603E-02	4.4496E+00	3.6889E+01	8.4212E-03	7.4460E+01	3.5690E+00	8.3807E-01	1.2947E+00	3.7422E-03
4.0E-02	3.0606E+00	2.4127E-02	4.0438E+00	2.2893E+01	1.2822E-02	4.7829E+01	3.0566E+00	8.0509E-01	1.2689E+00	5.5485E-03
5.0E-02	2.5139E+00	3.1932E-02	3.7394E+00	1.5615E+01	1.7585E-02	3.3584E+01	2.6829E+00	7.7648E-01	1.2462E+00	7.4306E-03
6.0E-02	2.1228E+00	3.9889E-02	3.4979E+00	1.1325E+01	2.2605E-02	2.4984E+01	2.3950E+00	7.5115E-01	1.2257E+00	9.3470E-03
7.0E-02	1.8287E+00	4.7916E-02	3.2988E+00	8.5773E+00	2.7812E-02	1.9357E+01	2.1649E+00	7.2839E-01	1.2070E+00	1.1272E-02
8.0E-02	1.5995E+00	5.5960E-02	3.1303E+00	6.7096E+00	3.3154E-02	1.5459E+01	1.9759E+00	7.0772E-01	1.1897E+00	1.3190E-02
9.0E-02	1.4159E+00	6.3986E-02	2.9848E+00	5.3820E+00	3.8593E-02	1.2638E+01	1.8173E+00	6.8877E-01	1.1737E+00	1.5088E-02

Space/Plane Radius s/α	Surface Luminosity $B^*(\alpha)$	Enclosed Luminosity $f^*(\alpha)$	Magnitude Difference $\Delta m(\alpha)$	Density $\rho^*(s)$	Mass $\mathcal{M}^*(s)$	Mean Density $\bar{\rho}^*(s)$	Force $F^*(s)$	Potential $\psi^*(s)$	Escape Velocity $w^*(s)$	Potential Energy $\Omega^*(s)$
1.0E-01	1.2658E+00	7.1967E-02	2.8572E+00	4.4047E+00	4.4102E-02	1.0528E+01	1.6821E+00	6.7129E-01	1.1587E+00	1.6961E-02
2.0E-01	5.5972E-01	1.4716E-01	2.0805E+00	1.0879E+00	9.9742E-02	2.9764E+00	9.5110E-01	5.4608E-01	1.0451E+00	3.3734E-02
3.0E-01	3.2385E-01	2.1273E-01	1.6804E+00	4.4711E-01	1.5245E-01	1.3479E+00	6.4608E-01	4.6796E-01	9.6743E-01	4.7043E-02
4.0E-01	2.1210E-01	2.6978E-01	1.4225E+00	2.2959E-01	2.0080E-01	7.4901E-01	4.7868E-01	4.1244E-01	9.0823E-01	5.7662E-02
5.0E-01	1.4947E-01	3.1981E-01	1.2378E+00	1.3394E-01	2.4488E-01	4.6769E-01	3.7362E-01	3.7019E-01	8.6045E-01	6.6275E-02
6.0E-01	1.1065E-01	3.6406E-01	1.0971E+00	8.4945E-02	2.8510E-01	3.1510E-01	3.0207E-01	3.3661E-01	8.2050E-01	7.3374E-02
7.0E-01	8.4875E-02	4.0352E-01	9.8534E-01	5.7165E-02	3.2188E-01	2.2403E-01	2.5056E-01	3.0911E-01	7.8627E-01	7.9308E-02
8.0E-01	6.6890E-02	4.3895E-01	8.9397E-01	4.0220E-02	3.5562E-01	1.6582E-01	2.1194E-01	2.8607E-01	7.5641E-01	8.4326E-02
9.0E-01	5.3855E-02	4.7094E-01	8.1758E-01	2.9296E-02	3.8668E-01	1.2663E-01	1.8209E-01	2.6643E-01	7.2998E-01	8.8614E-02
1.0E+00	4.4120E-02	5.0000E-01	7.5257E-01	2.1943E-02	4.1536E-01	9.9159E-02	1.5843E-01	2.4945E-01	7.0633E-01	9.2311E-02
1.1E+00	3.6671E-02	5.2651E-01	6.9649E-01	1.6817E-02	4.4191E-01	7.9262E-02	1.3930E-01	2.3460E-01	6.8498E-01	9.5524E-02
1.2E+00	3.0854E-02	5.5079E-01	6.4754E-01	1.3138E-02	4.6657E-01	6.4459E-02	1.2358E-01	2.2148E-01	6.6555E-01	9.8334E-02
1.3E+00	2.6233E-02	5.7312E-01	6.0440E-01	1.0434E-02	4.8952E-01	5.3193E-02	1.1048E-01	2.0979E-01	6.4776E-01	1.0081E-01
1.4E+00	2.2508E-02	5.9371E-01	5.6606E-01	8.4044E-03	5.1094E-01	4.4452E-02	9.9430E-02	1.9931E-01	6.3137E-01	1.0300E-01
1.5E+00	1.9468E-02	6.1277E-01	5.3175E-01	6.8537E-03	5.3096E-01	3.7558E-02	9.0010E-02	1.8985E-01	6.1620E-01	1.0495E-01
1.6E+00	1.6957E-02	6.3046E-01	5.0086E-01	5.6501E-03	5.4973E-01	3.2041E-02	8.1907E-02	1.8127E-01	6.0211E-01	1.0669E-01
1.7E+00	1.4865E-02	6.4691E-01	4.7289E-01	4.7030E-03	5.6735E-01	2.7569E-02	7.4880E-02	1.7344E-01	5.8896E-01	1.0825E-01
1.8E+00	1.3104E-02	6.6225E-01	4.4744E-01	3.9487E-03	5.8392E-01	2.3903E-02	6.8742E-02	1.6626E-01	5.7665E-01	1.0966E-01
1.9E+00	1.1612E-02	6.7659E-01	4.2419E-01	3.3412E-03	5.9953E-01	2.0867E-02	6.3345E-02	1.5966E-01	5.6509E-01	1.1093E-01
2.0E+00	1.0338E-02	6.9001E-01	4.0287E-01	2.8471E-03	6.1426E-01	1.8331E-02	5.8574E-02	1.5357E-01	5.5421E-01	1.1208E-01
2.1E+00	9.2435E-03	7.0259E-01	3.8324E-01	2.4417E-03	6.2818E-01	1.6193E-02	5.4332E-02	1.4793E-01	5.4393E-01	1.1313E-01
2.2E+00	8.2972E-03	7.1442E-01	3.6511E-01	2.1063E-03	6.4134E-01	1.4379E-02	5.0542E-02	1.4269E-01	5.3421E-01	1.1409E-01
2.3E+00	7.4749E-03	7.2555E-01	3.4833E-01	1.8268E-03	6.5382E-01	1.2829E-02	4.7142E-02	1.3781E-01	5.2499E-01	1.1496E-01
2.4E+00	6.7566E-03	7.3604E-01	3.3274E-01	1.5922E-03	6.6565E-01	1.1495E-02	4.4079E-02	1.3325E-01	5.1624E-01	1.1576E-01
2.5E+00	6.1264E-03	7.4595E-01	3.1823E-01	1.3942E-03	6.7688E-01	1.0342E-02	4.1309E-02	1.2898E-01	5.0791E-01	1.1650E-01
2.6E+00	5.5710E-03	7.5530E-01	3.0470E-01	1.2260E-03	6.8756E-01	9.3391E-03	3.8795E-02	1.2498E-01	4.9996E-01	1.1718E-01
2.7E+00	5.0797E-03	7.6416E-01	2.9204E-01	1.0824E-03	6.9772E-01	8.4626E-03	3.6506E-02	1.2122E-01	4.9238E-01	1.1780E-01
2.8E+00	4.6435E-03	7.7255E-01	2.8018E-01	9.5909E-04	7.0740E-01	7.6931E-03	3.4416E-02	1.1767E-01	4.8513E-01	1.1838E-01
2.9E+00	4.2548E-03	7.8051E-01	2.6905E-01	8.5282E-04	7.1663E-01	7.0148E-03	3.2502E-02	1.1433E-01	4.7818E-01	1.1892E-01
3.0E+00	3.9074E-03	7.8807E-01	2.5859E-01	7.6078E-04	7.2544E-01	6.4143E-03	3.0744E-02	1.1117E-01	4.7152E-01	1.1941E-01
3.1E+00	3.5959E-03	7.9525E-01	2.4874E-01	6.8075E-04	7.3385E-01	5.8807E-03	2.9127E-02	1.0818E-01	4.6513E-01	1.1988E-01
3.2E+00	3.3158E-03	8.0208E-01	2.3945E-01	6.1089E-04	7.4189E-01	5.4050E-03	2.7634E-02	1.0534E-01	4.5899E-01	1.2030E-01
3.3E+00	3.0633E-03	8.0859E-01	2.3068E-01	5.4969E-04	7.4958E-01	4.9795E-03	2.6254E-02	1.0264E-01	4.5309E-01	1.2070E-01
3.4E+00	2.8350E-03	8.1479E-01	2.2238E-01	4.9589E-04	7.5694E-01	4.5976E-03	2.4975E-02	1.0008E-01	4.4740E-01	1.2108E-01
3.5E+00	2.6282E-03	8.2071E-01	2.1453E-01	4.4844E-04	7.6399E-01	4.2540E-03	2.3788E-02	9.7646E-02	4.4192E-01	1.2143E-01
3.6E+00	2.4404E-03	8.2636E-01	2.0708E-01	4.0647E-04	7.7075E-01	3.9438E-03	2.2684E-02	9.5324E-02	4.3663E-01	1.2175E-01
3.7E+00	2.2695E-03	8.3175E-01	2.0001E-01	3.6923E-04	7.7724E-01	3.6632E-03	2.1655E-02	9.3107E-02	4.3153E-01	1.2206E-01
3.8E+00	2.1136E-03	8.3691E-01	1.9330E-01	3.3610E-04	7.8346E-01	3.4086E-03	2.0695E-02	9.0990E-02	4.2659E-01	1.2234E-01
3.9E+00	1.9712E-03	8.4185E-01	1.8691E-01	3.0656E-04	7.8944E-01	3.1771E-03	1.9797E-02	8.8966E-02	4.2182E-01	1.2261E-01
4.0E+00	1.8408E-03	8.4658E-01	1.8083E-01	2.8013E-04	7.9518E-01	2.9662E-03	1.8956E-02	8.7029E-02	4.1720E-01	1.2287E-01
4.1E+00	1.7213E-03	8.5111E-01	1.7504E-01	2.5645E-04	8.0071E-01	2.7735E-03	1.8168E-02	8.5173E-02	4.1273E-01	1.2310E-01
4.2E+00	1.6114E-03	8.5545E-01	1.6951E-01	2.3518E-04	8.0602E-01	2.5972E-03	1.7428E-02	8.3394E-02	4.0840E-01	1.2333E-01
4.3E+00	1.5103E-03	8.5962E-01	1.6424E-01	2.1602E-04	8.1114E-01	2.4356E-03	1.6733E-02	8.1686E-02	4.0419E-01	1.2354E-01
4.4E+00	1.4172E-03	8.6362E-01	1.5920E-01	1.9874E-04	8.1606E-01	2.2871E-03	1.6078E-02	8.0046E-02	4.0011E-01	1.2374E-01
4.5E+00	1.3312E-03	8.6746E-01	1.5438E-01	1.8312E-04	8.2081E-01	2.1504E-03	1.5461E-02	7.8469E-02	3.9615E-01	1.2393E-01
4.6E+00	1.2518E-03	8.7115E-01	1.4977E-01	1.6898E-04	8.2539E-01	2.0244E-03	1.4878E-02	7.6953E-02	3.9231E-01	1.2410E-01
4.7E+00	1.1782E-03	8.7469E-01	1.4536E-01	1.5614E-04	8.2980E-01	1.9081E-03	1.4328E-02	7.5492E-02	3.8857E-01	1.2427E-01
4.8E+00	1.1101E-03	8.7811E-01	1.4113E-01	1.4448E-04	8.3406E-01	1.8005E-03	1.3808E-02	7.4086E-02	3.8493E-01	1.2443E-01
4.9E+00	1.0468E-03	8.8139E-01	1.3708E-01	1.3386E-04	8.3817E-01	1.7008E-03	1.3315E-02	7.2730E-02	3.8139E-01	1.2458E-01

Space/Plane Radius s/α	Surface Luminosity $B^*(\alpha)$	Enclosed Luminosity $f^*(\alpha)$	Magnitude Difference $\Delta m(\alpha)$	Density $\rho^*(s)$	Mass $\mathcal{M}^*(s)$	Mean Density $\bar{\rho}^*(s)$	Force $F^*(s)$	Potential $\psi^*(s)$	Escape Velocity $w^*(s)$	Potential Energy $\Omega^*(s)$
5.0E+00	9.8804E-04	8.8455E-01	1.3319E-01	1.2417E-04	8.4214E-01	1.6084E-03	1.2848E-02	7.1422E-02	3.7795E-01	1.2473E-01
6.0E+00	5.7877E-04	9.1063E-01	1.0165E-01	6.2200E-05	8.7532E-01	9.6744E-04	9.2741E-03	6.0505E-02	3.4787E-01	1.2582E-01
7.0E+00	3.6115E-04	9.2927E-01	7.9640E-02	3.4000E-05	8.9962E-01	6.2615E-04	7.0028E-03	5.2446E-02	3.2387E-01	1.2651E-01
8.0E+00	2.3647E-04	9.4304E-01	6.3680E-02	1.9848E-05	9.1793E-01	4.2801E-04	5.4706E-03	4.6256E-02	3.0416E-01	1.2696E-01
9.0E+00	1.6084E-04	9.5345E-01	5.1759E-02	1.2200E-05	9.3202E-01	3.0522E-04	4.3888E-03	4.1356E-02	2.8760E-01	1.2727E-01
1.0E+01	1.1283E-04	9.6149E-01	4.2643E-02	7.8165E-06	9.4308E-01	2.2514E-04	3.5971E-03	3.7383E-02	2.7343E-01	1.2748E-01
1.1E+01	8.1206E-05	9.6780E-01	3.5535E-02	5.1828E-06	9.5188E-01	1.7073E-04	3.0006E-03	3.4097E-02	2.6114E-01	1.2764E-01
1.2E+01	5.9731E-05	9.7283E-01	2.9904E-02	3.5371E-06	9.5899E-01	1.3249E-04	2.5402E-03	3.1336E-02	2.5034E-01	1.2776E-01
1.3E+01	4.4761E-05	9.7689E-01	2.5381E-02	2.4741E-06	9.6479E-01	1.0484E-04	2.1775E-03	2.8984E-02	2.4077E-01	1.2785E-01
1.4E+01	3.4091E-05	9.8021E-01	2.1704E-02	1.7678E-06	9.6958E-01	8.4355E-05	1.8868E-03	2.6957E-02	2.3220E-01	1.2791E-01
1.5E+01	2.6336E-05	9.8294E-01	1.8684E-02	1.2868E-06	9.7356E-01	6.8866E-05	1.6504E-03	2.5193E-02	2.2447E-01	1.2796E-01
2.0E+01	8.5458E-06	9.9123E-01	9.5614E-03	3.2585E-07	9.8594E-01	2.9422E-05	9.4016E-04	1.8963E-02	1.9475E-01	1.2810E-01
2.5E+01	3.3702E-06	9.9505E-01	5.3903E-03	1.0600E-07	9.9184E-01	1.5154E-05	6.0530E-04	1.5191E-02	1.7430E-01	1.2815E-01
3.0E+01	1.5147E-06	9.9701E-01	3.2511E-03	4.0702E-08	9.9497E-01	8.7974E-06	4.2167E-04	1.2664E-02	1.5915E-01	1.2817E-01
3.5E+01	7.4828E-07	9.9810E-01	2.0622E-03	1.7600E-08	9.9675E-01	5.5500E-06	3.1035E-04	1.0855E-02	1.4735E-01	1.2818E-01
4.0E+01	3.9724E-07	9.9875E-01	1.3607E-03	8.3251E-09	9.9782E-01	3.7220E-06	2.3787E-04	9.4969E-03	1.3782E-01	1.2819E-01
4.5E+01	2.2322E-07	9.9915E-01	9.2685E-04	4.2253E-09	9.9849E-01	2.6159E-06	1.8807E-04	8.4394E-03	1.2992E-01	1.2819E-01
5.0E+01	1.3137E-07	9.9940E-01	6.4821E-04	2.2701E-09	9.9893E-01	1.9078E-06	1.5241E-04	7.5928E-03	1.2323E-01	1.2819E-01
5.5E+01	8.0338E-08	9.9957E-01	4.6355E-04	1.2784E-09	9.9923E-01	1.4338E-06	1.2599E-04	6.9000E-03	1.1747E-01	1.2820E-01
6.0E+01	5.0752E-08	9.9969E-01	3.3790E-04	7.4903E-10	9.9943E-01	1.1046E-06	1.0589E-04	6.3224E-03	1.1245E-01	1.2820E-01
8.0E+01	1.0332E-08	9.9990E-01	1.1061E-04	1.1888E-10	9.9981E-01	4.6618E-07	5.9586E-05	4.7338E-03	9.7301E-02	1.2820E-01
1.0E+02	2.7715E-09	9.9996E-01	4.2957E-05	2.6282E-11	9.9992E-01	2.3871E-07	3.8139E-05	3.7803E-03	8.6952E-02	1.2820E-01
1.2E+02	8.9438E-10	9.9998E-01	1.8775E-05	7.2420E-12	9.9997E-01	1.3815E-07	2.6487E-05	3.1446E-03	7.9305E-02	1.2820E-01
1.4E+02	3.2991E-10	9.9999E-01	8.9566E-06	2.3372E-12	9.9998E-01	8.7000E-08	1.9460E-05	2.6906E-03	7.3356E-02	1.2820E-01
1.6E+02	1.3473E-10	1.0000E+00	4.5725E-06	8.5012E-13	9.9999E-01	5.8284E-08	1.4899E-05	2.3500E-03	6.8557E-02	1.2820E-01
1.8E+02	5.9632E-11	1.0000E+00	2.4650E-06	3.3971E-13	1.0000E+00	4.0935E-08	1.1772E-05	2.0851E-03	6.4578E-02	1.2820E-01
2.0E+02	2.8175E-11	1.0000E+00	1.3897E-06	1.4649E-13	1.0000E+00	2.9841E-08	9.5356E-06	1.8732E-03	6.1209E-02	1.2820E-01
2.2E+02	1.4055E-11	1.0000E+00	8.1355E-07	6.7273E-14	1.0000E+00	2.2420E-08	7.8807E-06	1.6999E-03	5.8307E-02	1.2820E-01
2.4E+02	7.3407E-12	1.0000E+00	4.9182E-07	3.2580E-14	1.0000E+00	1.7269E-08	6.6219E-06	1.5554E-03	5.5774E-02	1.2820E-01
2.6E+02	3.9881E-12	1.0000E+00	3.0572E-07	1.6512E-14	1.0000E+00	1.3583E-08	5.6424E-06	1.4331E-03	5.3538E-02	1.2820E-01

$\alpha = r/r_e$ = reduced radius in projection
$s = R/r_e$ = reduced radius in space (note s is in units of the effective radius in projection r_e).
$B^*(\alpha)$ = reduced surface brightness (specific intensity).
$f^*(\alpha)$ = reduced luminosity enclosed within (circular) isophote of radius α.
$\Delta m(\alpha)$ = difference in magnitudes between luminosity enclosed within α and the total luminosity.

$\rho^*(s)$ = reduced space density.
$\mathcal{M}^*(s)$ = reduced mass contained within sphere of radius s.
$\bar{\rho}^*(s)$ = reduced mean space density within sphere of radius s.
$F^*(s)$ = reduced force.
$\psi^*(s)$ = reduced potential, with $\psi^*(0) = 1$ and $\psi^*(\infty) = 0$.
$w^*(s)$ = reduced escape velocity.
$\Omega^*(s)$ = reduced potential energy contained within radius s.

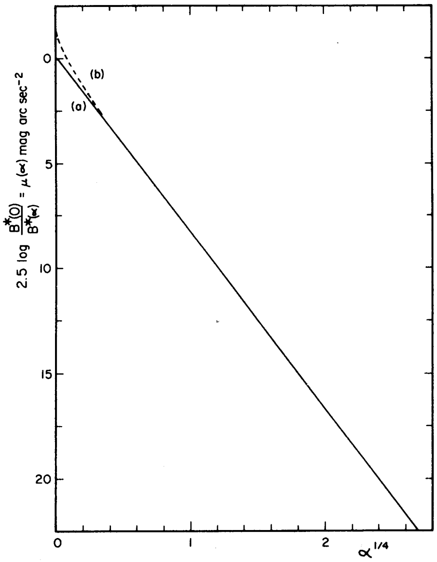

FIG. 1. Surface brightnesses of (a) the $r^{1/4}$ law, and (b) the projection of the asymptotic approximation for $\rho^*(s)$ (as $s \to \infty$). The curves have been normalized to approach each other as $\alpha \to \infty$.

$$\rho(s) \sim \frac{\exp(-bj)}{2j^3} \frac{1}{2} \int_0^\infty \frac{du}{[(u+1)^8 - 1]^{1/2}} = I_0 \frac{\exp(-bj)}{j^3}, \quad (29)$$

where
$$I_0 = \frac{1}{2} \int_1^\infty \frac{dt}{(t^8 - 1)^{1/2}} = \frac{1}{8} \int_0^{\pi/2} \sec^{1/4}\theta \, d\theta = 0.2409903. \quad (30)$$

The approximation resulting from Eq. (29) is plotted in Fig. 1. The logarithmic error at $s = 10^{-6}$ is 0.0335 and at $s = 10^{-4}$ is 0.0878. The density distribution becomes a power law $\tilde{\rho}(s) \propto s^{-3/4}$ for $s \lesssim 10^{-5}$.

The fact that the density of the model becomes infinite at the origin is not physically significant because it evidently does not apply to scales less than the average separation between stars which at the center is on the order of $s \simeq 10^{-6}$.

This expansion illustrates the central "spike" predicted by the $r^{1/4}$ law, but probably has little practical use because the empirical law may not hold precisely this close to the center. Statistical mechanics does not predict the above density profile with or without the presence of a massive object at $s = 0$ (Lynden-Bell 1967; Bahcall and Wolf 1976). However, a central spike inconsistent with all models approximating the isothermal sphere does seem to be observed when proper allowance is made for the convolution effects of seeing, perhaps even in excess of that of Eq. (29) as was mentioned in the Introduction.

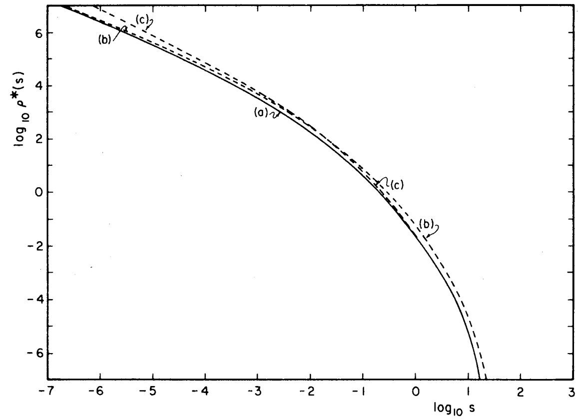

FIG. 3. Space density $\rho^*(s)$ for (a) the $r^{1/4}$ law, (b) the asymptotic expansion at $s = 0$, and (c) the asymptotic expansion for $s \to \infty$.

B. Asymptotic Expansion for $s \to \infty$

From Eq. (27),

$$\tilde{\rho}(s) = \frac{\exp(-bj)}{2j^3} \int_0^\infty \frac{\exp(-bju)}{[(u+1)^8 - 1]^{1/2}} du. \quad (31)$$

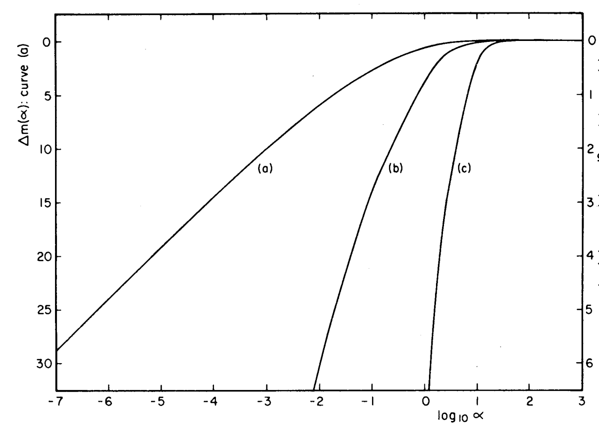

FIG. 2. Magnitude difference $\Delta m(\alpha)$ between luminosity enclosed by radius α and total luminosity. Three curves on different scales are given.

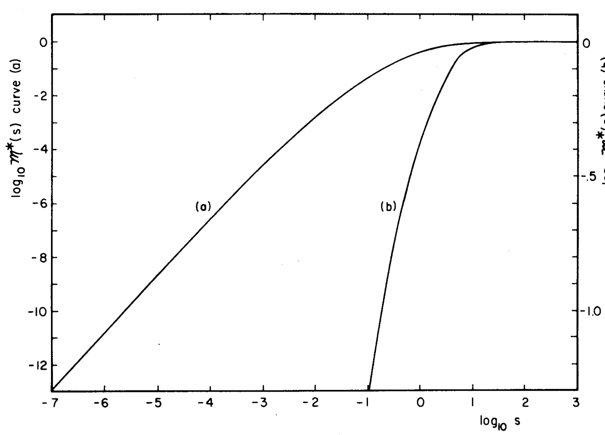

FIG. 4. Mass $\mathcal{M}^*(s)$ enclosed within radius $s = R/r_e$ for the $r^{1/4}$ law.

Thus,

$$\tilde{\rho}(s) \sim \frac{\exp(-bj)}{2j^3}\left(\frac{\pi}{8bj}\right)^{1/2}\left[1 - \frac{7}{8bj} + \cdots\right]. \quad (32)$$

The approximation

$$\tilde{\rho}(s) \sim \frac{\exp(-bj)}{2j^3}\left(\frac{\pi}{8bj}\right)^{1/2} \quad (33)$$

is plotted in Fig. 1 for comparison with the actual density corresponding to the deprojected $r^{1/4}$ law. Note that

$$\rho^*(s) = 4.6132 \times 10^{-2}\, \tilde{\rho}(s).$$

The logarithmic error of the approximation (33) is only 0.093 at $s = 0.1$, and so is valid over a large range of radial distances.

The total mass predicted by Eq. (33) is

$$\tilde{\mathcal{M}}'_{\mathrm{T}} = 2.4082 \times 10^{-3}$$

compared with the precise value

$$\tilde{\mathcal{M}}_{\mathrm{T}} = 2.1674 \times 10^{-3}.$$

The 10% overestimate is due to the neglect of the (negative) second term, and higher terms, but it is still small compared with observational errors.

The enclosed mass from integration of Eq. (33) is

$$\tilde{\mathcal{M}}(s) = 4\pi \int_0^s s^2 \tilde{\rho}(s)\, ds = \frac{(8\pi^3)^{1/2}}{b^9}\gamma(17/2,j), \quad (34)$$

where $\gamma(n,x)$ is the incomplete gamma function. This is in error by 10% as $s \to \infty$, and for $s \gtrsim 10^{-1}$ gives errors comparable to the density approximation (33).

C. Projected Surface Brightness of Asymptotic Expansion

It is of interest to project the asymptotic approximation of Sec. III-B to obtain the resultant surface brightness distribution. The projected surface brightness $B''(\alpha)$ of Eq. (33) normalized to satisfy $B''(\alpha) \sim B^*(\alpha)(\alpha \to \infty)$ is

$$B''(\alpha) = \left(\frac{b^8}{\pi 8!}\right)\left(\frac{8b}{\pi}\right)^{1/2} \int_{\alpha^{1/4}}^{\infty} \frac{\exp(-bu)u^{7/2}}{(u^8 - \alpha^2)^{1/2}}\, du$$

$$= \left(\frac{b^8}{\pi 8!}\right)\left(\frac{8b}{\pi}\right)^{1/2} \cdot 2 \int_A^{\infty} G(u)H(u)\, du, \quad (35)$$

where

$$G(u) = (u-1)^{1/2}u^{7/2}\exp(-Aub)/[(u+1)(u^2+1)(u^4+1)]^{1/2},$$

$$H(u) = bAu + \frac{u}{2(u+1)} + \frac{u^2}{u^2+1} + \frac{2u^4}{u^4+1} - \frac{7}{2},$$

$$A^4 = \alpha.$$

The results of the projection compared with the $r^{1/4}$ law are shown in Fig. 1. The surface brightness distribution given by $B''(\alpha)$ has a slight excess of luminosity in the central region $\alpha \lesssim 0.05$ over the $r^{1/4}$ law fitted asymptotically as $\alpha \to \infty$. The total luminosity is 11.1% or 0.114 mag brighter than the total luminosity of the $r^{1/4}$ law, and the central brightness, $B''(0) = 2^{3/2}B^*(0)$ is $2.5 \log B''(0)/B^*(0) = 1.129$ mag higher. Slight adjustments in the fit between the $r^{1/4}$ law and the surface brightness distribution (35) will allow $B''(\alpha)$ to mimic an $r^{1/4}$ law (for $\alpha \gtrsim 0.05$) but with a different central excess. Since there is observational evidence for an excess of luminosity above that predicted by the $r^{1/4}$ law near the center (de Vaucouleurs and Capaccioli 1977), the expression (33) may be a better approximation to $\tilde{\rho}(s)$ than Eq. (7). If $\mathcal{M}/\mathcal{L}$ varies in the galaxy, then the luminosity per unit volume may still be approximated by the functional form of Eq. (33).

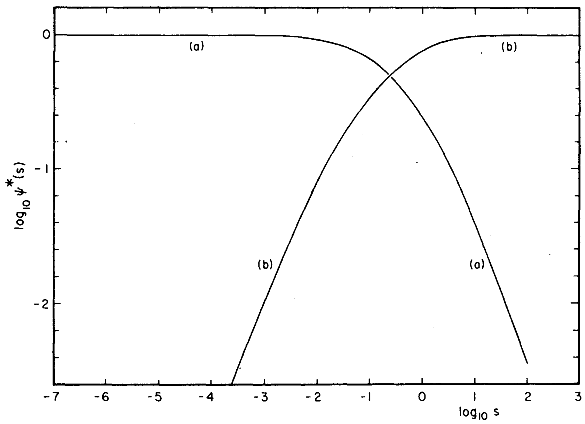

FIG. 5. Potential due to $r^{1/4}$ law, (a) $\psi^*(s)$ with $\psi^*(0) = 1$, $\psi^*(\infty) = 0$ and (b) $[1 - \psi^*(s)]$.

IV. APPLICATIONS TO ELLIPTICAL GALAXIES

(1) Assuming the galaxy to be spherical and in stable statistical equilibrium, then by the virial theorem,

$$2T' = \Omega'.$$

From Eq. (26),

$$\sigma_v^2 = 0.33611\, G\mathcal{M}'_{\mathrm{T}}/r'_e \quad (\mathrm{cm\ sec^{-1}})^2, \quad (36)$$

where σ_v is the mean velocity dispersion in space. Converting Eq. (36) to $\mathcal{M}_\odot$, pc, km sec^{-1} units,

$$\sigma_v^2 = 1.4456 \times 10^{-3}\, \mathcal{M}_{\mathrm{T}}/r_e \quad (\mathrm{km\ sec^{-1}})^2. \quad (37)$$

Approximating the central velocity dispersion of σ_v gives the total mass of the galaxy in terms of the observed σ_v, r_e (Poveda 1957). If the velocity distribution is isotropic, the line-of-sight velocity dispersion is

$$\sigma_r = \sigma_v/3^{1/2}. \quad (38)$$

In $\mathcal{M}_\odot$, pc, km sec^{-1} units the central escape velocity is

TABLE III. Numerical examples.

Object	$\mathcal{M}_T$[a] ($\mathcal{M}_\odot$)	r_e[a] (pc)	$\mathcal{L}_T$[a] ($\mathcal{L}_\odot$)	$K = \mathcal{M}_T$[a]$/\mathcal{L}_T$	$w(0)$ (km sec^{-1})	σ_V^2 (km sec^{-1})	$\mu(0)$[b]	$B(0)$ ($\mathcal{L}_\odot$ pc^{-2})	ρ_c ($\mathcal{M}_\odot$ pc^{-3})	$\mathcal{M}$(1pc) ($\mathcal{M}_\odot$)	$\mathcal{M}_N$ ($\mathcal{M}_\odot$)	R_N (pc)
Compact Dwarf E	3×10^9	150	3×10^8	10	672	170	10.82	1.3×10^6	5.5×10^7	2×10^6	3×10^6	1.2
Average E	3×10^{10}	700	2×10^9	15	983	249	12.10	3.9×10^5	5.4×10^6	2×10^6	3×10^7	5.7
Normal Giant E	3×10^{11}	2000	1.5×10^{10}	20	1840	466	12.19	3.5×10^5	2.3×10^6	2×10^6	3×10^8	16.2
Supergiant E	3×10^{12}	7000	7×10^{10}	30	3110	787	13.24	1.4×10^5	5.4×10^5	2×10^6	3×10^9	57

[a]Data for these columns from de Vaucouleurs (1974a).
[b]Units of mag arcsec^{-2}.

$$w(0) = 0.1502(\mathcal{M}_T/r_e)^{1/2} \quad (\text{km sec}^{-1}) \tag{39}$$

and

$$\sigma_v/w(0) = 0.253. \tag{40}$$

(2) Now, define the central density of a galaxy obeying exactly the $r^{1/4}$ law to be

$$\rho_c^* = \bar{\rho}^*(10^{-4}). \tag{41}$$

This corresponds to the central 1 pc and contains $\simeq 10^6$ stars, in a typical E supergiant (e.g., M87) with $r_e \simeq 10^4$ pc. The definition loses meaning if $r_e \lesssim 100$ pc for then the radius $s = 10^{-4}$ is of the order of interstellar distances (but note that for M32 $r_e = 150$ pc).

From Table II, converting to $\mathcal{M}_\odot$, pc units,

$$\rho_c = 6.1962 \times 10^4 \mathcal{M}_T r_e^{-3} \quad (\mathcal{M}_\odot\ \text{pc}^{-3}). \tag{42}$$

(3) An elliptical galaxy has no recognizable nucleus, nevertheless it may be of interest to define as "nucleus" the region which central stars could reach on the average under the influence of the gravitational potential of the $r^{1/4}$ law (without interactions, since the dynamical time scale is much shorter than the relaxation time in such conditions).

A central star with initial velocity v will be able to travel out a radial distance $\tilde{s}$ given by

$$v^2/w^2(0) = 1 - \psi^*(\tilde{s}). \tag{43}$$

A graph of $\tilde{s}$ as a function of $v/w(0)$ is shown in Fig. 6.

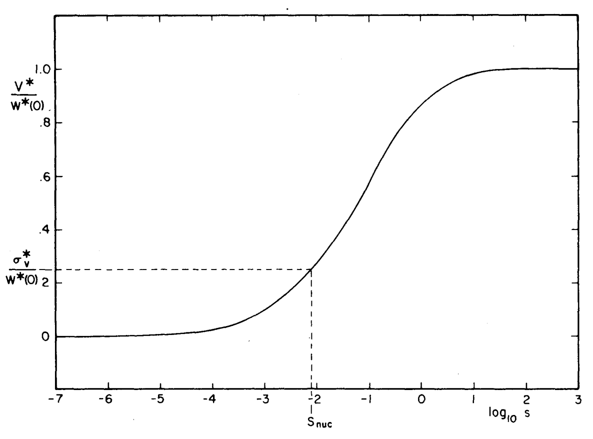

FIG. 6. Radius within which an initially central star with velocity v^* is bound by the gravitational potential of the $r^{1/4}$ law. $w^*(0)$ is the velocity of escape from the center of the galaxy.

Now consider a central star whose velocity is numerically equal to the velocity dispersion σ_v, then this star will stay within a radius s_N, where

$$\sigma_v^2/w^2(0) = 1 - \psi^*(s_N), \tag{44}$$

which defines a "nuclear" radius s_N.

From Eq. (40) and Table II,

$$s_N = 8.1 \times 10^{-3} \text{ and } \mathcal{M}^*(s_N) = 1.1 \times 10^{-3}. \tag{45}$$

$\mathcal{M}_N^* = \mathcal{M}^*(s_N) \simeq 10^{-3}\mathcal{M}_T$ may be used as a measure of the "nuclear" mass of a spherical galaxy obeying the $r^{1/4}$ law. The quantities $\mathcal{M}_N^*$ and s_N are useful in estimating the growth rate of a central massive black hole in the galaxy (Young, Shields, and Wheeler 1976).

(4) If the galaxy departs moderately from a sphere and has a small ellipticity ϵ, following King (1961) σ_v^2 is replaced by $\sigma_v^2(1+\gamma)$, where γ is a factor allowing for rotational energy

$$\gamma \simeq 8\epsilon/(5 - 8\epsilon).$$

For $\langle\epsilon\rangle = 0.36$ (de Vaucouleurs 1974a), $\gamma = 1.36$ and the mean velocity dispersion σ_v is reduced by a factor 0.651. Using the potential calculated for a spherical galaxy in this case of moderate ellipticity gives

$$\sigma_v/w(0) = 0.165;\ s_N = 3 \times 10^{-3};\ \mathcal{M}_N^* = 2 \times 10^{-4},$$

which suggests that the "nucleus" of a rotating galaxy is much smaller than for a spherical galaxy with the same total mass and effective radius.

(5) Table III gives a set of numerical examples for a plausible range of galaxy masses corresponding to (1) a supergiant E (e.g., M87), (2) a normal giant E (e.g., NGC 3379), (3) an average E, (4) a compact dwarf E system (e.g., M32).

Note the decrease of $\mu(0)$ and ρ_c with increasing mass. A consequence is that despite the change by two orders of magnitude of the central density ρ_c from compact E to supergiant E systems, the mass contained within a radius of 1 pc is essentially constant at $\mathcal{M}(1 \text{ pc}) \simeq 2 \times 10^{-6} \mathcal{M}_\odot$.

ACKNOWLEDGMENT

I thank Professor G. de Vaucouleurs for stimulating discussions and help in the preparation of this work for publication.

REFERENCES

Bahcall, J. N., and Wolf, R. A. (1976). Astrophys. J. To be published.
de Vaucouleurs, G. (1953). Mon. Not. R. Astron. Soc. **136,** 101.
de Vaucouleurs, G. (1958). Astrophys. J. **128,** 465.
de Vaucouleurs, G. (1959). *Handbuch der Physik* (Springer-Verlag, Berlin, Gottingen), Vol. 53.
de Vaucouleurs, G. (1962). *Classification of Galaxies by Form, Luminosity and Color,* IAU Symp. No. 15, edited by G. C. McVittie (MacMillan, New York), p. 3.
de Vaucouleurs, G. (1974a). *The Formation and Dynamics of Galaxies,* IAU Symp. No. 58, edited by J. R. Shakeshaft (Reidel, Dordrecht), p. 1.
de Vaucouleurs, G. (1974b). *The Formation and Dynamics of Galaxies,* IAU Symp. No. 58, edited by J. R. Shakeshaft (Reidel, Dordrecht), p. 335.
de Vaucouleurs, G. (1975). Astrophys. J. Suppl. **29,** No. 284, 193.
de Vaucouleurs, G., and Capaccioli, M. (1977). In preparation.
King, I. R. (1961). Astron. J. **66,** 68.
Lynden-Bell, D. (1967). Mon. Not. R. Astron. Soc. **136,** 101.
Poveda, A. (1957). Bol. Obs. Tonantzintla No. 17.
Poveda, A., Iturriaga, R., and Orozco, I. (1960). Bol. Obs. Tonantzintla No. 20, p. 3.
Young, P. J., Shields, G. A., and Wheeler, J. C. (1976). Preprint.