THE DYNAMIC PROPERTIES OF A SIX-DIMENSIONAL, PSEUDO-RIEMANNIAN MANIFOLD

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1. ABSTRACT

Here we describe a six-dimensional, pseudo-Riemannian manifold with three dimensions of time and three dimensions of space, we assign it some simple rules, and then describe the resulting properties. Time is imaginary, space is real. Any two temporal dimensions form an imaginary plane. Each spatial dimension is orthogonal to the imaginary plane and the size (extent) of the spatial dimension is a function of the temporal coordinates and the angle between the temporal coordinate axes. We then discuss the properties that this manifold possesses when projected onto the spatial dimensions and given a single evolution parameter. Here we demonstrate that this three-dimensional projection expands with time and even accelerates as it evolves, independent of any energy or momentum contained therein.

Key words: cosmology: theory, cosmology: large-scale structure of the universe, cosmology: cosmological parameters, galaxies: kinematics and dynamics

2. REAL AND IMAGINARY

For the scope of this paper, real is that which can be measured directly: space, acceleration, and the square root of a positive area, for example, while imaginary is that which isn't real and, thus, can only be inferred: time, velocity, and the square root of a negative area, for example.

3. TIME

Imagine a line. Give it a distinct origin. This is time.

4. SQUARED TIME

Time, by itself, is unremarkable, but if you have two dimensions of time in an imaginary plane, then some interesting properties emerge. A third set of coordinates – a dimension of squared time – exists as a function of the temporal coordinates and the angle between the temporal coordinate axes. This can be expressed as:

$$y^M = (y_1^R i\tau^R + y_1^B i\tau^B + i\tau^R i\tau^B)\cos\theta^M$$
$$y^M = (y_1^B i\tau^B + y_1^R i\tau^R - \tau^B\tau^R)\cos\theta^M \tag{1}$$

Because squared time is a secondary dimension that depends on two primary dimensions, a chromatic index is used. For example, magenta squared time is the product of red time and blue time.

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The extent of the squared time dimension is y^M, where τ^R is the red time coordinate and τ^B is the blue time coordinate, y_1^B and y_1^R are constants with units of time, τ, and the angle between the temporal coordinate axes is θ^M. Note also that the cosine of this angle is the derivative $d\tau^R/d\tau^B$.

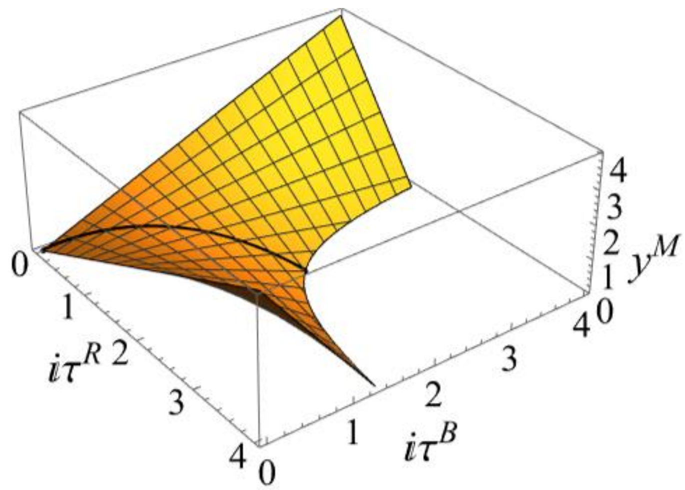

Figure 1 - The relationship between red time, blue time, and magenta squared time. The vertical axis is the size of the squared time dimension and is a function of the temporal coordinates and the angle between them. The black line describes the evolution of the magenta squared time dimension from the reference frame where red time and blue time advance at the same rate.

Squared time is real. It is the measurable surface of a deeper, more complex geometry that can only be inferred. We can see in Figure 1 that squared time will cycle once through an expansion and then a contraction. The surface of this manifold begins and ends in a singularity.

Also note that the expansion and contraction are quadratic. Objects at rest, in free fall, on the surface of this manifold, accelerate.

5. A METER

Even though squared time in Eq. (1) is real and can be measured directly, it still must be converted to a practical unit to make practical measurements.

$$y_1^R \equiv -\frac{i2v_1^R}{a_1}$$

$$y_1^B \equiv -\frac{i2v_1^B}{a_1}$$

$$x^M = \frac{a_1}{2} y^M$$

$$x^M = (v_1^B\tau^B + v_1^R\tau^R - \frac{a_1}{2}\tau^B\tau^R)\cos\theta^M \tag{2}$$

The extent of the magenta dimension, x^M is enumerated in a unit that we will generically refer to as a meter, where v_1^R and v_1^B are conversion factors between magenta space and red time and magenta space and blue time, respectively, and $\frac{a_1}{2}$ is the conversion factor between magenta space and squared time. A meter, then, is a practical method of measuring squared time.

6. A SECOND

Hold a meter stick in your hand. Space on this manifold can be measured with a meter stick. However, being pseudo-Riemannian, we must also measure the imaginary part. Now, imagine a ruler. Is it bigger or smaller than the meter stick in your hand? Is it twice as big, half as big? Exactly how do we objectively compare something real with something imaginary?

The general case is beyond the scope of this document, but we can simplify the problem by considering a special case where the two temporal axes are aligned. In this reference frame, the time coordinates advance synchronously such that $\tau^R = \tau^B = \tau$, $v_1^R = v_1^B = \frac{v_1}{2}$, and $\cos\theta^M = 1$.

We can now construct a coordinate system that is a proxy for blue time by giving it an extent that is, by definition, the same as magenta space, $x^B = x^M = x$. This will allow us to compare a change in spatial coordinates to a change in temporal coordinates provided we have an objective relationship between x and τ.

The extent of the spatial dimensions from Eq. (2) reduces to:

$$x = v_1\tau - \frac{a_1}{2}\tau^2 \tag{3}$$

From this, we obtain the relationship between space and time:

$$\frac{dx}{id\tau} = v_1 - a_1\tau$$

$$\frac{dx}{d\tau} = i(v_1 - a_1\tau) \tag{4}$$

$$\Delta\tau = \tau_1 - \tau_0$$

$$\Delta x = i\int_{\tau_0}^{\tau_1}(v_1 - a_1\tau)d\tau \tag{5}$$

A second, then, is the difference between a set of temporal coordinates, $\Delta\tau$, when projected onto space (or a proxy for space) yielding a change in spatial coordinates, Δx. This projection is a function of the tangent velocity of the manifold, $\frac{dx}{d\tau}$, at a given reference time, τ.

With this choice of a coordinate system, meters and seconds can be objectively compared to one another. In addition, meters are constant, seconds are variable and the length of a second decreases as a function of time.

This coordinate system possesses neither time symmetry nor Lorentz invariance.

7. TANGENT VELOCITY

We can define a distance in this manifold using the Pythagorean theorem.

$$\Delta s^2 = g_{\mu\nu} \Delta x^\mu \Delta x^\nu$$

Where Δs is the distance between two points, $g_{\mu\nu}$ is the metric tensor and Δx is the difference between a set of coordinates in each dimension. If we assume that, in the absence of energy and momentum, the meter and second coordinate axes are orthogonal to each other, then the magenta-blue distance, Δs^{MB}, simply expands to:

$$(\Delta s^{MB})^2 = (\Delta x^M)^2 + (\Delta x^B)^2 \tag{6}$$

The tangent velocity describes how space in the manifold changes with time. If we assume that all points in any frame on this manifold at a given time share the same tangent velocity, then we can describe the relationship between coordinate changes in a reference frame and coordinate changes in an arbitrary frame as:

$$\left(\frac{ds^{MB}}{d\tau}\right)^2 = -(v_1 - a_1\tau)^2 = \left(\frac{\partial x^M}{\partial \tau}\right)^2 + \left(\frac{\partial x^B}{\partial \tau}\right)^2 \tag{7}$$

Next, we assign the red axis to be the reference frame such that $\tau^R = \tau$, and allow the angle to the blue axis to vary as θ^M. Substituting Eq. (4) into Eq. (7) for the blue space partial derivative, and remembering that $\cos\theta^M = \frac{\partial \tau^R}{\partial \tau^B} = \frac{\partial \tau}{\partial \tau^B}$, we can solve for the magenta space term, ∂x^M:

$$\partial x^M = (v_1 - a_1\tau)\sqrt{\tan^2\theta^M}\,\partial\tau$$

The angle, θ^M, can now be expressed as a function of the relative velocity, v:

$$-iv \equiv \frac{dx^M}{id\tau^B}$$

$$v = \frac{dx^M}{d\tau^B}$$

$$\theta^M = \sin^{-1}\frac{v}{v_1 - a_1\tau}$$

From this we find the relationship between a change in time in an arbitrary frame and a change in time in the reference frame as a function of relative velocity:

$$d\tau^B = \frac{1}{\sqrt{1 - \frac{v^2}{(v_1 - a_1\tau)^2}}}\,d\tau \tag{8}$$

All points on the surface of this manifold at a given time share the same absolute tangent velocity, $i(v_1 - a_1\tau)$. The sum of the squares of the velocities of meters and seconds will always be equal to square of the tangent velocity.

8. TWO-DIMENSIONAL METRIC FORMULA

We can construct a metric formula by substituting Eq. (5) into Eq. (6) for the blue space term, Δx^B, and then deriving an expression for the magenta space term, Δx^M.

$$(\Delta s^{MB})^2 = (\Delta x^M)^2 - \left(\int_{\tau_0}^{\tau_1} (v_1 - a_1\tau)d\tau\right)^2 \tag{9}$$

Where τ_1 is the time of an observation, and τ_0 is the arbitrary time of some event. However, we have no way to directly measure Δx^M in expanding space. The schematic of Figure 2 illustrates that the only values that can be measured directly are Δx_0^M and Δx_1^M. We need a method to relate the coordinate distance, Δx_1^M, to the line element, Δx^M.

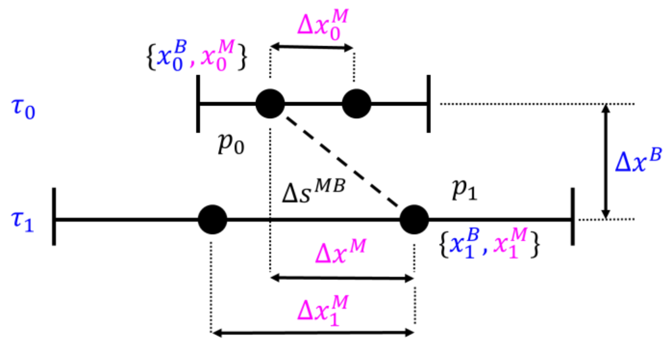

Figure 2 - The interval, Δs^{MB}, from p_0 to p_1 in quadratically expanding space showing the line elements, Δx^M and Δx^B.

Using Eq. (2), the relation between Δx_0^M and Δx_1^M, as the manifold expands, is:

$$\frac{\Delta x_0^M}{x_0^M} = \frac{\Delta x_1^M}{x_1^M}$$

$$\Delta x_0^M = -\frac{(2v_1\tau_0 - a_1\tau_0^2)}{\tau_1(-2v_1 + a_1\tau_1)}\Delta x_1^M \tag{10}$$

Where x_0^M is the spatial extent at τ_0 and x_1^M is the spatial extent at τ_1. Next, we will observe that the line element, Δx^M, is the coordinate distance, Δx_1^M, less one half of expansion (expressed as $\Delta x_1^M - \Delta x_0^M$):

$$\Delta x^M = \Delta x_1^M - \frac{1}{2}(\Delta x_1^M - \Delta x_0^M)$$

$$\Delta x^M = \frac{a_1(\tau_0^2 + \tau_1^2) - 2v_1(\tau_0 + \tau_1)}{2\tau_1(a_1\tau_1 - 2v_1)}\Delta x_1^M \tag{11}$$

The metric formula for a two-dimensional distance using practical coordinates is then:

$$(\Delta s^{MB})^2 = \left(\frac{a_1(\tau_0^2 + \tau_1^2) - 2v_1(\tau_0 + \tau_1)}{2\tau_1(a_1\tau_1 - 2v_1)}\right)^2 (\Delta x_1^M)^2 - \left(\int_{\tau_0}^{\tau_1} (v_1 - a_1\tau)d\tau\right)^2 \tag{12}$$

9. FOUR-DIMENSIONAL METRIC FORMULA

We now consider the permutations of space and time. Two dimensions of time result in only one spatial dimension, so any discussion would be of limited value. Three dimensions of time result in three dimensions of space, one for each imaginary plane. This option agrees with the observed inverse-square law, so let us put a pin in it. If we had four dimensions of time, then we would observe six dimensions of space. We do not, so this option also shows little promise.

Having considered the permutations, we will focus the discussion on a manifold with three imaginary dimensions of time and three real dimensions of space. We are going to label them as red time, τ^R, green time, τ^G, and blue time, τ^B due to the way they produce secondary dimensions. The product of blue time and green time is cyan space, x^C. The product of green time and red time is yellow space, x^Y. The additional line elements are:

$$(\Delta s^{CG})^2 = \left(\frac{a_1(\tau_0^2 + \tau_1^2) - 2v_1(\tau_0 + \tau_1)}{2\tau_1(a_1\tau_1 - 2v_1)}\right)^2 (\Delta x_1^C)^2 - \left(\int_{\tau_0}^{\tau_1} (v_1 - a_1\tau)d\tau\right)^2 \tag{13}$$

$$(\Delta s^{YR})^2 = \left(\frac{a_1(\tau_0^2 + \tau_1^2) - 2v_1(\tau_0 + \tau_1)}{2\tau_1(a_1\tau_1 - 2v_1)}\right)^2 (\Delta x_1^Y)^2 - \left(\int_{\tau_0}^{\tau_1} (v_1 - a_1\tau)d\tau\right)^2 \tag{14}$$

These line elements combine to give us the metric formula for the six-dimensional distance:

$$\Delta s^2 = (\Delta s^{MB})^2 + (\Delta s^{CG})^2 + (\Delta s^{YR})^2$$

$$\begin{aligned}\Delta s^2 = &\left(\frac{a_1(\tau_0^2 + \tau_1^2) - 2v_1(\tau_0 + \tau_1)}{2\tau_1(a_1\tau_1 - 2v_1)}\right)^2 (\Delta x_1^M)^2 - \left(\int_{\tau_0}^{\tau_1} (v_1 - a_1\tau)d\tau\right)^2 \\ &+ \left(\frac{a_1(\tau_0^2 + \tau_1^2) - 2v_1(\tau_0 + \tau_1)}{2\tau_1(a_1\tau_1 - 2v_1)}\right)^2 (\Delta x_1^C)^2 - \left(\int_{\tau_0}^{\tau_1} (v_1 - a_1\tau)d\tau\right)^2 \\ &+ \left(\frac{a_1(\tau_0^2 + \tau_1^2) - 2v_1(\tau_0 + \tau_1)}{2\tau_1(a_1\tau_1 - 2v_1)}\right)^2 (\Delta x_1^Y)^2 - \left(\int_{\tau_0}^{\tau_1} (v_1 - a_1\tau)d\tau\right)^2\end{aligned} \tag{15}$$

We can simplify this formula by recognizing that v_1 and a_1 can be combined into three-plane aggregates (the subscript indicates the number of temporal planes):

$$a_3 \equiv \sqrt{3}a_1$$

$$v_3 \equiv \sqrt{3}v_1$$

In addition, the constraints of our reference frame result in only four independent parameters to this metric formula, not six. With these constraints and substitutions, the metric formula for a three-dimensional projection of a six-dimensional manifold with a single evolution parameter is:

$$\begin{aligned}\Delta s^2 = &\left(\frac{a_3(\tau_0^2 + \tau_1^2) - 2v_3(\tau_0 + \tau_1)}{2\tau_1(a_3\tau_1 - 2v_3)}\right)^2 ((\Delta x_1^M)^2 + (\Delta x_1^Y)^2 + (\Delta x_1^C)^2) \\ &- \left(\int_{\tau_0}^{\tau_1} (v_3 - a_3\tau)d\tau\right)^2\end{aligned} \tag{16}$$

Where Δs is the distance between two points, τ_0 and τ_1 are the temporal coordinates of those points, and Δx_1^M, Δx_1^Y, and Δx_1^C, are the spatial distances.

10. INITIAL CONDITIONS

There are three initial conditions in Eq. (16): the constant acceleration, a_3, the initial tangent velocity, v_3, and the age of the manifold at the time of observation, τ_1.

If we chose a line-of-sign path along a null geodesic (that is, a path having $(\Delta x^M)^2 + (\Delta x^B)^2 = 0$ and $\Delta x_1^Y = \Delta x_1^C = 0$) then the distance between two points can be found by solving Eq. (16) for Δx_1^M:

$$\mathrm{D}(\tau_0, \tau_1) = \Delta x_1^M = i\,\frac{(\tau_0 - \tau_1)\tau_1\big(-2v_3 + a_3(\tau_0 + \tau_1)\big)||-2v_3 + a_3\tau_1||}{\sqrt{-(-2v_3(\tau_0 + \tau_1) + a_3(\tau_0^2 + \tau_1^2))^2}}$$

This formula can be simplified by encoding the time coordinates as:

$$z = \frac{x_1^M - x_0^{\mathrm{M}}}{x_0^M}$$

$$\tau_0 = \frac{v_3 + v_3 z - \sqrt{(1 + z)(v_3^2 + v_3^2 z - 2a_3 v_3 \tau_1 + a_3^2 \tau_1^2)}}{a_3 + a_3 z}$$

$$\mathrm{D}(z) = \frac{z\tau_1(2v_3 - a_3\tau_1)}{2 + z} \tag{17}$$

Where z, the redshift, is the change in a unit length (wavelength) from τ_0 to τ_1. This redshift value is encoded in photons, which we assume to travel along the null geodesic, making it possible to use a volume of photons from a known source (of luminosity) as proxies for luminous distance markers. As a function of redshift, the luminous distance is:

$$\mathrm{D_L}(z) = \frac{z\tau_1(2v_3 - a_3\tau_1)}{2 + z}(1 + z) \tag{18}$$

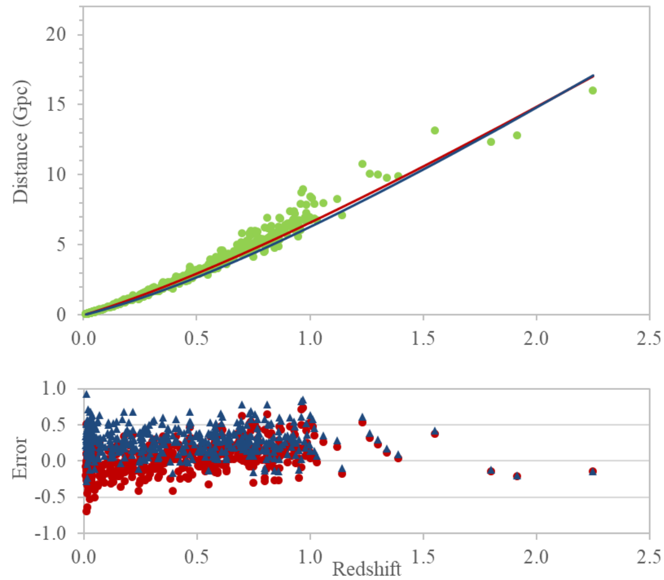

Figure 3 - Top: The luminous distances to a selection of 482 SNe Ia supernovae (green) and the distance predicted by the quadratically expandig space metric formula (red) and, for comparison, the distance predicted by the FLRW metric formula (blue)[2]. Bottom: The difference between the predicted luminous distance and the observed value in Gpc.

Using the combined data from (Conley et al. 2010), (Rodney et al. 2012), (Jones et al. 2013), (Rodney et al. 2016) and the parameters from (Rodney et al. 2016) to normalize the sets, we can extract the initial conditions using a chi-square minimization algorithm. The supernovae data demonstrates that this model is a better match to the observed universe than FLRW, with fewer free parameters, and provides us with a fiduciary model with which we can continue our discussion.

a_3	$3.65 \times 10^{-11}\ m\ s^{-2}$
v_3	$3.18 \times 10^{8}\ m\ s^{-1}$
τ_1	$4.95 \times 10^{17} s$

Table 1– The initial conditions of the fiduciary model.

11. MOTION

A particle on the surface of this manifold in free fall will accelerate. The equation describing this motion can be derived from Eq. (4):

$$\frac{d^2\vec{x}}{d\tau^2} = \frac{d\vec{u}}{d\tau} = \overrightarrow{a_3} \tag{19}$$

$$\frac{d(u^\alpha \overrightarrow{e_\alpha})}{d\tau} = a_3^\alpha \overrightarrow{e_\alpha}$$

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[2] As per the parameters found in (Planck Collaboration et al. 2019)

This can be expanded with the chain rule:

$$u^\alpha \frac{d\overrightarrow{e_\alpha}}{d\tau} + \frac{du^\alpha}{d\tau}\overrightarrow{e_\alpha} = a_3^\alpha \overrightarrow{e_\alpha}$$

$$\frac{d\overrightarrow{e_\alpha}}{d\tau} = \Gamma_{\alpha\beta}^{\gamma} u^\beta \overrightarrow{e_\gamma}$$

$$\Gamma_{\alpha\beta}^{\gamma} u^\alpha u^\beta \overrightarrow{e_\gamma} + \frac{du^\alpha}{d\tau}\overrightarrow{e_\alpha} = a_3^\alpha \overrightarrow{e_\alpha}$$

$$\Gamma_{\mu\nu}^{\alpha} u^\mu u^\nu + \frac{du^\alpha}{d\tau} = a_3^\alpha \tag{20}$$

Where $\frac{du^\alpha}{d\tau}$ is the proper acceleration of an object, and $\Gamma_{\mu\nu}^{\alpha} u^\mu u^\nu$ is the acceleration caused by a fictitious force (that is, the change in the basis vector with time). Our geodesic equation is:

$$\frac{du^\alpha}{d\tau} = a_3^\alpha - \Gamma_{\mu\nu}^{\alpha} u^\mu u^\nu \tag{21}$$

In non-relativistic domains, the equation for the motion of a free-falling particle with a mass of m in the presence of a collection of fictitious forces, $\sum F$, is:

$$\sum_{\mathrm{i}} F_i^\alpha \overrightarrow{e_\alpha} = \frac{dP^\alpha}{d\tau}\overrightarrow{e_\alpha}$$

$$\sum_{\mathrm{i}} F_i^\alpha \overrightarrow{e_\alpha} = \mathrm{m}\frac{du^\alpha}{d\tau}\overrightarrow{e_\alpha}$$

$$\sum_{\mathrm{i}} F_i^\alpha \overrightarrow{e_\alpha} = \mathrm{m}(a_3^\alpha - \Gamma_{\mu\nu}^{\alpha} u^\mu u^\nu)\overrightarrow{e_\alpha} \tag{22}$$

Where P is the momentum. We can derive a formula for orbital motion in a gravitational field by replacing the general terms of Eq. (22) with more specific terms.

$$-\frac{GMm}{r^2}\vec{r} = m\left(a_3 - \frac{v^2}{r}\right)\vec{r}$$

$$\frac{v^2}{r} = a_3 + \frac{GM}{r^2}$$

$$v = \sqrt{a_3 r + \frac{GM}{r}} \tag{23}$$

Where G is the gravitational constant, M is the mass within a radius, r, and v is the tangential velocity of an object at the given radius.

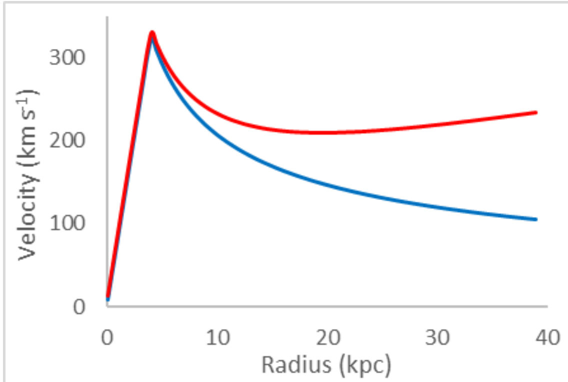

Figure 4 –The velocity curves of orbital motion around a spherical mass of $10^{11} M_{\odot}$ having a radius of 4 kpc and constant density, for our fiducial model in quadratically expanding space (red), and Newtonian dynamics (that is, $a_3 = 0$) (blue).

12. FUNDAMENTAL PLANE

Eq. (23) defines relationship between the tangential velocity, the radius of orbiting objects and the enclosed mass. This formula can be rearranged to predict the maximum mass possible in gravitationally bound orbit.

$$M = \frac{(v^2 - a_3 r)r}{G} \tag{24}$$

The radius where the maximum mass will be found is:

$$M' = \frac{(v^2 - a_3 r)r}{G}\frac{d}{dr} = 0$$

$$\frac{v^2 - 2a_3 r}{G} = 0$$

$$r = \frac{v^2}{2a_3} \tag{25}$$

Substituting the Eq. (25) back into Eq. (24) yields the formula for the maximum mass given the velocity:

$$M_{MAX} = \frac{\left(v^2 - a_3 \frac{v^2}{2a_3}\right)\frac{v^2}{2a_3}}{G}$$

$$M_{MAX} = \frac{1}{4a_3 G} v^4 \tag{26}$$

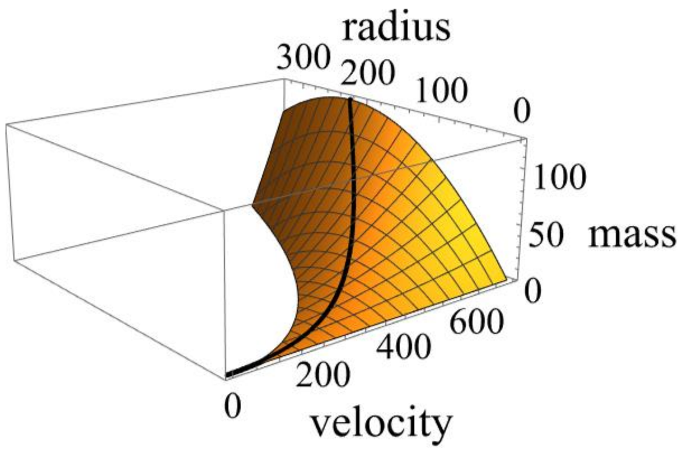

Figure 5 - The fundamental plane where the radius is in kpc, the velocity is in $km\ s^{-1}$, and the mass is in $10^{11}\ M_{\odot}$. Solid line is the maximum mass allowed for orbital motion in quadratically expanding space.

A study of the relation between velocity and mass was conducted in (McGaugh 2012). Employing a χ^2 minimization algorithm on Eq. (26) and solving for a_3, we find a value of $3.64 \times 10^{-11}\ m\ s^{-2}$, which we used earlier to define our fiducial model. The combined stellar and gas masses of a collection of gas-rich spiral galaxies are displayed in Figure 6 and overlaid with Eq. (26) using our fiducial model.

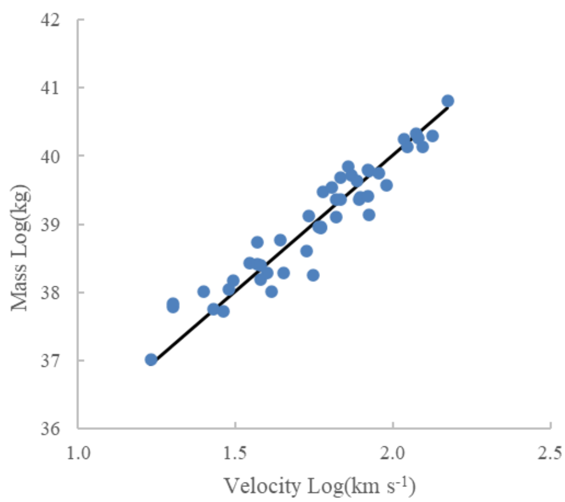

Figure 6 - The relationship between tangential velocity and mass. The blue circles are the combined gas and stellar mass of the gas-rich galaxies and the solid line is the maximum mass allowed by quadratically expanding space for orbital motion.

13. CONCLUSION

Here we have described a complex geometry with dynamic properties that are independent of any content. Some of those properties are:

- Space begins and ends in a singularity

- Space expands quadratically, free-falling objects accelerate at a rate of a_3
- The tangent velocity of the manifold is the absolute limit of the relative velocity
- Orbital rotation curves will appear to flatten in the domain of a_3
- The maximum mass is proportional to the fourth power of velocity in orbital systems

Here we have also provided evidence that this is more than an academic exercise. The observed universe possesses these properties suggesting we occupy a six-dimensional, pseudo-Riemannian manifold with three observable dimensions of space and a single evolution parameter corresponding the three hidden dimensions of time.

14. DATA AVAILIBILITY

The data and Mathematica notebooks underlying this article are available at https://github.com/DonaldAirey/quadratically-expanding-space.

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